

Homotopical intersection theory, II: equivariance

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Abstract This paper is a sequel to (Klein and Williams in Geom Topol 11:939–977, 2007). We develop here an intersection theory for manifolds equipped with an action of a finite group. As in Klein and Williams (2007), our approach will be homotopy theoretic, enabling us to circumvent the specter of equivariant transversality. We give applications of our theory to embedding problems, equivariant fixed point problems and the study of periodic points of self maps.

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Contents

1	Introduction
2	Preliminaries
3	The Proof of Theorem A
4	Naive versus equivariant stabilization
5	Parametrized G -spectra over a complete universe
6	Poincaré duality
7	The equivariant complement formula
8	The proof of Theorem B and Addendum C
9	The proof of Theorems D and E
10	The Proof of Theorem H
11	The Proof of Theorems I and J

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1 Introduction

Intersection problems Suppose N is a compact smooth manifold equipped with a closed submanifold $Q \subset N$. An *intersection problem* for (N, Q) consists of a map $f: P \rightarrow N$, where P is a closed manifold. A *solution* to the problem consists of a homotopy of f to a map g satisfying $g(P) \cap Q = \emptyset$. We depict the situation by

$$\begin{array}{ccc} & & N - Q \\ & \nearrow & \downarrow \\ P & \xrightarrow{f} & N, \end{array}$$

in which we seek to find the dotted arrow making the diagram homotopy commute. One also has a version of the above when P has a boundary whose image under f is disjoint from Q . We then require the deformation of f to hold the boundary fixed. Let $i_Q: Q \subset N$ be the inclusion. We will often denote the data by (f, i_Q) .

In [19], we produced an obstruction $\chi(f)$ living in a certain bordism group whose vanishing is necessary for the existence of a solution. Furthermore, the obstruction was shown to be sufficient in the range $p \leq 2n - 2q - 3$, where $\dim N = n$, $\dim Q = q$ and $\dim P = p$. We also gave a version of the obstruction for families.

Here, we will consider *equivariant* intersection problems. Suppose G is a finite group and the above manifolds are equipped with smooth G -actions.

In the equivariant setting, $i_Q: Q \subset N$ is a G -submanifold and $f: P \rightarrow N$ is an equivariant map. We now seek a deformation of f through G -maps to an equivariant map whose image is disjoint from Q .

The partial answers we will give to such questions are phrased in terms of isotropy data. If X is a G -space, we let

$$\mathcal{I}(G; X)$$

denote the conjugacy classes of subgroups of G which appear as stabilizer groups of points of X .

Indexing functions An *indexing function* ϕ_\bullet on a G -space X assigns to a subgroup $H \subset G$ a locally constant function ϕ_H with domain X^H , the fixed point set of H acting on X , and codomain given by the extended integers $\mathbb{Z} \cup \pm\infty$. It is also required to be conjugation invariant: if $K = gHg^{-1}$ and $h: X^H \rightarrow X^K$ is the homeomorphism $x \mapsto gx$, then $\phi_H = \phi_K \circ h$.

If ψ_\bullet is another indexing function on X , and $H \subset G$ is a subgroup, we write

$$\phi_H \leq \psi_H$$

if $\phi_H(x) \leq \psi_H(x)$ for all $x \in X^H$. If $\phi_H \leq \psi_H$ for all H , then we write $\phi_\bullet \leq \psi_\bullet$.

Here are some examples:

Dimension If M is a locally smooth G -manifold, then for any subgroup $H \subset G$ the components of the fixed point set M^H are manifolds [1, Ch. 4]. The dimensions of the components can vary. If $x \in M^H$, then the dimension of the component containing x defines a locally constant function m^H . The collection $m^\bullet := \{m^H\}_{H \subset G}$ is called the *dimension function* of M . If M^H is empty, our convention is to set $m^H = -\infty$.

Codimension Let $i_Q: Q \subset N$ be as above. Let

$$\mathrm{cd}_\bullet(i_Q)$$

be the indexing function on Q in which $\mathrm{cd}_H(i_Q)(x)$ is the dimension of the normal space to the embedding $Q^H \subset N^H$ at $x \in Q^H$ (if Q^H is empty but N^H isn't, our convention is to set $\mathrm{cd}_H(i_Q) = +\infty$).

Pullback Suppose $f: X \rightarrow Y$ is a G -map. Given an indexing function α_\bullet on Y , we obtain an indexing function $f^*\alpha_\bullet$ on X which is given by $f^*\alpha_H(x) = \alpha_H(f(x))$.

Pushforward Given $f: X \rightarrow Y$ as above, let β_\bullet be an indexing function on X . For $y \in Y$ we let $[y]$ denote the associated path component. Let $I_{f,y}$ be the set of those $[x]$ for which $[f(x)] = [y]$. That is, $I_{f,y}$ is the inverse image of $f_*: \pi_0(X) \rightarrow \pi_0(Y)$ at $[y]$.

Define an indexing function $f_!\beta_\bullet$ on Y by the rule

$$f_!\beta_H(y) = \begin{cases} \inf_{I_{f,y}} \beta_H(x) & \text{if } I_{f,y} \text{ is nonempty,} \\ \infty & \text{otherwise.} \end{cases}$$

Note $f_!f^*\alpha_\bullet \geq \alpha_\bullet$, with equality holding when f_* is a surjection, whereas $f^*f_!\beta_\bullet \leq \beta_\bullet$ with equality holding when f_* is an injection.

Stable intersections Just as in the unequivariant case, equivariant intersections can be removed when the codimension is sufficiently large. The equivariant intersection problem (f, i_Q) is said to be *stable* if

$$p^H \leq f^*(i_Q)_! \mathrm{cd}_H(i_Q) - 1$$

for every $(H) \in \mathcal{I}(G; P)$. (Roughly, this means the dimension of the transverse intersection of $f(P^H)$ and Q^H is negative).

If the intersection problem is stable, one can use elementary equivariant obstruction theory to show f equivariantly deforms off of Q , yielding a solution.

A “cohomological” result Our first main result gives a complete obstruction to solving equivariant intersection problems in the equivariant *metastable* range. The obstruction lies in the cohomology of P with coefficients in a certain *parametrized equivariant spectrum* over N . A reader who is not familiar with this technology should consult Sect. 3.

Theorem A *To an equivariant intersection problem (f, i_Q) , there is a naive parametrized G -spectrum $\mathcal{E}(i_Q)$ over N , which is constructed from the inclusion $i_Q: Q \subset N$, and an obstruction*

$$e_G(f) \in H_G^0(P; \mathcal{E}(i_Q))$$

which vanishes when the intersection problem has a solution.

Conversely, if $e_G(f) = 0$ and

$$p^H \leq 2f^*(i_Q)_! \mathrm{cd}_H(i_Q) - 3$$

for all $(H) \in \mathcal{I}(G; P)$, then the intersection problem has a solution.

Remark Theorem A is an equivariant version of [19, Cor. 3.5]. The word *naïve* is used here indicate that the parametrized spectrum is indexed over a trivial universe; the equivariant cohomology theory of the theorem is therefore of “Bredon type”.

The inequalities of Theorem A define the *equivariant metastable range*. When G is the trivial group, one has the sole inequality $2p \leq 2n - 2q - 3$, which is just the unequivariant metastable range (cf. [19]).

Homotopical equivariant bordism Since naive equivariant cohomology theories are not indexed over representations, they are not fully “stable.” From our viewpoint, a crucial deficiency of naive theories is their lack of Poincaré duality.

To get around this, we impose additional conditions to get a more tractible invariant residing in a theory which does possess Poincaré duality. We will map the equivariant cohomology theory of Theorem A into a similarly defined $\mathrm{RO}(G)$ -graded one. The additional constraints will insure the map is injective. Applying duality to our pushed-forward invariant, we obtain another invariant living in $\mathrm{RO}(G)$ -graded homology. We then identify the homology theory with the homotopical G -bordism groups of a certain G -space.

To a G -space X equipped with real G -vector bundle ξ , one has an associated *equivariant Thom spectrum*

$$X^\xi,$$

whose spaces X_V^ξ are indexed by representations V ranging over a complete G -universe \mathcal{U} (compare [23, Chap. XV]). Here, X_V^ξ denotes the Thom space of $\xi \oplus V$. Equivalently, X^ξ is the equivariant suspension spectrum of the Thom space of ξ . More generally, X^ξ is defined whenever ξ is an *virtual* G -bundle over X (see [22, Ch. 9] for details).

For a virtual G -representation $\alpha = V - W$, the *homotopical G -bordism group* of (X, ξ) in degree α is given by

$$\Omega_\alpha^G(X; \xi) := \operatorname{colim}_U [S^{V+U}, X_{W+U}^\xi]^G,$$

where $[S^{V+U}, X_{W+U}^\xi]^G$ denotes the homotopy classes of based G -maps $S^{V+U} \rightarrow X_{W+U}^\xi$ (in which S^{V+U} is the one point compactification of the direct sum of V and U), and the colimit is indexed over the finite dimensional subrepresentations U of \mathcal{U} using the partial ordering defined by inclusion. Actually, we will only need consider the case when $\alpha = 0$ is the trivial representation of rank zero.

Remarks (1) There is a related object, $\mathcal{N}_\alpha^G(X; \xi)$, called the *geometric bordism group* of (X, ξ) . It is generated by G -manifolds M equipped with G -map $u: M \rightarrow X$ and a stable G -bundle isomorphism

$$u^*\xi \oplus \tau_M \oplus \epsilon_W \cong \epsilon_V,$$

where ϵ_V denotes the G -bundle whose total space is $X \times V$.

The Pontryagin–Thom construction defines a homomorphism

$$\mathcal{N}_\alpha^G(X; \xi) \rightarrow \Omega_\alpha^G(X; \xi).$$

In contrast with the unequivariant case, this map can fail to be an isomorphism because of the lack of equivariant transversality (see [4], [23, p. 156], [25]).

- (2) When ξ is a G -vector bundle (not virtual), then X^ξ is the equivariant suspension spectrum of the Thom space of ξ . In particular, when ξ is trivial of rank zero, we get $\Sigma_G^\infty(X_+)$, the equivariant suspension spectrum of $X \amalg *$. In this case, the map from the equivariant

geometric bordism group to the homotopical one is an isomorphism [12, 20]. The k -th homotopy group of $\Sigma_G^\infty(X_+)$ coincides with $\Omega_k^{G, \text{fr}}(X)$, the k -dimensional equivariant framed bordism group of X .

- (3) When $G = e$ is the trivial group, and ξ has virtual rank n , $\Omega_0^e(X; \xi) = \Omega_0(X; \xi)$ is the bordism group generated by maps $\alpha: M \rightarrow X$, with M a compact n -manifold, together with a (stable) isomorphism $\alpha^*\xi$ with the stable normal bundle of M . Note the indexing convention used here is different from the one of [19] (the latter implicitly ignored the rank of ξ but indicated the dimension of the manifolds in the degree of the bordism group; thus the group $\Omega_n(X; \xi)$ of [19] coincides with the current $\Omega_0(X; \xi)$).

We now specialize to the equivariant bordism groups arising from intersection problems. Given an equivariant intersection problem (f, i_Q) , define

$$E(f, i_Q)$$

to be the *homotopy fiber product* (a.k.a. homotopy pullback) of f and i_Q . A point in $E(f, i_Q)$ is a triple (x, λ, y) in which $x \in P$, $y \in Q$ and $\lambda: [0, 1] \rightarrow N$ is a path such that $\lambda(0) = f(x)$ and $\lambda(1) = y$. There is an evident action of G on $E(f, i_Q)$.

There are forgetful maps $j_P: E(f, i_Q) \rightarrow P$ and $j_Q: E(f, i_Q) \rightarrow Q$, both equivariant. There is also an equivariant map $j_N: E(f, i_Q) \rightarrow N$ given by $(x, \lambda, y) \mapsto \lambda(1/2)$. Using these, we obtain an equivariant virtual bundle over $E(f, i_Q)$ by

$$\xi := j_N^* \tau_N - j_Q^* \tau_Q - j_P^* \tau_P.$$

If $Q \subset N$ is held fixed, then ξ is completely determined by $f: P \rightarrow N$.

A “homological” result

Theorem B *Given an equivariant intersection problem (f, i_Q) , there is an invariant*

$$\chi_G(f) \in \Omega_0^G(E(f, i_Q); \xi)$$

which vanishes when f is equivariantly homotopic to a map whose image is disjoint from Q . Conversely, assume $\chi_G(f) = 0$ and

- *for each $(H) \in \mathcal{I}(G; P)$, we have*

$$p^H \leq 2f^*(i_Q)_! \text{cd}_H(i_Q) - 2;$$

- *for each $(H) \in \mathcal{I}(G; P)$ and each proper subgroup $K \subsetneq H$, we have*

$$p^H \leq f^*(i_Q)_! \text{cd}_K(i_Q) - 2.$$

Then f is equivariantly homotopic to a map whose image is disjoint from Q .

Remarks (1) The assignment $f \mapsto \chi_G(f)$ is a *global section* of a locally constant sheaf over the equivariant mapping space $\text{map}(P, N)^G$. The stalk of this sheaf at f is $\Omega_0^G(E(f, i_Q); \xi)$. This explains the sense in which $\chi_G(f)$ is an invariant: an equivariant homotopy from f to another map $f': P \rightarrow N$ gives rise to an isomorphism of stalks over f and f' , and the isomorphism transfers $\chi_G(f)$ to $\chi_G(f')$.

- (2) The second set of inequalities of Theorem B can be regarded as a *gap condition*.
- (3) An advantage that Theorem B enjoys over Theorem A is that the obstruction group appearing in the former is defined directly in terms of the maps $f: P \rightarrow N$ and $Q \rightarrow N$. It turns out that obstruction group of Theorem A is defined in terms of f and the map $N - Q \rightarrow N$, which is not as easy to identify in terms of the input

data. Furthermore, the equivariant bordism group appearing in Theorem B arises from a Thom spectrum indexed over a complete universe, so more machinery is at hand for the purpose of making calculations (see [23]).

Boundary conditions There is also a version of Theorem B when N is compact, possibly with boundary, and P is compact with boundary $\partial P \neq \emptyset$ satisfying $f(\partial P) \cap Q = \emptyset$. In this instance one seeks an equivariant deformation of f , fixed on ∂P , to a new map whose image is disjoint from Q .

Addendum C Theorem B also holds when P and N are compact manifolds with boundary, where it is assumed $f(\partial P) \cap Q = \emptyset$ and Q is embedded in the interior of N .

Sparse isotropy When the action of G on P has few isotropy types, the inequalities in Theorem B unravel somewhat.

Free actions Suppose the action of G on P is free. Then the trivial group is the only isotropy group and the inequalities of Theorem B reduce to a single inequality

$$p \leq 2(n - q) - 3 = 2n - 2q - 3.$$

Furthermore, the equivariant bordism group of Theorem B is isomorphic to the unequivariant bordism group

$$\Omega_0(EG \times_G E(f, i_Q); \text{id}_{EG} \times_G \xi),$$

where $EG \times_G E(f, i_Q)$ is the Borel construction. This bordism group is generated by maps $u: M \rightarrow EG \times_G E(f, i_Q)$ together with a stable isomorphism $\nu_M \cong u^*(\text{id}_{EG} \times_G \xi)$, where M has dimension $p + q - n$ and ν_M denotes the stable normal bundle. The identification of these groups is obtained using a transfer construction (we omit the details).

Trivial actions If P has a trivial G -action, then the only isotropy group is G . In this instance, P has image in N^G and the intersection problem becomes an unequivariant one, involving the map $f: P \rightarrow N^G$ and the submanifold $Q^G \subset N^G$. Assume for simplicity that N^G and Q^G are connected. Then by [19], the intersection problem admits a solution when $\chi(f) \in \Omega_0(E(f, i_{Q^G}); \xi)$ is trivial and $p \leq 2n^G - 2q^G - 3$.

Prime order groups Let G be a cyclic group of prime order. By the above, we can assume both the trivial group and G appear as stabilizer groups. Then $\emptyset \neq P^G \subsetneq P$.

For simplicity, assume Q^G and N^G are connected. Then the first set of inequalities of Theorem B becomes

$$p \leq 2n - 2q - 3, \quad p^G \leq 2n^G - 2q^G - 3,$$

and the second set amounts to the single inequality

$$p^G \leq n - q - 2.$$

Local intersection theory Suppose an equivariant intersection problem (f, i_Q) has been partially solved in the following sense: there is G -subspace $U \subset P$ such that $f(U)$ is disjoint from Q . One can then ask whether the solution extends to a larger subspace of P . A *local equivariant intersection problem* amounts to these data. A systematic approach to such questions provided by the isotropy stratification of P .

The isotropy stratification The relation of *subconjugacy* describes a partial ordering $\mathcal{I}(G; P)$: we will write

$$(H) < (K)$$

if K is properly subconjugate to H . We then choose a total ordering which is compatible with the partial ordering. Let

$$(H_1) < (H_2) < \cdots < (H_\ell)$$

be the maximal chain coming from the total ordering of $\mathcal{I}(G; P)$.

Let $P_i \subset P$ be the set of points x having stabilizer group G_x in which $(G_x) \leq (H_i)$. Then we have a filtration of G -spaces

$$\emptyset = P_0 \subsetneq P_1 \subsetneq P_2 \subsetneq \cdots \subsetneq P_\ell = P,$$

where each inclusion $P_i \subset P_{i+1}$ possesses the equivariant homotopy extension property (cf. [5, 15]).

The local obstruction Suppose (f, i_Q) is an equivariant intersection problem with $f(P_{i-1}) \cap Q = \emptyset$ for some $i \geq 1$. We seek a deformation of f relative to P_{i-1} to a new map f' such that $f'(P_i) \cap Q = \emptyset$. The map f' is then a solution to the local problem.

Let H be a representative of (H_i) and let $f_H: P_H \rightarrow N$ denote the restriction of f to P_H . The Weyl group $W(H) = N(H)/H$ acts on P^H and freely on P_H . Let ${}^H\xi$ be the virtual $W(H)$ -bundle over $E(f_H, i_Q)$ defined by

$$j_N^* \tau_N - j_Q^* \tau_Q - j_{P_H}^* \tau_{P_H}.$$

Theorem D *There is an invariant*

$$\chi_G^i(f) \in \Omega_0^{W(H)}(E(f_H, i_Q); {}^H\xi)$$

which is trivial when the local problem at P_i relative to P_{i-1} can be solved.

Conversely, assume $\chi_G^i(f) = 0$ and

$$p^H \leq 2f^*(i_Q)! \operatorname{cd}_H(i_Q) - 3$$

for $(H) = (H_i)$. Then the local problem admits a solution.

Descent The global invariant $\chi_G(f)$ is an assemblage of all the local invariants. Although the local invariants may contain more information, they can fail to provide a solution to the global question. To address this point, we will give criteria for deciding when the vanishing of the global invariants implies the vanishing of the local ones. In combination with Theorem D the criteria yield a kind of *descent* theory for equivariant intersection problems.

Let $H \in \mathcal{I}(G; P)$ be and consider the inclusion

$$P_H \subset P^H.$$

Denote the corresponding inclusion $E(P_H, Q) \subset E(P^H, Q)$ by t_H . The map $f^H: P^H \rightarrow N$ will denote the restriction of f to P^H . Define a virtual $W(H)$ -bundle ${}^H\xi$ over $E(f^H, i_Q)$ by $j_N^* \tau_N - j_Q^* \tau_Q - j_{P_H}^* \tau_{P_H}$. Since the pullback of ${}^H\xi$ along t_H is ${}^H\xi$, we get an induced homomorphism

$$(t_H)_*: \Omega_0^{W(H)}(E(f_H, i_Q); {}^H\xi) \rightarrow \Omega_0^{W(H)}(E(f^H, i_Q); {}^H\xi).$$

Theorem E (Global-to-local) *Assume*

- $f(P_{i-1}) \cap Q = \emptyset$ for some $i \geq 1$ (so $\chi_G^i(f)$ is defined).
- $(t_H)_*$ is injective for $(H) = (H_i)$.

Then $\chi_G(f) = 0$ implies $\chi_G^i(f) = 0$.

Corollary F (Descent) *Let (f, i_Q) be an equivariant intersection problem. Assume*

- $\chi_G(f) = 0$,
- $(t_H)_*$ is injective,
- $p^H \leq 2f^*(i_Q)_! \text{cd}_H(i_Q) - 3$,

for every $(H) \in \mathcal{I}(G; P)$. Then there is an equivariant deformation of f to a map whose image is disjoint from Q .

Applications

Embeddings Suppose $f: P \rightarrow N$ is a smooth immersion. Equipping P with a Riemannian metric, we identify the total space of the unit tangent disk bundle of P with a compact tubular neighborhood of the diagonal $\Delta_P \subset P \times P$. With respect to this identification, the involution of $P \times P$ corresponds to the one on the tangent bundle that maps a tangent vector to its negative. Let $S(2)$ be the total space of the unit spherical tangent bundle of P , and let $P(2)$ be the effect of deleting the interior of the tubular neighborhood from $P \times P$. Then $(P(2), S(2))$ is a free \mathbb{Z}_2 -manifold with boundary.

If we rescale the metric, then $f \times f$ determines an equivariant map

$$(f(2), f(2)|_{S(2)}): (P(2), S(2)) \rightarrow (N^{\times 2}, N^{\times 2} - \Delta_N),$$

which yields relative \mathbb{Z}_2 -equivariant intersection problem with free domain. The fiber product $E(f(2), i_{\Delta_N})$ in this case coincides with the space of triples (x, γ, y) with $x, y \in P(2)$ and γ a path from $f(x)$ to $f(y)$. The involution is given by $(x, \gamma, y) \mapsto (y, \bar{\gamma}, x)$, where $\bar{\gamma}(t) := \gamma(1 - t)$. We set $E'(f, f) := E(f(2), i_{\Delta_N})$.

Applying Addendum B and observing the action is free, we have an obstruction

$$\mu(f) \in \Omega_0(E\mathbb{Z}_2 \times_{\mathbb{Z}_2} E'(f, f); \text{id} \times_{\mathbb{Z}_2} \xi)$$

whose vanishing suffices for finding an equivariant deformation of $f(2)$, fixed on $S(2)$, to a map whose image is disjoint from Δ_N , provided $3p + 3 \leq 2n$.

By a theorem of Haefliger [11], f is regularly homotopic to an embedding in the metastable range $3p + 3 \leq 2n$ if and only if the above equivariant intersection problem admits a solution. Consequently,

Corollary G (compare [14, Th. 2.3]) *If f is regularly homotopic to an embedding, then $\mu(f)$ is trivial.*

Conversely, in the metastable range, the vanishing of $\mu(f)$ implies f is regularly homotopic to an embedding.

Equivariant fixed point theory Let M be a closed smooth manifold equipped a smooth action of a finite group G . Let

$$\text{map}^b(M, M)^G$$

denote the space of fixed point free G -maps from M to itself. Equivariant fixed point theory studies the extent to which the inclusion

$$\text{map}^b(M, M)^G \rightarrow \text{map}(M, M)^G$$

is a surjection on path components.

For an equivariant self map $f: M \rightarrow M$, let

$$L_f M$$

be the space of paths $\lambda: [0, 1] \rightarrow M$ satisfying the constraint $f(\lambda(0)) = \lambda(1)$. Then G acts on $L_f M$ pointwise. Finally, let

$$\Omega_0^{G, \text{fr}}(L_f M)$$

be the G -equivariant framed bordism of $L_f M$ in dimension zero.

Theorem H *There is an invariant*

$$\ell_G(f) \in \Omega_0^{G, \text{fr}}(L_f M)$$

which vanishes when f is equivariantly homotopic to a fixed point free map.

Conversely, assume $\ell_G(f) = 0$. If

- $m^H \geq 3$ for all $(H) \in \mathcal{I}(G; M)$.
- $m^H \leq m^K - 2$ for proper inclusions $K \subsetneq H$ with $K, H \in \mathcal{I}(G; M)$,

then f is equivariantly homotopic to a fixed point free map.

Remarks (1) The above can be regarded as an equivariant analog of a classical theorem of Wecken [28].

- (2) A formula of tom Dieck splits $\Omega_0^{G, \text{fr}}(L_f M)$ into a direct sum of unequivariant framed bordism groups indexed over the conjugacy classes of subgroups of G . The summand corresponding to a conjugacy class (H) is

$$\Omega_0^{\text{fr}}(EW(H) \times_{W(H)} L_{f^H} M),$$

where $EW(H) \times_{W(H)} L_{f^H} M$ is the Borel construction of the Weyl group $W(H)$ acting on $L_{f^H} M$ (see [23, 26]). Consequently, $\ell_G(f)$ decomposes as a sum of invariants indexed in the same way. We conjecture the Nielsen number $N(f^H)$ can be computed from the projection of $\ell_G(f)$ onto the displayed summand.

- (3) Our result bears close similarity to a theorem of Fadell and Wong [7] (see also [8, 27]). Their result uses the Nielsen numbers $N(f^H)$ with $(H) \in \mathcal{I}(G; M)$ in place of our $\ell_G(f)$.

Periodic points A fundamental problem in discrete dynamics is to enumerate the periodic orbits of a self map $f: M \rightarrow M$, where M is a closed manifold.

Let $n \geq 2$ be an integer. A point $x \in M$ is said to be n -periodic if x is a fixed point of the n -th iterate of f , i.e., $f^n(x) = x$. The set of n -periodic points of f is denoted

$$P_n(f).$$

The cyclic group \mathbb{Z}_n acts on $P_n(f)$: if $t \in \mathbb{Z}_n$ is a generator, then the action is defined by $t \cdot x := f(x)$.

The homotopy n -periodic point set of f is the \mathbb{Z}_n -space

$$\text{ho}P_n(f)$$

consisting of n -tuples

$$(\lambda_1, \lambda_2, \dots, \lambda_n),$$

in which $\lambda_i : [0, 1] \rightarrow M$ is a path and the data are subject to the constraints

$$f(\lambda_{i+1}(0)) = \lambda_i(1), \quad i = 1, 2, \dots$$

Here we interpret the subscript i as being taken modulo n . The action of \mathbb{Z}_n on $\text{ho}P_n(f)$ is given by cyclic permutation of factors.

There is a map $P_n(f) \rightarrow \text{ho}P_n(f)$ given by sending an n -periodic point x to the n -tuple

$$(c_x, c_{f(x)}, c_{f^2(x)}, \dots, c_{f^{n-1}(x)})$$

in which c_x denotes the constant path with value x .

For a self map $f : M \rightarrow M$, as above, let

$$\Omega_0^{\mathbb{Z}_n, \text{fr}}(\text{ho}P_n(f))$$

be the \mathbb{Z}_n -equivariant framed bordism group of $\text{ho}P_n(f)$ in dimension 0.

Theorem I *There is a homotopy theoretically defined invariant*

$$\ell_n(f) \in \Omega_0^{\mathbb{Z}_n, \text{fr}}(\text{ho}P_n(f))$$

which is an obstruction to deforming f to an n -periodic point free self map.

Remarks At the time of writing, we do not know the extent to which $\ell_n(f)$ is the complete obstruction to making f n -periodic point free. When $\dim M \geq 3$, Jezierski [16] has shown the vanishing of the Nielsen numbers $N(f^k)$ for all divisors $k|n$ implies f is homotopic to an n -periodic point free map (here f^k denotes the k -fold composition of f with itself). We conjecture that $\ell_n(f)$ determines $N(f^k)$.

Periodic points and the fundamental group Let π be a group equipped with endomorphism $\rho : \pi \rightarrow \pi$. Consider the equivalence relation on π generated by the elementary relations

$$x \sim gx\rho^n(g)^{-1} \quad \text{and} \quad x \sim \rho(x)$$

for $x, g \in \pi$. Let

$$\pi_{\rho, n}$$

be the set of equivalence classes. Let

$$\mathbb{Z}[\pi_{\rho, n}]$$

denote the free abelian group with basis $\pi_{\rho, n}$.

Let $f : M \rightarrow M$ be a self map of a connected closed manifold M . Fix a basepoint $*$ in M . Choose a homotopy class of path $[\alpha]$ from $*$ to $f(*)$. Then $[\alpha]$ defines an isomorphism

$$\pi_1(M, *) \cong \pi_1(M, f(*)).$$

Furthermore, f and $[\alpha]$ together define a homomorphism

$$\rho : \pi_1(M, *) \xrightarrow{f_*} \pi_1(M, f(*)) \cong \pi_1(M, *).$$

Let $\pi = \pi_1(M, *)$.

Theorem J *The data consisting of the self map $f: M \rightarrow M$, the choice of basepoint $* \in M$ and the homotopy class of path $[\alpha]$ from $*$ to $f(*)$ determine an isomorphism of abelian groups*

$$\Omega_0^{\mathbb{Z}_n, \text{fr}}(\text{ho}P_n(f)) \cong \bigoplus_{k|n} \mathbb{Z}[\pi_{\rho, k}].$$

With respect to this isomorphism, there is a decomposition

$$\ell_n(f) = \bigoplus_{k|n} \ell_n^k(f),$$

in which $\ell_n^k(f) \in \mathbb{Z}[\pi_{\rho, k}]$.

2 Preliminaries

G -Universes The G -representations of this paper are assumed to come equipped with a G -invariant inner product. A G -universe \mathcal{U} is a countably infinite dimensional real representation of G which contains the trivial representation and which contains infinitely many copies of each of its finite dimensional subrepresentations.

We will be interested in two kinds of universes. A *complete* universe is one that contains infinitely many copies of representatives for the irreducible representations of G (in this instance one can take \mathcal{U} to be the countable direct sum of the regular representation). A *trivial* universe contains only trivial representations.

Spaces We work in the category of compactly generated topological spaces. The empty space is (-2) -connected and every non-empty space is (-1) -connected. A map $A \rightarrow B$ of spaces (with B nonempty) is r -connected if for any choice of basepoint in B , the homotopy fiber with respect to this choice of basepoint is an $(r-1)$ -connected space. In particular, any map $A \rightarrow B$ is (-1) -connected. A weak homotopy equivalence is an ∞ -connected map.

G -spaces Let G be a finite group. A G -space is a space X equipped with a left action of G . A map of G -spaces is a G -equivariant map.

Let T be a transitive G -set. The T -cell of dimension j is the G -space

$$T \times D^j,$$

where G acts diagonally with trivial action on D^j .

Remark 2.1 If a choice of basepoint $t \in T$ is given, then one has a preferred isomorphism $G/H \cong T$, where $H = G_t$ is the stabilizer of t . Given another choice of basepoint t' , the stabilizer group $G_{t'}$ is conjugate to H . We will call the conjugacy class (H) the *type* of T . Two transitive G -sets are isomorphic if and only if they have the same type.

Given a G -map $f: T \times S^{j-1} \rightarrow Y$, one may form

$$Y \cup_f (T \times D^j).$$

This is called a T -cell attachment. If $j = 0$, we interpret the above as a disjoint union.

A *relative G -cell complex* (X, Y) is a pair in which X is obtained from Y by iterated equivariant cell attachments (where we allow T to vary over different transitive G -sets; the collection of attached cells is allowed to be a class). The order of attachment defines a partial

ordering on the collection of cells. If this order is dimension preserving (i.e., no cell of dimension j is attached after a cell of dimension j' when $j < j'$), then (X, Y) is a *relative G -CW complex*. When the collection of such attachments is finite, one says (X, Y) is *finite*. When Y is the empty space, X is a *G -cell complex* and when the attachments are self-indexing, X is a *G -CW complex*.

The *cellular dimension function* d_\bullet for (X, Y) is the indexing function whose value at H is the maximal dimension of the cells of type (H) appearing in the collection of attached cells. We set $d_H = -\infty$ if (X, Y) has no cells of type (H) .

Remark 2.2 Let M be a closed smooth G -manifold with dimension function m^\bullet . A result of Illman [15] shows that M possesses an equivariant triangulation. If d_\bullet is the cellular dimension function of this triangulation, then $d_H = m^H$ for all $H \in \mathcal{I}(G; M)$.

Quillen model structure Let $T(G)$ be the category of G -spaces. A morphism $f: X \rightarrow Y$ is a *weak equivalence* if for every subgroup $H \subset G$ the induced map of fixed points

$$f^H: X^H \rightarrow Y^H$$

is a weak homotopy equivalence. Similarly, a morphism f is a *fibration* if f^H is a Serre fibration for every H . A morphism $f: X \rightarrow Y$ is a *cofibration* if there is a relative G -cell complex (Z, X) such that Y is a retract of Z relative to X .

Let $R(G)$ be the category of *based* G -spaces. A morphism $X \rightarrow Y$ of $R(G)$ is a weak equivalence, cofibration or fibration if and only if it is so when considered as a morphism of $T(G)$.

Proposition 2.3 ([6], [23, Ch. VI, Sect. 5]) *With respect to the above structure, both $T(G)$ and $R(G)$ are Quillen model categories.*

Connectivity One says an indexing function r_\bullet is a *connectivity function* for a G -space Y if Y^H is r_H -connected for $H \subset G$ a subgroup (if Y^H is empty, we set $r_H = -2$). If $f: Y \rightarrow Z$ is a morphism of $T(G)$, then a connectivity function for f is an indexing function r_\bullet such that $f^H: Y^H \rightarrow Z^H$ is an r_H -connected map of spaces (one can always assume $r_H \geq -1$ since every map of spaces is at least (-1) -connected).

Lemma 2.4 *Let $Y \rightarrow Z$ be a fibration of $T(G)$ with connectivity function r_\bullet . Suppose (X, A) is a relative G -cell complex with cellular dimension function d_\bullet . Assume $d_H \leq r_H$ for all subgroups $H \subset G$. Then given a factorization problem of the form*

$$\begin{array}{ccc} A & \longrightarrow & Y \\ \downarrow & \nearrow & \downarrow \\ X & \longrightarrow & Z \end{array}$$

we can find an equivariant lift $X \rightarrow Y$ such that the diagram commutes.

Remark 2.5 The condition $d_H \leq r_H$ is automatically satisfied if no cells of type (H) occur in (X, A) .

Proof of Lemma 2.4 The proof proceeds by induction on the equivariant cells which are attached to A to form X . The inductive step is reduced to solving an equivariant lifting

problem of the kind

$$\begin{array}{ccc} G/H \times S^{j-1} & \longrightarrow & Y \\ \downarrow & \nearrow & \downarrow f \\ G/H \times D^j & \longrightarrow & Z, \end{array}$$

where the horizontal maps are allowed to vary in their equivariant homotopy class.

Now, a G -map $G/H \times U \rightarrow Z$ when U has a trivial action is the same thing as specifying a map $U \rightarrow Z^H$. This means the lifting problem reduces to an unequivariant one of the form

$$\begin{array}{ccc} S^{j-1} & \longrightarrow & Y^H \\ \downarrow & \nearrow & \downarrow f^H \\ D^j & \longrightarrow & Z^H \end{array}$$

The latter lift exists because f^H is r_H -connected and $j \leq r_H$. \square

Corollary 2.6 *Consider the lifting problem*

$$\begin{array}{ccc} A & \longrightarrow & Y \\ \downarrow & \nearrow & \downarrow \\ X & \longrightarrow & Z \end{array}$$

of G -spaces in which

- $Y \rightarrow Z$ is a map with connectivity function r_\bullet ,
- Y is cofibrant,
- (X, A) is a relative G -cell complex with dimension function d_\bullet ,
- $d_H \leq r_H$ for each subgroup $H \subset G$.

Then there is a G -map $X \rightarrow Y$ making the top triangle of the diagram commute and the bottom triangle homotopy commute.

Proof Factorize the map $Y \rightarrow Z$ as

$$Y \rightarrow Y^c \rightarrow Z$$

in which the map $Y \rightarrow Y^c$ is a cofibration and a weak equivalence and the map $Y^c \rightarrow Z$ is a fibration. Apply Lemma 2.4 to the diagram with Y^c in place of Y . To get a map $X \rightarrow Y^c$ making the diagram commute.

Since every object is fibrant, the acyclic cofibration $Y \rightarrow Y^c$ is a retract; let $r: Y^c \rightarrow Y$ be a retraction. Let $f: X \rightarrow Y$ be the map $X \rightarrow Y^c$ followed by the retraction. Then f satisfies the conclusion stated in the corollary. \square

Fiberwise G -spaces Fix a G -space B . A G -space over B is a G -space X equipped with G -map $X \rightarrow B$, usually denoted p_X . A morphism $X \rightarrow Y$ of G -spaces over B is a G -map which commutes with the structure maps p_X and p_Y . Let

$$T(B; G)$$

be the category of G -spaces over B .

We also have a “retractive” version of this category, denoted

$$R(B; G).$$

An object of the latter consists of a G -space X and maps $p_X: X \rightarrow B$, $s_X: B \rightarrow X$ such that $p_X s_X = \text{id}_B$. A morphism $X \rightarrow Y$ is an equivariant map compatible with both structure maps.

In either of these categories, a morphism $X \rightarrow Y$ is said to be a weak equivalence/cofibration/fibration if it is so when considered as a morphism of $T(G)$ by means of the forgetful functor. With respect to these definitions, $T(B; G)$ and $R(B; G)$ are Quillen model categories.

An object X in either of these categories is said to be r_\bullet -connected if the structure map $X \rightarrow B$ is $(r_\bullet + 1)$ -connected. A morphism is said to be r_\bullet -connected if the underlying map of $T(G)$ is.

The category $R(B; G)$ has *internal smash products*, constructed as follows: let $X, Y \in R(B; G)$ be objects. Then

$$X \wedge_B Y \in R(B; G)$$

is the object given by the pushout of the diagram

$$B \longleftarrow X \cup_B Y \xrightarrow{\subset} X \times_B Y$$

where $X \times_B Y$ is the fiber product of X and Y .

Since $R(B; G)$ is a model category, one can form homotopy classes of morphisms. If $X, Y \in R(B; G)$, we let

$$[X, Y]_{R(B; G)}$$

denote the set of homotopy classes of morphisms. Recall the definition requires us to replace X by its cofibrant approximation and Y by its fibrant approximation.

3 The Proof of Theorem A

Unreduced fiberwise suspension Let $E \in T(B; G)$ be an object. The *unreduced fiberwise suspension* of E over B is the object $S_B E \in T(B; G)$ given by the double mapping cylinder

$$S_B E := B \times 0 \cup E \times [0, 1] \cup B \times 1.$$

The two evident inclusions $s_-, s_+: B \rightarrow S_B E$ are morphisms of $T(B; G)$. Using s_- , we will consider $S_B E$ as an object of $R(B; G)$.

Obstruction to sectioning Let

$$B^+$$

denote $B \sqcup B$ considered as an object of $R(B; G)$ using the left summand to define a section. Then

$$s := s_- \sqcup s_+: B^+ \rightarrow S_B E$$

is a morphism of $R(B; G)$. We consider the associated homotopy class

$$[s] \in [B^+, S_B E]_{R(B; G)}.$$

The following proposition is an equivariant version of results of Larmore ([21, Th. 4.2–4.3]; see also [19, Prop. 3.1]).

Proposition 3.1 *Assume $E \in T(B; G)$ is fibrant. If $E \rightarrow B$ admits an equivariant section, then $[s]$ is trivial.*

Conversely, assume

- $[s]$ is trivial,
- B is a G -cell complex with dimension function b_\bullet ,
- the object $E \in T(B; G)$ is r_\bullet -connected, and
- $b_\bullet \leq 2r_\bullet + 1$.

Then $E \rightarrow B$ admits an equivariant section.

Proof Let $\sigma : B \rightarrow E$ be a section. Apply the functor S_B and note $S_B B = B \times [0, 1]$. We then get a map

$$S_B \sigma : B \times [0, 1] \rightarrow S_B E$$

which gives a homotopy from s_- to s_+ through morphisms of $T(B; G)$. This is the same thing as establishing the triviality of $[s]$.

Conversely, the diagram

$$\begin{array}{ccc} E & \longrightarrow & B \\ \downarrow & & \downarrow s_+ \\ B & \xrightarrow{s_-} & S_B E \end{array}$$

is preferred homotopy commutative in the category $T(B; G)$. As a diagram of G -spaces it is a homotopy pushout. Let $H \subset G$ be a subgroup. Taking H -fixed points, we obtain a homotopy pushout

$$\begin{array}{ccc} E^H & \longrightarrow & B^H \\ \downarrow & & \downarrow s_+^H \\ B^H & \xrightarrow{s_-^H} & S_{B^H} E^H \end{array}$$

in the category $R(B^H; e)$ where e is the trivial group. Since E is an r_\bullet -connected object, the map $E^H \rightarrow B^H$ is $(r_H + 1)$ -connected.

By the Blakers–Massey theorem (see, e.g., [10, p. 309]), the second diagram is $(2r_H + 1)$ -cartesian. Consequently, the first diagram is $(2r_\bullet + 1)$ -cartesian. Let P denote the homotopy inverse limit of the diagram

$$B \xrightarrow{s_-} S_B E \xleftarrow{s_+} B.$$

Then we conclude the map $E \rightarrow P$ is $(2r_\bullet + 1)$ -cartesian. If we assume $[s] = 0$, then the map $P \rightarrow B$ admits a section up to homotopy (using the universal property of the homotopy pullback). By the assumptions on B and Corollary 2.6, the G -map $E \rightarrow B$ admits a section up to homotopy. Since E is fibrant, this homotopy section can be converted into a strict section. \square

Naive stabilization The *reduced fiberwise suspension* $\Sigma_B E$ of an object $E \in R(B; G)$ is given by considering E as an object of $T(B; G)$, taking its unreduced fiberwise suspension $S_B E$ and taking the pushout of the diagram

$$B \leftarrow S_B B \rightarrow S_B E$$

where $S_B B \rightarrow S_B E$ arises by applying S_B to the structure map $B \rightarrow E$.

A *naive parametrized G -spectrum* \mathcal{E} is a collection of objects

$$\mathcal{E}_n \in R(B; G)$$

equipped with maps $\Sigma_B \mathcal{E}_n \rightarrow \mathcal{E}_{n+1}$.

Example 3.2 Let $Y \in R(B; G)$ be an object. Its naive parametrized suspension spectrum $\Sigma_B^\infty Y$ has n -th object $\Sigma_B^n Y$, the n -th iterated fiberwise suspension of Y .

Definition 3.3 Let $X \in T(B; G)$ be an object. The zeroth *cohomology* of X with coefficients in \mathcal{E} is the abelian group given by

$$H_G^0(X; \mathcal{E}) := \operatorname{colim}_{n \rightarrow \infty} [\Sigma_B^n X^+, \mathcal{E}_n]_{R(B; G)}$$

where $X^+ = X \amalg B$ and the maps in the colimit arise from the structure maps of \mathcal{E} .

Remark 3.4 Assuming the maps $\mathcal{E}_n \rightarrow B$ are fibrations, one can take the pullbacks $f^* \mathcal{E}_n \rightarrow X$. These form a naive G -spectrum over X , and an unraveling of the definitions gives

$$H_G^0(X; f^* \mathcal{E}) = H_G^0(X; \mathcal{E}).$$

Definition 3.5 Let $X, E \in T(B; G)$ be objects with E fibrant and X cofibrant. Let $f: X \rightarrow B$ be the structure map. Let

$$e(f, E) \in H_G^0(X; \Sigma_B^\infty S_B E)$$

be the class defined by the map

$$X^+ \xrightarrow{f^+} B^+ \xrightarrow{s} S_B E.$$

Proposition 3.6 Let X, E and f be as above. If $E \rightarrow B$ admits an equivariant section along f , then $e(f, E)$ is trivial.

Conversely, assume

- $e(f, E)$ is trivial,
- X is a G -cell complex with dimension function k_\bullet ,
- $E \in T(B; G)$ is r_\bullet -connected, and
- $k_\bullet \leq 2r_\bullet + 1$.

Then $E \rightarrow B$ admits an equivariant section along f .

Proof By Proposition 3.1, it will be enough to prove the maps

$$\Sigma_B: [\Sigma_B^n X^+, \Sigma_B^n S_B E]_{R(B; G)} \rightarrow [\Sigma_B^{n+1} X^+, \Sigma_B^{n+1} S_B E]_{R(B; G)}$$

are isomorphisms in the stated range. We will do this when $n = 0$. The case $n > 0$ is similar.

We have a map

$$E \rightarrow \Omega_B \Sigma_B E$$

which is adjoint to the identity. By Corollary 2.6, it will be enough to show this morphism is $2r_\bullet + 1$ -connected. Let $H \subset G$ be a subgroup and consider the map

$$E^H \rightarrow \Omega_{B^H} \Sigma_{B^H} E^H$$

of $R(B^H; e)$. If $b \in B^H$ is any point, we have an induced map of fibers

$$E_b^H \rightarrow \Omega \Sigma E_b^H.$$

Since E_b^H is r_H -connected, the Freudenthal suspension implies the last map is $(2r_H + 1)$ -connected. We infer that the map $E^H \rightarrow \Omega_B \Sigma_B E^H$ is $(2r_H + 1)$ -connected, which is what we needed to show. \square

Proof of Theorem A

Lemma 3.7 *The map $(N - Q) \rightarrow N$ is $(i_Q)!, \text{cd}_H(i_Q) - 1$ -connected.*

Proof Let $H \subset G$ be a subgroup. Then $(N - Q)^H = N^H - Q^H$, and we need to compute the connectivity of the inclusion

$$N^H - Q^H \rightarrow N^H.$$

This will be done using transversality.

Consider a map of pairs

$$\gamma: (K, A) \rightarrow (N^H, N^H - Q^H)$$

$(K, A) = (D^j, S^{j-1})$ or (S^j, \emptyset) . We can assume γ is transverse to Q^H . If $y \in \gamma(K) \cap Q^H$, then it must be the case $j \geq \text{cd}_H(i_Q)_H(y)$. Therefore, $\gamma(K)$ is disjoint from Q^H whenever $j < \text{cd}_H(i_Q)_H(y)$ for all $y \in Q^H$. This is equivalent to requiring $j < (i_Q)!, \text{cd}_H(i_Q)$, so the conclusion follows. \square

Let

$$(N - Q) \rightarrow E \rightarrow N$$

be the effect of factorizing $N - Q \rightarrow N$ as an acyclic cofibration followed by a fibration. By Lemma 3.7, $E \in T(N; G)$ is an $(i_Q)!, \text{cd}_H(i_Q) - 2$ -connected object.

Since $N - Q$ is cofibrant, it will suffice to show $E \rightarrow N$ admits a section along f .

We set $\mathcal{E}(i_Q)$ equal to the naive fiberwise suspension spectrum

$$\Sigma_N^\infty S_N E$$

and

$$e_G(f) := e(f, E) \in H_G^0(P; \mathcal{E}(i_Q)).$$

By the first part of Proposition 3.1, if E admits a section along f , then $e_G(f)$ is trivial.

Conversely, assume $e_G(f)$ is trivial. Then

$$e(\text{id}_P, f^*E) \in H_G^0(P; f^*\mathcal{E}(i_Q))$$

is also trivial. One easily checks f^*E is an $(f^*(i_Q)!, \text{cd}_\bullet(i_Q) - 2)$ -connected object. By the second part of Proposition 3.1, the fibration $f^*E \rightarrow P$ admits a section when

$$p_\bullet \leq 2f^*(i_Q)!, \text{cd}_\bullet(i_Q) - 3.$$

This completes the proof of Theorem A.

4 Naive versus equivariant stabilization

The unfibred case If $Y \in R(G) = R(*; G)$ is a cofibrant object, we define

$$Q_G Y = \operatorname{colim}_V \Omega^V \Sigma^V Y,$$

where V ranges over the finite dimensional subrepresentations of a complete G -universe \mathcal{U} partially ordered with respect to inclusion, and $\Omega^V \Sigma^V Y$ is the space of unequivariant based maps $S^V \rightarrow S^V \wedge Y$, where S^V is the one point compactification of V . This is a G -space by conjugating maps by group elements.

Consider the natural G -map

$$QY \rightarrow Q_G Y.$$

Proposition 4.1 *Assume Y has connectivity function r_\bullet . Then the map $QY \rightarrow Q_G Y$ is s_\bullet -connected, where*

$$s_H = \inf_{K \subsetneq H} r_K.$$

Proof Let $H \subset G$ be a subgroup. We must show the map of fixed points

$$Q(Y^H) = (QY)^H \rightarrow (Q_G Y)^H.$$

is s_H -connected. By the tom Dieck splitting ([23, p. 203, Th. 1.3], [26, Th. 7.7]),

$$(Q_G Y)^H \simeq \prod_{(K)} QEW(K)_+ \wedge_{W(K)} Y^K,$$

where (K) varies over the conjugacy classes of subgroups of H and $W(K)$ denotes the Weyl group. The factor corresponding to $(K) = (H)$ gives the inclusion $Q(Y^H) \rightarrow (Q_G Y)^H$. As $QEW(K)_+ \wedge_{W(K)} Y^K$ is r_K -connected, it follows the inclusion is $(\inf_{K \subsetneq H} r_K)$ -connected. \square

Equivariant stabilization Let V be a finite dimensional G -representation equipped with invariant inner product. We let $D(V)$ be its unit disk and $S(V)$ its unit sphere.

Let $X \in T(B; G)$ be an object. The *unreduced V -suspension* of X over B is the object $S_B^V X$ given by

$$S(V) \times B \cup_{S(V) \times E} D(V) \times Y.$$

Note the case of the trivial representation $V = \mathbb{R}$ recovers $S_B X$.

If $Y \in R(B; G)$ is an object, then a reduced version of the construction is given by

$$\Sigma_B^V Y = B \cup_{D(V) \times B} S_B^V Y.$$

The *fiberwise V -loops* of $Y \in R(B; G)$ is the object

$$\Omega_B^V E$$

given by the space of pairs (b, λ) in which $b \in B$ and $\lambda: S^V \rightarrow p^{-1}(b)$ is a based map. The action of $g \in G$ on (b, λ) is given by $(gb, g\lambda)$.

Then (Σ_B^V, Ω_B^V) is an adjoint functor pair.

Definition 4.2 For an object $Y \in R(B; G)$, define

$$Q_B^G Y := \operatorname{colim}_V \Omega_B^V \Sigma_B^V Y,$$

where the colimit is indexed over the finite dimensional subrepresentations of a complete G -universe \mathcal{U} .

Proposition 4.3 Assume $Y \in R(B; G)$ is fibrant and cofibrant, with connectivity function r_\bullet . Then

$$Q_B Y \rightarrow Q_B^G Y$$

is s_\bullet -connected, where

$$s_H := \inf_{K \subsetneq H} r_K.$$

Proof Let $H \subset G$ be a subgroup. Then the H -fixed points of $Q_B Y$ is $Q_{B^H} Y^H$. Consider the evident map

$$Q_{B^H} Y^H \rightarrow (Q_B^G Y)^H.$$

Let $b \in B^H$ be a point. Then the associated map of homotopy fibers at b is identified with

$$QY_b^H \rightarrow (Q_G Y_b)^H,$$

where Y_b is the fiber of $Y^H \rightarrow B^H$ at b . By Proposition 4.1, the map of homotopy fibers is s_H -connected. We conclude $Q_B Y^H \rightarrow (Q_B^G Y)^H$ is also s_H -connected. \square

5 Parametrized G -spectra over a complete universe

Let \mathcal{U} be a complete G -universe. A (parametrized) G -spectrum \mathcal{E} over B indexed on \mathcal{U} is a collection of objects

$$\mathcal{E}_V \in R(B; G)$$

indexed over the finite dimensional subrepresentations V of \mathcal{U} together with maps

$$\Sigma_B^{V^\perp} \mathcal{E}_V \rightarrow \mathcal{E}_W$$

for $V \subset W$, where V^\perp is the orthogonal complement of V in W .

Example 5.1 Let $X \in R(B; G)$ be an object. The fiberwise equivariant suspension spectrum of X , denoted $\Sigma_B^{\infty, G} X$ has V -th space

$$Q_B^G(\Sigma_B^V X).$$

These give rise to unreduced $\operatorname{RO}(G)$ -graded cohomology theories on $T(B; G)$. In order to get the details of the construction right, it is helpful to know a Quillen model structure is lurking in the background. For this exposition, it will suffice to explain what the weak equivalences, fibrant and cofibrant objects are in this model structure. The reference for this material is the book [24].

One says \mathcal{E} is *fibrant* if each of the adjoint maps $\mathcal{E}_W \rightarrow \Omega_B^W \mathcal{E}_{V \oplus W}$ is a weak equivalence of $R(B; G)$ and moreover, each of the maps $\mathcal{E}_W \rightarrow B$ is a fibration of $R(B; G)$. Any

object \mathcal{E} can be converted into a fibrant object \mathcal{E}^f by a natural construction, called fibrant approximation.

A map $\mathcal{E} \rightarrow \mathcal{E}'$ is given by compatible maps $\mathcal{E}_V \rightarrow \mathcal{E}'_V$. A map is a *weak equivalence* if after applying fibrant approximation, it becomes an equivalence at each V . An object \mathcal{E} is *cofibrant* if it is the retract of an object which is obtained from the zero object by attaching cells. Any \mathcal{E} can be functorially replaced by a cofibrant object within its weak homotopy type; this is called cofibrant approximation.

Cohomology and homology Let \mathcal{E} be as above, and assume it is both fibrant and cofibrant. Let $X \in T(B; G)$ be a cofibrant object. The equivariant *cohomology* of X with coefficients in \mathcal{E} is the $\mathrm{RO}(G)$ -graded theory on $T(B; G)$, denoted

$$h_G^\bullet(X; \mathcal{E}),$$

and defined as follows: if $\alpha = V - W$ is a virtual representation, we set

$$h_G^\alpha(X; \mathcal{E}) := [S^W \wedge X^+, \mathcal{E}_V]_{R(B; G)}$$

Similarly, the *homology* of X with coefficients in \mathcal{E} , denoted

$$h_\bullet^G(X; \mathcal{E}),$$

is defined by

$$h_\alpha^G(X; \mathcal{E}) := \mathrm{colim}_U [S^{V+U}, (\mathcal{E}_{W+U} \wedge_B X^+)/B]_{R(*; G)},$$

where $(\mathcal{E}_{W+U} \wedge_B X^+)/B$ is the effect of taking the mapping cone of the section $B \rightarrow \mathcal{E}_{W+U} \wedge_B X^+$.

Using fibrant and cofibrant approximation, the above extends in a straightforward way to the case of all objects X and all \mathcal{E} a G -spectrum over B (we omit the details).

Remark 5.2 Here is an alternative approach to the above, based on [13]. A map of G -spaces $f : X \rightarrow Y$ induces a pullback functor

$$f^* : R(Y; G) \rightarrow R(X; G)$$

given by $Z \mapsto Z \times_Y X$, with evident structure map.

The functor f^* has a right adjoint f_* given by

$$T \mapsto \underline{\mathrm{sec}}_Y(X \rightarrow T),$$

where $\underline{\mathrm{sec}}_Y(X \rightarrow T)$ has total space

$$\{(y, s) \mid y \in Y, s : X_y \rightarrow T_y\}.$$

Here, X_y denotes the fiber of $X \rightarrow Y$ at y , T_y is the fiber of the composite $T \rightarrow X \rightarrow Y$ at y and $s : X_y \rightarrow T_y$ is a based (unequivariant) map.

The functor f^* also admits a left adjoint, denoted f_\sharp , which is defined by

$$T \mapsto T \cup_X Y.$$

Let $\mathrm{Sp}(Y; G)$ be the category of G -spectra over Y . If we make these constructions level-wise, we obtain functors

$$f_*, f_\sharp : \mathrm{Sp}(X; G) \rightarrow \mathrm{Sp}(Y; G).$$

Now take $f: X \rightarrow *$ to be the constant map to a point, and replace these functors by their derived versions (using the Quillen model structure). Let \mathcal{E} be a fibred G -spectrum over B . If $X \in T(B; G)$ is an object with structure morphism $p_X: X \rightarrow B$, we can take the (derived) pullback $p_X^* \mathcal{E}$, which is a G -spectrum over X . Then the $\mathrm{RO}(G)$ -graded homotopy groups of the G -spectra

$$f_* p_X^* \mathcal{E} \quad \text{and} \quad f_{\#} p_X^* \mathcal{E}$$

yield the above cohomology and homology theories.

6 Poincaré duality

The orientation bundle Let M be a G -manifold and TM its tangent bundle. Let $S^{\tau} \in R(M; G)$ defined by taking the fiberwise one point compactification of $TM \rightarrow M$ (the section $M \rightarrow S^{\tau}$ is given by the zero section of TM).

Define $S^{-\tau}$ to be the fiberwise functional dual of S^{τ} . Alternatively, one can define an unstable version of $S^{-\tau}$ as follows: equivariantly embed M in a G -representation V and let ν denote its normal bundle. Its fiberwise one point compactification S^{ν} then represents $S^{-\tau}$ up to a degree shift by V , i.e.,

$$S^{-\tau} \simeq S^{\nu-V}.$$

We call $S^{-\tau}$ the *orientation bundle* of M .

If \mathcal{E} is a fibrant and cofibrant G -spectrum over M , we set

$$^{-\tau} \mathcal{E} := S^{-\tau} \wedge_M \mathcal{E}.$$

Using the diagonal action, this is a fibred G -spectrum over M , called the *twist* of \mathcal{E} by the orientation bundle.

Remark 6.1 The reader may object to this construction since we haven't defined internal smash products of parametrized G -spectra. An *ad hoc* way to define $^{-\tau} \mathcal{E}$ is to use the normal bundle ν . Let ${}^{\nu} \mathcal{E}$ be the parametrized G -spectrum given by ${}^{\nu} \mathcal{E}_W = S^{\nu} \wedge_M \mathcal{E}_W$, where we are using the fiberwise smash product in $R(M; G)$. Then $^{-\tau} \mathcal{E}$ can be defined as the parametrized G -spectrum whose W -th space is $\Omega_M^V {}^{\nu} \mathcal{E}_W$.

Alternatively, the reader is referred to [24, Ch. 13] for the construction of the internal smash product.

Poincaré duality The following is a special case of [13, Th. 4.9] and also a special case of [24, Th. 19.6.1].

Theorem 6.2 (Fiberwise Poincaré duality) *Let \mathcal{E} be a G -spectrum over a closed smooth G -manifold M . Then there is an isomorphism*

$$h_{\bullet}^G(M; {}^{-\tau} \mathcal{E}) \cong h_{\bullet}^G(M; \mathcal{E}).$$

Remarks 6.3 (1) Here it is essential that \mathcal{E} be indexed over a complete G -universe.

(2) Here is how to recover Theorem 6.2 from [13, Th. 4.9]. Take $f: M \rightarrow *$ to be the constant map to a point. Then, using the notation of Remark 5.2, we have an equivalence of G -spectra

$$f_{\#}(\mathcal{E} \wedge_M C_f^{-1}) \simeq f_* \mathcal{E},$$

where C_f^{-1} is the orientation bundle $S^{-\tau}$. Hence,

$$f_{\sharp}^{-\tau} \mathcal{E} \simeq f_{*} \mathcal{E}.$$

Now take the equivariant homotopy groups of both sides and use Remark 5.2 to obtain Theorem 6.2.

- (3) Here is how to recover Theorem 6.2 from [24, Th. 19.6.1]. Using their notation, take $M = E = B$ and $J = {}^{-\tau} \mathcal{E}$. Then one has an equivalence of equivariant fibred G -spectra over M

$$J \simeq S_p \triangleright (J \wedge_M \mathbb{P}_M S^{\tau}).$$

After applying homology $h_{\bullet}^G(M; -)$ to both sides, the left side becomes, in our notation, $h_{\bullet}^G(M; {}^{-\tau} \mathcal{E})$, whereas the right side, after some unraveling of definitions and rewriting, becomes $h_G^{\bullet}(M; \mathcal{E})$.

7 The equivariant complement formula

As in the introduction, let N be a G -manifold and let $i: Q \subset N$ be a closed G -submanifold. Then $N - Q \rightarrow N$ is an object of $T(N; G)$. Let

$$S_N(N - Q) \in T(N; G)$$

denote its fiberwise suspension. This has the equivariant homotopy type of the complement of Q in $N \times [0, 1]$.

Let ν denote the normal bundle of Q in N . We let $D(\nu)$ be its unit disk bundle and $S(\nu)$ its unit sphere bundle.

Lemma 7.1 *There is an equivariant weak equivalence*

$$D(\nu) \cup_{S(\nu)} N \simeq S_N(N - Q).$$

Proof Identify $D(\nu)$ with a closed equivariant tubular neighborhood of Q . Then we have an equivariant pushout

$$\begin{array}{ccc} S(\nu) & \longrightarrow & N - \text{int } D(\nu) \\ \downarrow & & \downarrow \\ D(\nu) & \longrightarrow & N, \end{array}$$

where $\text{int } D(\nu)$ is identified with the interior of the tubular neighborhood and the inclusion $N - Q \subset N - \text{int } D(\nu)$ is an equivariant equivalence.

So we have an equivariant homeomorphism

$$D(\nu) \cup_{S(\nu)} N \cong N \cup_{N - \text{int } D(\nu)} N$$

and the right side has the equivariant homotopy type of $S_N(N - Q)$, considered as an object of $R(N; G)$. \square

The object $D(\nu) \cup_{S(\nu)} N$ is called the *fiberwise equivariant Thom space* of ν over N . We denote it by

$$T_N(\nu).$$

More generally, for ξ a virtual G -bundle over Q , one has a *fiberwise equivariant Thom spectrum* $T_N(\xi)$ over N .

The virtual G -vector bundle over Q defined by $i^*\tau_N - \tau_Q$ is represented unstably by v . Substituting this and taking fiberwise suspension spectra of the right side of Lemma 7.1, we obtain

Corollary 7.2 (Complement formula) *There is an weak equivalence of G -spectra over N*

$$T_N(i^*\tau_N - \tau_Q) \simeq \Sigma_N^{\infty, G} S_N(N - Q).$$

Note the left side of Corollary 7.2 depends only on the underlying homotopy class of the map $i: Q \rightarrow N$.

8 The proof of Theorem B and Addendum C

Proof of Theorem B Consider the equivariant intersection problem

$$\begin{array}{ccc} & N - Q & \\ & \downarrow & \\ P & \xrightarrow{f} & N, \end{array}$$

from Sect. 1. Recall $E \rightarrow N$ is the effect of converting $N - Q \rightarrow N$ into a fibration.

Consider

$$e(\text{id}_P, f^*E) \in H_G^0(P; f^*\mathcal{E}(i_Q)).$$

An unraveling of definitions shows $f^*\mathcal{E}(i_Q)$ is weak equivalent to the naive fiberwise suspension spectrum of $S_P f^*E$.

Since the object $S_P f^*E$ is $(f^*(i_Q)_! \text{cd}_\bullet(i_Q) - 1)$ -connected (cf. Lemma 3.7), by Proposition 4.3 the map

$$Q_P S_P f^*E \rightarrow Q_P^G S_P f^*E$$

is s_\bullet -connected, where

$$s_H = \inf_{K \subseteq H} f^*(i_Q)_! \text{cd}_K(i_Q) - 1.$$

Using Corollary 2.6, we infer that the evident homomorphism

$$H_G^0(P; \mathcal{E}(i_Q)) \cong H_G^0(P; \Sigma_P^\infty S_P f^*E) \rightarrow h_G^0(P; \Sigma^{\infty, G} G_P S_P f^*E)$$

from the naive theory to the complete one is injective when

$$p^\bullet < s_\bullet.$$

In this range, it follows the image of $e(\text{id}_P, f^*E)$ in $h_G^0(P; \Sigma_P^{\infty, G} f^*E)$ is trivial if and only if $e(\text{id}_P, f^*E)$ was trivial to begin with.

The next step is to identify $h_G^\bullet(P; \Sigma_P^{\infty, G} S_P f^*E)$. Using Lemma 7.1, there is a weak equivalence of objects

$$S_N(N - Q) \simeq T_N(v) \in R(N; G)$$

where ν is the normal bundle of Q in N . Consequently, there is an isomorphism

$$h_G^\bullet(P; \Sigma_P^{\infty, G} \Sigma_P f^* E) \cong h_G^\bullet(P; \Sigma_P^{\infty, G} f^* T_N(\nu)).$$

By Theorem 6.2 the group on the right is naturally isomorphic to

$$h_G^\bullet(P; {}^{-\tau_P} \Sigma_P^{\infty, G} f^* T_N(\nu)).$$

An unraveling of the construction shows the latter coincides with the equivariant homotopy groups of the equivariant Thom spectrum of the virtual bundle ξ over $E(f, i_Q)$ appearing in the introduction. In particular,

$$\Omega_0^G(E(f, i_Q); \xi) \cong h_0^G(P; {}^{-\tau_P} \Sigma_P^{\infty, G} f^* T_N(\nu)).$$

With respect to these identifications, we define the *equivariant stable homotopy Euler characteristic*

$$\chi_G(f) \in \Omega_0^G(E(f, i_Q); \xi)$$

to be the unique element that corresponds to $e(\text{id}_P, f^* E)$. By the above and Theorem A, $\chi_G(f)$ fulfills the statement of Theorem B. \square

Proof of Addendum C When N has a boundary and P is closed, the above proof extends without modification. When P has a boundary, one only needs to replace Poincaré duality (6.2) in the closed case with a version of Poincaré duality for manifolds with boundary.

To formulate this, let $(M, \partial M)$ is a compact smooth manifold with boundary. Then duality in this case gives an isomorphism

$$h_G^\bullet(M; {}^{-\tau_M} \mathcal{E}) \cong h_G^\bullet(M, \partial M; \mathcal{E}).$$

The right side is defined as follows: for $\alpha = V - W$ and \mathcal{E} fibrant and cofibrant, define

$$h_G^\bullet(M, \partial M; \mathcal{E}) := [\Sigma_M^W(M // \partial M), \mathcal{E}_V]_{R(M; G)}$$

where $M // \partial M$ is the double

$$M \cup_{\partial M} M \in R(M; G)$$

(the section $M \rightarrow M // \partial M$ is defined using the left summand). \square

9 The proof of Theorems D and E

Proof of Theorem D Recall the factorization $(N - Q) \rightarrow E \rightarrow N$ in which $(N - Q) \rightarrow E$ is an acyclic cofibration and $E \rightarrow N$ is a fibration of $T(N; G)$.

By construction we have an isomorphism

$$H_G^0(X; \Sigma_N^\infty S_N E) \cong [X^+, Q_N S_N E]_{R(B; G)}$$

and an isomorphism

$$h_G^0(X; \Sigma_N^{\infty, G} S_N E) \cong [X^+, Q_N^G S_N E]_{R(B; G)}$$

for any object $X \in T(B; G)$.

With respect to these identifications, the homomorphism $H_G^0(X; \Sigma_N^\infty S_N E) \rightarrow h_G^0(X; \Sigma_N^{\infty, G} S_N E)$ arises from the map

$$Q_N S_N E \rightarrow Q_N^G S_N E$$

by applying homotopy classes $[X^+, -]_{R(B;G)}$.

Consider the commutative diagram of abelian groups

$$\begin{array}{ccccc} [P_i // P_{i-1}, Q_N S_N E]_{R(N;G)} & \xrightarrow{j_1} & [P_i^+, Q_N S_N E]_{R(N;G)} & \xrightarrow{k_1} & [P_{i-1}^+, Q_N S_N E]_{R(N;G)} \\ \ell_1 \downarrow & & \downarrow \ell_2 & & \downarrow \ell_3 \\ [P_i // P_{i-1}, Q_N^G S_N E]_{R(N;G)} & \xrightarrow{j_2} & [P_i^+, Q_N^G S_N E]_{R(N;G)} & \xrightarrow{k_2} & [P_{i-1}^+, Q_N^G S_N E]_{R(N;G)} \end{array} \quad (1)$$

with exact rows, where the object $P_i // P_{i-1}$ is given by $P_i \cup_{P_{i-1}} N$.

Definition 9.1 Let $f_i: P_i \rightarrow N$ be the restriction of f to P_i . Let

$$e_G^i(f) \in [P_i // P_{i-1}, Q_N S_N E]_{R(N;G)}$$

be the class determined by the composite

$$P_i \xrightarrow{f_i} N \xrightarrow{s_+} \Sigma_N E$$

together with the observation that its restriction to P_{i-1} has a preferred homotopy over N to the composite

$$P_{i-1} \xrightarrow{f_{i-1}} N \xrightarrow{s_-} \Sigma_N E.$$

By essentially the same argument which proves Theorem A, the map f_i is equivariantly homotopic to a map whose image is disjoint from Q , relative to P_{i-1} , provided

- $e_G^i(f) = 0$ and
- $p^H \leq 2f^*(i_Q)_! \text{cd}_H(i_Q) - 3$ for all $(H) \in \mathcal{I}(G; P)$.

If this is indeed the case, the equivariant homotopy extension property can be used to obtain a new G -map f' , coinciding with f on P_{i-1} , and satisfying $f'(P_i) \cap Q = \emptyset$.

In order to complete the proof of Theorem D we will apply a version of Poincaré duality. Set $H = H_i$ and $P_s^H = P^H - P_H$. Then the inclusion of pairs

$$(G \cdot P^H, G \cdot P_s^H) \rightarrow (P_i, P_{i-1})$$

is a relative G -homeomorphism. Recall the Weyl group $W(H)$ acts on P^H and restricts to a free action on P_H . The following result follows from the existence of equivariant tubular neighborhoods.

Lemma 9.2 ([5, §IV]) *The open $W(H)$ -manifold P_H is the interior of a compact free $W(H)$ -manifold \bar{P}_H with corners. Furthermore, the inclusion $P_H \subset \bar{P}_H$ is an equivariant weak equivalence.*

Consider the left square of diagram 1. By Lemma 9.2, and “change of groups” it maps to the square

$$\begin{array}{ccc} [\bar{P}_H // \partial \bar{P}_H, Q_N S_N E]_{R(N;W(H))} & \xrightarrow{j'_1} & [(P^H)^+, Q_N S_N E]_{R(N;W(H))} \\ \ell'_1 \downarrow & & \downarrow \ell'_2 \\ [\bar{P}_H // \partial \bar{P}_H, Q_N^G S_N E]_{R(N;W(H))} & \xrightarrow{j'_2} & [(P^H)^+, Q_N^G S_N E]_{R(N;W(H))}. \end{array} \quad (2)$$

The proof of Theorem D is completed in two steps.

Step 1 The homomorphism ℓ'_1 is an isomorphism, since $W(H)$ acts freely on $\bar{P}_H // \partial \bar{P}_H$ in the “based” sense. This can be proved by an induction argument using an equivariant cell decomposition, together with the observation that the map $Q_N S_N E \rightarrow Q_N^G S_N E$ is a weak homotopy equivalence of underlying topological spaces.

Step 2 There is a relative $W(H)$ -homeomorphism

$$(\bar{P}_H, \partial \bar{P}_H) \cong (P^H, P_s^H)$$

which, together with change of groups, gives an isomorphism

$$[\bar{P}_H // \partial \bar{P}_H, Q_N S_N E]_{R(N; W(H))} \cong [P_i // P_{i-1}, Q_N S_N E]_{R(N; G)}.$$

We will consider $e_G^i(f)$ to be an element of the left hand side. Then $\ell'_1(e_G^i(f))$ can be regarded as an element of relative cohomology group

$$h_{W(H)}^0(\bar{P}_H, \partial \bar{P}_H; \Sigma_N^{\infty, G} S_N E).$$

Define $\chi_G^i(f)$ to be its Poincaré dual. Using the equivariant equivalence $P_H \simeq \bar{P}_H$, we can regard $\chi_G^i(f)$ as living in the homology group

$$h_0^{W(H)}(P_H; {}^{-\tau_{P_H}}(\Sigma_{P_H}^{\infty, G} S_{P_H} f_H^* E)).$$

As in the proof of Theorem B this homology group is isomorphic to the equivariant bordism group

$$\Omega_0^{W(H)}(E(f_H, i_Q); {}_H\xi).$$

□

Proof of Theorem E The proof uses diagrams (1) and (2). The homomorphism $(t_H)_*$ is identified with the Poincaré dual of the homomorphism j'_2 of diagram (2). Therefore $(t_H)_*$ is injective if and only if j'_2 is. Let

$$\ell: H_G^0(P; \Sigma_N^\infty S_N E) \rightarrow h_G^0(P; \Sigma_N^{\infty, G} S_N E)$$

be the canonical homomorphism. Recall $\chi_G(f)$ is the Poincaré dual of $\ell(e_G(f))$.

The class $j'_1(e_G^i(f))$ is the one associated with the composition

$$(P^H)^+ \subset P^+ \xrightarrow{f} N \xrightarrow{s} S_N E,$$

i.e., the restriction of $e_G(f)$ to P^H . By hypothesis, $\chi_G(f)$ is trivial, so $\ell'_2 j'_1(e_G^i(f))$ must also be trivial since the latter is the restriction to P^H of the trivial class $\ell(e_G(f))$.

Hence

$$j'_2 \ell'_1(e_G^i(f)) = \ell'_2 j'_1(e_G^i(f)) = 0.$$

Furthermore, since j'_2 is identified with $(t_H)_*$, and the latter is by hypothesis injective, the vanishing of $j'_2 \ell'_1(e_G^i(f))$ implies $\ell'_1(e_G^i(f)) = 0$. Hence, $\chi_G^i(f)$ vanishes too, as it is the Poincaré dual of $\ell'_1(e_G^i(f))$. The result is now concluded by induction on i and Theorem D.

□

10 The Proof of Theorem H

Given a closed smooth G -manifold M , we have a commutative square of equivariant mapping spaces

$$\begin{array}{ccc} \text{end}^b(M)^G & \xrightarrow{\quad \subset \quad} & \text{end}(M)^G \\ \downarrow & & \downarrow \\ \text{map}(M, M \times M - \Delta)^G & \longrightarrow & \text{map}(M, M \times M)^G \end{array} \quad (3)$$

where

- $\Delta := \Delta_M \subset M \times M$ is the diagonal,
- $M \times M$ is given the diagonal G -action,
- $\text{end}(M)^G$ is the space of equivariant self maps of M ,
- $\text{end}^b(M)^G$ is the subspace of fixed point free equivariant self maps, and
- the vertical maps of the square are given by taking graphs and the horizontal ones are inclusions.

Lemma 10.1 *The square (3) is ∞ -cartesian, i.e., it is a homotopy pullback.*

Proof The following idea is used in the proof. Suppose $X \rightarrow Y$ is a map of fibrations over B . Let X_b be the fiber of $X \rightarrow B$ at $b \in B$ and similarly let Y_b be the fiber of $Y \rightarrow B$. Then the diagram

$$\begin{array}{ccc} X_b & \longrightarrow & Y_b \\ \downarrow & & \downarrow \\ X & \longrightarrow & Y \end{array}$$

is ∞ -cartesian.

We claim the first factor projection map

$$M \times M - \Delta \rightarrow M \quad (4)$$

is a fibration of $T(G)$. For, let $H \subset G$ be a subgroup. Taking the induced map of fixed point spaces yields the projection map

$$M^H \times M^H - \Delta_{M^H} \rightarrow M^H. \quad (5)$$

Since M^H is a manifold, the map (5) is a Serre fibration of spaces. It follows that the map (4) is a fibration of $T(G)$.

Applying the functor $\text{map}(M, -)^G$ to the projection map, we infer

$$\text{map}(M, M \times M - \Delta)^G \rightarrow \text{map}(M, M)^G$$

is a fibration whose fiber at the identity map of M is $\text{map}^b(M)^G$.

Similarly, the first factor projection $M \times M \rightarrow M$ is an equivariant fibration, so the induced map

$$\text{map}(M, M \times M)^G \rightarrow \text{map}(M, M)^G$$

is a fibration whose fiber at the identity is $\text{map}(M, M)^G$.

It now follows easily from the first paragraph of the proof that the square (3) is ∞ -cartesian. \square

From Lemma 10.1, the obstruction to deforming an equivariant self map

$$f: M \rightarrow M$$

to a fixed point free map coincides with equivariantly deforming its graph $\Gamma_f: M \rightarrow M \times M$ off of the diagonal.

Consequently, we are reduced to the equivariant intersection problem

$$\begin{array}{ccc} & M \times M - \Delta & \\ & \downarrow & \\ M & \xrightarrow{\Gamma_f} & M \times M. \end{array}$$

We will prove Theorem H using Corollary F. We will need to compute the codimension function the diagonal.

Lemma 10.2 *Let $i_\Delta: \Delta \subset M \times M$ be the inclusion. For $(H) \in \mathcal{I}(M; G)$, we have*

$$\text{cd}_H(i_\Delta) = m^H.$$

Proof If $x \in \Delta = M$ is a point then the codimension of the diagonal inclusion $M_{(x)}^H \subset M_{(x)}^H \times M_{(x)}^H$ is clearly $m^H(x)$. \square

By Lemma 10.2, the inequality of Corollary F amounts to the condition

$$m^H \leq 2(\Gamma_f)^*(i_\Delta)_! m^H - 3.$$

By a straightforward argument which we omit, $(\Gamma_f)^*(i_\Delta)_! m^H$ coincides with m^H , so the inequality becomes

$$m^H \geq 3$$

for $(H) \in \mathcal{I}(M; G)$.

We now turn to the problem of deciding when the homomorphisms $(t_H)_*$ are injective. What is special about the fixed point case is that the virtual $W(H)$ -bundle ${}^H\xi$, which sits over the space

$$M^H \times_M L_f M,$$

is represented by an actual vector bundle. This vector bundle is just the pullback of the normal bundle of the embedding $M^H \subset M$ along the (projection) map $M^H \times_M L_f M \rightarrow M^H$. Henceforth, we identify ${}^H\xi$ with this vector bundle.

Therefore $(t_H)_*$ is identified with the homomorphism of equivariant framed bordism groups

$$\Omega_0^{W(H)}(M_H \times_M L_f M; {}_H\xi) \rightarrow \Omega_0^{W(H)}(M^H \times_M L_f M; {}^H\xi) \quad (6)$$

induced by the inclusion

$$t_H: M_H \times_M L_f M \rightarrow M^H \times_M L_f M,$$

where both ξ_H and ξ^H are $W(H)$ -vector bundles and the pullback $t_H^* \xi^H$ is isomorphic to ξ_H .

Hence, $(t_H)_*$ arises by taking the $W(H)$ -fixed spectra of the map of equivariant suspension spectra

$$\Sigma_{W(H)}^\infty(M_H \times_M L_f M)^{H\xi} \rightarrow \Sigma_{W(H)}^\infty(M^H \times_M L_f M)^{H\xi} \quad (7)$$

and then applying π_0 .

The inclusion $M_H \subset M^H$ is 1-connected, since by hypothesis $M_s^H := M^H - M_H$ has co-dimension at least two in M^H . Consequently, the inclusion $M_H \times_M L_f M \rightarrow M^H \times_M L_f M$ is also 1-connected. Furthermore, $M_H \times_M L_f M$ is $W(H)$ -free.

If we apply the tom Dieck splitting to the $W(H)$ -fixed points of the map (7), we obtain maps of summands of the form

$$\Sigma^\infty((M_H \times_M L_f M)^{H\xi})_{hW'(K)}^K \rightarrow \Sigma^\infty((M^H \times_M L_f M)^{H\xi})_{hW'(K)}^K, \quad (8)$$

where (K) ranges through the conjugacy classes of subgroups of $W(H)$, $W'(K)$ denotes the Weyl group of K in $W(H)$ and the subscript “ $_{hW'(K)}$ ” is an abbreviation for the Borel construction (in the notation above, we are first Thomifying, then taking fixed points and thereafter taking the Borel construction).

If K is not the trivial group, then the freeness of the action implies the domain of (8) is contractible, and therefore this map induces an injection on π_0 . If K is trivial, then the map takes the form

$$\Sigma^\infty(M_H \times_M L_f M)_{hW(H)}^{H\xi} \rightarrow \Sigma^\infty(M^H \times_M L_f M)_{hW(H)}^{H\xi}$$

which is evidently 1-connected. Assembling these injections, one sees the homomorphism (6) is also injective. Therefore, the homomorphism $(t_H)_*$ appearing in the statement of Corollary F is injective for every $(H) \in \mathcal{I}(M; G)$.

The Proof of Theorem H is now completed by applying Corollary F.

11 The Proof of Theorems I and J

Let $f: M \rightarrow M$ be a self map of a closed smooth manifold M . The *Fuller map* of f is the \mathbb{Z}_n -equivariant self map of $M^{\times n}$ given by

$$(x_1, \dots, x_n) \mapsto (f(x_n), f(x_1), f(x_2), \dots, f(x_{n-1}))$$

(compare Fuller [9]). Here $n \geq 2$ and \mathbb{Z}_n acts by cyclic permutation of factors. The assignment $x \mapsto (x, f(x), \dots, f^{n-1}(x))$ defines a \mathbb{Z}_n -equivariant bijective correspondence between the n -periodic point set of f and the fixed point set of $\Phi_n(f)$. In particular, f is n -periodic point free if and only if $\Phi_n(f)$ is fixed point free. We wish to know whether this statement is true up to homotopy.

Let $\text{end}(M)$ be the space of self maps of M , and $\text{end}(M^{\times n})^{\mathbb{Z}_n}$ the space of \mathbb{Z}_n -equivariant self maps of $M^{\times n}$. The *Fuller transform*

$$\Phi_n: \text{end}(M) \rightarrow \text{end}(M^{\times n})^{\mathbb{Z}_n}$$

is defined by $f \mapsto \Phi_n(f)$.

Let

$$\text{end}_n^b(M) \subset \text{end}(M)$$

be the subspace of self maps having no n -periodic points. Let

$$\mathrm{end}^b(M^{\times n})^{\mathbb{Z}_n} \subset \mathrm{end}(M^{\times n})^{\mathbb{Z}_n}$$

be the subspace of equivariant self maps of $M^{\times n}$ which are fixed point free.

Then there is a commutative diagram of spaces

$$\begin{array}{ccc} \mathrm{end}^{b_n}(M) & \longrightarrow & \mathrm{end}(M) \\ \downarrow & & \downarrow \\ \mathrm{end}^b(M^{\times n})^{\mathbb{Z}_n} & \longrightarrow & \mathrm{end}(M^{\times n})^{\mathbb{Z}_n} \end{array} \quad (9)$$

where the vertical maps are given by the Fuller transform and the horizontal ones are inclusions. The square is cartesian, i.e., it is a pullback. We wish to understand the extent to which it is a homotopy pullback.

Question Is the above square 0-cartesian?

That is, is the map from $\mathrm{end}^{b_n}(M)$ to the corresponding homotopy pullback a surjection on components? If yes, it would reduce the problem of studying the n -periodic points of f to the \mathbb{Z}_n -equivariant fixed point theory of $\Phi_n(f)$. At the time of writing we do not know this to be the case. Nevertheless, we can still use the diagram to get an invariant of self maps which is trivial when the self map is homotopic to an n -periodic point free one.

Definition 11.1 Set

$$\ell_n(f) := \ell_{\mathbb{Z}_n}(\Phi_n(f)).$$

By a straightforward calculation we omit, $\ell_n(f)$ lives in the group

$$\Omega_0^{\mathbb{Z}_n, \mathrm{fr}}(\mathrm{ho}P_n(f))$$

appearing in the statement of Theorem I. It is clear that $\ell_n(f)$ vanishes when f is homotopic to an n -periodic point free map. If we apply Theorem H to $\Phi_n(f)$ we obtain

Corollary 11.2 Assume $\dim M \geq 3$ and $\ell_n(f) = 0$. Then $\Phi_n(f)$ is equivariantly homotopic to a fixed point free map.

As mentioned in Sect. 1, a result of Jezierski [16] asserts f is homotopic to an n -periodic point free map if $\dim M \geq 3$ and the Nielsen numbers $N(f^k)$ vanish for each k a divisor of n . Conjecturally, the invariant $\ell_n(f)$ contains at least as much information as these Nielsen numbers (additional evidence for this is provided below in Theorem J). If one assumes this to be the case, then Jezierski's theorem tends to suggest that the square (9) is 0-cartesian. However, we do not see any homotopy theoretic reason why that should be true.

Proof of Theorem J The tom Dieck splitting yields a decomposition of $\Omega_0^{\mathbb{Z}_n, \mathrm{fr}}(\mathrm{ho}P_n(f))$ into summands of the form

$$\Omega_0^{\mathrm{fr}}(E\mathbb{Z}_k \times_{\mathbb{Z}_k} \mathrm{ho}P_k(f))$$

for k a divisor of n , where we are using the fact $\mathrm{ho}P_n(f)^{\mathbb{Z}_k} = \mathrm{ho}P_k(f)$.

Since the zeroth framed bordism of a space is the free abelian group on its path components, it will suffice to show $\pi_0(E\mathbb{Z}_k \times_{\mathbb{Z}_k} \mathrm{ho}P_k(f))$ is isomorphic to $\pi_{\rho, k}$. We first compute the set of components of $\mathrm{ho}P_k(f)$.

Recall from Sect. 1 that a point of $\text{ho}P_k(f)$ is given by a k -tuple of paths

$$(\lambda_1, \dots, \lambda_k)$$

subject to the constraint $f(\lambda_{i+1}(0)) = \lambda_i(1)$ where i is taken modulo k . Two points $(\lambda_1, \dots, \lambda_k)$ and $(\gamma_1, \dots, \gamma_k)$ are in the same component if and only if there are paths α_i having initial point $\lambda_i(0)$ and terminal point $\gamma_i(0)$ such that the concatenated paths

$$f(\alpha_i) * \lambda_{i+1} \quad \text{and} \quad \gamma_{i+1} * \alpha_{i+1}$$

are homotopic relative to their endpoints, for $i = 1, 2, \dots, k$.

Since M is connected, each component of $\text{ho}P_k(f)$ has a point of the form $(\lambda_1, \lambda_2, \dots, \lambda_k)$ satisfying $\lambda_i(0) = *$ and $\lambda_i(1) = f(*)$. Let π_f denote the set of homotopy classes of paths in M joining the basepoint $*$ to the point $f(*)$. A choice of element $[\alpha]$ of π_f determines an isomorphism with π . Using this isomorphism the set of path components of $\text{ho}P_k(f)$ is a quotient of the k -fold cartesian product

$$\pi \times \dots \times \pi$$

with respect to the equivalence relation

$$(x_1, x_2, \dots, x_k) \sim (g_1 x_1 \rho(g_2)^{-1}, g_2 x_2 \rho(g_3)^{-1}, \dots, g_k x_k \rho(g_1)^{-1})$$

for $x_i, g_i \in \pi$. Using this relation, the k -tuple (x_1, \dots, x_k) is equivalent to the k -tuple

$$(y, 1, \dots, 1)$$

where $y = x_1 \rho(x_2) \rho^2(x_3) \dots \rho^{k-1}(x_k)$. Furthermore, any two elements of the form $(y, 1, \dots, 1)$ and $(z, 1, \dots, 1)$ are related precisely when $z = g y \rho^k(g)^{-1}$ for some element $g \in \pi$. Summarizing thus far, we have shown $\pi_0(\text{ho}P_k(f))$ is the quotient of π with respect to the equivalence relation

$$y \sim g y \rho^k(g)^{-1}$$

for $g, y \in \pi$.

To complete the proof of Theorem J, one notes the set of path components of the Borel construction coincides with the coinvariants of \mathbb{Z}_k acting on $\pi_0(\text{ho}P_k(f))$. With respect to the k -tuple description of $\pi_0(\text{ho}P_k(f))$, the action is induced by cyclic permutation of factors: $(x_1, x_2, \dots, x_k) \mapsto (x_k, x_1, \dots, x_{k-1})$. If we identify this element with $(y, 1, \dots, 1)$ with y as above, then the result of acting by a generator of the cyclic group results in an element equivalent to $(\rho(y), 1, \dots, 1)$. Consequently, $\pi_0(E\mathbb{Z}_k \times_{\mathbb{Z}_k} \text{ho}P_k(f))$ is obtained from $\pi_0(\text{ho}P_k(f))$ by imposing the additional relation $y \sim \rho(y)$. Hence, the set of path components of the Borel construction is isomorphic to $\pi_{\rho, k}$. \square

Conjecture 11.3 *Let $\mathcal{N}_k(f)$ be the number of non-zero terms in $\ell_n^k(f)$ expressed as a linear combination of the basis elements of $\mathbb{Z}[\pi_{\rho, k}]$. Then $\mathcal{N}_k(f)$ equals the Nielsen number of f^k .*

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