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# Homotopical intersection theory I

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We give a new approach to intersection theory. Our "cycles" are closed manifolds mapping into compact manifolds and our "intersections" are elements of a homotopy group of a certain Thom space. The results are then applied in various contexts, including fixed point, linking and disjunction problems. Our main theorems resemble those of Hatcher and Quinn [17], but our proofs are fundamentally different.

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## 1 Introduction

In this article an intersection theory on manifolds is developed using the techniques of algebraic topology. The "cycles" in our theory will be maps between manifolds, whereas "intersections" will live in the homotopy groups of a certain Thom space. In order to give the obstructions a geometric interpretation, one must identify the homotopy of this Thom space with a suitable bordism group. Making this identification requires transversality. Consequently, if one is willing to forgo a geometric interpretation and work exclusively with Thom spaces, transversality can be dispensed with altogether. We emphasize this point for the following reason: although we will not pursue the matter here, our methods straightforwardly extend to give an intersection theory for Poincaré duality spaces, even though the usual transversality results are known to fail in this wider context.

Suppose $N$ is an $n$–dimensional compact manifold equipped with a closed submanifold

$$Q \subset N$$

of dimension $q$. Given a map $f\colon P \to N$, where $P$ is a closed manifold of dimension $p$, we ask for necessary and sufficient conditions insuring that $f$ is homotopic to a map $g$ whose image is disjoint from $Q$. We call these data an *intersection problem*.

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The situation is depicted by the diagram

$$\begin{array}{ccc} & & N - Q \\ & \nearrow & \downarrow \\ P & \xrightarrow[f]{} & N \end{array}$$

where we seek to fill in the dotted arrow by a map making the diagram homotopy commute. Note that transversality implies the above problem always has a solution when $p + q < n$.

When $f$ happens to be an embedding (or immersion), one typically requires a deformation of $f$ through isotopies (resp. regular homotopies). This version of the problem was studied by Hatcher and Quinn [17], who approached it geometrically using the methods of bordism theory (see also the related papers by Dax [11], Laudenbach [27] and Salomonsen [35]). Since many of the key proofs in [17] are just sketched, we feel it is useful to give independent homotopy theoretic proofs of their results. Also, some of our steps, such as the *Complement Formula* in Section 5, should be of independent interest.

We now summarize our approach. Let $E \to P$ be the fibration given by converting the inclusion $N - Q \to N$ into a fibration and then pulling the latter back along $P$. Then the desired lift exists if and only if $E \to P$ admits a section. Step one is to produce an obstruction whose vanishing guarantees the existence of such a section.

In a certain range of dimensions, it turns out that the complete obstruction to finding a section has been known for at least 35 years: it goes by the name of *stable cohomotopy Euler class* (see eg Crabb [9, Chapter 2], who attributes the ideas to various people, notably Becker [2; 3] and Larmore [26]).

The second step is to equate the stable cohomotopy Euler class, a "cohomological" invariant, with a *stable homotopy Euler characteristic*, a "homological" one. This is achieved using a version of Poincaré duality which appeared in Klein [22]. The characteristic lives in a homotopy group of a certain spectrum.

The third step of the program is to identify the spectrum in step two as a Thom spectrum. The idea here, which we believe is new, is to give an explicit homotopy theoretic model for the complement of the inclusion $Q \subset N$ in a certain stable range. We exhibit this model in Theorem 5.1 (this is the "Complement Formula" alluded to above).

The final step, which is optional, is to relate the Thom spectrum of step three with a twisted bordism theory (cf the next paragraph). This is a standard application of the Thom transversality theorem. As pointed out above, this step is omitted in the case of an intersection problem involving Poincaré duality spaces. This completes our outline of the program. We now proceed to state our main results in the manifold case. This will require some preparation.

We first make some well-known remarks about the relationship between bordism and Thom spectra. Suppose $X$ is a space equipped with a vector bundle $\xi$ of rank $k$. Consider triples

$$(M, g, \phi),$$

in which $M$ is a closed smooth manifold of dimension $n$ equipped with map $g\colon M \to X$ and $\phi$ is stable vector bundle isomorphism between the normal bundle of $M$ and the pullback $g^*\xi$. The set of equivalence classes of these under the relation of bordism defines an abelian group

$$\Omega_n(X; \xi),$$

in which addition is given by disjoint union. This is the *bordism group of $X$ with coefficients in $\xi$* in degree $n$.

The *Thom space* $T(\xi)$ is the one point compactification of the total space of $\xi$. If $\epsilon^j$ denotes the rank $j$ trivial bundle over $X$, then there is an evident map

$$\Sigma T(\xi \oplus \epsilon^j) \to T(\xi \oplus \epsilon^{j+1})$$

which gives the collection $\{T(\xi \oplus \epsilon^j)\}_{j\geq 0}$ the structure of a (pre-)spectrum. Its associated $\Omega$–spectrum is called the *Thom spectrum* of $\xi$, which we denote by

$$X^{\xi}.$$

The Thom–Pontryagin construction defines a homomorphism

$$\Omega_n(X; \xi) \to \pi_{n+k}(X^{\xi})$$

which is an isomorphism by transversality. These remarks apply equally as well in the more general case when $\xi$ is a virtual vector bundle of rank $k$.

More generally, if $\xi$ is a stable spherical fibration, the Thom–Pontryagin homomorphism is still defined, but can fail to be an isomorphism. The deviation from it being an isomorphism is detected by the surgery theory $L$–groups of $(\pi_1(X), w_1(\xi))$ (cf Levitt [28], Quinn [33], Jones [20] and Hausmann–Vogell [18]). In essence, the $L$–groups detect the failure of transversality.

We now return to our intersection problem. Let $i_Q\colon Q \subset N$ denote the inclusion. Define

$$E(f, i_Q)$$

to be *homotopy fiber product* of $f$ and $i_Q$ (also known as the *homotopy pullback*). Explicitly, a point of $E(f, i_Q)$ is a triple

$$(x, \lambda, y)$$

in which $x \in P$, $y \in Q$ and $\lambda\colon [0,1] \to N$ is a path such that $\lambda(0) = f(x)$ and $\lambda(1) = y$.

Define a virtual vector bundle $\xi$ over $E(f, i_Q)$ as follows: let

$$j_P\colon E(f, i_Q) \to P \text{ and } j_Q\colon E(f, i_Q) \to Q$$

be the forgetful maps and let

$$j_N\colon E(f, i_Q) \to N$$

be the map given by $(p, \lambda, q) \mapsto \lambda(1/2)$. Then $\xi$ is defined as the rank $n - p - q$ virtual bundle

$$(j_N)^*\tau_N - (j_P)^*\tau_P - (j_Q)^*\tau_Q,$$

where, for example, $\tau_N$ denotes the tangent bundle of $N$ and $(j_N)^*\tau_N$ is its pullback along $j_N$.

**Theorem A** *Given an intersection problem, there is an obstruction*

$$\chi(f, i_Q) \in \Omega_{p+q-n}(E(f, i_Q); \xi),$$

*which vanishes when $f$ is homotopic to a map with image disjoint from $Q$.*

*Conversely, if $p + 2q + 3 \le 2n$ and $\chi(f, i_Q) = 0$, then the intersection problem has a solution: there is a homotopy from $f$ to a map with image disjoint from $Q$.*

Our second main result identifies the homotopy fibers of the map of mapping spaces

$$\mathrm{map}(P, N - Q) \to \mathrm{map}(P, N)$$

in a range. Choose a basepoint $f \in \mathrm{map}(P, N)$. Then we have a homotopy fiber sequence

$$\mathcal{F}_f \to \mathrm{map}(P, N - Q) \to \mathrm{map}(P, N),$$

where $\mathcal{F}_f$ denotes the homotopy fiber at $f$.

**Theorem B** *Assume $\chi(f, i_Q)$ is trivial. Then there is a $(2n-2q-p-3)$–connected map*

$$\mathcal{F}_f \to \Omega^{\infty+1} E(f, i_Q)^{\xi},$$

*where the target is the loop space of the zeroth space of the Thom spectrum $E(f, i_Q)^{\xi}$.*

Theorem A and Theorem B are the main results of this work. In Sections Section 9–Section 11, we give applications of these results to fixed point theory, embedding theory and linking problems.

## Families

Theorem B yields an obstruction to removing intersections in families. Let

$$F\colon P \times D^j \to N$$

be a $j$–parameter family of maps whose restriction to $P \times S^{j-1}$ has disjoint image with $Q$, and whose restriction to $P \times *$ is denoted by $f$, where $* \in S^{j-1}$ is the basepoint. Assume $j > 0$.

The adjoint of $F$ determines a based map of pairs

$$(D^j, S^{j-1}) \to (\mathrm{map}(P, N), \mathrm{map}(P, N - Q))$$

whose associated homotopy class gives rise to an element of $\pi_{j-1}(\mathcal{F}_f)$. By Theorem B, this class is determined by its image in the abelian group

$$\pi_{j-1}(\Omega^{\infty+1} E(f, i_Q)^{\xi}) \cong \Omega_{j+p+q-n}(E(f, i_Q); \xi)$$

provided $j + p + 2q + 3 \le 2n$. Denote this element by $\chi^{\mathrm{fam}}(F)$.

**Corollary C** *Assume $0 < j \le 2n - 2q - p - 3$. Then the family $F\colon P \times D^j \to N$ is homotopic rel $P \times S^{j-1}$ to a family having disjoint image with $Q$ if and only if $\chi^{\mathrm{fam}}(F)$ is trivial.*

## Additional Remarks

(1) Some of the machinery developed below has recently been applied by M Aouina [1] to identify the homotopy type of the moduli space of thickenings of a finite complex in the metastable range.

(2) We plan two sequels to this paper. The first will develop tools to study equivariant intersection problems (which, for example, can be applied to the study of periodic points of dynamical systems). The second paper will consider a multirelative version in which we give an obstruction theory for deforming maps off of more than one submanifold.

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### Outline

Section 2 sets forth language. Section 3 contains various results about section spaces, and we define the stable cohomotopy Euler class. Most of this material is probably classical. In Section 4 we use a version of Poincaré duality to define the stable homotopy Euler class. Section 5 contains a "Complement Formula" which identifies the homotopy type of the complement of a submanifold in a stable range. Section 6 contains the proof of Theorem A and Section 7 the proof of Theorem B. Section 8 gives an alternative definition of the main invariant which doesn't require $i_Q\colon Q \to N$ to be an embedding. Section 9 describes a generalized linking invariant based on our intersection invariant. Section 10 applies our results to fixed point problems. Section 11 and Section 12 show how our results, in conjunction with a result of Goodwillie and first author, can be used to deduce the intersection theory of Hatcher and Quinn. In Appendix A, we outline a corresponding theory of self intersections, whose proofs are deferred to the second paper of this series.

## 2 Language

### Spaces

All spaces will be compactly generated, and products are to be re-topologized using the compactly generated topology. Mapping spaces are to be given the compactly generated, compact open topology. A *weak equivalence* of spaces denotes a (chain of) weak homotopy equivalence(s).

Some connectivity conventions: the empty space is $(-2)$–connected. Every nonempty space is $(-1)$–connected. A nonempty space $X$ is $r$–connected for $r \geq 0$ if $\pi_j(X, *)$ is trivial for $j \leq r$ for all choices of basepoint $* \in X$. A map $X \to Y$ of nonempty spaces is $(-1)$–connected and is 0–connected if it is surjective on path components. It is $r$–connected for $r > 0$ if all of its homotopy fibers are $(r-1)$–connected.

When speaking of manifolds, we work exclusively in the smooth ($C^\infty$) category. However, all results of the paper hold equally well in the PL and topological categories.

### Spaces as classifying spaces

Suppose that $Y$ is a connected based space. The simplicial total singular complex of $Y$ is a based simplicial set. Take its Kan simplicial loop group. Define $G_Y$ to be its geometric realization. Then $G_Y$ is a topological group and there is a functorial weak equivalence

$$Y \simeq BG_Y,$$

where $BG_Y$ is the classifying space of $G_Y$.

**Notation** *For the rest of the paper we resort to the following alternative notation: set*

$$\mathit{\Omega} Y := G_Y,$$

*to emphasize that $G_Y$ is a group model for the loop space. Observe we are using here the* slanted *variant "$\mathit{\Omega}$" of the standard symbol "$\Omega$" to avoid potential notational confusion between $\mathit{\Omega} Y$ and the actual loop space $\Omega Y$. (We will continue to use $\Omega Y$ for the usual loop space.)*

*Recall that $Y$ is identified with $B\mathit{\Omega} Y$ up to a chain of natural weak equivalences.*

## The thick fiber of a map

Let

$$f\colon X \to B$$

be a map, where $B = BG$ is the classifying space of a topological group $G$ which is the realization of a simplicial group.

The *thick fiber* of $f$ is the space

$$F := \mathrm{pullback}(EG \to B \leftarrow X)$$

where $EG \to B$ is a universal principal $G$–bundle. Let $G$ act on the product $EG \times X$ by means of its action on the first factor. This action leaves the subspace $F \subset EG \times X$ set-wise invariant, so $F$ comes equipped with a $G$–action.

Furthermore, as $EG$ is contractible, $F$ has the homotopy type of the homotopy fiber of $f$. Observe that $X$ has the weak homotopy type of the Borel construction $EG \times_G F$ and $f\colon X \to B$ is then identified up to homotopy as the fibration $EG \times_G F \to B$.

## Naive equivariant spectra

We will be using a low tech version of equivariant spectra, which are defined over any topological group.

Let $G$ be as above. A *(naive) $G$–spectrum* $E$ consists of based (left) $G$–spaces $E_i$ for $i \geq 0$, and equivariant based maps $\Sigma E_i \to E_{i+1}$ (where we let $G$ act trivially on the suspension coordinate of $\Sigma E_i$). A *morphism* $E \to E'$ of $G$–spectra consists of maps of based spaces $E_i \to E'_i$ which are compatible with the structure maps. A *weak equivalence* of $G$–spectra is a map inducing an isomorphism on homotopy groups. $E$ is an $\Omega$*–spectrum* if the adjoint maps $E_i \to \Omega E_{i+1}$ are weak equivalences. We will for the most part assume that our spectra are $\Omega$–spectra. If $E$ isn't an $\Omega$–spectra, we can functorially approximate it by one: $E \xrightarrow{\sim} E'$, where $E'_i$ is the homotopy colimit of the diagram of $G$–spaces $\{\Omega^k E_{i+k}\}_{k\geq 0}$. We use the notation $\Omega^\infty E$ for $E'_0$, and by slight abuse of language, we call it the *zeroth space* of $E$.

If $X$ is a based $G$–space, then its *suspension spectrum* $\Sigma^\infty X$ is a $G$–spectrum with $j$–th space $Q(S^j \wedge X)$, where $Q = \Omega^\infty\Sigma^\infty$ is the stable homotopy functor.

The *homotopy orbit spectrum*

$$E_{hG}$$

of $G$ acting on $E$ is the spectrum whose $j$th space is the orbit space of $G$ acting diagonally on the smash product $E_j \wedge EG_+$. The structure maps in this case are evident.

## 3 The stable cohomotopy Euler class

Suppose $p\colon E \to B$ is a Hurewicz fibration over a connected space $B$. We seek a generalized cohomology theoretic obstruction to finding a section.

The *fiberwise suspension* of $E$ over $B$ is the double mapping cylinder

$$S_B E := B \times 0 \cup E \times [0,1] \cup B \times 1.$$

This comes equipped with a map $S_B p\colon S_B E \to B$ which is also a fibration (cf [16, page 363], [36, page 80]) If $F_b$ denotes a fiber of $p$ at $b \in B$, then the fiber of $S_B p$ at $b$ is $SF_b$, the unreduced suspension of $F_b$.

Let

$$s_-, s_+\colon B \to S_B E$$

denote the sections given by the inclusions of $B \times 0, B \times 1$ into the double mapping cylinder. The following basic result is due to Larmore [26, Theorems 4.2–4.3].

**Proposition 3.1** *If $p\colon E \to B$ admits a section, then $s_-$ and $s_+$ are vertically homotopic.*

*Conversely, assume $p\colon E \to B$ is $(r+1)$–connected and $B$ homotopically a retract of a cell complex with cells in dimensions $\leq 2r+1$. If $s_-$ and $s_+$ are vertically homotopic, then $p$ has a section.*

**Proof** Assume $p\colon E \to B$ has a section $s\colon B \to E$. Apply the functor $S_B$ to $s$ to get a map

$$S_B s\colon S_B B \to S_B E$$

and note $S_B B = B \times [0,1]$. Then $S_B s$ is a vertical homotopy from $s_-$ to $s_+$.

To prove the converse, consider the square

$$\begin{array}{ccc} E & \xrightarrow{p} & B \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle s_+} \\ B & \xrightarrow[s_-]{} & S_B E \end{array}$$

which is commutative up to preferred homotopy. The square is homotopy cocartesian. Since the maps out of $E$ are $(r+1)$–connected, we infer via the Blakers–Massey theorem that the square is $(2r+1)$–cartesian.

Let $\mathcal{P}$ be the homotopy pullback of $s_-$ and $s_+$. Then the map $E \to \mathcal{P}$ is $(2r+1)$–connected. Furthermore, a choice of vertical homotopy from $s_-$ to $s_+$ yields a map $B \to \mathcal{P}$. Using the dimensional constraints on $B$, we can find a map $s\colon B \to E$ which factorizes $B \to \mathcal{P}$ up to homotopy. Then $ps$ is homotopic to the identity. The homotopy lifting property then enables us to deform $s$ to an actual section of $p$. $\square$

**Remark 3.2** Larmore relies on the Serre spectral sequence rather than the Blakers–Massey theorem to prove Proposition 3.1.

## Section spaces

For a fibration $E \to B$, let

$$\sec(E \to B)$$

denote its space of sections. Proposition 3.1 gives criteria for deciding when this space is non-empty. Another way to formulate it is to consider $\sec(S_B E \to B)$ as a space with basepoint $s_-$. Then the obstruction of Proposition 3.1 is given by asking whether the homotopy class

$$[s_+] \in \pi_0(\sec(S_B E \to B))$$

is that of the basepoint.

## Stabilization

As remarked above, $s_-$ equips the section space

$$\sec(S_B E \to B)$$

with a basepoint, and the fibers $SF_b$ of $S_B E \to B$ are based spaces (with basepoint given by the south pole).

Let

$$Q_B S_B E \to B$$

be the effect of applying the stable homotopy functor $Q = \Omega^\infty \Sigma^\infty$ to each fiber $SF_b$ of $S_B E \to B$.

**Lemma 3.3** *Assume $p$: $E \to B$ is $(r+1)$–connected and $B$ homotopically the retract of a cell complex with cells in dimensions $\le k$. Then the evident map*

$$\sec(S_B E \to B) \to \sec(Q_B S_B E \to B)$$

*is $(2r-k+3)$–connected.*

**Proof** For each $b \in B$, the space $SF_b$ is $(r+1)$–connected. The Freudenthal suspension theorem implies that the map $SF_b \to QSF_b$ is $(2r+3)$–connected. From this we infer that the map $S_B E \to Q_B S_B E$ is $(2r+3)$–connected. The result now follows from elementary obstruction theory. □

We call $\sec(Q_B S_B E \to B)$ the *stable section space* of $S_B E \to B$ and we change its notation to

$$\sec^{\mathrm{st}}(S_B E \to B).$$

In fact, the stable section space is the zeroth space of a spectrum whose $j$ th space is the stable section space of a fibration $E_j \to B$ in which the fiber at $b \in B$ is $Q\Sigma^j SF_b$. In particular, the set of path components of $\sec^{\mathrm{st}}(S_B E \to B)$ has the structure of an abelian group. In what follows, we regard $s_-, s_+$ as points of $\sec^{\mathrm{st}}(S_B E \to B)$.

**Definition 3.4** The *stable cohomotopy Euler class* of $p$: $E \to B$ is given by

$$e(p) := [s_+] \in \pi_0(\sec^{\mathrm{st}}(S_B E \to B)).$$

**Corollary 3.5** *If $p$ has a section, then $e(p)$ is trivial. Conversely, assume $p$: $E \to B$ is $(r+1)$–connected and $B$ is homotopically the retract of a cell complex with cells in dimensions $\le 2r+1$. If $e(p) = 0$, then $p$ has a section.*

We will also require an alternative description of the homotopy type of $\sec(E \to B)$ in a range. For a space $X$ equipped with two points $-, + \in X$, let

$$\Omega^\pm X$$

be the space of paths $\lambda$: $[0,1] \to X$ such that $\lambda(0) = -$ and $\lambda(1) = +$. When $X = SY$, and $\pm$ are the poles of $SY$, we obtain a natural map

$$Y \to \Omega^\pm QSY$$

which maps a point $y$ to the path $[0,1] \to QSY$ by $t \mapsto t \wedge y$, where $t \wedge y \in SY$ is considered as a point of $QSY$ in the evident way.

Next, suppose that $E \to B$ is a fibration. Then the associated fibration

$$Q_B S_B E \to B$$

has $QSF_b$ as its fiber at $b \in B$. So we have a map

$$F_b \to \Omega^{\pm} QSF_b.$$

Let

$$\Omega^{\pm}_B Q_B S_B E \to B$$

be the fibration whose fiber at $b$ is the space $\Omega^{\pm} QSF_b$. Then the above yields a map of section spaces

$$\sec(E \to B) \to \sec(\Omega^{\pm}_B Q_B S_B E \to B).$$

**Lemma 3.6** *Assume $E \to B$ is $(r+1)$–connected and $B$ is homotopically the retract of a cell complex with cells in dimensions $\le k$. Then the map*

$$\sec(E \to B) \to \sec(\Omega^{\pm}_B Q_B S_B E \to B)$$

*is $(2r+1-k)$–connected.*

**Proof** For each $b \in B$, the map of fibers

$$F_b \to \Omega^{\pm} QSF_b$$

factors as a composite

$$F_b \to \Omega^{\pm} SF_b \to \Omega^{\pm} QSF_b.$$

The first map in the composite is $(2r+1)$–connected (by the Blakers–Massey theorem) and the second map is is $(2r+2)$–connected (by Freudenthal's suspension theorem). Hence, the composed map is $(2r+1)$–connected. We infer by the five lemma that the map $E \to \Omega^{\pm}_B Q_B S_B E$ is also $(2r+1)$–connected. Taking section spaces then reduces the connectivity by $k$. □

**Lemma 3.7** *Fix a section $s$ of the fibration $E \to B$. Then with respect to this choice, there is a preferred weak equivalence*

$$\sec(\Omega^{\pm}_B Q_B S_B E \to B) \simeq \Omega \sec(Q_B S_B E \to B).$$

**Proof** The fiber of $(\Omega^{\pm}_B Q_B S_B E \to B)$ at $b \in B$ is the space $\Omega^{\pm} QSF_b$. The hypothesis that $E \to B$ is equipped with a section shows that $F_b$ is based and therefore $\Omega^{\pm} QSF_b$ is also based using the map $F_b \to \Omega^{\pm} QSF_b$.

A point of $\Omega^{\pm} QSF_b$ is a path in $QSF_b$ having fixed endpoints. Given another point of $\Omega^{\pm} QSF_b$, we get another path having the same endpoints. Now form the loop which starts by traversing the first path and returns by means of the second path. So we get a map

$$\Omega^{\pm} QSF_b \to \Omega QSF_b$$

which is a weak equivalence (an inverse weak equivalence is given by mapping a loop in $QSF_b$ to the path given by concatenating the base path with the given loop). This weak equivalence then induces a weak equivalence

$$\sec(\Omega^{\pm}_B Q_B S_B E \to B) \to \sec(\Omega_B Q_B S_B E \to B)$$

where $\Omega_B$ is the fiberwise loop space functor. Now use the evident homeomorphism

$$\sec(\Omega_B Q_B S_B E \to B) \cong \Omega \sec^{\mathrm{st}}(S_B E \to B)$$

to complete the proof. □

Assembling the above lemmas, we obtain the following Corollary.

**Corollary 3.8** *Let $E \to B$ be a fibration equipped with section. Assume $E \to B$ is $(r+1)$–connected and that $B$ is homotopically the retract of a cell complex whose cells have dimension $\leq k$. Then the map*

$$\sec(E \to B) \to \Omega \sec^{\mathrm{st}}(S_B E \to B)$$

*is $(2r+1-k)$–connected.*

## Further variants

This section, which is independent of the rest of the paper, gives another families variant of Proposition 3.1. For a space $X$ let $B^X$ be the space of maps from $X$ to $B$. Given a fixed map $p: E \to B$, we obtain a fibration

$$E(X; p) \to B^X$$

whose fiber at $f \in B^X$ is the homotopy pullback $f^*E$. Explicitly, $E(X, p)$ is the space of 4–tuples

$$(x, e, \lambda, f) \in X \times E \times B^{[0,1]} \times B^X$$

such that $\lambda\colon [0,1] \to B$ is a path from $f(x)$ to $p(e)$. Note the case when $X$ is a point recovers the standard way of converting $p$ into a fibration.

Now consider this construction for the fiberwise suspended map $S_B p\colon S_B E \to B$. The associated fibration

$$E(X, S_B p) \to B^X$$

comes equipped with two sections $s_\pm$ arising from the given sections of $S_B p$.

**Lemma 3.9** *Assume $p\colon E \to B$ is $(r+1)$–connected and that $X$ has the homotopy type of a cell complex of dimension $k$. Then the diagram*

$$\begin{array}{ccc} E^X & \longrightarrow & B^X \\ \downarrow & & \downarrow{\scriptstyle s_+} \\ B^X & \xrightarrow[s_-]{} & E(X, S_B p), \end{array}$$

*together with its preferred homotopy, is $(2r+1-k)$–cartesian.*

**Proof** When $X$ is a point, the space $E(\mathrm{pt}, S_B p)$ coincides up to homotopy with $S_B p$ and the result reduces to Proposition 3.1 in this case. The general case is proved by induction on cells, working relatively (we omit the details). □

Let $P^p$, $Q^q$ and $N^n$ be as in the introduction, and let $i_{N-Q}\colon N - Q \to N$ be the inclusion.

**Corollary 3.10** *The diagram*

$$\begin{array}{ccc} \mathrm{map}(P, N-Q) & \longrightarrow & \mathrm{map}(P, N) \\ \downarrow & & \downarrow{\scriptstyle s_+} \\ \mathrm{map}(P, N) & \xrightarrow[s_-]{} & E(P, S_N i_{N-Q}) \end{array}$$

*is $(2n-2q-p-3)$–cartesian.*

**Proof** The map $i_{N-Q}\colon N-Q \to N$ is $(n-q-1)$–connected and $P$ has the homotopy type of a $p$–dimensional cell complex. Now apply Lemma 3.9. □

# 4 The stable homotopy Euler characteristic

Let $B$ be a connected based space. We identify $B$ with the classifying space of the topological group $\Omega B$ described in Section 2. Let $E \to B$ be a fibration, and let $F$ be its thick fiber (Section 2). Take its unreduced suspension $SF$. Then $SF$ is a based $\Omega B$–space.

Assume now that $B$ is a closed manifold of dimension $d$. Let $\tau_B$ be its tangent bundle, and let $S(\tau_B + \epsilon)$ be the fiberwise one point compactification of $\tau_B$. Define

$$S^{\tau_B}$$

to be its thick fiber. This is a based $\Omega B$–space.

Define

$$S^{-\tau_B}$$

to be the mapping spectrum map$(S^{\tau_B}, S^0)$, ie, the spectrum whose $j$ th space consists of the stable based maps from $S^{\tau_B}$ to $S^j$. Then $S^{\tau_B}$ is an $\Omega B$–spectrum whose underlying unequivariant homotopy type is that of a $(-d)$–sphere.

Give the smash product $S^{-\tau_B} \wedge SF$ the diagonal action of $\Omega B$. Let $S^{-\tau_B} \wedge_{h\Omega B} SF$ be its homotopy orbit spectrum.

**Theorem 4.1** ("Poincaré Duality") *There is a preferred weak equivalence of infinite loop spaces*

$$\Omega^\infty(S^{-\tau_B} \wedge_{h\Omega B} SF) \simeq \sec^{\mathrm{st}}(S_B E \to B).$$

*In particular, there is an preferred isomorphism of abelian groups*

$$\pi_0(S^{-\tau_B} \wedge_{h\Omega B} SF) \cong \pi_0(\sec^{\mathrm{st}}(S_B E \to B)).$$

**Remark 4.2** A form of this statement which resembles classical Poincaré duality is given by thinking of the right side as "cohomology" with coefficients in the "cosheaf" of spectra $\mathcal{E}$ over $B$ whose stalk at $b \in B$ is the spectrum $\Sigma^\infty(SF_b)$. Then symbolically, the result identifies cohomology with twisted homology:

$$H_\bullet(B; S^{-\tau_B} \otimes \mathcal{E}) \simeq H^\bullet(B; \mathcal{E}),$$

where we interpret the displayed tensor product as fiberwise smash product.

**Proof of Theorem 4.1** The theorem is actually a special case of the main results of [22]. There, for any $\Omega B$–spectrum $W$, we constructed a weak natural transformation

$$S^{-\tau_B} \wedge_{h\Omega B} W \to W^{h\Omega B}$$

called the *norm map*, which was subsequently shown to be a weak equivalence for every $W$ (cf [22, Theorem D and Corollary 5.1]). The target of the norm map is the homotopy fixed points of $\Omega B$ acting on $W$. Recall that when $W$ is an $\Omega$–spectrum, $W^{h\Omega B}$ is the spectrum whose $j$ th space has the homotopy type of the section space of the fibration $E\Omega B \times_{\Omega B} W_j \to B$.

Specializing the norm map to the $\Omega B$–spectrum $W = \Sigma^\infty SF$, the source of the norm map is identified with $S^{-\tau_B} \wedge_{h\Omega B} SF$ whereas its target is identified with the spectrum whose associated infinite loop space is $\mathrm{sec}^{\mathrm{st}}(S_B E \to B)$. □

**Definition 4.3** The *stable homotopy Euler characteristic* of $p\colon E \to B$ is the class

$$\chi(p) \in \pi_0(S^{-\tau_B} \wedge_{h\Omega B} SF)$$

which corresponds to the stable cohomotopy Euler class $e(p)$ via the isomorphism of Theorem 4.1. That is, $\chi(p)$ is the Poincaré dual of $e(p)$.

**Corollary 4.4** *Assume $p\colon E \to B$ is $(r+1)$–connected, $B$ is a closed manifold dimension $d$ and $d \le 2r+1$. Then $p$ has a section if and only if $\chi(p)$ is trivial.*

## 5 The complement formula

Suppose

$$i_Q\colon Q^q \subset N^n$$

is the inclusion of a closed connected submanifold. We will also assume that $N$ is connected.

Choose a basepoint $* \in Q$. Then $N$ gets a basepoint. Fix once and for all identifications

$$Q \simeq B\Omega Q \quad N \simeq B\Omega N.$$

We also have a homomorphism $\Omega Q \to \Omega N$, such that application of the classifying space functor yields $i_Q\colon Q \to N$ up to homotopy.

Let

$$F := \text{thick fiber}(N - Q \to N)$$

be the thick fiber of the inclusion $N - Q \to N$ taken at the basepoint. Then $F$ is a $\Omega N$–space whose unreduced suspension $SF$ is a based $\Omega N$–space. Consequently, the suspension spectrum $\Sigma^\infty SF$ has the structure of an $\Omega N$–spectrum. We will identify the equivariant homotopy type of this spectrum.

Let $\nu_{Q\subset N}$ denote the normal bundle of $i_Q\colon Q \to N$. Form the fiberwise one-point compactification $S(\nu_{Q\subset N})$ of its total space to obtain a sphere bundle over $Q$ equipped with a preferred section at $\infty$. Let

$$S^{\nu_{Q\subset N}}$$

denote the thick fiber of $S(\nu_{Q\subset N}) \to Q$. This is, up to homotopy, a sphere whose dimension coincides with the rank of $\nu_{Q\subset N}$. Then $S^{\nu_{Q\subset N}}$ comes equipped with an $\Omega Q$–action.

Another way to construct $S^{\nu_{Q\subset N}}$ which emphasizes its dependence only on the homotopy class of $i_Q$ is as follows: let $S^{\tau_N}$ be the thick fiber of the fiberwise one point compactification of the tangent bundle of $N$. This is an $\Omega N$–spectrum, and therefore and $\Omega Q$–spectrum by restricting the action.

Similarly, using the tangent bundle $\tau_Q$ of $Q$, we obtain a based space $S^{\tau_Q}$ equipped with an $\Omega Q$–action. Let

$$S^{\tau_N - \tau_Q}$$

be the spectrum whose $j$th space consists of the stable based maps from $S^{\tau_Q}$ to the $j$–fold reduced suspension of $S^{\tau_N}$. Then $S^{\tau_N-\tau_Q}$ is a an $\Omega Q$–spectrum.

Using the stable bundle isomorphism

$$\nu_{Q\subset N} \cong i_Q^* \tau_N - \tau_Q$$

it is elementary to check that the suspension spectrum of $S^{\nu_{Q\subset N}}$ (an $\Omega Q$–spectrum) has the same weak equivariant homotopy type as $S^{\tau_N-\tau_Q}$.

**Theorem 5.1** (Complement Formula) *There is a preferred equivariant weak equivalence of $\Omega N$–spectra*

$$\Sigma^\infty SF \simeq S^{\tau_N-\tau_Q} \wedge_{h\Omega Q} (\Omega N)_+,$$

*where the right side is the homotopy orbits of $\Omega Q$ acting diagonally on the smash product $S^{\tau_N-\tau_Q} \wedge (\Omega N)_+$. Alternatively, it is the effect of inducing $S^{\tau_N-\tau_Q}$ along the homomorphism $\Omega Q \to \Omega N$ in a homotopy invariant way.*

**Remark 5.2** This result recovers the homotopy type of $S_N(N-Q)$ as a space over $N$ in the stable range. Namely, the Borel construction applied to $S^{\tau_N-\tau_Q} \wedge_{h\Omega Q} (\Omega N)_+$ gives a family of spectra over $N$, and the result says that this family coincides up to homotopy with the fiberwise suspension spectrum of $S_N(N-Q)$ over $N$.

**Proof of Theorem 5.1** Let $\nu := \nu_{Q\subset N}$ denote the normal bundle of $Q$ in $N$. Using a choice of tubular neighborhood, we have a homotopy cocartesian square

$$\begin{array}{ccc} S(\nu) & \longrightarrow & N - Q \\ \downarrow & & \downarrow \\ D(\nu) & \longrightarrow & N. \end{array}$$

There is then a weak equivalence of homotopy colimits

$$\text{hocolim}\,(D(\nu) \leftarrow S(\nu) \to N) \xrightarrow{\sim} \text{hocolim}\,(N \leftarrow (N-Q) \to N). \tag{1}$$

Each space appearing in (1) is a space over $N$. Take the thick fiber over $N$ of each of these spaces to get an equivariant weak equivalence

$$\text{hocolim}\,(\widetilde{D}(\nu) \to \widetilde{S}(\nu) \leftarrow *) \xrightarrow{\sim} \text{hocolim}\,(* \to F \leftarrow *) =: SF, \tag{2}$$

where $F$ is the thick fiber of $N - Q \to N$. The proof will be completed by identifying the domain of (2).

If $\widetilde{Q}$ denotes the thick fiber of $i\colon Q \to N$, then the domain of (2) is, by definition, the Thom space of the pullback of $\nu$ along the map $\widetilde{Q} \to Q$.

We can also identify $\widetilde{Q}$ with the homotopy orbits $\Omega Q$ acting on $\Omega N$:

$$\widetilde{Q} \simeq E\Omega Q \times_{\Omega Q} \Omega N.$$

This space comes equipped with a spherical fibration given by

$$E\Omega Q \times_{\Omega Q} (\Omega N \times S^{\nu}) \to E\Omega Q \times_{\Omega Q} (\Omega N \times *) = \widetilde{Q} \tag{3}$$

where $S^{\nu}$ denotes the thick fiber of the spherical fibration given by fiberwise one point compactifying $\nu$. The spherical fibration (3) comes equipped with a preferred section (coming from the basepoint of $S^{\nu}$. It is straightforward to check that this fibration coincides with the fiberwise one point compactification of the pullback of $\nu$ to $\widetilde{Q}$.

Consequently, the Thom space of the pullback of $\nu$ to $\widetilde{Q}$ coincides up to homotopy with the effect collapsing the preferred section of (3) to a point. But the effect of this collapse this yields

$$(\Omega N)_+ \wedge_{h\Omega Q} S^{\nu}.$$

Hence, what we've exhibited is an $\Omega N$–equivariant weak equivalence of based spaces

$$SF \simeq (\Omega N)_+ \wedge_{h\Omega Q} S^{\nu}.$$

The proof is completed by taking the suspension spectra of both sides and recalling that $\Sigma^\infty S^{\nu}$ is $S^{\tau_N - \tau_Q}$. □

# 6 Proof of Theorem A

Returning to the situation of the introduction, suppose

$$\begin{array}{ccc} & & N - Q \\ & \nearrow & \downarrow \\ P & \xrightarrow[f]{} & N \end{array}$$

is an intersection problem. As already mentioned, the obstructions to lifting $f$ up to homotopy coincide with the obstructions to sectioning the fibration

$$p\colon E \to P$$

where $E$ is the homotopy fiber product of $P \to N \leftarrow N - Q$.

Choose a basepoint for $P$. Then $N$ gets a basepoint via $f$. The thick fiber $p$ at the basepoint is identified with the thick fiber of $N - Q \to N$. Call the thick fiber of the latter $F$. Then $F$ is an $\Omega N$–space; using the homomorphism $\Omega P \to \Omega N$, we see that $F$ is also an $\Omega P$–space.

**Lemma 6.1** *The map $N - Q \to N$ is $(n - q - 1)$–connected.*

**Proof** Using the tubular neighborhood theorem, $N - Q \to N$ is the cobase change up to homotopy of the spherical fibration $S(\nu_{Q \subset N}) \to Q$ of the normal bundle of $Q$. The fibers of this fibration are spheres of dimension $n - q - 1$, so the fibration is that much connected. Then $N - Q \to N$ is also $(n-q-1)$–connected because cobase change preserves connectivity. □

**Corollary 6.2** *The map $E \to P$ is also $(n - q - 1)$–connected.*

**Proof** $E \to P$ is the base change of the the $(n - q - 1)$–connected map $N - Q \to N$ converted into a fibration. The result follows from the fact that base change preserves connectivity. □

We will now apply Corollary 4.4. For this we note that the manifold $P$ is homotopically a cell complex of dimension $p$, Consequently, if

$$p \le 2(n - q - 2) + 1 = 2n - 2q - 3,$$

Corollary 4.4 implies that a section exists if and only if the stable homotopy Euler characteristic

$$\chi(p) \in \pi_0(S^{-\tau_P} \wedge_{h\Omega P} SF)$$

is trivial.

To complete the proof, we will need to identify the homotopy type of the spectrum

$$S^{-\tau_P} \wedge_{h\Omega P} SF. \tag{4}$$

By the Complement Formula (Theorem 5.1) there is a preferred weak equivalence of $\Omega N$–spectra

$$\Sigma^\infty SF \simeq S^{\tau_N - \tau_Q} \wedge_{h\Omega Q} (\Omega N)_+.$$

Substituting this identification into (4), we get a weak equivalence of spectra

$$S^{-\tau_P} \wedge_{h\Omega P} SF \simeq S^{-\tau_P} \wedge_{h\Omega P} S^{\tau_N - \tau_Q} \wedge_{h\Omega Q} (\Omega N)_+. \tag{5}$$

To identify the right side of (5) as a Thom spectrum, rewrite it again as

$$S^{\tau_N - \tau_P - \tau_Q} \wedge_{h(\Omega P \times \Omega Q)} (\Omega N)_+.$$

Here, the action of $\Omega P \times \Omega Q$ on

$$S^{\tau_N - \tau_P - \tau_Q} = S^{\tau_N} \wedge S^{-\tau_P} \wedge S^{-\tau_Q}$$

is given by having $\Omega P$ act trivially on $S^{-\tau_Q}$, having $\Omega Q$ act trivially on $S^{-\tau_P}$, and having $\Omega P$ and $\Omega Q$ act on $S^{\tau_N}$ by restriction of the $\Omega N$ action.

Clearly, this is the Thom spectrum associated to the (stable) spherical fibration

$$(S^{\tau_N - \tau_P - \tau_Q} \times E\Omega(P \times Q)) \times_{\Omega P \times \Omega Q} \Omega N \to E\Omega(P \times Q) \times_{\Omega(P \times Q)} \Omega N.$$

It is straightforward to check that the base space of this fibration is weak equivalent to the homotopy fiber product $E(f, i_Q)$ described in the introduction.

Hence, the right side of (5) is just a Thom spectrum of the virtual bundle $\xi$ over $E(f, i_Q)$ (where $\xi$ is defined as in the introduction). Therefore, we get an equivalence of Thom spectra,

$$S^{-\tau_P} \wedge_{h\Omega P} SF \simeq E(f, i_Q)^\xi.$$

Consequently, the stable homotopy Euler characteristic becomes identified with an element of the group

$$\pi_0(E(f, i_Q)^\xi)$$

which, by transversality, coincides with the bordism group

$$\Omega_{p+q-n}(E(f, i_Q); \xi).$$

Therefore, $\chi(p)$ can be regarded as an element of this bordism group. The proof of Theorem A is then completed by applying Corollary 4.4.

## 7 Proof of Theorem B

Recall the weak equivalence

$$\mathcal{F}_f \simeq \sec(E \to P),$$

where $\mathcal{F}_f$ is the homotopy fiber of

$$\operatorname{map}(P, N - Q) \to \operatorname{map}(P, N)$$

at $f\colon P \to N$, and

$$E \to P$$

is the fibration in which $E$ is the homotopy pullback of

$$P \xrightarrow{\ f\ } N \xleftarrow{\ \supset\ } N - Q.$$

Using Corollary 6.2, we have that $E \to P$ is $(n{-}q{-}1)$–connected. So by Corollary 3.8, there is a $(2n - 2q - p - 3)$–connected map

$$\mathcal{F}_f \simeq \sec(E \to P) \to \Omega \sec^{\mathrm{st}}(S_P E \to P).$$

By Theorem 4.1 and the Complement Formula (Theorem 5.1) there is a weak equivalence

$$\sec^{\mathrm{st}}(S_P E \to P) \simeq \Omega^\infty E(f, i_Q)^\xi$$

where the right side is the Thom spectrum of associated with the bundle $\xi$ appearing in the introduction. Looping this last map, and using the previous identifications, we obtain a $(2n - 2q - p - 3)$–connected map

$$\mathcal{F}_f \to \Omega^{\infty+1} E(f, i_Q)^\xi.$$

This completes the proof of Theorem B.

## 8 A symmetric description

It is clear from its construction that the stable homotopy Euler characteristic depends only upon the homotopy class of $f\colon P \to N$ and the isotopy class of $i_Q\colon Q \to N$. Using a different description of the invariant, we will explain why it is an invariant of the *homotopy class* of $i_Q$.

The new description is more general in that it is defined not just for inclusions $i_Q\colon Q \to N$ but for any map, and it is symmetric in $P$ and $Q$.

Given maps $f\colon P \to N$ and $g\colon Q \to N$, consider the intersection problem

$$\begin{array}{ccc} & & N\times N-\Delta \\ & \nearrow & \downarrow \\ P\times Q & \xrightarrow[f\times g]{} & N\times N \end{array}$$

which asks to find a deformation of $f \times g$ to a map missing the diagonal $\Delta$ of $N \times N$. Then we have a stable homotopy Euler characteristic

$$\chi(f \times g, i_\Delta) \in \Omega_{p+q-n}(E(f \times g, i_\Delta); \xi')$$

for a suitable virtual bundle $\xi'$ defined as in the introduction. A straightforward chasing of definitions shows there to be a homeomorphism of spaces

$$E(f \times g, i_\Delta) \cong E(f, g).$$

Furthermore, if $\xi$ is the virtual bundle on $E(f, g)$ defined as in the introduction, it is clear that $\xi'$ and $\xi$ are equated via this homeomorphism. The upshot of these remarks is that we can think of $\chi(f \times g, i_\Delta)$ as an element of the bordism group

$$\Omega_{p+q-n}(E(f, g); \xi).$$

When $g = i_Q$ is an embedding, this is the same place where the our originally defined invariant $\chi(f, i_Q)$ lives.

**Theorem 8.1** *The invariants $\chi(f \times i_Q, i_\Delta)$ and $\chi(f, i_Q)$ are equal.*

**Remark 8.2** Theorem 8.1 immediately shows that $\chi(f, i_Q)$ depends only on the homotopy classes of $f$ and $i_Q$. It also extends our intersection invariant to the case when $i_Q$ isn't an embedding.

**Proof of Theorem 8.1** (Sketch). The way to compare the invariants is to consider the commutative diagram

$$\begin{array}{ccccc} & & N-Q & \longrightarrow & \mathrm{map}(Q, N^{\times 2}-\Delta) \\ & \nearrow & \downarrow & \nearrow & \downarrow \\ P & \xrightarrow[f]{} & N & \longrightarrow & \mathrm{map}(Q, N^{\times 2}) \end{array}$$

where the vertical maps are inclusions and the horizontal maps are described by mapping a point $x$ to the map $y \mapsto (x, y)$. The obstruction associated with the short dotted arrow is $\chi(f, i_Q)$, whereas the one associated with the long one is $\chi(f \times i_Q, i_\Delta)$. The result now follows from naturality. □

# 9 Linking

Let $P$, $Q$ and $N$ be compact manifolds where $P$ and $Q$ are closed. Let

$$\mathcal{D} \subset \mathrm{map}(P, N) \times \mathrm{map}(Q, N)$$

denote the *discriminant locus* consisting of those pairs of maps

$$(f, g)$$

such that $f(P) \cap g(Q)$ is non-empty. Then

$$\mathrm{map}(P, N) \times \mathrm{map}(Q, N) - \mathcal{D}$$

is the space of pairs $(f, g)$ such that $f$ and $g$ have disjoint images.

Consider the commutative diagram

$$\begin{array}{ccc} \mathrm{map}(P, N) \times \mathrm{map}(Q, N) - \mathcal{D} & \longrightarrow & \mathrm{map}(P \times Q, N^{\times 2} - \Delta) \\ \downarrow & & \downarrow \\ \mathrm{map}(P, N) \times \mathrm{map}(Q, N) & \longrightarrow & \mathrm{map}(P \times Q, N^{\times 2}), \end{array}$$

where the horizontal maps are defined by $(f, g) \mapsto f \times g$, and the vertical ones are inclusions.

Fix a basepoint $(f, g) \in \mathrm{map}(P, N) \times \mathrm{map}(Q, N)$, and define

$$F_{f,g} := \mathrm{hofiber}(\mathrm{map}(P, N) \times \mathrm{map}(Q, N) - \mathcal{D} \to \mathrm{map}(P, N) \times \mathrm{map}(Q, N)),$$

where the homotopy fiber is taken at the basepoint. Likewise, let

$$L_{f,g} := \mathrm{hofiber}(\mathrm{map}(P \times Q, (N \times N) - \Delta) \to \mathrm{map}(P \times Q, N \times N))$$

where the homotopy fiber is taken at the basepoint $f \times g$. Then we have a map of homotopy fibers

$$F_{f,g} \to L_{f,g}.$$

According to Theorem B and the remarks of the previous section, there is a $(2n - p - q - 3)$–connected map

$$L_{f,g} \to \Omega^{\infty+1} E(f, g)^{\xi}$$

where $E(f, g)$ is the homotopy fiber product of the maps $f$ and $g$ and $\xi$ is the virtual bundle or rank $n - p - q$ described as in the introduction.

**Definition 9.1** The composite map

$$\mathfrak{L}_{f,g}\colon F_{f,g} \to L_{f,g} \to \Omega^{\infty+1} E(f,g)^{\xi}$$

is called the *generalized linking map* of the pair $(f,g)$.

The induced map on path components

$$\pi_0(\mathfrak{L}_{f,g})\colon \pi_0(F_{f,g}) \to \Omega_{p+q+1-n}(E(f,g);\xi)$$

are called the *generalized linking function* of $f$ and $g$.

**Example 9.2** Take $N = D^n$, $P = S^p$ and $Q = S^q$. Take $f$ and $g$ to be the constant maps. Then $\pi_0(F_{f,g})$ is identified with the set of link homotopy classes of *link maps*

$$F\colon S^p \amalg S^q \to \mathbb{R}^n,$$

ie, maps such that $F(S^p) \cap F(S^q) = \varnothing$. This set is often denoted by

$$L^n_{p,q}.$$

The space $E(f,g)$ is identified with $S^p \times S^q$, and the virtual bundle $\xi$ is trivial. If we additionally assume that $p, q \le n-2$, then the bordism group $\Omega_{p+q+1-n}(E(f,g);\xi)$ is isomorphic to the stable homotopy group

$$\pi^{\mathrm{st}}_{p+q+1-n}(S^0).$$

With respect to these identifications, the generalized linking function takes the form

$$\pi_0(\mathfrak{L}_{f,g})\colon L^n_{p,q} \to \pi^{\mathrm{st}}_{p+q+1-n}(S^0),$$

and is just Massey and Rolfsen's "$\alpha$–invariant" (see [30] and Koschorke [25]). When $p+q+1 = n$, the invariant is an integer and is just the classical Hopf linking number.

**Remark 9.3** When $p+q+1 = n$, the generalized linking function is closely related to the affine linking invariants defined in a recent paper of Chernov and Rudyak [7]. The difference between our invariants and theirs has to do with indeterminacy.

Namely, a point in $F_{f,g}$ consists of a pair of maps $f_1\colon P \to N$ and $g_1\colon Q \to N$ having disjoint images, together with a choice of homotopies from $f_1$ to $f$ and $g_1$ to $g$. The choice of homotopies allows us to get an invariant living in the full bordism group.

By contrast, Chernov and Rudyak do not choose the homotopies, but this comes at the cost of obtaining an invariant landing in a certain *quotient* of the bordism group. The subgroup which one quotients out by measures the indeterminacy arising from the choice the homotopies.

# 10 Fixed point theory

Suppose we are given a closed manifold $M$ of dimension $m$ and a self map $f\colon M \to M$. Then the *graph* of $f$ is the map

$$G_f = (\mathrm{id}, f)\colon M \to M \times M.$$

Then $G_f$ is homotopic to a map missing the diagonal $\Delta \subset M \times M$, if and only if $f$ is homotopic to a fixed-point free map.

Actually, quite a bit more is true. Let

$$\mathrm{map}^\flat(M, M) \subset \mathrm{map}(M, M)$$

be the subspace of fixed point free self maps of $M$. Then we have a commutative square

$$\begin{array}{ccc} \mathrm{map}^\flat(M, M) & \longrightarrow & \mathrm{map}(M, M) \\ \downarrow & & \downarrow \\ \mathrm{map}(M, M^{\times 2} - \Delta) & \longrightarrow & \mathrm{map}(M, M^{\times 2}) \end{array} \tag{6}$$

whose horizontal arrows are inclusions and whose vertical ones are given by taking the graph of a self map. Since the first factor projection $M^{\times 2} - \Delta \to M$ is a fibration, the induced map of mapping spaces

$$\mathrm{map}(M, M^{\times 2} - \Delta) \to \mathrm{map}(M, M)$$

is also a fibration whose fiber at the identity is $\mathrm{map}^\flat(M, M)$. Similarly, the first factor projection map $M^{\times 2} \to M$ gives a fibration

$$\mathrm{map}(M, M^{\times 2}) \to \mathrm{map}(M, M)$$

whose fiber at the identity is $\mathrm{map}(M, M)$. Consequently, the square (6) is homotopy cartesian.

We infer that the homotopy fiber of the map

$$\mathrm{map}^\flat(M, M) \to \mathrm{map}(M, M)$$

taken at $f$ coincides with the homotopy fiber of the map

$$\mathrm{map}(M, M^{\times 2} - \Delta) \to \mathrm{map}(M, M^{\times 2})$$

taken at $G_f$.

Therefore, solutions of the intersection problem

$$\begin{array}{ccc} & & M^{\times 2} - \Delta \\ & \nearrow & \downarrow \\ M & \xrightarrow[G_f]{} & M^{\times 2} \end{array}$$

correspond to deformations of $f$ to a fixed point free map. Furthermore, the relevant virtual bundle $\xi$ is this case is trivial of rank $0$.

Hence, applying Theorem A and Theorem B, we immediately obtain

**Theorem 10.1** *Let $f\colon M \to M$ be a self map of a closed manifold $M$. Then there is a well-defined class*

$$\ell(f) := \chi(G_f, \Delta) \in \Omega_0^{\mathrm{fr}}(L_f M),$$

*where $L_f M$ denotes the space of paths $\gamma\colon [0,1] \to M$ such that $\gamma(1) = f(\gamma(0))$, and $\Omega_0^{\mathrm{fr}}(L_f M)$ is its framed bordism group in degree zero.*

1. *When $f$ is homotopic to a fixed-point free map, then $\ell(f)$ is trivial. Conversely, if $m \geq 3$, and $\ell(f)$ is trivial, then $f$ is homotopic to a fixed-point free map.*
2. *Assume $\ell(f)$ is trivial. Let $\mathcal{F}_f$ be denote the homotopy fiber of the map*

$$\mathrm{map}^\flat(M, M) \xrightarrow{\subset} \mathrm{map}(M, M),$$

*taken at $f$. Then there is an $(m-3)$–connected map*

$$\mathcal{F}_f \to \Omega Q(L_f M)_+.$$

*where the target is the loop space of the stable homotopy functor applied to $L_f M$ with the union of a disjoint basepoint.*

**Remarks 10.2**

1. The map $L_f M \to \mathrm{pt}$ induces a homomorphism

$$\Omega_0^{\mathrm{fr}}(L_f M) \to \Omega_0^{\mathrm{fr}}(\mathrm{pt}) \cong \mathbb{Z}.$$

It is possible to show that image of $\ell(f)$ under this homomorphism is just the classical *Lefschetz trace* of $f$.
In fact, the group $\Omega_0^{\mathrm{fr}}(L_f M)$ is isomorphic to the coinvariants of $\pi = \pi_1(M, *)$ acting on its integral group ring by means of *twisted conjugation*: $(g, x) \mapsto gx\phi(g)^{-1}$, for $g, x \in \pi$ where $\phi\colon \pi \to \pi$ is the homomorphism induced by $f$ and a choice of path from $*$ to $f(*)$.

Using our Index Theorem 12.1, $\ell(f)$ can be identified with the *Reidemeister trace of* $f$ [34]. With respect to this identification, the first part of Theorem 10.1 is essentially equivalent to the homotopy converse to the Lefschetz fixed point theorem. (cf Wecken [37] and Brown [4], [5, Corollary 3])

(2) The self map $f$ describes an action of the natural numbers $\mathbb{N}$ on $M$. The space $L_f M$ is just the homotopy fixed point set $M^{h\mathbb{N}}$ of this action, and the bordism class of the theorem is the one associated with the evident inclusion

$$M^{\mathbb{N}} \subset M^{h\mathbb{N}}$$

from fixed points to homotopy fixed points.

(3) The space $Q(L_f M)_+$ has an alternative description as the *topological Hochschild homology* of the associative ring (spectrum)

$$S^0[\Omega M]$$

(= the suspension spectrum of $(\Omega M)_+$) with coefficients in the bimodule given by $S^0[\Omega M]$, where the action is defined by twisted conjugation of loops.

## Families over a disk

$$F\colon M \times D^j \to M$$

be a $j$–parameter family of self maps of $M$ such that the restriction of $F$ to $M \times S^{j-1}$ is fixed point free. Let $f\colon M \to M$ be the the self map associated with the basepoint of $S^{j-1}$.

Then $F$ defines a map of pairs

$$(D^j, S^{j-1}) \to (\mathrm{map}(M, M), \mathrm{map}^\flat(M, M))$$

which by Theorem 10.1 yields an invariant

$$\ell^{\mathrm{fam}}(F) \in \Omega_j^{\mathrm{fr}}(L_f M).$$

**Corollary 10.3** *Assume $j \geq 1$. If $F\colon M \times D^j \to M$ is homotopic rel $M \times S^{j-1}$ to a fixed point free family, the $\ell^{\mathrm{fam}}(F) = 0$.*

*If $j \leq m-3$ and $\ell^{\mathrm{fam}}(F) = 0$, then the converse also holds.*

**Remark 10.4** Using different methods, an invariant of this type was discovered by Geoghegan and Nicas (see [13]), who have intensively studied the $j = 1$ case.

## General families

The previous situation can be generalized as follows: suppose that $p\colon E \to B$ is a smooth fiber bundle over a connected space $B$ having closed manifold fibers. Given a fiberwise self map $f\colon E \to E$, one asks whether $f$ is fiberwise homotopic to a map having no fixed points (the case when $B$ is disk coincides with the previous situation). This type of fixed point problem was studied by Dold [12].

For $b \in B$, let $f_b\colon E_b \to E_b$ be the induced self map of the fiber at $b$. Then we have a fibration

$$q\colon \mathcal{E}_f \to B$$

whose fiber at $b$ is $Q(L_{f_b} E_b)_+$. Note that the space of sections of $q$ is an infinite loop space. So if we let

$$\Omega_0^{B,\mathrm{fr}}(\mathcal{E}_f)$$

denote the set of homotopy classes of sections of $q$, we see that the latter has the structure of an abelian group.

**Theorem 10.5** *There is a well-defined element*

$$\ell_B(f) \in \Omega_0^{B,\mathrm{fr}}(\mathcal{E}_f),$$

*which vanishes when $f$ is fiberwise homotopic to a fixed point free self map.*

*Conversely, assume the fibers of $p$ have dimension $m$, and $B$ is homotopically the retract of a cell complex with cells of dimension $\le m-3$. Then $\ell_B(f) = 0$ implies that $f$ is fiberwise homotopic to a fixed point free self map.*

A version of this theorem was proved by V Coufal [8], relying on the fibered theory of Hatcher and Quinn [17, Theorem 4.2].

We will not give the proof of Theorem 10.5 here because a substantial revision of our methods would be needed. Rather, we will be content to explain the main issue. The situation at hand is represented by a family of intersection problems parameterized by points of some parameter space $B$. Hence, Theorem 10.5 would follow from a suitable generalization of Theorem B.

The required generalization is this: we are given a parameter space $B$ and a (continuous) family of intersection problems

$$\begin{array}{ccc} & & N_b - Q_b \\ & \nearrow & \downarrow \\ P_b & \xrightarrow[f_b]{} & N_b \end{array}$$

parameterized by $b \in B$. The approach of this paper is to *choose a basepoint* in each $P_b$. Once a continuously varying family of basepoints is chosen, the invariant can be defined, and furthermore, the parameterized intersection problem can be attacked using the machinery which proves Theorem B. The problem with this approach is: *a continuously varying family of basepoints might not exist*.

The reason we needed to choose basepoints is that we converted a *fiberwise* problem—finding a section of some fibration—into an *equivariant* one, in which the group is a loop space of the base space of the fibration (the conversion was made by taking a thick fiber). The basepoint was needed to form the loop space.

Hence, to avoid basepoint issues, we should not have passed to the equivariant category at all. That is, we should have developed all the technology in the fiberwise setting. A review of our constructions shows that the crux of the matter is to reformulate Theorem 4.1. Recall that this result relies heavily on the main results of [22], and the latter paper is written in the equivariant context. Fortunately, the relevant results of that paper can indeed be recast in the fiberwise setting (see eg Hu [19]). Rather than deluge the reader with technical details, we will leave it to him/her to transform this discussion into a proof of Theorem 10.5.

**Remark 10.6** Considering the map $E \to B$ as a map of spaces over $B$, we get an induced homomorphism

$$\Omega_0^{B,\mathrm{fr}}(\mathcal{E}_f) \to \Omega_0^{B,\mathrm{fr}}(\mathrm{pt}) \cong \pi^0_{\mathrm{st}}(B_+)$$

where the target is the stable cohomotopy of $B$ in degree zero.

Let $\mathrm{map}_B(E, E)$ denote the space of fibered self maps of $E$. Then Theorem 10.5, followed by above, yields a function

$$I\colon \pi_0(\mathrm{map}_B(E, E)) \to \pi^0_{\mathrm{st}}(B_+),$$

which presumably coincides with Dold's fixed point index [12] (compare Crabb–James [10]).

## 11 Disjunction

A *disjunction problem* is an intersection problem

$$\begin{array}{ccc} & & N - Q \\ & \nearrow & \downarrow \\ P & \xrightarrow[f]{} & N \end{array}$$

in which $f\colon P \to N$ is a smooth embedding. In this instance we ask for an isotopy of $f$ to another embedding $g$ such that the image of $g$ is disjoint from $Q$. Let

$$\mathrm{emb}(P, N)$$

be the space of embeddings of $P$ in $N$.

The following is a special instance of the main results of [14]. Its proof involves a mixture of homotopy theory, surgery, and Morlet's disjunction lemma (Morlet [32], Burghelea–Lashof–Rothenburg [6]).

**Theorem 11.1** Goodwillie–Klein [14] *Assume $p, q \le n-3$. Then the commutative square*

$$\begin{array}{ccc} \mathrm{emb}(P, N-Q) & \longrightarrow & \mathrm{emb}(P, N) \\ \downarrow & & \downarrow \\ \mathrm{map}(P, N-Q) & \longrightarrow & \mathrm{map}(P, N) \end{array}$$

*is $(n-2p-q-3)$–cartesian.*

(A square is *$j$–cartesian* if the map from its initial term to the homotopy pullback of the other terms is $j$–connected.)

**Remark 11.2** Hatcher and Quinn state a version of Theorem 11.1 (cf [17, Theorem 4.1]), but give a proof which is only a few lines long, and is therefore too scant to be regarded as complete. Additionally, we expected to find a parameterized "concordance implies isotopy" problem in their argument, but nothing of the sort is mentioned.

**Corollary 11.3** Hatcher–Quinn [17, Theorem 1.1] *Assume $2p+q+3 \le n$. Then an embedding $f\colon P \to N$ has a isotopy to an embedding with image disjoint from $Q$ if and only if there is a homotopy of $f$ to a map having image disjoint from $Q$.*

**Corollary 11.4** Compare [17, Theorem 2.2] *Assume $p+2q, 2p+q \le n-3$. Then $f$ is isotopic to an embedding with image disjoint from $Q$ if and only if $\chi(f, i_Q) = 0$.*

Let $\mathcal{E}_f$ be the homotopy fiber at $f$ of

$$\mathrm{emb}(P, N-Q) \to \mathrm{emb}(P, N).$$

Then Theorem 11.1 gives an $(n-2p-q-3)$–connected map

$$\mathcal{E}_f \to \mathcal{F}_f .$$

Compose this with the map of Theorem B to get a map

$$\mathcal{E}_f \to \Omega^{\infty+1} E(f, i_Q)^{\xi}.$$

**Corollary 11.5** *The map $\mathcal{E}_f \to \Omega^{\infty+1} E(f, i_Q)^{\xi}$ is*

$$\min(2n - 2q - p - 3, 2n - 2p - q - 3)\text{–connected.}$$

# 12 An index theorem

## The Hatcher–Quinn invariant

Given an intersection problem

$$\begin{array}{ccc} & & N - Q \\ & \nearrow & \downarrow \\ P & \xrightarrow[f]{} & N, \end{array}$$

we will assume $f$ is transverse to $i_Q \colon Q \subset N$. Then the intersection manifold

$$D := f^{-1}(Q)$$

has dimension $p + q - n$ and comes equipped with a map $D \to E(f, i_Q)$ given by $x \mapsto (x, c_{f(x)}, f(x))$, where $c_{f(x)}$ is the constant path. The pullback of $\xi$ along this map coincides with the stable normal bundle of $D$. The bordism class of these data in $\Omega_{p+q-n}(E(f, i_Q); \xi)$ is called the *Hatcher–Quinn invariant*, and is denoted

$$[f \pitchfork i_Q].$$

Recall that $E \to P$ is the fibration given by taking the homotopy pullback of the inclusion $N - Q \to N$ along the map $f \colon P \to N$. It has a stable cohomotopy Euler class $e(p) \in \pi_0(\sec^{\mathrm{st}}(S_P E \to P))$.

**Theorem 12.1** ("Index Theorem") *There is a bijection*

$$\mathfrak{pd} \colon \pi_0(\sec^{\mathrm{st}}(S_P E \to P)) \xrightarrow{\cong} \Omega_{p+q-n}(E(f, i_Q); \xi)$$

*such that*

$$\mathfrak{pd}(e(p)) = [f \pitchfork i_Q].$$

**Proof** Consider the fiberwise suspended fibration $S_P E \to P$. It is identified with the homotopy pullback of

$$S_N(N - Q) \to N$$

along $f\colon P \to N$. Consequently, specifying a section of $S_P E \to P$ is equivalent to choosing a map $s\colon P \to S_N(N-Q)$ and also a homotopy from the composite

$$P \xrightarrow{s} S_N(N-Q) \to N$$

to $f$.

Let $\nu$ be the normal bundle of $Q$ in $N$, and identify its unit disk bundle $D(\nu)$ with a tubular neighborhood of $Q$. Let $C$ denote the closure of the complement of this neighborhood. The we have a homeomorphism

$$N \cup_{S(\nu)} D(\nu) \cong N \cup_C N.$$

Furthermore, $N \cup_C N$ is identified with $S_N(N-Q)$ up to weak equivalence.

Let $\pi\colon N \cup_{S(\nu)} D(\nu) \to N$ be the evident map. Then up to homotopy, choosing a section of $S_P E \to P$ is the same thing as specifying a map

$$\sigma\colon P \to N \cup_{S(\nu)} D(\nu)$$

and a homotopy from $\pi\sigma$ to $f$.

Let $z_0\colon Q \to D(\nu)$ denote the zero section. If $\sigma$ is transverse to $z_0$ then

$$\mathcal{D} := P \cap \sigma^{-1}(z_0(Q))$$

is a submanifold of $P$ of dimension $p+q-n$ which comes equipped with a map $\mathcal{D} \to E(f, i_Q)$. The normal bundle of $\mathcal{D}$ in $P$ is isomorphic to the pullback of $\nu$. Therefore the stable normal bundle of $\mathcal{D}$ is a Whitney sum of the pullbacks of $\nu$ and the stable normal bundle of $P$. This implies that the stable normal bundle of $\mathcal{D}$ is given by taking the pullback of $\xi$. Consequently, these data determine an element of $\Omega_{p+q-n}(E(f, i_Q); \xi)$.

If $\sigma$ isn't transverse to $z_0$, we can still make it transverse without altering its vertical homotopy class. Therefore, we get a well-defined function

$$\pi_0(\sec(S_P E \to P)) \to \Omega_{p+q-n}(E(f, i_Q); \xi).$$

By definition, the vertical homotopy class of the section $s_+$ maps to $[f \pitchfork i_Q]$.

If we are now given a stable section of $S_P E \to P$, the above construction is now applied to an associated map

$$\sigma\colon P \times D^j \to N \cup_{S(\nu \oplus \epsilon^j)} D(\nu \oplus \epsilon^j)$$

for sufficiently large $j$. This produces a submanifold $\mathcal{D} \subset P \times D^j$ together with a map $\mathcal{D} \to E(f, i_Q)$ compatible with the bundle data. So we get a function

$$\mathfrak{pd}\colon \pi_0(\sec^{\mathrm{st}}(S_P E \to P)) \to \Omega_{p+q-n}(E(f, i_Q); \xi).$$

which maps $[s_+]$ to $[f \pitchfork i_Q]$.

We now show that $\mathfrak{pd}$ is onto. Let $(V, g, \phi)$ represent an element of

$$\Omega_{p+q-n}(E(f, i_Q); \xi).$$

Assume first that the composite

$$V \xrightarrow{g} E(f, i_Q) \to P$$

is an embedding. Notice that $\phi$ provides a stable isomorphism of the normal bundle $\nu_{V \subset P}$ of this embedding with the pullback of normal bundle $\nu$ of $Q \subset N$ along $f$. For the moment, we also assume that $\phi$ arises from an unstable isomorphism of these bundles. By choosing a tubular neighborhood, we get an embedding $D(\nu_{V \subset P}) \subset P$. Let $C$ denote the closure of the complement of this neighborhood. Then we have a decomposition

$$P = C \cup D(\nu_{V \subset P}),$$

so we get a map

$$C \cup D(\nu_{V \subset P}) \xrightarrow{f \cup \phi} N \cup D(\nu)\cdot$$

By the remarks at the beginning of the proof, we see that this determines a homotopy class of section of $S_P E \to P$.

In the general case, we choose an embedding of $V$ in $D^j$. Then $V$ is embedded in the product $P \times D^j$, and if $j$ is sufficiently large the normal bundle of this embedding is identified with the pullback of the $\nu \oplus \epsilon^j$ using $\phi$. The construction above now gives a homotopy class of stable section of $S_P E \to P$. It is straightforward to check that $\mathfrak{pd}$ applied to this homotopy class recovers the original bordism class. Consequently, $\mathfrak{pd}$ is onto.

The proof that $\mathfrak{pd}$ is one-to-one is similar. In this case however, one starts with a bordism whose boundary arises from two stable sections of $S_P E \to P$. Then, working relatively, one constructs a homotopy between these stable sections. We leave these details to the reader. This completes the proof of Theorem 12.1. □

# Appendix A Self intersection

Let $f\colon P \to N$ be a map, where $P$ and $N$ are compact smooth manifolds, and $P$ is without boundary. Then one obtains an intersection problem

$$\begin{array}{ccc} & & N \times N - \Delta_N \\ & \nearrow & \downarrow \\ P \times P & \xrightarrow[f\times f]{} & N \times N \end{array}$$

where $\Delta_N \subset N \times N$ is the diagonal.

Let $E(f,f)$ denote the homotopy pullback of $f$ with itself. A point in $E(f,f)$ consists of a triple

$$(x, \gamma, y)$$

where $x$ and $y$ are points of $P$ and $\gamma$ is a path from $f(x)$ to $f(y)$. Observe that $E(f,f)$ coincides with the homotopy pullback of $\Delta_N$ along $f \times f$.

Define a map

$$e\colon E(f,f) \to P \times N \times P$$

by $e(x,\gamma,y) = (x, \gamma(1/2), y)$. Define a virtual bundle $\xi$ over $E(f,f)$ by taking the pullback along $e$ of $(-\tau_P) \times \tau_N \times (-\tau_P)$.

**Definition A.1** The *self intersection class*

$$\lambda(f) \in \Omega_{2p-n}(E(f,f), \xi),$$

is given by $\chi(f \times f, i_{\Delta_N})$, ie, the stable homotopy Euler characteristic of $f \times f$ and the inclusion $i_{\Delta_N}\colon \Delta_N \subset N \times N$.

## Immersions

Suppose now that $f\colon P \to N$ is an immersion. Equipping $P$ with a Riemannian metric, we identify the total space $D(\tau_P)$ of unit tangent disk bundle of $P$ with a compact tubular neighborhood of the diagonal $\Delta_P \subset P \times P$. With respect to this identification, the involution of $P \times P$ corresponds to the one on $D(\tau_P)$ that maps a tangent vector to its negative. Let $S(\tau_P)$ be the total space of the unit sphere bundle of $\tau_P$, and let $P(2)$ be the effect of deleting the interior of the tubular neighborhood from $P \times P$. If we rescale the metric, then $f \times f$ determines an equivariant map of pairs

$$f(2)\colon (P(2), S(\tau_P)) \to (N^{\times 2}, N^{\times 2} - \Delta_N).$$

Consider the *equivariant* lifting problem

$$\begin{array}{ccc} S(\tau_P) & \longrightarrow & N \times N - \Delta_N \\ \Big\downarrow & \nearrow & \Big\downarrow \\ P(2) & \xrightarrow[f(2)]{} & N \times N. \end{array}$$

The above is a special instance of what we call a *(relative) equivariant intersection* problem. Equivariant intersection theory will be extensively studied in the second paper of this series (Klein–Williams [24]). For this reason, our intention here will be to state the main result about equivariant intersection theory in the self intersection case and defer its proof to the second paper.

Let

$$E'(f,f) \subset E(f,f)$$

denote the subspace of those points $(x,\gamma,y)$ in which $x \neq y$. Define an involution on $E'(f,f)$ by $(x,\gamma,y) \mapsto (y,\bar{\gamma},x)$, where $\bar{\gamma}(t) = \gamma(1-t)$.

Let $\xi'$ denote the pullback of $(-\tau_P) \times \tau_N \times (-\tau_P)$ to $E'(f,f)$. The involution $(u,v,w) \mapsto (w,-v,u)$ induces an involution of $\xi'$. The Thom spectrum

$$E'(f,f)^{\xi'}$$

is a $\mathbb{Z}_2$–spectrum indexed over a complete $\mathbb{Z}_2$–universe (see May [31, IX, Section 2]). Hence, the fixed point spectrum

$$(E'(f,f)^{\xi'})^{\mathbb{Z}_2}$$

is defined (see May [31, XVI, Section 1]).

**Definition A.2** The *homotopical equivariant bordism* group

$$\Omega^{\mathbb{Z}_2}_{2p-n}(E'(f,f);\xi')$$

is the homotopy group of $(E'(f,f)^{\xi'})^{\mathbb{Z}_2}$ in degree $0$.

**Remark A.3** As the involution on $E'(f,f)$ is free, the transfer determines an isomorphism

$$\Omega^{\mathbb{Z}_2}_{2p-n}(E'(f,f);\xi') \cong \Omega_{2p-n}(E'(f,f)_{h\mathbb{Z}_2};\xi'_{h\mathbb{Z}_2}).$$

where $E'(f,f)_{h\mathbb{Z}_2} = E'(f,f) \times_{\mathbb{Z}_2} E\mathbb{Z}_2$ is the Borel construction.

The proof of the following result is deferred to (Klein–Williams [24]).

**Theorem A.4** *To each immersion $f\colon P \to N$ there is an associated class*

$$\mu(f) \in \Omega^{\mathbb{Z}_2}_{2p-n}(E'(f,f);\xi')$$

*which is trivial whenever $f(2)$ is equivariantly homotopic rel $S(\tau_P)$ to a map whose image is disjoint from $\Delta_N$.*

*Conversely, if $p \le n-2$ and $\mu(f) = 0$, then $f(2)$ is equivariantly homotopic rel $S(\tau_P)$ to a map with image disjoint from $\Delta_N$.*

According to Haefliger [15], in the metastable range $3p \le 2n-3$, if $f(2)$ is equivariantly homotopic rel $S(\tau_P)$ to a map with image disjoint from $\Delta_N$, then $f$ is regularly homotopic to a smooth embedding. We infer the following Corollary.

**Corollary A.5** *Assume $3p \le 2n-3$. Then $f$ is regularly homotopic to an embedding if and only if $\mu(f) = 0$.*

## A formula relating $\lambda(f)$ and $\mu(f)$

Forgetting involutions together with the inclusion $E'(f,f) \subset E(f,f)$ defines a homomorphism

$$\phi\colon \Omega^{\mathbb{Z}_2}_{2p-n}(E'(f,f);\xi') \to \Omega_{2p-n}(E(f,f);\xi).$$

Let

$$j\colon P \to E(f,f)$$

be given by $j(x) = (x, c_x, x)$, where $c_x$ denotes the constant path at $x$. Then $j$ gives a homomorphism

$$j_*\colon \Omega_{2p-n}(P; f^*\tau_N - 2\tau_P) \to \Omega_{2p-n}(E(f,f);\xi).$$

The virtual bundle $f^*\tau_N - 2\tau_P$ appearing in the domain is obtained by pulling back $\xi$ along the map $P \to P \times N \times P$ given by $x \mapsto (x, f(x), x)$.

**Definition A.6** The *local characteristic* of the immersion $f$ is the class

$$\chi(\nu_f) \in \Omega_{2p-n}(P; f^*\tau_N - 2\tau_P)$$

given by the stable homotopy Euler characteristic of the sphere bundle of $\nu_f =$ the normal bundle of $f$.

**Theorem A.7** *For an immersion $f\colon P \to N$, we have*

$$\lambda(f) = \phi(\mu(f)) + j_*\chi(\nu_f).$$

**Proof** We give a geometric proof using Theorem 12.1 (a completely homotopy theoretic proof is also possible). Using the multijet transversality theorem, we may assume that $f\times f$ is transverse to the diagonal of $N$. Furthermore, we may assume that $N$ and $P$ are equipped with Riemannian metrics in such a way that $f$ is a local isometry.

According to Theorem 12.1, the invariant $\lambda(f)$ is the transverse intersection of $f\times f$ with the diagonal $\Delta_N$. Because $f$ is an immersion, the inverse image $(f\times f)^{-1}(\Delta_N)$ is a disjoint union

$$\mathcal{J} \amalg \mathcal{D}_2,$$

in which $\mathcal{J}$ is the inverse image of the zero section $\zeta_{\tau_N}\colon N \to D(\tau_N)$ with respect to the map $Df\colon D(\tau_P) \to D(\tau_N)$, and $\mathcal{D}_2$, the double point manifold, is the inverse image of $\Delta_N$ inside $P(2)$. The evident map $\mathcal{D}_2 \to E(f,f)$ gives a bordism class representing $\phi(\mu(f))$. It therefore suffices to prove that $\mathcal{J} \to E(f,f)$ represents $j_*\chi(\nu_f)$.

By Theorem 12.1, $\chi(\nu_f)$ is represented by the transverse self intersection of the zero section $\zeta_{\nu_f}\colon P \to D(\nu_f)$. Let $\eta$ be any vector bundle over $P$. Then by a straightforward argument which we omit, this self intersection coincides with the transversal intersection of the composite

$$P \overset{\zeta_{\nu_f}}{\to} D(\nu_f) \overset{i}{\subset} D(\nu_f \oplus \eta)$$

with the inclusion $i'\colon D(\eta) \to D(\nu_f \oplus \eta)$. Here we are using the fact that the evident weak equivalence $E(\zeta_{\nu_f}, \zeta_{\nu_f}) \to E(i \circ \zeta_{\nu_f}, i')$ to identify bordism groups.

Set $\eta = \tau_P$ and identify $\tau_P \oplus \nu_f = (\tau_N)_{|P}$. With respect to this identification, $E(i \circ \zeta_{\nu_f}, i')$ coincides with $E(\zeta_{\tau_N|P}, Df) =$ the homotopy pullback of the zero section $\zeta_{\tau_N|P}\colon P \to D(\tau_N)_{|P}$ and $Df\colon D(\tau_P) \to D(\tau_N)_{|P}$.

Finally, the transverse self intersection of $\zeta_{\tau_N|P}$ with $Df$ coincides identically with $\mathcal{J}$. The point here is that $D(\tau_N) \subset N\times N$ has codimension zero, $\mathcal{J}$ is the part of $(f\times f)^{-1}(\Delta_N)$ which maps into $D(\tau_N)$ and $f\times f$ is identified with $Df$ near $\Delta_P$.

Clearly, $j\colon P \to E(f,f)$ factorizes as $P \to E(\zeta_{\tau_N|P}, Df) \to E(f,f)$, so the proof is complete. □

**Remark A.8** In special cases, the literature contains many variants of Theorem A.7 (see eg, Crabb [9, page 56], Hatcher–Quinn [17, page 338]), Kalman–Szucs [21, Proposition 1], and Li [29, Theorem 2]).

For an alternative homotopical approach to self intersection, see (Klein [23]).

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