Computer Science Department of The University of Auckland
CITR at Tamaki Campus (http://www.citr.auckland.ac.nz)

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Cell Complexes through Time

Reinhard Klette [1]

Abstract

The history of cell complexes is closely related to the birth and development of topology in general. Johann Benedict Listing (1802-1882) introduced the term "topology" into mathematics in a paper published in 1847, and he also defined cell complexes for the first time in a paper published in 1862. Carl Friedrich Gauss (1777-1855) is often cited as the one who initiated these ideas, but he did not publish either on topology or on cell complexes. The pioneering work of Leonhard Euler (1707-1783) on graphs is also often cited as the birth of topology, and work was cited by Listing in 1862 as a stimulus for his research on cell complexes. There are different branches in topology which have little in common: point set topology, algebraic topology, differential topology etc. Confusion may arise if just "topology" is specified, without clarifying used concept. Topological subjects in mathematics are often related to continuous models, and therefore quite irrelevant to computer solutions in image analysis. Compared to this, only a minority of topology publications in mathematics addresses discrete spaces which appropriate for computer-based image analysis. In these cases, often the notion of a cell complex plays a crucial role. This paper briefly reports on a few of these publications, which might be helpful or at least of interest for recent studies in topological issues in image analysis. It is not a balanced review, due to a certain randomness in the selection process of cited work. This paper is also not intended to cover the very lively progress in cell complex studies within the context of image analysis during the last two decades. Basically it stops its historic review at the time when this subject in image analysis research gained speed in 1980-1990. As a general point of view, the paper indicates that image analysis contributes to a fusion of two topological concepts, the geometric or abstract cell complex approach and point set topology, which leads to an in-depth study of topologies defined on geometric or abstract cell complexes.

[1] Centre for Image Technology and Robotics, The University of Auckland, Tamaki Campus, Auckland, New Zealand

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Reinhard Klette

CITR Tamaki, University of Auckland
Tamaki Campus, Building 731, Auckland, New Zealand

ABSTRACT

The history of cell complexes is closely related to the birth and development of topology in general. Johann Benedict Listing (1802-1882) introduced the term "topology" into mathematics in a paper published in 1847, and he also defined cell complexes for the first time in a paper published in 1862. Carl Friedrich Gauss (1777-1855) is often cited as the one who initiated these ideas, but he did not publish either on topology or on cell complexes. The pioneering work of Leonhard Euler (1707-1783) on graphs is also often cited as the birth of topology, and Euler's work was cited by Listing in 1862 as a stimulus for his research on cell complexes.

There are different branches in topology which have little in common: point set topology, algebraic topology, differential topology etc. Confusion may arise if just "topology" is specified, without clarifying the used concept. Topological subjects in mathematics are often related to continuous models, and therefore quite irrelevant to computer based solutions in image analysis. Compared to this, only a minority of topology publications in mathematics addresses discrete spaces which are appropriate for computer-based image analysis. In these cases, often the notion of a cell complex plays a crucial role. This paper briefly reports on a few of these publications. This paper is not intended to cover the very lively progress in cell complex studies within the context of image analysis during the last two decades. Basically it stops its historic review at the time when this subject in image analysis research gained speed in 1980-1990. As a general point of view, the paper indicates that image analysis contributes to a fusion of topological concepts, the geometric and the abstract cell structure approach and point set topology, which may lead towards new problems for the study of topologies defined on geometric or abstract cell complexes.

Keywords: Cell complexes, digital topology, Listing, Steinitz, Reidemeister, Tucker, Rinow

1. INTRODUCTION

Theories of cell complexes emerged with the rise of topology, well before 1900. Traditionally they are a subject in *combinatorial* or *algebraic topology*. In digital image analysis, cell complexes are often used to model sets of elementary geometric units such as "pixels" or "voxels". We like to make a distinction between such cell complexes having cells in Euclidean spaces, and cell complexes where cells are only defined in an abstract (axiomatic) sense.

The definition of cells in Euclidean spaces $\mathcal{E}^n = [\mathcal{R}^n, d_2]$ is based on point set topologies of these spaces. For a set $A \subset \mathcal{E}^n$, let δA, $A^c = A \cup \delta A$, and $A^0 = A \setminus \delta A$ denote the boundary, the closure, and the interior of A, respectively. The *n-ball* B_n in $\mathcal{E}^n$ is the set of all points having a Euclidean distance of less than or equal 1 to the origin. The origin is the 0-ball B_0. A subset of $\mathcal{E}^n$ is an *open m-cell*, $m \leq n$, if it is topologically equivalent to the open m-ball B_m^0. A *closed m-cell* is the closure (in $\mathcal{E}^n$) of an open m-cell.

[GC definition] A *geometric cell structure* X, or *GC-structure* in short, is defined to be a set of countable many open or closed m-cells in $\mathcal{E}^n$, with $0 \leq m \leq n$.* This (very broad) definition of a GC-structure is introduced in

*Such a GC-structure might be interpreted following ideas of ZENO of ELEA (about 450 B.C.) that the real world is made up of indivisible units,[8] which also may be called *Democritus' atoms*. The objects are not infinitely divisible anymore. For example, a parameter may define the scale of the atoms (i.e. grid resolution in digital geometry), and there are only finitely many atoms for any bounded set. A *Zeno representation* of a given set is related to a specific geometric feature (area, perimeter etc.) and depends upon the grid resolution only. Historically (see Gauss, Jordan, Peano and others) cell complex based Zeno representations of sets in Euclidean space were directed on the contents of a set. Further features such as perimeter or surface area are of interest in digital image analysis.

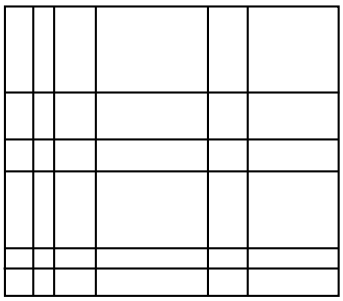

Figure 1. A rectangular grating formed by a finite number of segments across a rectangle, parallel to its sides.

this paper to make the distinction from abstract cell complexes more transparent. For geometric cell structures see [12,16,27,30,36,37,42,47]. Cells of GC-structures may be used to define *grid continua*,[41] or different types of cell-sequences[7] where a metric is important for defining concepts such as minimum-length polygons.

Textbooks by A.T. Lundell and S. Weingram[28] and by R. Fritsch and R.A. Piccinini[12] discuss GC-structures in topology as a discipline in mathematics. The fundamental notion in these books is that of a *CW-complex*, as introduced by J.H.C. Whitehead[47] in 1949. The C stands for "closure finite" (a restriction on how many cells of lower dimension a cell could meet) and the W stands for "weak topology"(the right topology for the union of the cells). Intuitively, a CW-complex can be considered as a union of disjoint 'open cells' of the Euclidean space.

[CW definition] Let S be a subset of a GC-structure X and $X(S)$ be the intersection of all subcomplexes (defined using the notion of a *skeleton*) of X containing S. A CW-complex is a cell complex satisfying the following two axioms:

- CW_1: *For any $x \in X$ it holds that $X(x)$ contains a finite number of cells only* (closure finiteness).
- CW_2: *If F is a subset of X such that the intersection $F \cap x^c$ is closed in x^c for any cell x in X, then F is a closed subset of X* (weak topology).

A forthcoming book by R. Geoghegan[13] on topological methods in group theory has an emphasis on locally finite CW-complexes (for example simplicial complexes, and regular CW-complexes).

Newman[30] † considered two-dimensional cell complexes in the form of rectangular gratings, see Fig. 1. His 0-, 1- and 2-cells may be specified to define a partition of the given rectangle into disjoint sets of vertices, edges without endpoints, and open rectangles. The grating indicates that the specific geometric size of the cells is unimportant for his studies of cell complexes in the Euclidean plane. He also discussed "curved complexes" at the end of his book.

[AC definition] Let S be an arbitrary set. Assume a binary relation between constituents of S which is denoted by $x \rho y$. A non-negative integer $[x]$, also denoted by $dim(x)$, is assigned for each $x \in S$. The following axioms are satisfied:

- AC_1: *If $x \rho y$ and $y \rho z$ then $x \rho z$ follows* (transitivity).
- AC_2: *If $x \rho y$ then $[x] < [y]$* (monotonicity).

Then $K = (S, \rho, dim)$ is an abstract cell complex, or *AC-complex* in short. The constituents of S are named *cells* of the complex. If $[x] = n$ then n is the *dimension* of x, and x is called an *n-cell*. 0-cells are named *vertices*. If $x \rho y$ then x *bounds* y, and x is a *proper side* of y. If $[x] = m$ then x is an *m-side* of y. We may also write $x \in K$ or $A \subseteq K$ for $x \in S$ or $A \subseteq S$. Let $x \leq y$ iff $x \rho y$ or $x = y$. This defines an ordering relation in K: the reflexivity is obvious by definition, the transitivity follows from AC_1. From AC_2 it follows: if $x \leq y$ and $y \leq x$ then $x = y$. Two cells are *incident* if $x \leq y$ or $y \leq x$. This relation is reflexive and symmetric. We write $(x, y) = 1$ or $inc(x, y)$ if x and y are incident. For abstract cell complexes see [36,37,42,45]. Abstract simplicial complexes, cellular imbeddings, cartesian products of graphs etc. are examples of subjects in a book by J.L. Gross and T.W. Tucker[14] on topological graph theory. See also the book by M. Henle[16] containing one chapter on cell complexes.

† He thanked J.H.C. Whitehead in the preface for many suggestions, which draws a line between his work and the CW-complexes cited earlier.

A *topology* on a set X is a system Z of subsets of X such that a union of any family of sets in Z, and an intersection of any finite family of sets in Z are sets in Z as well. The point set topology of CW-complexes can be summarized by saying that they are paracompact and Hausdorff (due to the weak topology), they are locally contractible, and a locally finite CW-complex is metrizable, and each of its components is countable. For topologies on AC-complexes see, for example, the definition and study of *open* and *closed subcomplexes* in [2,36,37,45]. These topologies of AC-complexes are not locally contractible.

AC-complexes are defined as axiomatic theories, and GC-structures may be models of such theories. For relations between theories and models see, for example, [15]. Axiomatic definitions may also be used for GC-structures, see the CW-definition above. However, in these cases the cells normally possess a fixed geometric interpretation.

Cell complexes are relevant for modelling discrete sets and their inter-relationships within the context of digital image analysis: see for example contributions on recent conferences in digital geometry and topology such as [1,4,29], or in earlier publications such as [21,23]. Publications by A. Rosenfeld[39] (see, e.g., the chapter on digital geometry in [38]), T. Pavlidis,[32] or J. Serra[40] have been important for the establishment of digital geometry and topology in image analysis. One-dimensional (or linear) cell complexes are discussed in a book by K. Voss.[46] For a more explicit treatment of (geometric) cell complexes see a recent book by G. Herman.[17] Just to cite a few authors of recent publications on cell complexes in the context of image analysis - L. DeFloriani, P. Magillo, E. Puppo,[10] Y. Kenmochi, A. Imiya, A. Ichikawa,[19] V. Kovalevsky,[24] and F. Sloboda, B. Zaťko, R. Klette[41] define and use such cell complexes, but all with (slightly) different definitions and intentions.

Publications on cell complexes often show that notations are still in flux, especially within the image analysis community, related to different "schools", and a wider agreement on fundamentals in this field could contribute to avoid repetitions of (and confusion in) basic definitions in different papers. A profound theoretical foundation of digital geometry and topology would be of eminent importance for progress in image analysis. Many researchers have contributed to this field: see, e.g., the extensive bibliography by A. Rosenfeld in [22]. It is interesting to state that a non-uniform approach to cell complexes is quite typical in the reviewed literature. In comparison, words such as "simplex", "isomorphism", or "Jordan surface" have been well-established in mathematics for about one hundred years, and do not need to be redefined. The term "cell complex" has been used since 1862, but it is one of the more "dynamic" notions in the mathematical literature, as it will be illustrated below.

The history of topology is documented by J.J. O'Connor and E.F. Robertson on the website [31]. That provides a very valuable context for the more narrowed discussion in this paper which focusses on cell complexes (combinatorial topology) and an increasing fusion with concepts in point set topology in the context of digital image analysis.

2. LISTING

J. B. Listing[27] expressed a theory of geometric cell complexes for the first time (with an earlier introduction of ideas on *complexes* in [26]). He is well known in physiological optics (*Listings's Law*, see [43]). He was also the first to use the word *topology*[‡] since 1837 in his correspondence. Being first a student and then a close friend of C.F. Gauss [6] it seems not unlikely that his research followed the advice or example of Gauss himself. The term topology (replacing Leibniz's "geometria situs" or "analysis situs") was introduced to distinguish qualitative geometry from those geometric studies focussing on quantitative relations.

In his 1862 paper he started with the famous theorem by L. Euler[11] about the relationship

$$v - e + f = 2 \tag{1}$$

between the numbers of vertices v, edges e and faces f of a simple polyhedron. As far as we know, Euler was the first mathematician able to think about polyhedra without limiting his studies to measurements. This step towards abstraction allowed him to build up fundamentals of a new discipline of topology by combinatorial studies of geometric objects. The first paper cited in [11] contains only an incomplete verification, but a complete proof is contained in the second paper.

[‡]He wrote in [27], page 109, that "*topologische* Eigenschaften (solche sind), die sich nicht auf die Quantität und das Maass der Ausdehnung, sondern auf den Modus der Anordnung und Lage beziehen." (Translation: Topological properties are those which are not related to quantity or contents, but on the mode of spatial order and position.)

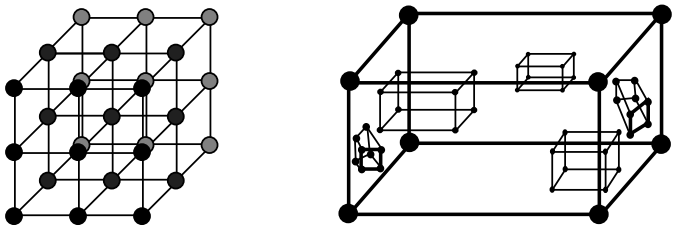

Figure 2. Left: A polyhedral aggregate as studied by Cauchy: a cube partitioned into eight subcubes has $v = 27$ vertices, $e = 54$ edges, $f = 36$ faces and $D = 8$ subpolyhedrons. Right: A polyhedral aggregate as studied by Lhuilier: a cube with $b = 3$ bubbles (all of the shape of a cube), $t = 2$ tunnels, $p = 4$ polygons within the original cube's faces, and it holds $v = 48$, $e = 72$, and $f = 32$.

A. CAUCHY[9] generalized this theorem (1) by introducing intercellular faces into the given simple polyhedron which replaces 2 by $D + 1$,

$$v - e + f = D + 1 \tag{2}$$

where D is the number of subpolyhedrons forming a *polyhedral aggregate*, see Fig. 2, left. Euler and Cauchy considered convex polyhedrons only.

A.-J. LHUILIER[25] suggested a generalization for topologically more general polyhedral aggregates, allowing tunnels and bubbles in them. He claimed that

$$v - e + f = 2(b - t + 1) + p \tag{3}$$

where b denotes the number of bubbles within a given simple polyhedron, t denotes the number of tunnels, and p is the number of polygons ("exits of tunnels") within faces of the given simple polyhedron. However, his (simplifying) induction about the number of tunnels does not cover the full range of possible topological complexity. See [3,5,18] for recent results on Euler characteristics.

[SC definition] Listing took this process of studying polyhedral aggregates of increasing structural complexity as motivation for introducing *spatial cell complexes* in $\mathcal{E}^3$ defined as arbitrary aggregates of points, lines and faces: Lines or faces may be plane or curved, open or closed, bounded or unbounded. All elements of a spatial cell complex (*SC-complex* for short) have to be connected with respect to the topology of the Euclidean space $\mathbb{E}^3$, and all elements define a partition of $\mathbb{E}^3$ into disjoint sets.

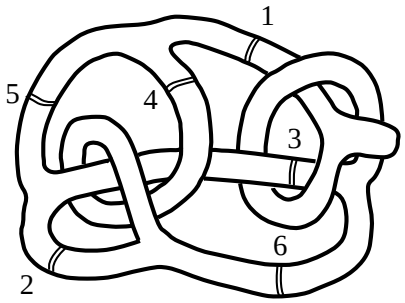

Figure 3. Listing[27]: Assume that we cut the shown solid at the labeled six positions. There are 720 different orders of such cuts. In most cases, the resulting solid is simply-connected after three cuts. In 24 cases (start sequences 136, 145, 235, 246 and their permutations), the third cut separates the solid into two parts where one is not yet simply-connected. The number of cuts to transform this solid into a simply-connected set is 3, and it is the topological genus of this set.

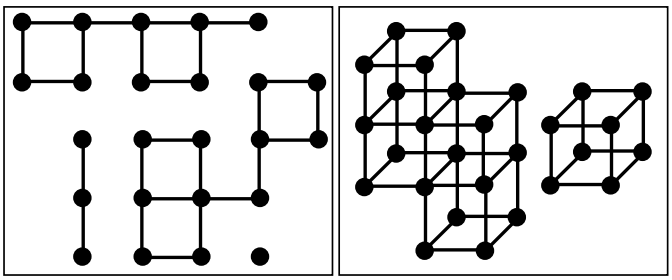

Figure 4. The four CL-complexes on the left have $v = 24$ points, $e = 25$ lines, $f = 5$ faces and $s = 1$ (note: no part of the threedimensional space has been separated) regions of space in total. The two CL-complexes on the right have $v = 28$, $e = 48$, $f = 27$ and $s = 6$.

It follows that any SC-complex is a GC-structure, symbolized by the proper inclusion $\mathbf{SC} \subset \mathbf{GC}$. Listing defined a *constituent* of a SC-complex to be one of its defining elements (points, lines, or faces) or one of the regions of space separated by elements of the spatial cell complex from one another. Thus there are four types of constituents. He ruled out that a line is considered to be an infinite union of points, a face to be an infinite union of lines, and a region of space to be an infinite union of faces - otherwise any spatial complex would convert into a set of points. He denoted the four types of constituents as *curies* (singular "Curie", plural "Curien"). Examples of SC-complexes are: a single point ($v = s = 1$ and $e = f = 0$), two points and an arc connecting both points but without containing these two end points ($v = 2$, $e = s = 1$ and $f = 0$), a closed circular line with a point on this line ($v = e = s = 1$ and $f = 0$), and the surface of a sphere or an infinite halfspace (both with $f = 1$, $s = 2$, and $v = e = 0$). For discussing such a spatial cell complex we may discriminate *effective elements* (which are counted in its curie), or *virtual elements* (which only have a temporary or auxiliary meaning).

Listing's topological studies of spatial cell complexes might be of interest for modern object analysis applications in computer vision. He introduced the *linear skeleton* of a constituent K by continuous contraction on points or lines, and called it the *cyclomatic diagram* ("cyclomatische Diagramm") of K, see page 116 in [27]. For example, the linear skeleton of a *simply-connected set* is a point, i.e. not the *medial axis* as suggested in pattern analysis [33], and that of a torus is a closed loop. Listing's work on skeletons is, for example, briefly discussed in [44].

Repeated cuts of regions in space may be discussed for specifying the topological genus, i.e. the degree of connectedness of a given set, see Fig. 3. Listing discussed such cuts within his notation of spatial complexes, i.e. cells of his SC-complexes may be of arbitrary shape. Cuts introduce new faces. Simply-connected sets are also called *acyklodic*, and these sets are finally used as constituents for combinatorial studies. An extensive discussion of this work on topological invariants, as on Listing's contributions to (algebraic) topology in general, may be found in [35].

[CL definition] Furthermore, Listing also discussed in [27] the *neighborhood relation* and the *bounding relation* for constituents in a very systematic way. His definitions are already a very important step towards the abstract boundary definition in AC-complexes. The boundary of a vertex is empty. The boundary of an edge may consist either of two, one or zero vertices, etc. A spatial cell complex is *closed with respect to the boundary relation* if for any non-empty boundary of a constituent of this complex there are constituents in this complex whose union is exactly this boundary, and if the union of a set of constituents is a closed boundary B of a bounded set then there is exactly one constituent in this complex whose boundary is identical to B. A *cellular Listing complex*, or *CL-complex* in short, is a spatial cell complex closed with respect to the boundary relation. It follows that $\mathbf{CL} \subset \mathbf{SC}$.

It follows that each constituent with a non-empty boundary, which is an element in a CL-complex, does not contain any subset of its boundary (i.e. it is open with respect to the Euclidean space). In this sense, a CL-complex is a special CW-complex in three-dimensional Euclidean space.

Listing's geometric-combinatorial studies are directed on similar problems as discussed in recent publications, such as [46] with a broad coverage of geometric-combinatorial problems, or [20] with a specific analysis of numbers of m-cells contained in the boundary of n-cells, for $0 \leq m \leq n$. Listing stated on page 131 in [27] that any *linear*

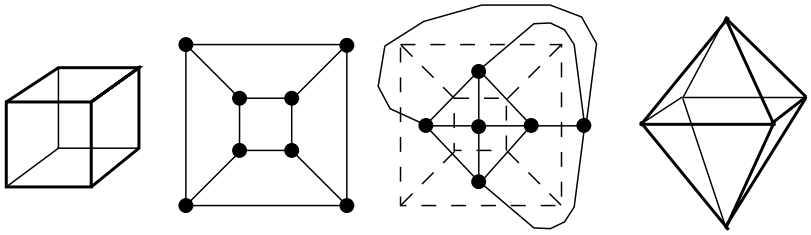

Figure 5. An example of a pair of reciprocal closed polyhedra: a cube, its planar graph, the dual graph, and the dual, an octahedron.

CL-complex (i.e. having only constituents which are points or lines) in the plane or in the surface of a sphere satisfies the equation that the number of points plus the number of faces equals the number of lines plus 2, a generalization of Euler's polyhedra theorem. A more general example is his theorem on page 151 (see Fig. 4) in [27]:

THEOREM 2.1. (Listing 1862) *Assume p CL-complexes of simply-connected constituents where the total number of points is v, of lines is e, of faces is f, and of regions of space is s. Then it holds that $v - e + f - s = p - 1$.*

See Fig. 4 for two examples of such sets of CL-complexes. Note that these sets of CL-complexes are considered with respect to the $\mathcal{E}^3$. Equation (1) is covered by this theorem for $s = 2$ and $p = 1$, and Equ. (2) is contained with $s = D + 1$ and $p = 1$. The situation of Equ. (3) is not addressed by this theorem because (3) was also intended to include situations with cyklodic faces. Some examples of CL-complexes: A closed circular line is not yet a CL-complex as required, but a cut by one point on this line is sufficient ($v = 1$, $e = 1$, $f = 1$, $s = 1$ and $p = 1$). For an (infinite) straight line we have $e = 1$, $v = 1$ (one vertex on this line is sufficient to cut this line), $f = 1$ (one half-plane bounded by this line is sufficient to cut the 3D space), $s = 1$ and $p = 1$. For the surface of a sphere we have $e = 1$ and $v = 1$ (one circle in this surface and one point on this circle are sufficient to cut the surface of the sphere, and the circle), $f = 2$, $s = 2$ and $p = 1$. For the surface of a torus we have $v = 1$, $e = 2$, $f = 3$, $s = 2$ and $p = 1$ as minimum set of resulting constituents.

3. STEINITZ

The digitization models introduced and discussed by Jordan, Peano, Minkowski and others in the late 19th and the early 20th century are further important studies on cellular spaces. Partitions into cells play a fundamental role in the topology of surfaces. H. POINCARÉ[34] used partitions of n-dimensional manifolds into cells for definitions of homology invariants (Betti numbers, torsion numbers). We select the work by E. STEINITZ[42] published at that time, for a brief discussion. Steinitz complained in [42] about the missing *axiomatic foundation* of the *analysis situs* [§] and he suggested *k-dimensional cells* as basic elements (of an axiomatic theory). He considered such cells as sets of unspecified geometric shape. However, he started with discussing simple polyhedra as being examples of such cells (i.e. as being a model of his axiomatic theory).

THEOREM 3.1. (Steinitz 1908) *For any closed n-dimensional simple polyhedron $\mathcal{P}$ there exists a reciprocal simple polyhedron $\mathcal{P}'$ such that any k-dimensional cell of $\mathcal{P}$ corresponds one-to-one to an $(n-k)$-dimensional cell of $\mathcal{P}'$ where pairs of incident cells remain incident.*

For example, the tetrahedron is reciprocal to itself, and a dodecahedron is reciprocal to an icosahedron. See Fig 5 for the regular polyhedra, cube and octahedron. Steinitz called this theorem *the first law of duality*, and he cited Poincaré[34] for a *second law of duality*:

THEOREM 3.2. (Poincaré 1895) *Reciprocal polyhedra are homeomorphic manifolds.*

Two subsets of an Euclidean space are called *homeomorphic* (or *topologically equivalent*) iff there exists a one-to-one mapping of one set onto the other which is continuous in both directions. Steinitz pointed out that the equivalence of polyhedral manifolds, following a combinatorial point of view, requires us to replace the continuity

[§] Leibniz's name for *topology*, used by Poincaré[34] for an early systematic treatment of topology.

constraint by "cell divisions". His study of such cell complexes was motivated by the goal to identify all topological invariants of n-dimensional manifolds such as *genus* or *characteristic*, the *number of boundaries*, and the *behavior of the indicatrix*.

The work of mathematicians in the 20th century on cell complexes is reviewed by W. RINOW.[37] He cites [42] as the historic source of the following definition of *boundary-finite AC-complexes*, specified by axioms AC_1, AC_2, and

- LF_1: *There is only a finite number of elements y with $y \rho x$, for each element x.*

This axiom corresponds to Listing's informal exclusions in [27] such as that a line should not be considered to be an infinite union of points. The main intention of this definition of boundary-finite AC-complexes in [37] is a topological specification of the notion of dimension. Another option would be to use the axiom

- LF_2: *There is only a finite number of elements y with $x \rho y$, for each element x.*

Note that axioms LF_1 and LF_2 are not equivalent, i.e. theories defined either by axioms AC_1, AC_2, LF_1 (boundary-finite AC-complexes) or by axioms AC_1, AC_2, LF_2 (*joint-finite AC-complexes*) do not only have different sets of theorems, but also different models. A *locally finite AC-complex* satisfies axioms AC_1, AC_2, LF_1, and LF_2.

Steinitz[42] defined *polyhedral manifolds* and did not use the term "abstract cell complexes". Steinitz reformulated the Listing complex in [42] in a more abstract, i.e. more formalized way, as follows:

Let S_n be a finite set, and $n \geq 0$. A non-negative integer $dim(x) = [x]$, $0 \leq [x] \leq n$, is assigned for each $x \in S$ specifying n-cells as defined above. Any two elements x, y are either *incident*, denoted as $(x, y) = 1$ or $(y, x) = 1$, or *non-incident*, denoted as $(x, y) = 0$ or $(y, x) = 0$. We also use a relational notation $inc(x, y)$ in case of $(x, y) = 1$. A *path* is an ordered sequence $x_1, x_2, ..., x_p$ of elements of S_n such that $(x_1, x_2) = 1$, $(x_2, x_3) = 1$,...,$(x_{p-1}, x_p) = 1$. A subset of S_n is *connected* if for any two elements x, y in this subset there exists a path in this subset having x as start and y as terminal element.

[CS definition] The structure (S_n, inc, dim) is called a *polyhedral manifold* or a *cellular Steinitz complex*, *CS-complex* in short, if the following axioms are satisfied:

- CS_1: *If $[x] = [y]$ then $(x, y) = 1$ iff $x = y$* (irreflexivity).
- CS_2: *If $(x, y) = 1$, $(y, z) = 1$, and $[x] \geq [y] \geq [z]$, then $(x, z) = 1$* (transitivity).
- CS_3: *Every 1-cell of S_n is incident with two 0-cells of S_n, and every $(n-1)$-cell is incident with one or two n-cells.*
- CS_4: *If $(x, y) = 1$ and $[x] = [y]+2$ then there are exactly two elements z in S_n satisfying $[z] = [y]+1$, $(x, z) = 1$ and $(y, z) = 1$* (boundary elements).
- CS_5: *S_n is connected.*
- CS_6: *For any element $x \in S_n$ with $[x] \geq 2$ ($[x] \leq n-2$) there is at least one element $y \in S_n$ such that $[y] < [x]$ ($[y] > [x]$) and $(x, y) = 1$, and all these elements y define a connected subset of S_n.*

In cases $n > 2$ Steinitz also postulated that the following axiom needs to be satisfied:

- CS_7: *If $(x, y) = 1$ and $[x] \geq [y] + 3$ then there is at least one element $z \in S_n$ such that $(x, z) = 1$, $(y, z) = 1$ and $[x] > [z] > [y]$, and all these elements z define a connected subset of S_n.*

Any CS-complex $P = (S_n, inc, dim)$ satisfies the axioms of a boundary-finite AC-complex, symbolized by the proper inclusion $\mathbf{CS} \subset \mathbf{AC}$. Note that the bounding relation ρ may be specified by the incidence relation inc. An AC-complex is allowed to have an infinite number of cells, a CS-complex is not. CS-complexes are examples of connected AC-complexes. In general, AC-complexes are not necessarily connected. This simplifies to consider subsets of AC-complexes as being AC-complexes again.

In the following we also say $x \in P$ instead of $x \in S_n$. The *characteristic* $C(P)$ of a polyhedral manifold P is defined as

$$C(P) = \sum_{x \in P} (-1)^{[x]} \tag{4}$$

Equations (1), (2), and (3) are historic results for such characteristics. Steinitz cited Poincaré as being the first who showed that any closed manifold of dimension 3 has characteristic zero, and that Dehn-Heegard showed that this is

actually true for closed manifolds of any odd dimension. Note that the characteristic may also be defined for the more general class of AC-complexes.

Steinitz introduced the *indicatrix* of a CS-complex as follows: At first consider a cell. We may discriminate two different indicatrices for each cell of dimension greater than zero. The indicatrix of a line segment is identical with its directional sense. For a face, the indicatrix is identical with contour orientation. The bounding faces of a three-dimensional cell obtain a specific indicatrix, which has to satisfy Möbius' edge law. Similarly, indicatrices may be defined for cells of higher dimensions, because the Möbius edge law is easy to generalize for any dimension higher than 3.

Steinitz pointed out that the theory of homeomorphisms of two-dimensional polyhedral manifolds culminates in the theorem saying that a complete system of invariants of such a manifold is given by its characteristic, its number of boundaries, and the behavior of its indicatrix. This is not the case anymore for polyhedral manifolds of higher dimensions. Further invariants have been studied by Riemann and Betti, known as Betti-numbers or coefficients of torsion. Steinitz discussed in [42] such invariants for the polyhedral manifolds as defined above. For example, he defined products of polyhedral manifolds of different dimensions for studying relationships between invariants (characteristic, indicatrix, number of boundaries) of a product and its factors.

4. REIDEMEISTER AND TUCKER

We briefly discuss two fundamental contributions published in the 30's of the 20th century: a book on the topology of polyhedra by K. Reidemeister,[36] first edition in 1938 and second edition in 1953, and an extensive paper by A.W. Tucker[45] on an abstract approach to manifolds.

Reidemeister introduced in [36] at first polyhedra as sets of points in a linear space. A set $M \subset \mathcal{R}^n$ is *polyhedron-like* if it is a convex region of space, or a sum $C_1 + ... + C_k$ of finitely many convex regions of space. Note that a polyhedron-like set needs not to be connected. A *polyhedron* is a closed polyhedron-like set of points. It follows that every polyhedron is a finite sum of closed convex regions of space. On the other hand, any finite sum of closed convex regions is a polyhedron. It follows that a finite sum of polyhedra is a polyhedron again, and a finite intersection of polyhedra is also a polyhedron again. Altogether, polyhedra are an algebraic ring with respect to the operations of sum and intersection.

Reidemeister studied then the following representation problem of polyhedra: how to define a standard representation of a polyhedron as a sum of its faces? He used cell complexes to answer this question. He showed that there are *boundary-sound* segmentations of polyhedra into convex regions of space. Such a segmentation is modelled as a cell complex. He showed that polyhedra possess properties which are invariant with respect to the chosen boundary-sound segmentation. A segmentation $\mathcal{Z}$ of a set into convex regions of space C_i is called *boundary-sound* if the following constraint is satisfied: if a region of space C_i of the segmentation $\mathcal{Z}$ contains a boundary point of another region of space C_k of $\mathcal{Z}$, then it holds that C_i is contained in the boundary δC_k of C_k. A point r is a *boundary point* of a convex open set of points C, if r is not in C and if there exists a point $p \in C$ such that the open straight line segment rp is in C. The set of all boundary points of C is its boundary δC.

Note that the definition of boundary points is an important step towards the definition of a topology on an AC-complex. Tucker[45] was probably the first who defined and studied topologies on AC-complexes, and Reidemeister's topology definition follows [45].

Theorem 4.1. (Reidemeister 1938) *Any polyhedron-like set of points possesses a boundary-sound segmentation.*

[CR definition] Reidemeister cited Tucker[45] for his definition of cell complexes. Let S be a set of cells x having dimensions $[x] = dim(x) \in \{0, 1, 2, ...\}$. We consider a binary relation ρ between cells. Reidemeister considers cell complexes containing finitely, or enumerable-infinitely many cells of bounded dimensions. A *cellular Reidemeister complex*, *CR-complex* in short, (S, ρ, dim) satisfies the axioms AC_1, AC_2, and

- CR: *If $x \rho z$ and $[z] - [x] > 1$ then there exists a cell $y \in S$ such that $x \rho y$ and $y \rho z$.*

The general AC-definition does not imply such type of completeness of a cell complex.

Theorem 4.2. (Reidemeister 1938) *Any boundary-sound segmentations of a polyhedron is a CR-complex.*

We cite a few more basic definitions and results from [36] which complete an introduction of a topology (of closed sets) on an AC-complex: a subset of a CR-complex is not necessarily a CR-complex again because axiom CR may not be satisfied. A subset of S defining a CR-complex again is called *CR-subcomplex* of S. A sum or an intersection of two CR-complexes is not necessarily a CR-complex. The *boundary complex* or *boundary* of a cell x is the set δx of all cells y with $y \rho x$, i.e. all cells which bound x. Such a boundary is a CR-complex. The *hull* or *closure* x^c of a cell x consists of x and all cells in δx. Such a hull is a CR-complex. A *pencil* of a cell x consists of x and all cells y with $x \rho y$, i.e. all cells which are bounded by x. Also such a pencil is a CR-complex. The intersection of a pencil and a hull of a cell is a CR-complex. A subset of cells is *closed* if it also contains all cells in δx for any of its cells x. A closed subset of a CR-complex S is a CR-complex, i.e. a CR-subcomplex of S. The sum or the intersection of two closed CR-subcomplexes is a closed CR-subcomplex. The *neighborhood* of a cell is the smallest closed CR-subcomplex which contains the cell and its pencil.

The topological characterization of polyhedra is the main intention of Reidemeister in [36]. Tucker motivated his research in [45] with the following two questions: "under what general conditions can aggregates of cells be regarded for combinatorial purposes as single cells? To what extent are the criteria defining combinatorial manifolds forced upon the topologist?" His text is more formalized (and harder to read) than Reidemeister's book. A definition of AC-complexes may be found on pages 193–198 in [45]. Tucker already specified open and closed subsets of (abstract) cell complexes.

Also note that Tucker and Reidemeister were more explicit (compared to Steinitz) that the cells in their AC-complexes do not necessarily have any geometric meaning anymore. The definition of cell complexes by Tucker and Reidemeister allows a study at a more abstract level. Here we do not have to ask for "disjoint" cells anymore. Cells are considered like different types (specified by the dimension) of "points" in a point set topology, and GC-structures may be models of such theories of AC-complexes.

5. RINOW

The definition of boundary-finite AC-complexes, see above, has been cited from a book by W. Rinow[37] on topology in general. In chapter VIII of his book, Rinow considered the task to show that the dimension of a set is an absolute topological invariant, and the discussed solution is based on contributions by Brouwer and Lebesgue using partitions of $\mathcal{E}^n$ into subspaces as suggested by Poincaré.[34] The introduction of cell complexes in [37] is motivated by this study of the topological invariance of dimensions.

Rinow begins his discussion of cell complexes with *Euclidean cell complexes* having convex sets as its elements. Let $C \subset \mathcal{E}^n$ be a convex region of space, and let P be a hyperplane in $\mathcal{E}^n$. If $dim(P \cap C) = n-1$ then $P \cap C$ is an $(n-1)$-*side* of C, which is also a convex region of space. The intersection of finitely many $(n-1)$-sides of an n-dimensional convex region of space C is, if not empty, a *proper side* of C. If a proper side has dimension r it is also called an r-side of C. The 0-sides are the vertices of C. The set C itself is an *improper side* and thus also a side of C. Every $(n-2)$-side of an n-dimensional convex region of space C is a side of exactly two $(n-1)$-sides of C. A *convex cell* is a bounded convex region of space.

[EC definition] An *EC-complex* K is defined to be a nonempty, at most enumerable set of convex cells of a Euclidean space $\mathcal{E}^n$, satisfying the following axioms:

- EC_1: If $z \in K$ and z' is a side of z then $z' \in K$.
- EC_2: The intersection of two cells of K is either empty or a joint side of both cells.
- EC_3: Each point in a cell of K has a neighborhood which has points in only a finite number of cells of K.

See Fig. 6 for an illustration of this definition. An EC-complex is a GC-structure where different cells are not assumed to be disjoint. However, this is just a formal aspect and could be resolved if desired. An EC-complex is *simplicial* if all of its cells are simplexes. Rinow introduced EC-complexes for the discussion of polyhedral cell complexes, and they are called *convex cell complexes* in [13]. Note that an EC-complex needs not to be connected. Analogous to the (Reidemeister) CR-complexes, EC-complexes specify another way to define partitions of polyhedra into convex sets. A subcomplex of an EC-complex is not necessarily an EC-complex again.

Two results from [37]: Each point of the Euclidean space $\mathcal{E}^n$ is contained in only a finite number of cells of such an EC-complex K. Each cell of an EC-complex K is a side of finitely many cells of K.

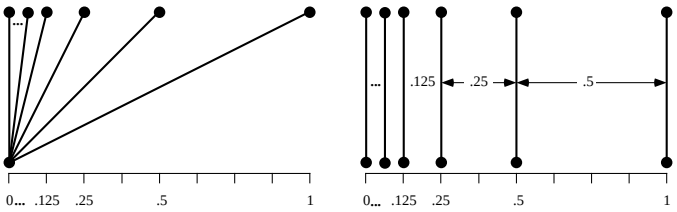

Figure 6. Rinow[37]: Both sketched sets, consisting of infinitely many vertices and edges, are not EC-complexes. Axioms EC_1 and EC_2 are satisfied but EC_3 is not. The set on the right would satisfy EC_3 if the limit edge (on the left) and its vertices are omitted.

Rinow's next step is to introduce boundary-finite AC-complexes. His motivation is to generalize from the geometric shape of cells to support a discussion of combinatorial structures of complexes. EC-complexes are models of locally finite AC-complexes (i.e. boundary-finite and joint-finite).

More definitions and results from [37]: An AC-complex $K = (S, \rho, dim)$ is *finite* if S is a finite set of cells. If K is not empty and $dim(x)$ bounded, for $x \in K$, then $dim(K)$ is the maximum of all values $dim(x)$. A cell $x \in K$ is a *basic cell* of K iff it does not bound any other cell in K. If $dim(K) = n$ then it follows that any n-cell is a basic cell. An AC-complex K is *homogeneous n-dimensional* if all of its basic cells do have the same dimension n. An undirected graph may be seen as a homogeneous 1-dimensional AC-complex where every 1-cell of the graph is bounded by two vertices (case of an *edge*), or by one vertex (case of a *loop*).

Of course, any finite AC-complex is also locally finite. Because digital geometry and topology (in the context of image analysis) is oriented towards the study of finite discrete structures we may restrict related studies on finite AC-complexes.

Consider a subset $A \subseteq S$ of a (finite) AC-complex $K = (S, \rho, dim)$. Let ρ_A and dim_A be the limitation of relation ρ and of function dim on A. Then (A, ρ_A, dim_A) is also an AC-complex, called *AC-subcomplex* of K. The relation "A is an AC-subcomplex of K" is an ordering relation in the class of all finite AC-complexes.

Reidemeister (following Tucker's work) defined already a topology on a CR-complex, see above, and his definition may be extended for AC-complexes. Rinow's definition of a topology on an AC-complex is as follows: subset A of an AC-complex K is *open* iff $x \in A$ and $x \leq y$ then $y \in A$, for all $x, y \in K$. Subset A of an AC-complex K is *closed* iff $x \in A$ and $y \leq x$ then $y \in A$, for all $x, y \in K$. This is exactly the same definition as given by Tucker and Reidemeister. Note that $\leq$ denotes the ordering relation introduced in Section 1 of this paper.

6. CONCLUSIONS

Spaces being a countable collection of geometric cells are studied in algebraic topology, and they are useful models for computer-based image analysis. Abstract cell complexes allow to discuss properties of cell complexes without depending on geometric shapes of cells. Concepts in the topology of sets of points (open, closed, topological definitions of curves or surfaces etc.) may be used to specify analogous topological objects in abstract or geometric cell complexes. It seems to be more suitable to do such studies first in abstract cell complexes, and apply results to geometric cell complexes being models of these abstract cell complexes. Topological definitions of curves and surfaces are of basic importance for two - or higher-dimensional image analysis, and these theories provide the proper tools. Recent research in image analysis goes towards a fusion of algebraic and point set topology, see, for example, publications in [1,4,22,29]. The given historic examples illustrate that this development was already initiated before digital geometry and topology have been established in the context of computer-based image analysis. These mathematical studies are suggested to be used in recent research in digital geometry and topology. This paper was not intended to review these studies, but to provide a "rough sketch" of what can be found in these publications.

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