Correspondence Kervaire ↔ Milnor about surgery found in Kervaire Nachlass in February 2009

Abstract

Transcribed by Shu Otsuka

1. Letter Milnor → Kervaire dated August 22 (1958).

August 22

Dear Kervaire,

Enclosed is a first draft of the lecture I gave in Edinburgh. If you would like to make a joint paper, why don't you work it over, and send it to me at Rorschach . It was supposed to be handed in yesterday; but I don't suppose they were serious about that.

Best regards
John

2. Letter Milnor → Kervaire dated September 8 (1958).

Rorschach , September 8

Dear Michel,

Could you straighten out the references (in particular the numbering) in the manuscript? I don't have a library here, and it will take a while till I get to work in Princeton. I think the paper is in very good shape otherwise. If you are satisfied you might as well send it on to England. A covering letter to Todd is enclosed.

Is Whitehead's proof that (tangent bundle trivial $\Longrightarrow$ normal bundle trivial) readable? I have forgotten.

As to von Staudt there are two theorems involved, each of which was discovered independently by someone else. The first theorem is found, for example, in Hardy and Wright. I hope you don't have trouble locating the second (concerning the numerators of B_n).

Wouldn't it be a good idea to have this manuscript mimeographed in Princeton or in Geneva if you have facilities? It will be a long time before the Congress proceedings come out. I hope that you have some carbon copies. (Otherwise perhaps you could have a photo copy made, to send to Princeton). Enclosed are copies of two pages I retyped.

Best wishes
John

Fine Hall, Princeton N.J.

3. Letter Milnor → Kervaire dated September 23 (1958).

Princeton
Sept. 23, 1958

Dear Michel,

The manuscript looks fine.

The theorem that a Π-manifold $M^k \subset \mathbb{R}^{2k}$ has trivial normal bundle is new to me. In any case there is no point in bringing that in.

As to the references:

[6]: ... AJM 80, 632-638 (1958).

[11]: ... Classification of mappings of an $(n+3)$-dimensional sphere into an n-dimensional one. … 19-22

[13]: ... Beweis eines Lehrsatzes, die Bernouillischen Zahlen betreffend. ...

Could you also send mimeographed copies to Hirzebruch (Mathematisches institut der Universität Bonn) and Rohlin (*КОЛОМНА*, ПЕДАГОГИЧЕСКИЙ ИНСТИТУТ)? Thanks a lot for having it mimeographed.

Sincerely
John

4. Letter Kervaire → Milnor dated October 7 1959.

Oct. 7, 1959

Dear Milnor,

I need the following statement which should be an easy extension of the surgery theorem you proved in “Differentiable manifolds which are homotopy spheres”.

Let M^n be a closed, differentiable manifold imbedded in $\mathbb{R}^{n+m}$ with m large. Assume the normal bundle v is almost trivial. Let $o(v, f)$ be the obstruction to extend some given x-section f of $v|_{M \setminus x}$.

Then surgery in M^n yields a manifold M_1^n in $\mathbb{R}^{n+m}$ which is r-connected, $r < n/2$. The normal bundle v_1 of M_1^n is almost trivial and there exists a x-section f_1 of $v_1|_{M_1 \setminus x_o}$ such that $o(v, f) = o(v_1, f_1)$. From this $I(M) = I(M_1)$ is a corollary. Moreover, if $I(M) = 0$, then surgery can make M_1 to be $[n/2]$-connected, still with existence of x-section f_1 of $v_1|_{M_1 \setminus x_o}$ such that $o(v, f) = o(v_1, f_1)$.

1°) Do you think the above statement is true?

It would imply that if $n \equiv 1, 2 \pmod 8$, then $o(v, f)$ does not depend on f. Can you prove this last statement a priori?

2°) If your answer to the first question is yes, do you intend to publish a surgery theorem including the statement on the obstructions and the case $r = [n/2]$?

If there is anything true in the above beyond your statements in the mimeographed notes on homotopy spheres, it would be very useful, I think, to have it in the literature.

I apologize for keeping the manuscript of your paper with Spanier such a long time. I'll make an effort to return it soon.

Very sincerely yours,

5. Letter Milnor → Kervaire dated October 15 1959.

Berkeley 4, California
October 15, 1959

Dear Michel,

Unfortunately I do not know how to prove as much as you need.

1) The assertion that $o(v, f)$ is unchanged by "surgery" can be proved by a slight modification of the argument used in 5.4 of my note "Differentiable manifolds which are homotopy spheres". Namely it is necessary to work with the Whitney sum (tangent bundle) $\oplus$ (trivial bundle). Do you have an idea for a better proof using the normal bundle? My proof is certainly hard to follow.

2) Suppose that $n = 2k$. Then it is easy to obtain a manifold M_1 which is $(k-1)$-connected using surgery. In order to obtain a manifold which is k-connected it is necessary to assume something further. For k even the assumption $I(M) = 0$ is sufficient, but for k odd there is an "obstruction" coming form the kernel of

$$\pi_{k-1}(SO_k) \to \pi_{k-1}(SO)$$

which is usually cyclic of order 2. (Compare 5.11 and 5.12. of my note.)

However the assertion that $o(v_1, f_1)$ is independent of f_1 follows in an easier way if $n = 2k$ with $k \equiv 5 \pmod 8$. Given a second cross section f_1', the only obstruction to a homotopy lies in

$$H^k(M_1; \pi_k(SO)) = 0.$$

Hence $o(v_1, f_1) = o(v_1, f_1')$.

Unfortunately there is a catch in this argument which I just noticed. Namely the specific partial cross-section f of v (or of $\tau \oplus$ trivial) is used in the construction of M_1 from M: namely it is used in deciding which product structure to give to the normal bundle of a sphere $f(S^r) \subset M$. (See 5.4). Thus starting with a different cross-section f' we may arrive at a different M_1. My ideas run out at this point.

3) For $n = 2k+1$ it is again possible to make M_1 $(k-1)$-connected; but it seems very difficult to go any further. (Compare 5.13.) Again it follows that $o(v_1, f_1)$ is independent of f_1 providing that $k \equiv 4 \pmod 8$; but again this does not imply anything for M.

I am hoping to write a paper on surgery, but haven't started yet.

There is no hurry in retuning the Spanier papers. I hope that you are enjoying New York.

Best regards
John Milnor

6. Letter Milnor → Kervaire dated November 19 1959.

Berkeley 4, California
November 19, 1959

Dear Michel,

Glad to hear that you are still thinking about these problems. Your last letters inspired me to get get to work, and I now have a manuscript being typed. I will send you a copy.

Both of your conjectures sound correct. In fact the second one is contained in my manuscript, as part of the proof of the proof of the following: M_1 can be obtained from M_2 by iterated surgery $\Longleftrightarrow$ M_1 and M_2 belong to the same cobordism class. [M_1 and M_2 must be closed manifolds of course. Actually I have switched terminology and am using the phrase "χ-construction" for surgery.]

However I do not follow your applications of theses conjectures. First consider two k-spheres in M^{2k} with one "clean" intersection point. Let $\alpha, \beta \in \mathbb{Z}_2 \subset \pi_{k-1}(SO_k)$ be the homotopy classes which correspond to their normal bundle. Then replacing these two imbedded spheres by a third, with homotopy class in $\pi_k(M^{2k})$ corresponding to the sum, I claim that the new normal bundle corresponds to the element $\alpha+\beta+1 \in \mathbb{Z}_2$ (rather than $\alpha+\beta$ as you claimed). Consider for example the spheres $S^k \times 0$ and $0 \times S^k$ in $S^k \times S^k$, with $\alpha = \beta = 0$. Then the new sphere which you construct would be isotopic to the diagonal, and therefore have non-trivial normal bundle.

More generally I claim the following. There is a function $\varphi : H_k(M^{2k}; \mathbb{Z}_2) \to \mathbb{Z}_2$ defined by

$\varphi(x) = \left\{ \begin{matrix} 1 \\ 0 \end{matrix} \right\}$ if the normal bundle of an imbedded sphere representing the homology class x is $\left\{ \begin{matrix} \text{non-trivial} \\ \text{trivial} \end{matrix} \right\}$.

This function φ satisfies the identity

$$\varphi(x+y) = \varphi(x) + \varphi(y) + (\text{Intersection number } \langle x, y \rangle).$$

Thus one obtains a quadratic form over the field $\mathbb{Z}_2$. Such a form is completely characterized by the middle Betti numbers, together with its "Arf invariant" which has only two possible values. One can kill $H_k(M^{2k}; \mathbb{Z}_2)$ by this method if and only if the Arf invariant is trivial. The proofs which I have for these statements are rather involved.

As for the use of Morse theory, didn't Morse make use of the sets $\varphi \leq$ constant rather than $\varphi =$ constant? (where $\varphi : M \to \mathbb{R}$). Unfortunately I don't have your thesis with me.

The following is the analysis which I have in mind for a $(2k+1)$-manifold. Consider an imbedding $S^k \times D^{k+1} \subset M$ which represents a homology class $\alpha \in H_k(M)$ of order r ; $1 < r < \infty$. Let $M_0 = M \setminus \text{Interior}(S^k \times D^{k+1})$ and let $\lambda, \mu \in H_k(M_0)$ correspond to the standard generators of $H_k(S^k \times S^k)$. Thus $H_k(M)$ is obtained from $H_k(M_0)$ by adding the relation $\mu = 0$. Since $\lambda \to \alpha$ of order r we have $r\lambda + s\mu = 0$ for some $s \in \mathbb{Z}$. This must be the only relation between λ and μ.

Now performing the "χ-construction" we must add the relation $\lambda = 0$. Thus the cyclic group of order r is replaced by a group of order s. The construction is successful only if $|s| < r$. (The case $s = 0$ means that we obtain an infinite cyclic group which can be eliminated, as you indicated.)

6. Letter Milnor → Kervaire dated November 19 1959 (continued).

The integer s itself seems rather hard to control, however the residue class of s modulo r is a familiar object: namely the self-linking number of α.

Now consider the extent to which the picture can be changed by choosing a new trivialization for the normal bundle of $S^k \times 0$.

Case 1. $k = 1, 3$ or 7. Then λ can be replaced by any $\lambda' = \lambda + i\mu$. Hence s can be replaced by any $s' = s - ir$. Choosing i so that $0 \leq s' < r$ the construction simplifies $H_k(M)$.

Case 2. k odd, $\neq 1, 3, 7$. Then λ can be replaced only by classes of the form $\lambda + 2i\mu$. Hence the best we can do is to choose $2i$ so that $-r < s' \leq r$. Thus the construction is successful unless $s \equiv r$ (mod $2r$). In particular it is always successful unless the self linking numbers

$$L(\alpha ,\alpha) = \text{residue class of } \pm s/r \text{ mod } 1 \in \mathbb{Q}/\mathbb{Z}$$

is zero.

If $L(\alpha ,\alpha) = 0$ for all $\alpha \in H_k(M)$ then the identity $L(\alpha + \beta,\alpha + \beta) = L(\alpha ,\alpha) + L(\beta ,\beta) + 2L(\alpha, \beta)$ implies that $L(\alpha, \beta) = 0$ or ½ for all α, β. This is only possible if $H_k(M) = \mathbb{Z}_2 + \cdots + \mathbb{Z}_2$. Thus one can replace M to a manifold having only 2-torsion. What now?

Case 3. k even. Then λ cannot be changed at all. Do you see some reason to believe that s must be zero? I don't know any examples and don't have any ideas here.

Best regards
John

7. Letter Kervaire → Milnor dated November 22 1959.

100 Bank Street
New York 14, N.Y.

Nov. 22, 1959

Dear John:

Thanks for correcting my last letter. I believe I can answer your last question, assuming that the χ-construction (explain to me your reason for this terminology, please) is equivalent to passing from one level surface to another with just one non-degenerate critical point in-between.

Set $r = k+1$, and let V^{2r} be a manifold with boundary $\partial V^{2r} = M' - M$. (dim M = dim M' = $2k+1$.) Let $f : V \to \mathbb{R}$ be differentiable with just one non-degenerate critical point 0 of index r in the interior of V. Assume $M = f^{-1}(-1)$, $M' = f^{-1}(+1)$, $-1 \leq f(x) \leq +1$ for every $x \in V$, and $f(0) = 0$. I am only interested in the case where the element of $H_k(M)$ killed by crossing 0 is a torsion element, and since $p_k \leq p_k' \leq p_k + 1$, where p_k = rank $H_k(M; \mathbb{Q})$, p_k' = rank $H_k(M'; \mathbb{Q})$, it follows that in order to prove that the disturbing element introduced in $H_k(M')$ is of infinite order, it is sufficient to prove that $p_k' \neq p_k$.

The theorem of Morse, concerning $p_i' - p_i$, I was referring to, is contained in his paper: "Homology relations on regular orientable manifolds" Proc. Nat. Acad. Sciences 38 (1952), 247-258. I want to use a refinement of this theorem which runs as follows. (The following is contained in my thesis §9. Sorry I have no more reprints.) Let χ^* denote the semi-characteristic, then *modulo 2*:

$$\chi^*(\partial V^{2r}) = \chi(V^{2r}) + \varphi,$$

where φ is the rank of the cup-product matrix of $H^r(V^{2r}, \partial V^{2r}; \mathbb{Q})$. (There is a better proof of this formula in "Relative characteristic classes".)

If r is odd, φ is congruent to 0 modulo 2 because $u \cdot u = 0$ for every $u \in H^r(V^{2r}, \partial V^{2r}; \mathbb{Q})$. From the existence of the gradient field of f over V, it follows that $\chi(V) = 1$ modulo 2, and since $p_i' = p_i$ for $i < k$, one has $p_k' \neq p_k$.

If r is even, you have reduced the problem to the case where

$$H_k(M) \cong H_k(M; \mathbb{Z}_2) \cong \mathbb{Z}_2 + \cdots + \mathbb{Z}_2.$$

What I have said before is, I believe, still true, regarding p_i, p_i' as being rank $H_k(M; \mathbb{Z}_2)$, $H_k(M'; \mathbb{Z}_2)$ and replacing "of infinite order" by "non-zero", and $p_k - 1 \leq p_k' \leq p_k$.

We still have to prove that $p_k' \neq p_k$, and this is apparently sufficient. This is equivalent to proving $\rho = 0$ modulo 2, where ρ is now the rank of the cup-product matrix of $H^r(V^{2r}, \partial V^{2r}; \mathbb{Z}_2)$.

Conjecture: If M^{2k+1} is a π-manifold, then V is also a π-manifold ???? If this is true, the statement $\rho = 0$ mod 2 follows from §5 of my thesis, page 239.

If the conjecture is wrong, I don't know how to prove $\rho = 0$ mod 2.

Best regards.

8. Letter Milnor → Kervaire dated December 15 1959.

Berkeley 4, California
December 15, 1959

Dear Michel,

Your argument sounds good. One thing bothers me: does it only apply to a compact manifold without boundary? It is known that every compact π-manifold without boundary represents the trivial cobordism class. Hence a series of χ-constructions can be used to reduce it to a sphere.

The conjecture which you mention is correct and will be included in the paper, which I am still trying to get into shape. If $2p+1 \leq n$ and if the imbedding $f: S^p \times D^{n-p} \to M^n$ is correctly chosen within its homotopy class, where M^n is a π-manifold without boundary, then the construction yields a parallelizable $(n+1)$-dimensional manifold with boundaries M^n and $\chi(M^n, f)$.

I am afraid that I have no good reason for the terminology "χ-construction" (χ can be taken as an abbreviation for Chirurgie.) It seemed to be convenient for such notation as $\chi(V, f)$ (= the manifold obtained from V by the χ-construction using the imbedding f) or "χ-equivalent". It didn't occur to me that it conflicted with the notation for the characteristic or semi-characteristic.

What do you have in mind as application for the argument in your letter? Is it possible to prove that the groups $\Theta^{2r-2}(\partial\pi)$ (which I defined in "Differentiable manifolds which are homotopy spheres.") are zero? Is it possible to prove that there exist a homotopy sphere M^{8k+1} which is not a π-manifold, assuming that the appropriated J-homomorphism is zero?

Sincerely
John

9. Letter Kervaire → Milnor dated December 26 1959.

Dec. 26, 1959

Dear John:

The argument in my last letter is I think OK for a manifold with boundary provided the boundary is a homotopy sphere. Let M_1^{2k+1} be the manifold with boundary Σ, and M_2 the mirror image. Perform the constructions on $M = M_1 \cup M_2$ leaving M_2 alone. If Σ is a homotopy sphere, there will be no "interaction" between the homology of M_1 and the homology of M_2 in $H_*(M)$.

I did have in mind that $Jc_{8s} = 0$ should imply existence of a $(8s+1)$-homotopy sphere which is not a π-manifold. It seems OK now, as well as $\Theta^{2r}(\partial\pi) = 0$.

There is a series of more or less conjectural statements as follows:

Case I. $\pi_{n+2k}(S^n)$ stable, S^k parallelizable.

For every $\alpha \in \pi_{n+2k}(S^n)$ take $f \in \alpha$ such that $f^{-1}(a) = M^{2k}$ is $(k-1)$-connected. Let $A_1, \ldots, A_q$, $B_1, \ldots, B_q$ be a "canonical" basis of $H_k(M^{2k}; \mathbb{Z})$. I.e. $A_i{\cdot}A_j = B_i{\cdot}B_j = 0$, $A_i{\cdot}B_j = \delta_{ij}$. Represent A_i, A_j by imbedded spheres $\alpha_i : S^k \to M^{2k}$, $\beta_j : S^k \to M^{2k}$. Take fields of normal k-frames τ_i, σ_j over $\alpha_i(S^k)$, $\beta_j(S^k)$ respectively. *Define* λ_i (resp. μ_j) to be the Steenrod-Hopf invariant of $\{\alpha_i(S^k) : \tau_i \times F_n\}$ (resp. $\{\beta_j(S^k) : \sigma_j \times F_n\}$), where F_n is the field of normal n-frames over M^{2k} in S^{n+2k}.

Since the sequence

$$\pi_k(SO(k)) \xrightarrow{i_*^2} \pi_k(SO(k+2)) \to \mathbb{Z}_2 \to 0$$

is exact if S^k is parallelizable, it follows that λ_i, μ_j are well defined modulo 2.

Define

$$\pi_{n+2k}(S^n) \xrightarrow{\gamma} \mathbb{Z}_2$$

by $\gamma(\alpha) = \sum_i \lambda_i{\cdot}\mu_i$. For $k = 1$, Pontryagin shows that this is indeed well defined, and a homomorphism.

Lemma. *If* $\gamma(\alpha) = 0$, *there exists* $f \in \alpha$ *such that* $f^{-1}(a)$ = *homotopy sphere for some* $a \in S^n$.

Corollary. *There exists an exact sequence*

$$0 \to \Theta^{2k} \to \pi_{n+2k}(S^n) \to \mathbb{Z}_2 \to 0$$

for $k = 1, 3,$ *and* 7. (*n large*.)

Corollary. $\Theta^6 = 0$. (I don't have Yamanoshita on hand to see what this means for Θ^{14}.)

Case II. $\pi_{n+2k}(S^n)$ stable, k odd, S^k not parallelizable.

For every $\alpha \in \pi_{n+2k}(S^n)$ pick $f \in \alpha$ with $f^{-1}(a) = M^{2k}$ $(k-1)$-connected. Use your function $\varphi : H_k(M^{2k}; \mathbb{Z}_2) \to \mathbb{Z}_2$ to define $h = \sum_i \varphi(A_i){\cdot}\varphi(B_i)$, where $A_1, \ldots, A_q, B_1, \ldots, B_q$ is a canonical basis. This expression does not depend on the choice of the basis (provided it is a canonical basis). Is this

9. Letter Kervaire → Milnor dated December 26 1959 (continued).

the Arf invariant?

Do you know whether or not h is a homotopy invariant $\pi_{n+2k}(S^n) \to \mathbb{Z}_2$? Also if γ (case I) is homotopy invariant, it is certainly surjective (it takes value 1 on the composition of a Hopf map with itself). Do you know whether h is surjective? If h is homotopy invariant, then

$$0 \to \Theta^{2k}(\pi) \to \pi_{n+2k}(S^n) \to \mathbb{Z}_2 \to \Theta^{2k+1}(\partial\pi)$$

is exact.

Case III. $\Theta^{2k+1}(\pi)/\Theta^{2k+1}(\partial\pi) \cong \pi_{n+2k+1}(S^n)/J\pi_{2k+1}(SO(n))$.

Case IV. $\Theta^{4r} \cong \pi_{n+4r}(S^n)/J\pi_{4r}(SO(n))$.

Best regards,

10. Letter Kervaire → Milnor dated January 2 1960.

Jan. 2, 1960

Dear John:

Enclosed are some more details about the proof of the statements in my last letter in Case I. At the end I have listed the χ-theorems which are needed.

As far as Case II is concerned, one should be able to prove that there exists an exact sequence

$$0 \to \Theta^{2k}(\pi) \to \pi_{2k} \to \mathbb{Z}_2 \to \Theta^{2k-1}(\pi) \to \pi_{2k-1}/\mathrm{Im}\,J \to 0$$

for k odd and S^k not parellelizable.

The homomorphism $\mathbb{Z}_2 \to \Theta^{2k-1}(\pi)$ being defined as follows: Let U, U' be two copies of the tubular neighborhood of the diagonal in $S^k \times S^k$. Let X be obtained from the disjoint union $U\ \ U'$ [sic] by identification of a coordinate neighborhood $\mathbb{R}^k_1 \times \mathbb{R}^k_2$ with its copy $\mathbb{R}_1' \times \mathbb{R}_2'$ under $\mathbb{R}^k_1 \times \mathbb{R}^k_2 \leftrightarrow \mathbb{R}_2' \times \mathbb{R}_1'$. The boundary of X is a homotopy sphere, image of $1 \in \mathbb{Z}_2$ under $\mathbb{Z}_2 \to \Theta^{2k-1}(\pi)$.

In may opinion, the main problem now would be to decide for which values of k the boundary of X represents the zero J-equivalence class.

Best wishes for the new year.

[The enclosure follows.]

Let V be a finite dimensional vector space over $\mathbb{Z}_2$ with a commutative bilinear product $V \times V \to \mathbb{Z}_2$ satisfying

(1) $x{\cdot}x = 0$ for every $x \in V$,
(2) $a{\cdot}x = 0$ for every $x \in V$ implies $a = 0$.

It follows that dim V is even; dim $V = 2q$. A basis $a_1, \ldots, a_q, b_1, \ldots, b_q$ of V is said to be *canonical* if $a_i{\cdot}a_j = b_i{\cdot}b_j = 0$, $a_i{\cdot}b_j = \delta_{ij}$. $(1 \le i, j \le q.)$ There exists at least one canonical basis.

Let $\varphi : V \to \mathbb{Z}_2$ be a function satisfying

$$\varphi(x + y) = \varphi(x) + \varphi(y) + x \cdot y.$$

LEMMA 1. *Let* $a_1, \ldots, a_q, b_1, \ldots, b_q$ *and* $a_1', \ldots, a_q', b_1', \ldots, b_q'$ *be two canonical bases of V. Then*

$$\Gamma = \textstyle\sum_1^q \varphi(a_i){\cdot}\varphi(b_i) = \sum_1^q \varphi(a_i'){\cdot}\varphi(b_i').$$

Proof. (Compare L. Pontryagin [1].) One proves that successive transformation of the basis a_i', b_j' not altering $\sum_i \varphi(a_i'){\cdot}\varphi(b_i')$ bring a_i', b_j' into a_i, b_j. Assume by induction that $a_k' = a_k$ and $b_k' = b_k$ for $r < k \le q$. Then, a_r is a linear combination of a_i', b_j' with $i, j \le r$,

$$a_r = \alpha_1 a_1' + \cdots + \alpha_r a_r' + \beta_1 b_1' + \cdots + \beta_r b_r'.$$

One of the coefficients is $\neq 0$. After possible permutation of the indices $1, \ldots, r$ and interchange of a and b, we can assume $\alpha_r = 1$. Define a new basis $u_1, \ldots, u_q, v_1, \ldots, v_q$ by

10. Letter Kervaire → Milnor dated January 2 1960 (continued).

$$u_i = a_i' + \beta_i b_r' \quad , \quad v_i = b_i' + \alpha_i b_r' \qquad \text{for } 1 \le i \le r-1$$
$$u_r = a_r \quad , \quad v_r = b_r'$$
$$u_k = a_k \quad , \quad v_k = b_k \qquad \text{for } r < k \le q.$$

The new basis is canonical, and

$$\begin{aligned}\textstyle\sum_1^q \varphi(u_i)\cdot\varphi(v_i) &= \textstyle\sum_1^{r-1}\varphi(a_i' + \beta_i b_r')\cdot\varphi(b_i' + \alpha_i b_r') + \varphi(a_r)\cdot\varphi(b_r') + \cdots \\ &= \textstyle\sum_1^q \varphi(a_i')\cdot\varphi(b_i') + A,\end{aligned}$$

where

$$A = \varphi(b_r')[\textstyle\sum_1^{r-1}(\beta_i\varphi(b_i') + \alpha_i\varphi(a_i') + \alpha_i\beta_i) + \varphi(a_r) + \varphi(a_r')]$$

The expression in brackets is zero because

$$\varphi(a_r) = \textstyle\sum_1^{r-1}(\alpha_i\varphi(a_i') + \beta_i\varphi(b_i') + \alpha_i\beta_i)) + \varphi(a_r') + \beta_t(1 + \varphi(b_r')),$$

and

$$\beta_t\varphi(b_r')(1 + \varphi(b_r')) = 0.$$

Claim:

$$b_r = \tau_1 u_1 + \cdots + \tau_r u_r + \sigma_1 v_1 + \cdots + \sigma_{r-1} v_{r-1} + v_r.$$

Indeed, the coefficient of v_r in the expansion of b_r is given by $b_r\cdot u_r = b_r\cdot a_r = 1$.

Interchanging u and v and applying the same procedure leads to a new canonical basis $u_1', \ldots, u_q', v_1', \ldots, v_q'$ such that

$$u_k' = a_k \qquad \text{and } v_k' = b_k \qquad \text{for } r \le k \le q,$$

and

$$\textstyle\sum_1^q \varphi(u_i')\cdot\varphi(v_i') = \sum_1^q \varphi(a_i')\cdot\varphi(b_i'). \quad \text{Q.E.D.}$$

Let π_{2k} be the stable homotopy group $\pi_{n+2k}(S^n)$, $2k+2 \le n$, and Θ^{2k} as in J. Milnor [2].

THEOREM 1. *For* $k = 1, 3, 7$ *there is an exact sequence*

$$0 \to \Theta^{2k} \to \pi_{2k} \xrightarrow{\Gamma} \mathbb{Z}_2 \to 0.$$

By [2], Corollary 6.8, $\Theta^{2k}(\pi)/\Theta^{2k}(\partial\pi)$ is naturally isomorphic to a subgroup of $\pi_{n+2k}(S^n)/J\pi_{2k}(SO(n))$. For $k = 1, 3$, or 7, $\Theta^{2k} = \Theta^{2k}(\pi)$ and $\Theta^{2k}(\partial\pi) = 0$ by [2], Theorem 5.13. Since $\pi_{2k}(SO(n)) = 0$ for $k = 1, 3$, or 7, we have exactness of

$$0 \to \Theta^{2k} \to \pi_{2k}.$$

We proceed to the definition of the homomorphism

$$\Gamma : \pi_{2k} \to \mathbb{Z}_2.$$

10. Letter Kervaire → Milnor dated January 2 1960 (continued).

Let $\alpha \in \pi_{n+2k}(S^n)$. Let $f : S^{n+2k} \to S^n$ be a C^∞-map representing α and $M^{2k} = f^{-1}$(regular value), F_n a field of normal n-frames over M^{2k} such that α is associated with $(M^{2k}; F_n)$.

Applying Theorem A, we obtain a $(k-1)$-connected π-manifold of dimension $2k$ imbedded in $\mathbb{R}^{n+2k}$ and a field of normal n-frames over it associated with the same α.

I.e. we may assume M^{2k} to be $(k-1)$-connected. Then $H_k(M^{2k}; \mathbb{Z})$ is a finitely generated free abelian group. Set $V = H_k(M^{2k}; \mathbb{Z}_2)$ and define $x{\cdot}y$ to be the intersection coefficient of $x, y \in V$. The axioms (1) and (2) of page 01 [= the beginning of this enclosure.] are satisfied.

Define a function $\varphi : V \to \mathbb{Z}_2$ as follows: For every $x \in V$ let $X \in H_k(M^{2k}; \mathbb{Z})$ be such that $X \equiv x$ modulo 2, and let $J_x : S^k \to M^{2k}$ be a completely regular immersion representing X. The normal bundle (in M^{2k}) of J_x is trivial (S^k is parellelizable). Let τ be a field of normal k-frames. The imbedding of M^{2k} in $\mathbb{R}^{n+2k}$ induces an immersion of S^k into $\mathbb{R}^{n+2k}$ with a field $\tau \times F_n$ of normal $(k{+}n)$-frames. Let ω_x be the "degree" of the induced map $S^k \to V_{n+2k,\, n+k}$. Define

$$\varphi(x) = \omega_x + S(J_x) + 1$$

where $S(J_x)$ is the self-intersection coefficient of the immersion $J_x : S^k \to M^{2k}$. To be proved:

(a) $\varphi(x)$ *does not depend on the choice of* τ (*under fixed* X *and* J_x);

(b) $\varphi(x)$ *does not depend on* J_x (*under fixed* X).

Clearly then, $\varphi(x)$ does not depend on the choice of X.

It is easily seen that if $J_x, J_y : S^k \to M^{2k}$ are immersions representing x an y respectively, there exists an immersion $J_{x+y} : S^k \to M^{2k}$ such that

$$\omega_{x+y} = \omega_x + \omega_y + 1,$$

and

$$S(J_{x+y}) = S(J_x) + S(J_y) + x{\cdot}y.$$

It follows that φ satisfies

$$\varphi(x{+}y) = \varphi(x) + \varphi(y) + x{\cdot}y.$$

Proof of (a). Let $X \in H_k(M^{2k}; \mathbb{Z})$ and $J_x : S^k \to M^{2k}$ representing X be fixed. Let τ, τ' be two fields of normal k-frames over $J_x(S^k)$ in M^{2k}. There exists a map $\delta: S^k \to SO(k)$ such that $\tau'(u) = \delta(u){\cdot}\tau(u)$ for every $u \in S^k$. If $\delta \in \pi_k(SO(k))$ also denotes the homotopy class of δ, and

$$i_*^n : \pi_k(SO(k)) \to \pi_k(SO(n{+}k))$$

is induced by the natural inclusion, then

$$\omega(\tau') = \omega(\tau) + j_* i_*^n \delta,$$

where $j_* : \pi_k(SO(n{+}k)) \to \pi_k(V_{n+2k,\, n+k})$ is natural.

If S^k is parallelizable, $i_*^n\delta$ is divisible by 2. Therefore $\omega(\tau') = \omega(\tau)$.

10. Letter Kervaire → Milnor dated January 2 1960 (continued).

Proof of (b). Let $T_k(M^{2k})$ be the space of the bundle of tangent k-frames on M^{2k}. The imbedding $f : M^{2k} \to \mathbb{R}^{n+2k}$ induces a map $f^\bullet : T_k(M^{2k}) \to V_{n+2k,\, n+k}$ given by $\tau \to f^\bullet(\tau) \times F_n$. We have a diagram

$$\begin{array}{ccccc} \pi_k(V_{2k,\,k}) & \xrightarrow{i_*} & \pi_k(T_k(M^{2k})) & \xrightarrow{p_*} & \pi_k(M^{2k}) \\ & & \downarrow f^\bullet_* & & \\ & & \pi_k(V_{n+2k,\,n+k}). & & \end{array}$$

Let s_k be a fixed field of tangent k-frames over S^k. With every immersion $j : S^k \to M^{2k}$ is associated a lifting $l_j : S^k \to T_k(M^{2k})$ given by s_k and j.

Let $j_0,\ j_1 : S^k \to M^{2k}$ be respectively a trivial immersion and a Whitney immersion (with precisely one self-intersection point). Define $\tau(j) = l_j - l_{j_0}$. If j is obtained as a sum of j' and j'', then $\tau(j) = \tau(j') + \tau(j'')$.

One has $f^\bullet_*(\tau(j)) = \omega_j + 1$.

Let j' and j'' be homotopic (as maps), then $\tau(j') - \tau(j'')$ is the kernel of p_*. Since Im i_* is generated by $\tau(j_1)$, it follows

$$\tau(j') = \tau(j'') + a{\cdot}\tau(j_1) = \tau(j'' + a{\cdot}j_1).$$

By M. Hirsch, this means that j' is regularly homotopic to $j'' + a{\cdot}j_1$. Thus $S(j') = S(j'' + a{\cdot}j_1) = S(j'') + a$.

Applying $f^\bullet_*$ to the equation $\tau(j') = \tau(j'') + a{\cdot}\tau(j_1)$ and using $f^\bullet_*(\tau(j_1)) = 1$, we get

$$\omega_{j'} + 1 + S(j') = \omega_{j''} + 1 + S(j'') \quad \text{modulo } 2.$$

Q.E.D.

Since Γ is well defined for a pair $(M^{2k}; F_n)$, where M^{2k}; is the disjoint union of $(k–1)$-connected closed manifolds, and clearly additive with respect to the disjoint union of manifolds in $\mathbb{R}^{n+2k}$ with fields of normal n-frames, the proof of the homotopy invariance of Γ amounts to proving that $\Gamma(M^{2k}; F_n) = 0$ if $(M^{2k}; F_n)$ is the restriction over the boundary of some $(W^{2k+1}; F_n)$.

There exists a canonical basis of $H_k(M^{2k}; \mathbb{Z})$ such that $A_1, \ldots, A_q$ is a basis of $H_k(M^{2k}) \to H_k(W^{2k+1})$.

By theorem χ_2, we can make W to be $(k–1)$-connected without changing the filed F_n on the boundary. It follows that $J_X : S^k \to M^{2k}$, immersion representing $X \in [A_1, \ldots, A_q]$ is homotopic to zero in W^{2k+1}. Let A be anyone of the classes $A_1, \ldots, A_q$, and $J_X : S^k \to M^{2k}$ an *imbedding* representing A. (Compare J. Milnor [2], Theorem 5.9.) Let τ be a field of normal k-frames over $J(S^k)$. Since $\varphi(a) = \omega_a + 1$ is a homotopy invariant of the sphere map associated with $J(S^k)$ and $\tau \times F_n$, and since F_n is extended all over W, it is sufficient to show that the map $M^{2k} \to S^k$ associated with $J(S^k)$ and τ can be extended to a map $W^{2k+1} \to S^k$. The only obstruction to such an extension lies in
$H^{k+1}(W, M; \mathbb{Z})$. The Poincaré dual in $H_k(W; \mathbb{Z})$ is the image of A under $H_k(M^{2k}; \mathbb{Z}) \to H_k(W^{2k+1}; \mathbb{Z})$. It follows that the obstruction is zero. Q.E.D.

If $\alpha, \beta \in \pi_k$ and $h(\alpha)$, $h(\beta)$ is the Steenrod-Hopf invariant of α, β respectively. Then $\Gamma(\alpha \circ \beta) = h(\alpha){\cdot}\, h(\beta)$. Therefore Γ [Missing in the original manuscript.] is surjective.

10. Letter Kervaire → Milnor dated January 2 1960 (continued).

Let $\alpha \in \pi_{2k}$ be an element in Ker Γ. Represent α by a manifold M^{2k} imbedded in $\mathbb{R}^{n+2k}$ with a filed of normal n-frames F_n. We can assume that M^{2k} is (k–1)-connected. Since $\Gamma(M^{2k}; F_n) = 0$, there exists a canonical basis $A_1, \ldots, A_q, B_1, \ldots, B_q$ of $H_k(M^{2k}; \mathbb{Z})$ such that $\varphi(A_1) = \varphi(A_2) = \cdots \varphi(B_q) = 0$. By Theorem χ_3, $(M^{2k}; F_n)$ is homotopic to $(\Sigma^{2k}; G_n)$ where Σ^{2k} is a homotopy sphere.

Theorem χ_1 : *Let M^d be a closed differentiable manifold imbedded in $\mathbb{R}^{d+n}$, where n is to be large. Let F_n be a filed of normal n-frames over M^d. There exists $M'^{\,d}$ in $\mathbb{R}^{d+n}$ with a field F'_n of normal n-frames such that $M'^{\,d}$ is* $[(d–1)/2]$*-connected and $(M^d; F_n)$ is homotopic to $(M'^{\,d}; F'_n)$.*

Theorem χ_2 : *If $(W^{d+1}; F_n)$ is a homotopy between $(M'^{d}; F'_n)$ and $(M''^{d}; F''_n)$, i.e. $\partial W = M'' – M'$ and $F'_n = F_n | M'$, $F''_n = F_n | M''$ and if M', M'' are* $[(d–1)/2]$*-connected, then there exists a homotopy $(W^{d+1}; F_n)$ between $(M'^{\,d}; F'_n)$ and $(M''^{\,d}; F''_n)$* such that *W^{d+1} is* $[(d–1)/2]$*-connected.*

Theorem χ_3 : *Given $(M^{2k}; F_n)$ where M^{2k} is* (k–1)*-connected.* Then *$(M^{2k}; F_n)$ is homotopic to some* $(M'; F'_n)$ where *M' is a homotopy sphere iff $\Gamma(M^{2k}; F_n) = 0$. If S^k is parallelizable Γ is defined in the text* (page 03) [= page 13, bottom.]. *If S^k is not parallelizable Γ is as in your letter of Nov. 19.*

[1] L. Pontryagin, *Smooth manifolds and their applications in homotopy theory.* Translations A.M.S. Vol. 11, Series 2, p, *101*.

[2] Differentiable manifolds which are homotopy spheres.

[3] M. Hirsch, Transactions paper. (Probably Hirsch's theorem is not really needed here.)

N.B. To the proof of homotopy invariance of Γ. (Case I, bottom of page 07.) [= page15, bottom.] The map $M^{2k} \to S^k$ associated with $J(S^k)$ and τ can be extended to $W \backslash U \to S^k$, where U is a spherical neighborhood of some point $\in$ Int W. Thus the map associated with $J(S^k)$ and $\tau \times F_n$ is homotopic to the n-th suspension of a map $S^{2k} \to S^k$. The Steenrod-Hopf invariant of such an animal is zero.

Case II.

Definition of $\Gamma : \pi_{2k}(S^n) \to \mathbb{Z}_2$ for k odd, and S^k not parallelizable.

According to M. Hirsch [3], the map $J \to \pi_k(T_k(M^{2k}))$ copied from the definition of the Smale invariant is bijective. (M^{2k} unbounded compact manifold; $T_k(M^{2k})$, the space of the bundle of tangent k-frames over M^{2k}, and J the set of regular homotopy classes of immersions $S^k \to M^{2k}$.)

If $j \in J$, denote by $[j]$ the corresponding element in $\pi_k(T_k(M^{2k}))$. The argument on page 125 of [4] yields

$$[j] = [j'] + [j'']$$

If j is constructed as sum of j' and j''. Let j_1 be a Whitney immersion.

LEMMA 2. *Let $f : \pi_k(T_k(M^{2k})) \to \mathbb{Z}_2$ be any homomorphism such that $f[j_1] = 1$, then there is a function $\varphi : \pi_{2k}(M^{2k}) \to \mathbb{Z}_2$ defined by $\varphi(\alpha) = f[j] + S[j]$, where j is an immersion representing α.*

If M^{2k} is not parellelizable, there is $f : \pi_k(T_k(M^{2k})) \to \mathbb{Z}_2$ given by normal bundle. f is a homeomorphism. If M^{2k} is (k–1)-connected this yields a function $\varphi : H_k(M^{2k}; \mathbb{Z}_2) \to \mathbb{Z}_2$ satisfying

10. Letter Kervaire → Milnor dated January 2 1960 (continued).

$\varphi(x + y) = \varphi(x) + \varphi(y) + x{\cdot}y.$

Proof of Lemma 2. Since $p_*[j]$ = homotopy class of j, where $p_* : \pi_k(T_k(M^{2k})) \to \pi_{2k}(M^{2k})$, it follows that if j' and j'' are homotopic immersions $S^k \to M^{2k}$, then

$$[j'] - [j''] = a[j_1],$$

for some $a \in \mathbb{Z}_2$, where j_1 is a Whitney immersion. ($S(j_1) = 1$ and $p_*[j_1] = 0$.) Thus j' and $j'' + aj_1$ are regularly homotopic. Therefore $S(j') = S(j'') + a$. It follows

$$f[j'] + S(j') = f[j''] + S(j'').$$

Γ is thus well defined and additive on pairs $(M^{2k}; F_n)$, where M^{2k} is a $(k–1)$-connected unbounded manifold in $\mathbb{R}^{n+2k}$ and F_n is a field of normal n-frames over M^{2k}. To prove the homotopy invariance of Γ it is sufficient to prove that $\Gamma(M^{2k}; F_n) = 0$ if $M^{2k} = \partial W^{2k+1}$ where W^{2k+1} is a manifold in $\mathbb{R}^{n+2k+1}$ over which F_n can be extended as a field of *normal* n-frames. It is sufficient to prove $\varphi(A) = 0$ for A in the kernel of $H_k(M^{2k}; \mathbb{Z}) \to H_k(W^{2k+1}; \mathbb{Z})$. Let $j : S^k \to M^{2k}$ be an *imbedding* representing A. If the normal bundle of j were nontrivial we would get a map $f : M^{2k} \to S^k \cup e^{2k}$ (where e^{2k} is attached $[i_k, i_k]$) such that $f_* : H_{2k}(M^{2k}; \mathbb{Z}) \to H_{2k}(S^k \cup e^{2k})$ is an isomorphism.

Again, the extension of f is possible over M except possibly in some spherical neighborhood. The boundary of this neighborhood being S^{2k} we get that the top cycle of $S^k \cup e^{2k}$ is spherical. I.e. $[i_k, i_k] = 0$. This contradicts J. F. Adams if $k \neq 1, 3, 7$. (Of course the χ-construction, theorem χ_2, has to be used again to make W $(k–1)$-connected and $H^{q+1}(W, M; G) = 0$ for $k < q < 2k$.)

Theorem 2. *For k odd and* $\neq 1, 3, 7$, *there is an exact sequence*

$$0 \to \Theta^{2k}(\pi) \to \pi_{2k} \to \mathbb{Z}_2 \to \Theta^{2k-1}(\pi) \to \pi_{2k-1}/J \to 0.$$

If Σ^{2k-1} is a homotopy sphere which bounds a π-manifold V^{2k}, then theorem χ_2 yields a V^{2k} which is $(k–1)$-connected. Further χ-construction leaves us either with V^{2k} having the homotopy type of a disk, or $H_k(V^{2k}; \mathbb{Z}) \cong \mathbb{Z} + \mathbb{Z}$ with generators represented by imbeddings $j' : S^k \to V^{2k}$, $j'' : S^k \to V^{2k}$ with $S(j', j'') = 1$ and both normal bundles nontrivial. If U is a neighborhood of $j'(S^k) \cup j''(S^k)$, contractible on $j'(S^k) \cup j''(S^k)$, then $\dot{U}$ is a homotopy sphere which is J-equivalent to Σ^{2k-1}. This proves exactness at $\Theta^{2k-1}(\pi)$.

11. Letter Milnor → Kervaire dated March 15 1960.

Berkeley 4, California
March 15, 1960

Dear Michel,

I am still trying to study your letter; but keep getting sidetracked on other things.

There are two new developments since I wrote last. C. T. C. Wall (The Loft Malting Lane, Cambridge England) has written to me indicating that he is also working on these questions, and that he can prove the assertion $\Theta^{2k}(\pi) = 0$, as well as the assertion $\Theta^6 = 0$. He included some details in his letter, but not enough for me to follow. I told him that you had also proved these assertions.

A. H. Wallace (Indiana University, Bloomington) sent em a copy of a manuscript which should appear in the Canadian Journal in April. This overlaps a great deal with the manuscript which I sent you a few weeks ago. (You probably have received it by now.) However there is no overlap with what you have done. Wallace uses the term "spherical modification". This does seem better to me than "surgery" or "χ-construction". What do you think? Wallace was led to the concept via a forthcoming paper by Aeppli, dealing with modifications of algebraic varieties. In any case I plan to publish my manuscript, more or less as it stands, in the proceedings of the conference on differential geometry which was recently held in Tucson.

I will try to write a more mathematical letter later.

Sincerely
John

12. Letter Milnor → Kervaire dated March 20 1960.

Berkeley 4, California
March 20, 1960

Dear Michel,

The manuscript which you sent me is very nice. I had tried to prove the existence of a manifold without differentiable structure for a ling time, without success.

Smale has announced the same result (in dimensions 8, 12, …) by a completely different argument. He claims to have proved that, for $n \neq 3, 4$, every C^∞ n-manifold which is a homotopy sphere is $\begin{cases} \text{homeomorphic to } S^n \text{ for all } n \neq 3, 4 \\ \text{combinatorially equivalent to } S^n \text{ for } n \text{ even.} \end{cases}$

Using my example of a homotopy 7-sphere which bounds a 3-connected 8-manifold with index 8, it follows that there exists an 8-manifold without differentiable structure.

However your example is simpler, and is also sharper in a way. The 10-manifold can be triangulated so that the star of each vertex is a combinatorial cell, whereas this is not known in Smale's examples.

Wall has sent me a mimeographed note proving that $\Theta^{2m}(\partial\pi) = 0$.

Sincerely
John

13. Letter Milnor → Kervaire dated June 19 1961.

Rorschach June 29, 1961

Dear Michel,

Unfortunately I haven't gotten too far with our manuscript. The following absurd difficulty came up. It seems to me that the relation of f-cobordism as defined is not symmetric. At least for 1-dimensional manifolds there is a definite asymmetry. In higher dimensions I don't really know what happens. In any case some patchwork seems to be needed. There are many possibilities, none of which really appeals to me. (E.g. using $(n+2)$-frames or dropping the concept of ∞-frames in place of f-cobordism completely.) Perhaps you will have a good idea by the time I get to Berkeley. (Circa July 16.)

I have been trying to work on the conjecture that the various exact sequences:

$$\begin{tikzcd}
P_{n+1} \arrow[rr] \arrow[dr] & & \Theta_n \arrow[rr] \arrow[dr] & & \pi_{n-1}(SO) \\
& F\Theta_n \arrow[ur] \arrow[dr] & & A_n \arrow[ur] \arrow[dr] & \\
\pi_n(SO) \arrow[rr] \arrow[ur] & & \pi_n \arrow[rr] \arrow[ur] & & P_n
\end{tikzcd}$$

are isomorphic to those of a triple

$$SO_N \subset \text{Combinatorial automorphism group} \subset \text{Homotopy equivalence of } S^{n-1}.$$

The following seems to be a promising candidate for the middle object. Let Comb_N be the c.s.s. Group where k-simplexes are piecewise linear maps

$$(\text{standard } k\text{-simplex}) \times (\text{neighborhood of } 0 \text{ in } \mathbb{R}^N) \to \mathbb{R}^N$$

such that, for each fixed coordinate in the simplex, one obtains a PL-imbedding

$$(\text{neighborhood of } 0, 0) \to (\mathbb{R}^N, 0).$$

Two such are to be identified if they coincide over a smaller neighborhood.

Then given any combinatorial n-manifold one can define a c.s.s. “tangent bundle” with Comb_n as structure group.

With best regards
Jack