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A functional relation in stable knot theory

John R. Klein

Universität Gesamthochschule Siegen, Fachbereich Mathematik, Hölderlinstrasse 3, W-5900 Siegen, Federal Republic of Germany

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Introduction

Let $V^{n+1} \subset S^{n+2}$ be a Seifert surface for a knot $K: S^n \subset S^{n+2}$, and let C denote the exterior of V. Pushing V into C along the two different unit normal frames defines a pair of maps $p_+, p_-: V \to C$. It is an easy consequence of the Mayer-Vietoris sequence of the triple (S^{n+2}, V, C) that $p_{+*} - p_{-*}: H_*(V) \to H_*(C)$ is an isomorphism. Let $\theta: V \wedge V \to S^{n+1}$ be the homotopy Seifert pairing of $V \subset S^{n+2}$, which is defined to be the composite of $\mathrm{id}_V \wedge p_+: V \wedge V \to V \wedge C$ followed by the canonical Spanier-Whitehead duality map $d: V \wedge C \to S^{n+1}$. The map θ is the homotopy theoretic version of the Seifert form. Farber proves that if V is r-connected, with $r \geqq (n+1)/3$ and $n \geqq 5$, then the isotopy class of $V \subset S^{n+2}$ is determined by the "isometry class" of θ [F].

For example, if $S^n \subset S^{n+2}$ is a fibred knot (i.e., the exterior fibres smoothly over the circle), then its fibre is a canonical Seifert surface, and there is consequently a canonical homotopy Seifert pairing for it. Hence, Farber's theorem is a complete classification of fibred n-knots whose fibres are $(n+1)/3$-connected with $n \geqq 5$. This result was later extended to one dimension better ($3r \geqq n$) by Richter [R]. We shall say that V is *stable* if the condition $3r \geqq n$ holds.

It is the purpose of this paper to establish a formula expressing the homotopy class of the inclusion $S^n \subset V$ in terms of the homotopy Seifert pairing of stable Seifert surfaces (Theorem 3.1). The same result was obtained by Richter using a different argument involving the addition/composition formulae for generalized Hopf invariants (in fact, Richter appealed to these formulae in his homotopy theoretic proof of Farber's theorem). After deducing the main result, we state a general conjecture which we hope is valid outside of the stable range. We then interpret this conjecture in the *metastable* range: $4r \geqq n+1$.

We remark that there is nothing sacred about the assumption that the boundary of V is a standard sphere; all of our results hold for homotopy spherical boundaries as well.

By Poincaré duality, if V is stable then the homology dimension of V is less than or equal to $2r$. Consequently, the Freundenthal suspension theorem implies that V desuspends uniquely up to homotopy. In particular, V has a unique comultiplication (up to homotopy) which we shall denote by $+: V \to V \vee V$. It therefore makes sense to speak of the map $p_+ - p_- : V \to C$.

Lemma 0.1. *If* $\operatorname{conn}(V) = r \geqq 1$, *then* $p_+ - p_-$ *is a homotopy equivalence.*

Proof. Since C is also simply connected, this is just the Whitehead theorem. □

1 The flat product

We shall work entirely within the category of spaces which are the homotopy type of a CW complex.

For pointed spaces X and Y, let $X \flat Y$ be the homotopy fibre of the inclusion $X \vee Y \subset X \times Y$. Let $W_{X,Y}: X \flat Y \to X \vee Y$ denote the canonical map of the homotopy fibre into the total space. We will call $W_{X,Y}$ the *universal Whitehead product*. If $f: X \to X'$ and $g: Y \to Y'$ are pointed maps, then the universal Whitehead product satisfies the naturality property

$$(f \vee g) \circ W_{X,Y} = W_{X',Y'} \circ f \flat g, \tag{1.1}$$

where $f \flat g: X \flat Y \to X' \flat Y'$ is the obvious map. If $X = Y$, we let $w_X: X \flat X \to X$ denote the composition of the fold map $X \vee X \to X$ with $W_{X,X}$. We shall need the following fact, which is a special case of the Blakers-Massey excision theorem.

Lemma 1.2 (cf. [G]). *There is a natural map* $N_{X,Y}: X \flat Y \to \Omega(X \wedge Y)$ *which is*

$$\min(\operatorname{conn}(X), \operatorname{conn}(Y)) + \operatorname{conn}(X) + \operatorname{conn}(Y) + 1$$

connected.

2 The dual homotopy Seifert pairing

Let $\theta: V \wedge V \to S^{n+1}$ be the homotopy Seifert pairing associated with a Seifert surface $V \subset S^{n+2}$. Up to sign, the S-dual to θ is the map $\theta^*: S^{n+1} \to C \wedge C$ which may be described as follows (cf. [K-S]):

Let $D(V) = V_+ \cup V_-$ be the double of V. Consider the map $t: D(V) \to V \times C$ given by the rule

$$t(v) = \begin{cases} (v, p_+(v)), & \text{if } v \in V_+, \text{ and} \\ (v, p_-(v)), & \text{if } v \in V_-. \end{cases}$$

Let $D_0(V)$ be the space obtained from $D(V)$ by removing the top cell. Since the inclusion $V \vee C \subset V \times C$ is $(2r+1)$-connected and since $D_0(V)$ has homology dimension $\leqq 2r$ it follows from elementary obstruction theory that we can homotop t to a map $t_1: D(V) \to V \times C$ such that t_1 maps $D_0(V)$ into $V \vee C$. Let $d^*: S^{n+1} \to V \wedge C$ denote the induced map of quotients,

$$S^{n+1} = D(V)/D_0(V) \xrightarrow{t_1} V \times C / V \vee C = V \wedge C.$$

(In fact, d^* is a Spanier-Whitehead duality [K-S, K].) Then $\theta^*: S^{n+1} \to C \wedge C$ is defined to be the composition $(p_- \wedge \mathrm{id}_C) \circ d^*$.

Let $\operatorname{adj}(d^*): S^n \to \Omega(V \wedge C)$, $\operatorname{adj}(\theta^*): S^n \to \Omega(C \wedge C)$ be the adjoints to d^* and θ^*. By Lemma 1.2, there are maps (unique up to homotopy) $D: S^n \to V \flat C$ and $\Theta: S^n \to C \flat C$ such that $N_{X,Y} \circ D \simeq \operatorname{adj}(d^*)$ and $N_{X,Y} \circ \Theta \simeq \operatorname{adj}(\theta^*)$. Note by definition the relation which exists between D and Θ:

(2.1) $$\Theta \simeq (p_- \flat \mathrm{id}_C) \circ D,$$

where $p_- \flat \mathrm{id}_C: V \flat C \to C \flat C$.

3 The inclusion of the boundary

Let $\alpha: S^n \subset V$ denote the inclusion of the boundary.

Theorem 3.1. *If V is stable ($3r \geqq n$), then the following relation holds:*

$$\alpha \simeq (p_+ - p_-)^{-1} \circ w_C \circ (-\Theta),$$

where $(p_+ - p_-)^{-1}$ is a homotopy inverse for $p_+ - p_-$.

Proof. Identify $D(V)$ with the boundary of a tubular neighborhood of V in S^{n+2}. The attaching map for the top cell of $D(V)$ is then the composite

$$S^n \xrightarrow{(1,-1)} S^n \vee S^n \xrightarrow{\alpha \vee \alpha} V \vee V \simeq D_0(V),$$

where $(1, -1)$ is the map which is degree one on the first factor and degree minus one on the second factor. Note that

$$\mathrm{pr}_1 \circ (\alpha \vee \alpha) \circ (1, -1) = \alpha,$$

where $\mathrm{pr}_1: V \vee V \to V$ is the projection onto the first factor.

By [K, 5.3], $D(V)$ may be also identified with $(V \vee C) \cup_\varrho D^{n+1}$, where $\varrho: S^n \to V \vee C$ is the composite

$$S^n \xrightarrow{D} V \flat C \xrightarrow{w_{V,C}} V \vee C.$$

We shall for the reader's convenience prove part of this assertion: Since S^{n+2} may be identified with the pushout of

$$V \xleftarrow{\mathrm{id}_V \cup \mathrm{id}_V} D(V) \xrightarrow{p_+ \cup p_-} C,$$

it follows that the pushout of

(*) $$V \xleftarrow{\mathrm{id}_V \vee \mathrm{id}_V} V \vee V \xrightarrow{p_+ \vee p_-} C,$$

is contractible, since $V \vee V$ may be identified with the punctured double, $D_0(V)$. Now a pair of maps $X \leftarrow A \to Y$ of 1-connected spaces has contractible pushout iff A is *homologically* the wedge of X and Y by the Mayer-Vietoris sequence. In the above case, the connectivity hypothesis furthermore implies, in fact, that A is *homotopically* the wedge of X and Y.

To see this, consider the map $\sigma: V \vee V \to V \times C$ which on the first factor of the wedge is

$$(\mathrm{id}_V, p_+): V \to V \times C$$

and which on the second factor of the wedge is (id_V, p_-). Since $V \vee V$ has homology dimension $2r$ ($r = \operatorname{conn}(V)$), and since the inclusion $V \vee C \subset V \times C$ is $(2r+1)$-connected, it follows by obstruction theory that σ is homotopic to a map which factors through $V \vee C$, moreover this map is unique up to homotopy. Denote the factorization by $\tau: V \vee V \to V \vee C$. Now τ upon taking homology yields the isomorphism which appears in the Mayer-Vietoris sequence associated to the diagram (*) above. Hence, τ is a homotopy equivalence by the Whitehead theorem.

Since $D(V)$ is obtained from $D_0(V)$ by attaching a top cell, it follows from the splitting argument that $D(V) \simeq (V \vee C) \cup D^{n+1}$. We leave it as an excercise to the reader to prove that the top cell is actually attached along the map $\varrho: S^n \to V \vee C$ given above. The basic idea is that ϱ followed by the inclusion $i: V \vee C \subset V \times C$ has a canonical null homotopy, since the $i \circ \tau$ must extend to the double $D(V)$. Therefore ϱ factors as $W_{V,C} \circ ?$, and the reader must show that $? = D$.

We now use the comultiplication on $V \vee V$ to invert τ; this will lead to the formula for α: An easy calculation shows that the two splittings $D_0(V) \simeq V \vee V$ and $D_0(V) \simeq V \vee C$ are equated by the homotopy equivalence $A: V \vee C \to V \vee V$ (the homotopy inverse of τ) which is given by the 2×2 matrix of maps

$$A = \begin{pmatrix} -(p_+ - p_-)^{-1} \circ p_- & (p_+ - p_-)^{-1} \\ (p_+ - p_-)^{-1} \circ p_+ & -(p_+ - p_-)^{-1} \end{pmatrix}.$$

Under this identification ϱ and α are easily seen to satisfy the relation

$$\mathrm{pr}_1 \circ A \circ \varrho \simeq \alpha.$$

But $\mathrm{pr}_1 \circ A$ is just the 1×2 matrix of maps

$$(-(p_+ - p_-)^{-1} \circ p_- \; (p_+ - p_-)^{-1}) = (p_+ - p_-)^{-1} \circ (-p_- \; \mathrm{id}_C),$$

and hence,

$$\alpha \simeq (p_+ - p_-)^{-1} \circ (-p_- \; \mathrm{id}_C) \circ \varrho = (p_+ - p_-)^{-1} \circ (-p_- \; \mathrm{id}_C) \circ W_{V,C} \circ D.$$

On the other hand, $(-p_- \; \mathrm{id}_C) \circ W_{V,C} = w_C \circ ((-p_-) \flat \mathrm{id}_C)$ by (1.1). Substituting this into the above, we get

$$\alpha \simeq (p_+ - p_-)^{-1} \circ w_C \circ ((-p_-) \flat \mathrm{id}_C) \circ D.$$

Finally, we have by (2.1), $\Theta \simeq (p_- \flat \, \mathrm{id}_C) \circ D$, and therefore,

$$-\Theta \simeq ((-p_-) \flat \mathrm{id}_C) \circ D,$$

as the reader may easily check. This yields the desired relation,

$$\alpha \simeq (p_+ - p_-)^{-1} \circ w_C \circ (-\Theta). \quad \square$$

4 A conjecture

Suppose that K is a space and that a map $\phi: S^{n+1} \to K \wedge K$ is given. We say that f is a *(dual) homotopy Seifert pairing* for K if

$$d := \phi + (-1)^{n+1} \circ T \circ \phi$$

is an S-duality, where $T: K \wedge K \to K \wedge K$ is the map which interchanges factors (cf. [F, 1.4]).

By a *Ganea-Seifert triad* (of dimension n) for K, we mean a triple (q_+, q_-, Θ) where,

(1) $q_{\pm}: K \to K$ are maps;

(2) $\Theta: S^n \to K \flat K$ is a map such that $N_{K,K} \circ \Theta: S^n \to \Omega(K \wedge K)$ is adjoint to a homotopy Seifert pairing for K;

(3) with respect to the S-duality $d: S^{n+1} \to K \wedge K$ associated with (2), the S-dual of q_- is

$$\mathrm{adj}(N_{K,K} \circ \Theta): S^{n+1} \to K \wedge K,$$

and the S-dual of q_+ is

$$(-1)^{n+1} T \circ \mathrm{adj}(N_{K,K} \circ \Theta),$$

[i.e., $(q_- \wedge \mathrm{id}_K) \circ d \simeq \mathrm{adj}(N_{K,K} \circ \Theta)$, etc.].

There is also the notion of *isometry* of Ganea-Seifert triads – the definition of this we leave to the reader.

Lemma 4.1. *If $V^{n+1} \subset S^{n+2}$ is a 1-connected Seifert surface having the structure of a* co-H space, then there is a canonical Ganea-Seifert triad *(of dimension n) up to isometry associated with V.*

Proof. We set $K = V$, $q_+ = (p_+ - p_-)^{-1} \circ p_+$, and $q_- = (p_+ - p_-)^{-1} \circ p_-$. We define $\Theta: S^n \to K \flat K$ as follows:

As in Theorem 3.1, identify the double $D(V)$ with the boundary of a tubular neighborhood of V and let $D_0(V) \simeq V \vee V$ be the punctured double, i.e. the space obtained from $D(V)$ by removing the top cell. Then $D_0(V)$ is a co-H space. Let C be the exterior of V. Then there is a canonical equivalence $D_0(V) \simeq V \vee C$ defined by co-adding the fold map $V \vee V \to V$ with the map $p_+ \vee p_-: V \vee V \to C$. The attaching map $S^n \to D_0(V)$ for the top cell of $D(V)$ with respect to this equivalence factors as $W_{V,C} \circ D$, where $D: S^n \to V \flat C$ satisfies the condition that $N_{V,C} \circ D$ is adjoint to an S-duality [K, 5.3]. We then define $\Theta: S^n \to V \flat V$ to be the composition

$$S^n \xrightarrow{D} V \flat C \xrightarrow{q_- \flat (p_+ - p_-)^{-1}} V \flat V.$$

We now sketch a proof that the triple (q_+, q_-, Θ) has the desired properties. To prove (2), note that

$$\begin{aligned} N_{V,V} \circ \Theta &= N_{V,V} \circ q_- \flat (p_+ - p_-)^{-1} \circ D, \\ &\simeq \Omega(q_- \wedge (p_+ - p_-)^{-1}) \circ N_{V,C} \circ D, \quad \text{by Lemma 1.2}. \end{aligned}$$

Hence, by taking adjoints we infer that

$$\begin{aligned} \mathrm{adj}(N_{V,V} \circ \Theta) &\simeq (q_- \wedge (p_+ - p_-)^{-1}) \circ N_{V,C} \circ D \\ &= ((p_+ - p_-)^{-1} \wedge (p_+ - p_-)^{-1}) \circ (p_- \wedge \mathrm{id}_C) \circ D \\ &= ((p_+ - p_-)^{-1} \wedge (p_+ - p_-)^{-1}) \circ \tilde{\Theta}, \end{aligned}$$

where $\tilde{\Theta} := (p_- \wedge \mathrm{id}_C) \circ D$ is the dual homotopy Seifert pairing of $V \subset S^{n+2}$ in the sense of Sect. 2. Since $(p_+ - p_-)^{-1}$ is a homotopy equivalence, condition (2) will be satisfied if

$$\tilde{\Theta} + (-1)^{n+1} \circ T \circ \tilde{\Theta}$$

is an S-duality map. But this in fact follows from [F, 1.4] (or rather its S-dual version).

To prove (3), we may simplify things and identify V again with C using the equivalence $p_+ - p_-$. Under this identification, the first part of (3) is equivalent to showing that

$$\mathrm{adj}(N_{C,C}\circ\Theta):=\mathrm{adj}(N_{C,C}\circ(p_-\flat\mathrm{id}_C)\circ D)$$

is S-dual to p_- (with respect to the duality map $d_{V,C}:=\mathrm{adj}(N_{V,C}\circ D):S^{n+1}\to\Sigma V\wedge C$).

By naturality (1.2), this is the same as

$$\mathrm{adj}(\Omega(p_-\wedge\mathrm{id}_C)\circ N_{V,C}\circ D)=(p_-\wedge\mathrm{id}_C)\circ d_{V,C},$$

and hence the first part of (3) is established. The last part of (3) follows by a similar argument using the fact that p_+ and p_- satisfy the equation

$$(-1)^{n+1}\circ T\circ(p_-\wedge\mathrm{id}_C)\circ d_{V,C}\simeq(p_+\wedge\mathrm{id}_C)\circ d_{V,C},$$

(see [F, 1.4]). □

We now propose the following conjecture:

Conjecture 4.2. (1) (Existence). *If K is a 1-connected* co-H *space and $S=(q_+,q_-,\Theta)$ is a Ganea-Seifert triad of dimension $n\geqq 5$ for K, then there is a Seifert surface $V^{n+1}\subset S^{n+2}$ with ∂V a homotopy sphere Σ^n, such that $V\simeq K$, and such that the Ganea-Seifert triad associated with V* (cf. 4.1) *is isometric to S.*

(2) (Uniqueness). *Let V^{n+1} and W^{n+1} be Seifert surfaces (with homotopy spherical boundaries) in the sphere S^{n+2}. Additionally, assume that V and W are 1-connected and have the structure of a* co-H *space. Then V and W are isotopic in S^{n+2} if and only if their associated Ganea-Seifert triads are isometric.*

We remark that this conjecture generalizes the statements of the theorems of Farber [F].

5 Interpretation of Conjecture 4.2 in the metastable range

It is our intention in this section to give the data of Sect. 4 a simpler description under a metastable type connectivity restriction.

If K is an r-connected co-H space, then we say that a Ganea-Seifert triad $S=(q_+,q_-,\Theta)$ of dimension n for K is *metastable* if $4r\geqq n+1$. If the conditions of Conjecture 4.2(1) hold and if S is metastable, then Poincaré duality implies that K has homology dimension $\leqq 3r-1$. Consequently, by [G, 3.6], it follows that there is a space Y and a primitive equivalence of co-H spaces $\Sigma Y\simeq K$, i.e., K desuspends.

We now use the Hilton-Milnor decomposition of $K\flat K$ as an infinite wedge (see e.g. [G] for this computation):

$$K\flat K\simeq\Sigma((\Sigma^{-1}K)^{\wedge 2}\vee 2(\Sigma^{-1}K)^{\wedge 3}\vee 3(\Sigma^{-1}K)^{\wedge 4}\vee\ldots\vee j(\Sigma^{-1}K)^{\wedge(j+1)}\vee\ldots)).$$

By obstruction theory, the inclusion of terms of smash order $\leqq 3$ is $(4r+3)$-connected. As $4r\geqq n+1$ by hypothesis, we infer that $\Theta:S^n\to K\flat K$ is determined by a map

$$\Theta':S^n\to\Sigma^{-1}K\wedge K\vee\Sigma^{-2}K\wedge K\wedge K\vee\Sigma^{-2}K\wedge K\wedge K.$$

By obstruction theory again, the inclusion

$$\Sigma^{-1}K \wedge K \vee \Sigma^{-2}K \wedge K \wedge K \vee \Sigma^{-2}K \wedge K \wedge K$$
$$\subset \Sigma^{-1}K \wedge K \times \Sigma^{-2}K \wedge K \wedge K \times \Sigma^{-2}K \wedge K \wedge K$$

is more than n-connected. Consequently, $\Theta : S^n \to K \flat K$ is determined by three maps

$$\Theta_1 : S^n \to \Sigma^{-1}K \wedge K, \quad \Theta_2 : S^n \to \Sigma^{-2}K \wedge K \wedge K, \quad \text{and}$$
$$\Theta_3 : S^n \to \Sigma^{-2}K \wedge K \wedge K.$$

Note that each of these maps is in the stable range. It can be shown that $\Sigma\Theta_1$ is the dual homotopy Seifert pairing. The other maps are possibly a "tri-linear" analogue of the homotopy Seifert pairing. It would be interesting to know what these maps mean geometrically. Is there a functional relationship between Θ_2 and Θ_3?

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