NONFIBERWISE SIMPLY CONNECTED POINCARÉ SURGERY

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ABSTRACT. Klein showed that a metastable Poincaré embedding $M\times I \subset X\times I$ with complement W compresses to an embedding $M \subset X$ iff an invariant $\nu^+ \in [M^+, W]$ vanishes. Klein conjectured that ν is Poincaré dual to the Hopf invariant of the unstable normal invariant $\rho \in [\Sigma(X^+), \Sigma(M/\partial M)]$, for a 1-connected middle dimensional embedding. We prove this and deduce Poincaré surgery in the simply connected case.

1. INTRODUCTION

Let X be an n-dimensional 1-connected Poincaré complex, and let $(M, \partial M)$ be an n-dimensional Poincaré pair, where M is p-dimensional as a CW complex. Assume that $n = 2p$, so we study embeddings in the middle dimension. The most interesting case is $(M, \partial M) = (S^p\times D^p, S^p\times S^{p-1})$, but we we consider the general case, since our proofs do not simplify in the case of framed emdeddings of spheres. Define

$$A = M \cup (\partial M)\times I \cup M$$

so $(M\times I, A)$ is an $(n+1)$-dimensional Poincaré pair. Suppose we have an interior Poincaré embedding $(M\times I, A) \subset X\times I$, that is, a homotopy pushout square

$$\begin{array}{ccccc} A & \xrightarrow{\ \alpha\ } & W & \xleftarrow{\ \sigma\ } & X \amalg X \\ {\scriptstyle p}\big\downarrow & & \big\downarrow & \swarrow & \\ M\times I & \xrightarrow[\ f\ ]{} & X\times I & & \end{array} \tag{1}$$

where $(W, A \amalg X \amalg X)$ is an $(n+1)$-dimensional Poincaré pair. Klein's result [Kl1, Thm. A] specialized to the middle dimension (Theorem 2.3 below) shows the only obstruction to compressing the embedding above to an embedding $(M, \partial M) \subset X$ is a Seifert-type linking invariant ν defined by pushing M into the complement in the positive direction. Defined in Composite (4) below, ν is essentially a cohomology class. We prove a conjecture of Klein (Theorem 2.2 below): the Hopf invariant of the normal invariant is Poincaré dual to the linking invariant ν. The proof is similar to the proof of Richter's earlier Seifert surface result [Ri1, Thm. 3.1]. We then use Theorems 2.3 and 2.2 together as Klein envisioned to deduce Poincaré surgery.

Our assumption that $W \rightarrowtail X\times I$ is a cofibration is unjustified, but it simplifies the argument below so we keep it for now.

2. THE HOPF INVARIANT OF THE NORMAL INVARIANT

We need some preliminaries in order to state Theorem 2.2.

Let Top_* be the category of compactly generated weak Hausdorff based spaces, which we will refer to as *pointed spaces*. Homotopy theory is done in Top_* as opposed to Top, and yet we cannot assume that our Poincaré complexes have basepoints, since we will

Thanks to Paul Burchard for the commutative diagram package, which uses XY-pic arrows.

later handle the nonsimply connected case, where we will employ Ranicki's pointed CW-π complexes. Of course, the quotient spaces like $W/X\times 1$ are naturally pointed. For an unpointed space $A \in$ Top, CA we mean the *unreduced cone* $CA := A\times I/A\times 0$. For pointed spaces, we will use the usual reduced cones and suspensions, and the usual homotopy classes of pointed maps.

Following Klein [Kl1], we will denote $M\times 0 \subset A$ by M_0 and $X\times 0 \subset X\times I$ by X_0, similarly for M_1 and X_1. W is a space over X and under $X \amalg X$, and we denote the two inclusions $X_i \subset W$ by $\sigma_i\colon X \longrightarrow W$. Similarly, A is a space over M and under $M \amalg M$, and we donote the two inclusions $M_i \subset A$ by $\sigma_i\colon M \longrightarrow A$.

Given pointed spaces A and B, we denote the inclusion of the wedge summand of $A \vee B$ by $x\colon A \longrightarrow A \vee B$ and $y\colon B \longrightarrow A \vee B$. For a pointed NDR pair (K, L), the *boundary map* $\partial\colon K/L \longrightarrow \Sigma L$ is the unique homotopy class such that the composite $K \cup C(L) \longrightarrow K/L \xrightarrow{\partial} \Sigma L$ is the natural map pinching out K.

Following Boardman and Steer [B-S], suspension will mean smashing on the right with the circle, so $\Sigma Y := Y \wedge S^1$, for any pointed space Y. Therefore by the associativity of smash products we have the canonical isomorphism $Y \wedge (\Sigma Z) \cong \Sigma(Y \wedge Z)$, which by abuse of notation we will refer to as equality. We will denote by τ the flip map on a smash power $Y \wedge Y$ taking $x \wedge y$ to $y \wedge x$. There is a selfmap of ΣY which we will call (-1), the degree minus one map, meaning that we smash id_Y with the degree minus one map on S^1. Furthermore we will write products in the group $[\Sigma A, Y]$ additively, even if the group is nonabelian. So e.g. we have $-(f + g) = -g - f$. Then the "additive inverse" is given by prefixing with the degree -1 map on ΣA, so $-f = f \cdot (-1) \in [\Sigma A, \Sigma Y]$. The group $[\Sigma^2 A, X]$ is abelian and we can reorder our "sums" arbitrarily.

We define the *normal invariant* $\rho\colon \Sigma(X^+) \longrightarrow \Sigma(M/\partial M)$ via the excision equivalence $\mathfrak{e}\colon (M\times I)/A \xrightarrow{\sim} (X\times I)/W$ and the homeomorphism $\Sigma(M/\partial M) = (M\times I)/A$. Thus, ρ is the unique homotopy class making the following diagram homotopy commute.

$$\begin{array}{ccc} \Sigma(X^+) = (X\times I)/(X \amalg X) & \longrightarrow & (X\times I)/W \\ \rho\big\downarrow & & \sim\big\uparrow \mathfrak{e} \\ \Sigma(M/\partial M) & = & (M\times I)/A \end{array} \qquad (2)$$

We need the well known lemma:

Lemma 2.1. *Given an unpointed map $p\colon E \longrightarrow B$ with a section $\sigma : B \longrightarrow E$, the composite is a homotopy equivalence:*

$$B \cup CE \xrightarrow{\partial} \Sigma(E^+) \to \Sigma(E \cup CB).$$

Proof. Consider the commutative diagram of pointed cofibration sequences:

$$\begin{array}{ccccccc} E \cup CB & \longrightarrow & * & \longrightarrow & \Sigma(E \cup CB) & = & \Sigma(E \cup CB) \\ \big\uparrow & & \big\uparrow & & \big\uparrow & & \big\uparrow \\ E^+ & \xrightarrow{p^+} & B^+ & \longrightarrow & B \cup CE & \xrightarrow{\partial} & \Sigma(E^+) \\ \sigma\big\uparrow & & \big\uparrow & & \big\uparrow & & \\ B^+ & = & B^+ & \longrightarrow & * & & \end{array}$$

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We now define the (pointed) pinch maps $h\colon A^+ \longrightarrow A/(M_0 \cup \partial M \times I) \cong M/\partial M$ and $k\colon W^+ \longrightarrow W/X_0$. We denote by σ_1' and α' the composites $X^+ \xrightarrow{\sigma_1^+} W^+ \xrightarrow{k} W/X_0$ and $A^+ \xrightarrow{\alpha^+} W^+ \xrightarrow{k} W/X_0$.

Using Lemma 2.1 twice, we define $\chi\colon \Sigma(M/\partial M) \xrightarrow{\sim} \Sigma(W/X_0)$ to be the unique homotopy class making the following diagram homotopy commute.

$$\begin{array}{ccccccc} \Sigma(X+) & \longrightarrow & (X\times I)/W & \xrightarrow{\partial} & \Sigma(W^+) & \xrightarrow{\Sigma(k)} & \Sigma(W/X_0) \\ {\scriptstyle\rho}\downarrow & & {\scriptstyle\mathfrak{e}}\uparrow{\scriptstyle\sim} & & & & {\scriptstyle\sim}\uparrow{\scriptstyle\chi} \\ \Sigma(M/\partial M) & = & (M\times I)/A & \xrightarrow{\partial} & \Sigma(A^+) & \xrightarrow{\Sigma(h)} & \Sigma(M/\partial M) \end{array} \quad (3)$$

We define the linking invariant ν to be the homotopy class of the composite

$$\nu\colon M \xrightarrow{\sigma_1} A \xrightarrow{\alpha} W \xrightarrow{k} W/X_1. \quad (4)$$

Now we can state Klein's conjecture.

Theorem 2.2. *The Hopf invariant of $\rho\colon \Sigma(X^+) \longrightarrow \Sigma(M/\partial M)$ vanishes if and only if $\nu^+ \in [M^+, W/X_0]$ vanishes. Furthermore, $\chi^{(2)} \cdot \lambda_2(\rho)$ is homotopic to the composite*

$$\Sigma^2(X^+) \xrightarrow{\Sigma\rho} \Sigma^2(M/\partial M) \xrightarrow{\Sigma^2\Delta} \Sigma(M^+) \wedge \Sigma(M/\partial M) \xrightarrow{\Sigma(\nu)\wedge\chi} (\Sigma(W/X_0))^{(2)}.$$

Klein's conjectured this result, as well as the nonsimply connected analogue, after proving his key result [Kl1, Thm. A], which we know describe.

The Poincaré embedding in Diagram (1) is said to *compress*, if there exists a Poincaré embedding $(M, \partial M) \subset X$, called the *compression*, such that the original embedding is *concordant* to the *decompression* $(M, \partial M)\times I \subset X\times I$. See [Kl1] for more information, including the definition of *concordance*, which we will not use in this paper. Now we state our middle dimensional version of Klein's compression theorem.

Theorem 2.3. *The Poincaré embedding in Diagram* (1) *compresses iff the linking invariant* (4) $\nu \in [M^+, W/X_0]$ *is trivial.*

We defer the proof to section §3.

Remark 2.4 (The idea of the proof of Theorem 2.2). Since we wish to extend our proof later to the nonsimply connected case, we assume that M is a CW-complex with 0-skeleton $K \rightarrowtail M$. In the nonsimply connected case, we will replace M/K by the CW-$\pi_1(X)$ complex $\tilde{M}/\tilde{K}$. Via $M_0 \rightarrowtail A$, we have a cofibration $K \rightarrowtail A$, and we call $j\colon A^+ \longrightarrow A/K$ the pinch map. Note that the earlier pinch maps h factors through j, by a pinch map we will call $\bar{h}\colon A/K \longrightarrow M/\partial M$, and the map p factors through $\bar{p}\colon A/K \longrightarrow M/K$. Since K is 0-dimensional, and W/X_0 is highly connected, our maps α' and ν factors through j uniquely up to homotopy, by maps $\bar{\alpha}\colon A/K \rightarrow W/X_0$ and $\bar{\nu}\colon M/K \rightarrow W/X_0$.

Following the "Massey-Mahowald-Williams" splitting of [Ri2], we have a stable splitting of $\Sigma(A/K)$ into the wedge of two spaces $\Sigma(W/X_0)$ and $\Sigma(M/K)$ which are essentially Poincaré dual. Then following the original Seifert surface intuition [Ri1, Thm. 3.1], we calculate the Hopf invariant of the composite

$$\zeta\colon \Sigma(X^+) \xrightarrow{\rho} \Sigma(M/\partial M) \xrightarrow{\partial} \Sigma(A^+) \xrightarrow{\Sigma(j)} \Sigma(A/K), \quad (5)$$

in the same way that one calculates the Hopf invariant of the Hopf construction. That is, for pointed connected spaces A and B, we define the Hopf construction $\mathfrak{h}\colon \Sigma(A \wedge B) \longrightarrow$

$\Sigma(A\times B)$ so that composite of $\mathfrak{h}$ and the homotopy equivalence

$$\Pi = x\cdot\Sigma(\pi_1) + y\cdot\Sigma(\pi_2) + z\cdot\Sigma(\pi_{1,2})\colon \Sigma(A\times B) \xrightarrow{\sim} \Sigma(A)\vee\Sigma(B)\vee\Sigma(A\wedge B)$$

is the inclusion of the third summand, hence a suspension. Applying the Cartan formula to this equation then calculates $\lambda_2(\mathfrak{h})$, or really $\Pi^{[2]}\cdot\lambda_2(\mathfrak{h})$. We calculate our Hopf invariant similarly, understanding that we need to invert our analogue of the equivalence Π.

Then by naturality of the Hopf invariant we can calculate $\lambda_2(\rho)$ from the Hopf invariant of composite (5), using the suspended retraction $\Sigma(\bar{h})\colon \Sigma(A/K)\longrightarrow\Sigma(M/\partial M)$.

We need several Lemmas for the proof of Theorem 2.2. First we make two explicit calculations of boundary maps.

Lemma 2.5. *In Diagram* (3) *above, the long horizontal composite of the*

(a) *top row is homotopic to the negative of the suspension* $\sigma_1'\colon X^+\longrightarrow W/X_0$.
(b) *bottom row is homotopic to the negative of the identity map of* $\Sigma(M/\partial M)$.

Proof. By the naturality of the boundary map, the top row is homotopic to the composite

$$\Sigma(X^+) = (X\times I)/(X_0 \amalg X_1) \xrightarrow{\partial} \Sigma(X_0\amalg X_1) \longrightarrow \Sigma(X_1^+) \xrightarrow{\sigma_1'} \Sigma(W/X_0).$$

But the composite $\Sigma(X^+) \xrightarrow{\partial} \Sigma(X_0\amalg X_1) \longrightarrow \Sigma(X_1^+)$ is the degree -1 map, by the definition of the boundary map. We see this by choosing the inverse homotopy equivalence $\Sigma(X^+)\longrightarrow (X\times I)\cup C(X_0\amalg X_1)$ to the natural pinch map $(X\times I)\cup C(X_0\amalg X_1)\longrightarrow \Sigma(X^+)$ which is degree $+1$ on $C(X_0)$ and degree -1 on $C(X_1)$. This proves Part (a).

For any pointed NDR pair (N,B) with N contractible, with nullhomotopy $H\colon C(N)\longrightarrow N$, the degree -1 map on ΣB is homotopic to the composite

$$\Sigma(B) \underset{\sim}{\xrightarrow{\bar{H}_B}} N/B \xrightarrow{\partial} \Sigma B.$$

Consider the map $j\colon (M\times I, A)\longrightarrow (N,B)$, where

$$N = (M\times I)/(M_0\cup\partial M\times I), \qquad B = A/(M_0\cup\partial M\times I) = M/\partial M.$$

The nullhomotopy $H_s[m,t] = [m,st]$ then yields as above the homotopy equivalence $\bar{H}_B\colon\Sigma(B)\xrightarrow{\sim} N/B$, which is in fact the identity map on $\Sigma(M/\partial M)$ under the two homeomorphic identifications. Part (b) now follows from the naturality of the boundary map. The long horizontal composite of the bottom row is homotopic to the composite

$$\Sigma(M/\partial M) = N/B \xrightarrow{\partial} \Sigma B = \Sigma(M/\partial M),$$

which by the above is homotopic to the negative of the identity map of $\Sigma(M/\partial M)$. □

We now list some corollaries of Lemma 2.5.

Lemma 2.6. *The composite*

(a) $\Sigma(X^+)\xrightarrow{\rho}\Sigma(M/\partial M)\xrightarrow{-\chi}\Sigma(W/X_0)$ *is homotopic to* $-\Sigma(\sigma_1')$,
(b) $\Sigma(M/\partial M) = (M\times I)/A \xrightarrow{\partial}\Sigma(A^+)\xrightarrow{\Sigma(\alpha')}\Sigma(W/X_0)$ *is homotopic to* $-\chi$,
(c) $\Sigma(X^+)\xrightarrow{\rho}(M\times I)/A\xrightarrow{\partial}\Sigma(A^+)\xrightarrow{\Sigma(\alpha')}\Sigma(W/X_0)$ *is homotopic to* $-\Sigma(\sigma_1')$.

Proof. Since $-\chi$ is the composite $\Sigma(M/\partial M)\xrightarrow{-1}\Sigma(M/\partial M)\xrightarrow{\chi}\Sigma(W/X_0)$, Part (a) follows from Lemma 2.5(a–b) and Diagram (3).

Part (b) follows from Lemma 2.5(b) and Diagram (3), including part that is not shown,

$$\begin{array}{ccccc} (X{\times}I)/W & \xrightarrow{\ \partial\ } & \Sigma(W^+) & \xrightarrow{\Sigma(k)} & \Sigma(W/X_0) \\ \mathfrak{e}\uparrow\sim & & \Sigma(\alpha^+)\uparrow & \nearrow_{\Sigma(\alpha')} & \\ (M{\times}I)/A & \xrightarrow[\partial]{} & \Sigma(A^+) & & \end{array}$$

(This homotopy commutative square was not inscribed in Diagram (3) because the square to the right of it is not necessarily homotopy commutative, due to the linking invariant ν.)

Part (c) follows from combining Part (b) and Part (c). □

Now we require a result about Hopf invariant, which we first deduced from Boardman and Steer's results [B-S] relating James-Hopf and Hilton-Hopf invariants.

Lemma 2.7. *For pointed CW complexes A and X and a map $f\colon \Sigma A \longrightarrow \Sigma X$, we have*

$$\lambda_2((-1)\cdot f\cdot(-1)) = -\tau\cdot\lambda_2(f) \in [\Sigma^2 A, (\Sigma X)^{[2]}].$$

Proof. Let $\mu(f) = -\tau\cdot\lambda_2((-1)\cdot f\cdot(-1))$. It suffices to show that $\mu(f) = \lambda_2(f)$. It follows from the proof of Boardman and Steer's main theorem [B-S] that it suffices to show that μ vanishes on the image of the suspension map $E\colon [A,X] \longrightarrow [\Sigma A, \Sigma X]$, and that μ satifies the *Cartan formula* $\mu(f+g) = \mu(f) + f{\smile}g + \mu(g)$, where the *cup product* $f{\smile}g$ is defined to be the composite

$$f{\smile}g\colon \Sigma^2 A \xrightarrow{\Sigma^2(\Delta)} \Sigma^2 A\wedge A \xrightarrow{\mathcal{S}} (\Sigma A)^{[2]} \xrightarrow{f\wedge g} (\Sigma X)^{[2]},$$

where $\mathcal{S}\colon \Sigma^2(Y\wedge Z) = Y\wedge Z\wedge S^1\wedge S^1 \longrightarrow (\Sigma Y)\wedge(\Sigma Z)$ is the *shuffle permutation* (23).

We note that for any two maps $f,g\colon \Sigma A \longrightarrow \Sigma X$, we have

$$\tau\cdot(f{\smile}g) = -g{\smile}f \in [\Sigma^2 A, (\Sigma X)^{[2]}]. \qquad (6)$$

Boardman and Steer's proof is that the flip map τ on $S^2 := S^1\wedge S^1$ has degree -1. For a suspended element $f = \Sigma(\phi) \in [\Sigma A, \Sigma X]$, we have $(-1)\cdot f\cdot(-1) = f$, so μ vanishes on suspended elements because λ_2 does. For the Cartan formula, notice that

$$(-1){\cdot}(f{+}g) = (-1){\cdot}f{+}(-1){\cdot}g \qquad \text{and} \qquad (f{+}g){\cdot}(-1) = -(f{+}g) = g{\cdot}(-1){+}f{\cdot}(-1),$$

because the group operation is given by the comultiplication in the source. So we have

$$(-1)\cdot(f+g)\cdot(-1) = (-1)\cdot g\cdot(-1) + (-1)\cdot f\cdot(-1) \in [\Sigma A, \Sigma X].$$

By the Cartan formula for λ_2, we have

$$\mu(f+g) = \mu(f) - \tau\cdot[(-1)\cdot g\cdot(-1) \smile (-1)\cdot f\cdot(-1)] + \mu(g) \in [\Sigma^2 A, (\Sigma X)^{[2]}].$$

The minus signs cancel out, since $(-1)\wedge(-1)$ is homotopic to the identity. Hence by Equation (6), we have our Cartan formula for μ. □

Remark 2.8. We sketch another proof. One can show that λ_2 is characterized by the property that the map $f_k = \Sigma(\pi_1) + \cdots + \Sigma(\pi_k)\colon \Sigma(X^k) \longrightarrow \Sigma(X)$ has Hopf invariant

$$\lambda_2(f_k) = \sum_{i<j} \mathcal{S}\cdot\Sigma^2(\pi_{i,j}) \in [\Sigma^2(X^k), (\Sigma X)^{[2]}], \qquad \forall k\in\mathbb{N}.$$

This follows from the proof of Boardman and Steer's recognition principle. But

$$(-1)\cdot f_k\cdot(-1) = -(-1)\cdot f_k = \Sigma(\pi_k) + \cdots + \Sigma(\pi_1) \in [\Sigma(X^k), \Sigma(X)]$$

is the same "sum" in reverse order, so

$$\lambda_2((-1)\cdot f_k\cdot(-1)) = \sum_{1\le i<j\le k} \mathcal{S}\cdot\Sigma^2(\pi_{j,i}) = -\tau\cdot\lambda_2(f_k)\in[\Sigma^2 X^k, (\Sigma X)^{[2]}]. \quad \square$$

Now we are ready to prove Klein's conjecture.

Proof of Theorem 2.2. First define $\eta = (-1)\cdot\rho\cdot(-1)$. This is suggested by reversing the orientation of both $X\times I$ and $M\times I$. By Lemma 2.7, we have

$$\lambda_2(\eta) = -\tau\cdot\lambda_2(\rho)\in[\Sigma^2(X^+), (\Sigma(W/X_0)^{[2]}]. \tag{7}$$

By pinching out X_0 and K in Diagram (1), we have a homotopy cocartesian diagram

$$\begin{array}{ccc} A/K & \xrightarrow{\bar\alpha} & W/X_0 \\ {\scriptstyle\bar p}\downarrow & & \downarrow \\ (M\times I)/K & \longrightarrow & *. \end{array}$$

Hence we have an equivalence

$$\Pi = x\cdot\Sigma(\bar\alpha) + y\cdot\Sigma(\bar p)\colon \Sigma(A/K)\xrightarrow{\sim}\Sigma(W/X_0)\vee\Sigma(M/K).$$

By the naturality of the coaction map and the boundary map, we have

$$\begin{array}{ccccccc} \Sigma(X^+) \xrightarrow{\rho} & (M\times I)/A & \xrightarrow{\text{coaction}} & \Sigma(A^+)\vee(M\times I)/A & \xrightarrow{\pi_1} & \Sigma(A^+) \\ \quad\searrow^{\zeta} & \downarrow{\scriptstyle\Sigma(j)\cdot\partial} & & \downarrow{\scriptstyle \mathrm{id}\vee\Sigma(j)\cdot\partial} & & \downarrow{\scriptstyle x\cdot\Sigma(\alpha')} \\ & \Sigma(A/K) & \xrightarrow{(x+y)} & \Sigma(A/K)\vee\Sigma(A/K) & \xrightarrow{\Sigma\bar\alpha\vee\Sigma\bar p} & \Sigma(W/X_0)\vee\Sigma(M/K). \end{array}$$

So in particular, $\Pi\cdot\Sigma(j)\cdot\partial$ is homotopic to $x\cdot\Sigma(\alpha')\cdot\partial$. By Lemma 2.6(c), we have

$$\Pi\cdot\zeta = x\cdot\Sigma(\bar\alpha)\cdot\partial\cdot\rho = -x\cdot\Sigma(\sigma_1')\in[\Sigma(X^+), \Sigma(W/X_0)\vee\Sigma(M/K)].$$

By pushing the minus signs around, we have, $\Pi\cdot\Sigma(j)(-\partial)\eta = x\cdot\Sigma(\sigma_1')$.

So by Boardman & Steer's composition formula and Cartan formula [B-S], we have

$$\Pi^{(2)}\cdot\lambda_2(\Sigma(j)(-\partial)\eta) = -\lambda_2(\Pi)\cdot\Sigma(\Sigma(j)(-\partial)\eta) = [x\cdot\Sigma(\bar\alpha)\smile y\cdot\Sigma(\bar p)]\cdot\Sigma(\zeta). \tag{8}$$

If we inverted $\Pi^{(2)}$, we could postfix the λ_2 term with $(\Sigma(\bar h))^{(2)}$, and we'd have

$$(\Sigma(\bar h))^{(2)}\cdot\lambda_2(\Sigma(j)(-\partial)\eta) = \lambda_2(\Sigma(\bar h)\Sigma(j)(-\partial)\eta) = \lambda_2(\Sigma(h)(-\partial)\eta) = \lambda_2(\eta), \tag{9}$$

by naturality of λ_2, Lemma 2.5(b), and $\Sigma(h)\cdot(-\partial) = \Sigma(h)\cdot\partial\cdot(-1) = -\Sigma(h)\cdot\partial$.

To invert $\Pi^{(2)}$, we factor $\Sigma(\bar h)$ through the equivalence Π. It's more convenient to factor the composite $\chi\cdot\Sigma(\bar h)\colon\Sigma(A/K)\to\Sigma(W/X_0)$ through Π. That is, there exists a map $\theta\vee\phi\colon\Sigma(W/X_0)\vee\Sigma(M/K)\to\Sigma(W/X_0)$ such that

$$(\theta\vee\phi)\cdot\Pi = \chi\cdot\Sigma(\bar h)\in[\Sigma(A/K), \Sigma(W/X_0)]. \tag{10}$$

Before solving for θ and ϕ, let's postfix Equation (8) with $(\theta\vee\phi)^{(2)}$. We obtain, by (9),

$$\chi^{(2)}\cdot\lambda_2(\eta) = [\theta\cdot\Sigma(\bar\alpha)\smile\phi\cdot\Sigma(\bar p)]\cdot\Sigma(\zeta). \tag{11}$$

Now we solve for the maps θ and ϕ. By prefixing Equation (10) by $\Sigma(j)\partial$, and using $\Pi\cdot\Sigma(j)\partial = x\cdot\Sigma(\alpha')\cdot\partial$, as well as Lemmas 2.5(b) & 2.6(b), we have

$$\theta\cdot\Sigma(\alpha')\cdot\partial = \chi\cdot\Sigma(\bar h)\cdot\Sigma(j)\partial = \chi\cdot\Sigma(h)\cdot\partial = -\chi = \Sigma(\alpha')\cdot\partial\in[(M\times I)/A, \Sigma(W/X_0)],$$

and hence $\theta = \mathrm{id}$, since $\Sigma(\alpha')\cdot\partial = -\chi$ is a homotopy equivalence.

Now prefix Equation (10) by $\Sigma(\bar{\sigma}_0)\colon \Sigma(M_0/K) \longrightarrow \Sigma(A/K)$. Since the composite $M_0/K \xrightarrow{\bar{\sigma}_0} A/K \xrightarrow{\bar{h}} M/\partial M$ is trivial, the RHS vanishes, and we obtain $\phi = -\Sigma(\bar{\nu})$.

So we've factored $\chi \cdot \Sigma(\bar{h})$ through Π. Plugging θ and ϕ into (11), we have

$$\chi^{(2)} \cdot \lambda_2(\eta) = [\Sigma(\bar{\alpha}) \smile -\Sigma(\bar{\nu}) \cdot \Sigma(\bar{p})] \cdot \Sigma(\zeta).$$

This gives a calculation of $\lambda_2(\rho)$, since the LHS above is $\chi^{(2)} \cdot \lambda_2(\eta) = -\tau \cdot \chi^{(2)} \cdot \lambda_2(\rho)$, by Equation (7). Postfixing the equation with $-\tau$, and using Equation (6), we have

$$\chi^{(2)} \cdot \lambda_2(\rho) = -[\Sigma(\bar{\nu}) \cdot \Sigma(\bar{p}) \smile \Sigma(\bar{\alpha})] \cdot \Sigma(\partial \cdot \rho),$$

since the minus sign passes through a suspension. That is, $(-1) \cdot f = -f$, if f desuspends.

Now the cup product term above involves the suspension of the composite

$$(M{\times}I)/A \xrightarrow{\partial} \Sigma(A/K) \xrightarrow{\Sigma\Delta} \Sigma(A/K \wedge A/K) \xrightarrow{\Sigma(\bar{p}\wedge\bar{\alpha})} \Sigma(M/K \wedge W/X_0),$$

which is homotopic to the composite

$$(M{\times}I)/A \xrightarrow{\partial} \Sigma(A^+) \xrightarrow{\Sigma\Delta} \Sigma(A^+ \wedge A^+) \xrightarrow{\Sigma(p^+\wedge\alpha')} \Sigma(M^+ \wedge W/X_0).$$

By "Browder-style" compatibility of cup products and diagonal maps diagrams (cf. [Ri2, diagram p. 439]) and Lemma 2.6(b), we have the homotopy commutative diagram:

$$\begin{array}{ccccccc}
(M{\times}I)/A & \xrightarrow{\tilde{\Delta}} & M^+ \wedge (M{\times}I)/A & \xrightarrow{\cong} & M^+ \wedge \Sigma(M/\partial M) & & \\
\downarrow{\scriptstyle\partial} & & & \searrow{\scriptstyle \mathrm{id}\wedge\partial} & & \searrow{\scriptstyle \mathrm{id}\wedge(-\chi)} & \\
\Sigma(A^+) & \xrightarrow[\Sigma\Delta]{} & \Sigma(A^+ \wedge A^+) & \xrightarrow[\Sigma(p^+\wedge\mathrm{id})]{} & M^+ \wedge \Sigma(A^+) & \xrightarrow[\mathrm{id}\wedge\Sigma\alpha']{} & M^+ \wedge \Sigma(W/X_0)
\end{array}$$

To see this, replace $(M{\times}I)/A$ by $(M{\times}I) \cup CA$, which we need to do in order to define the boundary map, and then the quadrilateral commutes strictly. The suspension of the bottom row followed by a shuffle is the composite $[\Sigma(p^+) \smile \Sigma(\alpha')] \cdot \Sigma(\partial)$. Furthermore the top row is strictly equal to the composite $(M{\times}I)/A = \Sigma(M/\partial M) \xrightarrow{\Sigma\tilde{\Delta}} M^+ \wedge \Sigma(M/\partial M)$.

Now we define a "homology S-duality map," the composite

$$\mathcal{D}\colon \Sigma(X^+) \xrightarrow{\rho} \Sigma(M/\partial M) \xrightarrow{\Sigma\tilde{\Delta}} M^+ \wedge \Sigma(M/\partial M) \xrightarrow[\sim]{\mathrm{id}\wedge\chi} M^+ \wedge \Sigma(W/X_0).$$

We now have that $\chi^{(2)} \cdot \lambda_2(\rho)$ is the composite

$$\Sigma^2(X^+) \xrightarrow{\Sigma(\mathcal{D})} \Sigma^2(M^+ \wedge W/X_0) \xrightarrow{\mathcal{S}} \Sigma(M^+) \wedge \Sigma(W/X_0) \xrightarrow{\Sigma(\nu^+)\wedge\mathrm{id}} (\Sigma(W/X_0))^{(2)},$$

since we can move the minus sign in $-\chi$ to the front, since it's a suspended element. □

3. Klein's Compression Theorem in the Middle Dimension

Suppose we have spaces and maps $\sigma\colon X \rightarrowtail E$ and $p\colon E \longrightarrow X$, with composite $p \cdot \sigma = \mathrm{id}_X$. In fiberwise parlance, E is called a cofibrant object in $X\backslash\mathrm{TOP}/X$.

Lemma 3.1. *Assume X and E are 1-connected, and (E, X) is r-connected. Then the map $p\colon E \longrightarrow X$ is $(r+1)$-connected, and the map*

$$p{\times}h\colon E \longrightarrow X{\times}E/X$$

is $(r+2)$-connected.

Proof. The first assertion follows from the dual of Lemma 2.1, which implies that the homotopy fiber of σ is the loop space of the homotopy fiber of p.

The connectivity of the map $p\times h$ is seen by projecting onto E/X to equal the connectivity of the map $F \longrightarrow X$, where F is the homotopy fiber of the map $E \longrightarrow E/X$. There is a natural map $\mathfrak{e}\colon X \longrightarrow F$ lifting σ, which makes F into a space over and under X. By Blakers-Massey excision, $\mathfrak{e}$ is $(r+1)$-connected, since X is 1-connected. Then the map $F \longrightarrow X$ is $(r+2)$-connected by the first assertion applied to F. □

Proof of Theorem 2.3. Klein's result [Kl1, Thm. A'] is that the Poincaré embedding in Diagram (1) compresses iff the following diagram is homotopy commutative.

$$\begin{array}{ccc} M & \xrightarrow{f} & X_0 \\ {\scriptstyle \sigma_0}\downarrow & & \downarrow{\scriptstyle \sigma_0} \\ A & \xrightarrow{\alpha} & W \end{array}$$

We recall for the reader that the engine of Klein's proof is his relative embedded thickening theorem (unpublished), which relativizes the main result [Kl2, Thm. A]. The main result [Kl2, Thm. A] is Poincaré analogue of the PL embedded thickening theorem of Wall [Wa], whereas the relative result is the Poincare analogue of the relativation of Wall's theorem given by Hodgson(insert ref). Both results of Klein alluded to here admit non-fiberwise proofs (cf. [Kl2, Rem. 4.8].

(W, X_0) is $(p-1)$-connected, since $\chi\colon \Sigma(M/\partial M) \longrightarrow \Sigma(W/X_0)$ is a homotopy equivalence and $(M, \partial M)$ is $(p-1)$-connected by Poincaré duality. Hence by Lemma 3.1, the map $W \longrightarrow X\times W/X_0$ is $(p+1)$-connected. So we have an injection

$$[M, W] \rightarrowtail [M, X]\times[M, W/X_0].$$

But the 1st components of the maps $\alpha\cdot\sigma_0, \sigma_0\cdot f \in [M, X]$ are homotopic by Diagram (1). Thus, the diagram commutes up to homotopy iff the 2nd components are equal. □

4. Simply Connected Poincaré Surgery

Given a stable Poincaré embedding (Klein's definition of a *Poincaré immersion*)

$$M\times D^m \subset X\times D^m$$

with normal invariant $r\colon \Sigma^m(X^+) \longrightarrow \Sigma^m(M/\partial M))$, suppose the stable Hopf invariant

$$H_2(r) \in \{X^+, D_2(M/\partial M)\}$$

is zero in the above cohomology group. Poincaré surgery amounts to the assertion, which we will demonstrate, that the stable embedding compresses to an embedding $(M, \partial M) \subset X$. This involves Klein's further definition of *regular homotopy* of Poincaré immersion as stable concordance, which the second author admits he does not yet understand :-0.

Using Klein's Theorem A inductively, we can compress the stable embedding to a Poincaré embedding $M\times I \longrightarrow X\times I$, say with normal invariant $\rho\colon \Sigma(X^+) \longrightarrow \Sigma(M/\partial M)$, which is a desuspension of r. If we knew that the cohomology class

$$\lambda_2(\rho) \in [\Sigma^2(X^+), (\Sigma(M/\partial M))^{[2]}]$$

was zero, then Theorem 2.2 asserts that we can compress the embedding, since $\lambda_2(\rho)$ is determined by the linking invariant ν, and by Klein's Theorem A, if $\nu = 0$, there exists a compression. But we cannot deduce $\lambda_2(\rho) = 0$ from $H_2(r) = 0$.

However, by the interlocking EHP sequences, we know there is a short exact sequence

$$[\Sigma^2(X^+),(\Sigma(M/\partial M))^{[2]}] \xrightarrow{(1+\tau)} [\Sigma^2(X^+),(\Sigma(M/\partial M))^{[2]}] \twoheadrightarrow \{X^+, D_2(M/\partial M)\},$$

and that $\lambda_2(\rho)$ projects to $H_2(r)$. Furthermore, since $H_2(r) = 0$, there exists a map $\beta\colon \Sigma(X^+) \to \Sigma(M/\partial M)^{[2]}$ such that $\rho' = \rho + [1,1]\cdot\beta$ desuspends. We must show that there also exists a compression $M\times I \to X\times I$ with normal invariant ρ'. We do this in the next section.

5. Whitehead Products and Normal Invariants

Theorem 5.1. *Given a Poincaré embedding $M\times I \to X\times I$, with normal invariant $\rho\colon \Sigma(X^+) \to \Sigma(M/\partial M)$ and a cohomology class $\beta\colon \Sigma(X^+) \to \Sigma(M/\partial M)^{[2]}$, there exists a compression of the decompression $M\times I^2 \to X\times I^2$ with normal invariant $\rho' = \rho + [1,1]\cdot\beta\colon \Sigma(X^+) \to \Sigma(M/\partial M)$.*

Proof. The decompressed embedding looks like:

$$\begin{array}{ccccc} A' & \xrightarrow{\alpha} & W' & \xleftarrow{\sigma} & X\times S^1 \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle \iota} & \swarrow{\scriptstyle \iota} & \\ M & \xrightarrow{f} & X\times I^2 & & \end{array} \qquad (12)$$

where $A' = M\times S^1 \cup (\partial M)\times I^2 = \Sigma^2_M(\partial M) = \Sigma_M(A)$.

The link $\nu\colon M \xrightarrow{\sigma_0} A' \xrightarrow{\alpha} W' \to W'/X_0$ is nullhomotopic by dimension reasons, and we have a canonical nullhomotopy from the compressed embedding. The proof of Klein's Theorem B shows that for a given nullhomotopy of ν, the normal invariant of the compressed embedding is the composite

$$\rho\colon \Sigma(X^+) = X\times S^1/X \to W'/X \xleftarrow{k} A'/M = M/A = \Sigma(M/\partial M)$$

where k is an equivalence determined by the nullhomotopy. Different nullhomotopies amount to coacting

$$A'/M \to A'/M \vee \Sigma(M) \to W'/X$$

over all maps $\Sigma(M) \to W'/X$, which are on the top cell and so detected in homology. Let's back up by k and prefix with the excision equivalence $A'/M = M/A$ to get instead

$$M/A \to M/A \vee \Sigma(A) \to M/A \vee \Sigma(M) \xrightarrow{1\vee\Sigma(\alpha)} M/A \xrightarrow{k} W'/X$$

for $\alpha \in [M, M/\partial M] \cong [\Sigma(M), \Sigma(M/\partial M)] = [\Sigma(M), M/A]$. And as I used in my Duke paper, the 1st part is the composite

$$M/A \xrightarrow{\Delta} M/A \wedge M^+ \to M/A \wedge \Omega\Sigma(M)^+ \to M/A \vee \Sigma(M),$$

where the last map is the homotopy fiber of the projection $M/A \vee \Sigma(M) \to \Sigma(M)$.

Since $M/A = \Sigma(M/\partial M)$ is a suspension and we're barely in the metastable range, we can write our composite as the sum of k and the composite

$$M/A \xrightarrow{\Delta} M/A \wedge M \xrightarrow{1\wedge\alpha} M/A \wedge (M/\partial M) \xrightarrow{[1,1]} M/A \xrightarrow{k} W'/X.$$

But since M/A is $(n+1)$-dimensional, this factors as

$$M/A \xrightarrow{p} S^{n+1} \xrightarrow{\beta} M/A \wedge (M/\partial M) \xrightarrow{[1,1]} M/A \xrightarrow{k} W'/X,$$

where β is an arbitrary of this homology group. By the usual Hilton-Hopf Barcus-Barratt nonsense, we see that the equivalence

$$1 + [1,1] \cdot b \cdot p\colon M/A \to M/A$$

has homotopy inverse

$$1 - [1,1] \cdot b \cdot p\colon M/A \to M/A,$$

and hence the new normal invariant is

$$\Sigma(X^+) \xrightarrow{\rho} M/A \xrightarrow{(1-[1,1]\cdot b\cdot p)} M/A$$

which by more Hilton-Hopf Barcus-Barratt stuff is ρ minus the composite

$$\Sigma(X^+) \to S^{n+1} \xrightarrow{b} M/A \wedge (M/\partial M) \xrightarrow{[1,1]} M/A.$$

By the EHP sequence, that picks up all possible desuspensions of the normal invariant

$$\Sigma^2(X^+) \to \Sigma M/A$$

that we started with coming from the embedding $M \to X \times I^2$. □

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