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The refined transfer, bundle structures, and algebraic K-theory

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Abstract

We give new homotopy theoretic criteria for deciding when a fibration with homotopy finite fibres admits a reduction to a fibre bundle with compact topological manifold fibres. The criteria lead to an unexpected result about homeomorphism groups of manifolds. A tool used in the proof is a surjective splitting of the assembly map for Waldhausen's functor $A(X)$. We also give concrete examples of fibrations having a reduction to a fibre bundle with compact topological manifold fibres but which fail to admit a compact fibre smoothing. The examples are detected by algebraic K-theory invariants. We consider a refinement of the Becker–Gottlieb transfer. We show that a version of the axioms described by Becker and Schultz uniquely determines the refined transfer for the class of fibrations, admitting a reduction to a fibre bundle with compact topological manifold fibres. In the Appendix, we sketch a theory of characteristic classes for fibrations. The classes are primary obstructions to finding a compact fibre smoothing.

1. *Introduction*

Let

$$p\colon E \to B$$

be a fibration whose base space B and whose fibres have the homotopy type of a finite complex. The transfer construction of Becker and Gottlieb [**2**] associates to p a 'wrong way' stable homotopy class

$$\chi(p)\colon B_+ \to E_+$$

such that the assignment $p \mapsto \chi(p)$ is homotopy invariant and natural with respect to base change (here B_+ denotes B with the addition of a disjoint basepoint). The transfer has shown itself to be an important tool in algebraic topology. For example, one of its early applications was a simple proof of the Adams Conjecture [**3**].

A refinement of the transfer, also considered by Becker and Gottlieb [**2**, top of p. 115], has recently surfaced in the Dwyer, Weiss, and Williams approach to fibrewise smoothing problems and the theory of higher Reidemeister torsion (see [**1**, **15**, **18**]).

Let E^+ denote the disjoint union $E \amalg B$. Then E^+ is a retractive space over B. The category of such spaces is the subject of fibrewise homotopy theory (cf. [**12**]). The associated stable homotopy category is thus the study of fibrewise stable phenomena (cf. [**24**]).

The *refined transfer* of p is a certain fibrewise stable homotopy class

$$t(p)\colon B^+ \to E^+.$$

The Becker–Gottlieb transfer $\chi(p)$ is obtained from $t(p)$ by collapsing the preferred sections $B \to E^+$ and $B \to B^+$ to a point.

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Becker and Schultz [4] gave an axiomatic characterization of the Becker–Gottlieb transfer under the assumption that the fibration p is fibre homotopy equivalent to a fibre bundle with compact topological manifold fibres. Their axioms involve naturality, normalization, compatibility with products and additivity of the transfer.

Definition 1.1. If $p\colon E \to B$ admits a fibre homotopy equivalence to a fibre bundle with compact topological manifold fibres, then p is said to have a *compact* TOP *reduction*.

If p is fibre homotopy equivalent to a fibre bundle with compact smooth manifold fibres, then p is said to have a *compact* DIFF *reduction* or a *compact fibre smoothing*.

Remark 1.2. The fibres of these bundles are permitted to have non-empty boundary. Our terminology in the smooth case differs from that of Casson and Gottlieb [11], who instead use the term *closed fibre smoothing*. Our preference is to use 'compact' instead of 'closed' so as to avoid potential confusion.

The following, communicated to us by Goodwillie, gives fibrations with homotopy finite fibres that fail to admit a compact TOP reduction.

Example 1.3. Let F be a connected based finite complex equipped with a based self-homotopy equivalence $\theta\colon F \to F$. Assume θ induces the identity map on fundamental groups and has non-trivial Whitehead torsion. Then the mapping torus $F \times_\theta S^1 \to S^1$, converted into a fibration, does not admit a compact TOP reduction.

The example is verified by contradiction. A compact TOP reduction would yield a homotopy equivalence $k\colon F \to M$, with M a compact topological manifold, and a homotopy inverse $k^{-1}\colon M \to F$ such that the composite $k \circ \theta \circ k^{-1}\colon M \to M$ is homotopic to a homeomorphism. But this would show that the torsion of $k \circ \theta \circ k^{-1}$ is trivial [9, Theorem 1]. Since θ induces the identity on π_1, the composition formula [10, 22.4], shows that the torsion of θ is also trivial. This gives the contradiction. For specific homotopy equivalences θ satisfying Example 1.3, see [10, 24.4].

The main result of this paper is to give explicit homotopy theoretic criteria for deciding when a fibration admits a compact TOP reduction. Our approach is to use the recent work of Dwyer, Weiss, and Williams, specifically, the 'Converse Riemann–Roch Theorem', which gives an abstract characterization of when such a reduction exists [15], and entails an understanding of how the refined transfer relates to Waldhausen's algebraic K-theory of spaces. Along the way, we will extend the Becker–Schulz axioms to the fibrewise setting and show how the axioms characterize the refined transfer for those fibrations admitting a compact TOP reduction. The proof of this characterization follows along the lines of Becker–Schulz, and we do not claim any originality in this direction. As to whether the axioms characterize the refined transfer for all fibrations with homotopy finite fibres is an interesting open question.

The axiomatic characterization of the refined transfer is independent of the rest of the paper and is included because of Igusa's recent progress on axiomatizing higher Reidemeister torsion invariants [19]. There is a close relationship between higher torsion and the refined transfer: when the fibration is fibre homotopy equivalent to a smooth fibre bundle with compact fibres, then the refined transfer admits a lift into a group closely associated with algebraic K-theory, and this lift coincides with the higher torsion invariant of Dwyer, Weiss, and Williams. A currently unsolved problem is to determine whether the Dwyer–Weiss–Williams torsion coincides with Igusa's torsion. The problem would be solved if one could verify that Igusa's axioms hold for the Dwyer–Weiss–Williams torsion. Some evidence in favour of this is that the axioms we will shortly give for the refined transfer seem to be close in spirit to Igusa's axioms, although further effort will be needed to pin down the exact relationship.

Here are the axioms.

DEFINITION 1.4. A *refined transfer* is a rule, which assigns to fibrations $p\colon E \to B$ with homotopy finite base and fibres, a fibrewise stable homotopy class

$$t(p)\colon B^+ \to E^+$$

that is subject to the following axioms.

- **A1** (Naturality). For commutative homotopy pullback diagrams

$$\begin{tikzcd} E' \arrow[r, "\tilde{f}"] \arrow[d, "p'"'] & E \arrow[d, "p"] \\ B' \arrow[r, "f"'] & B \end{tikzcd}$$

in which p' and p are fibrations, we have

$$\tilde{f}^+ \circ f_* t(p') \;=\; t(p) \circ f^+,$$

where f^+ denotes $f \amalg \mathrm{id}_B$, $\tilde{f}^+$ denotes $\tilde{f} \amalg \mathrm{id}_B$, and $f_* t(p')$ is the effect of making $t(p')$ into a fibrewise stable homotopy class B by taking cobase change along f.

- **A2** (Normalization). Let $1\colon B \to B$ be the identity. Then

$$t(1)\colon B^+ \to B^+$$

is the identity.

- **A3** (Products). For a product fibration $p \times p'\colon E \times E' \to B \times B'$, we have

$$t(p \times p') = t(p) \wedge t(p'),$$

where $\wedge$ means external fibrewise smash product.

- **A4** (Additivity). If

$$\begin{tikzcd} E_\emptyset \arrow[r, "j_2"] \arrow[d, "i_1"'] & E_2 \arrow[d, "i_2"] \\ E_1 \arrow[r, "j_1"'] & E \end{tikzcd}$$

is a commutative homotopy pushout diagram of fibrations over B, then

$$t(p) = (j_1)_* t(p_1) + (j_2)_* t(p_2) - (j_\emptyset)_* t(p_\emptyset),$$

where for $S \subsetneq \{1,2\}$, $p_S\colon E_S \to B$ denotes the projection and $(j_S)_*\colon E_S^+ \to E^+$ is the evident map.

In Section 3, we explain Becker and Gottlieb's construction of a refined transfer. Their version will be called *the* refined transfer, employing the definite article to distinguish it from other constructions satisfying the axioms.

THEOREM A. *Let t and t' be refined transfers defined on the class of fibrations having homotopy finite fibres. Then $t = t'$ for those fibrations that admit a compact* TOP *reduction.*

We now give homotopy theoretic criteria for deciding when a fibration admits a compact TOP reduction. One should regard this as the main result of the paper.

THEOREM B. *Let $p\colon E \to B$ be a fibration with homotopy finite base and fibres. Assume*
- *p comes equipped with a section,*
- *p is $(r+1)$-connected and*
- *B has the homotopy type of a cell complex of dimension $\leqslant 2r$.*

Then p admits a preferred compact TOP *reduction.*

Consequences. Combining Theorems A and B, we immediately obtain the following.

COROLLARY C. *Let t and t' be refined transfers. The $t = t'$ for the fibrations appearing in Theorem B.*

Here is a way to construct examples satisfying Theorem B. Start with any Hurewicz fibration $p\colon E \to B$ with homotopy finite base and fibres. The (*unreduced*) *fibrewise suspension* of p is the fibration $S_B p\colon S_B E \to B$ whose total space is the double mapping cylinder of the map p:

$$S_B E = (B \times 0) \cup (E \times [0,1]) \cup (B \times 1)$$

(cf. [**28**]). The fibre of $S_B p$ at $b \in B$ is given by the unreduced suspension of the fibre of p at b. Consequently, the connectivity of the map $S_B p$ is one more than that of p, so iteration of the fibrewise suspension construction eventually yields a fibration that satisfies the conditions of Theorem B.

COROLLARY D. *Stably, any fibration $p\colon E \to B$ with homotopy finite base and fibres admits a compact* TOP *reduction. That is, there is an iterated fibrewise suspension $S_B^j p\colon S_B^j E \to B$ that admits a compact* TOP *reduction.*

The method of proof of Theorem B yields a new and unexpected result about automorphism groups of manifolds. For a compact connected manifold M with basepoint $*$ in its interior, let $\mathrm{TOP}(M,*)$ be the simplicial group whose k-simplices are the homeomorphisms of $\Delta^k \times M$ that commute with projection to Δ^k and are the identity when restricted to $\Delta^k \times *$. Let $G(M,*)$ be defined similarly, using homotopy equivalences in place of homeomorphisms. The forgetful homomorphism induces a map of classifying spaces

$$B\mathrm{TOP}(M,*) \to BG(M,*).$$

The surprise will be that this map has a section up to homotopy along the $2r$-skeleton of $BG(M,*)$ when $M \subset \mathbb{R}^m$ is an r-connected compact codimension zero manifold with a sufficiently low dimensional spine (the exact dimensions will be spelled out in Section 11).

More precisely, define the *stable homeomorphism group*

$$\mathrm{TOP}^{\mathrm{st}}(M,*),$$

to be colimit of $\mathrm{TOP}(M \times I^k,*)$ via stabilization given by taking cartesian product with the unit interval.

Similarly, one can define $G^{\mathrm{st}}(M,*)$, but in this case the associated inclusion $G(M,*) \to G^{\mathrm{st}}(M,*)$ is a homotopy equivalence. It follows that one has a map of classifying spaces

$$B\mathrm{TOP}^{\mathrm{st}}(M,*) \to BG(M,*).$$

Theorem E. *Let $M \subset \mathbb{R}^m$ be a compact codimension zero smooth submanifold. Assume M is r-connected.*

Then there is a space X_M and a map $X_M \to B\mathrm{TOP}^{\mathrm{st}}(M,*)$ such that the composite

$$X_M \to B\mathrm{TOP}^{\mathrm{st}}(M,*) \to BG(M,*)$$

is $2r$-connected.

Furthermore, there is a preferred decomposition of homotopy groups

$$\pi_*(\mathrm{TOP}^{\mathrm{st}}(M,*)) \cong \pi_*(G(M,*)) \oplus \pi_*(\mathrm{map}(M,\mathrm{TOP})) \oplus \pi_{*+1}(Wh^{\mathrm{top}}(M)),$$

which is valid in degrees $* \leqslant 2r-2$.

In the above, $Wh^{\mathrm{top}}(M)$ is the topological Whitehead space ([**17**, **29**, Section 3]), TOP is the group of homeomorphisms of euclidean space stabilized with respect to dimension, and map (M, TOP) is the function space of maps $M \to \mathrm{TOP}$.

Examples. We now give examples of fibrations that fail to have a compact fibre smoothing but do admit a compact TOP reduction.

Theorem F. *There exist fibrations $p\colon E \to B$ that admit a compact* TOP *reduction but do not have a compact fibre smoothing.*

The fibres of these fibrations have the homotopy type of a finite wedge of spheres $\vee_k S^n$, for suitable choice of k and n.

Furthermore, these examples are detected in the rationalized algebraic K-theory of the integers.

Theorem G. *Let*

$$S^3 \to E \xrightarrow{p} S^3$$

be the spherical fibration corresponding to the generator of

$$\pi_3(BF_3) \cong \pi_5(S^3) = \mathbb{Z}_2,$$

where F_3 is the topological monoid of based self-homotopy equivalences of S^3.

Then p admits a compact TOP *reduction but does not admit a compact fibre smoothing.*

The following result shows that the obstructions to compact fibre smoothing are killed when taking the cartesian product with finite complex having trivial Euler characteristic.

Theorem H. *Let $p : E \to B$ be a fibration with homotopy finite fibres. Let X be a finite complex with zero Euler characteristic. Then the fibration*

$$q\colon E \times X \to B$$

given by $q(e,x) = p(e)$ admits a compact fibre smoothing.

Remark 1.5. At the time the first draft of this paper was written, it was forgotten by both authors that this result was already stated in [**33**, Corollary 5.2.5] with a sketched proof. This paper contains a different proof.

The trace map. Given a fibration $p\colon E \to B$, let $p^+ : E^+ \to B^+$ be the associated map of retractive spaces over B.

Given a refined transfer $t(p)\colon B^+ \to E^+$, we take its composition with p^+ to obtain a fibrewise stable homotopy class

$$p^+ \circ t(p)\colon B^+ \to B^+.$$

A straightforward unravelling of definitions shows that $p^+ \circ t(p)$ is equivalent to specifying an ordinary stable cohomotopy class

$$\mathrm{tr}_t(p)\colon B_+ \to S^0$$

(because B^+ coincides with $B \times S^0$). The latter is called the *trace* of the fibration p (compare Brumfiel and Madsen [**8**, p. 137]).

The following is a uniqueness result about the trace.

Theorem I. *Let t and t' be refined transfers. Then $\mathrm{tr}_t = \mathrm{tr}_{t'}$ on the class of fibrations whose base and fibre have the homotopy type of a finite complex.*

Remark 1.6. For further results, see Douglas [**14**] and Dorabiała and Johnson [**13**].

Assembly. The proof of Theorem B uses the assembly map for Waldhausen's algebraic of K-theory spaces functor $A(X)$. If f is a homotopy functor from spaces to spectra, the assembly map is a natural transformation

$$f^{\%}(X) \to f(X),$$

which best approximates f by an excisive functor $f^{\%}$ in the homotopy category of functors (recall that a functor is excisive if it preserves homotopy pushouts).

The crucial result used in the proof of Theorem B is a functorial stable range splitting for the assembly map for $A(X)$ considered as a functor on the category of based spaces.

Theorem J. *For based spaces X, the assembly map*

$$A^{\%}(X) \to A(X)$$

is stably split.

More precisely, there is a homotopy functor $X \mapsto B(X)$ from based spaces to spectra, and a natural transformation $B(X) \to A^{\%}(X)$ such that the composite map

$$B(X) \to A^{\%}(X) \to A(X)$$

is $2r$-connected whenever X is r-connected.

Given what is already known about $A(X)$, this result is not hard to prove. However, it is worth stating here since it is one of our main tools. The role of the basepoint here is crucial; the result is false on the category of unbased spaces.

2. *Conventions*

Spaces. We work in the category of compactly generated spaces. A map of spaces $f\colon X \to Y$ is a *weak homotopy equivalence* when it induces an isomorphism on homotopy in each degree. A space X is *r-connected*, if for every integer k such that $-1 \leqslant k \leqslant r$, every map $S^k \to X$ extends to map $D^{k+1} \to X$. In particular, every non-empty space is (-1)-connected. The empty space is considered to be (-2)-connected. A map of spaces is r-connected if its homotopy fibres (with respect to all basepoints) are $(r-1)$-connected. A space is *homotopy finite* if it has the homotopy type of a finite cell complex.

Although Quillen model categories are barely mentioned in this paper, we will be implicitly working in the model structure for spaces whose weak equivalences are the weak homotopy equivalences, whose fibrations are Serre fibrations, and whose cofibrations are the Serre cofibrations.

There is one notable exception to this policy: it is not known whether the fibrewise suspension of a Serre fibration is again a fibration, but the analogous statement is true in the Hurewicz fibration case. So, unless otherwise mentioned, we usually work with Hurewicz fibrations.

A commutative square of spaces (or spectra)

$$\begin{tikzcd} A \arrow[r] \arrow[d] & C \arrow[d] \\ B \arrow[r] & D \end{tikzcd}$$

is said to be *r-cartesian* if the map from A to the homotopy pullback $P := \operatorname{holim}(B \to D \leftarrow C)$ is r-connected. More generally, suppose that the square only commutes up to homotopy. Given a choice of homotopy $A \times [0,1] \to D$, one gets a preferred map $A \to P$. In this instance, we say that the square together with its commuting homotopy is r-cartesian provided that $A \to P$ is r-connected.

Fibrewise spaces. We will be assuming familiarity with fibrewise homotopy theory in its unstable and stable contexts. The book of Crabb and James [**12**] gives the foundational material on this subject.

For a cofibrant space B, let $T(B)$ denote the category of *spaces over* B. An object of $T(B)$ consists of a space X and a reference map $X \to B$ (where the latter is typically suppressed from the notation). A morphism $X \to Y$ is a map of spaces that is compatible with reference maps. A morphism is a fibration or weak equivalence if and only if it is one when considered as a map of topological spaces. This comes from a model structure on $T(B)$, where the cofibrations are defined using the lifting property [**25**].

The 'pointed' version of $T(B)$ is the category $R(B)$ of *retractive spaces over* B. This category has objects consisting of a space Y together with maps $B \to Y$ and $Y \to B$ such that the composite $B \to Y \to B$ is the identity map. A morphism is a map of underlying spaces that is compatible with the structure maps. Again $R(B)$ is a model category by appealing to the forgetful functor to spaces. An object of $R(B)$ is said to be *finite* if it is obtained from the zero object by attaching a finite number of cells.

The (*reduced*) *fibrewise suspension* functor $\Sigma_B \colon R(B) \to R(B)$ is given by mapping an object Y to the object

$$\Sigma_B Y = (Y \times [0,1]) \cup_{X \times [0,1]} X,$$

where $X \times [0,1] \to Y \times [0,1]$ arises from the structure map $X \to Y$ by taking cartesian product with the identity map and $X \times [0,1] \to X$ is first factor projection.

$R(B)$ also has smash products. If Y and Z are objects, then their fibrewise smash product is given by

$$Y \wedge_B Z \ := \ (Y \times_B Z) \cup_{(Y \cup_X Z)} X,$$

where $Y \times_B Z$ is the fibre product and $Y \cup_B Z \to Y \times_B Z$ is the evident map. Note that the special case of $Z = S^1 \times B$ gives $\Sigma_B Y$.

Schwede [27] has shown that the category of fibred spectra over B, that is, spectra formed using objects of $R(B)$, again forms a model category, where the weak equivalences in this case are the 'stable weak homotopy equivalences'.

The recent book of May and Sigurdsson equips the category of fibred spectra over B with a well-behaved internal smash product [24, Section 11.2].

3. *Construction of a refined transfer*

Becker and Gottlieb define a refined transfer in their paper [2, Section 5]. The purpose of this section is to sketch the idea behind their construction.

First, consider the case when B is a point. Let F be a homotopy finite space, which, for convenience, we take to be cofibrant. We let F_+ denote the effect of adding a disjoint basepoint to F. The *S-dual* of F_+ is the spectrum $D(F_+)$, which is the mapping spectrum $\mathrm{map}(F_+, S^0)$, where S^0 is the sphere spectrum. Explicitly, it is the spectrum whose kth space is the space of maps $F_+ \to QS^k$, where $Q = \Omega^\infty\Sigma^\infty$ is the stable homotopy functor. More generally, we use the notation $D(E)$ for the function spectrum of maps $E \to S^0$ whenever E is a homotopy finite spectrum.

There is a map of spectra

$$d\colon F_+ \wedge D(F_+) \to S^0,$$

which is defined as the adjoint to the identity map of $D(F_+)$. The canonical stable map

$$F_+ \to D(D(F_+))$$

is a weak equivalence. Furthermore, we have a preferred weak equivalence

$$F_+ \wedge D(F_+) \;\simeq\; D(F_+ \wedge D(F_+)),$$

which shows that $F_+ \wedge D(F_+)$ is *self-dual*. Hence the dualization of the map d above yields a map

$$d^*\colon S^0 = D(S^0) \to D(F_+ \wedge D(F_+)) \simeq F_+ \wedge D(F_+).$$

The map d^* is well defined in the homotopy category of spectra.

Now form the homotopy class

$$t(F)\colon S^0 \xrightarrow{d^*} F_+ \wedge D(F_+) \xrightarrow{\Delta_{F_+}\wedge \mathrm{id}} F_+ \wedge F_+ \wedge D(F_+) \xrightarrow{\mathrm{id}\wedge d} F_+ \wedge S^0.$$

Then $t(F)$ is identified with a stable homotopy class $S^0 \to Q(F_+)$. This gives a refined transfer in the case when B is a point.

Proceeding to the case of a general fibration $E \to B$, one appeals to a fibrewise version of the above to get a refined transfer. As in Section 1, let E^+ denote $E \amalg B$, and define $D_B(E^+)$ to be the fibrewise mapping spectrum of maps $E^+ \to B \times S^0$. Then, analogous to the above, one has a fibrewise stable map,

$$d\colon E^+ \wedge_B D_B(E^+) \to B^+,$$

which is adjoint to the identity. One then continues in the same way as above, and the outcome is a fibrewise stable homotopy class

$$B^+ \to E^+.$$

This gives our rough description of the refined transfer in the general case.

Verification of the axioms. The only axiom that is not straightforward to verify is the additivity axiom A4. Becker and Schultz remark that this axiom follows from formal considerations involving S-duality. In our context, the crucial points are that the map of fibred spectra

$$E^+ \to D_B D_B(E^+)$$

is a natural transformation and the double dual $D_B D_B$ preserves homotopy pushouts.

4. *Characterization when B is a point*

We show how the axioms characterize the refined transfer for the constant fibration $F \to *$, where F is a homotopy finite cell complex. This case actually follows from the work of Becker and Schultz. However, it will be useful for what comes later to recast their proof in a more coordinate-free language. The case $B = *$ captures the main features of the proof in the general case.

We first digress with an observation about the axioms in the case of a trivial fibration. For a trivial fibration

$$F \to F \times B \overset{p_B}{\to} B,$$

a refined transfer can be regarded as a fibrewise stable homotopy class

$$t(p_B)\colon B^+ \to (F_+) \times B,$$

and the associated ordinary transfer can be regarded as the associated stable homotopy class

$$t_F(B)\colon B_+ \to (F \times B)_+,$$

which is obtained from $t(p_B)$ by collapsing the preferred copy of B to a point (note: $(F \times B)_+ = ((F_+) \times B)/B$).

More generally, let (B, A) be a cofibration pair. Choose an actual stable map $\hat{t}(p_B)\colon B^+ \to (F_+) \times B$ representing the refined transfer $t(p_B)$. The naturality axiom implies that the fibrewise homotopy class of the composite stable map

$$A^+ \longrightarrow B^+ \xrightarrow{\hat{t}(p_B)} (F_+) \times B$$

coincides with inclusion $(F_+) \times A \to (F_+) \times B$ composed with the refined transfer $t(p_A)$ for the trivial fibration $p_A : F \times A \to A$. Furthermore, since the diagram

$$\begin{tikzcd} F \times A \arrow[r] \arrow[d] & F \times B \arrow[d] \\ A \arrow[r] & B \end{tikzcd}$$

is a pullback, the space of choices consisting of a choice of representative $\hat{t}(p_A)$ of $t(p_A)$ together with a choice of homotopy making the diagram of axiom A1 commute is a contractible space. This shows that once the representative $\hat{t}(p_B)$ is chosen then, for any cofibration $A \subset B$, one obtains a preferred contractible choice of representatives for all $t(p_A)$ equipped with a commuting homotopy.

In particular, the representative $\hat{t}(p_B)$ determines a preferred stable homotopy class of pairs

$$(B_+, A_+) \to ((F \times B)_+, (F \times A)_+)$$

whose components are the transfers $t_F(B)$ and $t_F(A)$. The latter in turn induces a stable homotopy class on quotients

$$t_F(B, A)\colon B/A \to (F_+) \wedge (B/A).$$

Note the special case when A is the empty space gives $t_F(B, \emptyset) = t_F(B)$.

Axioms A2 and A3 straightforwardly imply that

$$t_F(B,A) \;=\; t_F \wedge \mathrm{id}_{B/A},$$

where $t_F : S^0 \to F_+$ coincides with $t_F(*,\emptyset)$. Also, note that $t_F(B,A) = t_F(B/A,*)$.

With this observation, we can now return to the problem of characterizing the refined transfer when the base space is a point. In this instance, a refined transfer is represented by a homotopy class of stable map $t_F : S^0 \to F_+$, where $t_F = t_F(*,\emptyset)$.

Because F is a homotopy finite space, there is a codimension zero compact *smooth* manifold

$$M \subset \mathbb{R}^d$$

and a homotopy equivalence $F \simeq M$. By homotopy invariance (that is, axiom A1 when f is the identity map of a point), it will suffice to characterize the homotopy class

$$t_M := t_M(*,\emptyset) \colon S^0 \to M_+ .$$

Consider the commutative pullback diagram of pairs

$$\begin{tikzcd}
(S^d \times M, M) \arrow[r, "\alpha\times 1"] \arrow[d, "\pi_1"'] & ((M/\partial M)\times M, M) \arrow[d, "\pi_1"] \\
(S^d, *) \arrow[r, "\alpha"'] & (M/\partial M, *).
\end{tikzcd} \tag{1}$$

The vertical maps of these diagrams are fibrations. Applying the relative transfer construction and using naturality, we see that the associated diagram of stable maps

$$\begin{tikzcd}
S^d \wedge M_+ \arrow[r, "\alpha\wedge 1"] & M/\partial M \wedge M_+ \\
S^d \arrow[u, "t_M(S^d,*)"] \arrow[r, "\alpha"'] & M/\partial M \arrow[u, "t_M(M/\partial M,*)"']
\end{tikzcd} \tag{2}$$

homotopy commutes. The product axiom for S^d and F implies that

$$t_*(S^d,*) \wedge t_M(*,\emptyset) = t_M(S^d,*).$$

The normalization axiom implies that $t_*(S^d,*)\colon S^d \to S^d$ is the identity, and $t_M = t_M(*,\emptyset)$ by definition. Consequently, we see that the diagram of stable maps

$$\begin{tikzcd}
S^d \wedge M_+ \arrow[r, "\alpha\wedge 1"] & M/\partial M \wedge M_+ \\
S^d \arrow[u, "1_{S^d}\wedge t_M"] \arrow[r, "\alpha"'] & M/\partial M \arrow[u, "t_M(M/\partial M,*) \simeq 1_{M/\partial M}\wedge t_M"']
\end{tikzcd} \tag{3}$$

is homotopy commutative.

Consider the diagonal embedding

$$\Delta\colon (M, \partial M) \to (M \times M, (\partial M) \times M).$$

The associated map of quotients $M/\partial M \to (M/\partial M) \wedge (M_+)$ will also be denoted by Δ. The diagonal embedding has a compact tubular neighbourhood isomorphic to the total space of the unit tangent disk bundle of M, which is a trivial bundle since M is a codimension zero submanifold of euclidean space. Let D denote the unit disk bundle and let C denote the complement of the interior of the tubular neighbourhood. Then we have a pushout square

$$\begin{tikzcd} S \arrow[r] \arrow[d] & C \arrow[d] \\ D \arrow[r] & M \times M. \end{tikzcd}$$

The inclusion $(\partial M) \times M \to M \times M$ admits a factorization up to homotopy through C. The factorization is given by choosing an internal collar of ∂M and letting M_0 denote the result of removing the open collar. Then $M_0 \subset M$ is a homotopy equivalence and the inclusion $\partial M \times M_0 \to M \times M$ has image in C provided that the tubular neighbourhood has been chosen sufficiently small.

The inclusion

$$(M \times M_0, \partial M \times M_0) \to (M \times M, C)$$

then gives rise to a map of quotients

$$M/\partial M \wedge M_+ \cong M/\partial M_0 \wedge (M_0)_+ \to (M \times M)/C \cong D/S \cong S^d \wedge M_+ .$$

Denote it by $c\colon M/\partial M \wedge M_+ \to S^d \wedge M_+$.

Lemma 4.1 (Compare [4, Lemma 2.5]). *The composite*

$$S^d \wedge M_+ \xrightarrow{\alpha \wedge 1} (M/\partial M) \wedge M_+ \xrightarrow{c} S^d \wedge M_+$$

is homotopic to the identity.

Proof. We can assume that M is embedded in the unit disk D^d in such a way that M meets $\partial D^d = S^{d-1}$ transversely in ∂M. Then we have an embedding $(M, \partial M) \subset (D^d, S^{d-1})$. Let $(W, \partial_0 W)$ be the closure of the complement of $(M, \partial M)$ in (D^d, S^{d-1}).

Consider the associated embedding

$$(M \times M, \partial M \times M) \subset (D^d \times M, S^{d-1} \times M)$$

given by taking the cartesian product with M. Then the effect of collapsing $W \times M$ in $D^d \times M$ gives rise to the map $\alpha \wedge 1$.

Consider the composite embedding

$$(M, \partial M) \xrightarrow{\Delta} (M \times M, \partial M \times M) \subset (D^d \times M, S^{d-1} \times M).$$

The composite $c \circ (\alpha \wedge 1)$ is the effect of collapsing the complement of a tubular neighbourhood M in $D^d \times M$ to a point. But the complement is contractible, so this collapse map is homotopic to the identity. $\square$

Proposition 4.2 (Compare [4, 2.9]). *The map c homotopically coequalizes $t_M(M/\partial M, *)$ and Δ, that is,*

$$c \circ t_M(M/\partial M, *) \;\simeq\; c \circ \Delta.$$

Proof. The argument will use the commutative pushout diagram of pairs

$$\begin{tikzcd} (S,S_0) \arrow[r] \arrow[d] & (C,C_0) \arrow[d] \\ (D,D_0) \arrow[r] & (M\times M,(\partial M)\times M), \end{tikzcd}$$

where (D,D_0) denotes the unit tangent disk bundle of $(M,\partial M)$, (S,S_0) is the unit sphere bundle, and (C,C_0) is the complement. Let

$$\pi_1\colon (M\times M,(\partial M)\times M)\to (M,\partial M)$$

be the first factor projection. Let $p_D\colon (D,D_0)\to (M,\partial M)$ be its restriction to (D,D_0); this is a fibration pair with fibre D^d. Similarly, let $p_C\colon (C,C_0)\to (M,\partial M)$ and $p_S\colon (S,S_0)\to (M,\partial M)$ be the restrictions to (C,C_0) and (S,S_0), respectively. Each of these is also a fibration pair. The fibre of p_S is S^{d-1} and the fibre of p_C is M_0, where M_0 is the effect of removing an open ball from the interior of M. We therefore have a pushout square of fibres

$$\begin{tikzcd} S^{d-1} \arrow[r] \arrow[d] & M_0 \arrow[d] \\ D^d \arrow[r] & M. \end{tikzcd}$$

Then the additivity axiom implies that

$$t_M(M,\partial M)=j_1t_{D^d}(M,\partial M)+j_2t_{M_0}(M,\partial M)-j_{12}t_{S^{d-1}}(M,\partial M),$$

where j_S for $S\subset\{1,2\}$ is induced by the evident inclusion map into M.

Then by the homotopy invariance and normalization axioms

$$t_{D^d}(M,\partial M)=i\circ t_*(M,\partial M)=i,$$

where $i\colon M/\partial M\to D/D_0$ arises from the zero section. By definition, j_1i is the reduced diagonal map $\Delta\colon M/\partial M\to M/\partial M\wedge M_+$. Consequently,

$$j_1t_{D^d}(M,\partial M)=\Delta.$$

In particular,

$$c\circ j_1t_{D^d}(M,\partial M)=c\circ\Delta.$$

To complete the proof of the proposition, it will suffice to show that c applied to each of the terms $j_2t_{M_0}(M,\partial M)$ and $j_{12}t_{S^{d-1}}(M,\partial M)$ is trivial, for this will yield $c\circ t_M(M,\partial M)=c\Delta$.

To see why $c\circ j_2t_{M_0}(M,\partial M)$ is trivial, recall that c is defined by collapsing $C\subset M\times M$ to a point, whereas $j_2t_{M_0}(M,\partial M)$ is given by a composite of the form

$$M/\partial M\xrightarrow{t_{M_0}(M,\partial M)} C/C_0\longrightarrow (M\times M)/(\partial M\times M).$$

The triviality of $cj_2t_{M_0}(M,\partial M)$ therefore follows from the fact that it factors through C/C_0. A similar argument shows that $cj_{12}t_{S^{d-1}}(M,\partial M)$ is trivial. □

Proof of Theorem A when B is a point. By Lemma 4.1 and Proposition 4.2 and diagram (3), we have

$$t_M\;=\;c\circ(\alpha\wedge 1)\circ t_M\;=\;c\circ t_M(M/\partial M,*)\circ\alpha\;=\;c\circ\Delta\circ\alpha.$$

This shows that t_M is determined by the maps c, Δ, and α, whose definition is independent of t_M. □

5. *Interpretation*

Motivated by Peter May's paper on the Euler characteristic in the setting of derived categories [23], we give an alternative interpretation of what we have just shown in terms of the algebra of the stable homotopy category. This section is independent of the rest of the paper.

Given F as above, recall that $D(F_+)$ denotes the S-dual of of F_+. Then $D(F_+)$ is a ring spectrum with unit $u\colon S^0 \to D(F_+)$, which represents a desuspension of the map α appearing above.

In what follows, we consider F_+ as an object of the stable homotopy category. Then we have an action

$$\mu\colon D(F_+) \wedge F_+ \to F_+,$$

which is defined as the formal adjoint to the map $D(F_+) \to \hom(F_+, F_+)$ given by mapping a stable map $f\colon F_+ \to S^0$ to $f \wedge 1_{F_+}$ composed with the diagonal of F_+. The map μ is a homotopy theoretic version of the collapse map c described above. Lemma 4.1 in this language asserts that F_+ is a $D(F_+)$-module.

Furthermore, F_+ is a coalgebra in the stable category, and one has a co-action map

$$\kappa\colon D(F_+) \to D(F_+) \wedge F_+,$$

which expresses $D(F_+)$ as an F_+-comodule: it can be defined as the linear dual of μ (= maps into S^0). Then κ is a homotopy theoretic version of the diagonal map Δ.

Corollary 5.1. *With respect to above, we have:*

- *the diagram*

$$\begin{tikzcd} S^0 \wedge F_+ \arrow[r, "u\wedge 1"] & D(F_+) \wedge F_+ \\ S^0 \arrow[u, "t_F"] \arrow[r, "u"'] & D(F_+) \wedge S^0 \arrow[u, "1_{D(F_+)}\wedge t_F"'] \end{tikzcd}$$

  is homotopy commutative (*cf. diagram* (3))*;*
- *the composite*

$$S^0 \wedge F_+ \xrightarrow{u\wedge 1} D(F_+) \wedge F_+ \xrightarrow{\mu} F_+ = S^0 \wedge F_+$$

  is homotopic to the identity;
- *the map μ homotopically coequalizes $1_{D(F_+)} \wedge t_F$ and κ* (cf. 4.2).

From Corollary 5.1, we immediately infer

$$t_F = \mu \circ \kappa \circ u.$$

6. *Proof of Theorem A*

Let $p\colon E \to B$ be a fibration with homotopy finite fibres. Assume that p admits a compact TOP reduction $q\colon W \to B$. When B is homotopy finite, one can replace q with its topological stable normal bundle along the fibres to obtain a new compact TOP reduction, which is a codimension zero sub-bundle of the trivial bundle $B \times \mathbb{R}^j$ for j sufficiently large (cf. [4, p. 599], [26]).

Consequently, we can assume without loss in generality that q comes equipped with a fibrewise codimension zero topological embedding $W \subset B \times \mathbb{R}^d$. We let

$$\partial^v W \to B$$

be the *fibrewise boundary* of q. This is the bundle, the fibre of which at $b \in B$ is given by ∂W_b.

The idea of the proof of Theorem A will be to adapt the method of Section 4 to the fibrewise topological setting.

The proof will hinge upon the following structure, which is assumed to vary continuously in $b \in B$:

- the fibres W_b come equipped with a degree one collapse map $S^d \to W_b/\partial W_b$ and
- the diagonal map $(W_b, \partial W_b) \to (W_b \times W_b, (\partial W_b) \times W_b)$ has a compact tubular neighbourhood.

The first of these properties is given by taking the Thom–Pontryagin collapse of the embedding $W_b \subset \{b\} \times \mathbb{R}^d$. The second property is discussed in [**4**, p. 599].

Proof of Theorem A. As in Section 4, the proof begins by considering a commutative diagram (the fibred analogue of diagram (1)):

$$\begin{tikzcd}
(S^d \times W, W) \arrow[r, "(\alpha \times_B 1, 1)"] \arrow[d, "(1\times q, q)"'] & ((W/\!\!/\partial^v W) \times_B W, W) \arrow[d, "(q^*, q)"] \\
(S^d \times B, B) \arrow[r, "(\alpha, 1)"'] & (W/\!\!/\partial^v W, B).
\end{tikzcd}$$

Here $W/\!\!/\partial^v W$ denotes the pushout of the diagram $B \leftarrow \partial^v W \subset W$, and q^* denotes the pullback of $q\colon W \to B$ along the map $W/\!\!/\partial^v W \to B$. Note that the fibres of $W/\!\!/\partial^v W \to B$ are given by $W_b/\partial W_b$. The map α is given by the fibrewise Thom–Pontryagin collapse map of the codimension zero embedding $W \subset \mathbb{R}^d$.

Given a refined transfer t, we apply it to the above and appeal to the naturality and product axioms to obtain a homotopy commutative diagram,

$$\begin{tikzcd}
\Sigma^d_B W^+ \arrow[r, "\alpha \wedge_B 1"] & (W/\!\!/\partial^v W) \wedge_B W^+ \\
\Sigma^d_B B^+ \arrow[r, "\alpha"'] \arrow[u, "\Sigma^d_B t(q)"] & W/\!\!/\partial^v W, \arrow[u, "t(q^*, q)"']
\end{tikzcd}$$

where $\Sigma^d_B W^+$ denotes the d-fold fibrewise suspension of $W^+ \to B$ (note that $\Sigma^d_B B^+ = B \times S^d$). The vertical maps are refined transfer maps associated with fibration pairs. This is the fibred analogue of diagram (3).

Let

$$c\colon (W/\!\!/\partial^v W) \wedge_B W^+ \to \Sigma^d_B W^+$$

be the fibrewise collapse map that, on each fibre, maps the complement data of the fibrewise embedding of the diagonal to a point.

To complete the proof, we appeal to the following two assertions, which are fibrewise versions of Lemma 4.1 and Proposition 4.2, and are proved similarly (alternatively, these are proved in the smooth case in [**4**, Lemma 2.5, Equation (2.9)] and their proof adapts in our case).

ASSERTION 1. The composition $c \circ (\alpha \wedge_B 1)\colon \Sigma^d_B W^+ \to \Sigma^d_B W^+$ is fibrewise homotopic to the identity.

ASSERTION 2. The map c homotopically coequalizes the $t(q^*, q)$ and the fibrewise reduced diagonal map $\Delta\colon W/\!\!/\partial^v W \to (W/\!\!/\partial^v W) \wedge_B W^+$. That is, $c \circ t(q^*, q) \simeq c \circ \Delta$.

Given these assertions, we obtain the equation

$$\Sigma^d_B t(q) \;\simeq\; c \circ \Delta \circ \alpha,$$

which uniquely determines $t(q)$. □

7. *Proof of Theorems* B *and* J

We first give the definition of the assembly map. Given a homotopy functor

$$f\colon \mathrm{Top} \to \mathrm{Spectra}$$

from spaces to spectra, the assembly map is an associated natural transformation of homotopy functors,

$$f^{\%}(X) \to f(X),$$

giving the best approximation to f on the left by a homology theory. The simplest construction of $f^{\%}(X)$ is to take the homotopy colimit

$$\operatorname*{hocolim}_{\Delta^k \to X} f(\Delta^k),$$

where the indexing is given by the category whose objects are the singular simplices $\Delta^k \to X$ and whose morphisms are given by restriction to faces.

The natural transformation $f^{\%}(X) \to f(X)$ is induced from the evident maps $f(\Delta^k) \to f(X)$ associated with each singular simplex $\Delta^k \to X$. Using the weak equivalence $f(\Delta^k) \to f(*)$, we obtain a weak equivalence of functors:

$$f^{\%}(X) \xrightarrow{\sim} X_+ \wedge f(*).$$

Consequently, we may think of the assembly map as a natural transformation:

$$X_+ \wedge f(*) \to f(X).$$

For more details, see [**32**].

Proof of Theorem J. We will give two proofs. The first, suggested by a referee, shows that the assembly map is $(2r - c)$-split for some constant $c \geqslant 0$.

Recalling the decomposition

$$A(X) \;\simeq\; \Sigma^\infty(X_+) \times Wh^{\mathrm{diff}}(X)$$

[**31**], it will suffice to split the assembly map of each factor. The assembly map for $\Sigma^\infty(X_+)$ is the identity map, so it clearly splits. The assembly map for $Wh^{\mathrm{diff}}(X)$ has the form

$$X_+ \wedge Wh^{\mathrm{diff}}(*) \to Wh^{\mathrm{diff}}(X).$$

There is a natural map

$$Wh^{\mathrm{diff}}(*) \to X_+ \wedge Wh^{\mathrm{diff}}(*),$$

which arises from the basepoint of X. The composite with the assembly map yields the map

$$Wh^{\mathrm{diff}}(*) \to Wh^{\mathrm{diff}}(X)$$

that is induced by the inclusion of the basepoint. As noted by Waldhausen [**30**, p. 153] for r-connected X, this map is approximately $2r$-connected. This completes the first proof.

Our second proof shows that the assembly map for $A(X)$ is $2r$-split. It uses the commutative diagram of spectra

$$\begin{tikzcd} A(X) \arrow[r] \arrow[d] & X_+ \wedge S^0 \arrow[d] \\ A(*) \arrow[r] & S^0 \end{tikzcd} \tag{4}$$

where the vertical maps are induced by the map $X \to *$, and the horizontal ones are given using Waldhausen's splitting of $A(X)$ mentioned above.

Suppose that X is r-connected. Then Goodwillie has shown that the square (4) is $(2r+1)$-cartesian [**16**, Corollary 3.3]. It follows that the square

$$\begin{tikzcd} X_+ \wedge S^0 \arrow[r] \arrow[d] & A(X) \arrow[d] \\ S^0 \arrow[r] & A(*) \end{tikzcd}$$

is $2r$-cartesian, where the horizontal maps are given by the natural transformation from stable homotopy to the algebraic K-theory of spaces induced from the inclusion functor from finite sets to finite spaces.

Using the basepoint for X, it follows that

$$X \wedge S^0 \to A(X) \to A(*)$$

is a homotopy fibre sequence up through dimension $2r$ in the sense that the map from $X \wedge S^0$ to the homotopy fibre of the map $A(X) \to A(*)$ is $2r$-connected. Furthermore, we have a homotopy fibre sequence

$$A(*) \wedge X \to A^{\%}(X) \to A(*).$$

Consider the diagram

$$\begin{tikzcd} X \wedge A(*) \arrow[r] \arrow[d] & A^{\%}(X) \arrow[r] \arrow[d] & A(*) \arrow[d, equal] \\ X \wedge S^0 \arrow[r] & A(X) \arrow[r] & A(*) \end{tikzcd}$$

where the middle vertical map is the assembly map, and the left vertical map is given by smashing the map $A(*) \to S^0$ with the identity map of X. Since the lower row is a homotopy fibre sequence up through dimension $2r$, and the map $A(*) \wedge X \to S^0 \wedge X$ is homotopically split, it follows that the assembly map

$$A^{\%}(X) \to A(X)$$

is $2r$-split. The functor $B(X)$ in this case is given by the wedge

$$(X \wedge S^0) \vee A(*),$$

and the map $B(X) \to A^{\%}(X)$ is given using the evident map $X \wedge S^0 \to X \wedge A(*) \simeq A^{\%}(X)$ together with the map $A(*) \to A^{\%}(X)$ coming from the basepoint of X. □

REMARK 7.1. There is no such stable splitting of the assembly map for $A(X)$ on the category of *un*based spaces. For suppose that there were a homotopy functor $B(X)$ defined on unbased spaces equipped with a natural transformation $B(X) \to A^{\%}(X)$ such that the composite $B(X) \to A^{\%}(X) \to A(X)$ is $(2r-c)$-connected for r-connected X. Taking the first stage of the Goodwillie tower of these functors yields maps

$$B(X) \to P_1 B(X) \to P_1 A^{\%}(X) \to P_1 A(X)$$

such that the composite $B(X) \to P_1 A(X)$ is a weak equivalence. Since $A^{\%}(X) = P_1 A^{\%}(X)$, we infer that $A^{\%}(X) \to P_1 A(X)$ has a section up to homotopy. But this is impossible when $X = \emptyset$ is the empty space, since $A^{\%}(\emptyset) = *$ whereas $P_1 A(\emptyset)$ is not contractible. We are indebted to a referee for explaining this argument to us.

Proof of Theorem B. Consider the fibration $p\colon E \to B$ with section whose underlying map is $(r+1)$-connected. Then the fibrewise version of the assembly map

$$A^{\%}_B(E) \to A_B(E)$$

is defined and is a map of fibred spectra over B (cf. [**15**, p. 51]). Applying the method of our second proof of Theorem J in a fibrewise manner, we obtain a fibred spectrum $\mathcal{B}_B(E)$ and a map $\mathcal{B}_B(E) \to A^{\%}_B(E)$ such that the composite

$$\mathcal{B}_B(E) \to A^{\%}_B(E) \to A_B(E)$$

is $2r$-connected (note: the fibre of $\mathcal{B}_B(E)$ at a point $b \in B$ is identified with $B(E_b)$).

By slight abuse of notation, consider the fibrewise assembly map as a map of (associated) fibrewise infinite loop spaces. Then, assuming that B has the homotopy type of a cell complex of dimension $\leqslant 2r$, the previous paragraph shows that the induced map of section spaces

$$\sec(A^{\%}_B(E) \to B) \to \sec(A_B(E) \to B)$$

is surjective on path components.

By the 'Converse Riemann–Roch Theorem' of Dwyer, Weiss, and Williams [**15**, Corollary 10.18], it follows that $p\colon E \to B$ admits a compact TOP reduction. □

8. *Proof of Theorems* F *and* G

Proof of Theorem F. Recall that Waldhausen's space $A(*)$ has the same rational homotopy type as $K(\mathbb{Z})$, the algebraic K-theory space of the integers. Borel showed that the rational cohomology of $K(\mathbb{Z})$ is an exterior algebra on classes b_{4k+1} in degree $4k+1$ for $k > 0$ [**6**]. Therefore, $H^{4k+1}(A(*);\mathbb{Z})$ has a non-trivial torsion-free summand for $k > 0$.

Let $\vee_k S^n$ be a k-fold wedge of n-spheres and let $\mathcal{H}^n_k$ denote the topological monoid of based homotopy equivalences of $\vee_k S^n$ (cf. [**29**, p. 385]). Then we have a homomorphism $\mathcal{H}^n_k \to \mathcal{H}^{n+1}_k$ given by suspension and a homomorphism $\mathcal{H}^n_k \to \mathcal{H}^n_{k+1}$ given by wedging on a single copy of the identity map. One of the standard definitions of $A(*)$ is

$$\mathbb{Z} \times \lim_{k,n} B\mathcal{H}^{n\ +}_k,$$

where B denotes the classifying space functor, and $+$ denotes Quillen's plus construction. In particular, the natural map

$$\iota\colon \mathbb{Z} \times \lim_{k,n} B\mathcal{H}^n_k \to A(*)$$

(from the space to its plus construction) is a rational cohomology isomorphism. Let $x \in H^{4i+1}(A(*);\mathbb{Z})$ be any non-torsion element. Then the restriction $\iota^* x \in H^{4i+1}(B\mathcal{H}^n_k;\mathbb{Z})$ is also non-torsion when k and n are chosen sufficiently large. Finally, let

$$B \subset B\mathcal{H}^n_k$$

be a connected, finite subcomplex that supports the class $\iota^* x$. This inclusion can be thought of as a classifying map for a fibration

$$p\colon E \to B,$$

whose fibre at the basepoint is $\vee_k S^n$. □

Theorem F is a consequence of the following assertion.

Claim. *The fibration $p\colon E \to B$ does not admit a compact fibre smoothing. However, p admits a compact* TOP *reduction provided n is sufficiently large.*

The second part of the claim follows directly from Theorem B. To prove the first part, it will be sufficient by the work of Dwyer, Weiss, and Williams to prove that the *A-valued trace map*

$$\chi_A(p)\colon B \to A(*)$$

(cf. below) does *not* admit a factorization up to homotopy as

$$B \to Q(S^0) \to A(*).$$

This is sufficient because the theory (cf. [**15**, Section 12]) shows that the existence of a compact fibre smoothing would imply the existence of such a factorization.

Since $Q(S^0)$ has trivial rational cohomology in positive degrees, it suffices to show that the A-valued trace is rationally non-trivial in cohomology in degree $4i+1$.

We first give a quick sketch of the construction of the A-valued trace map using the alternative definition of $A(*)$ as the algebraic K-theory of the category of homotopy finite based spaces (with cofibrations and weak equivalences). Deferring to Waldhausen's notation, let $wR^{\mathrm{hf}}(*)$ be the category whose objects are based, homotopy finite cofibrant topological spaces, and whose morphisms are weak homotopy equivalences. In particular, a homotopy finite space F determines an object of $wR^{\mathrm{hf}}(*)$, namely F_+.

Waldhausen also gives a '1-skeleton' inclusion map

$$j\colon |wR^{\mathrm{hf}}(*)| \to A(*)$$

[**29**, p. 329], so we can regard F_+ as a point of either the realization $|wR^{\mathrm{hf}}(*)|$ or of $A(*)$. Apply this construction to each fibre of the fibration p. This gives for each $x \in B$ a point $(F_x)_+ \in A(*)$ that can be arranged so as to vary continuously in x (the reader is referred to [**15**, Section 1.6] for the details). This yields the desired trace map $\chi_A(p)\colon B \to A(*)$.

From the construction we have given, it is immediate that it gives a factorization of $\chi_A(p)$ as

$$B \xrightarrow{\ v\ } |wR^{\mathrm{hf}}(*)| \xrightarrow{\ j\ } A(*),$$

where v is a continuous rectification of the map $x \mapsto (F_x)_+$.

On the other hand, since the fibration $p\colon E \to B$ comes equipped with a preferred section (arising from the wedge point), each fibre F_x is automatically a based space. We therefore have another object $F_x \in wR^{\mathrm{hf}}(*)$, so we have another map

$$u\colon B \to |wR^{\mathrm{hf}}(*)|$$

given by $x \mapsto F_x$. Waldhausen has also shown [**29**, Proposition 2.2.5] that the component of $|wR^{\mathrm{hf}}(*)|$, which contains the fibres F_x, is homotopy equivalent to the classifying space $B\mathcal{H}_k^n$, and, with respect to this identification, u can be regarded as the classifying map of the fibration p. In particular, the composition

$$B \xrightarrow{\ \subset\ } B\mathcal{H}_k^n \to |wR^{\mathrm{hf}}(*)| \xrightarrow{\ j\ } A(*)$$

coincides with $j \circ u$ up to homotopy and is therefore rationally non-trivial in degree $4i+1$.

Now, using Waldhausen's additivity theorem [**29**, Proposition 1.3.2], $j \circ v$ coincides with the map $x \mapsto (F_x) \vee S^0$, which is the wedge sum of the map $j \circ u$ and the constant map with value $S^0 \in A(*)$, and recall that wedge sum gives the H-space structure on $A(*)$ [**29**, p. 330]. In particular, the maps $j \circ v$ and $j \circ u$ coincide on rational cohomology in positive degrees, so $j \circ v$ is rationally non-trivial in degree $4i+1$.

This completes both the proof of the claim and also the proof of Theorem F.

Proof of Theorem G. Waldhausen [**30**, Section 3] constructs a map

$$BF \to A(*),$$

where F is the topological monoid of based stable self-homotopy equivalences of the sphere.

Let F_k denote the topological monoid of based (unstable) self-homotopy equivalences of S^k. Then, up to homotopy, the composite

$$BF_k \to BF \to A(*)$$

can be conveniently described as follows. Recall that

$$wR^{\mathrm{hf}}(*)$$

is the category of weak equivalences of cofibrant based spaces. Let $wR^{\mathrm{hf}}(*)_{(S^k)}$ denote the component of $wR^{\mathrm{hf}}(*)$ that contains the sphere S^k. Then the realization

$$|wR^{\mathrm{hf}}(*)_{(S^k)}|$$

has the homotopy type of BF_k [**29**, Proposition 2.2.5], and with respect to this identification, the map $BF_k \to A(*)$ is the '1-skeleton' inclusion

$$j\colon |wR^{\mathrm{hf}}(*)_{(S^k)}| \subset A(*)$$

(cf. [**29**, p. 329]).

Bökstedt and Waldhausen [**5**, p. 419] have shown that the composite map

$$BF \to A(*) \to Wh^{\mathrm{diff}}(*)$$

is non-trivial on homotopy groups in degree three. The second map in the composite is the splitting map for $A(*) \simeq Q(S^0) \times Wh^{\mathrm{diff}}(*)$.

By the Freudenthal suspension theorem, the homomorphism

$$\pi_3(BF_3) \to \pi_3(BF) = \mathbb{Z}_2$$

is an isomorphism. On the level of spherical fibrations, this group is generated by the clutching construction of a map $S^2 \times S^3 \to S^3$ whose associated Hopf construction $S^5 \to S^3$ represents η^2, where $\eta \in \pi_1^{\mathrm{st}}(S^0)$ is represented by the Hopf map $S^3 \to S^2$. The clutching construction produces the spherical fibration $S^3 \to E \to S^3$ stated in Theorem G.

It follows from the computation of Bökstedt and Waldhausen that the image of this generator under the homomorphism

$$\pi_3(BF_3) \to \pi_3(A(*))$$

is not an element of the subgroup

$$\mathbb{Z}_{24} = \pi_3(QS^0) \subset \pi_3(A(*)).$$

From the theory of Dwyer, Weiss, and Williams [**15**, Section 12], we infer that the fibration fails to have a compact fibre smoothing. On the other hand, the fibration admits a compact TOP reduction by Theorem B. Since this fibration represents a torsion element, it is not detected rationally. □

9. *Proof of Theorem* I

Let $t(p), t'(p)\colon B^+ \to E^+$ be refined transfers associated to a fibration $p\colon E \to B$. We will show that the traces

$$\mathrm{tr}_t(p), \mathrm{tr}_{t'}(p)\colon B_+ \to S^0$$

coincide. Let $S_B p\colon S_B E \to B$ be the fibrewise suspension of p. Let $C_B p\colon C_B E \to B$ be the mapping cone of p. Then the inclusion $B \to C_B E$ is a fibre homotopy equivalence.

Apply the additivity and normalization axioms to the pushout

$$\begin{tikzcd} E \arrow[r] \arrow[d] & C_B E \arrow[d] \\ C_B E \arrow[r] & S_B E \end{tikzcd}$$

and then take the associated traces to get

$$\mathrm{tr}_t(S_B p) = 1 + 1 - \mathrm{tr}_t(p),$$

where $1\colon B_+ \to S^0$ is the unit map.

Applying fibrewise suspension again, we obtain

$$\mathrm{tr}_t(S_B^2 p) = \mathrm{tr}_t(p).$$

Iterating this last equation j times, we get

$$\mathrm{tr}_t(S_B^{2j} p) = \mathrm{tr}_t(p).$$

If j is sufficiently large, the fibration $S_B^{2j} p$ admits a compact TOP reduction by Theorem B. By Theorem A, t and t' agree on $S_B^{2j} p$. Taking traces, we conclude $\mathrm{tr}_t(p) = \mathrm{tr}_{t'}(p)$.

10. *Proof of Theorem* H

Let m be the dimension of the finite complex X. Let X^k denote the k-skeleton, and let $X^{(k)}$ denote the quotient X^k/X^{k-1}.

Consider the fibration $p\colon E \to B$. At each fibre E_x, there is a cofibration sequence of retractive spaces over E_x of the form

$$(E_x \times X^k) \amalg E_x \to (E_x \times X^{k+1}) \amalg E_x \to E_x \times X^{(k+1)}$$

for $k \geqslant 0$. Denote the sum operation in the category of retractive spaces by $+$; this is given by fibrewise wedge. By the additivity theorem [**29**, Proposition 1.3.2], we obtain a preferred homotopy class of path in $A(E_x)$ from the sum

$$(E_x \times X^k) \amalg E_x \; + \; (E_x \times X^{(k+1)})$$

to $(E_x \times X^{k+1}) \amalg E_x$.

Summing over k, we get a preferred homotopy class of path in $A(E_x)$ connecting the points

$$(E_x \times X) \amalg E_x \quad \text{and} \quad \sum_{k=0}^{m} E_x \times X^{(k)}. \tag{5}$$

Let T_k denote a based set having cardinality one more than the number of k-spheres in $X^{(k)}$. Then we get an identification $T_k \wedge S^k \cong X^{(k)}$.

As fibrewise suspension induces the homotopy inverse to the H-multiplication defined by the sum (see [**29**, Proposition 1.6.2]), and the sum operation is homotopy commutative, there is a preferred path between the above and the sum

$$\sum_k E_x \times T_{2k} + \sum_k E_x \times \Sigma T_{2k+1}.$$

The latter can be rewritten as

$$E_x \times (T_0 \vee T_2 \vee \cdots) + E_x \times \Sigma(T_1 \vee T_3 \vee \cdots),$$

where the based sets $(T_0 \vee T_2 \vee \cdots)$ and $(T_1 \vee T_3 \vee \cdots)$ have the same cardinality under the assumption that the Euler characteristic of X is zero. Consequently, if we let T denote $(T_0 \vee T_2 \vee \cdots)$, the above is identified with

$$(E_x \times T) \vee \Sigma_{E_x}(E_x \times T)$$

which, by the additivity theorem, has a preferred homotopy class of path to the zero object.

Now the assignment $x \mapsto (E_x \times X) \amalg E_x$ gives rise to the generalized Euler characteristic of the fibration $q\colon E \times X \to B$, which is a section of the fibration $A_B(E \times X) \to B$ (see [**15**, I.1]). The above argument shows that this section is vertically homotopic to the constant section given by the basepoint of each fibre $A(E_x)$. But the basepoint section clearly factors through the map $Q_B E \to A_B E$ via the basepoint section of the fibration $Q_B E \to B$. We now apply the converse Riemann–Roch theorem in the smooth case [**15**, Section 12] to conclude that q admits a compact fibre smoothing.

This completes the proof of Theorem H.

REMARK 10.1. When $X = (S^1)^{\times k}$ is a torus of sufficiently large dimension, Theorem H becomes the 'closed fiber smoothing theorem' of Casson and Gottlieb [**11**, p. 160].

11. *Proof of Theorem* E

By replacing M by $M \times D^2$ if necessary, we can assume that $M \subset \mathbb{R}^m$ is a codimension zero compact connected smooth submanifold such that $\partial M \subset M$ is 2-connected.

Let $\mathcal{E}(M, *)$ denote the geometric realization of the simplicial monoid of which the k-simplices are families of topological embeddings

$$e\colon \Delta^k \times M \to \Delta^k \times M$$

such that e commutes with projection to Δ^k, e is a homotopy equivalence and is the identity when restricted to $\Delta^k \times *$.

Similarly, let $\mathrm{TOP}(M, *)$ be the geometric realization of the simplicial group whose k-simplices are self-homeomorphism of $\Delta^k \times M$ that preserve $\Delta^k \times *$. Then one has a forgetful homomorphism

$$\mathrm{TOP}(M, *) \to \mathcal{E}(M, *)$$

of topological monoids, which induces a map of classifying spaces

$$B\mathrm{TOP}(M, *) \to B\mathcal{E}(M, *),$$

whose homotopy fibre is identified with the Borel construction

$$E\mathrm{TOP}(M, *) \times_{\mathrm{TOP}(M,*)} \mathcal{E}(M, *).$$

The latter may also be identified with the orbit space $\mathcal{E}(M, *)/\mathrm{TOP}(M, *)$, because the action of $\mathrm{TOP}(M, *)$ on $\mathcal{E}(M, *)$ is free.

The orbit space may also be identified with $\mathcal{H}(\partial M)$, the space of topological h-cobordisms of ∂M. This can be seen as follows: let $\mathcal{E}'(M, *)$ be defined just as $\mathcal{E}(M, *)$ but where we now require the embedding e to have an image in $\Delta^k \times \mathrm{int}(M)$, where $\mathrm{int}(M)$ is the interior of M. Using a choice of collar neighbourhood of ∂M, one sees that the inclusion $\mathcal{E}'(M,*) \subset \mathcal{E}(M,*)$ is a deformation retract. Therefore, the orbit space is also identified with the Borel construction of $\mathrm{TOP}(M,*)$ acting on $\mathcal{E}'(M,*)$. Define a map $\mathcal{E}'(M,*) \to \mathcal{H}(\partial M)$ by sending an embedding $e\colon \Delta^k \times M \to \Delta^k \times \mathrm{int}(M)$ to the k-parameter family of h-cobordisms

$$(\Delta^k \times M) \; - \; e(\Delta^k \times \mathrm{int}(M)).$$

Then

$$\mathrm{TOP}(M,*) \to \mathcal{E}'(M,*) \to \mathcal{H}(\partial M)$$

is a homotopy fibre sequence (compare [**33**, p. 170]), so the assertion follows.

Taking classifying spaces, we extend to the right to obtain a homotopy fibre sequence

$$\mathcal{H}(\partial M) \to B\mathrm{TOP}(M,*) \to B\mathcal{E}(M,*).$$

Let $B = B\mathcal{E}(M,*)$, and let $E \to B$ be the associated universal fibration with fibre M, obtained as follows: the tautological action of $\mathcal{E}(M,*)$ on M gives a Borel construction $E\mathcal{E}(M,*) \times_{\mathcal{E}(M,*)} M \to B\mathcal{E}(M,*)$ which is a quasifibration. Then $E \to B$ is the effect of converting the quasifibration into a fibration.

Using this fibration, a fibrewise generalized Euler characteristic can be obtained:

$$B \xrightarrow{\chi} A_B(E).$$

The restriction of χ to $B\mathrm{TOP}(M,*)$ factors through the fibrewise assembly map

$$A_B^{\%}(E) \to A_B(E)$$

via an 'excisive characteristic' $\chi^{\%}$ ([**15**, 7.11]; here we are considering the fibrewise assembly map as a map of fibrewise infinite loop spaces). The resulting diagram

$$\begin{tikzcd}
B\mathrm{TOP}(M,*) \arrow[r, "\chi^{\%}"] \arrow[d] & A_B^{\%}(E) \arrow[d] \\
B \arrow[r, "\chi"'] & A_B E
\end{tikzcd} \tag{6}$$

is preferred homotopy commutative.

Taking vertical homotopy fibres, we obtain a map of spaces

$$\mathcal{H}(\partial M) \to \Omega Wh^{\mathrm{top}}(M).$$

This map is a composite of the form

$$\mathcal{H}(\partial M) \xrightarrow{(a)} \Omega Wh^{\mathrm{top}}(\partial M) \xrightarrow{(b)} \Omega Wh^{\mathrm{top}}(M),$$

where the map (a) is an equivalence in the topological concordance stable range (which is approximately $m/3$ by [**7**] and [**20**]). In particular, by taking the cartesian product of M with a disk of sufficiently large dimension, we can assume that the map (a) is highly connected. The map (b) is induced by applying the functor ΩWh^{top} to the inclusion $\partial M \to M$. If we replace M by $M \times D^k$, it too becomes highly connected when k grows because $\partial(M \times D^k) \to M \times D^k$ is at least $(k-1)$-connected, and after applying the functor the result is also approximately k-connected (since the same is true for the functor $A^{\%}$ and also the functor A using, say, [**16**, Corollary 3.3]). The upshot of this is that we can, by replacing M by $M \times D^k$, assume that the composite map $(b) \circ (a)$ is a weak equivalence up through any given dimension. We will assume this to be the case.

Now let M be r-connected. It follows with respect to our assumptions that the diagram (6) is $2r$-cartesian. By the method of proof of Theorem B, we know that the fibrewise assembly map $A_B^{\%}(E) \to A_B(E)$ is $2r$-split in a preferred way. This shows that the map

$$B\mathrm{TOP}(M,*) \to B\mathcal{E}(M,*)$$

is also $2r$-split. It follows that we have a preferred decomposition of homotopy groups

$$\pi_*(\mathrm{TOP}(M,*)) \;\cong\; \pi_*(\mathcal{E}(M,*)) \oplus \pi_{*+1}(Wh^{\mathrm{top}}(M))$$

for $* < 2r-1$.

To complete the proof we need to identify $\mathcal{E}(M,*)$. Let $\mathcal{I}(M,*)$ be the realization of the simplicial monoid defined just as $\mathcal{E}(M,*)$ but now with immersions in place of embeddings. By topological transversality [**21**], the inclusion map

$$\mathcal{E}(M,*) \to \mathcal{I}(M,*)$$

is a weak equivalence in our range after replacing M with $M \times D^k$ for k sufficiently large. Finally, let τ_M be the topological tangent microbundle of M, which is a trivial fibre bundle $M \times \mathbb{R}^m \to M$ since M is a codimension zero submanifold of euclidean space. Let $G(\tau_M,*)$ be the (simplicial) monoid whose zero simplices are pairs (f,ϕ) such that $f\colon M \to M$ is a based self-homotopy equivalence and $\phi\colon \tau_M \to \tau_M$ is a fibre bundle isomorphism covering f. The k-simplices of $G(\tau_M)$ are families of such pairs parameterized by the standard k-simplex. Then we have an identification

$$G(\tau_M,*) \;=\; G(M,*) \times \mathrm{maps}(M,\mathrm{TOP}_m),$$

and the evident map

$$\mathcal{I}(M,*) \to G(\tau_M,*)$$

is known to be a weak equivalence by immersion theory [**22**, p. 137]. Assembling this information completes the proof of Theorem E.

Remark 11.1. A more careful statement of Theorem E is as follows. Let M have dimension m and spine dimension d. Let c be the concordance stable range of M (this is the connectivity of the stabilization map $C(M) \to C(M \times I)$, where $C(M)$ is the smooth concordance space of M; by [**20**] one has $c \geqslant \max(2m+7, 3m+4)$). Then the map

$$B\mathrm{TOP}(M,*) \to BG(M,*)$$

has a section up to homotopy on the $(2r)$-skeleton provided that both $m-d$ and c are greater than $2r$. Consequently, if the homotopy type of M is held fixed, one needs the dimension of M to approximately exceed both $6r$ and $d+2r$ for there to be a section.

Appendix. *Characteristic classes for fibrations*

This appendix, which might be of independent interest, sketches a theory of characteristic classes for fibrations with homotopy finite base and fibres. These classes were implicitly used in Section 8.

Let B be a connected finite complex. Then, as in Section 8, a fibration $p\colon E \to B$ with homotopy finite fibres gives an A-valued trace map

$$\chi_A(p)\colon B \to A(*).$$

Pulling back the Borel classes $y_{4k+1} \in H^{4k+1}(A(*);\mathbb{Q})$, we obtain rational cohomology classes

$$y_{4k+1}(p) \in H^{4k+1}(B;\mathbb{Q}), \quad k > 0.$$

These classes vanish whenever p admits a compact fibre smoothing. Furthermore, they satisfy the following axioms:

- (Naturality). The classes $y_{4k+1}(p)$ are natural with respect to base change.
- (Products). For a product fibration $p \times p' \colon E \times E' \to B \times B'$ with fibre $F \times F'$, we have
$$y_{4k+1}(p \times p') \;=\; y_{4k+1}(p) \otimes \chi(F') \;+\; \chi(F) \otimes y_{4k+1}(p'),$$
where $\chi(F) \in H^0(B) \cong \mathbb{Z}$ is the Euler characteristic.
- (Additivity). If
$$\begin{array}{ccc} E_\emptyset & \longrightarrow & E_2 \\ \downarrow & & \downarrow \\ E_1 & \longrightarrow & E \end{array}$$
is a homotopy pushout of fibrations over B having homotopy finite fibres, then
$$y_{4k+1}(p) \;=\; y_{4k+1}(p_1) + y_{4k+1}(p_2) - y_{4k+1}(p_{12}),$$
where $p_S \colon E_S \to B$ for $S \subsetneq \{1,2\}$.

Remark 11.2. (1) The classes $y_{4k+1}(p) \in H^{4k+1}(B;\mathbb{Q})$ are primary obstructions to finding a compact fibre smoothing. When there is a compact fibre smoothing, one has the higher Reidemeister torsion classes
$$\tau_{4k}(p) \in H^{4k}(B;\mathbb{Q})$$
defined by Igusa [**18**]. One can view the latter as a corresponding theory of secondary characteristic classes of the fibration p that depend on the specific choice of compact fibre smoothing.

(2) Given $p \colon E \to B$, let $q \colon E \times X \to B$ be the effect of taking cartesian product with the map $X \to *$, where X is a finite complex having zero Euler characteristic. Then $y_{4k+1}(q)$ vanishes by the product axiom (compare Theorem H).

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