BERNOULLI NUMBERS, HOMOTOPY GROUPS, AND A THEOREM OF ROHLIN

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A homomorphism $J\colon \pi_{k-1}(\mathbf{SO}_m) \to \pi_{m+k-1}(S^m)$ from the homotopy groups of rotation groups to the homotopy groups of spheres has been defined by H. Hopf and G. W. Whitehead[16]. This homomorphism plays an important role in the study of differentiable manifolds. We will study its relation to one particular problem: the question of possible Pontrjagin numbers of an 'almost parallelizable' manifold.

Definition. A connected differentiable manifold M^k with base point x_0 is *almost parallelizable* if $M^k - x_0$ is parallelizable. If M^k is imbedded in a high-dimensional Euclidean space R^{m+k} $(m \geqslant k+1)$ then this is equivalent to the condition that the normal bundle ν, restricted to $M^k - x_0$, be trivial (compare the argument given by Whitehead[17], or Kervaire ([5], § 8)).

The following theorem was proved by Rohlin in 1952 (see Rohlin[11,12], Kervaire[6]).

Theorem (Rohlin). Let M^4 be a compact oriented differentiable 4-manifold with Stiefel–Whitney class w_2 equal to zero. Then the Pontrjagin number $p_1[M^4]$ is divisible by 48.

Rohlin's proof may be sketched as follows. It may be assumed that M^4 is a connected manifold imbedded in R^{m+4}, $m \geqslant 5$.

Step 1. It is shown that M^4 is almost parallelizable.

Let f be a cross-section of the normal $\mathbf{SO}_m$-bundle ν restricted to $M^4 - x_0$. The obstruction to extending f is an element

$$\mathfrak{o}(\nu, f) \in H^4(M^4; \pi_3(\mathbf{SO}_m)) \approx \pi_3(\mathbf{SO}_m).$$

Step 2. It is shown that $J\mathfrak{o}(\nu, f) = 0$.

Since J carries the infinite cyclic group $\pi_3(\mathbf{SO}_m)$ onto the cyclic group $\pi_{m+3}(S^m)$ of order 24, this implies that $\mathfrak{o}(\nu, f)$ is divisible by 24. Now identify the group $\pi_3(\mathbf{SO}_m)$ with the integers.

Step 3. It is shown that the Pontrjagin class $p_1(\nu)$ is equal to $\pm 2\mathfrak{o}(\nu, f)$.

Since by Whitney duality $p_1(\nu) = -p_1$ (tangent bundle), it follows that $p_1[M^4]$ is divisible by 48.

The first step in this argument does not generalize to higher dimensions. However Step 2, the assertion that $J\mathfrak{o}(\nu, f) = 0$, generalizes immediately. In fact we have:

Lemma 1. *Let $\alpha \in \pi_{k-1}(\mathbf{SO}_m)$; then $J\alpha = 0$ if and only if there exists an almost parallelizable manifold $M^k \subset R^{m+k}$ and a cross-section f of the induced normal $\mathbf{SO}_m$-bundle ν over $M^k - x_0$ such that $\alpha = \mathfrak{o}(\nu, f)$.*

Step 3 can be replaced by the following. Identify the group $\pi_{4n-1}(\mathbf{SO}_m)$, $m > 4n$, with the integers (compare Bott[2]). *Define a_n to be equal to* 2 for n odd and 1 for n even.

Lemma 2. *Let ξ be a stable $\mathbf{SO}_m$-bundle over a complex K $(\dim K < m)$, and let f be a cross-section of ξ restricted to the skeleton $K^{(4n-1)}$. Then the obstruction class $\mathfrak{o}(\xi, f) \in H^{4n}(K; \pi_{4n-1}(\mathbf{SO}_m))$ is related to the Pontrjagin class $p_n(\xi)$ by the identity $p_n(\xi) = \pm a_n . (2n-1)!\, \mathfrak{o}(\xi, f)$.*

Combining Lemmas 1 and 2, we obtain the following theorems.

Define j_n as the order of the finite cyclic group $J\pi_{4n-1}(\mathbf{SO}_m)$ in the stable range $m > 4n$.

Theorem 1. *The Pontrjagin number $p_n[M^{4n}]$ of an almost parallelizable $4n$-manifold is divisible by $j_n a_n (2n-1)!$.*

(For $n = 1$, this gives Rohlin's assertion, since $j_1 = 24$, $a_1 = 2$.)

Proof. This follows since $\mathfrak{o}(\nu, f)$ must be divisible by j_n.

Conversely:

Theorem 2. *There exists an almost parallelizable manifold M_0^{4n} with*

$$p_n[M_0^{4n}] = j_n a_n . (2n-1)!.$$

The proof is clear.

Proof of Lemma 1. Given an imbedding $i\colon V^{k-1} \to R^{m+k-1}$ of a compact differentiable manifold V^{k-1} into Euclidean space, and given a cross-section f of the normal $\mathbf{SO}_m$-bundle over V^{k-1}, a well-known procedure due to Thom associates with i and f a sphere mapping $\phi\colon S^{m+k-1} \to S^m$ (compare Kervaire[5], p. 223).

The map ϕ is homotopic to zero if and only if there exists a bounded manifold Q^k with boundary V^{k-1} imbedded in R^{m+k} on one side of R^{m+k-1} such that:

(i) the restriction to V^{k-1} of the imbedding of Q^k is the given imbedding of V^{k-1} in R^{m+k-1};

(ii) Q^k meets R^{m+k-1} orthogonally so that the restriction to V^{k-1} of the normal bundle of Q^k is just the normal bundle of V^{k-1} in R^{m+k-1}; and

(iii) the cross-section f can be extended throughout Q^k as a cross-section f' of the normal $\mathbf{SO}_m$-bundle.

These facts follow from Thom [15], ch. I, § 2 and Lemmas IV, 5, IV.5'.

To obtain Lemma 1 above, take $V^{k-1} = S^{k-1}$ and take $i(S^{k-1})$ to be the unit sphere in $R^k \subset R^{m+k-1}$. Since the normal m-plane at each point of $i(S^{k-1})$ in R^{m+k-1} admits a natural basis (consisting of the radius vector followed by the vectors of a basis for R^{m+k-1}/R^k), the cross-section f

provides a mapping $a: S^{k-1} \to \mathbf{SO}_m$. Let $\alpha \in \pi_{k-1}(\mathbf{SO}_m)$ be its homotopy class. It is easily seen (compare Kervaire[7], §1.8) that the map $\phi: S^{m+k-1} \to S^m$ associated with i and f represents $J\alpha$ up to sign.

If $J\alpha = 0$ then there exists a bounded manifold $Q^k \subset R^{m+k}$ satisfying conditions (i), (ii) and (iii). Let $M^k \subset R^{m+k}$ denote the unbounded manifold obtained from Q^k by adjoining a k-dimensional hemisphere, which lies on the other side of R^{m+k-1} and has the same boundary $i(S^{k-1})$. Since the normal bundle ν restricted to Q^k has a cross-section f', it follows that M^k is almost parallelizable. Clearly the obstruction class $\mathfrak{o}(\nu, f')$ is equal to α.

Conversely, let M^k be a manifold imbedded in R^{m+k} and let f be a cross-section of the normal bundle ν restricted to $M^k - x_0$. After modifying this imbedding by a diffeomorphism of R^{m+k} we may assume that some neighborhood of x_0 in M^k is a hemisphere lying on one side of the hyperplane R^{m+k-1}, and that the rest of M^k lies on the other side. Removing this neighborhood we obtain a bounded manifold $Q^k \subset R^{m+k}$ just as above, having the unit sphere $S^{k-1} \subset R^k \subset R^{m+k-1}$ as boundary. The cross-section f restricted to S^{k-1} gives rise to a map $a: S^{k-1} \to \mathbf{SO}_m$ which represents the homotopy class $\mathfrak{o}(\nu, f)$. The argument above shows that $J\mathfrak{o}(\nu, f) = 0$; which completes the proof of Lemma 1.

Remark. Lemma 1 could also be proved using the interpretation of J given in Milnor[9].

Proof of Lemma 2. (Compare Kervaire[8].) The $\mathbf{SO}_m$-bundle ξ induces a $\mathbf{U}_m$-bundle ξ' and hence a $\mathbf{U}_m/\mathbf{U}_{2n-1}$-bundle ξ''. Similarly, the partial cross-section f induces partial cross-sections f' and f''. By definition the obstruction class $\mathfrak{o}(\xi'', f'')$ is equal to the Chern class $c_{2n}(\xi')$ and hence to the Pontrjagin class $\pm p_n(\xi)$. Therefore $p_n(\xi)$ equals $\pm q_* h_* \mathfrak{o}(\xi, f)$, where

$$h: \pi_{4n-1}(\mathbf{SO}_m) \to \pi_{4n-1}(\mathbf{U}_m) \quad \text{and} \quad q: \pi_{4n-1}(\mathbf{U}_m) \to \pi_{4n-1}(\mathbf{U}_m/\mathbf{U}_{2n-1})$$

are the natural homomorphisms and h_*, q_* are the homomorphisms in the cohomology of K induced by the coefficient homomorphisms h, q.

Using the following computations of Bott[2]:

$$\pi_{4n-1}(\mathbf{U}_m) \approx \mathbf{Z}, \quad \pi_{4n-1}(\mathbf{U}_m/\mathbf{SO}_m) \approx \mathbf{Z}_{a_n}, \quad \pi_{4n-2}(\mathbf{SO}_m) = 0,$$

it follows that h carries a generator into a_n times a generator. Similarly, using the fact that

$$\pi_{4n-2}(\mathbf{U}_{2n-1}) \approx \mathbf{Z}_{(2n-1)!} \text{ (see [3]) and } \pi_{4n-2}(\mathbf{U}_m) = 0,$$

it follows that q carries a generator into $(2n-1)!$ times a generator. Therefore $p_n(\xi) = \pm a_n (2n-1)!\, \mathfrak{o}(\xi, f)$. This completes the proof of Lemma 2.

Hirzebruch's index theorem[4] states that the index $I(M^{4n})$ of any $4n$-manifold is equal to

$$2^{2n}(2^{2n-1}-1)\, B_n p_n[M^{4n}]/(2n)! + (\text{terms involving lower Pontrjagin classes}).$$

Here B_n denotes the nth Bernoulli number. For an almost parallelizable manifold the lower Pontrjagin classes are zero. Therefore

Corollary. The index $I(M_0^{4n})$ *is equal to* $2^{2n-1}(2^{2n-1}-1)\, B_n j_n a_n / n$; *and the index of any almost parallelizable* $4n$*-manifold is a multiple of this number.*

The fact that $I(M_0^{4n})$ is an integer can be used to estimate the number j_n (compare Milnor[9]). However, a sharper estimate, which includes the prime 2, can be obtained as follows, using a new generalization of Rohlin's theorem.

Borel and Hirzebruch ([1], §§23.1 and 25.4) define a rational number

$$\hat{A}[M^{4n}] = -B_n p_n[M^{4n}]/2(2n)! + (\text{terms involving } p_1, \ldots, p_{n-1});$$

and prove that the denominator of $\hat{A}[M^{4n}]$ is a power of 2.

Theorem 3. *If the Stiefel–Whitney class* w_2 *of* M^{4n} *is zero then* $\hat{A}[M^{4n}]$ *is actually an integer.*†

The proof will be given in a subsequent paper by Borel and Hirzebruch. It is based on the methods of [1], together with the assertion that the Todd genus of a generalized almost complex manifold is an integer (Milnor[10]).

Applying this theorem to the manifold M_0^{4n} of Theorem 2 it follows that $B_n j_n a_n / 4n$ is an integer. Therefore:

Theorem 4. *The order* j_n *of the stable group* $J\pi_{4n-1}(\mathbf{SO}_m)$ *is a multiple of the denominator of the rational number* $B_n a_n / 4n$.

As examples, for $n = 1, 2, 3$, the number $B_n a_n / 4n$ is equal to $1/12$, $1/240$, and $1/252$ respectively. Since $\pi_{m+7}(S^m)$ is cyclic of order 240, it follows that $j_2 = 240$. Since $\pi_{m+11}(S^m)$ is cyclic of order 504, it follows that j_3 is either 252 or 504. It may be conjectured that j_n is always equal to the denominator of $B_n/4n$.

The theorems of von Staudt[13,14] can be used to compute such denominators (compare Milnor[9]).

Lemma 3. *The denominator of* $B_n/2n$ *can be described as follows. A prime power* p^{i+1} *divides this denominator if and only if*

$$2n \equiv 0 \mod p^i(p-1).$$

† See note at the end of the paper.

Combining Lemma 3 with Theorem 4, we see that the stable homotopy groups of spheres contain elements of arbitrary finite order. In fact:

Corollary. If $2n$ is a multiple of the Euler Φ function $\Phi(r)$, then the stable group $\pi_{m+4n-1}(S^m)$ contains an element of order r.

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[*Added in proof.*] For the case n odd, Hirzebruch has since sharpened Theorem 3, showing that $\hat{A}[M^{4n}]$ is an even integer. Thus the factor a_n in Theorem 4 can be cancelled.

ON THE FOURTEENTH PROBLEM OF HILBERT

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The purpose of the present paper is to show that the answer to the 14th problem of Hilbert[1] is negative, even in the following restricted case, which may be called the original 14th problem of Hilbert:

Let G be a subgroup of the full linear group of the polynomial ring in indeterminates $x_1, \ldots, x_n$ over a field k, and let $\mathfrak{o}$ be the set of elements of $k[x_1, \ldots, x_n]$ which are invariant under G. Is $\mathfrak{o}$ finitely generated?

Our construction of a counter-example is independent of the characteristic of k, and k can be the field of complex numbers.

1. The construction of a counter-example

Let $\{a_{ij}\}$ $(i = 1, 2, 3;\ j = 1, 2, \ldots, 16)$ be algebraically independent elements over the prime field π of arbitrary characteristic, and let k be a field containing the a_{ij}. Let V be the vector space of dimension 16 over k and let V^* be the set of vectors in V which are orthogonal to the vectors $(a_{i1}, a_{i2}, \ldots, a_{i16})$ $(i = 1, 2, 3)$. (V^* is a subspace of dimension 13.)

Let $x_1, \ldots, x_{16}, t_1, \ldots, t_{16}$ be algebraically independent elements over k and let G be the set of linear transformations σ such that (i) $\sigma(t_i) = t_i$ for any i and (ii) $\sigma(x_i) = x_i + b_i t_i$ with $(b_1, \ldots, b_{16}) \in V^*$. Then:

The set $\mathfrak{o}$ of elements of $k[x_1, \ldots, x_{16}, t_1, \ldots, t_{16}]$ which are invariant under G is not finitely generated.

2. A lemma on plane curves

In order to prove the example, we need the following lemma on plane curves:

Fundamental lemma. Let $P_1, \ldots, P_{16}$ be independent generic points of the projective plane S over the prime field π. For any curve C of degree d, the sum of the multiplicities of P_i on C is less than $4d$.

Proof. Assume that there exists a curve C of degree d such that $\Sigma m_i \geqslant 4d$, where m_i is the multiplicity of P_i on C. Since the P_i are independent generic points, the P_i can be specialized to any permutation of the P_i and therefore we see that there exists a curve of degree d' such that the multiplicity of the P_i is equal to m for every i and $d' \leqslant 4m$. Therefore it is sufficient to prove the following lemma (which is equivalent to the fundamental lemma):