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Homology of Linear Groups

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To Ellen, for inspiration;
and to Gus, for distraction

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Preface

With his definition of the higher algebraic K-groups of a ring, Daniel Quillen launched a new branch of mathematics. These groups are defined as the homotopy groups of a rather complicated space $BGL(R)^+$. This space is a modification of the classifying space of the infinite general linear group $GL(R) = \bigcup_{n\geq 1} GL_n(R)$; its homology groups coincide with those of the discrete group $GL(R)$. Quillen's calculation of the (co)homology of $GL_n(\mathbb{F}_q)$ and the resulting computation of the K-groups of $\mathbb{F}_q$ were the first results in this new field.

In the intervening 25 years, some of the world's best mathematicians have devoted their energies to the study of Quillen's K-groups. From the very beginning, the importance of the unstable homology groups $H_*(GL_n(R))$ was apparent. Explicit computations have been hard to come by, though. For example, a group as simple as $SL_4(\mathbb{Z})$ has resisted the complete calculation of its homology. Not surprisingly, then, the K-groups of $\mathbb{Z}$ are still mysterious (although a great deal is known; for example, the 2-torsion in $K_*(\mathbb{Z})$ has been determined completely).

This monograph presents the current state of affairs in the study of the homology of linear groups. I have tried to trace the development of the theory chronologically, beginning with Quillen's results and proceeding to the present. This linearity is interrupted occasionally, however.

Chapter 1 is an overview of the early results of the subject. Quillen's calculation of $H^*(GL_n(\mathbb{F}_q))$ is presented along with certain conjectures about the structure of $H^*(GL_n(\Lambda), \mathbb{F}_p)$(due to Quillen and Lichtenbaum), where Λ is a $\mathbb{Z}[1/p]$-algebra. These conjectures have spurred the development of a great deal of interesting mathematics, such as the étale K-theory of W. Dwyer and E. Friedlander. Chapter 1 also includes a discussion of A. Borel's calculation of the stable cohomology of arithmetic groups (e.g., $SL(\mathbb{Z})$) and the resulting consequences for the K-theory of rings of integers in number fields. There is also a brief discussion of congruence subgroups. A thorough discussion of much of the material in Chapter 1 would require an entire book; for this reason, I have chosen simply to outline or omit proofs of several results. The major exception to this is the calculation of $H^*(GL_n(\mathbb{F}_q))$. When proofs are truncated, a reference is always provided.

Chapter 2 presents the known stability results for $H_*(GL_n)$. The basic idea is to study the extent to which the groups $H_i(GL_n)$ stabilize as n increases. We

first discuss a theorem of W. van der Kallen, a very general result in this area, and we prove a special case, due to Y. Nesterenko and A. Suslin, for so-called rings with many units. This stability result is then used to glean information about the K-groups of local rings. Chapter 2 also contains some results about the rank conjecture for infinite fields.

Chapter 3 is concerned with low-dimensional homology groups, especially the group $H_3(GL_2)$. This group has a surprising connection to the study of scissors congruence classes of polytopes in hyperbolic space, an idea studied extensively by J. Dupont and C.-H. Sah. The relation between H_3 and the so-called Bloch group (due to Suslin) is also discussed, along with generalizations and extensions due to the author, S. Yagunov, and P. Elbaz–Vincent. A connection between the Bloch group and hyperbolic 3-manifolds, discovered by W. Neumann and J. Yang, is also presented.

Rank one groups are the focus of Chapter 4. It turns out that the homology of SL_2 and PGL_2 over certain rings is computable via actions of these groups on trees. The homology of the following groups is computed completely: $SL_2(\mathbb{Z}[1/p])$ (Adem–Naffah), $SL_2(k[t])$, $SL_2(k[t,t^{-1}])$ (Knudson), $PGL_2(k[C])$ where C is a smooth affine curve of the form $X - \{p\}$, where X is projective over an infinite field k (Suslin, Knudson). The generalization of these results to groups of higher rank (due to Henn, Soulé, and the author) is also discussed, along with various applications.

The final chapter (Chapter 5) provides a comprehensive account of the Friedlander–Milnor conjecture concerning the homology of algebraic groups made discrete. All known cases—solvable Lie groups (Milnor); solvable algebraic groups, tori, normalizers of maximal tori (Jardine); the stable groups GL (Suslin, Jardine); arbitrary groups over $\overline{\mathbb{F}}_p$ (Friedlander–Mislin)—are discussed. The low-dimensional cases H_1, H_2 (Sah, Milnor) and $H_3(GL_2)$ (Sah, Knudson) are also proved.

There are three appendices. The first provides a brief overview of the homology of discrete groups. The second recalls the basic notions of classifying spaces and the definition of K-theory (topological and algebraic). The third covers the fundamentals of étale cohomology.

Also, I have included exercises at the end of each chapter. These vary from the routine to the very challenging. I hope they will prove to be useful to graduate students.

This book should be accessible to graduate students who have a good working knowledge of algebraic topology and the fundamentals of group cohomology. Indeed, many of the calculations presented here are interesting applications of the spectral sequence techniques introduced in Chapter VII of K. Brown's excellent book [**21**]. An acquaintance with the basics of algebraic groups is helpful, but if the reader prefers, the following translation may be

used: reductive group: GL_n, semi-simple group: SL_n, Borel subgroup: upper triangular matrices, torus: diagonal subgroup, unipotent group: upper triangular matrices with 1's on the diagonal, Weyl group: the symmetric group on n letters. Spectral sequences are used extensively; the construction of the spectral sequence associated to a filtered complex is reviewed in Appendix A. No extensive knowledge of algebraic geometry or étale cohomology is required; however, we do assume that the reader understands the basics of scheme theory.

I would like to thank all those with whom I have had the opportunity to discuss the material in this book. These include Dick Hain, Andrei Suslin, Eric Friedlander, Rick Jardine, Howard Garland, John Harer, Jun Yang, Marc Levine, Chuck Weibel, Mark Walker, Serge Yagunov, Philippe Elbaz–Vincent, Burt Totaro, Rob de Jeu, and Hans-Werner Henn. I owe a special debt to Professor Suslin for freely sharing his ideas and for being an inspirational influence overall. I am also grateful to a pair of anonymous referees whose comments significantly enhanced the quality of this monograph.

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Detroit, Michigan K. K.
April 2000

Chapter 1

Topological Methods

In this chapter, we discuss the topological methods available for attacking linear group homology. Most of these techniques have their foundations in the work of D. Quillen and A. Borel.

One of Quillen's major contributions to mathematics was his definition of higher algebraic K-theory and subsequent calculation of the K-theory of finite fields [**99**]. This computation was a consequence of a complete description of $H^\bullet(GL_n(k), \mathbb{F}_l)$ for k finite and $l \neq$ char k, and a partial description of $H^\bullet(GL_n(k), \mathbb{F}_p)$ for $p =$ char k. These results provided a complete description of the integral cohomology of the stable group $GL(k)$ which in turn yielded a homotopy equivalence $BGL(k)^+ \simeq F\Psi^q$, where $F\Psi^q$ is a certain space with easily computed homotopy groups. The method of proof was purely topological (with some representation theory thrown in).

Borel took a more analytical approach [**14, 15**]. He studied complexes of differential forms on symmetric spaces to compute the (stable) cohomology of arithmetic subgroups of $GL_n(\mathbb{C})$ (such as $SL_n(\mathbb{Z})$). This led to Borel's famous calculation of the ranks of the K-groups of rings of integers in number fields, a theorem which should be viewed as a higher dimensional analogue of the Dirichlet unit theorem.

These influential papers spawned a great deal of interesting mathematics including the introduction of étale K-theory by Dwyer and Friedlander and the study of the relationship between cuspidal cohomology and the cohomology of arithmetic groups. We discuss this and other related topics below. Most of the proofs in this chapter, especially those in Section 1.4, are deliberately sketchy. This chapter is included mostly for completeness.

1.1. Finite Fields

In this section we survey Quillen's remarkable paper [**99**]. Let k be a finite field with $q = p^d$ elements and let l be a prime distinct from p. We shall describe the structure of $H^\bullet(GL_n(k), \mathbb{F}_l)$.

Denote by BU the infinite complex Grassmannian (see Appendix B). This space represents complex topological K-theory; i.e., for a compact space X

$$\tilde{K}^0(X) = [X, BU]$$

where $\tilde{K}^0(X)$ is the kernel of the dimension map $K^0(X) \to K^0(\mathrm{pt}) = \mathbb{Z}$ and $[A, B]$ denotes the set of homotopy classes of maps from A to B. For a positive integer q there is an endomorphism ψ^q of $K^0(X)$ called the qth Adams operation. It is characterized by the properties that ψ^q is a ring homomorphism and if $x \in K^0(X)$ is the class of a line bundle, then $\psi^q(x) = x^q$. The map ψ^q is induced by a map $\Psi^q : BU \to BU$. This essentially follows from Yoneda's Lemma: the functor $\tilde{K}^0$ is represented by $[X, BU]$ and hence any natural transformation on $\tilde{K}^0$ is represented by a map $BU \to BU$. Of course, this argument does not quite work since BU is not compact. However, BU can be realized as an increasing union of finite CW-complexes X_m with cells only in even dimensions. In such a case, one can show that $\tilde{K}^0(BU) = \varprojlim \tilde{K}^0(X_m)$, so that ψ^q is representable by Ψ^q. Denote by $F\Psi^q$ the fiber of the map

$$1 - \Psi^q : BU \longrightarrow BU$$

which represents $\mathrm{id} - \psi^q$.

LEMMA 1.1.1. *The homotopy groups of $F\Psi^q$ are*

$$\begin{aligned} \pi_{2i}(F\Psi^q) &= 0 \\ \pi_{2i-1}(F\Psi^q) &= \mathbb{Z}/(q^i - 1). \end{aligned}$$

PROOF. Consider the long exact homotopy sequence

$$\cdots \longrightarrow \pi_k(BU) \xrightarrow{1-\Psi^q} \pi_k(BU) \longrightarrow \pi_{k-1}(F\Psi^q) \longrightarrow \cdots .$$

By Bott periodicity, $\pi_{2i-1}(BU) = 0$ and $\pi_{2i}(BU) = \mathbb{Z}$. One checks easily that $1 - \Psi^q$ acts as multiplication by $1 - q^i$ on π_{2i}. □

Quillen then computes the mod l cohomology of $F\Psi^q$ by describing a set of classes which form a basis. Fix $l \neq p$ and let $r \geq 1$ be the least integer with $q^r \equiv 1 \mod l$.

THEOREM 1.1.2. *The monomials*

$$c_r^{\alpha_1} c_{2r}^{\alpha_2} \cdots e_r^{\beta_1} e_{2r}^{\beta_2} \cdots$$

with $0 \leq \alpha_j$ and $0 \leq \beta_j \leq 1$ form a basis for $H^*(F\Psi^q, \mathbb{F}_l)$. Here $\deg(c_{jr}) = 2jr$ and $\deg(e_{jr}) = 2jr - 1$. If $l \neq 2$, or if $l = 2$ and $q \equiv 1 \mod 4$ (the typical case*), then $e_{jr}^2 = 0$ so that*

$$H^*(F\Psi^q, \mathbb{F}_l) \cong P[c_r, c_{2r}, \ldots] \otimes \bigwedge [e_r, e_{2r}, \ldots]$$

where P denotes a polynomial algebra and $\bigwedge$ denotes an exterior algebra over $\mathbb{F}_l$.

PROOF. The idea of the proof is as follows. The space $F\Psi^q$ is obtained via a cartesian square

$$\begin{array}{ccc} F\Psi^q & \xrightarrow{\gamma} & BU^I \\ {\scriptstyle\phi}\downarrow & & \downarrow{\scriptstyle\Delta} \\ BU & \xrightarrow[(\mathrm{id},\Psi^q)]{} & BU \times BU \end{array}$$

where BU^I is the path space of BU and Δ sends a path to its endpoints (this is the general construction of the space of homotopy fixed points of a map $\phi : X \to X$). Associated to this square is an Eilenberg–Moore spectral sequence (see e.g., [111])

$$E_2^{s,t} = \mathrm{Tor}_{-s}^{H^\bullet(BU\times BU)}(H^\bullet(BU), H^\bullet(BU^I)) \Longrightarrow H^\bullet(F\Psi^q).$$

Let $c_i \in H^{2i}(BU)$ denote the ith universal Chern class mod l. Then we have

$$\begin{aligned} H^\bullet(BU) &= P[c_1, c_2, \ldots] \\ H^\bullet(BU \times BU) &= P[c_1', c_2', \ldots, c_1'', c_2'', \ldots] \end{aligned}$$

where $c_i' = \mathrm{pr}_1^*(c_i)$ and $c_i'' = \mathrm{pr}_2^*(c_i)$. Set $A = H^\bullet(BU \times BU)$. Then as A-modules, one has $H^\bullet(BU^I) = A/M$ and $H^\bullet(BU) = A/J$, where M (resp. J) is the ideal with generators $c_i' - c_i''$ (resp. $q^i c_i' - c_i''$). Denote by A_j, $1 \le j \le 4$ the subrings of A with the following sets of generators

$$\begin{array}{lll} (1) & c_i' - c_i'' & \text{for } i \equiv 0 \mod r \\ (2) & c_i' - c_i'' & \text{for } i \not\equiv 0 \mod r \\ (3) & q^i c_i' - c_i'' & \text{for } i \not\equiv 0 \mod r \\ (4) & c_i' & \text{for } i \equiv 0 \mod r. \end{array}$$

By the Künneth formula

$$E_2 = \mathrm{Tor}^A(A/J, A/M) \cong \mathrm{Tor}^{A_1}(\mathbb{F}_l, \mathbb{F}_l) \otimes \mathbb{F}_l \otimes \mathbb{F}_l \otimes A_4.$$

Thus we have an isomorphism

$$E_2 \cong P[c_r, c_{2r}, \ldots] \otimes \bigwedge[e_r, e_{2r}, \ldots]$$

where $c_{jr} \in E_2^{0,2jr}$ and $e_{jr} \in E_2^{-1,2jr-1}$. Since E_2 is generated by $E_2^{0,\bullet}$ and $E_2^{-1,\bullet}$ which are killed by the differentials (this is a second quadrant spectral sequence), it follows from the multiplicative structure of the spectral sequence that $E_2 = E_\infty$.

This provides an isomorphism

$$\mathrm{gr}H^\bullet(F\Psi^q) \cong P[c_r, c_{2r}, \ldots] \otimes \bigwedge[e_r, e_{2r}, \ldots]$$

for some filtration of $H^\bullet(F\Psi^q)$.

This gives the additive structure of $H^\bullet(F\Psi^q, \mathbb{F}_l)$. The multiplicative structure is also given by this decomposition except that instead of having $e_{jr}^2 = 0$, we only know that e_{jr}^2 is in the subring generated by $c_r, c_{2r}, \ldots$.

We now compute the multiplicative structure of $H^\bullet(F\Psi^q, \mathbb{F}_l)$ while simultaneously computing $H^\bullet(BGL_n(k), \mathbb{F}_l)$. To do this, we first need a map $BGL_n(k) \to F\Psi^q$; equivalently, we must construct an element of the reduced K-theory of $BGL_n(k)$.

Given a representation $\sigma : G \to GL_N(\mathbb{C})$ of a finite group G, there is an associated vector bundle on BG given as the fiber product $EG \times_G \mathbb{C}^N$, where G acts on $\mathbb{C}^N$ via σ and EG is the universal cover of BG. Thus, we need a virtual representation of $GL_n(k)$; i.e., a formal difference of representations of $GL_n(k)$ in the complex representation ring $R(GL_n(k))$. This is achieved via *Brauer lifting*.

Let $\overline{k}$ be an algebraic closure of k and choose an embedding

$$\rho : \overline{k}^\times \longrightarrow \mathbb{C}^\times.$$

Let G be a finite group and let $\sigma : G \to GL(E)$ be a finite dimensional representation of G over $\overline{k}$. Let $\{\lambda_i(g)\}$ be the eigenvalues (counted with multiplicity) of $\sigma(g) \in GL(E)$. Define the Brauer character of E to be the complex-valued function

$$\chi_E(g) = \sum_i \rho(\lambda_i(g)).$$

THEOREM 1.1.3 ([48],Theorem 1). *The function χ_E is the character of a unique virtual complex representation $\rho E \in R(G)$.* □

Now suppose that E is a representation of G over k and set $\overline{E} = E \otimes_k \overline{k}$. One can define Adams operations on the representation ring $R(G)$ (see, for example, [8], Section 5.9). Since the set of eigenvalues $\{\lambda_i(g)\}$ is stable under the Frobenius map $x \mapsto x^q$ and since $(\psi^q\chi)(g) = \chi(g^q)$, we see that $\psi^q(\rho\overline{E}) = \rho\overline{E}$. Thus Brauer lift gives rise to a map

$$R_k(G) \longrightarrow R(G)^{\psi^q}$$

sending E to $\rho\overline{E}$. Now, the composition

$$R(G) \to K^0(BG) \to \tilde{K}^0(BG) = [BG, BU]$$

commutes with the operation ψ^q and hence we have a well-defined map

$$R(G)^{\psi^q} \longrightarrow [BG, BU]^{\psi^q}.$$

By the Atiyah Completion Theorem (see, e.g., [9], p. 48), we have $[BG, U] = 0$, and hence, by Lemma 1 of [99],

$$[BG, F\Psi^q] \cong [BG, BU]^{\psi^q}.$$

Thus, given a representation E of G over k, we have associated a homotopy class of maps

$$E^{\#} : BG \longrightarrow F\Psi^q.$$

Define mod l classes $c_{jr}(E), e_{jr}(E) \in H^\bullet(G, \mathbb{F}_l)$ by

$$c_{jr}(E) = (E^{\#})^*(c_{jr}), \qquad e_{jr}(E) = (E^{\#})^*(e_{jr})$$

where $c_{jr}, e_{jr} \in H^\bullet(F\Psi^q, \mathbb{F}_l)$ are the generators.

Now let k^n denote the standard representation of $GL_n(k)$ on k^n. Recall that $q^r \equiv 1 \mod l$.

THEOREM 1.1.4. *The classes $c_{jr}(k^n), e_{jr}(k^n)$ satisfy the following properties.*

1. $c_{jr}(k^n) = e_{jr}(k^n) = 0$ *for* $jr > n$.
2. *The monomials*

$$c_r(k^n)^{\alpha_1} \cdots c_{mr}(k^n)^{\alpha_m} e_r(k^n)^{\beta_1} \cdots e_{mr}(k^n)^{\alpha_m}$$

with $0 \le \alpha_j$ *and* $0 \le \beta_j \le 1$, $m = [n/r]$, *form a basis for the ring* $H^\bullet(GL_n(k), \mathbb{F}_l)$.

In the typical case, then, we have an algebra isomorphism

$$H^\bullet(GL_n(k), \mathbb{F}_l) \cong P[c_r(k^n), \ldots, c_{mr}(k^n)] \otimes \Lambda[e_r(k^n), \ldots, e_{mr}(k^n)].$$

PROOF. If C denotes a cyclic group of order $q^r - 1$, then C has an irreducible representation of dimension r over $\mathbb{F}_q$ via the isomorphism $C \cong \mathbb{F}_{q^r}^\times$. Write $n = mr + e$ where $e < r$. We can then embed C^m into $GL_n(k)$ as r copies of C along the diagonal together with an e-dimensional trivial representation. We claim that the restriction map

$$H^\bullet(GL_n(k), \mathbb{F}_l) \longrightarrow H^\bullet(C^m, \mathbb{F}_l)$$

is injective.

Assume first that l is odd. If G is a group, recall that the *wreath product* $\Sigma_s \wr G$ consists of elements $(\sigma; g_1, \ldots, g_s)$ with multiplication defined by

$$(\sigma'; g_1', \ldots, g_s')(\sigma; g_1, \ldots, g_s) = (\sigma'\sigma; g'_{\sigma(1)} g_1, \ldots, g'_{\sigma(s)} g_s).$$

Now factor the embedding of C^m in $GL_n(k)$ as

$$C^m \hookrightarrow \Sigma_m \wr GL_r(\mathbb{F}_q) \hookrightarrow GL_n(k).$$

Since l is odd, the index of the wreath product in $GL_n(k)$ is prime to l and hence the restriction map in mod l cohomology is injective. By a theorem of Nakaoka [**90**], the cohomology of a wreath product $\Sigma_m \wr G$ is detected on G^m and $\Sigma_m \times G$. Also, the Sylow l-subgroups of Σ_m are iterated wreath products of cyclic groups of order l. It follows that the cohomology of $GL_n(k)$ is detected on l-subgroups of exponent dividing $q^r - 1$. Since these are conjugate to subgroups of C^m, the result follows.

Now if $l = 2$, then $r = 1$ and $m = n$ (recall that r is the multiplicative order of q modulo l). The index of the above wreath product is not odd, so we must find a different subgroup. In this case, consider the subgroup $\Sigma_{[n/2]} \wr GL_2(\mathbb{F}_q) \times (\mathbb{F}_q^\times)^{n-2[n/2]}$. This subgroup has odd index. To complete the proof, we need only check that the mod 2 cohomology of $GL_2(\mathbb{F}_q)$ is detected on abelian 2-subgroups of exponent dividing $q-1$. For this, one need only consider the Hochschild–Serre spectral sequence of the extension

$$1 \longrightarrow SL_2(k) \longrightarrow GL_2(k) \longrightarrow k^\times \longrightarrow 1$$

and note that since the Sylow 2-subgroup of $SL_2(k)$ is quaternion, the cohomology of $SL_2(k)$ is periodic of order 4. This gives an upper bound for the cohomology of $GL_2(k)$ which in turn gives the result.

Now, consider the maps

$$H^\bullet(F\Psi^q, \mathbb{F}_l) \longrightarrow H^\bullet(GL_n(k), \mathbb{F}_l) \longrightarrow H^\bullet(C^m, \mathbb{F}_l).$$

The normalizer of C in $GL_r(\mathbb{F}_q)$ contains a cyclic group of order r acting on C by raising elements to the qth power. Thus, the image of the second map above lies in the invariants of $\Sigma_m \wr \mathbb{Z}/r$. Thus we have

$$H^\bullet(F\Psi^q, \mathbb{F}_l) \longrightarrow H^\bullet(GL_n(k), \mathbb{F}_l) \longrightarrow ((H^\bullet(C, \mathbb{F}_q)^{\mathbb{Z}/r})^{\otimes m})^{\Sigma_m}.$$

The second map is injective, so we need only show that the composition is surjective to complete the proof of Theorems 1.1.2 and 1.1.4. The characteristic classes c_r and e_r of the r-dimensional representation of C over $\mathbb{F}_q$ are nonzero; this follows by considering the ring structure of $H^\bullet(C, \mathbb{F}_l)$. Using the formula for computing the characteristic classes of a direct sum of representations [**99**], one sees that the classes $c_r, \ldots, c_{mr}$ and $e_r, \ldots, e_{mr}$ of the n-dimensional representation of C^m generate the invariants of the $\Sigma_m \wr \mathbb{Z}/r$ action on $H^\bullet(C^m, \mathbb{F}_l)$. It follows that the composite map above is surjective and that the second map is an isomorphism. We also see that we may evaluate the squares e_{jr}^2 by restricting to C^m.

If l is odd, then $H^\bullet(C, \mathbb{F}_l)$ is generated by an element in degree 2 and an element in degree 1 which squares to 0. If $l = 2$, then C has order $q-1$. If $q \equiv 1 \mod 4$, then C has order divisible by 4 and the cohomology of C has the same structure as in the case of l odd. If $q \equiv 3 \mod 4$, then C is the product of a cyclic group of order 2 and a cyclic group of odd order so that $H^\bullet(C, \mathbb{F}_l)$ is a polynomial ring on a single generator of degree 1 [**8**], p. 67. Thus, in the typical case, we have $e_{jr}^2 = 0$ and in the exceptional case, we have $e_{jr}^2 = \sum_{b=0}^{jr} c_b c_{2jr-1-b}$. This completes the proof of Theorem 1.1.4. □

This also completes the proof of Theorem 1.1.2. □

By passing to the limit, we obtain a map (unique up to homotopy)

$$BGL(k) \longrightarrow F\Psi^q.$$

THEOREM 1.1.5. *The above map induces an isomorphism*

$$H^\bullet(F\Psi^q, \mathbb{F}_l) \xrightarrow{\cong} H^\bullet(BGL(k), \mathbb{F}_l).$$

□

The mod p cohomology of $GL_n(k)$ is much more difficult to compute. Quillen did prove the following vanishing results. Recall that k has $q = p^d$ elements.

THEOREM 1.1.6. $H^i(GL_n(k), \mathbb{F}_p) = 0$ *for* $0 < i < d(p-1)$ *and all* n.

PROOF. Let U be the subgroup of $GL_n(k)$ consisting of upper triangular matrices with ones on the diagonal and denote by T the diagonal subgroup of $GL_n(k)$. The group U is a Sylow p-subgroup of $GL_n(k)$, hence the restriction map $H^i(GL_n(k)) \to H^i(U)$ is injective. Moreover, U is normalized by T so that the image of the restriction map is contained in the subgroup $H^i(U)^T$ of invariants. Thus, it suffices to show that $H^i(U)^T = 0$ for $0 < i < d(p-1)$.

Let $\overline{k}$ be an algebraic closure of k. The Grothendieck group of representations of T over $\overline{k}$ may be identified with the integral group ring of $\mathrm{Hom}(T, \overline{k}^\times)$. Denote by $\mathrm{cl}(V)$ the class of a representation V and write $\mathrm{cl}(V) = \sum n_u u$, where n_u is the multiplicity of the character u in V. Define the Poincaré series of $H^\bullet(U)$ as a representation of T by

$$\mathrm{PS}(H^\bullet(U)) = \sum_{i \geq 0} \mathrm{cl}(H^i(U) \otimes_{\mathbb{F}_p} \overline{k}) z^i$$

where z is an indeterminate. This series has the form $\sum n_{iu} u z^i$ with $i \geq 0$ and $u \in \mathrm{Hom}(T, \overline{k}^\times)$. We must show that $n_{i\varepsilon} = 0$ for $0 < i < d(p-1)$, where ε is the trivial character.

Let $a : T \to \overline{k}^\times$ be a character and denote by k_a the abelian group k with T-action $t(x) = a(t)x$. As an abelian group k_a is an elementary abelian p-group; its cohomology is given by

$$H^\bullet(k_a) \cong \begin{cases} \bigwedge(k_a^*[1]) \otimes_{\mathbb{F}_p} S(k_a^*[2]) & p \neq 2 \\ S(k_a^*[1]) & p = 2 \end{cases}$$

where $k_a^*[i]$ denotes the subspace of $H^i(k_a)$ isomorphic to

$$k_a^* = \mathrm{Hom}_{\mathbb{F}_p}(k_a, \mathbb{F}_p)$$

and $\bigwedge$ and S denote the exterior and symmetric algebra functors over $\mathbb{F}_p$. We have a ring isomorphism $k \otimes_{\mathbb{F}_p} \overline{k} \xrightarrow{\cong} (\overline{k})^d$ with components $x \otimes y \mapsto x^{p^b} y$ where $0 \leq b < d$. So as a representation of T over $\overline{k}$, we have

$$k_a \otimes_{\mathbb{F}_p} \overline{k} = \bigoplus_b L(a^{p^b})$$

where $L(u)$ denotes the one-dimensional representation of T over $\overline{k}$ with character u. Hence, $k_a^* \otimes_{\mathbb{F}_p} \overline{k} \cong \bigoplus_b L(a^{-p^b})$ and therefore,

$$H^\bullet(k_a) \otimes_{\mathbb{F}_p} \overline{k} \cong \begin{cases} \bigotimes_b \bigwedge(L(a^{-p^b})[1]) \otimes S(L(a^{-p^b})[2]) & p \neq 2 \\ \bigotimes_b S(L(a^{-p^b})[1]) & p = 2 \end{cases}$$

where on the right hand side, all $\bigotimes$, $\bigwedge$ and S are over $\overline{k}$. Since the Poincaré series of a tensor product is the product of the Poincaré series of its factors, we see that for $p \neq 2$

$$\mathrm{PS}(H^\bullet(k_a)) = \prod_{b=0}^{d-1} (1 + a^{-p^b} z)/(1 - a^{-p^b} z^2)$$

while for $p = 2$ we have

$$\begin{aligned} \mathrm{PS}(H^\bullet(k_a)) &= \prod_b 1/(1 - a^{-2^b} z) \\ &= \prod_b (1 + a^{-2^b} z)/(1 - a^{-2^{b+1}} z^2) \\ &= \prod_b (1 + a^{-2^b} z)/(1 - a^{-2^b} z^2) \end{aligned}$$

since $a^{2^d} = a$.

We now use the fact that the group U is an iterated extension of the one-dimensional k_a to obtain an upper bound on $\mathrm{PS}(H^\bullet(U))$. Let Δ^+ be the set of positive roots. These are the homomorphisms $t \mapsto t_i/t_j$ for $1 \leq j < i \leq n$ where t_i is the ith entry of the diagonal matrix t. Order Δ^+ by $(i', j') \leq (i, j)$ if either $i' < i$ or if $i' = i$ and $j' \leq j$. For $a \in \Delta^+$, let U_a be the subgroup of U generated by the one-parameter subgroups corresponding to the roots greater than a. We have an extension of groups

$$1 \longrightarrow k_a \longrightarrow U/U_a \longrightarrow U/U_{a'} \longrightarrow 1$$

where a' denotes the element of Δ^+ immediately preceding a. This is a central extension and hence the associated spectral sequence has the form

$$E_2 = H^\bullet(U/U_{a'}) \otimes H^\bullet(k_a) \Longrightarrow H^\bullet(U/U_a).$$

This shows that

$$\mathrm{PS}(H^\bullet(U/U_a)) \ll \mathrm{PS}(H^\bullet(U/U_{a'}))\mathrm{PS}(H^\bullet(k_a))$$

where $\ll$ means that each coefficient on the left side is less than or equal to the corresponding coefficient on the right. Thus if we let a vary over Δ^+ and

$0 \le b < d$, then

$$\begin{aligned} \mathrm{PS}(H^\bullet(U)) &\ll \prod_a \mathrm{PS}(H^\bullet(k_a)) \\ &= \prod_{a,b}(1+a^{-p^b}z)\sum_{j\ge 0} a^{-jp^b}z^{2j} \\ &= \sum_I (\prod_a a^{-M_a(I)})z^{D(I)} \end{aligned}$$

where I runs over families (m_{ab}, n_{ab}) of integers with $0 \le m_{ab} \le 1$, $0 \le n_{ab}$ and

$$\begin{aligned} M_a(I) &= \sum_b (m_{ab}+n_{ab})p^b \\ D(I) &= \sum_{a,b}(m_{ab}+2n_{ab}). \end{aligned}$$

Now, to show that $H^i(U)$ does not contain the trivial character of T for $0 < i < d(p-1)$ it suffices to show that for any family I satisfying

$$\prod_a a^{-M_a(I)} = \varepsilon$$

that either $D(I) = 0$ or $D(I) \ge d(p-1)$. Let $a_i(t) = t_{i+1}/t_i$ be the simple roots. Then we may write

$$a = \prod_{i=1}^{n-1} (a_i)^{c_{ai}}$$

where $c_{ai} = 1$ for $h \le i < j$ and 0 otherwise ($a(t) = t_j/t_h$). Then the above product becomes

$$\prod_{i=1}^{n-1} a_i^{-e_i} = \varepsilon, \qquad e_i = \sum_{a,b} c_{ai}(m_{ab}+n_{ab})p^b.$$

The map $T \to (k^\times)^{n-1}$ with components a_i is surjective. Since $k^\times$ is cyclic of order $p^d - 1$, we have, for each $1 \le i \le n-1$,

$$\sum_{a,b} c_{ai}(m_{ab}+n_{ab})p^b \equiv 0 \mod (p^d-1).$$

We now need the following number-theoretic fact. Let j_b, $0 \le b < d$ be nonnegative integers such that

$$\sum_{b=0}^{d-1} j_b p^b \equiv 0 \mod (p^d-1).$$

If not all the j_b are zero, then $\sum_b j_b \geq d(p-1)$. Given this, suppose that the family $I = \{m_{ab}, n_{ab}\}$ satisfies $D(I) > 0$. Then for some a, b we have $m_{ab} + n_{ab} > 0$ so that for some b and i we have $\sum_a c_{ai}(m_{ab} + n_{ab}) > 0$. Thus by the statement above we have

$$D(I) \geq \sum_{a,b}(m_{ab} + n_{ab}) \geq \sum_{ab} c_{ai}(m_{ab} + n_{ab}) \geq d(p-1).$$

This completes the proof of Theorem 1.1.6. □

REMARK 1.1.7. An alternate proof of Theorem 1.1.6 is outlined in the exercises.

COROLLARY 1.1.8. *If k' is an infinite algebraic extension of $\mathbb{F}_p$, then*

$$H^i(GL_n(k'), \mathbb{F}_p) = 0$$

for all $i > 0$.

PROOF. We have

$$H^\bullet(GL_n(k'), \mathbb{F}_p) = \varprojlim_{k \subset k'} H^\bullet(GL_n(k), \mathbb{F}_p)$$

where k runs over all finite subfields of k'. □

COROLLARY 1.1.9. *$H^i(GL(k), \mathbb{F}_p) = 0$ for all $i > 0$.*

PROOF. Recall that the mod l cohomology of $GL_n(k)$ was computed via "characteristic classes" for the standard representation. Let $i > 0$ and suppose $x \in H^i(GL(k), \mathbb{F}_p)$. Interpret x as a characteristic class for representations over k as follows. A representation E of G with $\dim E = n$ determines a map $G \to GL_n(k)$. Hence, associated to any E we get a homomorphism $E^\# : G \to GL(k)$ (unique up to conjugation) and we set $x(E) = (E^\#)^*(x) \in H^i(G, \mathbb{F}_p)$. Since $H^i(GL(k), \mathbb{F}_p)$ is the inverse limit of the $H^i(GL_n(k), \mathbb{F}_p)$, it suffices to show that $x(E) = 0$ for all E to show that $x = 0$. We proceed by induction on i, the case $i = 1$ being obvious since $H^1(GL(k), \mathbb{F}_p) \cong \mathrm{Hom}(k^\times, \mathbb{F}_p) = 0$.

Suppose that $H^j(GL(k), \mathbb{F}_p) = 0$ for $0 < j < i$. Then x is a primitive class in $H^\bullet(GL(k), \mathbb{F}_p)$. If E and E' are two representations of G, then we have

$$x(E \oplus E') = x(E) + x(E').$$

Choose an extension k' of k of degree d' with $(d', p) = 1$ and $dd'(p-1) > i$. Then by Theorem 1.1.6, $H^i(GL_n(k'), \mathbb{F}_p) = 0$ for all n, hence there are no non-trivial i-dimensional characteristic classes for representations over k'. In particular, if a representation E' over k' is regarded as being over k, then $x(E') = 0$. Now if E is any representation over k, we have

$$0 = x(E \otimes_k k') = x(E^{\oplus d'}) = d'x(E).$$

Since d' is prime to p, $x(E) = 0$. □

This corollary allows us to compute the homotopy type of $BGL(k)^+$. Recall that there is a map

$$\varphi : BGL(k) \longrightarrow F\Psi^q.$$

Since $\pi_1(F\Psi^q)$ is abelian, this map can be extended to a map

$$\varphi^+ : BGL(k)^+ \longrightarrow F\Psi^q.$$

THEOREM 1.1.10. *The map φ^+ is a homotopy equivalence.*

PROOF. By the Whitehead theorem, it suffices to show that φ^+ induces an isomorphism on integral (co)homology. In turn, to show this it suffices to show that φ^+ induces a (co)homology isomorphism with $\mathbb{Q}$, $\mathbb{F}_p$, and $\mathbb{F}_l$ coefficients ($l \neq p$). Since $H_\bullet(BGL(k), \mathbb{Z}) \cong H_\bullet(BGL(k)^+, \mathbb{Z})$, it suffices to show this for φ. By Theorem 1.1.5, the result holds for $\mathbb{F}_l$-coefficients. By Theorem 1.1.6, the $\mathbb{F}_p$-(co)homology of $BGL(k)$ vanishes. The rational (co)homology of $BGL(k)$ is also trivial since $GL(k)$ is a union of finite groups. On the other hand, since $F\Psi^q$ is a simple space with finite homotopy groups of order prime to p, its rational and mod p (co)homology are also trivial via the p-local version of the Hurewicz theorem (see [**85**], Chapter 10). □

COROLLARY 1.1.11. *If k is a finite field with q elements, then for all $i \geq 1$,*

$$\begin{aligned} K_{2i}(k) &= 0 \\ K_{2i-1}(k) &= \mathbb{Z}/(q^i - 1). \end{aligned}$$

Moreover, if $k \to k'$ is a homomorphism of finite fields, then the induced map $K_\bullet(k) \to K_\bullet(k')$ is injective.

PROOF. The calculation of the K-groups is clear. For the second statement, we may assume that k' is a subfield of $\overline{k}$ and that the map $k \to k'$ is the standard inclusion. Let d be the degree of this field extension; then k' has $q' = q^d$ elements. Since

$$[F\Psi^q, F\Psi^{q'}] = [F\Psi^q, BU]^{\Psi^{q'}}$$

we see that there is at most one map $F\Psi^q \to F\Psi^{q'}$ lying over a given endomorphism of BU. Since the Brauer lift of a representation E depends only on $E \otimes \overline{k}$, the map $BGL_n(k) \to BGL_n(k')$ lies over the identity of BU. Since

$$[BGL(k)^+, BU] = [BGL(k), BU] = \varprojlim_n [BGL_n(k), BU],$$

the map $BGL(k)^+ \to BGL(k')^+$ corresponds to the unique map $t : F\Psi^q \to F\Psi^{q'}$ lying over the identity of BU. Consider the identity

$$1 - \psi^{q'} = \sum_{b=0}^{d-1} \psi^{q^b}(1 - \psi^q).$$

This implies that we have a map of fibrations

$$\begin{array}{ccccc} F\Psi^q & \longrightarrow & BU & \xrightarrow{1-\Psi^q} & BU \\ \downarrow & & \downarrow 1 & & \downarrow \sum_{b=0}^{d-1} \Psi^{q^b} \\ F\Psi^{q'} & \longrightarrow & BU & \xrightarrow{1-\Psi^{q'}} & BU. \end{array}$$

Then the long exact homotopy sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{Z} & \xrightarrow{1-q^i} & \mathbb{Z} & \longrightarrow & \pi_{2i-1}(F\Psi^q) & \longrightarrow & 0 \\ & & \downarrow 1 & & \downarrow \sum_b q^{bi} & & \downarrow t_* & & \\ 0 & \longrightarrow & \mathbb{Z} & \xrightarrow{1-q^{di}} & \mathbb{Z} & \longrightarrow & \pi_{2i-1}(F\Psi^{q'}) & \longrightarrow & 0 \end{array}$$

show that t_* is injective. □

COROLLARY 1.1.12. *Let F be an algebraic closure of $\mathbb{F}_p$. Then for $i \geq 1$,*

$$\begin{aligned} K_{2i}(F) &= 0 \\ K_{2i-1}(F) &= \bigoplus_{l \neq p} \mathbb{Q}_l/\mathbb{Z}_l. \end{aligned}$$

PROOF. Note that $K_\bullet(F)$ is the direct limit of $K_\bullet(k)$, where k runs over the finite subfields of F. Thus, $K_{2i}(F) = 0$ is clear. From the previous corollary, the l-primary component of $K_{2i-1}(F)$ $(l \neq p)$ is a union of cyclic groups of order l^n with $n \to \infty$, hence it is isomorphic to $\mathbb{Q}_l/\mathbb{Z}_l$. □

1.2. Quillen's Conjecture

The particularly simple description of the mod l cohomology of $GL_n(\mathbb{F}_q)$ is appealing, but it is unrealistic to expect the structure of $H^\bullet(GL_n(R))$ for other rings R to be as nice. Quillen's study of equivariant cohomology [**98**] led him to make a conjecture which spurred the development of a great deal of mathematics. We discuss this in this section.

In [**98**], Quillen proves that under certain hypotheses there is a one-to-one correspondence between the minimal prime ideals of the cohomology ring $H^\bullet(G, \mathbb{F}_p)$ and conjugacy classes of maximal elementary abelian p-subgroups of G. Here, G is a compact group or a group with finite $\mathbb{F}_p$ cohomological dimension. Typical examples are finite groups and S-arithmetic subgroups of reductive groups over number fields. This result has certain consequences which led to Quillen's conjectural description of $H^\bullet(\Gamma, \mathbb{F}_p)$ for $\Gamma = GL_n(\Lambda)$, where Λ is the ring of S-integers in a number field.

Now, the cohomology ring $H^\bullet(GL_n(\mathbb{F}_{l^d}), \mathbb{F}_p)$ is the tensor product of a polynomial ring and an exterior algebra. Suppose that $H^\bullet(\Gamma, \mathbb{F}_p)$ has this form. Then this ring has only one minimal prime ideal and hence the group Γ has only one conjugacy class of maximal elementary abelian p-subgroups. However, in many cases, this does not happen (e.g., $\Lambda = \mathbb{Z}$). Often there are maximal elementary abelian p-subgroups of different ranks and hence $H^\bullet(\Gamma, \mathbb{F}_p)$ *cannot*, in general, be a finitely generated free module over a polynomial subring.

However, if p is a unit in Λ, then things improve. For simplicity, assume that Λ is the ring of S-integers in a number field containing a primitive pth root of unity, ζ_p (e.g., $\Lambda = \mathbb{Z}[1/p, \zeta_p]$). If A is an elementary abelian p-subgroup of Γ, decompose Λ^n into eigenspaces

$$\Lambda^n = \bigoplus E_i$$

corresponding to the characters of A. Since Λ is a Dedekind domain and each E_i is a projective Λ-module, we see that E_i is a sum of invertible modules. If A is maximal, it follows that E_i is invertible and that $A \cong \mu_p^n$, where μ_p is the group of pth roots of unity. Moreover, the ith factor of A acts by multiplication on E_i and trivially on the other summands. Thus, all maximal A have rank n.

We can now state Quillen's conjecture. Let $c_i \in H^{2i}(\Gamma, \mathbb{F}_p)$ be the image of the ith universal Chern class in $H^{2i}(BGL_n(\mathbb{C})^{\text{top}}, \mathbb{F}_p)$ under the map induced by the embedding $\Lambda \to \mathbb{C}$. (This image is obtained via the map induced by the composition $GL_n(\Lambda) \to GL_n(\mathbb{C}) \xrightarrow{\varepsilon} GL_n(\mathbb{C})^{\text{top}}$, where ε is the continuous map $A \overset{\varepsilon}{\mapsto} A$ from the discrete group $GL_n(\mathbb{C})$ to the Lie group $GL_n(\mathbb{C})^{\text{top}}$. We recall that $H^\bullet(BGL_n(\mathbb{C})^{\text{top}}, \mathbb{F}_p) \cong H^\bullet(BU(n), \mathbb{F}_p) \cong H^\bullet(G(n), \mathbb{F}_p) \cong \mathbb{F}_p[c_1, \ldots, c_n]$, where $G(n)$ is the Grassmannian of n-planes in $\mathbb{C}^\infty$, see e.g., [**49**].) The elements $c_1, \ldots, c_n$ generate a polynomial subring of $H^\bullet(\Gamma, \mathbb{F}_p)$ over which $H^\bullet(\Gamma, \mathbb{F}_p)$ is a finitely generated module.

CONJECTURE 1.2.1 ([**98**], p. 591). *$H^\bullet(\Gamma, \mathbb{F}_p)$ is a free module over the subring $\mathbb{F}_p[c_1, \ldots, c_n]$.*

This implies the following conjecture in many cases. If G is a group with subgroup K, we say that the cohomology of G is *detected* on K if the induced map $H^\bullet(G, \mathbb{F}_p) \to H^\bullet(K, \mathbb{F}_p)$ is injective.

CONJECTURE 1.2.2 ([**57**], p. 51). *The mod p cohomology of $GL_n(\Lambda)$ is detected on the subgroup $D_n(\Lambda)$ of diagonal matrices.*

This conjecture has been proved in a few cases and disproved in infinitely many others, but remarkably little else is known. The case $\Lambda = \mathbb{Z}[1/2]$ was proved for $n = 2$ by Mitchell [**84**] and for $n = 3$ by Henn [**56**] via explicit computations of the cohomology of $GL_n(\Lambda)$. Voevodsky recently announced a proof of the mod 2 Lichtenbaum–Quillen conjecture which implies Conjecture 1.2.2 for $\Lambda = \mathbb{Z}[1/2]$ in the stable range.

A clever disproof of Conjecture 1.2.2 for $GL_n(\Lambda)$, $\Lambda = \mathbb{Z}[1/2]$, $n = 32$, was given by Dwyer [**32**]. We sketch the line of proof. Let P and G be groups. Two homomorphisms $\alpha, \beta : P \to G$ are *conjugate* if there is an element $g \in G$ with $g\alpha g^{-1} = \beta$. Let $\rho_{\mathbb{R}} : GL_n(\Lambda) \to GL_n(\mathbb{R})$ and $\rho_{\mathbb{F}_3} : GL_n(\Lambda) \to GL_n(\mathbb{F}_3)$ be the obvious maps. Two maps $\alpha, \beta : P \to GL_n(\Lambda)$ are said to *become conjugate over* $\mathbb{R}$ (resp. *over* $\mathbb{F}_3$) if $\rho_{\mathbb{R}}\alpha$ and $\rho_{\mathbb{R}}\beta$ (resp. $\rho_{\mathbb{F}_3}\alpha$ and $\rho_{\mathbb{F}_3}\beta$) are conjugate. Dwyer proves the following result.

THEOREM 1.2.3. *Suppose the mod* 2 *cohomology of* $GL_n(\Lambda)$ *is detected on diagonal matrices. Let* P *be a finite* 2*-group with homomorphisms* $\alpha, \beta : P \to GL_n(\Lambda)$. *Then* α *is conjugate to* β *if and only if* α *becomes conjugate to* β *over* $\mathbb{R}$ *and* $\mathbb{F}_3$.

This leads to a disproof of Conjecture 1.2.2 for $n = 32$ as follows. Let μ_n denote the group of 2^nth roots of unity. The smallest n such that the ideal class group of $\Lambda(\mu_n)$ (the integral closure of Λ in $\mathbb{Q}(\mu_n)$) is nontrivial is 6, and the rank of $\Lambda(\mu_6)$ as a Λ-module is $\phi(2^6) = 2^5 = 32$. Let $P = \mu_6 \subset \Lambda(\mu_6)^\times$ and let I be a nonprincipal ideal in $\Lambda(\mu_6)$. One uses the action of P on I and on $\Lambda(\mu_6)$ to construct two nonconjugate homomorphisms $P \to GL_{32}(\Lambda)$ which become conjugate over $\mathbb{R}$ and over $\mathbb{F}_3$ (this follows since $I \otimes k \cong \Lambda(\mu_6) \otimes k$ for $k = \mathbb{R}, \mathbb{F}_3$).

REMARK 1.2.4. Recently, Henn and Lannes have improved Dwyer's computation to $n = 14$. Moreover, Henn [**55**] proved that if Conjecture 1.2.2 is false for $n = n_0$, then it is false for all $n \geq n_0$. Also, using methods similar to Dwyer's, Anton [**3**] has disproved Conjecture 1.2.2 for $\Lambda = \mathbb{Z}[1/3, \zeta_3]$ and $n \geq 27$. He also provided a proof of 1.2.2 for $n = 2$.

The proof of Theorem 1.2.3 is homotopy theoretic in nature and relies on the construction of a space X_n and a map $\chi_n : BGL_n(\Lambda) \to X_n$ satisfying certain properties. The space X_n is constructed using *étale homotopy theory* [**40**].

1.3. Étale homotopy theory

A complete discussion of étale homotopy theory would lead us too far afield, but we can say a few things. This theory provides a covariant mechanism for assigning a (pro-)space $X_{\text{ét}}$ to a (simplicial) scheme X. We provide some examples of étale homotopy types.

EXAMPLE 1.3.1. Let k be a field and denote by π the Galois group over k of the separable algebraic closure of k. Then $\text{Spec}(k)_{\text{ét}}$ is a pro-space of type $K(\pi, 1)$. If S is a complete local ring with residue field k, then the map $\text{Spec}(k)_{\text{ét}} \to \text{Spec}(S)_{\text{ét}}$ is an equivalence. Thus $\text{Spec}(\mathbb{C})_{\text{ét}}$ is contractible, as

is $\mathrm{Spec}(\mathbb{C}[[t]])_{\text{ét}}$. The space $\mathrm{Spec}(\mathbb{R})_{\text{ét}}$ is equivalent to $B\mathbb{Z}/2 \cong \mathbb{RP}^\infty$. Both $\mathrm{Spec}(\mathbb{F}_p)_{\text{ét}}$ and $\mathrm{Spec}(\mathbb{Z}_p)_{\text{ét}}$ are equivalent to the profinite completion of S^1.

EXAMPLE 1.3.2. The étale homotopy type of a number ring is rather complicated in general. We give a partial description for the ring $\Lambda = \mathbb{Z}[1/2]$ which we shall use in the proof of Theorem 1.2.3. Choose an embedding $\mathbb{F}_3 \to \mathbb{C}$. Then we have a commutative diagram of pro-spaces

$$\begin{array}{ccc} \mathrm{spec}(\mathbb{C})_{\text{ét}} & \longrightarrow & \mathrm{spec}(\mathbb{R})_{\text{ét}} \\ \downarrow & & \downarrow \\ \mathrm{spec}(\mathbb{F}_3) & \longrightarrow & \mathrm{spec}(\Lambda). \end{array}$$

Since $\mathrm{spec}(\mathbb{C})_{\text{ét}}$ is contractible, we have an induced map

$$\mathrm{spec}(\mathbb{F}_3)_{\text{ét}} \vee \mathrm{spec}(\mathbb{R})_{\text{ét}} \longrightarrow \mathrm{spec}(\Lambda)_{\text{ét}}.$$

Choose an equivalence $B\mathbb{Z}/2 \to \mathrm{spec}(\mathbb{R})_{\text{ét}}$ and a map $S^1 \to \mathrm{spec}(\mathbb{F}_3)_{\text{ét}}$ which sends the generator of $\pi_1(S^1)$ to the Frobenius automorphism of $\overline{\mathbb{F}}_3$ over $\mathbb{F}_3$. Then we have a map

$$S^1 \vee B\mathbb{Z}/2 \longrightarrow \mathrm{spec}(\Lambda)_{\text{ét}}$$

which by [**34**] induces an isomorphism in mod 2 cohomology.

EXAMPLE 1.3.3. Let k be an algebraically closed field of characteristic zero. Then $(BGL_{n,k})_{\text{ét}}$ is equivalent to the profinite completion of $BGL_n(\mathbb{C}^{\text{top}})$.

EXAMPLE 1.3.4. If k is a field of characteristic zero with algebraic closure $\overline{k}$, then the sequence

$$(BGL_{n,\overline{k}})_{\text{ét}} \longrightarrow (BGL_{n,k})_{\text{ét}} \longrightarrow \mathrm{Spec}(k)_{\text{ét}}$$

is a fibration of pro-spaces.

This yields an étale approximation to BGL_n as follows. Let $R = \mathbb{Z}[1/l]$ and let A be an R-algebra. According to [**33**], 4.2, the space $BGL_n(A)$ can be identified with the basepoint component of the space of maps $\mathrm{spec}(A) \to BGL_{n,R}$ over $\mathrm{spec}(R)$. Denote by $BGL_n^{\text{ét}}(A)$ the basepoint component of the space of maps from $\mathrm{spec}(A)_{\text{ét}}$ to the fibrewise l-completion of $(BGL_{n,R})_{\text{ét}}$ over $\mathrm{spec}(R)_{\text{ét}}$. Functoriality yields a natural map

$$\phi_{n,A} : BGL_n(A) \longrightarrow BGL_n^{\text{ét}}(A).$$

In fact, one can perform the same construction over any affine group scheme G.

PROOF OF THEOREM 1.2.3. Define the space X_n to be the space

$$BGL_n^{\text{ét}}(\Lambda) \quad \text{and let} \quad \chi_n : BGL_n(\Lambda) \longrightarrow X_n$$

be the natural map. Denote by G_n the group $GL_n(\Lambda)$. Let $BD_\bullet$, $BG_\bullet$ and $X_\bullet$ denote the spaces $\coprod BD_n$, $\coprod BG_n$ and $\coprod X_n$, respectively. There are maps

$$BD_\bullet \xrightarrow{i} BG_\bullet \xrightarrow{\chi} X_\bullet.$$

Under matrix block sum, these spaces are homotopy associative H-spaces; the maps above respect the multiplications. Note that

$$BD_1 \simeq B(\mathbb{Z} \times \mathbb{Z}/2) \simeq S^1 \times B\mathbb{Z}/2.$$

Let e be the generator of $H_1(S^1)$ and β_k the generator of $H_k(B\mathbb{Z}/2)$ (all homology in this proof is with $\mathbb{Z}/2$ coefficients). Denote the classes $e \otimes \beta_{k-1}$ and $1 \otimes \beta_k$ in $H_k(BD_1)$ by a_k and b_k respectively. Set

$$a_k^G = i_*(a_k),\ b_k^G = i_*(b_k),\ a_k^X = (\chi i)_*(a_k),\ b_k^X = (\chi i)_*(b_k).$$

Since $H_\bullet(BD_\bullet)$ is the free $\mathbb{Z}/2$-algebra on the elements a_k, $k \geq 1$ and b_k, $k \geq 0$, the image of i_* is the subalgebra generated by the a_k^G and b_k^G, and that the image of $(\chi i)_*$ is generated by the classes a_k^X and b_k^X. By the work of Mitchell [**84**], we have the following result.

PROPOSITION 1.3.5. *The algebra $H_\bullet(X_\bullet)$ is the free commutative $\mathbb{Z}/2$-algebra on the classes a_k^X and b_k^X subject to the following relations:*

1. $(a_k^X)^2 = 0$ *for k odd, and*
2. $a_k^X b_0^X + a_{k-1}^X b_1^X + \cdots + a_1^X b_{k-1}^X = 0$ *for k even.* □

Mitchell's results also imply that the classes a_k^G and b_k^G in $H_\bullet(BG_\bullet)$ satisfy the analogous relations. From this we conclude the following.

LEMMA 1.3.6. *If the* mod 2 *cohomology of G_n is detected on D_n, then $\chi_n^* : H^\bullet(X_n) \to H^\bullet(BG_n)$ is an isomorphism.*

PROOF. The map $(\chi_n i_n)_*$ is surjective and its kernel is equal to the kernel of $(i_n)_*$. Dualizing, we see that the cohomology map $(\chi_n i_n)^*$ is injective with image equal to the image of i_n^*. Thus i_n^* is injective if and only if χ_n^* is an isomorphism. □

Now let P be a finite 2-group. If Γ is a discrete group, denote by $\{P, \Gamma\}$ the set of conjugacy classes of homomorphisms $P \to \Gamma$. There is an injective map [**32**], 3.6,

$$[BP, X_n] \longrightarrow [BP, \hat{B}GL_n(\mathbb{F}_3)] \times [BP, \hat{B}GL_n^{\mathrm{top}}(\mathbb{R})]$$

where $\hat{}$ denotes 2-completion. Moreover, a theorem of Carlsson [**24**] implies that the natural map

$$\{P, \Gamma\} \cong [BP, B\Gamma] \longrightarrow [BP, \hat{B}\Gamma]$$

is a bijection for Γ of virtually finite cohomological dimension.

LEMMA 1.3.7. *If χ_n^* is an isomorphism, then for any finite 2-group P, the map*

$$\chi_n \cdot (-) : [BP, BG_n] \longrightarrow [BP, X_n]$$

is a bijection.

PROOF. The space X_n is 2-complete, so if χ_n^* is an isomorphism, then X_n is equivalent to $\hat{B}G_n$. Since G_n has virtual finite cohomological dimension, the above result finishes the proof. □

LEMMA 1.3.8. *Let P be a finite 2-group with homomorphisms $\alpha, \beta : P \to G_n$. Then $\chi_n \cdot (B\alpha)$ is homotopic to $\chi_n \cdot (B\beta)$ if and only if α becomes conjugate to β over $\mathbb{R}$ and $\mathbb{F}_3$.*

PROOF. Consider the commutative diagram

$$\begin{array}{ccccc}
[BP, BGL_n(\mathbb{R})] & \xleftarrow{B\rho_{\mathbb{R}}} & [BP, BG_n] & \xrightarrow{B\rho_{\mathbb{F}_3}} & [BP, BGL_n(\mathbb{F}_3)] \\
\downarrow & & \downarrow & & \downarrow \\
[BP, BGL_n^{\text{ét}}(\mathbb{R})] & \longleftarrow & [BP, X_n] & \longrightarrow & [BP, BGL_n^{\text{ét}}(\mathbb{F}_3)].
\end{array}$$

The right and left vertical arrows are bijections and the maps in the bottom row are injective. The result follows. □

Lemmas 1.3.6, 1.3.7 and 1.3.8 together imply Theorem 1.2.3. □

DEFINITION 1.3.9. Let A be a finitely generated R-algebra of finite mod l étale cohomological dimension. The étale K-theory space $BGL^{\text{ét}}(A)$ is defined as

$$BGL^{\text{ét}}(A) = \text{colim}_n\{BGL_n^{\text{ét}}(A)\}$$

and the ith mod l^ν étale K-group of A is defined to be

$$K_i^{\text{ét}}(A, \mathbb{Z}/l^\nu) = \pi_i(BGL^{\text{ét}}(A), \mathbb{Z}/l^\nu), \quad i > 1.$$

Taking the limit of the maps $\phi_{n,A}$ defined above produces a map

$$\phi : BGL(A) \longrightarrow BGL^{\text{ét}}(A)$$

which can be lifted to a map

$$\phi : BGL(A)^+ \longrightarrow BGL^{\text{ét}}(A).$$

Then we have induced maps on homotopy:

$$\phi_i^\nu : K_i(A, \mathbb{Z}/l^\nu) \longrightarrow K_i^{\text{ét}}(A, \mathbb{Z}/l^\nu).$$

THEOREM 1.3.10 ([33]). *The maps ϕ_i^ν satisfy the following.*

1. *If A is a finite field of order q where l does not divide q, then ϕ_i^ν is an isomorphism for all $\nu > 0$.*

2. *If A is the ring of S-integers in a global field (and contains a primitive 4th root of unity if $l = 2$), then for $i > 0$ and $\nu > 0$ the map ϕ_i^ν is surjective.*

3. *If A is a local field or the coordinate ring of a smooth affine curve over a finite field, then ϕ_i^ν is surjective for $i > 0$ and $\nu > 0$.* □

In a certain sense, then, the étale K-theory space $BGL^{\text{ét}}(A)$ provides a plausible analogue of $F\Psi^q$ for the ring A. Conjecture 1.2.1 is equivalent to the conjecture that the maps ϕ_i^ν in 2 above are isomorphisms.

Dwyer and Friedlander [**35**] used the spaces $BGL_n^{\text{ét}}(A)$ to obtain a lower bound for the cohomology of $BGL_n(A)$ for the ring $A = \mathbb{Z}[1/l, \zeta_l]$. They prove the following result.

PROPOSITION 1.3.11. *Let A be a noetherian $\mathbb{Z}[1/l]$-algebra such that the Picard group* $\mathrm{Pic}(A)$ *has no infinitely l-divisible elements. Denote by T_n the diagonal subgroup of GL_n. Then the natural map*

$$BT_n(A) \to BT_n^{\text{ét}}(A)$$

induces an isomorphism

$$H^\bullet(BT_n^{\text{ét}}(A), \mathbb{Z}/l) \longrightarrow H^\bullet(BT_n(A), \mathbb{Z}/l).$$

Note that the cohomology algebra $H^\bullet(BT_n(A), \mathbb{Z}/l)$ is easily computable via the Künneth formula and the identification of $T_1(A)$ with the group $A^\times$. This result is then used to prove the following.

PROPOSITION 1.3.12. *Let l be an odd regular prime; that is, l does not divide the order of the class group of $\mathbb{Z}[\zeta_l]$. Let A denote the ring $\mathbb{Z}[1/l, \zeta_l]$. Then the natural map $BGL_n(A) \to BGL_n^{\text{ét}}(A)$ induces a monomorphism*

$$H^\bullet(BGL_n^{\text{ét}}(A), \mathbb{Z}/l) \longrightarrow H^\bullet(BGL_n(A), \mathbb{Z}/l).$$

The proof of this relies on the existence of a "good mod l model" of A; details may be found in [**35**].

Dwyer and Friedlander also compute the graded ring associated to a filtration of $H^\bullet(BGL_n^{\text{ét}}(A), \mathbb{Z}/l)$. It is isomorphic to the $(l+1)/2$-fold tensor power

$$\Omega_S^* \otimes_S \cdots \otimes_S \Omega_S^*$$

where Ω_S^* is the deRham complex of $S = \mathbb{F}_l[c_1, \ldots, c_n]$, $\deg(c_i) = 2i$. The above injectivity holds also for the stable group $BGL(\mathbb{Z}[1/l])$.

1.4. Analytical Methods

At around the same time as Quillen's calculation of $K_\bullet(\mathbb{F}_q)$, A. Borel, building upon previous work of Garland [**44, 46**], Matsushima [**78**], and Raghunathan [**101**], was studying the cohomology of arithmetic subgroups of semisimple groups defined over $\mathbb{Q}$. Let Γ be such a subgroup of the semisimple group G and denote by X the symmetric space G/K, where K is a maximal compact subgroup of the group $G(\mathbb{R})$ of real points of G. Then there is an isomorphism

$$H^\bullet(\Omega_X^\Gamma, \mathbb{R}) \cong H^\bullet(\Gamma, \mathbb{R}),$$

where Ω_X^Γ is the complex of Γ-invariant smooth differential forms on X. This follows from the deRham theorem, since Ω_X^Γ computes the real cohomology of $\Gamma\backslash X$, and since X is contractible the cohomology of Γ coincides with that of $\Gamma\backslash X$ up to torsion.

Now let I_G be the complex of forms on X which are invariant under the identity component $G(\mathbb{R})^\circ$ of $G(\mathbb{R})$. Denote by I_G^Γ the subcomplex consisting of Γ-invariant forms. Since I_G consists of closed (in fact harmonic) forms, we obtain a homomorphism

$$j^\bullet : I_G^\Gamma \longrightarrow H^\bullet(\Gamma, \mathbb{R}).$$

If $G(\mathbb{R})/\Gamma$ is compact, then standard results of Hodge theory imply that $j^\bullet$ is injective in all dimensions. If $G(\mathbb{R})/\Gamma$ is not compact, it is still possible to show that $j^\bullet$ is injective in a certain range explicitly computable from the algebraic structure of G. Results of Garland and Matsushima show that $j^\bullet$ is surjective in a suitable range so that we have an isomorphism

$$j^\bullet : I_G^\Gamma \longrightarrow H^\bullet(\Gamma, \mathbb{R})$$

for $\bullet$ sufficiently small.

THEOREM 1.4.1 ([**14**]). *There exist constants $c(G)$ and $m(G(\mathbb{R}))$ such that the map*

$$j^i : (I_G^\Gamma)^i \longrightarrow H^i(\Gamma, \mathbb{R})$$

is injective for $i \leq c(G)$ and surjective for $i \leq \min\{c(G), m(G(\mathbb{R}))\}$. □

The constant $c(G)$ is defined in terms of the weights of G and is easily computed from this data. The constant $m(G(\mathbb{R}))$ is defined using the Cartan decomposition of the Lie algebra $\mathfrak{g}$:

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p},$$

where $\mathfrak{k}$ is the Lie algebra of K, along with the Killing form and a certain quadratic form on $\bigwedge^2 \mathfrak{p}$ defined using the curvature tensor. Borel showed that $c(G), m(G(\mathbb{R})) \geq (\text{rank}_{\mathbb{Q}}(G))/4 - 1$.

This theorem allowed Borel to compute the real cohomology of the stable linear groups SL, O, Sp, etc. over rings of integers in number fields. Utilizing

Cartan's calculation of the homotopy of symmetric spaces he obtained the following result. Let k be a number field with ring of integers $\mathcal{O}_k$. Let S be the set of archimedean places of k and let $d = r_1 + 2r_2$ be the degree of k over $\mathbb{Q}$. For $v \in S$, denote by k_v the completion of k at v; we have $k_v = \mathbb{R}$ or $\mathbb{C}$ depending on whether v is real or complex. Let G_n be a semisimple group over $\mathbb{Q}$ with arithmetic subgroup Γ_n. Assume $G_n \hookrightarrow G_{n+1}$ and $\text{rank}_{\mathbb{Q}} G_n \to \infty$ as $n \to \infty$. Denote by G'_n, Γ'_n the corresponding group over k. Denote by G the direct limit $\varinjlim G_n$ and by Γ the limit $\varinjlim \Gamma_n$.

THEOREM 1.4.2. *There is an isomorphism*

$$H^\bullet(\Gamma', \mathbb{R}) \cong \bigotimes_{v \in S} I_v$$

for G' and Γ' as in the following table.

G'	Γ'	I_v v real	I_v v complex
$SL(k)$	$SL(\mathcal{O}_k)$	$\bigwedge(x_i), \deg(x_i) = 4i+1$	$\bigwedge(x_i), \deg(x_i) = 2i+1$
$Sp(k)$	$Sp(\mathcal{O}_k)$	$P(x_i), \deg(x_i) = 4i-2$	$\bigwedge(x_i), \deg(x_i) = 4i-1.$

Here, $\bigwedge$ denotes an exterior algebra and P denotes a polynomial algebra.

EXAMPLE 1.4.3. Suppose $k = \mathbb{Q}$, $\mathcal{O}_k = \mathbb{Z}$. Then we see that

$$H^\bullet(SL(\mathbb{Z}), \mathbb{R}) \cong \bigwedge(x_i), \deg(x_i) = 4i+1.$$

In [**15**], Borel considered the computation of $H^\bullet(\Gamma, E)$, where E is a nontrivial (complex) representation of G. The case H^1 had been considered previously by Raghunathan [**101**] who showed that $H^1(\Gamma, E) = 0$ provided $\text{rank}_{\mathbb{Q}}(G) > 1$.

Let E be a finite dimensional representation of G. The constants $c(G)$ and $m(G(\mathbb{R}))$ can be modified to obtain constants $c(G, E)$ and $m(G(\mathbb{R}), E)$. Borel proves the following.

THEOREM 1.4.4. *Suppose that E contains no nontrivial subspace on which $G(\mathbb{R})^\circ$ acts trivially. Then*

$$H^i(\Gamma, E) = 0$$

for $i \leq \min\{c(G, E), m(G(\mathbb{R}), E)\}$.

EXAMPLE 1.4.5. For the group $\Gamma = SL_n(\mathbb{Z})$, the constants c and m are bounded by $[(n-1)/4]$. Thus, $H^2(SL_n(\mathbb{Z}), E) = 0$ for $n \geq 9$.

Application: The real K-theory of number rings. Quillen defined the K-groups of a ring R as the homotopy groups of the H-space $BGL(R)^+$: $K_i(R) = \pi_i(BGL(R)^+)$. Note that the universal cover of the space $BGL(R)^+$ is (usually) $BSL(R)^+$. Thus, $K_i(R) \cong \pi_i(BSL(R)^+)$ for $i \geq 2$.

By the Milnor–Moore theorem[**82**], there is an injection

$$\pi_i(BGL(R)^+) \otimes \mathbb{R} \longrightarrow H_i(BGL(R)^+, \mathbb{R})$$

with image equal to the space of indecomposable elements. Thus, the calculation above gives the following.

THEOREM 1.4.6. *Let k be a number field with ring of integers $\mathcal{O}_k$. Then for $i \geq 2$, $\dim(K_i(\mathcal{O}_k) \otimes \mathbb{R})$ is periodic of period 4 and is equal to $0, r_1+r_2, 0, r_2$ depending on whether $i \equiv 0, 1, 2, 3 \mod 4$.* □

A complete description of the torsion subgroups of K-groups of number rings remains elusive. Recent calculations by Rognes–Weibel [**104**] give the 2-torsion.

1.5. Unstable Calculations

Twenty years after publishing [**14**], Borel, in collaboration with J. Yang, successfully computed the real cohomology algebras of $SL_n(k)$ for all number fields k. This calculation is carried out in [**17**] and relies on a hard analytical theorem of Blasius–Franke–Grunewald [**10**].

Before describing the result, we need to establish some notation. Let k be a number field, V_∞ (resp. V_f) the set of archimedean (resp. non-archimedean) places of k and $V = V_\infty \cup V_f$. If $S \subset V$, define $S_f = S \cap V_f$. The completion of k at v is denoted by k_v.

Let G be an affine algebraic group over k. If S is a finite subset of V, define $r(S, G)$ to be the sum of the k_v ranks of G for $v \in S$. Set

$$G_\infty = G(k \otimes_{\mathbb{Q}} \mathbb{R}) = \prod_{v \in V_\infty} G(k_v),$$

which is viewed as a real Lie group.

If M is a real Lie group and $E = \mathbb{R}$ or $\mathbb{C}$, then $H^i_{\mathrm{ct}}(M, E)$ denotes the ith continuous cohomology group of G with coefficients in E; i.e., cohomology based on continuous (or differentiable) cochains (see e.g., IX in [**16**]). Let M^δ be M viewed as a discrete group and let

$$f : H^\bullet_{\mathrm{ct}}(M, E) \longrightarrow H^\bullet(M^\delta, E)$$

be the natural map. The main result of [**17**] is the following.

THEOREM 1.5.1. *Let G be a connected, simply connected, almost absolutely simple k-group of strictly positive k-rank $r_k(G)$. Then the natural map*

$$\mu : H^\bullet_{\mathrm{ct}}(G_\infty, \mathbb{R}) \longrightarrow H^\bullet(G(k), \mathbb{R})$$

which is a composition of f with the restriction map

$$H^\bullet(G^\delta_\infty, \mathbb{R}) \longrightarrow H^\bullet(G(k), \mathbb{R})$$

is an isomorphism.

PROOF. Let S be a finite subset of V containing V_∞ and at least one finite place ($S_f \neq \emptyset$). Let Γ_S be a congruence S-arithmetic subgroup of $G(k)$; e.g., if $G \hookrightarrow GL_N$, we may take

$$\Gamma_S = G(\mathcal{O}_S) := G(k) \cap GL_N(\mathcal{O}_S),$$

where $\mathcal{O}_S$ is the ring of S-integers of k. By Theorem 1 of [**10**],

$$H^\bullet(\Gamma_S, \mathbb{R}) \cong H^\bullet_{\text{ct}}(G_\infty, \mathbb{R}) \oplus H^\bullet_{\text{cusp}}(\Gamma_S, \mathbb{R}).$$

The second summand is *cuspidal cohomology.*

The relevant property of $H^\bullet_{\text{cusp}}(\Gamma_S, \mathbb{R})$ is that it vanishes in degrees less than $r(S_f, G)$. As S runs through a strictly increasing sequence of subsets of V, $r(S_f, G) \to \infty$. Thus if $i \in \mathbb{N}$, we have

$$H^i(\Gamma_S, \mathbb{R}) \cong H^i_{\text{ct}}(G_\infty, \mathbb{R})$$

if S is sufficiently large. Since $k = \varinjlim_{S \subset V} \mathcal{O}_S$, we have $G(k) = \varinjlim_{S \subset V} G(\mathcal{O}_S)$. Thus, for a fixed i, duality implies that

$$H_i(G(\mathcal{O}_S), \mathbb{R}) \cong H_i(G(k), \mathbb{R})$$

for S sufficiently large (depending on i). Dualizing again, we see that

$$H^i(G(k), \mathbb{R}) \xrightarrow{\cong} H^i(G(\mathcal{O}_S), \mathbb{R})$$

for large S and hence

$$H^i(G(k), \mathbb{R}) \xrightarrow{\cong} H^i_{\text{ct}}(G_\infty, \mathbb{R}).$$

One checks easily that this isomorphism is the one claimed in the statement of the theorem. □

Theorem 1.5.1 can be used to compute the algebra $H^\bullet(SL_n(k), \mathbb{R})$ as follows. Let $G_n = SL_n$ be viewed as an algebraic group over k. Then

$$H^\bullet_{\text{ct}}(G_{n,\infty}, \mathbb{R}) \cong \bigotimes_{v \in V_\infty} H^\bullet_{\text{ct}}(SL_n(k_v), \mathbb{R}).$$

It is known, via [**16**], that

$$H^\bullet_{\text{ct}}(SL_n(k_v), \mathbb{R}) = \begin{cases} H^\bullet(SU_n/SO_n, \mathbb{R}) & v \text{ real} \\ H^\bullet(SU_n, \mathbb{R}) & v \text{ complex.} \end{cases}$$

The values of the right hand side were computed by Borel in [**13**]. They are

$$\begin{aligned} H^\bullet(SU_n/SO_n, \mathbb{R}) &= \begin{cases} \bigwedge\langle x_{n,5}, x_{n,9}, \ldots, x_{n,2n-1}\rangle & n \text{ odd} \\ \bigwedge\langle x_{n,5}, x_{n,9}, \ldots, x_{n,2n-3}, e_n\rangle & n \text{ even} \end{cases} \\ H^\bullet(SU_n, \mathbb{R}) &= \bigwedge\langle x_{n,3}, x_{n,5}, \ldots x_{n,2n-1}\rangle, \end{aligned}$$

where the second subscript of $x_{n,j}$ denotes the degree and $\deg(e_n) = n$. Also, if $m > n$, then $x_{m,i} \mapsto x_{n,i}$ under the induced map

$$H^i(SL_m(k_v), \mathbb{R}) \longrightarrow H^i(SL_n(k_v), \mathbb{R}).$$

By Theorem 1.5.1, $H^\bullet(SL_n(k), \mathbb{R}) \cong H^\bullet_{\mathrm{ct}}(SL_{n,\infty}, \mathbb{R})$. For example,

$$\begin{aligned} H^\bullet(SL_2(\mathbb{Q}), \mathbb{R}) &= \bigwedge\langle e_2\rangle \\ H^\bullet(SL_2(\mathbb{Q}(i)), \mathbb{R}) &= \bigwedge\langle x_{2,3}\rangle \\ H^\bullet(SL_3(\mathbb{Q}(\sqrt{2})), \mathbb{R}) &= \bigwedge\langle x_{3,5}\rangle. \end{aligned}$$

1.6. Congruence Subgroups

The methods used in Section 1.4 can be applied equally well to subgroups of finite index in $SL_n(\mathbb{Z})$. In particular, we can consider the principal congruence subgroups $\Gamma(n,k)$ defined by the short exact sequence

$$1 \longrightarrow \Gamma(n,k) \longrightarrow SL_n(\mathbb{Z}) \xrightarrow{\bmod k} SL_n(\mathbb{Z}/k) \longrightarrow 1.$$

With rational coefficients, the cohomology of the groups $\Gamma(n,k)$ and $SL_n(\mathbb{Z})$ are related by the simple formula

$$H^i(SL_n(\mathbb{Z}), \mathbb{Q}) \cong H^i(\Gamma(n,k), \mathbb{Q})^{SL_n(\mathbb{Z}/k)}$$

(this follows from the Hochschild–Serre spectral sequence). This is one motivation for studying the cohomology of these groups.

A thorough study of congruence subgroups could fill an entire book. There are deep connections with the theory of modular forms, for example, which would take us too far afield. Instead, we present two results which are in keeping with the general theme of this monograph.

1.6.1. Principal congruence subgroups and $K_3(\mathbb{Z})$. Let p be a prime and consider the principal congruence subgroup $\Gamma(n,p) \subset SL_n(\mathbb{Z})$. Denote by $\mathfrak{sl}_n(\mathbb{F}_p)$ the Lie algebra of trace-zero matrices over $\mathbb{F}_p$.

THEOREM 1.6.1 ([75]). *Let $n \geq 3$. There is a homomorphism*

$$\varphi : \Gamma(n,p) \longrightarrow \mathfrak{sl}_n(\mathbb{F}_p)$$

whose kernel is the commutator subgroup. Thus, $H_1(\Gamma(n,p), \mathbb{Z}) \cong \mathfrak{sl}_n(\mathbb{F}_p)$.

PROOF. If $X \in \Gamma(n,p)$, we may write $X = I + pA$, where A is an $n \times n$ integral matrix and I is the identity. Define $\varphi(X) = A \mod p$. This is a homomorphism:

$$\begin{aligned} \varphi((I+pA)(I+pB)) &= \varphi(I + p(A+B) + p^2AB) \\ &= A + B \mod p \\ &= \varphi(I+pA) + \varphi(I+pB). \end{aligned}$$

Moreover, if $X = I + pA$, then A has trace zero. Indeed, we have

$$\begin{aligned} 1 = \det X &= \det(I + pA) \\ &\equiv 1 + \operatorname{trace}(A) \mod p^2, \end{aligned}$$

from which it follows that $\operatorname{trace}(A) \equiv 0 \mod p$. The surjectivity of φ is shown as follows. Let e_{ij} be the matrix with 1 in the i, j position and zeroes elsewhere.

Then $\mathfrak{sl}_n(\mathbb{F}_p)$ has basis $\{e_{ij}\}_{i \neq j} \cup \{d_i\}_{i=1}^{n-1}$ where $d_i = e_{ii} - e_{i+1,i+1}$. Then

$$\varphi(I + pe_{ij}) = e_{ij}$$

and

$$\varphi(I + p(e_{i,i+1} - e_{i+1,i} + d_i)) = e_{i,i+1} - e_{i+1,i} + d_i.$$

Thus, φ hits all the basis elements. It follows that we have an exact sequence

$$1 \longrightarrow \Gamma(n, p^2) \longrightarrow \Gamma(n, p) \xrightarrow{\varphi} \mathfrak{sl}_n(\mathbb{F}_p) \longrightarrow 0.$$

Let $E_n(\mathbb{Z}) \subset SL_n(\mathbb{Z})$ be the subgroup generated by all $I + te_{ij}$, $i \neq j$, $t \in \mathbb{Z}$. Let $E(n, p)$ be the normal subgroup of $\Gamma(n, p)$ generated by all $I + te_{ij}$, $i \neq j$, $t \equiv 0 \mod p$. By the work of Bass–Milnor–Serre [**4**], we have

$$\Gamma(n, p^2) = E(n, p^2).$$

Moreover, we clearly have

$$E(n, p^2) \subseteq [E(n, p), E(n, p)], \quad n \geq 3.$$

Thus,

$$\Gamma(n, p^2) = E(n, p^2) \subseteq [E(n, p), E(n, p)];$$

that is, $\Gamma(n, p^2) = [\Gamma(n, p), \Gamma(n, p)]$. □

Lee and Szczarba also obtain an explicit description of the $SL_3(\mathbb{F}_3)$-module $H_2(\Gamma(3, 3))$. These results are applied in [**76**] to show that the group $K_3(\mathbb{Z})$ is cyclic of order 48. This stood as the largest n for which $K_n(\mathbb{Z})$ had been computed completely until 1998 when Rognes [**103**] showed that $K_4(\mathbb{Z}) = 0$.

1.6.2. Congruence subgroups over polynomial rings. Let R be a ring such that $SL_n(R)$ is generated by elementary matrices (e.g. R Euclidean or local) and consider the exact sequence

$$1 \longrightarrow K(R) \longrightarrow SL_n(R[t]) \xrightarrow{t=0} SL_n(R) \longrightarrow 1.$$

The group $K(R)$ consists of those matrices which are congruent to the identity modulo t. Define a sequence of groups

$$K(R) = K^1(R) \supset K^2(R) \supset \cdots$$

by

$$K^i(R) = \{A \in K(R) : A \equiv I \mod t^i\}.$$

One checks easily that $[K^i, K^j] \subseteq K^{i+j}$ and the graded quotients satisfy

$$K^i(R)/K^{i+1}(R) \cong \mathfrak{sl}_n(R).$$

This is proved exactly as in the case above. Thus, we have an exact sequence

$$1 \longrightarrow K^2(R) \longrightarrow K(R) \longrightarrow \mathfrak{sl}_n(R) \longrightarrow 0,$$

and hence, $K^2(R) \supseteq \Gamma^2 K(R)$, where $\Gamma^\bullet$ denotes the lower central series.

Suppose $R = F$ is a field. We studied the group $H_1(K(R))$ in [**68**]. The group $K(F)$ acts on the Bruhat–Tits building associated to $SL_n(F[t])$ (see Section 4.6) and a fundamental domain can be computed. Via this approach we proved the following.

THEOREM 1.6.2. *Let $F = \mathbb{F}_2$ or $\mathbb{F}_3$. If $n = 3$, we have an isomorphism $H_1(K(F), \mathbb{Z}) \cong \mathfrak{sl}_3(F)$.* □

The technique of proof is much too complicated to work for larger finite fields. Indeed, the case $n = 3$, $F = \mathbb{F}_3$ requires the computation of the rank of a 42×25 matrix. However, we expect that the following holds.

CONJECTURE 1.6.3. *If F is a finite field, then for all $n \geq 3$, there is an isomorphism $H_1(K(F), \mathbb{Z}) \cong \mathfrak{sl}_n(F)$.*

This is false in general. For most fields F, there is a surjective map

$$K^2/\Gamma^2 \longrightarrow \Omega^1_F,$$

where Ω^1_F is the module of differentials; this is nonzero in general. Thus, the group $H_1(K(F))$ is strictly larger than $\mathfrak{sl}_n(F)$ since we have an exact sequence

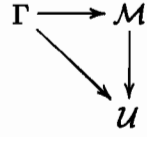
$$0 \longrightarrow K^2/\Gamma^2 \longrightarrow H_1(K(F)) \longrightarrow \mathfrak{sl}_n(F) \longrightarrow 0.$$

Since $\Omega^1_F = 0$ for finite fields, this obstruction is not an issue. Also, one can compute $H_1(K(F))$ when $n = 2$ via the free product decomposition [**66**]

$$K = \amalg_{s \in SL_2(F)/B(F)} sCs^{-1}$$

where $C \subset K$ is the upper triangular subgroup and $B(F)$ is the upper triangular subgroup of $SL_2(F)$. In particular, $H_1(K(F))$ is an infinite-dimensional F-vector space.

1.6.3. Malcev completions. Let Γ be a group. The Malcev completion of Γ is a prounipotent group $\mathcal{M}$ defined over $\mathbb{Q}$ together with a homomorphism $\Gamma \to \mathcal{M}$ satisfying the obvious universal mapping property: If $\Gamma \to \mathcal{U}$ is another map into a prounipotent group, then there is a unique map $\mathcal{M} \to \mathcal{U}$ making the diagram



commute. If G is nilpotent, the Malcev completion $\hat{G}$ can be identified with the set of "grouplike" elements of the completed group algebra $\hat{\mathbb{Q}}G$ (completed with respect to the augmentation ideal). Quillen [**96**] characterized $\hat{G}$ by the following three properties:

1. $\hat{G}$ is nilpotent and uniquely divisible;
2. The kernel of $j : G \to \hat{G}$ is the torsion subgroup of G;
3. If $x \in \hat{G}$, then $x^n \in \mathrm{im}(j)$ for some $n \neq 0$.

If G is an arbitrary group, denote by G_r the nilpotent group $G/\Gamma^r G$. Define the Malcev completion of G to be

$$\mathcal{M} = \varprojlim \hat{G}_r.$$

This group is easily seen to satisfy the universal mapping property.

A quick check of the definitions shows that if $H_1(\Gamma, \mathbb{Q}) = 0$, then the Malcev completion of Γ is trivial. Deligne suggested the following notion of relative completion to cover this situation. Suppose S is a semisimple group over $\mathbb{Q}$ and that $\rho : \Gamma \to S$ is a representation with Zariski dense image. The completion of Γ with respect to ρ is a proalgebraic group $\mathcal{G}$ over $\mathbb{Q}$ which is an extension of S by a prounipotent group $\mathcal{P}$, and a lifting $\tilde{\rho} : \Gamma \to \mathcal{G}$ of ρ with Zariski dense image. The group $\mathcal{G}$ is required to satisfy the obvious universal mapping property. If S is the trivial group, then this reduces to the usual Malcev completion.

The basic properties of relative completion were worked out by Hain [**52**]. He used this idea in [**53**] to study the relative completions of the mapping class groups $\Gamma_{g,r}$ of orientable surfaces. With this approach, he was able to provide a lower bound on the second cohomology of the Torelli group $T_{g,r}$, which is the subgroup of $\Gamma_{g,r}$ consisting of elements that act trivially on the first homology of the surface. While he was able to compute the completion $\mathcal{G}_{g,r}$ of $\Gamma_{g,r}$ with respect to the map $\Gamma_{g,r} \to Sp_{2g}(\mathbb{Z})$, the homomorphism $\Gamma_{g,r} \to \mathcal{G}_{g,r}$ is a bit mysterious.

In [**72**], we computed the completions of the groups $SL_n(\mathbb{Z}[t])$ and $SL_n(\mathbb{Z}[t, t^{-1}])$ with respect to the homomorphisms

$$SL_n(\mathbb{Z}[t]) \xrightarrow{t=0} SL_n(\mathbb{Z}) \longrightarrow SL_n(\mathbb{Q})$$

and

$$SL_n(\mathbb{Z}[t, t^{-1}]) \xrightarrow{t=1} SL_n(\mathbb{Z}) \longrightarrow SL_n(\mathbb{Q}).$$

The basic idea is to compute the ordinary Malcev completion of the kernel $K(R)$ ($R = \mathbb{Z}[t], \mathbb{Z}[t, t^{-1}]$); denote this completion by $\mathcal{M}(R)$. If we denote by

$\mathcal{G}(R)$ the relative completion of $SL_n(R)$, then we have a commutative diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & K(R) & \longrightarrow & SL_n(R) & \longrightarrow & SL_n(\mathbb{Z}) & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & \mathcal{P}(R) & \longrightarrow & \mathcal{G}(R) & \longrightarrow & SL_n(\mathbb{Q}) & \longrightarrow & 1. \end{array}$$

The universal mapping property of $\mathcal{M}(R)$ implies that we have a unique map $\mathcal{M}(R) \to \mathcal{P}(R)$ through which $K(R) \to \mathcal{P}(R)$ factors.

Note that there is an obvious candidate for $\mathcal{G}(R)$; namely, the proalgebraic group $SL_n(\mathbb{Q}[[T]])$. We have a commutative diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & K(R) & \longrightarrow & SL_n(R) & \longrightarrow & SL_n(\mathbb{Z}) & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & \mathcal{U} & \longrightarrow & SL_n(\mathbb{Q}[[T]]) & \xrightarrow{T=0} & SL_n(\mathbb{Q}) & \longrightarrow & 1 \end{array}$$

where $SL_n(R) \to SL_n(\mathbb{Q}[[T]])$ is induced by $t \mapsto T$ for $R = \mathbb{Z}[t]$ and by $t \mapsto 1 + T$ for $R = \mathbb{Z}[t, t^{-1}]$. The group $\mathcal{U}$ is the prounipotent subgroup of matrices congruent to I modulo T.

Using the basic properties of relative completion in [**52**] and some results of Bousfield [**19**], we obtain the following results.

PROPOSITION 1.6.4. *If $n \geq 4$, the map $K(R) \to \mathcal{U}$ is the Malcev completion.*

PROOF. The first step is to show that for each r, the map

$$K(R)/K^r(R) \longrightarrow \mathcal{U}/\mathcal{U}^r$$

(where $\mathcal{U}^\bullet$ is the T-adic filtration in $\mathcal{U}$) is the Malcev completion of the nilpotent group $K(R)/K^r(R)$. This is straightforward from Quillen's criteria. Now, the kernel of the map $K/\Gamma^r K \to \mathcal{U}/\mathcal{U}^r$ is the abelian group $K^r/\Gamma^r K$. It remains to show that this group is torsion (it certainly contains the torsion subgroup of $K/\Gamma^r K$). For $n \geq 4$, we have an isomorphism (see e.g., [**122**])

$$K^r/E_n(R, \mathfrak{m}^r) \xrightarrow{\cong} K_1(R, \mathfrak{m}^r),$$

where $E_n(R, \mathfrak{m}^r)$ is the normal subgroup of $SL_n(R)$ generated by the matrices $I + e_{ij}(x)$, $i \neq j$, $x \in \mathfrak{m}^r$ ($\mathfrak{m} = (t)$, resp. $(t-1)$). We have $E_n(R, \mathfrak{m}^r) \subseteq \Gamma^r K \subseteq K^r$. By the work of Roberts [**102**], the group $K_1(R, \mathfrak{m}^r)$ is torsion. This completes the proof. □

REMARK 1.6.5. This result is false for $n = 2$. The proof breaks down in this case because the Lie algebra $\mathfrak{sl}_2(\mathbb{Z})$ is not perfect. It is probably true for $n = 3$, but we do not have a proof.

COROLLARY 1.6.6. *If $n \geq 4$, then $H_1(K(R), \mathbb{Q}) = \mathfrak{sl}_n(\mathbb{Q})$.*

PROOF. We have an exact sequence

$$1 \longrightarrow K^2(R)/\Gamma^2 K(R) \longrightarrow H_1(K(R), \mathbb{Z}) \longrightarrow \mathfrak{sl}_n(\mathbb{Z}) \longrightarrow 0.$$

The kernel is torsion. □

THEOREM 1.6.7. *If $n \geq 4$, then $SL_n(R) \to SL_n(\mathbb{Q}[[T]])$ is the completion with respect to the map $SL_n(R) \to SL_n(\mathbb{Q})$.*

PROOF. Let $\mathcal{P}$ denote the prounipotent radical of the group $\mathcal{G}(R)$. The map $\mathcal{U} \to \mathcal{P}$ is surjective [**52**] and its kernel is central in $\mathcal{U}$. Since $\mathcal{U}$ has trivial center, the map is an isomorphism. Since $\mathcal{G}(R)$ is the semidirect product of $SL_n(\mathbb{Q})$ and $\mathcal{P}$, we must have $\mathcal{G}(R) \cong SL_n(\mathbb{Q}[[T]])$. □

These results can be used to obtain information about the groups $H^2(SL_n(R), \mathbb{Q})$. One expects that these groups vanish. Indeed, if $n \geq 5$, then using van der Kallen's stability theorem 2.1.3 and the fundamental theorem of algebraic K-theory, we see that $H^2(SL_n(R), \mathbb{Q}) = 0$. The cases $n = 3, 4$ were treated in [**72**] (the case $n = 3$ requires the conjectural result that $\mathcal{U}$ is the Malcev completion of K).

Consider the Hochschild–Serre spectral sequence associated to the extension

$$1 \longrightarrow K(R) \longrightarrow SL_n(R) \longrightarrow SL_n(\mathbb{Z}) \longrightarrow 1.$$

To show that $H^2(SL_n(R), \mathbb{Q}) = 0$, it suffices to show

1. $H^2(SL_n(\mathbb{Z}), \mathbb{Q}) = 0$;
2. $H^1(SL_n(\mathbb{Z}), H^1(K(R), \mathbb{Q})) = 0$;
3. $H^0(SL_n(\mathbb{Z}), H^2(K(R), \mathbb{Q})) = 0$.

The first statement is clear since $H_2(SL_n(\mathbb{Z}), \mathbb{Z})$ is torsion for all n. The second follows from [**101**] since $H^1(K(R), \mathbb{Q})$ is the adjoint representation $\mathfrak{sl}_n(\mathbb{Q})$.

To obtain partial information about the third statement, we use continuous cohomology [**51**]. Let π be a group and define

$$H^\bullet_{\text{cts}}(\pi, \mathbb{Q}) = \varinjlim H^\bullet(\pi/\Gamma^r\pi, \mathbb{Q}).$$

The group $H^k_{\text{cts}}(\pi, \mathbb{Q})$ coincides with $H^k(\pi, \mathbb{Q})$ for $k = 0, 1$ and the natural map

$$H^2_{\text{cts}}(\pi, \mathbb{Q}) \longrightarrow H^2(\pi, \mathbb{Q})$$

is injective. A group π is called *pseudonilpotent* if its continuous and ordinary cohomology coincide. Examples of pseudonilpotent groups include free groups and the pure braid groups.

If $\mathcal{P}$ is the Malcev completion of π and $\mathfrak{p}$ is the Lie algebra of $\mathcal{P}$, then if $H_1(\pi, \mathbb{Q})$ is finite dimensional, the natural map

$$H^\bullet_{\text{cts}}(\pi, \mathbb{Q}) \longrightarrow H^\bullet_{\text{cts}}(\mathfrak{p}, \mathbb{Q})$$

is an isomorphism. Note that this is the case for the group $K(R)$.

THEOREM 1.6.8. *If $n \geq 3$, then $H^0(SL_n(\mathbb{Z}), H^2_{\text{cts}}(K(R), \mathbb{Q})) = 0$.*

PROOF. Let $\mathfrak{u}$ be the Lie algebra of $\mathcal{U}$. The algebra $\mathfrak{u}$ is the inverse limit of the nilpotent Lie algebras $\mathfrak{u}/\mathfrak{u}^l$. To prove the theorem, it suffices to show that for each l,

$$H^0(SL_n(\mathbb{Z}), H^2(K(R)/K^l(R), \mathbb{Q})) = 0.$$

To do this, one uses Lie algebra cohomology and induction on l; the case $l = 2$ is obvious since

$$H^2(K(R)/K^2(R), \mathbb{Q}) = H^2(\mathfrak{sl}_n(\mathbb{Z}), \mathbb{Q}) = \bigwedge^2 \mathfrak{sl}_n(\mathbb{Q}),$$

and direct computation shows that this group has no invariant submodule. The general case follows by considering the Hochschild–Serre spectral sequence for Lie algebra cohomology associated to the central extension

$$0 \longrightarrow \mathfrak{u}^l/\mathfrak{u}^{l+1} \longrightarrow \mathfrak{u}/\mathfrak{u}^{l+1} \longrightarrow \mathfrak{u}/\mathfrak{u}^l \longrightarrow 0.$$

□

Exercises

1. Show that $1 - \Psi^q$ acts as multiplication by $1 - q^i$ on $\pi_{2i}(BU)$.
2. Let G be a finite group and let E be a representation over $k = \mathbb{F}_q$. Let $\overline{E} = E \otimes_k \overline{k}$. Show that the eigenvalues $\{\lambda_i(g)\}$ are stable under the Frobenius map $x \mapsto x^q$.
3. Show that if l is odd, then the index of $\Sigma_m \wr GL_r(\mathbb{F}_q)$ in $GL_n(\mathbb{F}_q)$ is prime to l.
4. Verify the inequality for $\mathrm{PS}(H^\bullet(U))$ given in the proof of Theorem 1.1.6.
5. Let j_b, $0 \leq b < d$ be nonnegative integers such that

$$\sum_{b=0}^{d-1} j_b p^b \equiv 0 \mod (p^d - 1).$$

If not all the j_b are zero, show that $\sum_b j_b \geq d(p-1)$. (Hint: choose a solution $\{j_b\}$ with $\sum j_b$ minimal and positive. Argue that each $j_b < p$ and use the uniqueness of the p-adic expansion to show that $j_b = p - 1$ for each b.)

6. (Quillen, [**39**]) This exercise provides an alternate proof of Theorem 1.1.6 which also works for the groups SO_{2m}, Sp_{2m} and U_{2m}. First prove the following. Let V be an $\mathbb{F}_q$-vector space and let λ in $\mathbb{F}_q^\times$ act by multiplication on V, where $\lambda^{1/b}$ generates $\mathbb{F}_q^\times$ and b divides $p-1$. Then the induced map

$$\lambda_* : H_n(V, \mathbb{F}_p) \longrightarrow H_n(V, \mathbb{F}_p)$$

has no invariants for $0 < n < d(p-1)/b$ (here, $q = p^d$). (Hint: The structure of $H_*(V, \mathbb{F}_p)$ is given in the proof of Theorem 1.1.6. If λ_* has a fixed point, then the induced map λ_* on $H_n(V \otimes_{\mathbf{F}_p} \overline{\mathbb{F}}_p, \mathbb{F}_p)$ has 1 as an eigenvalue. Examine the action of λ_* on $\bigwedge^i(V \otimes_{\mathbf{F}_p} \overline{\mathbb{F}}_p) \otimes S^j(V \otimes_{\mathbf{F}_p} \overline{\mathbb{F}}_p)$ $(i + 2j = n)$ to obtain a relation like the one in the previous exercise. The previous exercise finishes the proof.)

Now, to prove Theorem 1.1.6, it suffices to prove the vanishing of the $\mathbb{F}_p$-cohomology of $B_n(\mathbb{F}_q)$, the subgroup of upper triangular matrices. Proceed by induction on n, the case $n = 1$ being trivial since $B_1(\mathbb{F}_q) = GL_1(\mathbb{F}_q) = \mathbb{F}_q^\times$ is a group of order prime to p. For the general case, consider the short exact sequence

$$1 \longrightarrow R_n(\mathbb{F}_q) \longrightarrow B_n(\mathbb{F}_q) \longrightarrow B_{n-1}(\mathbb{F}_q) \longrightarrow 1$$

where $R_n(\mathbb{F}_q)$ is the upper row subgroup of $B_n(\mathbb{F}_q)$. One has

$$H^i(R_n(\mathbb{F}_q), \mathbb{F}_p) = H^0(\mathbb{F}_q^\times, H^i(V, \mathbb{F}_p))$$

since $R_n(\mathbb{F}_q)$ is the semidirect product of $\mathbb{F}_q^\times$ and $V \cong \mathbb{F}_q^{n-1}$. The action of $\mathbb{F}_q^\times$ is by multiplication, so the first part of the exercise finishes the proof.

Now, to prove vanishing for the other groups, one needs to construct similar exact sequences of maximal solvable subgroups. These take the form

$$1 \longrightarrow R(SO_{2m}(\mathbb{F}_q)) \longrightarrow \Delta(SO_{2m}(\mathbb{F}_q)) \longrightarrow B_m(\mathbb{F}_q) \longrightarrow 1$$
$$1 \longrightarrow R(Sp_{2m}(\mathbb{F}_q)) \longrightarrow \Delta(Sp_{2m}(\mathbb{F}_q)) \longrightarrow B_m(\mathbb{F}_q) \longrightarrow 1$$
$$1 \longrightarrow R(U_{2m}(\mathbb{F}_q)) \longrightarrow \Delta(U_{2m}(\mathbb{F}_q)) \longrightarrow B_m(\mathbb{F}_{q^2}) \longrightarrow 1$$

where $\Delta(G)$ denotes the maximal solvable subgroup of G. This group is defined by taking those matrices which preserve the standard maximal isotropic flag of $\mathbb{F}_q^{2m}$. The action of λ is by multiplication by λ^2. This gives vanishing of the $\mathbb{F}_p$ cohomology of these groups up to dimension $d(p-1)/2$ for SO_{2m} (p odd), and for Sp_{2m} for all p; and up to dimension $d(p-1)$ for U_{2m}. Verify all these assertions. (For U_{2m}, identify this group

with the subgroup of $GL_{2m}(\mathbb{F}_{q^2})$ consisting of matrices M such that

$$M \begin{pmatrix} 0 & I_m \\ I_m & 0 \end{pmatrix} (M^{(q)})^T = \begin{pmatrix} 0 & I_m \\ I_m & 0 \end{pmatrix}$$

where $M^{(q)}$ is obtained from M by raising each entry to the qth power.)

7. Show that for the filtration $K^\bullet(R)$ of the group $K(R)$ of matrices congruent to the identity modulo t, the graded quotients $K^i(R)/K^{i+1}(R)$ are all isomorphic to $\mathfrak{sl}_n(R)$.
8. Let G be a group, let G_r be the nilpotent group $G/\Gamma^r G$ and let $\hat{G}_r$ be the Malcev completion of G_r. Prove that $\mathcal{M} = \varprojlim \hat{G}_r$ is the Malcev completion of G.
9. Show that if $H_1(G, \mathbb{Q}) = 0$, then the Malcev completion of G is trivial.
10. Show that for each r, the map

$$K(R)/K^r(R) \longrightarrow \mathcal{U}/\mathcal{U}^r$$

is the Malcev completion (this is true for $n \geq 2$).

11. Show that free groups are pseudonilpotent.

Chapter 2

Stability

Suppose we have a sequence $G_1 \subset G_2 \subset \cdots \subset G_n \subset \cdots$ of groups. A natural question to ask is the following: for a fixed k, is there an integer $n(k)$ such that the map

$$H_k(G_i) \longrightarrow H_k(G_{i+1})$$

is an isomorphism for $i \geq n(k)$? The answer is certainly no if for example G_n is the free abelian group of rank n, but there are many examples for which stability does happen. For example, there are stability results for the sequence $\{\Sigma_n\}_{n\geq 1}$ of symmetric groups [**89**] and also for the mapping class groups $\{\Gamma_{g,r}\}_{g,r\geq 0}$ of orientable surfaces [**54**] (g = genus, r = number of boundary components); H_k stabilizes at $n = 2k$ in the first case and at $g = 3k$ in the second.

In this chapter we study the stability question for the sequence

$$GL_1(R) \subset GL_2(R) \subset \cdots \subset GL_n(R) \subset \cdots$$

where $GL_n(R)$ is included in $GL_{n+1}(R)$ via

$$A \mapsto \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}.$$

This question has obvious implications for K-theory. If one can show that H_k stabilizes at some $n(k)$:

$$H_k(GL_{n(k)}(R), \mathbb{Z}) \cong H_k(GL(R), \mathbb{Z}),$$

then one need only concentrate on computing $H_k(GL_{n(k)}(R))$. Granted, this may still be a very difficult problem, but in Chapter 4 we shall see that useful techniques are available for studying $H_\bullet(GL_n(R))$ for finite n.

Let us outline the general strategy for proving stability, since all such arguments are essentially the same. Suppose we have a sequence

$$G_1 \subset G_2 \subset \cdots \subset G_n \subset \cdots$$

of groups. The basic idea is to construct a simplicial complex T_n on which G_n acts such that (1) T_n is highly connected, (2) G_n acts transitively on simplices of fixed low dimension, and (3) for some $k < n$, G_k is the stabilizer in G_n of some 0-simplex. One then uses the spectral sequence associated to this set-up to relate the homology of G_n to the homology of the stabilizers. Often the

sequence gives an isomorphism $H_i(G_k) \xrightarrow{\cong} H_i(G_n)$ for i small (depending on n).

It is useful to keep the following example in mind. Consider the fibration $SO_n(\mathbb{R}) \longrightarrow SO_{n+1}(\mathbb{R}) \longrightarrow S^n$ and the resulting long exact homotopy sequence

$$\cdots \to \pi_{i+1}(S^n) \to \pi_i(SO_n(\mathbb{R})) \to \pi_i(SO_{n+1}(\mathbb{R})) \to \pi_i(S^n) \to \cdots .$$

Since S^n is $(n-1)$-connected, we have an isomorphism

$$\pi_i(SO_n(\mathbb{R})) \xrightarrow{\cong} \pi_i(SO_{n+1}(\mathbb{R}))$$

for $i \leq n-2$ and a surjection

$$\pi_{n-1}(SO_n(\mathbb{R})) \longrightarrow \pi_{n-1}(SO_{n+1}(\mathbb{R})).$$

To compute homology, we have to replace the long exact sequence by some sort of spectral sequence, but the philosophy is the same.

The most general result proved with this method is due to van der Kallen [**63**]. We shall not give the complete proof (which occupies more than 30 journal pages), but we shall provide a proof in the case of rings with many units due to Nesterenko–Suslin [**92**]. This special case actually yields a better range of stability than van der Kallen's general result. We then discuss applications of the latter result to the Milnor K-theory of local rings. We also provide some auxiliary stability theorems.

Another approach to proving homology stability is to consider the analogous question for homotopy groups. The homology groups then may be studied via the Hurewicz map. Suslin [**123**] obtained such a result which yields a better stability range than van der Kallen's theorem. We discuss this at the end of the chapter.

2.1. van der Kallen's Theorem

The starting point for van der Kallen's theorem is some unpublished work of Quillen, where a certain simplicial complex is introduced. Quillen showed that for local rings the complex is highly connected and he conjectured that the same is true for finite dimensional noetherian rings. The main result of [**63**] proves this conjecture. In this section we describe Quillen's complex and sketch the proof of van der Kallen's result.

Let R be an associative ring with unit. A vector $(a_1, \ldots, a_n)^T$ in R^n is *unimodular* if the ideal generated by the a_i is R.

DEFINITION 2.1.1. The stable rank of R, denoted $sr(R)$, is the least natural number n such that for every unimodular column $(a_0, \ldots, a_n)^T$ there are $b_1, \ldots, b_n \in R$ such that $(a_1 + a_0 b_1, \ldots, a_n + a_0 b_n)^T$ is also unimodular.

For example, any local ring has stable rank 1.

Let R^∞ denote the free right R-module with basis $e_1, e_2, \ldots$ and let R^n be the submodule generated by $e_1, \ldots, e_n$. A collection of vectors $v_1, \ldots, v_m$ in R^∞ is called *jointly unimodular* if $v_1, \ldots, v_m$ form a basis of a free direct summand.

DEFINITION 2.1.2. Let V be a set. Define $\mathcal{O}(V)$ to be the partially ordered set of ordered sequences of distinct elements of V with partial ordering defined in the following manner: $(v_1, \ldots, v_m) \leq (w_1, \ldots, w_n)$ if there is a strictly increasing function $\phi : \{1, \ldots, m\} \to \{1, \ldots, n\}$ with $v_i = w_{\phi(i)}$. A subset F of $\mathcal{O}(V)$ satisfies the *chain condition* if for each $(w_1, \ldots, w_n) \in F$, we have $(v_1, \ldots, v_m) \in F$ whenever $(v_1, \ldots, v_m) \leq (w_1, \ldots, w_n)$. If F is such a subposet, $F_\bullet$ denotes the simplicial set whose d-simplices are the $(v_0, \ldots, v_d)$ in F.

Let $\mathcal{U}$ be the subposet of $\mathcal{O}(R^\infty)$ consisting of jointly unimodular sequences. Such an element $(v_1, \ldots, v_k)$ is called a k-frame.

THEOREM 2.1.3. *Let R be a ring and set $r = sr(R)$. Let δ be 0 or 1. Then $\mathcal{O}(R^n + e_{n+1}\delta) \cap \mathcal{U}$ is $(n - r - 1)$-connected (that is, the geometric realization of the associated simplicial set is $(n - r - 1)$-connected).*

PROOF. Proving the acyclicity of this set (i.e., the vanishing of the reduced homology groups) is not difficult if R has "many units", but the theorem also asserts that the geometric realization is simply connected. While acyclicity suffices for the purpose of computing homology, we sketch the proof of the stronger statement.

The proof follows from some technical lemmas. We first introduce some terminology. If F is a poset and S a subset of F, then for $x \in F$ we write $S^+(x)$ for $\{y \in S : y \geq x\}$ and $S^-(x)$ for $\{y \in S : y \leq x\}$. The link of x in S, denoted $\mathrm{Link}_S(x)$ is

$$\mathrm{Link}_S(x) = \mathrm{Link}_S^-(x) \cup \mathrm{Link}_S^+(x),$$

where $\mathrm{Link}_S^-(x) = \{y \in S : y < x\}$ and $\mathrm{Link}_S^+(x) = \{y \in S : y > x\}$. The geometric realization of the link is the join:

$$|\mathrm{Link}_S(x)| = |\mathrm{Link}_S^-(x)| * |\mathrm{Link}_S^+(x)|.$$

We also have, for $x \notin S$, that $|S \cup \{x\}|$ is obtained from $|S|$ by adding a cone over $|\mathrm{Link}_S(x)|$.

LEMMA 2.1.4. *Let F be a poset, and let S be a subset such that for each $x \in F$, the poset $S^-(x)$ has a supremum in itself. Then S is a deformation retract of F.*

PROOF. Define $r : F \to S$ by $r(x) = \sup(S^-(x))$. Then $r(x) = x$ for $x \in S$. If $i : S \to F$ is the inclusion, then $i(r(x)) \leq x$ for all $x \in F$ so that we have a natural transformation of functors $ir \to \mathrm{id}_F$. This implies that on the level of geometric realizations, we have a homotopy equivalence $i_* r_* \simeq \mathrm{id}$, where $()_*$ denotes the induced map on realization. □

Recall that $\mathcal{U}$ is the set of frames in R^∞. Two frames $(v_1, \ldots, v_n)$ and $(w_1, \ldots, w_m)$ are *transversal* if $(v_1, \ldots, v_n, w_1, \ldots, w_m)$ is also a frame. If $F \subseteq \mathcal{U}$ and $(v_1, \ldots, v_n)$ is a frame, we write $F_{(v_1, \ldots, v_n)}$ for the set of frames $(w_1, \ldots, w_m)$ with $(w_1, \ldots w_m, v_1, \ldots, v_n) \in F$.

LEMMA 2.1.5. *Let $F \subseteq \mathcal{U}$ satisfy the chain condition. Let d be an integer and $(v_1, \ldots, v_m) \in F$ be such that for all $(w_1, \ldots, w_n) \in F^+((v_1, \ldots, v_m))$ the poset $F_{(w_1, \ldots, w_n)}$ is $(d-n)$-acyclic. Then the poset $\mathrm{Link}_F((v_1, \ldots, v_m))$ is $(d-1)$-acyclic.*

PROOF. By hypothesis, for $0 \leq k \leq m-2$ there are natural one-to-one correspondences between k-simplices of $\mathrm{Link}_F^-((v_1, \ldots, v_m))$, subsets of size $k+1$ of $\{1, \ldots, m\}$ and k-dimensional faces of the standard $(m-1)$-simplex. It follows that $|\mathrm{Link}_F^-((v_1, \ldots, v_m))|$ is the barycentric subdivision of the boundary of the standard $(m-1)$-simplex and is hence an $(m-2)$-sphere. Thus we have

$$\begin{aligned} |\mathrm{Link}_F((v_1, \ldots, v_m))| &= |\mathrm{Link}_F^-((v_1, \ldots, v_m))| * |\mathrm{Link}_F^+((v_1, \ldots, v_m))| \\ &= \Sigma^{m-1} |\mathrm{Link}_F^+((v_1, \ldots, v_m))|, \end{aligned}$$

where Σ denotes the suspension functor. To complete the proof, it remains to show that $\mathrm{Link}_F^+((v_1, \ldots, v_m))$ is $(d-m)$- acyclic. This is proved by induction on m. Denote by L the poset $\mathrm{Link}_F^+((v_1, \ldots, v_m))$. Let $P_0 = \{(w_1, \ldots, w_n) \in L : w_n = v_m\}$ and let $P_1 = \{(w_1, \ldots, w_r) \in F :$ for some n with $1 \leq n \leq r, (w_1, \ldots, w_n) \in P_0\}$. Evidently, P_0 is a deformation retract of P_1. If $m = 1$, then P_0 is isomorphic to $F_{(v_1)}$, which is $(d-1)$-acyclic by hypothesis. If $m > 1$, then P_0 is isomorphic to $\mathrm{Link}^+_{F_{(v_m)}}((v_1, \ldots, v_m))$. By the induction hypothesis, this is $((d-1)-(m-1))$-acyclic. Thus for $m \geq 1$, the poset P_1 is $(d-m)$-acyclic. The complement of P_1 in L consists of the elements $(v_1, \ldots, v_m, z_1, \ldots, z_q)$ in F with $q \geq 1$. Let $Q_r = P_1 \cup \{(v_1, \ldots, v_m, z_1, \ldots, z_q) \in F : 1 \leq q \leq r\}$. Then L is the union of the Q_r and it suffices to show that each Q_r is $(d-m)$-acyclic (the case $r = 0$ is already known). Observe that $Q_{r+1} - Q_r$ is discrete; hence one passes from $|Q_r|$ to $|Q_{r+1}|$ by adding a cone over the link in Q_r of each $(v_1, \ldots, v_m, z_1, \ldots z_{r+1})$ in F. By the argument above, this link is isomorphic to the r-fold suspension of $\mathrm{Link}^+_{Q_r}((v_1, \ldots, v_m, z_1, \ldots z_{r+1}))$. Call this latter poset L'. When $m = 1$, L' has as deformation retract

$$\{(w_1, \ldots, w_s, v_1, z_1, \ldots, z_{r+1}) : (w_1, \ldots, w_s) \in F_{(v_1, z_1, \ldots, z_{r+1})}\}.$$

This poset is isomorphic to $F_{(v_1,z_1,\ldots,z_{r+1})}$,and hence is $(d-r-2)$-acyclic (by hypothesis). Thus L' is $(d-r-2)$-acyclic. Similarly, one can show that if $m>1$, then L' is $(d-m-1)$-acyclic. Thus, one passes from $|Q_r|$ to $|Q_{r+1}|$ by adding cones over $(d-m-1)$-acyclic links. Since Q_0 is $(d-m)$-acyclic a Mayer–Vietoris argument shows that Q_r is $(d-m)$-acyclic. □

LEMMA 2.1.6. *Let $F \subseteq \mathcal{U}$ satisfy the chain condition and let $X \subseteq R^\infty$. Assume that $\mathcal{O}(X) \cap F$ is d-connected and that for all frames $(v_1,\ldots,v_k) \in F - \mathcal{O}(X)$, the poset $\mathcal{O}(X) \cap F_{(v_1,\ldots,v_k)}$ is $(d-k)$-connected. Then F is d-connected.*

PROOF. Let $P_0 = \{(v_1,\ldots,v_k) \in F : \text{at least one } v_i \in X\}$. Then by Lemma 2.1.4, $\mathcal{O}(X)\cap F$ is a deformation retract of P_0; hence P_0 is d-connected. Now let $P_q = P_0 \cup \{(w_1,\ldots w_r) \in F : r \leq q\}$. Consider $\mathrm{Link}_{P_q}((w_1,\ldots,w_{q+1}))$ for some $(w_1,\ldots,w_{q+1}) \in P_{q+1} - P_q$. Let

$$Q = \mathcal{O}(X \cup \{(w_1,\ldots,w_{q+1})\}) \cap F.$$

Then $\mathrm{Link}_{P_q}((w_1,\ldots,w_{q+1}))$ has $\mathrm{Link}_Q((w_1,\ldots,w_{q+1}))$ as a deformation retract and this retract is $(d-1)$-acyclic by Lemma 2.1.5.

Thus, one passes from $|P_q|$ to $|P_{q+1}|$ by adding cones over $(d-1)$- acyclic links. It follows easily that P_q is d-acyclic for all $q \geq 0$, and when $d \geq 1$ the van Kampen theorem shows that P_q is simply connected (hence d-connected). Now note that $F = \bigcup_{q\geq 0} P_q$. □

We now complete the proof of Theorem 2.1.3. Let $F = \mathcal{O}(R^n + e_{n+1}\delta) \cap \mathcal{U}$ and set $d = n-r-1$ $(r = sr(R))$. We must show that F is d-connected. When $n=0$, this is clear. If $n>0$, choose $X = (R^{n-1}+e_{n+1}\delta)\cup(R^{n-1}+e_n+e_{n+1}\delta)$. By the induction hypothesis $\mathcal{O}(X)\cap F$ is d-connected since X is the same as $(R^{n-1}+e_n\delta)\cup(R^{n-1}+e_n\delta+e_{n+1})$ up to a change of coordinates. One checks easily that all the conditions of Lemma 2.1.6 are satisfied and hence F is d-connected. □

Van der Kallen then uses this result to study the stabilization of the homology of $GL_n(R)$ and $E_n(R)$. His theorem is as follows.

THEOREM 2.1.7 ([**63**], Theorem 4.11). *Let $e = \max(1, sr(R)-1)$. Then the maps*

$$H_m(GL_n(R),\mathbb{Z}) \to H_m(GL_{n+1}(R),\mathbb{Z})$$

and

$$H_m(E_n(R),\mathbb{Z}) \to H_m(E_{n+1}(R),\mathbb{Z})$$

are surjective for $n \geq 2m+e-1$ and injective for $n \geq 2m+e$. □

We shall not prove this here. The proof involves a careful analysis of the spectral sequence obtained from the action of $GL_n(R)$ and $E_n(R)$ on the geometric realization of the poset $\mathcal{O}(R^n) \cap \mathcal{U}$. In the next section, we provide a proof for rings with many units which gives a better range of stability than the above result. The argument used to prove Theorem 2.1.7 is essentially the same as the proof given below except that it works in greater generality.

Van der Kallen also studied homology with twisted coefficients. For example, one has the standard representation R^n of $GL_n(R)$ or the adjoint representation $\mathfrak{gl}_n(R)$. If ρ_n is a representation of $GL_n(R)$, then van der Kallen proves that $H_m(GL_n(R), \rho_n)$ stabilizes at approximately $n = 2m + r - 1 + k$, where k is the "degree" of the coefficient system. For example $\rho_n = (R^n)^{\otimes k}$ has degree k. Details may be found in Section 5 of [**63**].

2.2. Stability for rings with many units

DEFINITION 2.2.1. A ring A is an $S(n)$-ring if there are $a_1, \ldots, a_n \in A^\times$ such that the sum of each nonempty subfamily is a unit. If A is an $S(n)$-ring for all n, then we say that A has many units.

Some examples of rings with many units are the following.

1. Infinite fields. Indeed, any field is clearly an $S(1)$-ring. Suppose F is an infinite field and assume inductively that F is an $S(n)$ ring. Let $a_1, \ldots, a_n$ be a family of nonzero elements of F such that the sum of each nonempty subfamily is also nonzero. Since F is infinite, we can choose a nonzero element a_{n+1} such that $-a_{n+1}$ is not equal to the sum of any subfamily of $\{a_1, \ldots, a_n\}$. Thus F is an $S(n+1)$-ring.
2. Local rings with infinite residue fields. For each n we construct a set $a_1, \ldots, a_n$ with the requisite property by lifting such a set from the residue field.
3. Algebras over rings with many units. This is left as an exercise.

In [**92**], Nesterenko and Suslin prove a strengthened version of Theorem 2.1.3 for rings with many units. This is based upon a computation of the homology of certain affine groups of the form

$$\mathrm{Aff}_{nm}(A) = \begin{pmatrix} GL_n(A) & M_{nm}(A) \\ 0 & I_m \end{pmatrix}$$

where $M_{nm}(A)$ denotes the ring of $n \times m$ matrices over A. We first prove this result and then prove the stability theorem.

2.2.1. Homology of affine groups.

THEOREM 2.2.2. *Let A be a ring with many units. Then the inclusion*

$$i : GL_n(A) \longrightarrow \mathrm{Aff}_{nm}(A)$$

induces an isomorphism on integral homology.

REMARK 2.2.3. This theorem is almost trivial if A has characteristic zero. Indeed, we have the Hochschild–Serre spectral sequence

$$E^2_{p,q} = H_p(GL_n(A), H_q(M_{nm}(A))) \Longrightarrow H_{p+q}(\mathrm{Aff}_{nm}(A))$$

and it suffices to show that $E^2_{p,q} = 0$ for $q > 0$. If $\mathrm{char}(A) = 0$, then $M_{nm}(A)$ is a torsion-free abelian group so that $H_q(M_{nm}(A)) = \bigwedge^q M_{nm}(A)$. The central subgroup $A^\times \subset GL_n(A)$ then contains an element a which acts as multiplication by a^q on $H_q(M_{nm}(A))$ and hence in the same way on $E^2_{p,q}$ for all $p \geq 0$. However, since $A^\times$ is central, this action must be the identity and so $H_p(GL_n(A), H_q(M_{nm}(A)))$ is annihilated by $a^q - 1$. Since this space is a rational vector space (A is a $\mathbb{Q}$-algebra) it must vanish for $q > 0$.

That this same argument works in positive characteristic is much more delicate. We need several lemmas.

LEMMA 2.2.4. *Let $\varphi_1, \ldots, \varphi_n : A \to B$ be ring homomorphisms such that $\prod_{i=1}^n \varphi_i(x) = 1$ for all $x \in A^\times$. Then A is not an $S(n+1)$-ring.*

PROOF. Suppose there are elements $a_1, \ldots, a_{n+1} \in A^\times$ such that for any $1 \leq j_1 \leq \cdots \leq j_k \leq n+1$ the element $a_{j_1} + \cdots + a_{j_k} \in A^\times$. Set $I = \{1, \ldots, n+1\}$. For any $s = (m_1, \ldots, m_n) \in I^n$, set $\pi(s) = \prod_{i=1}^n \varphi_i(a_{m_i})$ and $p(s) = \{m_1, \ldots, m_n\} \subseteq I$. Let $J = \{j_1 < \cdots < j_k\} \subseteq I$. Then by hypothesis

$$1 = \prod_{i=1}^n \varphi_i(a_{j_1} + \cdots + a_{j_k}) = \sum_{s \in J^n} \pi(s).$$

We prove by induction on $k = |J|$ that $\sum_{p(s)=J} \pi(s) = (-1)^{k+1}$. If $k = 1$, then $p(s) = J$ for any $s \in J^n$ and the formula above proves the result. In the general case, the induction hypothesis gives

$$\begin{aligned}
\sum_{p(s)=J} \pi(s) &= \sum_{s \in J^n} \pi(s) - \sum_{K \subset J} \sum_{p(s)=K} \pi(s) \\
&= 1 - \sum_{K \subset J} (-1)^{|K|+1} \\
&= 1 + \sum_{m=1}^{k-1} \binom{k}{m} (-1)^m \\
&= (1-1)^k - (-1)^k \\
&= (-1)^{k+1}.
\end{aligned}$$

Applying this to I, we obtain a contradiction:

$$0 = \sum_{p(s)=I} \pi(s) = (-1)^n.$$

(The first equality follows since no sequence $s = (m_1, \ldots, m_n) \in I^n$ satisfies $p(s) = I$.) □

From now on, assume that A has many units. If B is an abelian group, denote by $S_n(B)$ the group $(B^{\otimes n})^{\Sigma_n}$ of symmetric tensors (Σ_n denotes the symmetric group on n elements). Denote by $\Gamma(B)$ the algebra of divided powers in B ([21], Chapter V). The ring $\Gamma(B)$ is a graded commutative algebra concentrated in even dimensions. Moreover, $\Gamma_{2n}(B) = S_n(B)$ with multiplication

$$v \cdot w = \sum_{\sigma \in \Sigma_{n+m}/\Sigma_n \times \Sigma_m} \sigma(v \otimes w)$$

($v \in \Gamma_{2n}(B)$, $w \in \Gamma_{2m}(B)$) and the system of divided powers is given by

$$v^{(i)} = \sum_{\sigma \in \Sigma_{ni}/T} \sigma(v \otimes \cdots \otimes v)$$

where $T \subseteq \Sigma_{ni}$ is the subgroup of permutations which permutes the blocks $\{1, \ldots, n\}, \{n+1, \ldots, 2n\}, \ldots$.

LEMMA 2.2.5. *Let $\varphi : A \to B$ be a ring homomorphism and consider the induced $A^\times$-action on $B^{\otimes n}$: $a(b_1 \otimes \cdots \otimes b_n) = \varphi(a)b_1 \otimes \cdots \otimes \varphi(a)b_n$. Then $H_0(A^\times, B^{\otimes n}) = 0$.*

PROOF. Denote by I the ideal of $B^{\otimes n}$ generated by elements of the form $\varphi(a) \otimes \cdots \otimes \varphi(a) - 1$. Then $H_0(A^\times, B^{\otimes n}) = B^{\otimes n}/I$. Suppose I is a proper ideal and denote by φ_i the map $A \to B^{\otimes n}/I$ given by $\varphi_i(a) = 1 \otimes \cdots \otimes \varphi(a) \otimes \cdots \otimes 1$. Then for each $a \in A^\times$, $\prod_{i=1}^n \varphi_i(a) = 1$, which contradicts Lemma 2.2.4. □

Now assume that B is an algebra over $\mathbb{Q}$ or $\mathbb{F}_p$ so that the tensor and exterior powers of B over $\mathbb{Z}$ coincide with those over $\mathbb{Q}$ and $\mathbb{F}_p$.

LEMMA 2.2.6. *Let $\varphi : A \to B$ be a ring homomorphism. Then*

$$H_0(A^\times, S_n(B)) = 0.$$

PROOF. Let $k_1, \ldots, k_r$ be natural numbers with $\sum k_i = n$. Denote by $S_n^{k_1,\ldots,k_r}$ the subspace generated by tensors of the form

$$\sum_{\sigma \in \Sigma_n/\Sigma_{k_1} \times \cdots \times \Sigma_{k_r}} \sigma(\underbrace{b_1 \otimes \cdots \otimes b_1}_{k_1} \otimes \cdots \otimes \underbrace{b_r \otimes \cdots \otimes b_r}_{k_r}).$$

Then $S_n^{k_1,\ldots,k_r}$ is an $A^\times$-invariant subspace of $S_n(B)$ and $S_n(B) = \sum S_n^{k_1,\ldots,k_r}$. Set $S_n^{(i)} = \sum_{r \geq n-i} S_n^{k_1,\ldots,k_r}$. Then the $S_n^{(i)}$ form an $A^\times$-invariant filtration.

One checks easily that if $r = n - i$ and $n = k_1 + \cdots + k_r$ then the map $B^r \to S_n^{(i)}/S_n^{(i-1)}$ given by

$$(b_1, \ldots, b_r) \mapsto \sum_{\sigma \in \Sigma_n/\Sigma_{k_1} \times \cdots \times \Sigma_{k_r}} \sigma(b_1 \otimes \cdots \otimes b_1 \otimes \cdots \otimes b_r \otimes \cdots \otimes b_r) \mod S_n^{(i-1)}$$

is multilinear and hence gives an $A^\times$-equivariant map $B^{\otimes r} \to S_n^{(i)}/S_n^{(i-1)}$. Thus we obtain an $A^\times$-equivariant surjection

$$\bigoplus_{k_1 + \cdots + k_r = n} B^{\otimes r} \longrightarrow S_n^{(i)}/S_n^{(i-1)}.$$

Since H_0 is a right exact functor, we have a surjection

$$H_0(A^\times, \bigoplus B^{\otimes r}) \longrightarrow H_0(A^\times, S_n^{(i)}/S_n^{(i-1)});$$

Lemma 2.2.5 then shows that

$$H_0(A^\times, S_n^{(i)}/S_n^{(i-1)}) = 0.$$

By induction on i, we see that $H_0(A^\times, S_n^{(i)}) = 0$ for any i. □

LEMMA 2.2.7. *Let $n_1, n_2 \geq 0$ with $n = n_1 + n_2$. Let P and Q be B-modules. Then $\bigwedge^{n_1}(P) \otimes S_{n_2}(Q)$ has a natural $S_n(B)$-module structure.*

PROOF. Clearly, $P^{\otimes n_1} \otimes Q^{\otimes n_2}$ is a $B^{\otimes n}$-module, hence an $S_n(B)$-module. Moreover, if $\sigma \in \Sigma_{n_1} \times \Sigma_{n_2}$, then $\sigma(vw) = \sigma(v)\sigma(w)$ for all $v \in B^{\otimes n}$, $w \in P^{\otimes n_1} \otimes Q^{\otimes n_2}$. It follows that $P^{\otimes n_1} \otimes S_{n_2}(Q)$ is an $S_n(B)$-submodule of $P^{\otimes n_1} \otimes Q^{\otimes n_2}$. Since the kernel of the projection

$$P^{\otimes n_1} \otimes S_{n_2}(Q) \longrightarrow \bigwedge^{n_1}(P) \otimes S_{n_2}(Q)$$

is generated by tensors $w \in P^{\otimes n_1} \otimes S_{n_2}(Q)$ satisfying $w^{(ij)} = w$ for some transposition (ij), it is naturally an $S_n(B)$-submodule. This proves the lemma. □

LEMMA 2.2.8. *Let Λ be a ring, M a Λ-module, G an abelian group and $\varphi : G \to \Lambda^\times$ a homomorphism. If $H_0(G, \Lambda) = 0$, then $H_p(G, M) = 0$ for all $p \geq 0$.*

PROOF. We have $H_0(G, \Lambda) = \Lambda/I$ where I is generated by all $\varphi(g) - 1$, $g \in G$. By hypothesis, $I = \Lambda$. The group G acts on M via Λ-linear maps so that the groups $H_p(G, M)$ have a natural Λ-module structure. The map $(\tau_g, \sigma_g) : (G, M) \to (G, M)$, where τ_g is conjugation by g and $\sigma_g : M \to M$ is multiplication by g, induces the identity on homology. On the other hand, since G is abelian, $\tau_g = \mathrm{id}$ so that the induced map on homology is multiplication by $\varphi(g) \in \Lambda$. Thus, $H_p(G, M)$ is annihilated by the generators of the ideal I and hence vanishes since $I = \Lambda$. □

REMARK 2.2.9. This last proof is a generalization of the center kills argument given in Appendix A.

COROLLARY 2.2.10. *Under the above hypotheses,*

$$H_p(A^\times, \bigwedge^{n_1}(P) \otimes S_{n_2}(Q)) = 0.$$

PROOF. By Lemma 2.2.6, $H_0(A^\times, S_n(B)) = 0$. Since $\bigwedge^{n_1} P \otimes S_{n_2}(Q)$ is an $S_n(B)$-module by Lemma 2.2.7, the result follows. □

Recall the structure of the homology of an abelian group N (see Appendix A). With rational coefficients we have $H_p(N, \mathbb{Q}) = \bigwedge^p(N \otimes \mathbb{Q})$. With $\mathbb{F}_p$-coefficients, the situation is more complicated, but we have an isomorphism

$$\bigwedge^\bullet(N/p) \otimes \Gamma({}_pN) \longrightarrow H_\bullet(N, \mathbb{F}_p)$$

where ${}_pN$ denotes the p-torsion subgroup of N. Unfortunately, this isomorphism is not canonical since it depends on a choice of section of the map $H_2(N, \mathbb{F}_p) \to {}_pN$. Define a filtration on $H_j(N, \mathbb{F}_p)$ by setting

$$H_j^{(r)} = \bigoplus_{i \le r} \bigwedge^{j-2i}(N/p) \otimes \Gamma_{2i}({}_pN).$$

One checks easily that this filtration is independent of the choice of section and there is a canonical isomorphism

$$H_j^{(r)}/H_j^{(r-1)} \cong \bigwedge^{j-2r}(N/p) \otimes \Gamma_{2r}({}_pN).$$

PROPOSITION 2.2.11. *Let k be a prime field. Then for all $i \ge 0$ and all $j > 0$, we have $H_i(A^\times, H_j(A^s, k)) = 0$ where $A^\times$ acts on A^s diagonally.*

PROOF. Set $B = A^s \otimes k$. Then $H_j(A^s, k)$ is either $\bigwedge^j(B)$ or has an $A^\times$-invariant filtration whose successive factors are $\bigwedge^{j-2i}(B) \otimes \Gamma_{2i}(M)$, where $M = {}_pA^s$ is a $B = A^s/p$-module. In either case, the result follows from Corollary 2.2.10. □

We now prove Theorem 2.2.2. It suffices to check that i induces an isomorphism on homology with coefficients in any prime field k. For this it suffices to show that $H_p(GL_n(A), H_q(M_{nm}(A), k)) = 0$ for $q > 0$ (see Remark 2.2.3). Identify $A^\times$ with the scalar matrices in $GL_n(A)$. Then the Hochschild–Serre spectral sequence

$$H_j(GL_n(A)/A^\times, H_l(A^\times, H_q(M_{nm}(A)))) \Rightarrow H_{j+l}(GL_n(A), H_q(M_{nm}(A)))$$

along with Proposition 2.2.11 finishes the proof. □

2.2.2. The stability theorem. Denote by $U(A^n)$ the simplicial set whose k-simplices are the unimodular frames $(v_0, \dots, v_k)$, $v_i \in A^n$, with the usual face operators

$$\partial_i : (v_0, \dots, v_k) \mapsto (v_0, \dots, \hat{v}_i, \dots, v_k).$$

Let $SU(A^n)$ be the simplicial subset consisting of special unimodular frames; that is, those frames $(v_0, \dots, v_k)$ which can be completed to a basis of A^n.

The following result is easily proved by induction on k.

LEMMA 2.2.12. *Let $(v_1, \dots, v_k)$ be a unimodular frame in A^n with $n \geq k + sr(A)$. Then there is a matrix $\alpha \in E_n(R)$ with $\alpha(v_1, \dots, v_k) = (e_1, \dots, e_k)$.* □

Lemma 2.2.12 implies that for $k \leq n - sr(A) - 1$, the k-simplices of $U(A^n)$ coincide with those of $SU(A^n)$: $SU(A^n)_k = U(A^n)_k$.

LEMMA 2.2.13. *The simplicial set $SU(A^n)$ is $(n - sr(A) - 1)$-acyclic.*

PROOF. Set $r = sr(A)$. By Theorem 2.1.3, the set $U(A^n)$ is $(n - r - 1)$-acyclic; this immediately implies the $(n - r - 2)$-acyclicity of $SU(A^n)$. We need only show that $\tilde{H}_{n-r-1}(SU(A^n))$ vanishes. For this it suffices to show that if $u = (v_0, \dots, v_{n-r}) \in U(A^n)$ is a unimodular frame, then the $(n - r - 1)$-cycle du is a boundary in $SU(A^n)$. Using Lemma 2.2.12 we reduce to the following special case: $u = (e_1, \dots, e_{n-r}, e_1 + \cdots + e_{n-r} + v)$, where $v \in Ae_{n-r+1} + \cdots + Ae_n$ is unimodular. Denote by w the frame $(e_1, \dots, e_{n-r}, e_1 + \cdots + e_{n-r} + v, e_{n-r+1})$. This frame is not unimodular in general, but we do have $u = \partial_{n-r+1}(w)$. Moreover, for $i = 0, \dots, n - r$, $\partial_i(w)$ is clearly a special unimodular frame. Consider the formula

$$0 = d^2(w) = d\left(\sum_{i=0}^{n-r+1} (-1)^i \partial_i(w)\right).$$

From this we conclude that

$$du = (-1)^{n+r} d\left(\sum_{i=0}^{n-r} (-1)^i \partial_i(w)\right)$$

is a boundary in $SU(A^n)$. □

Denote by $C_\bullet(A^n)$ the augmented complex associated to $SU(A^n)$ with dimensions shifted by 1; thus, $C_0 = \mathbb{Z}$ and $C_k = \mathbb{Z}[SU(A^n)_{k-1}]$. By Lemma 2.2.13, we see that $H_i(C_\bullet(A^n)) = 0$ for $i \leq n - sr(A)$. The group $GL_n(A)$ acts naturally on $C_k(A^n)$ (on the left), turning $C_\bullet(A^n)$ into a complex of $GL_n(A)$-modules.

Consider the two spectral sequences associated to the action of $GL_n(A)$ on $C_\bullet(A^n)$ (see Appendix A). The first sequence has

$$E^2_{p,q} = H_p(GL_n(A), H_q(C_\bullet)) = 0$$

for $q \leq n - sr(A)$. Thus, the common limit of the spectral sequences is 0 in dimensions less than $n - sr(A)$.

From now on, assume that A has many units and set $r = sr(A)$. The (transposed) second spectral sequence has $E^1_{p,q} = H_p(GL_n(A), C_q)$. The group $GL_n(A)$ permutes transitively the elements of C_q and the stabilizer of the frame $(e_{n-q}, \ldots, e_n)$ is the affine group

$$\mathrm{Aff}_{n-q,q}(A) = \begin{pmatrix} GL_{n-q}(A) & 0 \\ M_{q,n-q}(A) & I \end{pmatrix}.$$

By Shapiro's lemma and Theorem 2.2.2, we obtain the following result.

LEMMA 2.2.14.

$$\begin{aligned} E^1_{p,q} &= H_p(GL_n(A), C_q) \\ &= H_p(\mathrm{Aff}_{n-q,q}(A), \mathbb{Z}) \\ &= H_p(GL_{n-q}(A), \mathbb{Z}). \end{aligned}$$

We now compute the differential $d^1 : E^1_{p,q} \to E^1_{p,q-1}$ (recall that we have transposed the spectral sequence; i.e., in Appendix A, we have $E^1_{p,q} = H_q(GL_n(A), C_p)$). This map has the form

$$H_p(GL_{n-q}(A), \mathbb{Z}) \longrightarrow H_p(GL_{n-q+1}(A), \mathbb{Z}).$$

Denote by i the embedding $GL_{n-q}(A) \to GL_{n-q+1}(A)$.

LEMMA 2.2.15. *The homomorphism d^1 is zero if q is even and equals i_* if q is odd.*

PROOF. The homomorphism d^1 is induced by $d : C_q \to C_{q-1}$. Since $d = \sum_{k=0}^{q-1}(-1)^k \partial_k$, it suffices to check that for any k, the map $(\partial_k)_*$ coincides with i_*. By construction the isomorphism

$$H_p(GL_n(A), C_q) \longrightarrow H_p(GL_{n-q}(A), \mathbb{Z})$$

is induced by the morphism

$$(GL_{n-q}(A), \mathbb{Z}) \overset{(i_{n-q}, u_q)}{\longrightarrow} (GL_n(A), C_q)$$

where i_{n-q} is the standard embedding and u_q maps 1 to the frame $(e_{n-q+1}, \ldots, e_n)$. The composition $(1_G, \partial_k)(i_{n-q}, u_q)$ coincides with the map $(i_{n-q}, \partial_k u_q)$, while the composition $(i_{n-q+1}, u_{q-1})(i, 1_{\mathbb{Z}})$ agrees with the map (i_{n-q}, u_{q-1}). We must prove that these two compositions induce the same map on homology. Observe that the frame $\partial_k u_q = (e_{n-q+1}, \ldots, \hat{e}_{n-q+k+1}, \ldots, e_n)$ can be obtained from u_{q-1} by multiplication by the permutation matrix σ corresponding to the cyclic permutation $(n-q+k+1, n-q+k, \ldots, n-q+1)$. Since σ commutes with i_{n-q}, we have

$$(i_{n-q}, \partial_k u_q) = (c(\sigma), \sigma)(i_{n-q}, u_{q-1}),$$

where $c(\sigma)$ is conjugation by σ. But conjugation induces the identity on homology. □

COROLLARY 2.2.16.

$$E^2_{p,q} = \begin{cases} \ker(H_p(GL_{n-q}(A),\mathbb{Z}) \longrightarrow H_p(GL_{n-q+1}(A),\mathbb{Z}) & q \text{ odd} \\ \operatorname{coker}(H_p(GL_{n-q-1}(A),\mathbb{Z}) \longrightarrow H_p(GL_{n-q}(A),\mathbb{Z}) & q \text{ even.} \end{cases}$$

PROPOSITION 2.2.17. *The differentials d^r, for $r \geq 2$, are trivial. Hence, $E^2_{p,q} = E^\infty_{p,q}$.*

PROOF. Proceed by induction on n. For $n = 0, 1$, the sequence is concentrated in the first two columns so that the higher differentials vanish. Let $n \geq 2$. If $C_\bullet$ is a complex, denote by $C_\bullet[m]$ the shifted complex with $C_k[m] = C_{k+m}$. Define a homomorphism of complexes $\psi : C_\bullet(A^{n-2})[-2] \to C_\bullet(A^n)$ by

$$\begin{aligned} \psi(v_1,\ldots,v_{q-2}) \;=\; & (v_1,\ldots,v_{q-2},e_{n-1},e_n) \\ & -(v_1,\ldots,v_{q-2},e_{n-1},e_{n-1}+e_n) \\ & +(v_1,\ldots,v_{q-2},e_n,e_{n-1}+e_n). \end{aligned}$$

The map ψ respects the embedding $GL_{n-2}(A) \to GL_n(A)$ and hence induces a map of spectral sequences $\psi_* : E(A^{n-2})[0,-2] \to E(A^n)$. Denote the first sequence by $\tilde{E}$. We have

$$\tilde{E}^1_{p,q} = E(A^{n-2})_{p,q-2} = H_p(GL_{n-q}(A),\mathbb{Z})$$

for $2 \leq q \leq n$ and $\tilde{E}^1_{p,q} = 0$ for $q = 0, 1$. An argument similar to the proof of Lemma 2.2.15 shows that for $2 \leq q \leq n$ the map

$$\psi_* : \tilde{E}^1_{p,q} = H_p(GL_{n-q}(A)) \longrightarrow E^1_{p,q} = H_p(GL_{n-q}(A))$$

is the identity. It follows that $\tilde{E}^2_{p,q} \xrightarrow{\cong} E^2_{p,q}$ for $2 \leq q \leq n$. We now prove that $d^i = 0$ by induction on $i \geq 2$. We know that $\tilde{d}^i$ vanishes by our induction hypothesis on n (i.e., we have assumed that $d^i = 0$ for $k < n$). Moreover, the differentials starting in the first two rows vanish for dimension reasons (i.e., they map into the fourth quadrant). If $q \geq 2$, then by the induction hypothesis on i, $E^i_{p,q} = E^2_{p,q} = \tilde{E}^2_{p,q} = \tilde{E}^i_{p,q}$. Consider the commutative diagram

$$\begin{array}{ccc} \tilde{E}^i_{p,q} & \xrightarrow{\tilde{d}^i=0} & \tilde{E}^i_{p+i-1,q-i} \\ {\scriptstyle\psi_*}\downarrow{\scriptstyle\cong} & & {\scriptstyle\psi_*}\downarrow{\scriptstyle\cong} \\ E^i_{p,q} & \xrightarrow{d^i} & E^i_{p+i-1,q-i}. \end{array}$$

This shows that $d^i = 0$. □

THEOREM 2.2.18. *If A is a ring with many units, then the natural map*

$$H_p(GL_{n-1}(A),\mathbb{Z}) \longrightarrow H_p(GL_n(A),\mathbb{Z})$$

is surjective for $p \leq n - sr(A)$ and bijective for $p \leq n - sr(A) - 1$.

PROOF. The limit of the spectral sequence is 0 in dimensions $\leq n - sr(A)$. Now using 2.2.16 and 2.2.17 we get

$$\operatorname{coker}(H_p(GL_{n-1}(A),\mathbb{Z}) \to H_p(GL_n(A),\mathbb{Z})) = E^2_{p,0} = E^\infty_{p,0} = 0$$

for $p \leq n - sr(A)$ and

$$\ker(H_p(GL_{n-1}(A),\mathbb{Z}) \to H_p(GL_n(A),\mathbb{Z})) = E^2_{p,1} = E^\infty_{p,1} = 0$$

for $p + 1 \leq n - sr(A)$. □

2.3. Local rings and Milnor K-theory

Suppose that A is a local ring with infinite residue field k. Since $sr(A) = 1$, Theorem 2.2.18 implies that $H_n(GL(A)) = H_n(GL_{n+1}(A))$ and that the map $H_n(GL_n(A)) \to H_n(GL_{n+1}(A))$ is surjective. In fact, the latter map is injective and the quotient $H_n(GL_n(A))/H_n(GL_{n-1}(A))$ has a particularly nice description.

Denote by $T_\bullet(A^\times)$ the tensor algebra of the abelian group $A^\times$. Note that $T_0(A^\times) = \mathbb{Z}$ and $T_1(A^\times) = A^\times$. Let I be the homogeneous ideal generated by all elements of the form $a \otimes (1-a)$ for $a, 1-a \in A^\times$. The algebra $T_\bullet(A^\times)/I$ is denoted by $K^M_\bullet(A)$ and is called the Milnor ring of A. We have $K^M_0(A) = \mathbb{Z}$ and $K^M_1(A) = A^\times$. If we denote the image of $a_1 \otimes \cdots \otimes a_n$ in $K^M_n(A)$ by $\{a_1, \ldots, a_n\}$, then multiplication is given by

$$\{a_1, \ldots, a_n\} \cdot \{b_1, \ldots, b_m\} = \{a_1, \ldots, a_n, b_1 \ldots, b_m\}.$$

Moreover, $K^M_n(A)$ can be presented as the abelian group with generators $\{a_1, \ldots, a_n\}$ and relations

1. $\{a_1, \ldots, a_i a'_i, \ldots, a_n\} = \{a_1, \ldots, a_i, \ldots, a_n\} + \{a_1, \ldots, a'_i, \ldots, a_n\}$
2. $\{a_1, \ldots, a_n\} = 0$ if $a_i + a_{i+1} = 1$ for some i.

For any n, m we have natural homomorphisms

$$\begin{aligned} H_n(GL_n(A)) \otimes H_m(GL_m(A)) &\longrightarrow H_{n+m}(GL_n(A) \times GL_m(A)) \\ &\longrightarrow H_{n+m}(GL_{n+m}(A)) \end{aligned}$$

induced by the composition of the homology product with the map induced by the inclusion $GL_n(A) \times GL_m(A) \hookrightarrow GL_{n+m}(A)$. In particular, we have a map

$$\Theta : A^\times \otimes \cdots \otimes A^\times = H_1(GL_1(A)) \otimes \cdots \otimes H_1(GL_1(A)) \longrightarrow H_n(GL_n(A)).$$

Composing with the quotient homomorphism, we obtain a map

$$\Theta : A^\times \otimes \cdots \otimes A^\times \longrightarrow H_n(GL_n(A))/H_n(GL_{n-1}(A)).$$

Moreover, if $a_i + a_j = 1$ for some $i \neq j$, then $\Theta(a_1 \otimes \cdots \otimes a_n) = 0$. Thus, Θ induces a map, still denoted by Θ,

$$K_n^M(A) \longrightarrow H_n(GL_n(A))/H_n(GL_{n-1}(A)).$$

THEOREM 2.3.1. *Let A be a local ring with infinite residue field. Then*

1. *The map $H_n(GL_n(A), \mathbb{Z}) \longrightarrow H_n(GL_{n+1}(A), \mathbb{Z})$ is an isomorphism;*
2. *The map $\Theta : K_n^M(A) \longrightarrow H_n(GL_n(A))/H_n(GL_{n-1}(A))$ is an isomorphism.*

PROOF. Recall the simplicial sets $SU(A^n) \subseteq U(A^n)$. Since A is local, every projective A-module is free, and hence every unimodular frame in A^n is special; that is, $SU(A^n) = U(A^n)$.

Let V, W be finitely generated free A-modules. If V has rank n, then we say that vectors $v_1, \ldots, v_s \in V$ are in *general position* if any $\min(n, s)$ of them are jointly unimodular. Define simplicial sets $U(V, W) \subset GP(V, W)$ as follows. The p-simplices of $U(V, W)$ (resp. $GP(V, W)$) are the frames

$$\begin{pmatrix} v_0 \\ w_0 \end{pmatrix}, \ldots, \begin{pmatrix} v_p \\ w_p \end{pmatrix}; \qquad v_i \in V, w_i \in W$$

such that the vectors $v_0, \ldots, v_p$ are jointly unimodular (resp. are in general position). Denote by $C_\bullet(V, W)$ (resp. $\tilde{C}_\bullet(V, W)$) the associated augmented chain complexes with dimensions shifted by 1. If $V \cong A^n$, then $C_i(V, W) = \tilde{C}_i(V, W)$ for $i \leq n$ and $C_i(V, W) = 0$ for $i > n$.

LEMMA 2.3.2. *For any V, W, the complex $\tilde{C}_\bullet(V, W)$ is acyclic.*

PROOF. Let

$$y = \sum_{j=1}^{N} n_j \left(\begin{pmatrix} v_1^j \\ w_1^j \end{pmatrix}, \ldots, \begin{pmatrix} v_p^j \\ w_p^j \end{pmatrix} \right)$$

be a p-cycle. Since the residue field k is infinite, we can find $v \in V$ such that (denoting by $\overline{x} \in V/\mathfrak{m} \cong k^n$ the image of $x \in V$ modulo the maximal ideal $\mathfrak{m}$) for any j, $(\overline{v}, \overline{v}_1^j, \ldots, \overline{v}_p^j)$ and hence $(v, v_1^j, \ldots, v_p^j)$ are in general position. Now pick any $w \in W$ and set

$$z = \sum_{j=1}^{N} n_j \left(\begin{pmatrix} v \\ w \end{pmatrix}, \begin{pmatrix} v_1^j \\ w_1^j \end{pmatrix}, \ldots, \begin{pmatrix} v_p^j \\ w_p^j \end{pmatrix} \right).$$

One checks easily that $dz = y$. □

COROLLARY 2.3.3. *If $V \cong A^n$, then $C_\bullet(V, W)$ is acyclic in dimensions $\leq n-1$.* □

REMARK 2.3.4. Note that this is a much simpler proof a weak version of Theorem 2.1.3 in this special case (Theorem 2.1.3 asserts the simple connectivity of the associated space). This approach does not work for general rings, however.

Now, if $W = 0$, denote by $C_\bullet(A^n)$ the complex $C_\bullet(A^n, 0)$ and by $S_n(A)$ the module $H_0(GL_n(A), H_n(A))$, where $H_n(A)$ is the nth homology group of $C_\bullet(A^n)$. Note that $H_n(A)$ is the only nonzero homology group of $C_\bullet(A^n)$. Evidently, we have $H_0(A) = \mathbb{Z}$. Thus, $S_0(A) = \mathbb{Z}$. Moreover, the group $H_1(A)$ is the augmentation ideal $I(A)$ of the group ring $\mathbb{Z}[A^\times]$. It follows that $S_0(A) = H_0(A^\times, I(A)) = A^\times$. Now, for each $n \geq 2$, there is an exact sequence

$$\tilde{C}_{n+2}(A^n) \xrightarrow{d} \tilde{C}_{n+1}(A^n) \xrightarrow{d} H_n(A) \longrightarrow 0.$$

Since the coinvariants functor is right exact, we obtain an exact sequence

$$H_0(GL_n(A), \tilde{C}_{n+2}(A^n)) \xrightarrow{\bar{d}} H_0(GL_n(A), \tilde{C}_{n+1}(A^n)) \xrightarrow{\bar{d}} S_n(A) \to 0.$$

The group $GL_n(A)$ acts freely on the sets of $(n+1)$-frames and $(n+2)$-frames in general position. Each orbit can be represented by an element of the form $(e_1, \ldots, e_n, v)$ and $(e_1, \ldots, e_n, v, w)$, respectively. The vectors in each frame are in general position if and only if $v = \sum a_i e_i$, where $a_i \in A^\times$ and $w = \sum \lambda_i a_i e_i$, where the $\lambda \in A^\times$ satisfy $\overline{\lambda}_i \neq \overline{\lambda}_j$ for $i \neq j$ (here, $\overline{\lambda}$ denotes the image of λ in the residue field).

Denote the orbit of the frame $(e_1, \ldots, e_n, \sum a_i e_i)$ by $p(a)$ and the orbit of $(e_1, \ldots, e_n, \sum a_i e_i, \sum \lambda_i a_i e_i)$ by $p(a, \lambda)$. Then

$$H_0(GL_n(A), \tilde{C}_{n+1}(A^n)) = \bigoplus_a \mathbb{Z} \cdot p(a)$$

and

$$H_0(GL_n(A), \tilde{C}_{n+2}(A^n)) = \bigoplus_{a,\lambda} \mathbb{Z} \cdot p(a, \lambda).$$

The induced map in the above exact sequence is then

$$\bar{d}p(a, \lambda) = (-1)^n p(a, \lambda) + (-1)^{n+1} p(a) \tag{2.1}$$
$$+ \sum_{i=1}^{n} (-1)^{i-1} p((\lambda_1 - \lambda_i)a_1, \ldots, \widehat{(\lambda_i - \lambda_i)a_i}, \ldots, (\lambda_n - \lambda_i)a_n, \lambda_i).$$

Hence, the group $S_n(A)$ is generated by elements $\overline{d}(p(a)) = [a_1, \ldots, a_n]$, $a_i \in A^\times$, subject to the relations

$$[\lambda_1 a_1, \ldots, \lambda_n a_n] - [a_1, \ldots, a_n] = \sum_{i=1}^{n} (-1)^{i+n} [(\lambda_1 - \lambda_i)a_1, \ldots, \widehat{(\lambda_i - \lambda_i)a_i}, \ldots, (\lambda_n - \lambda_i)a_n, \lambda_i]$$

where $\lambda_i \in A^\times$ and $\overline{\lambda}_i \neq \overline{\lambda}_j$ for $i \neq j$. Since the generators of $K_n^M(A)$ also satisfy these relations, we obtain a homomorphism $\varphi_n : S_n(A) \to K_n^M(A)$ sending $[a_1, \ldots, a_n]$ to $\{a_1, \ldots, a_n\}$.

Observe that we can multiply elements

$$S_n(A) \otimes S_m(A) \longrightarrow S_{n+m}(A)$$

as follows. Let V be a finitely generated free A-module. Say that $x = (v_1, \ldots, v_n) \in \tilde{C}_n(V)$ and $y = (w_1, \ldots, w_m) \in \tilde{C}_m(V)$ are in general position if the vectors $v_1, \ldots, v_n, w_1, \ldots, w_m$ are. In this case, set $xy = (v_1, \ldots, v_n, w_1, \ldots, w_m) \in \tilde{C}_{n+m}(V)$. This extends to a linear pairing

$$\tilde{C}_\bullet(V) \otimes \tilde{C}_\bullet(V) \longrightarrow \tilde{C}_\bullet(V).$$

Now, if V is decomposed into a direct sum $V = V_1 \oplus V_2$, then there are embeddings of complexes $C_\bullet(V_1, V_2) \to C_\bullet(V)$ and $C_\bullet(V_2) \to C_\bullet(V)$. If $x \in C_n(V_1, V_2)$ and $y \in C_m(V_2)$, then x and y are in general position and $xy \in C_{n+m}(V)$. This gives a pairing

$$C_\bullet(V_1, V_2) \otimes C_\bullet(V_2) \longrightarrow C_\bullet(V)$$

and an induced pairing

$$H_\bullet(C_\bullet(V_1, V_2)) \otimes H_\bullet(C_\bullet(V_2)) \longrightarrow H_\bullet(C_\bullet(V)).$$

All these pairings respect the $GL_n(A)$-action and hence we have an induced pairing

$$S(V_1) \otimes S(V_2) \longrightarrow S(V).$$

This pairing is associative, and we therefore obtain a graded ring

$$S_\bullet(A) = \bigoplus_{n \geq 0} S_n(A).$$

PROPOSITION 2.3.5. *The map*

$$\varphi_\bullet : S_\bullet(A) \longrightarrow K_\bullet^M(A)$$

is a homomorphism of graded rings.

PROOF. We first describe how to represent an arbitrary element of $S_n(A)$ as a sum of generators. Let

$$x = \sum_i n_i(v_1^i, \ldots, v_n^i) \in H_n(A)$$

be an arbitrary cycle. Find a vector v in general position with each v_j^i. Then

$$x = (-1)^n d(xv) = (-1)^n \sum n_i d(v_1^i, \ldots, v_n^i).$$

Each frame $(v_1^i, \ldots, v_n^i)$ defines a matrix $\alpha_i \in GL_n(A)$.

Write $\alpha_i^{-1} v = (a_1^i, \ldots, a_n^i)^T$. Then the frame $(v_1^i, \ldots, v_n^i, v)$ is $GL_n(A)$-equivalent to the frame $(e_1, \ldots, e_n, \sum_j a_j^i e_j)$. Thus

$$x \equiv (-1)^n \sum n_i [a_1^i, \ldots, a_n^i] \mod GL_n(A).$$

Now, let

$$x = \sum n_i s_i \in H_n(A), \qquad y = \sum m_j r_r \in H_m(A),$$

where the s_i are unimodular n-frames in A^n and the r_j are unimodular m-frames in A^m. Find vectors $v \in A^n$ and $w \in A^m$ in general position with all the s_i and r_j respectively. Each frame s_i (resp. r_j) determines an invertible matrix in $GL_n(A)$ (resp. $GL_m(A)$), which we denote by the same symbol. Set $s_i^{-1} v = (a_1^i, \ldots, a_n^i)^T$ and $r_j^{-1} w = (b_1^j, \ldots, b_m^j)^T$. The vector $\binom{v}{w}$ is in general position with the frames (s_i, r_j) and $(s_i, r_j)^{-1}\binom{v}{w} = (a_1^i, \ldots, a_n^i, b_1^j, \ldots, b_m^j)^T$. Thus we have

$$\begin{aligned}
x &\equiv (-1)^n \sum_i n_i [a_1^i, \ldots, a_n^i] \mod GL_n(A), \\
y &\equiv (-1)^m \sum_j m_j [b_1^j, \ldots, b_m^j] \mod GL_m(A), \\
xy &\equiv (-1)^{n+m} \sum_{i,j} n_i m_j [a_1^i, \ldots, a_n^i, b_1^j, \ldots, b_m^j] \mod GL_{n+m}(A),
\end{aligned}$$

and hence,

$$\begin{aligned}
\varphi_{n+m}(xy) &= (-1)^{n+m} \sum_{i,j} n_i m_j \{a_1^i, \ldots, a_n^i, b_1^j, \ldots, b_m^j\} \\
&= \varphi_n(x) \varphi_m(y).
\end{aligned}$$

This completes the proof. □

We now construct a homomorphism $\theta_\bullet : K_\bullet^M(A) \to S_\bullet(A)$. Since $K_1^M(A) = A^\times = S_1(A)$, the identity map $K_1^M(A) \to S_1(A)$ will extend to the desired map of graded rings, provided we show that the defining relations of $K_\bullet^M(A)$ hold in $S_\bullet(A)$. For this, it suffices to show the following

LEMMA 2.3.6. *If $a \in A^\times$ with $\overline{a} \neq 1$, then $[a][1-a] = 0$ in $S_2(A)$.*

PROOF. First consider the following fact: if $a_1, \ldots, a_n \in A^\times$, then in $S_n(A)$ we have

$$[a_1]\cdots[a_n] = \sum_{1 \leq i_1 < \cdots < i_k \leq n} (-1)^k [a_1, \ldots, 1, \ldots, 1, \ldots, a_n],$$

where the 1's are located in the $i_1, \ldots, i_k$ positions. To see this, note that $[a] = ((a) - (1)) \mod GL_1(A)$ so that the left hand side coincides with the image in $S_n(A)$ of the cycle

$$\begin{aligned}&((a_1e_1) - (e_1))((a_2e_2) - (e_2))\cdots((a_ne_n) - (e_n))\\ &\quad = \sum_{1 \leq i_1 < \cdots < i_k \leq n} (-1)^{n-k}(e_1, \ldots, a_{i_1}e_{i_1}, \ldots, a_{i_k}e_{i_k}, \ldots, e_n).\end{aligned}$$

To represent this as a sum of generators in $S_n(A)$, take the vector v to be $\sum_{i=1}^n a_i e_i$. This is in general position with all the vectors in the sum. The procedure above yields the result.

Now, to prove that $[a][1-a] = 0$ in $S_2(A)$, we express it as a sum

$$[a][1-a] = [a, 1-a] - [a, 1] - [1, 1-a] + [1, 1].$$

We now manipulate the defining relation of $S_2(A)$:

$$[\lambda_1 a_1, \lambda_2 a_2] - [a_1, a_2] = [(\lambda_1 - \lambda_2)a_1, \lambda_2] - [(\lambda_2 - \lambda_1)a_2, \lambda_1].$$

Via obvious choices for a_i and λ_i, we obtain the following relations:

$$[a, \lambda] - [a, 1] = [(1-\lambda)a, \lambda] - [\lambda - 1, 1], \quad \overline{\lambda} \neq 1$$

and

$$[(1-\lambda)a, \lambda] - [a, \lambda] = [-\lambda a, 1][\lambda^2, 1-\lambda], \quad \overline{\lambda} \neq 1.$$

Substituting the second into the first, we have

$$[a, \lambda] - [a, 1] = [a, \lambda] + [-\lambda a, 1] - [\lambda^2, 1-\lambda] - [\lambda - 1, 1];$$

that is, $[-\lambda a, 1] + [a, 1] = [\lambda^2, 1-\lambda] + [\lambda - 1, 1]$. Thus, the element $[a, 1] + [b, 1]$ with $\overline{a} \neq -\overline{b}$ depends only on ba^{-1}. Similarly, we have

$$[1-\lambda, \lambda] = [(\lambda - 1)a, 1] + [a, 1] - [\lambda a, 1].$$

Since $\overline{\lambda - 1} \neq -1$, we have

$$[(\lambda - 1)a, 1] + [a, 1] = [\lambda a, 1] + [\lambda(\lambda - 1)^{-1}a, 1].$$

It follows that $[1-\lambda, \lambda] = [\lambda(\lambda-1)^{-1}a, 1]$. Since a is arbitrary in this formula, we see that

$$[1-\lambda, \lambda] = [a, 1] = [1, 1].$$

Using the defining relation again, we find

$$[1, a\lambda] - [1, a] = [1-\lambda, \lambda] - [(\lambda - 1)a, 1].$$

This, along with the previous formula then shows that $[1, a\lambda] - [1, a] = 0$. It follows that $[1, a] = [1, 1]$ for any a. This proves the lemma. □

COROLLARY 2.3.7. *The composition*

$$K_\bullet^M(A) \xrightarrow{\theta_\bullet} S_\bullet(A) \xrightarrow{\varphi_\bullet} K_\bullet^M(A)$$

is the identity.

PROOF. The existence of $\theta_\bullet$ follows from Lemma 2.3.6. Moreover, it is clear that $\theta_n(\{a_1, \ldots, a_n\}) = [a_1] \ldots [a_n]$. By the proof of Lemma 2.3.6, we have $\varphi_n([a_1] \ldots [a_n]) = \{a_1, \ldots, a_n\}$. □

We identify $K_\bullet^M(A)$ with the subring of $S_\bullet(A)$ generated by $S_1(A) = A^\times$. Denote by $\tilde{S}_n(A)$ the subgroup of decomposable elements in $S_n(A)$. By definition, this is the image of the map

$$S_1(A) \otimes S_{n-1}(A) \oplus \cdots \oplus S_{n-1}(A) \otimes S_1(A) \longrightarrow S_n(A).$$

Clearly, for any nontrivial decomposition $A^n = V \oplus W$, the image of the map

$$H(V, W) \otimes H(W) \longrightarrow H_n(A) \longrightarrow S_n(A)$$

lies in $\tilde{S}_n(A)$. We now show the following.

LEMMA 2.3.8. *If $n \geq 2$, then for all $a_j, \lambda \in A^\times$ and all $1 \leq i \leq n$,*

$$[a_1, \ldots, \lambda a_i, \ldots, a_n] - [a_1, \ldots, a_n] \in \tilde{S}_n(A).$$

PROOF. This is clear for the subgroup of $\tilde{S}_n(A)$ consisting of elements in $K_\bullet^M(A)$. That it is true in general is more delicate. Suppose $1 < i < n$. Introduce the notation $\langle s \rangle$ for the class of a cycle $s \in H_n(A)$ modulo the action of $GL_n(A)$. Let $a = \sum a_i e_i$. Then

$$\begin{aligned}
&[a_1, \ldots, \lambda a_i, \ldots, a_n] - [a_1, \ldots, a_n] \\
&\quad = \overline{d}(\langle e_1, \ldots, e_n, a_1 e_1 + \cdots + \lambda a_i e_i + \cdots + a_n e_n \rangle - \langle e_1, \ldots, e_n, a \rangle) \\
&\quad = \overline{d}(\langle e_1, \ldots, \lambda^{-1} e_i, \ldots, e_n, a \rangle - \langle e_1, \ldots, e_n, a \rangle) \\
&\quad = \overline{d}(\langle (e_1, \ldots, e_{i-1})((\lambda^{-1} e_i) - (e_i))(e_{i+1}, \ldots, e_n, a) \rangle) \\
&\quad + \langle d(e_1, \ldots, e_{i-1})((\lambda^{-1} e_i) - (e_i))(e_{i+1}, \ldots, e_n, a) \\
&\quad + (-1)^i (e_1, \ldots, e_{i-1})((\lambda^{-1} e_i) - (e_i)) d(e_{i+1}, \ldots, e_n, a) \rangle \\
&\quad = \langle z \rangle.
\end{aligned}$$

Now, let $u = a_1 e_1 + \cdots + a_i e_i$. Then we have

$$(e_{i+1}, \ldots, e_n, a) = d(u, e_{i+1}, \ldots, e_n, a) + (u) d(e_{i+1}, \ldots, e_n, a),$$

and

$$(e_1, \ldots, e_{i-1}) = (-1)^{i-1} (d(e_1, \ldots, e_{i-1}, u) - d(e_1, \ldots, e_{i-1})(u)).$$

Substituting these formulæ into the above equation, we write

$$z = z_1 - z_2 + z_3.$$

We now show that each z_i is decomposable.

Denote by V_1 and V_2 the submodules of A^n generated respectively by $e_1, \ldots, e_i$ and $e_{i+1}, \ldots, e_n$. Then the cycles

$$d(e_1, \ldots, e_{i-1}, u)((\lambda^{-1}e_i) - (e_i))$$

and

$$d(e_1, \ldots, e_{i-1})((\lambda^{-1}e_i) - (e_i))(u) + (u)((\lambda^{-1}e_i) - (e_i))$$

are in $H(V_1)$ and $d(e_{i+1}, \ldots, e_n, a)$ is in $H(V_2, V_1)$. Thus, z_2 and z_3 are in the image of $H(V_2, V_1) \otimes H(V_1) \to H_n(A)$. This shows that $\langle z_2 - z_3 \rangle$ lies in $\tilde{S}_n(A)$. Also, if W_1 is the submodule of A^n generated by $u, e_{i+1}, \ldots, e_n$ and if W_2 is any direct complement of W_1, then

$$d(u, e_{i+1}, \ldots, e_n, a) \in H(W_1)$$

and

$$d(e_1, \ldots, e_{i-1})((\lambda^{-1}e_i) - (e_i)) \in H(W_2, W_1).$$

Thus, $\langle z_1 \rangle$ lies in $\tilde{S}_n(A)$.

The cases $i = 1, n$ are similar and are left to the reader. □

As a consequence we have the following result.

COROLLARY 2.3.9. *If $n \geq 3$, then $S_n(A) = \tilde{S}_n(A)$.*

PROOF. First observe that Lemma 2.3.8 implies that $[a_1,\ldots,a_n] - [1,\ldots,1]$ is decomposable for any $n \geq 2$. Thus to prove the corollary, it suffices to show that if $n \geq 3$, then the cycle $[1, \ldots, 1]$ is decomposable. Consider the defining relation of $S_n(A)$. If n is odd, the left hand side of the relation is decomposable and the right is congruent to $[1, \ldots, 1]$ modulo the decomposables. If n is even, consider the cycle

$$[1, \ldots, 1][1, 1] \in S_{n-2}(A) \cdot S_2(A) \subset \tilde{S}_n(A).$$

This cycle has the form $\sum n_i r_i$ where the r_i are unimodular n-frames and $\sum n_i = 1$. Expressing each r_i as a sum of generators, we obtain

$$[1, \ldots, 1][1, 1] = \sum n_i [a_1^i, \ldots, a_n^i].$$

But each term in the sum is congruent to $[1, \ldots, 1]$ modulo the decomposables and since $\sum n_i = 1$, the result follows. □

This corollary implies that the ring $S_\bullet(A)$ is generated over $K_\bullet^M(A)$ by $[1,1]$. It is easy to see that $[1,1]$ is central in $S_\bullet(A)$. It follows that we have a direct sum decomposition

$$S_n(A) = K_n^M(A) \oplus [1,1]S_{n-2}(A). \tag{2.2}$$

Now denote by $C_\bullet(G)$ the bar resolution of a group G (see Appendix A). Since resolutions are unique up to homotopy, we obtain a unique homomorphism of complexes

$$\begin{array}{ccccccccc} \cdots \to & C_n(GL_n(A)) & \to \cdots \to & C_0(GL_n(A)) & \to & \mathbb{Z} & \to 0 \\ & \downarrow & & \downarrow & & \| & \\ 0 \to & H_n(A) & \to \cdots \to & C_1(A^n) & \to & \mathbb{Z} & \to 0. \end{array}$$

Factoring out the $GL_n(A)$-action and passing to homology, we obtain a map

$$\varepsilon_n : H_n(GL_n(A), \mathbb{Z}) \longrightarrow H_0(GL_n(A), H_n(A)) = S_n(A).$$

Note that the map $C_1(A^n) \to C_0(A^n) = \mathbb{Z}$ has a $GL_{n-1}(A)$-equivariant section given by $m \mapsto m(e_n)$. Thus, the restriction of ε_n to $H_n(GL_{n-1}(A))$ is trivial. Thus we get a map

$$\varepsilon_n : H_n(GL_n(A))/H_n(GL_{n-1}(A)) \longrightarrow S_n(A).$$

Moreover, the following diagram commutes

$$\begin{array}{ccc} H_n(GL_n(A)) \otimes H_1(GL_1(A)) & \longrightarrow & H_{n+1}(GL_{n+1}(A)) \\ \downarrow{\scriptstyle \varepsilon_n \otimes \varepsilon_1} & & \downarrow{\scriptstyle \varepsilon_{n+1}} \\ S_n(A) \otimes S_1(A) & \longrightarrow & S_{n+1}(A). \end{array} \tag{2.3}$$

Since $\varepsilon_1 : A^\times \to A^\times$ is the identity, we see that the image of ε_n contains $K_n^M(A)$.

To finish the proof of Theorem 2.3.1, consider the action of $GL_n(A)$ on $C_\bullet(A^n)$ and the associated spectral sequences. The first sequence collapses to show that $E_m^\infty = 0$ for $m < n$ and

$$E_{n+p}^\infty = H_p(GL_n(A), H_n(A)).$$

Note that $E_n^\infty = H_0(GL_n(A), H_n(A)) = S_n(A)$ and there is an increasing filtration such that

$$S_n(A)^{(i)}/S_n(A)^{(i-1)} = E_{n-i,i}^\infty = E_{n-i,i}^2.$$

Now, by Corollary 2.2.16,

$$S_n(A)^{(0)} = H_n(GL_n(A))/H_n(GL_{n-1}(A)),$$

(the isomorphism being obviously induced by ε_n) and

$$S_n(A)^{(1)}/S_n(A)^{(0)} = E_{n-1,1}^2 = \ker(H_{n-1}(GL_{n-1}(A)) \to H_{n-1}(GL_n(A))).$$

We now proceed as in the proof of Proposition 2.2.17. Define a map $\psi : C_\bullet(A^{n-2})[-2] \to C_\bullet(A^n)$ by multiplying by the cycle $(e_{n-1}, e_n) - (e_{n-1}, e_{n-1} + e_n) + (e_n, e_{n-1} + e_n)$. This map commutes with the $GL_{n-2}(A)$ action and hence induces a map of spectral sequences. As before,

$$\psi_* : \tilde{E}^2_{pq} \xrightarrow{\cong} E^2_{pq}$$

for $q \geq 2$. This implies that $E^\infty = \tilde{E}^\infty \oplus (E^\infty)^{(1)}$. Now, $\tilde{E}^\infty_n = S_{n-2}(A)$ and the map

$$\psi_* : S_{n-2}(A) = \tilde{E}^\infty_n \longrightarrow E^\infty_n = S_n(A)$$

is obviously multiplication by $[1, 1]$. Thus

$$S_n(A) = [1, 1]S_{n-2}(A) \oplus S_n(A)^{(1)}.$$

But, we also have

$$S_n(A) = [1, 1]S_{n-2}(A) \oplus K^M_n(A)$$

and

$$K^M_n(A) \subset S_n(A)^{(0)}.$$

Thus, $K^M_n(A) = S_n(A)^{(0)} = S_n(A)^{(1)}$. This shows that

$$K^M_n(A) = H_n(GL_n(A))/H_n(GL_{n-1}(A))$$

and

$$0 = E^2_{n-1,1} = \ker\{H_{n-1}(GL_{n-1}(A)) \longrightarrow H_{n-1}(GL_n(A))\}.$$

This completes the proof of Theorem 2.3.1. □

REMARK 2.3.10. The statement of Theorem 2.3.1 asserts that the map

$$\Theta : K^M_n(A) \longrightarrow H_n(GL_n(A), \mathbb{Z})/H_n(GL_{n-1}(A), \mathbb{Z})$$

is an isomorphism. The proof actually asserts that

$$\varepsilon_n : H_n(GL_n(A), \mathbb{Z})/H_n(GL_{n-1}(A), \mathbb{Z}) \longrightarrow K^M_n(A)$$

is an isomorphism. These two maps are evidently inverses.

Theorem 2.3.1 has the following applications. For any ring A, the natural map $A^\times \to K_1(A)$ induces a homomorphism $K^M_\bullet(A) \to K_\bullet(A)$. If A is local with infinite residue field, then we define a map $\varphi : K_n(A) \to K^M_n(A)$ to be the composition

$$\begin{aligned} K_n(A) = \pi_n(BGL(A)^+) &\overset{\text{Hurewicz}}{\longrightarrow} H_n(BGL(A)^+) = H_n(GL(A)) \\ &\longrightarrow H_n(GL(A))/H_n(GL_{n-1}(A)) = K^M_n(A). \end{aligned}$$

THEOREM 2.3.11 ([**92**], Theorem 4.1). *The composition*

$$K^M_n(A) \to K_n(A) \to K^M_n(A)$$

coincides with multiplication by $(-1)^{n-1}(n-1)!$.

Consider the particular case $n = 2$.

LEMMA 2.3.12. *Let A be a commutative ring with $SK_1(A) = 0$ (e.g. any local ring). Then the natural map*

$$K_2(A) \to H_2(GL(A))/H_2(GL_1(A))$$

is an isomorphism.

PROOF. By hypothesis, $SL(A) = E(A)$ so that $K_2(A) = H_2(SL(A))$. The result now follows by considering the Hochschild–Serre spectral sequence associated to the (split!) extension

$$1 \longrightarrow SL(A) \longrightarrow GL(A) \longrightarrow A^\times \longrightarrow 1$$

and noting that the $A^\times$-action on $H_\bullet(SL(A))$ is trivial. □

COROLLARY 2.3.13. *If A is a local ring with infinite residue field, then $K_2(A) = K_2^M(A)$.*

PROOF. This follows since $H_2(GL(A))/H_2(GL_1(A)) = K_2^M(A)$. □

REMARK 2.3.14. Corollary 2.3.13 was proved originally by van der Kallen [**64**].

2.4. Auxiliary stability results

2.4.1. Orthogonal groups. In [**131**], K. Vogtmann considered the question of stability for the orthogonal groups $O_{n,n}(k)$, char $k = 0$, where $O_{n,n}(k)$ denotes the orthogonal group of the quadratic form $\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$ on k^{2n} ($I = n \times n$ identity matrix).

THEOREM 2.4.1. *If k is a field of characteristic zero, then the natural map*

$$i_* : H_m(O_{n,n}(k), \mathbb{Z}) \longrightarrow H_m(O_{n+1,n+1}(k), \mathbb{Z})$$

is surjective for $n \geq 3m + 1$ and bijective for $n \geq 3m + 3$. □

The proof of Theorem 2.4.1 is virtually identical to the proof of Theorem 2.2.18. The idea is to construct a complex X on which $G_n = O_{n,n}(k)$ acts and utilize the resulting spectral sequence. The stabilizers of the various simplices are analogous to the affine groups of Section 2.2. Since char $k = 0$, the usual center kills argument shows that homologically these groups are the same as $GL_p(k) \times G_{n-p}$. This allows one to use an induction procedure to prove the theorem. Moreover, one expects that by using a suitable modification of the arguments in Section 2.2, the theorem holds in positive characteristic as well.

The simplicial complex X is defined as follows. If k is any field, let V be a $2n$-dimensional vector space over k with quadratic form (,). Choose a

polar basis $\{e_1, \ldots, e_n, f_1, \ldots, f_n\}$ of V; that is, in this basis the matrix of the form is

$$\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}.$$

Let $\bullet$ denote the associated inner product.

DEFINITION 2.4.2. A subspace $W \subset V$ is totally isotropic if $v \bullet w = 0$ for all vectors $v, w \in W$.

The set of all nontrivial totally isotropic subspaces of V is partially ordered by inclusion; denote by X the geometric realization of this poset. A p-simplex of X is a chain of $(p+1)$ totally isotropic subspaces of V:

$$W_0 \subset W_1 \subset \cdots \subset W_p.$$

Since every maximal totally isotropic subspace of V has dimension n, the complex X has dimension $(n-1)$.

PROPOSITION 2.4.3. *The complex X is homotopy equivalent to a wedge of $(n-1)$-spheres.*

PROOF. Let $E \subset V$ be spanned by $e_1, \ldots, e_n$. For $0 \leq k \leq n$, define a subcomplex X_k by

$$\begin{aligned} X_k &= \text{union of all maximal simplices } A_1 \subset \cdots \subset A_n \\ &\quad \text{such that } \dim(A_n \cap E) \geq n-k \\ &= \bigcup_{\dim(A_n \cap E) \geq n-k} \overline{\mathrm{st}(A_n)} \end{aligned}$$

where $\overline{\mathrm{st}(A_n)}$ denotes the closure of the star of A_n. Thus, $X_0 = \overline{\mathrm{st}(E)}$ and $X_n = X$. Via a proof similar to that of Theorem 2.1.3, one shows that X_{k-1} is a deformation retract of X_k for $1 \leq k \leq n-1$. Since X_0 is contractible, X_{n-1} is contractible. Note that X_{n-1} is X minus the stars of maximal isotropic subspaces A_n such that $A_n \cap E = 0$. For such an A_n, the link, $\mathrm{Link}(A_n)$ is contained in X_{n-1}. Thus, by contracting X_{n-1} to a point we obtain

$$X \simeq \bigvee_{A_n \cap E = 0} \Sigma(\mathrm{Link}(A_n)).$$

But $\mathrm{Link}(A_n)$ is the Tits building for A_n since every subspace of a totally isotropic space is totally isotropic. By the Solomon–Tits theorem [45], $\mathrm{Link}(A_n)$ is homotopy equivalent to a wedge of $(n-2)$-spheres. Thus,

$$X \simeq \bigvee \Sigma(\bigvee S^{n-2}) \simeq \bigvee S^{n-1},$$

as claimed. □

2.4.2. Dedekind domains. A Dedekind domain is a noetherian normal domain of dimension 1. Typical examples include the ring of integers in a number field and the coordinate ring of a nonsingular irreducible affine curve over a field.

Theorem 2.1.3 applies to such rings, but R. Charney [**27**] proved a stability theorem for GL_n of such rings by contructing a different complex on which GL_n acts. The crucial step in the proofs of Theorems 2.2.18 and 2.4.1 is showing that the map

$$i_* : H_*(GL_n(A), \mathbb{Z}) \longrightarrow H_*(\mathrm{Aff}_{nm}(A), \mathbb{Z})$$

is an isomorphism (i.e., we can "kill" the unipotent piece of $\mathrm{Aff}_{nm}(A)$). If A is a Dedekind domain with many units, then there is no problem; the results of Section 2.2 apply. But if A is the ring of integers in a number field, then we are out of luck.

Charney circumvented this difficulty by inventing *split buildings*. Let A be a Dedekind domain and let W be a rank n projective A-module. Define a partially ordered set $S(W)$ by

$$S(W) = \{(P, Q) : P \oplus Q = W, P \neq 0, Q \neq 0\}$$

with ordering $(P, Q) \leq (P', Q')$ if $P \subseteq P'$ and $Q \supseteq Q'$. Denote by $[W]$ the geometric realization of $S(W)$.

THEOREM 2.4.4. *If W is a rank n projective A-module, then $[W]$ has the homotopy type of a wedge of $(n-2)$-spheres.* □

The proof of this is similar to Vogtmann's proof that the space built from total isotropic subspaces of k^{2n} is a wedge of $(n-1)$-spheres.

The usefulness of this construction is that the stabilizers of simplices in $[W]$ have the form

$$\begin{pmatrix} GL_p & 0 \\ 0 & GL_{n-p} \end{pmatrix}$$

so that we no longer need to get rid of a unipotent piece. The usual stability argument then goes through to yield the following result.

THEOREM 2.4.5. *If A is a principal ideal domain, then*

1. $H_k(GL_{n+1}(A), GL_n(A); \mathbb{Z}) = 0$ *for* $n \geq 3k$;
2. $H_k(GL_{n+1}(A), GL_n(A); \mathbb{Z}[1/2]) = 0$ *for* $n \geq 2k$.

The first statement is true also for SL_n. □

This range of stability is not quite as good as Theorem 2.1.3. However, if we ignore 2-torsion, then it is basically the same as van der Kallen's result.

2.5. Stability via Homotopy

The Hurewicz homomorphism

$$h_i : \pi_i(X) \longrightarrow H_i(X, \mathbb{Z})$$

is an important tool in the study of both homotopy and homology. Consider the particular case $X = BGL_n(R)^+$. Denote the ith homotopy group of X by $K^Q_{i,n}(R)$ ("Q" for Quillen). Then the Hurewicz map is a homomorphism

$$K^Q_{i,n}(R) \longrightarrow H_i(GL_n(R), \mathbb{Z}).$$

Thus, if one can prove stability for $K^Q_{i,n}(R)$, then the above map will yield stability for $H_i(GL_n(R), \mathbb{Z})$.

Unfortunately, the groups $K^Q_{i,n}(R)$ are difficult to work with. Moreover, for $i = 1, 2$, they need not agree with the "classical" definitions:

$$\begin{aligned} K_{1,n}(R) &= GL_n(R)/E_n(R) \\ K_{2,n}(R) &= \ker\{\mathrm{St}_n(R) \longrightarrow E_n(R)\} \end{aligned}$$

(here, $\mathrm{St}_n(R)$ is the nth Steinberg group of R; see [**80**]). To circumvent this, one can work with the unstable Volodin groups

$$K_{i,n} = \pi_{i-1}(V_n(R)),$$

where $V_n(R)$ is a certain simplicial complex defined below. Suslin [**123**] proved the following stability result for the groups $K_{i,n}(R)$.

THEOREM 2.5.1. *Let R be a ring and set $r = \mathrm{sr}(R)$. Then the canonical map $K_{i,n}(R) \to K_{i,n+1}(R)$ is surjective for $n \geq r + i - 1$ and bijective for $n \geq r + i$.*

Suslin then constructed maps

$$K_{i,n}(R) \longrightarrow K^Q_{i,n}(R)$$

for $n \geq 2i+1$ and showed that they are surjective for $n \geq \max(2i+1, r+i-1)$ and bijective for $n \geq \max(2i+1, r+i)$. Thus, we have the following result.

COROLLARY 2.5.2. *The canonical map*

$$K^Q_{i,n}(R) \longrightarrow K^Q_{i,n+1}(R)$$

is surjective for $n \geq \max(2i+1, r+i-1)$ and bijective for $n \geq \max(2i+1, r+i)$. □

COROLLARY 2.5.3. *The canonical map*

$$H_i(GL_n(R), \mathbb{Z}) \longrightarrow H_i(GL_{n+1}(R), \mathbb{Z})$$

is surjective for $n \geq \max(2i+1, r+i-1)$ and bijective for $n \geq \max(2i+1, r+i)$.

PROOF. Consider the commutative diagram

$$\begin{array}{ccccc} \pi_{i+1}(BGL^+_{n+1}, BGL^+_n) & \longrightarrow & K^Q_{i,n}(R) & \longrightarrow & K^Q_{i,n+1}(R) \\ \downarrow & & \downarrow & & \downarrow \\ H_{i+1}(BGL^+_{n+1}, BGL^+_n) & \longrightarrow & H_i(GL_n(R)) & \longrightarrow & H_i(GL_{n+1}(R)) \end{array}$$

The result follows from the fact that if the relative homotopy groups $\pi_j(X, Y)$ vanish, then the relative homology groups $H_j(X, Y)$ vanish as well. □

The proof of Theorem 2.5.1 rests on the construction of certain spaces. If G is a group and $\{G_i\}_{i\in I}$ is a family of subgroups, define a simplicial complex $V(G)$ as follows. The vertices of $V(G)$ are the elements of G and a p-simplex of $V(G)$ is a collection $g_0, \ldots, g_p$ of distinct elements of G such that all the $g_j g_k^{-1}$ lie in some G_i. This construction is functorial with respect to maps

$$(G, \{G_i\}_{i\in I}) \longrightarrow (H, \{H_j\}_{j\in J}).$$

Define a second simplicial complex $W(G)$ as follows. The p-simplices are collections $(g_0, \ldots, g_p)$ of (not necessarily distinct) elements of G such that the $g_j g_k^{-1}$ all lie in some G_i. Denote by $S_\bullet(V(G))$ the simplicial set of singular simplices of $V(G)$; the geometric realization of $S_\bullet(V(G))$ is homotopy equivalent to $V(G)$. Define a map $W(G) \to S_\bullet(V(G))$ by

$$(g_0, \ldots, g_p) \mapsto \{v_i \mapsto g_i\},$$

where v_i is the ith vertex of the standard p-simplex. This map is a homotopy equivalence.

Now, if R is a ring and σ is a linear ordering of $\{1, 2, \ldots, n\}$, define $T^\sigma_n(R)$ to be the subgroup of $GL_n(R)$ consisting of matrices (a_{kl}) with $a_{ii} = 1$ and $a_{ij} = 0$ for $i \overset{\sigma}{\not<} j$. Such subgroups are called *triangular*. Denote by $V_n(R)$ the space $V(GL_n(R), \{T^\sigma_n(R)\})$ and define

$$K_{i,n}(R) = \pi_{i-1}(V_n(R)).$$

The connected component of the identity matrix in $V_n(R)$ is the space $V(E_n(R))$; its universal cover is $V(\mathrm{St}_n(R), \{\tilde{T}^\sigma_n(R)\})$ ($\tilde{T}^\sigma_n(R)$ is the lift of $T^\sigma_n(R)$ to the Steinberg group). Thus we have

$$K_{i,n}(R) = \pi_{i-1}(V(\mathrm{St}_n(R))) = \pi_{i-1}(W(\mathrm{St}_n(R))), \quad i \geq 3.$$

Theorem 2.5.1 follows by proving homology stability for the groups $H_i(W(\mathrm{St}_n(R)))$: for then the relative homology groups vanish and the relative Hurewicz theorem shows that the relative homotopy groups vanish as well. The proof of stability for $H_i(W(\mathrm{St}_n(R)))$ is similar to the proofs above and we omit it. The interested reader is urged to consult Suslin's paper [**123**] for full details.

2.6. The Rank Conjecture

Let A be a commutative ring and denote by $K_n(A)_{\mathbb{Q}}$ the vector space $K_n(A) \otimes_{\mathbb{Z}} \mathbb{Q}$. By the Milnor–Moore theorem, we have an injection

$$K_n(A)_{\mathbb{Q}} \longrightarrow H_n(GL(A), \mathbb{Q})$$

with image equal to the space of primitive elements ($H_\bullet(GL(A), \mathbb{Q})$ is a Hopf algebra). Define the rank filtration of $K_n(A)_{\mathbb{Q}}$ by

$$r_m K_n(A)_{\mathbb{Q}} = \{\mathrm{Im}(H_n(GL_m(A), \mathbb{Q}) \longrightarrow H_n(GL(A), \mathbb{Q}))\} \cap K_n(A)_{\mathbb{Q}}.$$

This is an increasing filtration. By Theorem 2.2.18, if $A = F$ is an infinite field, then

$$r_n K_n(F)_{\mathbb{Q}} = K_n(F)_{\mathbb{Q}}$$

and

$$r_n K_n(F)_{\mathbb{Q}} / r_{n-1} K_n(F)_{\mathbb{Q}} = K_n^M(F).$$

There is another filtration, the γ-filtration [**115**], defined as follows. We first define λ-operations on the K-groups of A. Consider the Grothendieck group $K_0(\mathbb{Z}[GL_n])$. A representation of GL_n gives rise to a map $GL_n \to GL$, and using the H-space structure on $BGL(A)^+$, we get a map

$$K_0(\mathbb{Z}[GL_n]) \longrightarrow [BGL_n(A), BGL(A)^+] = [BGL_n(A)^+, BGL(A)^+].$$

The latter object is an abelian group, and the map above is a homomorphism of abelian groups. It is also compatible with the inclusion $GL_n \to GL_{n+1}$, so we get a homomorphism

$$\varprojlim_n K_0(\mathbb{Z}[GL_n]) \longrightarrow \varprojlim_n [BGL_n(A)^+, BGL(A)^+].$$

Note that $\varprojlim [BGL_n(A)^+, BGL(A)^+] = \varprojlim [BGL(A)^+, BGL(A)^+]$. Denote by id_n the standard n-dimensional representation of GL_n, and by n the trivial n-dimensional representation. The collection $\{\mathrm{id}_n - n\}_n$ is an element of $\varprojlim K_0(\mathbb{Z}[GL_n])$. Define λ-operations on $K_0(\mathbb{Z}[GL_n])$ by $\lambda^m(V) = \bigwedge^m V$ if V is a representation, and by demanding that the map $\lambda_t : K_0(\mathbb{Z}[GL_n]) \to K_0(\mathbb{Z}[GL_n])$ defined by

$$\lambda_t(x) = 1 + \lambda^1(x) + \lambda^2(x) + \cdots$$

be a homomorphism of abelian groups. These operations are compatible with the maps $K_0(\mathbb{Z}[GL_{n+1}]) \to K_0(\mathbb{Z}[GL_n])$. Then we can consider the maps λ^m as elements in $\varprojlim [BGL_n(A), BGL(A)^+]$ by taking $\{\lambda^m(\mathrm{id}_n - n)\}_n$ in this group. We then get induced operations, still denoted by λ^m on the homology groups $H_\bullet(GL(A), \mathbb{Z})$ and on the homotopy groups $K_\bullet(A)$.

We now use the λ-operations to define Adams operations. We have an involution on $K_0(\mathbb{Z}[GL_n])$ defined by sending a representation to its dual;

denote this map by $x \mapsto \hat{x}$. Then if k is a nonzero integer, we define the kth Adams operation by demanding

$$\psi^k(x) = \psi^{-k}(\hat{x}), \qquad k \geq 1$$

and

$$\psi^k - \lambda^1\psi^{k-1} + \cdots + (-1)^{k-1}\lambda^{k-1}\psi^1 + (-1)^k k\lambda^k = 0.$$

Now define

$$K_n^{(j)}(A) = \ker(\psi^k - k^j\psi^1 : K_n(A)_{\mathbb{Q}} \to K_n(A)_{\mathbb{Q}}),$$

the jth eigenspace of ψ^k. This gives us a decreasing filtration on $K_n(A)_{\mathbb{Q}}$,

$$F^m_{\gamma,n} = \bigoplus_{j=m}^{n} K_n^{(j)}(A),$$

called the γ-filtration.

CONJECTURE 2.6.1 (Rank Conjecture). *Let F be an infinite field. Then for all $n \geq 1$ and all m, we have a direct sum decomposition*

$$r_m K_n(F)_{\mathbb{Q}} \oplus F^{m+1}_{\gamma,n}(F) = K_n(F)_{\mathbb{Q}}.$$

A recent result of R. de Jeu [**28**] shows that for any commutative ring with identity, there is an inclusion

$$r_m K_n(A)_{\mathbb{Q}} \subseteq \bigoplus_{j=1}^{m} K_n^{(j)}(A),$$

and hence the rank conjecture is equivalent to the assertion that

$$r_m K_n(F)_{\mathbb{Q}} = \bigoplus_{j=1}^{m} K_n^{(j)}(F).$$

The rank conjecture for number fields. Let k be a number field. The rank conjecture for k is implied by two assertions about the homomorphism

$$j_{i,m,n} : H_i(GL_m(k), \mathbb{R}) \longrightarrow H_i(GL_n(k), \mathbb{R})$$

($m \leq n$) induced by the standard inclusion; namely,

(1) $\operatorname{im}(j_{2m-1,m,n}) \supset PH_{2m-1}(GL_n(k), \mathbb{R})$,
(2) $\operatorname{im}(j_{2m-1,m-1,n}) \cap PH_{2m-1}(GL_n(k), \mathbb{R}) = \{0\}$,

for $n \gg m \geq 2$, where PH_i is the space of primitive elements. In other words, we need to prove that

(1)′ $r_m K_{2m-1}(k)_{\mathbb{Q}} = K_{2m-1}(k)_{\mathbb{Q}}$, and
(2)′ $r_{m-1} K_{2m-1}(k)_{\mathbb{Q}} = 0$.

But there is an isomorphism

$$PH_{2m-1}(SL_n(k), \mathbb{R}) \xrightarrow{\cong} PH_{2m-1}(GL_n(k), \mathbb{R}) \qquad (n \gg m \geq 2)$$

so that it suffices to prove (1) and (2) with GL_n replaced by SL_n. This is easily obtained via the following result.

THEOREM 2.6.2. *For any field E of characteristic zero,*

1. $j^i_{m,n} : PH^i(SL_n(k), E) \longrightarrow PH^i(SL_m(k), E)$ *and*
2. $j_{i,m,n} : PH_i(SL_m(k), E) \longrightarrow PH_i(SL_n(k), E)$ *are isomorphisms for i odd, $i \leq 2m-1$.*
3. $j^{2m-1}_{m-1,n}(PH^{2m-1}(SL_n(k), E)) = \{0\}$.

PROOF. This is a consequence of the following formulæ:

$$\begin{array}{ll} j^i_{m,n}(x_{n,i}) = x_{m,i}, & i \leq 2m-1 \\ j^i_{m,n}(x_{n,i}) = 0, & i > 2m-1 \\ j^n_{m,n}(e_n) = 0, & n \text{ even.} \end{array}$$

The first two equations follow from the fact that if $m > n$, then $x_{m,i}$ maps to $x_{n,i}$ under the induced map

$$H^\bullet_{\mathrm{ct}}(SL_n(k_v), \mathbb{R}) \longrightarrow H^\bullet_{\mathrm{ct}}(SL_m(k_v), \mathbb{R})$$

(see Section 1.5). That $j^n_{m,n}(e_n) = 0$ follows from the fact that e_n is an even-dimensional cohomology class. □

The rank conjecture now may be deduced as follows. Assertion (1) follows from part (1) of the theorem. By duality, the second assertion is equivalent to

$$j^{2m-1}_{m,n}(PH^{2m-1}(SL_n(k), E)) = 0$$

which follows from (3).

Exercises

1. Prove that a local ring has stable rank 1.
2. Describe the structure of the partially ordered set $\mathcal{O}(\{1, 2, \ldots, n\})$ (see Definition 2.1.2).
3. Check that the conditions of Lemma 2.1.6 are satisfied by the poset $F = \mathcal{O}(R^n + e_{n+1}\delta) \cap \mathcal{U}$.
4. Modify the proof of Theorem 2.2.18 to obtain a proof of Theorem 2.1.7.
5. Prove that an algebra over an $S(n)$-ring is also an $S(n)$-ring.
6. Show that the map $B^r \to S^{(i)}_n / S^{(i-1)}_n$ given by

$$(b_1, \ldots, b_r) \mapsto \sum_{\sigma \in \Sigma_n / \Sigma_{k_1} \times \cdots \times \Sigma_{k_r}} \sigma(b_1 \otimes \cdots \otimes b_1 \otimes \cdots \otimes b_r \otimes \cdots \otimes b_r) \mod S^{(i-1)}_n$$

is multilinear.

7. Prove that the filtration of $H_j(N, \mathbb{F}_p)$, where N is an abelian group, defined by

$$H_j^{(r)} = \bigoplus_{i \leq r} \bigwedge^{j-2i}(N/p) \otimes \Gamma_{2i}({}_pN)$$

is independent of the choice of section of the map

$$H_2(N, \mathbb{F}_p) \longrightarrow {}_pN\,.$$

8. Prove Lemma 2.2.12.
9. Show that the map ψ_* defined in the proof of Proposition 2.2.17 is the identity.
10. Complete the proof of Lemma 2.3.2 by checking that $dz = y$.
11. Check the formula in equation (2.1).
12. Show that the generators of $K_n^M(A)$ satisfy the relation (2.1). (Hint: proceed by induction on n, the case $n = 1$ being an obvious consequence of the definition of $K_n^M(A)$. For the general case, make use of the facts that in $K_n^M(A)$:

$$\{a_1, \ldots, a_i, -a_i, \ldots, a_{n-1}\} = 0,$$

and

$$\{a_{\sigma(1)}, \ldots, a_{\sigma(n)}\} = \operatorname{sgn}(\sigma)\{a_1, \ldots, a_n\}.$$

To prove these last two facts it suffices to show that $\{a, -a\} = 0$ in $K_2^M(A)$ and $\{a_1, a_2\} = -\{a_2, a_1\}$; this can be done by direct computation.)
13. Finish the proof of Lemma 2.3.8.
14. Prove that if $z \in S_n(A)$, then $[a]z = (-1)^n z[a]$.
15. Prove that the diagram (2.3) commutes.
16. Prove that the maps

$$\Theta : K_n^M(A) \longrightarrow H_n(GL_n(A), \mathbb{Z})/H_n(GL_{n-1}(A), \mathbb{Z})$$

and

$$\varepsilon_n : H_n(GL_n(A), \mathbb{Z})/H_n(GL_{n-1}(A), \mathbb{Z}) \longrightarrow K_n^M(A)$$

are inverses.
17. Show that every maximal totally isotropic subspace of a $2n$-dimensional vector space V has dimension n.
18. Show that every subspace of a totally isotropic space is totally isotropic.
19. Show that $K_{1,n}(R) = GL_n(R)/E_n(R)$.
20. Show that the map $W(G) \longrightarrow S_\bullet(V(G))$ is a homotopy equivalence.
21. Show that the universal cover of $V(E_n(R))$ is $V(\mathrm{St}_n(R))$.

Chapter 3

Low-dimensional Results

The main focus of this chapter is the computation of low-dimensional homology groups of GL_n. The results of the previous chapter show, for example, that $H_2(GL_2(A),\mathbb{Z}) \cong H_2(GL(A),\mathbb{Z})$ for A local with infinite residue field. Thus, one need only consider the former group. In this case, Suslin completely described the structure of $H_2(GL_2(A),\mathbb{Z})$—it surjects onto the second Milnor K-group $K_2^M(A)$ and the kernel of this map is the image of $H_2(GL_1(A),\mathbb{Z})$ in $H_2(GL_2(A),\mathbb{Z})$.

That begs the question: What about $H_3(GL_2(A),\mathbb{Z})$? There is a natural map $H_3(GL_2(A)) \longrightarrow H_3(GL_3(A)) = H_3(GL(A))$, but what more can be said? More generally, what about $H_{n+1}(GL_n(A))$? There are no easy answers to these questions, but we can present a few results.

3.1. Scissors Congruence

We begin with a problem which is seemingly unrelated to the study of the homology of linear groups, namely the scissors congruence question. This is an extended version of Hilbert's third problem and has been studied primarily by Dupont, Sah, Wagoner and Cathelineau [**105, 106, 107, 31, 30, 25, 26**]. We present it in its most general terms.

Let X be a set with a specified family of distinguished subsets called cells (or n-cells if n is the dimension). Two cells A and B are *interior disjoint* if

1. $A \cap B$ contains no nonempty cells;
2. If C is a cell contained in $A \cup B$, then $C \subseteq A$ if and only if $C \cap B$ contains no nonempty cells.

A finite union of pairwise interior disjoint cells is called a polytope (the empty set is allowed). If $P = \coprod P_i$ where the P_i are cells (resp. polytopes), then the finite collection of P_i's is called a cell (resp. polytope) decomposition of P.

If $A = B \coprod C$ (A, B, C cells), then we say that B and C form a *simple pasting* of A. If $P = \coprod P_i$ is a cell decomposition, then a simple subdivision (resp. pasting) of P is understood to be one involving one (resp. two) of the finite number of cells P_i. Any finite iteration of simple subdivisions (resp. pastings) beginning with $P = \coprod P_i$ is called a subdivision (resp. pasting) of P. A finite sequence of subdivisions and pastings will be called a cut and

paste. This evidently defines an equivalence relation among cell decompositions $P = \coprod P_i$ of a given polytope P. We need to assure ourselves an ample supply of small cells. Thus we impose the following axiom:

> Let $P = \coprod P_i$ and $Q = \coprod Q_j$ be cell decompositions. Then there exist cut and paste processes leading to cell decompositions $P = \coprod_u R_u$ and $Q = \coprod_v S_v$ such that each R_u (resp. S_v) is either interior disjoint from Q (resp. P) or coincides with one of the S_v (resp. R_u).

EXAMPLE 3.1.1. Consider the classical Euclidean, spherical, or hyperbolic spaces. In these cases we can take cells to be convex closures of finite numbers of points. Then there is a well-defined notion of dimension.

We now define the notion of geometric scissors congruence. Fix a group G of bijective maps (called *motions*) of X. We assume further that the set of cells of X is closed under the action of G.

DEFINITION 3.1.2. A G-scissor between polytopes P and Q, denoted $P \sim_G Q$ or $P \sim Q \mod G$, consists of cell decompositions $P = \coprod_i P_i$, $Q = \coprod_i Q_i$, $1 \leq i \leq t$ such that P_i and Q_i are G-congruent for each i.

The G-scissors congruence problem is the following: Determine necessary and sufficient conditions for polytopes P and Q to be G-scissors congruent.

DEFINITION 3.1.3. An equivalence relation $\equiv$ on the set of polytopes of X is subtractive if whenever $P = P' \coprod P''$ and $Q = Q' \coprod Q''$ are polytope decompositions, then $P \equiv Q$ and $P' \equiv Q'$ together imply $P'' \equiv Q''$.

The geometric scissors congruence problem has an algebraic formulation. Define a group $\mathcal{P}(X, G)$ as follows. First define a group $\mathcal{P}(X, \{1\})$ as the abelian group with generators the cells in X subject to the relations

$$A - B - C = 0 \text{ when } A = B \amalg C \text{ is a simple subdivision.}$$

The class of A will be denoted by $[A]$. Now define $\mathcal{P}(X, G)$ as

$$\mathcal{P}(X, \{1\})/\langle [\sigma A] - [A] : \sigma \in G, A \text{ is a cell in } X \rangle.$$

The group $\mathcal{P}(X, G)$ is called the group of G-scissors congruence classes of polytopes on X.

THEOREM 3.1.4 ([**105**]). *Assume that the relation $\sim_G$ is subtractive. Then $P \sim_G Q$ if and only if $[P] = [Q]$ in $\mathcal{P}(X, G)$.* □

Thus, the study of G-scissors congruence classes is equivalent to the study of the group $\mathcal{P}(X, G)$. Let $\mathcal{H}^n$ denote n-dimensional hyperbolic space and denote by $\overline{\mathcal{H}}^n$ the extended hyperbolic space obtained by adding the ideal points on $\partial\mathcal{H}^n$. We write $\mathcal{P}(\mathcal{H}^n)$, $\mathcal{P}(\overline{\mathcal{H}}^n)$ for the scissors congruence groups where G is the group of all isometries of $\mathcal{H}^n$, $\overline{\mathcal{H}}^n$.

THEOREM 3.1.5 ([31], Theorem 2.1). *The natural inclusion $\mathcal{H}^n \to \overline{\mathcal{H}}^n$ induces an isomorphism*

$$\mathcal{P}(\mathcal{H}^n) \xrightarrow{\cong} \mathcal{P}(\overline{\mathcal{H}}^n).$$

PROOF. According to Dupont [29], there is an alternate description of $\mathcal{P}(\mathcal{H}^n)$ and $\mathcal{P}(\overline{\mathcal{H}}^n)$. Let $\mathcal{T}(\mathcal{H}^n)$ (resp. $\mathcal{T}(\overline{\mathcal{H}}^n)$) be the Tits complex of flags of proper geodesic subspaces of $\mathcal{H}^n$ (resp. $\overline{\mathcal{H}}^n$) (this is the simplicial complex whose k-simplices are flags $V_0 \subset V_1 \subset \cdots \subset V_k$, where each V_i is a proper geodesic subspace). Let $G(n)$ be the group of isometries of $\mathcal{H}^n$. The complexes $\mathcal{T}(\mathcal{H}^n)$ and $\mathcal{T}(\overline{\mathcal{H}}^n)$ have homology concentrated in dimension $n-1$ (see Appendix A). Put

$$\mathrm{St}(\mathcal{H}^n) = \tilde{H}_{n-1}(\mathcal{T}(\mathcal{H}^n), \mathbb{Z})$$

and

$$\mathrm{St}(\overline{\mathcal{H}}^n) = \tilde{H}_{n-1}(\mathcal{T}(\overline{\mathcal{H}}^n), \mathbb{Z}).$$

Then there are natural isomorphisms

$$\mathcal{P}(\mathcal{H}^n) \cong H_0(G(n), \mathrm{St}(\mathcal{H}^n)^t)$$

and

$$\mathcal{P}(\overline{\mathcal{H}}^n) \cong H_0(G(n), \mathrm{St}(\overline{\mathcal{H}}^n)^t),$$

where the upper index ^{t} indicates that the action is twisted by the determinant ($= \pm 1$).

For $p \in \partial\mathcal{H}^n$, let $\mathcal{T}(\mathcal{H}^n, p)$ be the complex of flags of proper subspaces of $\mathcal{H}^n$ passing through p. One checks that $\mathcal{T}(\mathcal{H}^n, p)$ is isomorphic to the affine Tits complex $A\mathcal{T}(\mathbb{R}^{n-1})$ of flags of proper affine subspaces of $\mathbb{R}^{n-1}$ (note: $\partial\mathcal{H}^n \cong \mathbb{R}^{n-1} \cup \{\infty\}$). The homology of this space is concentrated in dimension $n-2$. Set

$$\begin{aligned} \mathrm{St}(\mathcal{H}^n, p) &= \tilde{H}_{n-2}(\mathcal{T}(\mathcal{H}^n, p), \mathbb{Z}), n \geq 2 \\ \mathrm{ASt}(\mathbb{R}^{n-1}) &= \tilde{H}_{n-2}(A\mathcal{T}(\mathbb{R}^{n-1}), \mathbb{Z}), n \geq 2. \end{aligned}$$

The group $\mathrm{St}(\mathcal{H}^n, p)$ is a module over the isotropy group $G(n)_p$ and the group $\mathrm{ASt}(\mathbb{R}^{n-1})$ is a module for the group of affine transformations of $\mathbb{R}^{n-1}$. In our model, $G(n)_\infty$ acts on $\mathbb{R}^{n-1} \subset \partial\mathcal{H}^n$ as the group $\mathrm{Sim}(n-1)$ of similarities of $\mathbb{R}^{n-1}$.

One now checks that there is an exact sequence of $G(n)$-modules

$$0 \longrightarrow \mathrm{St}(\mathcal{H}^n) \longrightarrow \mathrm{St}(\overline{\mathcal{H}}^n) \longrightarrow \bigoplus_{p \in \partial\mathcal{H}^n} \mathrm{St}(\mathcal{H}^n, p) \longrightarrow 0$$

as follows.

Consider the long exact homology sequence for the pair $(\mathcal{T}(\overline{\mathcal{H}}^n), \mathcal{T}(\mathcal{H}^n))$. Clearly there is an isomorphism of complexes

$$C_\bullet(\mathcal{T}(\overline{\mathcal{H}}^n), \mathcal{T}(\mathcal{H}^n)) \cong \bigoplus_{p \in \partial \mathcal{H}^n} C_{\bullet-1}(\mathcal{T}(\mathcal{H}^n), p).$$

Each factor of the summand has homology only in dimension $n-2$, and each space in the pair $(\mathcal{T}(\overline{\mathcal{H}}^n), \mathcal{T}(\mathcal{H}^n))$ has homology concentrated in degree $n-1$. Thus the only relevant portion of the long exact sequence is

$$0 \longrightarrow H_{n-1}(\mathcal{T}(\mathcal{H}^n)) \longrightarrow H_{n-1}(\mathcal{T}(\overline{\mathcal{H}}^n)) \longrightarrow \bigoplus_{p \in \partial \mathcal{H}^n} H_{n-2}(\mathcal{T}(\mathcal{H}^n), p) \longrightarrow 0.$$

The group $\mathrm{Sim}(n)$ is a semidirect product

$$\mathrm{Sim}(n) = T(n) \cdot \mathrm{Sim}_0(n),$$

where $T(n) \cong \mathbb{R}^n$ is the subgroup of translations and $\mathrm{Sim}_0(n)$ is the group of linear similarities.

It is easy to see (via a center kills argument) that

$$H_\bullet(\mathrm{Sim}_0(n), \mathbb{Z}^t) \xrightarrow{\cong} H_\bullet(\mathrm{Sim}(n), \mathbb{Z}^t).$$

The crucial step in the proof is showing the following:

LEMMA 3.1.6. *For all $n > 0$,*

$$H_\bullet(\mathrm{Sim}(n), \mathrm{ASt}(\mathbb{R}^n)^t) = 0.$$

PROOF. The proof is by induction on n. For $n = 1$, we have the exact sequence of $\mathrm{Sim}(1)$-modules

$$0 \longrightarrow \mathrm{ASt}(\mathbb{R}^1) \longrightarrow \bigoplus_{p \in \mathbb{R}^1} \mathbb{Z}(p) \xrightarrow{\varepsilon} \mathbb{Z} \longrightarrow 0$$

where ε is the augmentation. Shapiro's lemma shows that

$$H_\bullet(\mathrm{Sim}(1), \{\bigoplus_{p \in \mathbb{R}^1} \mathbb{Z}(p)\}^t) \cong H_\bullet(\mathrm{Sim}_0(1), \mathbb{Z}^t),$$

the isomorphism being induced by ε and the inclusion $\mathrm{Sim}_0(1) \to \mathrm{Sim}(1)$. Since the right hand side of the above equation is isomorphic to $H_\bullet(\mathrm{Sim}(1), \mathbb{Z}^t)$, the long exact coefficient sequence associated to the above sequence of modules shows that

$$H_\bullet(\mathrm{Sim}(1), \mathrm{ASt}(\mathbb{R}^1)^t) = 0.$$

Now assume that $n > 1$. There is an exact sequence of $\mathrm{Sim}(n)$-modules

$$0 \longrightarrow \mathrm{ASt}(\mathbb{R}^n) \longrightarrow \bigoplus_{V^{n-1}} \mathrm{ASt}(V^{n-1}) \longrightarrow \cdots \longrightarrow \bigoplus_{V^0} \mathrm{ASt}(V^0) \xrightarrow{\varepsilon} \mathbb{Z} \longrightarrow 0$$

where the sum over V^j runs over all j-dimensional affine subspaces of $\mathbb{R}^n$. The stabilizer of $\mathbb{R}^j \subset \mathbb{R}^n$ is a product

$$\mathrm{Sim}(n)_{\mathbb{R}^j} \cong \mathrm{Sim}(j) \times O(n-j, \mathbb{R})$$

where $O(n-j, \mathbb{R})$ is the orthogonal group of $\mathbb{R}^{n-j}$. By Shapiro's lemma, we have

$$H_\bullet(\mathrm{Sim}(n), \{\bigoplus_{V^j} \mathrm{ASt}(V^j)\}^t) \cong H_\bullet(\mathrm{Sim}(j) \times O(n-j, \mathbb{R}), \mathrm{ASt}(\mathbb{R}^j)^t \otimes \mathbb{Z}^t).$$

Using the induction hypothesis and the Künneth formula, we see that the right hand side of this equation vanishes for $0 < j < n$. Similarly,

$$H_\bullet(\mathrm{Sim}(n), \{\bigoplus_{V^0} \mathrm{ASt}(V^0)\}^t) \cong H_\bullet(\mathrm{Sim}_0(n), \mathbb{Z}^t).$$

Now split the above exact sequence of modules into short exact sequences

$$0 \longrightarrow Z_0 \longrightarrow \bigoplus_{V^0} \mathrm{ASt}(V^0) \longrightarrow \mathbb{Z} \longrightarrow 0,$$

$$0 \longrightarrow Z_j \longrightarrow \bigoplus_{V^j} \mathrm{ASt}(V^j) \longrightarrow Z_{j-1} \longrightarrow 0,$$

$$0 \longrightarrow \mathrm{ASt}(\mathbb{R}^n) \longrightarrow \bigoplus_{V^{n-1}} \mathrm{ASt}(V^{n-1}) \longrightarrow Z_{n-1} \longrightarrow 0.$$

Then we see

$$\begin{aligned} H_k(\mathrm{Sim}(n), \mathrm{ASt}(\mathbb{R}^n)^t) &\cong H_{k+1}(\mathrm{Sim}(n), Z_{n-2}^t) \\ &\cong \cdots \cong H_{k+n-1}(\mathrm{Sim}(n), Z_0^t) \\ &= 0 \end{aligned}$$

for all $k \geq 0$. □

Given this, the proof of Theorem 3.1.5 is finished via Shapiro's lemma and the long exact coefficient sequence:

$$\begin{aligned} H_\bullet(G(n), \bigoplus_{p \in \partial \mathcal{H}^n} \mathrm{St}(\mathcal{H}^n, p)^t) &= H_\bullet(G(n)_\infty, \mathrm{St}(\mathcal{H}^n, \infty)^t) \\ &= H_\bullet(\mathrm{Sim}(n-1), \mathrm{ASt}(\mathbb{R}^{n-1})^t) \\ &= 0. \end{aligned}$$

□

Dupont and Sah [**31**] then prove that the groups $\mathcal{P}(\partial \mathcal{H}^3)$, $\mathcal{P}(\mathcal{H}^3)$, and $\mathcal{P}(\overline{\mathcal{H}^3})$ are divisible. This was later improved by Sah [**107**] who showed that $\mathcal{P}(\mathcal{H}^3)$ is uniquely divisible. The method of proof is the following. Let F be

a field and define an abelian group $\mathfrak{p}(F)$ as the group with generators $[x]$, $x \in F^\times - \{1\}$, and relations

$$[x] - [y] + [y/x] + [(1-x)/(1-y)] - [(1-x^{-1})/(1-y^{-1})]$$

for $x \neq y$. Using explicit technical calculations in $\mathfrak{p}(F)$, Dupont–Sah prove the following result.

THEOREM 3.1.7. *If F is algebraically closed, then the group $\mathfrak{p}(F)$ is divisible.* □

From this we deduce the divisibility of $\mathcal{P}(\mathcal{H}^3)$ and $\mathcal{P}(\overline{\mathcal{H}}^3)$. According to Sah [**107**], Theorem 4.16, there is an isomorphism

$$\mathcal{P}(\mathcal{H}^3) \cong \mathfrak{p}(\mathbb{C})^-,$$

where $\mathfrak{p}(\mathbb{C})^-$ is the negative eigenspace under the action of complex conjugation. The divisibility of $\mathcal{P}(\mathcal{H}^3)$ (and by Theorem 3.1.5 that of $\mathcal{P}(\overline{\mathcal{H}}^3)$) follows at once.

The study of $\mathfrak{p}(F)$ is intimately related to the computation of the low-dimensional homology of $SL_2(F)$. The following theorem is an unpublished result of Bloch and Wigner.

THEOREM 3.1.8 ([**31**]). *Let F be an algebraically closed field of characteristic zero. Then there is an exact sequence*

$$0 \to \mu_F(2) \to H_3(SL_2(F)) \to \mathfrak{p}(F) \xrightarrow{\lambda} \bigwedge^2_{\mathbb{Z}}(F^\times) \xrightarrow{sym} K_2(F) \to 0.$$

Here, $\mu_F(2)$ is the group μ_F of roots of unity in F with $Aut(F)$ acting via the quadratic character. For $z \in F^\times - \{1\}$, $\lambda([z]) = z \wedge (1-z)$ and for $u, v \in F^\times$, $sym(u,v) = \{u,v\}$, the Steinberg symbol in $K_2(F)$. □

The map $H_3(SL_2(F)) \to \mathfrak{p}(F)$ is induced by sending the homogeneous 3-chain (g_0, g_1, g_2, g_3), $g_i \in SL_2(F)$ to $[z]$, where z is the cross-ratio of the points $g_0(\infty), g_1(\infty), g_2(\infty), g_3(\infty) \in \mathbb{P}^1(F)$ (recall that the cross-ratio of the complex numbers z_0, z_1, z_2, z_3 is $(z_0 - z_1)(z_3 - z_2)/(z_0 - z_2)(z_3 - z_1)$). This map is independent of the choice of ∞ as a base point.

We omit the proof of this theorem. Instead we shall summarize Suslin's study of a certain subgroup $B(F) \subset \mathfrak{p}(F)$ in the next section.

3.2. The Bloch Group

3.2.1. Relation with $H_3(GL_2(F))$. Let F be any field and denote by $\mathcal{D}(F)$ the free abelian group with basis $[z]$, where $z \in F^\times - \{1\}$. Define a homomorphism

$$\varphi : \mathcal{D}(F) \longrightarrow F^\times \otimes F^\times$$

by $\varphi([z]) = z \otimes (1-z)$. An easy calculation shows that

$$\varphi\left([x] - [y] + \left[\frac{y}{x}\right] - \left[\frac{1-x^{-1}}{1-y^{-1}}\right] + \left[\frac{1-x}{1-y}\right]\right) = \tag{3.1}$$
$$x \otimes \left(\frac{1-x}{1-y}\right) + \left(\frac{1-x}{1-y}\right) \otimes x.$$

Let σ be the involution of $F^\times \otimes F^\times$ defined by $\sigma(x \otimes y) = -(y \otimes x)$ and denote by $(F^\times \otimes F^\times)_\sigma$ the quotient of $F^\times \otimes F^\times$ by the action of σ. Then (3.1) shows that φ induces a homomorphism

$$\lambda : \mathfrak{p}(F) \longrightarrow (F^\times \otimes F^\times)_\sigma.$$

Note that λ is the homomorphism λ in the exact sequence of Theorem 3.1.8.

DEFINITION 3.2.1. The Bloch group of F, denoted by $B(F)$, is the kernel of the homomorphism λ.

Thus, we have an exact sequence

$$0 \longrightarrow B(F) \longrightarrow \mathfrak{p}(F) \xrightarrow{\lambda} (F^\times \otimes F^\times)_\sigma \longrightarrow K_2(F) \longrightarrow 0.$$

In the case where F is an algebraically closed field of characteristic zero, Theorem 3.1.8 asserts that there is an exact sequence

$$0 \longrightarrow \mu_F(2) \longrightarrow H_3(SL_2(F), \mathbb{Z}) \longrightarrow B(F) \longrightarrow 0.$$

Suslin [**128**] produced an exact sequence involving the groups $B(F)$ and $H_3(GL_2(F), \mathbb{Z})$ for any infinite field F. We describe this result now.

Denote by T_n (resp. B_n) the subgroup of $GL_n(F)$ consisting of diagonal (resp. upper triangular) matrices and let $GM_n(F)$ be the subgroup of monomial matrices. We have an exact sequence

$$1 \longrightarrow T_n \longrightarrow GM_n(F) \longrightarrow \Sigma_n \longrightarrow 1,$$

where Σ_n denotes the symmetric group on n letters.

THEOREM 3.2.2 ([**128**], Theorem 2.1). *Let F be an infinite field. Then there is an exact sequence*

$$H_3(GM_2(F)) \longrightarrow H_3(GL_2(F)) \longrightarrow B(F) \longrightarrow 0.$$

PROOF. Let $C_\bullet$ be the complex with C_p the free abelian group with basis $(x_0, \ldots, x_p)$, where the x_i are distinct points of $\mathbb{P}^1(F)$ (this is the same as taking projective equivalence classes of unimodular vectors in F^2). By Lemma 2.3.2, the augmented complex $C_\bullet \to \mathbb{Z} \to 0$ is acyclic.

The group $GL_2(F)$ acts 3-transitively on points in $\mathbb{P}^1(F)$; that is, it acts transitively on C_0, C_1, C_2. Every orbit of the action on C_p for $p \geq 3$ has

a unique representative of the form $(0, \infty, 1, x_1, \ldots, x_{p-2})$, where the $x_i \in F^\times - \{1\}$ are distinct. Denote this orbit by $[x_1, \ldots, x_{p-2}]$. Then

$$(C_p)_{GL_2(F)} = \bigoplus_{x_i \neq x_j \in F^\times - \{1\}} \mathbb{Z}[x_1, \ldots, x_{p-2}],$$

and the differential $d : (C_p)_{GL_2(F)} \to (C_{p-1})_{GL_2(F)}$ is

$$\begin{aligned} d([x_1, \ldots, x_{p-2}]) &= \left[\frac{1-x_1}{1-x_2}, \ldots, \frac{1-x_1}{1-x_{p-2}}\right] \\ &\quad - \left[\frac{1-x_1^{-1}}{1-x_2^{-1}}, \ldots, \frac{1-x_1^{-1}}{1-x_{p-2}^{-1}}\right] \\ &\quad + \left[\frac{x_2}{x_1}, \ldots, \frac{x_{p-2}}{x_1}\right] \\ &\quad + \sum_{i=1}^{p-2} (-1)^i [x_1, \ldots, \hat{x}_i, \ldots, x_{p-2}]. \end{aligned} \tag{3.2}$$

Consider the spectral sequence associated to the action of $GL_2(F)$ on $C_\bullet$. This spectral sequence converges to $H_\bullet(GL_2(F))$ and has E^1-term

$$E^1_{p,q} = H_q(GL_2(F), C_p).$$

In particular, the row $q = 0$ is the complex $(C_\bullet)_{GL_2(F)}$. It has the form

$$\mathbb{Z} \leftarrow \mathbb{Z} \leftarrow \mathbb{Z} \leftarrow \bigoplus_{x \in F^\times - \{1\}} \mathbb{Z}[x] \leftarrow \bigoplus_{x_1 \neq x_2} \mathbb{Z}[x_1, x_2] \leftarrow \cdots.$$

Thus we have

$$E^2_{p,0} = \begin{cases} \mathbb{Z} & p = 0 \\ 0 & p = 1, 2; \end{cases}$$

and

$$\begin{aligned} E^2_{3,0} &= \bigoplus_x \mathbb{Z}[x] / \left\langle \left[\frac{1-x_1}{1-x_2}\right] - \left[\frac{1-x_1^{-1}}{1-x_2^{-1}}\right] + \left[\frac{x_2}{x_1}\right] - [x_2] + [x_1] \right\rangle \\ &= \mathfrak{p}(F). \end{aligned}$$

Now, the stabilizer of 0 is the subgroup B_2, the stabilizer of $(0, \infty)$ is $T_2 = F^\times \oplus F^\times$, and the stabilizer of $(0, \infty, 1)$ is the subgroup $F^\times$ of scalar matrices. Thus,

$$E^1_{0,q} = H_q(B_2), \quad E^1_{1,q} = H_q(T_2), \quad E^1_{2,q} = H_q(F^\times).$$

We know that the inclusion $T_2 \to B_2$ induces an isomorphism in homology; that is,

$$H_q(F^\times \oplus F^\times) = H_q(T_2) = H_q(B_2).$$

Let σ be the automorphism of $H_\bullet(F^\times \oplus F^\times)$ induced by $(a,b) \mapsto (b,a)$. Then an easy calculation shows that

$$d^1_{1,q} : H_q(T_2) \longrightarrow H_q(B_2)$$

is the map $\sigma - \mathrm{id}$. Also, if $\Delta : F^\times \to F^\times \oplus F^\times$ is the diagonal embedding, then

$$d^1_{2,q} : H_q(F^\times) = E^1_{2,q} \longrightarrow E^1_{1,q} = H_q(F^\times \oplus F^\times)$$

is the map Δ_*.

Since Δ has a left inverse, Δ_* is injective so that $E^2_{2,\bullet} = 0$. Also, $H_1(F^\times \oplus F^\times) = F^\times \oplus F^\times$ and σ transposes terms. Furthermore,

$$H_2(F^\times \oplus F^\times) = H_2(F^\times) \oplus H_2(F^\times) \oplus (F^\times \otimes F^\times),$$

σ interchanges the copies of $H_2(F^\times)$ and σ coincides with the involution $x \otimes y \mapsto -(y \otimes x)$ on $F^\times \otimes F^\times$. Thus the E^2-term of the spectral sequence is

$$\begin{array}{lll} H_3(T_2)_\sigma & & \\ H_2(F^\times) \oplus (F^\times \otimes F^\times)_\sigma & (F^\times \otimes F^\times)^\sigma & \\ F^\times & 0 & 0 \\ \mathbb{Z} & 0 & 0 \quad \mathfrak{p}(F). \end{array}$$

A long computation shows that

$$d^3 : E^3_{3,0} = \mathfrak{p}(F) \to H_2(F^\times) \oplus (F^\times \otimes F^\times)_\sigma = E^3_{0,2}$$

is defined as $d^3([x]) = x \wedge (1-x) - x \otimes (1-x)$. Thus,

$$E^\infty_{3,0} = \ker(d^3) = B(F).$$

Now, this spectral sequence determines a filtration of $H_3(GL_2(F))$ with $H_3(GL_2(F))^{(0)} = \mathrm{im}(H_3(T_2))$. There is a canonical surjection

$$H_2(T_2)^\sigma \twoheadrightarrow H_2(T_2)^\sigma / \Delta_*(H_2(F^\times)) = (F^\times \otimes F^\times)^\sigma \twoheadrightarrow H_3(GL_2(F))^{(1)/(0)}.$$

Moreover, we have an isomorphism

$$H_3(GL_2(F))/H_3(GL_2(F))^{(2)} \xrightarrow{\cong} B(F).$$

To finish the proof of the theorem, we must show that $H_3(GL_2(F))^{(2)} = \mathrm{im}(H_3(GM_2(F)))$.

Let $C_\bullet(G)$ be the standard resolution of the group G and let ρ_g be the homotopy denoted by $h_\bullet : C_\bullet(G,M) \to C_\bullet(G,M)$ in Proposition A.1.9.

LEMMA 3.2.3. *Let $u \in H_2(T_2)^\sigma$, and let h be a representing cycle for u. Denote by τ the automorphism of $H_2(T_2)^\sigma$ which transposes terms and write $\tau(h) - h = db$ for some 3-chain b of T_2. Then the image of u in $H_3(GL_2(F))/H_3(T_2)$ is the class of the 3-cycle $b - \rho_s(h)$, where $s = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$.*

PROOF. Our spectral sequence is the spectral sequence of the bicomplex $C_\bullet(GL_2(F)) \otimes_{GL_2(F)} C_\bullet$. The augmentation $\varepsilon : C_\bullet \to \mathbb{Z}$ induces a quasi-isomorphism

$$C_\bullet(GL_2(F)) \otimes_{GL_2(F)} C_\bullet \xrightarrow{\varepsilon} C_\bullet(GL_2(F))_{GL_2(F)}.$$

Consider the diagram

$$\begin{array}{ccc} b - \rho_s(h) & \xleftarrow{\varepsilon} & b \otimes (0) - \rho_s(h) \otimes (\infty) \\ & & \Big\downarrow{\scriptstyle d_{GL_2(F)}} \\ & & h \otimes ((\infty) - (0)) \xleftarrow{d_{C_\bullet}} h \otimes (0, \infty). \end{array}$$

Note that

$$\begin{aligned} d(b \otimes (0) - \rho_s(h) \otimes (\infty)) &= db \otimes (0) - d\rho_s(h) \otimes (\infty) \\ &= \tau(h) \otimes (0) - h \otimes (0) \\ &\quad + h \otimes (\infty) - shs^{-1} \otimes s(\infty) \\ &= \tau(h) \otimes (0) - h \otimes (0) \\ &\quad + h \otimes (\infty) - \tau(h) \otimes (0) \\ &= h \otimes ((\infty) - (0)). \end{aligned}$$

Thus, $h \mapsto b - \rho_s(h) \in H_3(GL_2(F))/H_3(T_2)$. □

Now consider the Hochschild–Serre spectral sequence associated to the extension

$$1 \longrightarrow T_2 \longrightarrow GM_2(F) \longrightarrow \Sigma_2 \longrightarrow 1.$$

This yields a filtration of $H_3(GM_2(F))$ satisfying

1. $H_3(GM_2(F))^{(0)} = \mathrm{im}(H_3(T_2))$;
2. $H_2(T_2)^\sigma \twoheadrightarrow H_1(\Sigma_2, H_2(T_2)) = H_2(T_2)^\sigma/(1+\sigma)H_2(T_2) \twoheadrightarrow H_3(GM_2(F))^{(1)/(0)}$;
3. $H_3(GM_2(F))^{(2)/(1)} = 0$ (since $H_\bullet(\Sigma_2, T_2) = 0$);
4. $H_3(GM_2(F)) = H_3(GM_2(F))^{(2)} \oplus H_3(\Sigma_2)$;

(the direct sum decomposition in 4 follows since the extension is split). The preceding remarks show that the homomorphism

$$H_3(GM_2(F)) \longrightarrow H_3(GL_2(F))$$

maps $H_3(GM_2(F))^{(0)}$ onto $H_3(GL_2(F))^{(0)}$. Lemma 3.2.3 tells us the image of $u \in H_2(T_2)^\sigma$ in $H_3(GM_2(F))/H_3(T_2)$. Thus, $H_3(GM_2(F))^{(1)}$ is mapped onto $H_3(GL_2(F))^{(1)}$ and we have

$$\mathrm{im}(H_3(GM_2(F))) = \mathrm{im}(H_3(\Sigma_2)) \oplus H_3(GL_2(F))^{(2)}$$

(since the first and second filtration levels coincide). Now, since the matrix s is similar to $\left(\begin{smallmatrix} 1 & 1 \\ 0 & -1 \end{smallmatrix}\right) \in B_2$, we have

$$\mathrm{im}(H_3(\Sigma_2)) \subset \mathrm{im}(H_3(B_2)) = \mathrm{im}(H_3(T_2)) = H_3(GL_2(F))^{(0)}.$$

Therefore, $\mathrm{im}(H_3(GM_2(F))) = H_3(GL_2(F))^{(2)}$, and the theorem is proved. □

REMARK 3.2.4. The proof of Theorem 3.2.2 follows the argument used by Dupont–Sah [**31**] to prove Theorem 3.1.8. In this form, it works for any infinite field. The proof in [**31**], while giving a slightly stronger statement, relies heavily on the fact that F is algebraically closed.

3.2.2. Relation with K-theory. Since $H_3(GL_2(F))$ influences $K_3(F)$, we might expect some relationship between $K_3(F)$ and $B(F)$. It was S. Bloch [**11**] who first noticed this connection (hence the name "Bloch group"). Suslin made this relationship precise and we briefly sketch his results.

The first step is to extend the homomorphism $H_3(GL_2(F)) \to B(F)$ to a map $H_3(GL_3(F)) \to B(F)$. This is accomplished by considering the complex $C^2_\bullet$, where C^2_p is the free abelian group with basis $(x_0, \ldots, x_p)$, where the $x_i \in \mathbb{P}^2(F)$ are in general position. Define two differentials $d, d' : C^2_p \to C^2_{p-1}$ by

$$d(x_0, \ldots, x_p) = \sum_{i=0}^{p} (-1)^i (x_0, \ldots, \hat{x}_i, \ldots, x_p)$$

and

$$d'(x_0, \ldots, x_p) = \sum_{i=1}^{p-1} (-1)^i (x_0, \ldots, \hat{x}_i, \ldots, x_p).$$

Then an easy calculation shows that the complexes $(C_\bullet, d) \xrightarrow{\varepsilon} \mathbb{Z} \to 0$ and $(C_\bullet, d') \xrightarrow{\varepsilon} \mathbb{Z} \to 0$ are acyclic.

Define a map $\lambda : C^2_p \to C^2_p$ by

$$\lambda(x_0, \ldots, x_p) = \sum_{k=0}^{p} (-1)^{pk} (x_k, x_{k+1}, \ldots, x_{k+p}).$$

One checks easily that $d'\lambda = \lambda d$. Consider the map of complexes

$$\begin{array}{ccccccc}
C^2_1 & \xleftarrow{d} & C^2_2 & \xleftarrow{d} & C^2_3 & \xleftarrow{d} & \cdots \\
\downarrow{\scriptstyle \lambda} & & \downarrow{\scriptstyle \lambda} & & \downarrow{\scriptstyle \lambda} & & \\
C^2_1 & \xleftarrow{d'} & C^2_2 & \xleftarrow{d'} & C^2_3 & \xleftarrow{d'} & \cdots .
\end{array}$$

Let $D_\bullet$ be the cone; that is, $D_0 = C_1^2$, $D_p = C_p^2 \oplus C_{p+1}^2$ with differential given by the matrix

$$\begin{pmatrix} -d & 0 \\ \lambda & d' \end{pmatrix}.$$

Define the augmentation $D_0 \to \mathbb{Z}$ as the composition

$$D_0 = C_1^2 \xrightarrow{d} C_0^2 \xrightarrow{\varepsilon} \mathbb{Z}.$$

Then it is easy to see that $D_\bullet \to \mathbb{Z} \to 0$ is acyclic. The group $GL_3(F)$ acts on $D_\bullet$ and determines a map $H_3(GL_3(F)) \to H_3((D_\bullet)_{GL_3(F)})$. Direct computation shows that

$$H_3((D_\bullet)_{GL_3(F)}) = \operatorname{coker}\left(\bigoplus \mathbb{Z}\left[\begin{smallmatrix} a \\ x \end{smallmatrix}\right] \oplus \bigoplus \mathbb{Z}\left[\begin{smallmatrix} a & b \\ x & y \end{smallmatrix}\right] \xrightarrow{\begin{pmatrix} -\bar{d} & 0 \\ \bar{\lambda} & \bar{d} \end{pmatrix}} \mathbb{Z} \oplus \bigoplus \mathbb{Z}\left[\begin{smallmatrix} a \\ x \end{smallmatrix}\right]\right)$$

where $a \neq x$, $b \neq y \in F^\times - \{1\}$ and $ay - bx \neq 0$. By setting $\rho([\begin{smallmatrix} a \\ x \end{smallmatrix}]) = [a] \in \mathfrak{p}(F)$, we get a well-defined homomorphism $H_3((D_\bullet)_{GL_3(F)}) \to \mathfrak{p}(F)$. Hence by composing with the canonical map $H_3(GL_3(F)) \to H_3((D_\bullet)_{GL_3(F)})$, we obtain a map $H_3(GL_3(F)) \to \mathfrak{p}(F)$.

The restriction of this map to $H_3(GL_2(F))$ coincides with the map $H_3(GL_2(F)) \to \mathfrak{p}(F)$ defined above. This is proved as follows (see Lemma 3.4 of [128]). Denote by C'_p the subgroup of C_{p+1}^2 generated by elements $(x_0, \ldots, x_{p+1})$, where $x_{p+1} = (0,0,1)^T$. Since d' does not affect the last coordinate, $C'_\bullet$ is a $GL_2(F)$-subcomplex of $D_\bullet$. Thus, the composition

$$H_3(GL_2(F)) \longrightarrow H_3(GL_3(F)) \longrightarrow H_3((D_\bullet)_{GL_3(F)})$$

can also be written as the composition

$$H_3(GL_2(F)) \longrightarrow H_3((C'_\bullet)_{GL_2(F)}) \longrightarrow H_3((D_\bullet)_{GL_3(F)}).$$

Induce a homomorphism of $GL_2(F)$-complexes $C'_\bullet \to C_\bullet$ by projecting $\mathbb{P}^2(F) \to \mathbb{P}^1(F)$ from $(0,0,1)^T$ and discarding the last coordinate. The composition

$$H_3(GL_2(F)) \longrightarrow H_3((C'_\bullet)_{GL_2(F)}) \longrightarrow H_3((C_\bullet)_{GL_3(F)}) = \mathfrak{p}(F)$$

is evidently the homomorphism ∂ defined above. The proof is finished by noting that the following diagram

$$\begin{array}{ccc} (C'_3)_{GL_2(F)} & \longrightarrow & (C_3)_{GL_2(F)} \\ \downarrow & & \downarrow \\ (D_3)_{GL_3(F)} & \xrightarrow{\rho} & \mathfrak{p}(F) \end{array} \tag{3.3}$$

commutes.

PROPOSITION 3.2.5. *We have an isomorphism*

$$H_3(GL_3(F))/(H_3(GM_2(F)) + H_3(T_3)) \xrightarrow{\cong} B(F).$$

PROOF. The group $H_3(T_3)$ lies in the kernel since the augmentation $\varepsilon_{D_\bullet} : C_1^2 \to \mathbb{Z}$ has a T_3-equivariant section:

$$n \longmapsto n \cdot \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

By [**124**], we know that $H_3(GL_3(F))$ is generated by $H_3(GL_2(F))$ and $H_3(T_3)$. Thus, the image of $H_3(GL_3(F)) \to \mathfrak{p}(F)$ coincides with that of $H_3(GL_2(F))$; that is, the image is $B(F)$. Since we have an isomorphism

$$H_3(GL_2(F))/H_3(GM_2(F)) \cong B(F),$$

the result follows. □

Denote the map $H_3(GL_3(F)) \to B(F)$ by ∂. Since

$$H_3(GL_3(F)) \cong H_3(GL(F))$$

(Theorem 2.3.1), we can view ∂ as a map $H_3(GL(F)) \to B(F)$. Denote by $H_3^0(GM_n(F))$ the kernel of the split surjection

$$H_3(GM_n(F)) \longrightarrow H_3(\Sigma_n).$$

LEMMA 3.2.6. *There is an exact sequence*

$$H_3^0(GM(F)) \longrightarrow H_3(GL(F)) \longrightarrow B(F) \longrightarrow 0.$$

PROOF. The first step is to show that $H_3(T)$ lies in the kernel of ∂. The Künneth formula implies that $H_3(T)$ is generated by the subgroups $H_3(T^{ijk})$, where T^{ijk} is the subgroup consisting of diagonal matrices with 1's everywhere except for possibly in positions i, j, k. Note, however, that T^{ijk} is conjugate to T_3 by a permutation matrix and hence the image of each $H_3(T^{ijk})$ coincides with that of $H_3(T_3)$. But $H_3(T_3)$ lies in the kernel of ∂.

Now let m, n be positive integers. For a pair G_1, G_2 of groups, denote by p_i the projection $G_1 \times G_2 \to G_i$. Denote by q_i the composition

$$GL_m(F) \times GL_n(F) \xrightarrow{p_i} GL_*(F) \longrightarrow GL(F).$$

Also, denote by j the embedding $GL_m(F) \times GL_n(F) \to GL_{m+n}(F) \to GL(F)$. Note that for any n, the embedding $T_n \to GL_n(F)$ induces a surjection on H_1 and H_2 (for H_1 this is clear; for H_2 this follows from Theorem 2.3.1 since we have a surjection $K_2^M(F) \longrightarrow H_2(GL_2(F))$ induced by the homology product $H_1(T_1) \otimes H_1(T_1) \to H_2(GL_2(F))$ and the domain of the latter map lies inside $H_2(T_2)$). Thus, since $j(T_m \times T_n) \subset T$, we see that

$$j_*(H_2(GL_m(F)) \otimes H_1(GL_n(F))) \subset H_3(T).$$

Now let n be sufficiently large and consider the commutative diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & T_2 & \longrightarrow & GM_2(F) \times \Sigma_{n-2} & \longrightarrow & \Sigma_2 \times \Sigma_{n-2} & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & T_n & \longrightarrow & GM_n(F) & \longrightarrow & \Sigma_n & \longrightarrow & 1. \end{array}$$

Denote the Hochschild–Serre spectral sequence of the top sequence by E' and of the lower sequence by E. We have

$$E^2_{pq} = H_p(\Sigma_n, H_q(T_n)).$$

Denote the ith summand of $T_n = (F^\times)^n$ by $F_i^\times$. Then

$$H_2(T_n) = \bigoplus_{i=1}^{n} \bigwedge^2 F_i^\times \oplus \bigoplus_{i<j} (F_i^\times \otimes F_j^\times).$$

The action of Σ_n is transitive on both terms of the summand with stabilizers

$$\bigwedge^2 (F_1^\times)_{\Sigma_n} = \Sigma_1 \times \Sigma_{n-1}$$

and

$$(F_1^\times \otimes F_2^\times)_{\Sigma_n} = \Sigma_2 \times \Sigma_{n-2}.$$

Thus we have

$$E^2_{p2} = H_p(\Sigma_{n-1}, \bigwedge^2 F^\times) \oplus H_p(\Sigma_2 \times \Sigma_{n-2}, F^\times \otimes F^\times).$$

Similarly, we have

$$E'^2_{p2} = H_p(\Sigma_{n-2}, \bigwedge^2 F^\times) \oplus H_p(\Sigma_2 \times \Sigma_{n-2}, F^\times \otimes F^\times).$$

The homology of Σ_n stabilizes for large n, and hence we see that for $p \leq 5$, the two spectral sequences coincide. The same is true for the $q = 1$ rows of the sequences. It follows that the $(1,2)$ and $(2,1)$ terms of the limits coincide and that

$$H_3^0(GM_n(F)) = H_3(T_n) + \ker\{H_3(GM_2(F) \times \Sigma_{n-2}) \longrightarrow H_3(\Sigma_2 \times \Sigma_{n-2})\}.$$

Passing to the limit, we see that the image of $H_3^0(GM_n(F))$ is contained in $H_3(T) + H_3(GM_2(F))$, which is precisely the kernel of ∂. The reverse inclusion is clear. □

We now have the following result.

THEOREM 3.2.7. *The Hurewicz map induces an isomorphism*

$$K_3(F)/\pi_3^0(BGM(F)^+) \xrightarrow{\cong} H_3(GL(F))/H_3^0(GM(F)),$$

where $\pi_3^0(BGM(F)^+)$ *is the kernel of the projection* $\pi_3(BGM(F)^+) \to \pi_3(\Sigma^+)$.

PROOF. This is based on a number of lemmas whose proofs may be found in Section 5 of [**128**]. Let $h : S^3 \to S^2$ be the Hopf bundle. This induces a map $h^* : \pi_2(X) \to \pi_3(X)$ for any space X. The map h^* is not a homomorphism in general, but if X is an H–space, then it is a homomorphism. One checks that if X is a simply-connected H -space, then there is an exact sequence

$$\pi_2(X) \xrightarrow{h^*} \pi_3(X) \longrightarrow H_3(X) \longrightarrow 0.$$

Taking $X = E(\Lambda) \subset GL(\Lambda)$, we obtain an exact sequence

$$K_2(\Lambda) \xrightarrow{h^*} K_3(\Lambda) \longrightarrow H_3(E(\Lambda)) \longrightarrow 0.$$

Now consider the group $GM(F)$ and its commutator subgroup M. The group M is the subgroup of monomial matrices having determinant 1 and an even permutation. This group is perfect. Consider the spaces $BSL(F)^+$ and BM^+. We have a commutative diagram

$$\begin{array}{ccccccc} \pi_2(BM^+) & \longrightarrow & \pi_3(BM^+) & \longrightarrow & H_3(M) & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow & & \\ K_2(F) & \longrightarrow & K_3(F) & \longrightarrow & H_3(SL(F)) & \longrightarrow & 0. \end{array}$$

Now, we have homotopy equivalences

$$B([G,G])^+ \times BG^{\mathrm{ab}} \xrightarrow{\simeq} BG^+ \tag{3.4}$$

for the groups $G = GL(F), GM(F)$ ([**128**], Lemma 5.3). Thus, for $i \geq 2$, $\pi_i(BM^+) = \pi_i(BGM(F)^+)$. Moreover, for $i \leq 2$, the map $\pi_i(BGM(F)^+) \to K_i(F)$ is surjective. Thus,

$$K_3(F)/\pi_3(BGM(F)^+) = H_3(SL(F))/H_3(M).$$

The same is true if we divide out by $\pi_3^0(BGM(F)^+)$ and by

$$H_3^0(M) = \ker(H_3(M) \to H_3(\Sigma^+)).$$

Denote by $SM(F)$ the group of monomial matrices in $SL(F)$. The projection $SM(F) \to SM(F)^{\mathrm{ab}} \cong \mathbb{Z}/2$ has a section. Using the Künneth formula and the equivalence (3.4) for $SM(F)$, we see that

$$H_3(SL(F))/H_3(M) = H_3(SL(F))/H_3(SM(F))$$

and similarly if we divide by H_3^0.

Now, applying the equivalence (3.4) to $G = GL(F)$, we find

$$H_3(GL(F)) = H_3(SL(F)) \oplus (H_2(SL(F)) \otimes F^\times) \oplus H_3(F^\times).$$

Since the last two terms lie in $H_3(GM(F))$, the map

$$H_3(SL(F))/H_3(SM(F)) \longrightarrow H_3(GL(F))/H_3(GM(F))$$

is surjective. We define a left inverse by $\alpha \mapsto \alpha \oplus \det(\alpha^{-1})$. (Note: the fact that $H_2(SL(F)) \otimes F^\times \subset H_3(GM(F))$ follows from the more general fact that $H_2(GL_m(F)) \otimes H_1(GL_n(F)) \subset H_3(T)$ for any m, n.) The same is evidently true if we divide out by H_3^0. □

It is possible to give a description of the image of $\pi_3^0(BGM(F)^+)$ in $K_3(F)$. There is a ring structure on $\pi_\bullet(BGM(F)^+)$ such that the maps

$$\pi_\bullet(BGM(F)^+) \longrightarrow K_\bullet(F)$$

and

$$\pi_\bullet(BGM(F)^+) \longrightarrow \pi_\bullet(\Sigma^+)$$

are compatible with the multiplication. We have

$$\pi_1(BGM(F)^+) = H_1(GM(F)) = F^\times \oplus \mathbb{Z}/2,$$

with $\pi_1^0(BGM(F)^+) = F^\times$ and $\pi_1(B\Sigma^+) = \mathbb{Z}/2$. The map

$$\pi_1(BGM(F)^+) \to K_1(F) = F^\times$$

is the identity on $F^\times$ and takes the generator of $\mathbb{Z}/2$ to $-1 \in F^\times$. Thus, the image of $\pi_3^0(BGM(F)^+)$ in $K_3(F)$ contains the decomposable part

$$\mathrm{im}(K_3^M(F) \to K_3(F)).$$

Since the image of the decomposable part of $\pi_\bullet(B\Sigma^+)$ is contained in the decomposable part of $K_\bullet(F)$, the image of $\pi_3^0(BGM(F)^+)$ in the indecomposable group $K_3(F)_{\mathrm{ind}} = K_3(F)/K_3^M(F)$ coincides with the image of

$$\pi_3^0(BGM(F)^+)_{\mathrm{ind}} = \ker(\pi_3(BGM(F)^+)_{\mathrm{ind}} \longrightarrow \pi_3(B\Sigma^+)_{\mathrm{ind}}).$$

Via some technical arguments using the Atiyah–Hirzebruch spectral sequence, homotopy theory, and Chern class maps on K-theory, we obtain the following results.

1. There are natural maps

   $$\mathrm{Tor}(\mu(F), \mu(F)) \longrightarrow \pi_3^0(BGM(F)^+)_{\mathrm{ind}} \longrightarrow \mathrm{Tor}(\mu(F), \mu(F)),$$

   and the composition is multiplication by 2.

2. The composition

   $$\mathrm{Tor}(\mu(F), \mu(F)) \longrightarrow \pi_3^0(BGM(F)^+)_{\mathrm{ind}} \longrightarrow K_3(F)_{\mathrm{ind}}$$

   is injective.

3. There is an exact sequence

   $$0 \longrightarrow \mu_2(F) \longrightarrow \pi_3^0(BGM(F)^+)_{\mathrm{ind}} \longrightarrow \mathrm{Tor}(\mu(F), \mu(F)) \longrightarrow 0.$$

   If $\mathrm{char}(F) \neq 2$, the extension is nontrivial. Denote this extension by $\mathrm{Tor}(\mu(F), \mu(F))^\sim$.

LEMMA 3.2.8. *The map* $\pi_3^0(BGM(F)^+)_{\mathrm{ind}} \to K_3(F)_{\mathrm{ind}}$ *is injective.*

PROOF. This follows from 2 above by passing to the algebraic closure of F (see [**128**], Lemma 5.8). □

Summarizing, we obtain the following.

THEOREM 3.2.9. *For any infinite field F, there is an exact sequence*

$$0 \longrightarrow \mathrm{Tor}(\mu(F), \mu(F))^{\sim} \longrightarrow K_3(F)_{\mathrm{ind}} \longrightarrow B(F) \longrightarrow 0.$$

COROLLARY 3.2.10. *If F is algebraically closed, then $B(F)$ is uniquely divisible.*

PROOF. By Suslin's solution to the Lichtenbaum–Quillen conjecture for algebraically closed fields [**126, 127**], the group $K_3(F)$ is divisible with torsion subgroup $\bigoplus_{\ell \neq \mathrm{char}(F)} \mathbb{Q}_\ell/\mathbb{Z}_\ell$. The same is true for $K_3(F)_{\mathrm{ind}}$ since $K_3^M(F)$ is known to be uniquely divisible by the work of Bass and Tate [**5**]. Now,

$$\mathrm{Tor}(\mu(F), \mu(F))^{\sim} = \bigoplus_{\ell \neq \mathrm{char}(F)} \mathbb{Q}_\ell/\mathbb{Z}_\ell.$$

Since every injective map of $\mathbb{Q}_\ell/\mathbb{Z}_\ell$ into itself is bijective, we see that the group $\mathrm{Tor}(\mu(F), \mu(F))^{\sim}$ is the torsion subgroup of $K_3(F)_{\mathrm{ind}}$; that is, the quotient $B(F)$ is uniquely divisible. □

This provides another proof of the unique divisibility of the scissors congruence group $\mathcal{P}(\mathcal{H}^3)$. Since $\mathcal{P}(\mathcal{H}^3) \cong \mathfrak{p}(\mathbb{C})^-$, it suffices to show that $\mathfrak{p}(\mathbb{C})$ is uniquely divisible. But this follows from the above corollary since we have an exact sequence

$$0 \longrightarrow B(F) \longrightarrow \mathfrak{p}(F) \longrightarrow (F^\times \otimes F^\times)_\sigma$$

and the group $(F^\times \otimes F^\times)_\sigma$ is uniquely divisible when F is algebraically closed.

If F is not algebraically closed, then $B(F)$ need not be uniquely divisible. Let $x, y \in F^\times$. Consider the elements $[x] + [1-x], [y] + [1-y] \in B(F)$.

LEMMA 3.2.11. *In $B(F)$, $[x] + [1-x] = [y] + [1-y]$.*

PROOF. We may assume $x \neq y$. The elements

$$[x] - [y] + [y/x] - [(1-x^{-1})/(1-y^{-1})] + [(1-x)/(1-y)]$$
$$[1-y] - [1-x] + [(1-x)/(1-y)] - [(1-x^{-1})/(1-y^{-1})] + [y/x]$$

are zero in $\mathfrak{p}(F)$. Subtracting one from the other, we obtain the result. □

Denote the element $[x] + [1-x]$ by c_F.

LEMMA 3.2.12. $6c_F = 0$.

PROOF. If $x \in F^\times - \{1\}$, denote by $\langle x \rangle$ the element $[x] + [x^{-1}]$. We have the relations

$$\begin{aligned} [x] - [y] + [y/x] - [(1-x^{-1})/(1-y^{-1})] + [(1-x)/(1-y)] &= 0 \\ [x^{-1}] - [y^{-1}] + [x/y] - [(1-x)/(1-y)] + [(1-x^{-1})/(1-y^{-1})] &= 0. \end{aligned}$$

Adding these, we see that $\langle x \rangle - \langle y \rangle + \langle y/x \rangle = 0$. Interchanging x and y, we see that $2\langle y/x \rangle = 0$ (since $\langle y/x \rangle = \langle x/y \rangle$). Thus, for any z, $2\langle z \rangle = 0$. Now,

$$\begin{aligned} 3c_F &= [x] + [1-x] + [x^{-1}] + [1-x^{-1}] + [(1-x)^{-1}] + [1-(1-x)^{-1}] \\ &= \langle x \rangle + \langle 1-x \rangle + \langle 1-x^{-1} \rangle \\ &= \langle x \rangle + \langle 1-x \rangle + \langle -(1-x)/x \rangle \\ &= \langle x \rangle + \langle 1-x \rangle + \langle -(1-x) \rangle - \langle x \rangle \\ &= \langle 1-x \rangle + \langle -(1-x) \rangle \\ &= \langle 1-x \rangle + \langle -(1-x)^2/(1-x) \rangle \\ &= \langle -(1-x)^2 \rangle. \end{aligned}$$

But for any z, $2\langle z \rangle = 0$. So $6c_F = 0$. □

Now, if the equations $x^2 + 1 = 0$ and $x^2 - x + 1 = 0$ are solvable in F, one sees that $c_F = 0$ ([**128**], Lemma 1.5). In particular, if F is algebraically closed, then $c_F = 0$. However, Suslin shows that the element c_F is nontrivial and has order exactly 6 for any $F \subseteq \mathbb{R}$. Thus $B(F)$ is not uniquely divisible since it has torsion.

REMARK 3.2.13. The group $B(\mathbb{Q})$ is cyclic of order 6, generated by $c_{\mathbb{Q}}$ (see [**128**], Corollary 5.3).

3.3. Extensions and Generalizations

3.3.1. Extensions. Many of the results in the previous section hold for rings other than fields. In particular, if A is a local ring with infinite residue field F, we can define the Bloch group $B(A)$ exactly as we did for F. It was shown by the author in [**70**] that the proof of Theorem 3.2.2 goes through without modification for A. Thus we have an exact sequence

$$H_3(GM_2(A)) \longrightarrow H_3(GL_2(A)) \longrightarrow B(A) \longrightarrow 0.$$

The proof of Theorem 3.2.7 requires a bit of modification, but we still have the same result; namely, there is an exact sequence

$$\pi_3^0(BGM(A)^+) \longrightarrow K_3(A) \longrightarrow B(A) \longrightarrow 0.$$

The only problem with the proof of Theorem 3.2.7 in this case is that it makes use of a projection $\mathbb{P}^2(F) \to \mathbb{P}^1(F)$ which is not well-defined over A. However, it is possible to construct a map on the appropriate chain complexes.

Unfortunately, the proof of Theorem 3.2.9 does not generalize to A. Thus, we *cannot* deduce the exact sequence

$$0 \longrightarrow \mathrm{Tor}(\mu(A), \mu(A))^{\sim} \longrightarrow K_3(A)_{\mathrm{ind}} \longrightarrow B(A) \longrightarrow 0. \tag{3.5}$$

However, if we assume that A is a Hensel local F-algebra, then we can say the following. Let p be a prime distinct from $\mathrm{char}(F)$. Since tensor product is right exact we obtain a commutative diagram

$$\begin{array}{ccccccc} \pi_3^0(BGM(F)^+) \otimes \mathbb{Z}/p & \to & K_3(F) \otimes \mathbb{Z}/p & \to & B(F) \otimes \mathbb{Z}/p & \to & 0 \\ \downarrow & & \downarrow & & \downarrow & & \\ \pi_3^0(BGM(A)^+) \otimes \mathbb{Z}/p & \to & K_3(A) \otimes \mathbb{Z}/p & \to & B(A) \otimes \mathbb{Z}/p & \to & 0 \end{array}$$

with exact rows. The map $B(F) \otimes \mathbb{Z}/p \to B(A) \otimes \mathbb{Z}/p$ is injective (since $B(F) \to B(A)$ splits) and the map $K_3(F) \otimes \mathbb{Z}/p \to K_3(A) \otimes \mathbb{Z}/p$ is an isomorphism [**127**]. A simple diagram chase gives the following result.

PROPOSITION 3.3.1. *Let F be an infinite field and let A be a Hensel local F-algebra. Then the natural map*

$$B(F) \otimes \mathbb{Z}/p \longrightarrow B(A) \otimes \mathbb{Z}/p$$

is an isomorphism for $p \neq \mathrm{char}(F)$. □

Since $K_\bullet^M(A) \otimes \mathbb{Z}/p \cong K_\bullet^M(F) \otimes \mathbb{Z}/p$ (this follows from the isomorphism $A^\times \otimes \mathbb{Z}/p \cong F^\times \otimes \mathbb{Z}/p$), and since $\mu(A) = \mu(F)$, we see that we do obtain the exact sequence (3.5) for A after tensoring with $\mathbb{Z}/p$.

The group $K_3(R)_{\mathrm{ind}}$ can be described explicitly for a large class of rings. A ring R is an $H1$-ring [**50**] if the following holds for all $l \geq 2, k \geq 1$: For any family of k surjective linear maps $f_i : R^l \to R$, there exists $v \in R^l$ such that $f_i(v) \in R^\times$ for $i = 1, \ldots k$. For example, semilocal rings with infinite residue fields are $H1$. Also, $H1$-rings have many units and have stable rank 1. We have the following result due to P. Elbaz–Vincent [**36**].

THEOREM 3.3.2. *Let R be a commutative $H1$-ring. Then there is an isomorphism*

$$K_3(R)_{\mathrm{ind}} \otimes \mathbb{Q} \cong H_0(R^\times, H_3(SL_2(R), \mathbb{Q})).$$

This relies on the following stability result proved in [**36**]:

The map $GL_2(R) \to GL_3(R)$ induces an injection

$$H_3(GL_2(R), \mathbb{Q}) \longrightarrow H_3(GL_3(R), \mathbb{Q}).$$

The proof of this result is much like the proof of Theorem 3.1.8. Indeed, the proof follows that of Sah's proof of the following result [**107**].

THEOREM 3.3.3. *Let D be one of the three real division algebras or any infinite field with $F^\times = (F^\times)^6$. Then the map*

$$H_3(SL_2(D), \mathbb{Z}) \longrightarrow H_3(SL_3(D), \mathbb{Z})$$

is injective. □

As a corollary, Sah proves that for any infinite field F, we have

$$K_3(F)_{\text{ind}} = H_0(F^\times, H_3(SL_2(F)))$$

modulo the class of abelian groups annihilated by powers of 6.

3.3.2. Generalizations. The derivation of the exact sequence

$$H_3(GM_2(F)) \longrightarrow H_3(GL_2(F)) \longrightarrow B(F) \longrightarrow 0$$

in Section 3.2 required a nontrivial amount of computation. One expects, however, that a similar approach to the study of $H_{n+1}(GL_n(F))$ might prove fruitful. Unfortunately, the resulting spectral sequence is very complicated.

In his thesis [**133**] S. Yagunov constructed a new resolution of $GL_n(F)$-modules and studied the resulting simpler spectral sequence. This leads to a definition of higher Bloch groups which are related to $H_{n+1}(GL_n(F), A)$. One (minor) drawback is that we must assume that $n!$ is invertible in A.

For each $n \geq 2$, define a group $\mathfrak{p}_n(F)$ as follows. Let $P^n_\bullet(F)$ be the acyclic complex with $P^n_k(F)$ the free abelian group with basis the $(k+1)$-tuples of rational points of $\mathbb{P}^{n-1}(F)$ which are in general position. This is clearly a complex of $GL_n(F)$-modules. Define

$$\mathfrak{p}_n(F) = H_{n+1}(P^n_\bullet(F)_{GL_n(F)}).$$

When $n = 2$, this is precisely the group denoted $\mathfrak{p}(F)$ in Section 3.2. It is clear that there is a surjective map

$$H_{n+1}(GL_n(F), \mathbb{Z}) \longrightarrow B_n(F)$$

for some subgroup $B_n(F) \subset \mathfrak{p}_n(F)$. Determining this group $B_n(F)$ seems rather difficult in general (certainly the complexity grows as n increases) since it involves the calculation of higher differentials in a spectral sequence.

Yagunov defines A-modules $P_n(F)$ in the following way. Let $\tilde{C}^n_\bullet$ be the complex with $\tilde{C}^n_k$ the free A-module with basis $(k+1)$-tuples of vectors in F^n in general position. This complex is clearly acyclic. Define a subcomplex $C^{n,p}_\bullet$ by taking $C^{n,p}_k$ to be generated by the elements

$$(x_0, \ldots, x_p, x_{p+1}, \ldots, x_k) - (\alpha_0 x_0, \ldots, \alpha_p x_p, x_{p+1}, \ldots, x_k)$$

if $k > p$ and by

$$(x_0, \ldots, x_k) - (\alpha_0 x_0, \ldots, \alpha_k x_k)$$

otherwise. Here, the $\alpha_i \in F^\times$. Then the quotient complex $C_\bullet^n = \tilde{C}_\bullet^n / C_\bullet^{n,n-1}$ is easily seen to be acyclic. The corresponding spectral sequence has

$$E^1_{p,q} = H_q(GL_n(F), C_p^n) = \begin{cases} H_q((F^\times)^{p+1} \times GL_{n-p-1}(F)) & p < n \\ 0 & p \geq n. \end{cases}$$

Here, $GL_0(F) = \{1\}$. The bottom row is the complex $(C_\bullet^n)_{GL_n(F)}$. One checks easily that

$$E^2_{p,0} = \begin{cases} A & p = 0 \\ 0 & 0 < p \leq n. \end{cases}$$

We say that a vector $v \in F^n$ is *monomial* if it has exactly one nonzero entry and *affine* if all its entries are nonzero. Define D_k^n to be the free A-module on the following basis. Consider all $n \times (k+1)$ matrices of the form $(v_0, \ldots, v_j, w_{j+1}, \ldots, w_k)$ where the v_i are independent monomial vectors and the submatrix $(w_{j+1}, \ldots, w_k)$ is such that any minor is non-zero. The positions of the entries in $(v_0, \ldots, v_j)$ uniquely define an ordered subset of $\{1, \ldots, n\}$. Two such matrices are *projectively equivalent* if they have the same affine parts and the same ordered subsets of $\{1, \ldots, n\}$. Now take the set of projective equivalence classes of such matrices as the basis of D_k^n. With the usual differential, the complex $D_\bullet^n$ is acyclic and there is a natural action of the monomial group $GM_n(F)$ on $D_\bullet^n$. There is an isomorphism

$$H_i((D_\bullet^n)_{GM_n(F)}) = \begin{cases} A & i = 0 \\ 0 & 0 < i < n+1 \end{cases}$$

and a surjection $A \otimes (F^\times)^{[n/2]} \twoheadrightarrow H_{n+1}((D_\bullet^n)_{GM_n(F)})$. Moreover, we have the following result.

PROPOSITION 3.3.4. *If $p > 0$ and $n!$ is invertible in A, then*

$$H_p(GM_n(F), D_q^n) = \begin{cases} H_p((F^\times)^{q+1} \times GM_{n-q+1}(F)) & 0 \leq q \leq n-1 \\ 0 & q \geq n. \end{cases}$$

Now consider the complex

$$\mathrm{Ind}_{GM_n(F)}^{GL_n(F)} D_\bullet^n = AGL_n(F) \otimes_{AGM_n(F)} D_\bullet^n.$$

There is a natural surjective map

$$\mathrm{Ind}_{GM_n(F)}^{GL_n(F)} D_\bullet^n \longrightarrow C_\bullet^n$$

defined by $a \otimes x \mapsto ax$. Denote the kernel by $K_\bullet^n$.

PROPOSITION 3.3.5. $H_p(GL_n(F), K_\bullet^n) = H_{p+1}(GL_n(F), GM_n(F); A)$.

Here, $H_\bullet(GL_n(F), GM_n(F); A)$ are the relative homology groups (see Appendix A).

On the other hand, we have the spectral sequence associated to the action of $GL_n(F)$ on $K^n_\bullet$. It has the following form.

PROPOSITION 3.3.6. *If $q > 0$, then*

$$\begin{aligned} E^1_{p,q} &= H_q(GL_n(F), K^n_p) \\ &= H_{q+1}(GL_{n-p-1}(F) \times (F^\times)^{p+1}, GM_{n-p-1}(F) \times (F^\times)^{p+1}; A). \end{aligned}$$

We also have an exact sequence of complexes

$$0 \longrightarrow H_0(GL_n(F), K^n_\bullet) \longrightarrow (D^n_\bullet)_{GM_n(F)} \longrightarrow (C^n_\bullet)_{GL_n(F)} \longrightarrow 0.$$

This follows from the fact that for each p, the map

$$H_1(GM_{n-p}(F) \times (F^\times)^p) \longrightarrow H_1(GL_{n-p}(F) \times (F^\times)^p)$$

is surjective so that we have an exact sequence

$$0 \longrightarrow H_0(GL_n(F), K^n_p) \longrightarrow (D^n_p)_{GM_n(F)} \longrightarrow (C^n_p)_{GL_n(F)} \longrightarrow 0$$

(this is the tail end of the long exact sequence associated to

$$0 \longrightarrow K^n_p \longrightarrow \mathrm{Ind}_{GM_n(F)}^{GL_n(F)} D^n_p \longrightarrow C^n_p \longrightarrow 0).$$

Thus, we obtain the following result.

LEMMA 3.3.7.

$$\begin{aligned} E^2_{p,0} &= H_0(GL_n(F), K^n_p) \\ &= \begin{cases} 0 & p < n \\ \mathrm{coker}(H_{n+1}((D^n_\bullet)_{GM_n(F)}) \longrightarrow H_{n+1}((C^n_\bullet)_{GL_n(F)})) & p = n. \end{cases} \end{aligned}$$

DEFINITION 3.3.8. The A-module $P_n(F)$ is defined as

$$P_n(F) = \mathrm{coker}(H_{n+1}((D^n_\bullet)_{GM_n(F)}) \longrightarrow H_{n+1}((C^n_\bullet)_{GL_n(F)})).$$

There is an obvious map

$$H_{n+1}(GL_n(F), GM_n(F); A) \longrightarrow P_n(F).$$

Yagunov makes the following conjecture.

CONJECTURE 3.3.9. *Let F be an infinite field, let $n \geq 2$ and assume that $n!$ is invertible in A. Then there is a natural isomorphism*

$$P_n(F) = \bigoplus_{i \geq 0} H_{n+1-2i}(GL_{n-2i}(F), \{GM_{n-2i}(F), GL_{n-2i-1}(F)\}; A)_{\mathrm{ind}}$$

where $_{\text{ind}}$ *denotes the indecomposable part of the homology. The product structure on homology is defined via the compositions*

$$\begin{aligned} H_i(GL_n(F), GM_n(F)) \otimes H_j(F^\times) &\longrightarrow H_{i+j}(GL_n \times F^\times, GM_n \times F^\times) \\ &\longrightarrow H_{i+j}(GL_{n+1}(F), GM_{n+1}(F)). \end{aligned}$$

Yagunov verifies this conjecture for $n \leq 4$ (the case $n = 4$ requires that F be algebraically closed).

It is natural to ask the relationship between $P_n(F)$ and $\mathfrak{p}_n(F) \otimes A$. Yagunov constructs a natural map $\mathfrak{p}_n(F) \otimes A \to P_n(F)$ for all $n \geq 2$.

PROPOSITION 3.3.10. *Suppose* 2 *is invertible in* A. *Then the map*

$$\mathfrak{p}(F) \otimes A \longrightarrow P_2(F)$$

is an isomorphism. □

For further generalizations of Bloch groups, the reader is referred to Zagier's paper [**134**] where a conjectural description of the groups $B_m(F)$ is given for m odd. Zagier focuses on the case where F is a number field and relates these groups with the Borel regulator map on $K_{2m-1}(F)$ and values of the zeta function $\zeta_F(s)$.

3.4. Invariants of hyperbolic manifolds

The connection between the Bloch group and scissors congruences in hyperbolic 3-space suggests that it may be possible to define invariants of manifolds in $B(\mathbb{C})$. W. Neumann and J. Yang [**93**] showed that this is indeed the case. We briefly discuss this here.

Let $\mathbb{H}^3$ be hyperbolic 3-space with boundary $\mathbb{CP}^1$. An *ideal simplex* Δ in $\overline{\mathbb{H}}^3 = \mathbb{H}^3 \cup \mathbb{CP}^1$ is one whose vertices z_1, z_2, z_3, z_4 lie in $\mathbb{CP}^1$. Such a Δ is determined up to congruence by the cross ratio

$$z = [z_1 : z_2 : z_3 : z_4] = \frac{(z_3 - z_2)(z_4 - z_1)}{(z_3 - z_1)(z_4 - z_2)}.$$

Take finitely many geometric 3-simplices and glue them together along 2-faces to obtain a cellular complex Y which is a manifold except possibly at isolated points. If the complement of these points is oriented, we call Y a geometric 3-cycle; the complement $Y - Y^{(0)}$ of the vertices is an oriented manifold.

Suppose $M = \mathbb{H}^3/\Gamma$ is a hyperbolic manifold of finite volume, where $\Gamma \subset PSL_2(\mathbb{C})$ is discrete. A degree one ideal triangulation of M consists of a geometric 3-cycle Y with a map $f : Y - Y^{(0)} \longrightarrow M$ satisfying

1. f is degree one almost everywhere in M;
2. For each 3-simplex S in Y, there is a map f_S to an ideal simplex in $\overline{\mathbb{H}}^3$, bijective on vertices, such that $f|_{S-S^{(0)}} : S - S^{(0)} \longrightarrow M$ is the composition $\pi \circ (f_S|_{S-S^{(0)}})$, where $\pi : \mathbb{H}^3 \longrightarrow M$ is the projection.

Thurston proved that compact hyperbolic 3-manifolds have degree one ideal triangulations with $|Y| \simeq M$. Most noncompact M also have such triangulations as well.

DEFINITION 3.4.1. Let $f : Y - Y^{(0)} \longrightarrow M$ be a degree one ideal triangulation of M. Each 3-simplex of Y maps to an ideal simplex in $\overline{\mathbb{H}}^3$; denote by $z_1, \ldots, z_n$ the cross ratio parameters of these ideal simplices. The Bloch invariant $\beta(M)$ is the element $\sum_{j=1}^{n}[z_j] \in \mathfrak{p}(\mathbb{C})$. If the z_j all lie in a subfield $K \subseteq \mathbb{C}$, we may consider $\beta(M)$ as an element of $\mathfrak{p}(K)$.

A priori, the invariant $\beta(M)$ depends on the triangulation. This is not the case, however.

PROPOSITION 3.4.2 ([**93**], Proposition 4.3). *The invariant $\beta(M)$ is independent of the triangulation.* □

This is proved by defining a fundamental class

$$[M] \in H_3(\Gamma, \mathbb{CP}^1) = \mathrm{Tor}_2^{\mathbb{Z}\Gamma}(J\mathbb{CP}^1, \mathbb{Z}),$$

where $J\mathbb{CP}^1$ is the kernel of the augmentation map $\mathbb{Z}[\mathbb{CP}^1] \longrightarrow \mathbb{Z}$. There is a homomorphism $H_3(\Gamma, \mathbb{CP}^1) \longrightarrow \mathfrak{p}(\mathbb{C})$ which maps $[M]$ to $\beta(M)$, and this implies the result.

THEOREM 3.4.3. *The invariant $\beta(M)$ lies in $B(\mathbb{C})$.* □

The proof of this is fairly straightforward, but lengthy and technical. The essential point is that the relation

$$\sum z_i \wedge (1 - z_i) = 0$$

holds in $\mathbb{C}^\times \wedge \mathbb{C}^\times$; that is, $\sum[z_j] \in B(\mathbb{C})$.

The invariant $\beta(M)$ is related to other invariants of M. Denote by $\mathrm{vol}(M)$ the volume of M. For any hyperbolic 3-manifold M, there is a well-defined invariant $CS(M)$, called the Chern–Simons invariant, which lies in $\mathbb{R}/\pi^2\mathbb{Z}$. It is defined as the integral of a certain 3-form built from the connection and curvature forms over a nonsingular section $s(M)$ of the positively oriented orthonormal frame bundle.

Define the Bloch regulator map

$$\rho : B(\mathbb{C}) \longrightarrow \mathbb{C}/\mathbb{Q}$$

as follows [**11**]. For $z \in \mathbb{C}^\times - \{1\}$, define

$$\rho(z) = \frac{\log z}{2\pi i} \wedge \frac{\log(1-z)}{2\pi i} + 1 \wedge \frac{\mathcal{R}(z)}{2\pi^2} \in \mathbb{C} \wedge \mathbb{C},$$

where

$$\mathcal{R}(z) = \frac{1}{2} \log z \log(1-z) - \int_0^z \frac{\log(1-t)}{t}\, dt.$$

This map ρ vanishes on the defining relation of $\mathfrak{p}(\mathbb{C})$ and hence induces a map

$$\rho : \mathfrak{p}(\mathbb{C}) \longrightarrow \mathbb{C} \wedge \mathbb{C}.$$

We have a commutative diagram

$$\begin{array}{ccccc} B(\mathbb{C}) & \longrightarrow & \mathfrak{p}(\mathbb{C}) & \xrightarrow{\lambda} & \mathbb{C}^\times \wedge \mathbb{C}^\times \\ \downarrow & & \downarrow \rho & & \downarrow \\ \mathbb{C}/\mathbb{Q} & \longrightarrow & \mathbb{C} \wedge \mathbb{C} & \xrightarrow{\epsilon} & \mathbb{C}^\times \wedge \mathbb{C}^\times \end{array}$$

where $\epsilon = 2(e \wedge e)$ with $e(z) = \exp(2\pi i z)$. So ρ restricts to give a map $\rho : B(\mathbb{C}) \longrightarrow \mathbb{C}/\mathbb{Q}$.

THEOREM 3.4.4 ([**93**], Theorem 1.3). *The Bloch invariant satisfies*

$$\rho(\beta(M)) = \frac{i}{2\pi^2}(\mathrm{vol}(M) + iCS(M)) \in \mathbb{C}/\mathbb{Q}.$$

It is possible to extend this idea to higher dimensions. Let $S_q(\partial\overline{\mathbb{H}}^n)$ be the abelian group generated by $(q+1)$-tuples of distinct points in $\partial\overline{\mathbb{H}}^n$ modulo the relations

$$(z_0, \ldots, z_q) = \mathrm{sgn}(\tau)(z_{\tau(0)}, \ldots, z_{\tau(n)})$$

for any permutation $\tau \in \Sigma_{q+1}$. With the usual boundary map this defines an acyclic complex $S_\bullet(\partial\overline{\mathbb{H}}^n)$. The group $\mathrm{Isom}^+(\mathbb{H}^n)$ acts on this complex. Define

$$\mathcal{P}_n = H_n(S_\bullet(\partial\overline{\mathbb{H}}^n)_{\mathrm{Isom}^+(\mathbb{H}^n)}).$$

If $n = 3$, then we recover the group $\mathfrak{p}(\mathbb{C})$. These groups are similar to the groups denoted by $\mathfrak{p}_n(F)$ in Section 3.3.

Let M be a complete hyperbolic n-manifold of finite volume and let $f : \pi_1(M) \longrightarrow \mathrm{Isom}^+(\mathbb{H}^n)$ be a homomorphism. Then one may define an invariant $\beta(f) \in \mathcal{P}_n$. If $n = 3$, then if f is the homomorphism corresponding to some "Dehn filling" M' of M, then $\beta(f) = \beta(M')$. Thus, this $\beta(f)$ is a generalization of $\beta(M)$.

Exercises

1. Prove that the inclusion $\mathrm{Sim}_0(n) \to \mathrm{Sim}(n)$ induces an isomorphism
$$H_\bullet(\mathrm{Sim}_0(n), \mathbb{Z}^t) \longrightarrow H_\bullet(\mathrm{Sim}(n), \mathbb{Z}^t).$$
2. Prove that
$$\mathrm{Sim}(n)_{\mathbb{R}^j} \cong \mathrm{Sim}(j) \times O(n-j, \mathbb{R}).$$
3. Prove the equality in Equation 3.1.
4. Verify the formula in Equation 3.2.
5. Verify that the maps (defined in the proof of Theorem 3.2.2)
$$d^1_{1,q} : H_q(T_2) \longrightarrow H_q(B_2)$$
and
$$d^1_{2,q} : H_q(F^\times) \longrightarrow H_q(F^\times \oplus F^\times)$$
are $\sigma - \mathrm{id}$ and Δ_*, respectively.
6. Prove that the map
$$d^3 : \mathfrak{p}(F) \longrightarrow E^3_{0,2}$$
is defined as $d^3([x]) = x \wedge (1-x) - x \otimes (1-x)$ (see the proof of Theorem 3.2.2).
7. Consider the extension
$$1 \longrightarrow T_2 \longrightarrow GM_2(F) \longrightarrow \Sigma_2 \longrightarrow 1.$$
Use the Hochschild–Serre spectral sequence to verify the assertions made about $H_3(GM_2(F))$ in the proof of Theorem 3.2.2.
8. Compute $H_1(\Sigma_2, H_2(T_2))$. What is the answer for F algebraically closed?
9. Verify the various formulas and assertions of acyclicity of the complexes defined in the discussion prior to Proposition 3.2.5.
10. Verify that the diagram (3.3) commutes.
11. Prove that a (semi)local ring with infinite residue field(s) is an $H1$-ring.

Chapter 4

Rank One Groups

The homology of rank one linear groups (SL_2, PGL_2) often can be computed via the action of the group on a suitable simplicial complex. There is the well-known tiling of the hyperbolic plane by $SL_2(\mathbb{Z})$-translates of an ideal triangle (see, e.g. [**21**], p. 215) and there is the Bruhat-Tits tree associated to a field with discrete valuation. Often, the action implies something about the structure of the group such as the existence of an amalgamated free product decomposition.

In this chapter we study the groups SL_2 and PGL_2 over the rings $\mathbb{Z}[1/p]$, p a prime, and $k[C]$, where C is a smooth affine curve over the field k. In these cases we are able to obtain a great deal of information about the homology (often, we can compute it completely). Applications of these results will be given in Chapter 5. We also discuss the generalization of these results to higher rank groups.

4.1. $SL_2(\mathbb{Z}[1/p])$

We begin with the following classical fact ([**110**], p. 11):

$$SL_2(\mathbb{Z}) \cong \mathbb{Z}/4 *_{\mathbb{Z}/2} \mathbb{Z}/6,$$

where the subgroups $\mathbb{Z}/2, \mathbb{Z}/4, \mathbb{Z}/6$ are generated by

$$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$$

respectively. There is a map $SL_2(\mathbb{Z}) \to \mathbb{Z}/12$ defined by sending the generator of $\mathbb{Z}/4$ to 3 mod 12 and the generator of $\mathbb{Z}/6$ to 2 mod 12.

THEOREM 4.1.1. *The map $SL_2(\mathbb{Z}) \longrightarrow \mathbb{Z}/12$ induces an isomorphism in integral homology:*

$$H_i(SL_2(\mathbb{Z}), \mathbb{Z}) \cong H_i(\mathbb{Z}/12, \mathbb{Z}) \cong \begin{cases} \mathbb{Z} & i = 0 \\ \mathbb{Z}/12 & i \text{ odd} \\ 0 & i \text{ even.} \end{cases}$$

PROOF. Associated to any decomposition $G = G_1 *_A G_2$, there is a Mayer-Vietoris sequence

$$\cdots \to H_i(A) \to H_i(G_1) \oplus H_i(G_2) \to H_i(G) \to H_{i-1}(A) \to \cdots .$$

The homology of a cyclic group $\mathbb{Z}/n$ is given by (see Appendix A)

$$H_i(\mathbb{Z}/n, \mathbb{Z}) = \begin{cases} \mathbb{Z} & i = 0 \\ \mathbb{Z}/n & i \text{ odd} \\ 0 & i \text{ even.} \end{cases}$$

Moreover, it is easy to see that if $\mathbb{Z}/n \to \mathbb{Z}/(nm)$ is the inclusion $1 \mapsto m$, then the map $H_{2i-1}(\mathbb{Z}/n) \to H_{2i-1}(\mathbb{Z}/(nm))$ is the same inclusion. It follows that the sequence for computing $H_\bullet(SL_2(\mathbb{Z}))$ has the following form:

$$0 \to H_{2i}(SL_2(\mathbb{Z})) \to \mathbb{Z}/2 \to \mathbb{Z}/4 \oplus \mathbb{Z}/6 \to H_{2i-1}(SL_2(\mathbb{Z})) \to 0,$$

since $H_{2i}(\mathbb{Z}/n) = 0$ for all n, $i \geq 1$. Note that the map $\mathbb{Z}/2 \to \mathbb{Z}/4 \oplus \mathbb{Z}/6$ is injective so that $H_{2i}(SL_2(\mathbb{Z})) = 0$ for $i \geq 1$ and $H_{2i-1}(SL_2(\mathbb{Z}))$ has order 12. One checks easily that $H_{2i-1}(SL_2(\mathbb{Z}))$ is in fact cyclic. □

The groups $SL_2(\mathbb{Z}[1/p])$, p prime, also admit a decomposition:

$$SL_2(\mathbb{Z}[1/p]) \cong SL_2(\mathbb{Z}) *_{\Gamma_0(p)} SL_2(\mathbb{Z}),$$

where $\Gamma_0(p)$ is the subgroup of $SL_2(\mathbb{Z})$ consisting of matrices which are upper triangular modulo p. The two copies of $SL_2(\mathbb{Z})$ are embedded as the standard copy and as the set of matrices of the form $\begin{pmatrix} a & pb \\ p^{-1}c & d \end{pmatrix}$ where $ad - bc = 1$. This decomposition is obtained in [**110**] using the action of $SL_2(\mathbb{Z}[1/p])$ on a certain tree; this is a special case of the construction in Section 4.2.

The cohomology of $SL_2(\mathbb{Z}[1/p])$ was completely calculated by A. Adem and M. Naffah in [**1**]. We shall not give full details here, but we will indicate the line of proof. Since the cohomology of $SL_2(\mathbb{Z})$ is the same as that of $\mathbb{Z}/12$, one need only compute $H^\bullet(\Gamma_0(p))$ and then use the Mayer-Vietoris sequence.

The proof is divided into cases: $p = 2, p = 3, p \geq 5$. The first computation is the following.

PROPOSITION 4.1.2. *If $p \geq 5$, then $H^1(\Gamma_0(p), \mathbb{Z}) \cong \mathbb{Z}^{N(p)}$, where*

$$N(p) = \begin{cases} (p-7)/6 & p \equiv 1 \mod 12 \\ (p+1)/6 & p \equiv 5 \mod 12 \\ (p-1)/6 & p \equiv 7 \mod 12 \\ (p+7)/6 & p \equiv 11 \mod 12. \end{cases}$$

PROOF. Let $\Gamma(p)$ be the level p congruence subgroup. We have an extension

$$1 \longrightarrow \Gamma(p) \longrightarrow \Gamma_0(p) \longrightarrow B \longrightarrow 1,$$

where B is the upper triangular subgroup of $SL_2(\mathbb{Z})$. The group $SL_2(\mathbb{Z})$ acts on a tree T with finite stabilizers and quotient an edge (this is one way to prove the amalgamated free product decomposition). Since $\Gamma(p)$ is torsion-free, it acts freely on T and thus $G = SL_2(\mathbb{F}_p)$ acts on $T/\Gamma(p)$ with isotropy groups $\mathbb{Z}/4$ and $\mathbb{Z}/6$ for the vertices and $\mathbb{Z}/2$ for the edge. Let EB be the universal cover of BB. Via the projection $\Gamma_0(p) \to B$, $\Gamma_0(p)$ acts diagonally on $EB \times T$; the isotropy groups are trivial. Since $EB \times T$ is contractible, we have $B\Gamma_0(p) \simeq EB \times_B T/\Gamma(p)$. Let $C^\bullet$ be the cellular cochain complex of $T/\Gamma(p)$. Then we have

$$C^0 \cong \mathbb{Z}/[G/(\mathbb{Z}/4)]|_B \oplus \mathbb{Z}[G/(\mathbb{Z}/6)]|_B,$$

and

$$C^1 \cong \mathbb{Z}[G/(\mathbb{Z}/2)]|_B,$$

where $|_B$ means restriction of modules (this follows from the restriction–induction formulas for modules). The associated spectral sequence has

$$E_1^{pq} = H^q(B, C^p) \Longrightarrow H^{p+q}(\Gamma_0(p), \mathbb{Z}).$$

This collapses to give a long exact sequence

$$\cdots \to H^i(\Gamma_0(p)) \to H^i(B, C^0) \to H^i(B, C^1) \to \cdots.$$

The result follows easily. □

One then uses the fact that the cohomology of $\Gamma_0(p)$ is periodic to reduce the computation to that of H^2 and H^3. The final result is the following.

THEOREM 4.1.3. *If $i \geq 1$, then*

$$H^{2i}(\Gamma_0(p), \mathbb{Z}) \cong \begin{cases} \mathbb{Z}/12 \oplus \mathbb{Z}/6 & p \equiv 1 \mod 12 \\ \mathbb{Z}/4 \oplus \mathbb{Z}/2 & p \equiv 5 \mod 12 \\ \mathbb{Z}/3 \oplus \mathbb{Z}/6 & p \equiv 7 \mod 12 \\ \mathbb{Z}/2 & p \equiv 11 \mod 12, \end{cases}$$

and

$$H^{2i+1}(\Gamma_0(p), \mathbb{Z}) \cong (\mathbb{Z}/2)^{N(p)}.$$

This is proved by first passing to the projective group

$$P\Gamma_0(p) = \Gamma_0(p)/\{\pm 1\}$$

and computing the cohomology of $P\Gamma_0(p)$ via its action on the tree defined above. One then uses the Hochschild–Serre spectral sequence associated to the extension

$$1 \longrightarrow \{\pm 1\} \longrightarrow \Gamma_0(p) \longrightarrow P\Gamma_0(p) \longrightarrow 1.$$

To complete the calculation, one uses the Mayer–Vietoris sequence along with some facts about the cohomological dimension of the group $SL_2(\mathbb{Z}[1/p])$. There is an extension problem to be solved, but it presents no real difficulty.

THEOREM 4.1.4. *Let $p \geq 5$. Then $H^1(SL_2(\mathbb{Z}[1/p]), \mathbb{Z}) = 0$ and*

$$H^2(SL_2(\mathbb{Z}[1/p]), \mathbb{Z}) \cong \begin{cases} \mathbb{Z}^{(p-7)/6} \oplus \mathbb{Z}/12 & p \equiv 1 \mod 12 \\ \mathbb{Z}^{(p+1)/6} \oplus \mathbb{Z}/12 & p \equiv 5 \mod 12 \\ \mathbb{Z}^{(p-1)/6} \oplus \mathbb{Z}/12 & p \equiv 7 \mod 12 \\ \mathbb{Z}^{(p+7)/6} \oplus \mathbb{Z}/12 & p \equiv 11 \mod 12. \end{cases}$$

For $i \geq 2$, we have

$$H^{2i}(SL_2(\mathbb{Z}[1/p]), \mathbb{Z}) \cong \begin{cases} (\mathbb{Z}/2)^{(p-7)/6} \oplus \mathbb{Z}/12 & p \equiv 1 \mod 12 \\ (\mathbb{Z}/2)^{(p+1)/6} \oplus \mathbb{Z}/12 \oplus \mathbb{Z}/3 & p \equiv 5 \mod 12 \\ (\mathbb{Z}/2)^{(p-7)/6} \oplus \mathbb{Z}/12 \oplus \mathbb{Z}/4 & p \equiv 7 \mod 12 \\ (\mathbb{Z}/2)^{(p+1)/6} \oplus \mathbb{Z}/12 \oplus \mathbb{Z}/12 & p \equiv 11 \mod 12, \end{cases}$$

and

$$H^{2i-1}(SL_2(\mathbb{Z}[1/p]), \mathbb{Z}) \cong \begin{cases} \mathbb{Z}/6 & p \equiv 1 \mod 12 \\ \mathbb{Z}/2 & p \equiv 5 \mod 12 \\ \mathbb{Z}/3 & p \equiv 7 \mod 12 \\ 0 & p \equiv 11 \mod 12. \end{cases}$$

The case $p = 3$ is completely analogous and yields

$$H^i(SL_2(\mathbb{Z}[1/3]), \mathbb{Z}) \cong \begin{cases} 0 & i \text{ odd} \\ \mathbb{Z} \oplus \mathbb{Z}/4 & i = 2 \\ \mathbb{Z}/12 \oplus \mathbb{Z}/4 & i = 2j, j > 1. \end{cases}$$

The case $p = 2$ is complicated by the fact that the principal congruence subgroup $\Gamma(2) \subset \Gamma_0(2)$ is not torsion-free. Still, it is possible to work around this and obtain

$$H^i(SL_2(\mathbb{Z}[1/2]), \mathbb{Z}) \cong \begin{cases} 0 & i \text{ odd} \\ \mathbb{Z} \oplus \mathbb{Z}/3 & i = 2 \\ \mathbb{Z}/24 \oplus \mathbb{Z}/3 & i = 2j, j > 1. \end{cases}$$

This last computation is interesting since $SL_2(\mathbb{Z}[1/2])$ has no subgroups of order eight.

The question of computing $H^*(SL_2(\mathbb{Z}[1/n]), \mathbb{Z})$ for an arbitrary integer n remains open. The rational cohomology was computed by K. Moss [**86**], and Naffah [**87**] has computed the 3-primary part.

4.2. The Bruhat–Tits Tree

The amalgamated free product decomposition of $SL_2(\mathbb{Z}[1/p])$ is a consequence of a general construction due to F. Bruhat and J. Tits [**23**]. In this section we describe the tree associated to a rank one group over a field with discrete valuation.

Let K be a field with valuation $v : K^\times \to \mathbb{Z}$. Denote by $\mathcal{O}$ the valuation ring, by $\mathfrak{m}$ the maximal ideal of $\mathcal{O}$, and by k the residue field $\mathcal{O}/\mathfrak{m}$. By a *lattice* we mean a free, rank two $\mathcal{O}$-submodule of K^2. Two lattices L, L' are *equivalent* if there is a nonzero x in K with $L' = xL$. We denote the equivalence class of a lattice L by $[L]$.

Let π be an element with $v(\pi) = 1$. Define a simplicial complex X as follows. The vertices of X are classes of lattices in K^2. Two vertices are adjacent if there exist representatives L, L' with $\pi L \subset L' \subset L$. This defines a graph and one has the following result (see Serre [**110**], p. 70).

THEOREM 4.2.1. *The graph X is a tree.* □

There is an obvious action of $GL_2(K)$ on X: if $g \in GL_2(K)$ and if $[L]$ is a vertex of X, then $g \cdot [L] = [gL]$. This action is obviously transitive and the center of $GL_2(K)$ acts trivially. If one considers the induced action of $SL_2(K)$, then one checks easily that there are two orbits of vertices with representatives $\mathcal{O} \oplus \mathcal{O}$ and $\mathcal{O} \oplus \pi\mathcal{O}$, and one edge orbit. The stabilizers of the vertices in $SL_2(K)$ are $SL_2(\mathcal{O})$ and $\mathrm{diag}(1,\pi)SL_2(\mathcal{O})\mathrm{diag}(1,\pi^{-1})$. The edge stabilizer Γ consists of those elements of $SL_2(\mathcal{O})$ which are upper triangular modulo π.

COROLLARY 4.2.2. *There is an amalgamated free product decomposition*

$$SL_2(K) \cong SL_2(\mathcal{O}) *_\Gamma SL_2(\mathcal{O}).$$

PROOF. This type of decomposition exists for any group G which acts on a tree with an edge as fundamental domain (see [**110**], p. 32). □

One also has the following result.

PROPOSITION 4.2.3. *Suppose that R is a dense subring of K (in the topology defined by v). Denote by $\mathcal{O}_R$ the intersection $\mathcal{O} \cap R$ and by $\Gamma(R)$ the subgroup $\Gamma \cap SL_2(R)$. Then there is a decomposition*

$$SL_2(R) \cong SL_2(\mathcal{O}_R) *_{\Gamma(R)} SL_2(\mathcal{O}_R).$$

PROOF. See [**110**], p. 78. □

EXAMPLE 4.2.4. Consider the field $\mathbb{Q}$ of rational numbers with the p-adic valuation. Since the ring $\mathbb{Z}[1/p]$ is dense in this topology we obtain the decomposition $SL_2(\mathbb{Z}[1/p]) \cong SL_2(\mathbb{Z}) *_{\Gamma(\mathbb{Z}[1/p])} SL_2(\mathbb{Z})$ mentioned in the previous section.

EXAMPLE 4.2.5. Denote by $\mathcal{L}$ the field of formal Laurent series over a field k with valuation

$$v\left(\sum_{n\geq n_0} a_n t^n\right) = n_0, \quad a_{n_0} \neq 0.$$

The ring $k[t,t^{-1}]$ is dense in $\mathcal{L}$ and $k[t,t^{-1}] \cap \mathcal{O} = k[t]$.

Thus we have $SL_2(k[t,t^{-1}]) \cong SL_2(k[t]) *_\Gamma SL_2(k[t])$.

4.3. $SL_2(k[t])$

In 1959, H. Nagao [**88**] published an elementary proof of the following fact. If k is any field, then there is an amalgamated free product decomposition

$$SL_2(k[t]) \cong SL_2(k) *_{B(k)} B(k[t]),$$

where $B(R)$ denotes the subgroup of upper triangular matrices over R. Using the tree X associated to $K = k(t)$, $v(f/g) = \deg(g) - \deg(f)$, Serre provided a new proof of this fact by finding a fundamental domain for the action of $SL_2(k[t])$ on X. It turns out that such a subtree is an infinite path $v_0, v_1, v_2, \ldots$, where $v_i = [t^i\mathcal{O} \oplus \mathcal{O}]$ [**110**], p. 87.

This decomposition allows one to compute the homology of $SL_2(k[t])$.

THEOREM 4.3.1. *Suppose the field k is infinite. Then the natural inclusion $SL_2(k) \to SL_2(k[t])$ induces an isomorphism*

$$H_*(SL_2(k), \mathbb{Z}) \longrightarrow H_*(SL_2(k[t]), \mathbb{Z}).$$

REMARK 4.3.2. This result is clearly false for finite fields. For example, one sees easily that $H_1(SL_2(\mathbb{F}_2[t]), \mathbb{Z})$ contains a copy of an infinite-dimensional $\mathbb{F}_2$-vector space while $H_1(SL_2(\mathbb{F}_2), \mathbb{Z})$ is finite.

However, if one uses $\mathbb{Z}/p$-coefficients, where $p \neq \text{char } k$, then we do have an isomorphism $H_*(SL_2(k[t]), \mathbb{Z}/p) \cong H_*(SL_2(k), \mathbb{Z}/p)$.

PROOF OF THEOREM 4.3.1. The decomposition yields a Mayer–Vietoris sequence:

$$\cdots \to H_i(B(k)) \to H_i(B(k[t])) \oplus H_i(SL_2(k)) \to H_i(SL_2(k[t])) \to \cdots.$$

Since the map $B(k) \to B(k[t])$ is split by evaluation at $t = 0$, the long exact sequence breaks into short exact sequences

$$0 \to H_i(B(k)) \to H_i(B(k[t])) \oplus H_i(SL_2(k)) \to H_i(SL_2(k[t])) \to 0.$$

We now need the following result. If the field k is infinite, then

$$H_*(B(k), \mathbb{Z}) \cong H_*(B(k[t]), \mathbb{Z}).$$

This is trivial if char $k = 0$ since one can use the Hochschild–Serre spectral sequence associated to the extension

$$0 \to R \to B(R) \to k^\times \to 1$$

for $R = k, k[t]$. A center kills argument shows that $H_i(k^\times, H_j(R)) = 0$ for $j \geq 1$ so that $H_\bullet(B(R),\mathbb{Z}) \cong H_\bullet(k^\times,\mathbb{Z})$. The positive characteristic case is more delicate, but it follows from a simple modification of the argument used to prove Theorem 2.2.2 (note also that this isomorphism was observed modulo p-torsion ($p = \text{char } k$) by Alperin [2]).

Given this fact, the theorem now follows easily by considering the above exact sequence. □

4.4. $SL_2(k[t,t^{-1}])$

The isomorphism $H_\bullet(SL_2(k[t]),\mathbb{Z}) \cong H_\bullet(SL_2(k),\mathbb{Z})$ reduces the computation of $H_\bullet(SL_2(k[t,t^{-1}]),\mathbb{Z})$ to that of $H_\bullet(\Gamma,\mathbb{Z})$, where Γ is the subgroup of matrices in $SL_2(k[t])$ which are upper triangular modulo t. Indeed the decomposition of $SL_2(k[t,t^{-1}])$ given in Example 4.2.5 yields a Mayer–Vietoris sequence

$$\to H_i(\Gamma) \to H_i(SL_2(k[t])) \oplus H_i(SL_2(k[t])) \to H_i(SL_2(k[t,t^{-1}])) \to .$$

Consider the split exact sequence

$$1 \longrightarrow K \longrightarrow \Gamma \xrightarrow{t=0} B(k) \longrightarrow 1.$$

Here K consists of those matrices which are congruent to the identity modulo t.

THEOREM 4.4.1. *If the field k is infinite, then the map $\Gamma \to B(k)$ induces an isomorphism in integral homology. Consequently, there is a natural isomorphism $H_\bullet(\Gamma,\mathbb{Z}) \cong H_\bullet(k^\times,\mathbb{Z})$.*

REMARK 4.4.2. As in the previous section, one can show that if k is finite, then the map $\Gamma \to B(k)$ induces a mod p homology isomorphism ($p \neq \text{char } k$).

PROOF OF THEOREM 4.4.1. An elementary proof (which works only in characteristic zero) was given in [**66**]. The idea is as follows. If T is the fundamental domain for the action of $SL_2(k[t])$ on the tree X associated to $k(t)$, then the subtree $D \subset X$ defined as

$$D = \bigcup_{s \in SL_2(k)/B(k)} sT$$

is a fundamental domain for the action of K on X. This is proved by first showing that D is a tree. To do this it suffices to show that D is connected (since D is a subgraph of a tree). A set of coset representatives of $SL_2(k)/B(k)$ is

$$\left\{\begin{pmatrix} 1 & 0 \\ s & 1 \end{pmatrix}\right\} \cup \left\{\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right\}.$$

Each of these fixes the initial vertex $[\mathcal{O} \oplus \mathcal{O}]$ of T; the connectivity of D follows immediately. One then shows that the vertices of D are inequivalent modulo

the action of K and that every edge of X is equivalent to an edge of D modulo K. This is left to the reader.

If C denotes the subgroup of matrices of the form

$$\left\{\begin{pmatrix} 1 & p(t) \\ 0 & 1 \end{pmatrix} : p(t) \in tk[t]\right\},$$

then we have a free product decomposition

$$K \cong \amalg_{s \in SL_2(k)/B(k)} sCs^{-1}.$$

This implies that for $i \geq 1$, $H_i(K) = \bigoplus_{s \in SL_2(k)/B(k)} H_i(sCs^{-1})$. Now, if char $k = 0$, C is a torsion-free abelian group and $H_i(sCs^{-1}) = \bigwedge^i_{\mathbb{Z}}(sCs^{-1})$. A straightforward (if tedious) calculation then shows that $H_p(B(k), H_q(K))$ vanishes for $q \geq 1$. Consequently, $H_p(\Gamma) = H_p(B(k))$ (via the Hochschild–Serre spectral sequence).

A more streamlined proof (which works in any characteristic) is as follows (see [**67**]). One uses the action of $B(k)$ on D to show that a fundamental domain for the action of Γ on X is the path D' consisting of the vertices $\ldots, u_2, u_1, v_0, v_1, v_2, \ldots$, where v_i is as in Section 4.3 and $u_i = [\mathcal{O} \oplus t^i\mathcal{O}]$. The stabilizer in Γ of each vertex and edge is homologically equivalent to $k^\times$. Since X is contractible, we have a spectral sequence converging to $H_\bullet(\Gamma)$ with E^1–term

$$E^1_{p,q} = \bigoplus_{\sigma^{(p)} \subset D'} H_q(\Gamma_\sigma)$$

where $\sigma^{(p)}$ is a p-simplex in D' and Γ_σ is the stabilizer of σ in Γ. The qth row of this spectral sequence is the chain complex $C_\bullet(D', \mathcal{H}_q)$, where $\mathcal{H}_q$ is the coefficient system $\sigma \mapsto H_q(\Gamma_\sigma)$. In this case, the system $\mathcal{H}_q$ is constant: $\mathcal{H}_q(\Gamma_\sigma) = H_q(k^\times)$ for all σ. Since D' is contractible, we have

$$E^2_{p,q} = H_p(D', \mathcal{H}_q) = \begin{cases} H_q(k^\times) & p = 0 \\ 0 & p > 0. \end{cases}$$

It follows that $H_q(\Gamma) = H_q(k^\times)$ for all $q \geq 0$. □

Even though the pieces are now in place, the final computation of the homology of $SL_2(k[t, t^{-1}])$ remains elusive. Clearly, $H_\bullet(SL_2(k), \mathbb{Z})$ lies in $H_\bullet(SL_2(k[t, t^{-1}]), \mathbb{Z})$ as a direct summand, but other than this, we only have the long exact sequence

$$\cdots \to H_i(k^\times) \to H_i(SL_2(k)) \oplus H_i(SL_2(k)) \to H_i(SL_2(k[t, t^{-1}])) \to \cdots.$$

At present, little is known about the map $H_i(k^\times) \to H_i(SL_2(k))$. A description of this homomorphism would simplify the calculation of the homology of $SL_2(k[t, t^{-1}])$ drastically.

Still, some things can be said. Suppose k is a number field. Denote by r the number of real embeddings and by s the number of conjugate pairs of complex embeddings.

THEOREM 4.4.3. *The map*

$$\partial : H_i(SL_2(k[t,t^{-1}]),\mathbb{Q}) \to H_{i-1}(k^\times,\mathbb{Q})$$

is injective for $i \geq 2r+3s+1$ *and bijective for* $i \geq 2r+3s+2$.

PROOF. By Theorem 1.5.1, $H_i(SL_2(k),\mathbb{Q}) = 0$ for $i \geq 2r+3s+1$. □

REMARK 4.4.4. With a bit more work (see [**66**]), one can show that

$$H_3(SL_2(\mathbb{Q}[t,t^{-1}]),\mathbb{Q}) \cong H_2(\mathbb{Q}^\times,\mathbb{Q}).$$

4.5. Curves of Higher Genus

The rings $k[t]$ and $k[t,t^{-1}]$ are the coordinate rings of $\mathbb{A}^1_k$ and $\mathbb{A}^1_k - \{0\}$, respectively. Suppose S is a smooth projective curve over k and let $q \in S$ be a closed point. Denote by C the affine curve $S - \{q\}$ and by A the coordinate algebra of C. The ring A is a Dedekind domain and its field of fractions, F, is the function field of S. We have $A^\times = k^\times$. For example, if $S = \mathbb{P}^1_k$, we have $C = \mathbb{A}^1_k$, $A = k[t]$, $F = k(t)$.

The point q gives rise to a discrete valuation, v_q, on F. Let X be the Bruhat–Tits tree associated to (F, v_q). The group $GL_2(A)$ acts on X as it is a subgroup of $GL_2(F)$. Serre [**110**], p. 106, provided a partial description of the quotient $GL_2(A)\backslash X$; it consists of a graph Y with *cusps* attached. (We will define a cusp below.) He also showed that the cusps are in one-to-one correspondence with the elements of $\mathrm{Pic}^0(S)$. In the case $S = \mathbb{P}^1_k$, we saw above that the quotient graph $GL_2(k[t])\backslash X$ consists of an edge Y with vertices $[\mathcal{O}\oplus\mathcal{O}]$ and $[t\mathcal{O}\oplus\mathcal{O}]$ and a single cusp $v_1, v_2, \ldots$, where $v_i = [t^i\mathcal{O}\oplus\mathcal{O}]$. Note that $\mathrm{Pic}^0(\mathbb{P}^1_k) = 0$. Recently, Suslin [**129**] gave a more complete description of the quotient graph along with the stabilizers of the vertices and edges. We discuss this below.

4.5.1. Elliptic Curves. In [**130**], S. Takahashi provided a complete description of $GL_2(A)\backslash X$ when S is an elliptic curve. In fact, he exhibited a subtree D of X such that $D \cong GL_2(A)\backslash X$. Since the center of $GL_2(A)$ acts trivially on X, we have $D \cong PGL_2(A)\backslash X$. This allows us to compute the integral homology of $PGL_2(A)$ provided the field k is infinite.

We shall describe D only informally. Assume the affine elliptic curve $S - \{\infty\}$ is defined via a Weierstrass equation $F(x, y) = 0$, where

$$F(x, y) = y^2 + a_1xy + a_3y - x^3 - a_2x^2 - a_4x - a_6.$$

If $l \in k$ and if $F(l, y) = 0$ has no k-rational solution, we denote by $k(\omega)$ the quadratic extension of k in which $F(l, \omega) = 0$.

The tree D consists of the following. There is a distinguished vertex o. For each $l \in k \cup \{\infty\}$, there is a vertex $v(l)$ adjacent to o. Denote by $D(l)$ the connected component of $D - \{o\}$ which contains $v(l)$. The $D(l)$ fall into three types.

(1) Suppose $F(x, y) = 0$ has no rational solution with $x = l$. Then $D(l)$ consists only of $v(l)$ (see Figure 4.1).



FIGURE 4.1. $F(l, y) = 0$ has no rational solutions

(2) Suppose $l = \infty$ or $F(x, y) = 0$ has a unique rational solution with $x = l$. Let p be the point at infinity of E or the rational point corresponding to the solution. Note that p is a point of order 2. Then $D(l)$ consists of an infinite path $c(p, 1), c(p, 2), \ldots$ and an extra vertex $e(p)$ (see Figure 4.2).

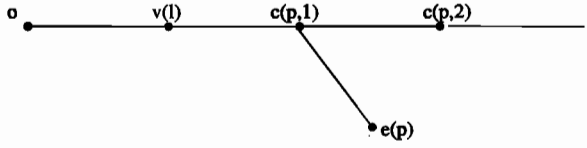


FIGURE 4.2. $F(l, y) = 0$ has a unique rational solution

(3) Suppose $F(x, y) = 0$ has two different solutions such that $x = l$. Let p, q be the corresponding points on E. Then $D(l)$ consists of two infinite paths $c(p, 1), c(p, 2), \ldots$ and $c(q, 1), c(q, 2), \ldots$ (see Figure 4.3).

The infinite path $c(p, 1), c(p, 2), \ldots$ is called a *cusp*. Note that there is a one-to-one correspondence between cusps and the rational points of S.

The stabilizers of the various vertices are given by the next result (see [**130**], Theorem 5).

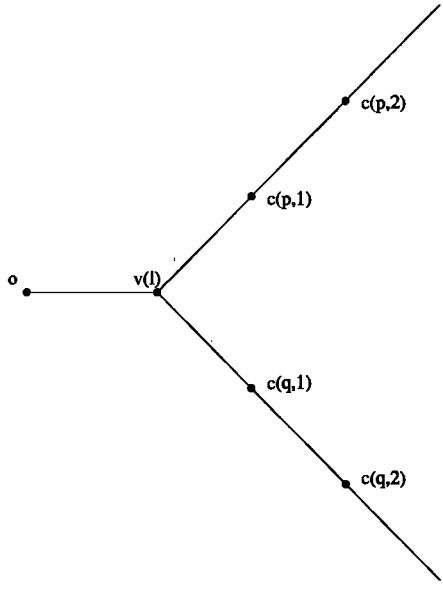


FIGURE 4.3. $F(l, y) = 0$ has two distinct solutions

PROPOSITION 4.5.1. *Up to isomorphism, the stabilizers in $PGL_2(A)$ of the vertices of D are as follows:*

$$\begin{aligned}
\Gamma_o &= \{1\} \\
\Gamma_{v(l)} &\cong \begin{cases} k(\omega)^\times/k^\times & \text{in case (1)} \\ k & \text{in case (2)} \\ k^\times & \text{in case (3)} \end{cases} \\
\Gamma_{c(p,n)} &\cong \left\{ \begin{pmatrix} p & v \\ 0 & q \end{pmatrix} : p, q \in k^\times, v \in k^n \right\}/k^\times \\
\Gamma_{e(p)} &\cong PGL_2(k).
\end{aligned}$$

The stabilizer of an edge is the intersection of its vertex stabilizers (one of which is contained in the other). □

The action of $PGL_2(A)$ on X yields a spectral sequence converging to $H_\bullet(PGL_2(A), \mathbb{Z})$ with E^1–term

$$E^1_{p,q} = \bigoplus_{\sigma^{(p)} \subset D} H_q(\Gamma_\sigma, \mathbb{Z}).$$

As pointed out in Section 4.3, we have $H_\bullet(\Gamma_{c(p,n)}, \mathbb{Z}) \cong H_\bullet(k^\times, \mathbb{Z})$, provided k is infinite. This leads to the following computation.

THEOREM 4.5.2. *If the field k is infinite, then for all $i \geq 1$,*

$$H_i(PGL_2(A),\mathbb{Z}) \cong \bigoplus_{\substack{l\in k\cup\{\infty\}\\ F(l,y)=0 \text{ has unique sol.}}} H_i(PGL_2(k),\mathbb{Z}) \oplus \bigoplus_{\substack{l\in k\\ F(l,y)=0 \text{ has two sol.}}} H_i(k^\times,\mathbb{Z}) \oplus \bigoplus_{\substack{l\in k\\ F(l,y)=0 \text{ has no sol.}}} H_i(k(\omega)^\times/k^\times,\mathbb{Z}).$$

PROOF. If $\mathcal{H}_q$ denotes the coefficient system $\sigma \mapsto H_q(\Gamma_\sigma)$, then $E^1_{\bullet,q} = C_\bullet(D,\mathcal{H}_q)$. Since the stabilizers of o and the edges adjacent to it are trivial, we see that

$$C_\bullet(D,\mathcal{H}_q) = \bigoplus_{l\in k\cup\{\infty\}} C_\bullet(D(l),\mathcal{H}_q).$$

If $F(l,y)=0$ has no rational solution then $C_\bullet(D(l),\mathcal{H}_q)$ consists of the group $H_q(k(\omega)^\times/k^\times)$ sitting in degree zero, whence the factor of $H_q(k(\omega)^\times/k^\times)$ in $H_q(PGL_2(A))$

If $F(l,y)=0$ has two solutions, then the stabilizer of each edge and vertex in $D(l)$ is homologically equivalent to $k^\times$. Thus, since $D(l)$ is contractible (Figure 4.3), $H_i(D(l),\mathcal{H}_q)$ vanishes for $i>0$ and equals $H_q(k^\times)$ for $i=0$.

If $F(l,y)=0$ has a unique solution, then the complex $D(l)$ is as in Figure 4.2. The vertex $v(l)$ and the edge e joining it to $c(p,1)$ have stabilizers isomorphic to k. The map $\Gamma_e \to \Gamma_{v(l)}$ is an isomorphism (this is clear). Since the group Γ_e has the form

$$\left\{\begin{pmatrix} 1 & v \\ 0 & 1 \end{pmatrix} : v \in k\right\},$$

the map $\Gamma_e \to \Gamma_{c(p,1)}$ factors through the inclusion $k \to \Gamma_{c(p,1)}$ and hence induces the zero map on homology Thus, if $D(l)'$ denotes the complex obtained by deleting $v(l)$ and e, we have $H_i(D(l),\mathcal{H}_q) = H_i(D(l)',\mathcal{H}_q)$ for all i. A straightforward computation shows that the latter group vanishes for $i>0$ and equals $H_q(PGL_2(k))$ for $i=0$. □

REMARK 4.5.3. Theorem 4.5.2 is valid for finite fields for $i \leq 2$ with integral coefficients, and in all degrees with $\mathbb{Z}/p$-coefficients ($p \neq$ char k).

4.5.2. Application: The K-theory of Elliptic Curves. Theorem 4.5.2 can be used to derive information about the K-groups of elliptic curves. There are many deep conjectures about the structure of these groups, especially if the ground field k is a number field. Perhaps the most famous, due to A. Beilinson [7] (and later modified by Bloch and Grayson [12]), asserts that if E is an elliptic curve over a number field k, then $K_2(E)\otimes\mathbb{Q}$ has dimension equal to

the number of infinite places of k plus the number of primes $\mathfrak{p} \subset \mathcal{O}_k$ where E has split multiplicative reduction modulo $\mathfrak{p}$. At this time, it is not known if $\dim(K_2(E) \otimes \mathbb{Q})$ is finite (even for a single curve).

Let E be an elliptic curve, $q \in E$ a closed point, and denote by A the affine coordinate algebra of $E - \{q\}$, as in the previous section. Denote by F the function field of E and consider the obvious embedding $i : GL_2(A) \to GL_2(F)$. We have the following result.

THEOREM 4.5.4. *The image of the map*

$$i_* : H_\bullet(GL_2(A), \mathbb{Z}) \longrightarrow H_\bullet(GL_2(F), \mathbb{Z})$$

coincides with the image of

$$(i|_{GL_2(k)})_* : H_\bullet(GL_2(k), \mathbb{Z}) \longrightarrow H_\bullet(GL_2(F), \mathbb{Z}).$$

PROOF. This is the main theorem of [71]. It suffices to show that $j : PGL_2(A) \to PGL_2(F)$ satisfies

$$\mathrm{im}(j_*) = \mathrm{im}((j|_{PGL_2(k)})_*).$$

One then uses the Hochschild–Serre spectral sequences associated to the extensions

$$1 \longrightarrow k^\times \longrightarrow GL_2(A) \longrightarrow PGL_2(A) \longrightarrow 1$$

and

$$1 \longrightarrow F^\times \longrightarrow GL_2(F) \longrightarrow PGL_2(F) \longrightarrow 1$$

along with the Universal Coefficient Theorem to finish the proof.

The isomorphism of Theorem 4.5.2 is induced by the various inclusions $\Gamma_{v(l)} \to PGL_2(A)$ and $\Gamma_{e(p)} \to PGL_2(A)$. The subgroup $\Gamma_{e(\infty)}$ is the usual copy of $PGL_2(k)$. All the other Γ's are *not* subgroups of $PGL_2(k)$. However, one can show (via a lengthy calculation) that each $\Gamma_{v(l)}$ and $\Gamma_{e(p)}$ is conjugate by an element of $PGL_2(F)$ to a subgroup of $PGL_2(k)$. Since conjugation induces the identity map in homology (Appendix A), it follows that $\mathrm{im}(j_*) = \mathrm{im}((j|_{PGL_2(k)})_*)$. □

Recall the *rank filtration* of the rational K-groups of a ring R: $K_i(R)_\mathbb{Q} := K_i(R) \otimes \mathbb{Q}$. This is an increasing filtration defined by

$$r_m K_n(R)_\mathbb{Q} = \mathrm{im}\{H_n(GL_m(R), \mathbb{Q}) \longrightarrow H_n(GL(R), \mathbb{Q})\} \cap K_n(R)_\mathbb{Q}.$$

Theorem 4.5.4 has the following consequence.

COROLLARY 4.5.5. *The image of the map $r_2K_n(A)_\mathbb{Q} \to r_2K_n(F)_\mathbb{Q}$ coincides with the image of $r_2K_n(k)_\mathbb{Q} \to r_2K_n(F)_\mathbb{Q}$.* □

Now suppose k is a number field. The localization sequences for A and E give rise to a commutative diagram

$$\begin{array}{ccc} K_2(E)_{\mathbb{Q}} & \longrightarrow & K_2(F)_{\mathbb{Q}} \\ & \searrow & \uparrow \\ & & K_2(A)_{\mathbb{Q}} \end{array}$$

where all three maps are injections.

COROLLARY 4.5.6. *If k is a number field, then $r_2K_2(A)_{\mathbb{Q}} = 0$.*

PROOF. The image of $r_2K_2(A)_{\mathbb{Q}}$ in $r_2K_2(F)_{\mathbb{Q}}$ coincides with the image of $r_2K_2(k)_{\mathbb{Q}} = 0$. But the map $K_2(A)_{\mathbb{Q}} \to K_2(F)_{\mathbb{Q}}$ is injective. □

Theorem 2.2.18 implies that $r_3K_2(A)_{\mathbb{Q}} = K_2(A)_{\mathbb{Q}}$. Thus, to show that $K_2(A)_{\mathbb{Q}}$ (and hence $K_2(E)_{\mathbb{Q}}$) is finite dimensional, it would suffice to show that the image of $H_2(GL_3(A), \mathbb{Q})$ in $H_2(GL_3(F), \mathbb{Q})$ is finite dimensional.

4.5.3. Genus Greater than One. A complete description of the structure of $PGL_2(k[C])\backslash X$ for curves $C = S - \{q\}$ of genus $g \geq 2$ remains elusive. However, A. Suslin computed $H_*(PGL_2(k[C]), \mathbb{Z}/p)$, where $p \neq \text{char } k$ and k is algebraically closed.

As mentioned above, Serre showed that $PGL_2(k[C])\backslash X$ consists of a graph Y with cusps attached (one for each element of the Jacobian of S) by interpreting the vertices of X as rank two vector bundles on S. Suslin showed (using some results of Seshadri [91]) that the vertices and edges of $PGL_2(k[C])\backslash X - \{\text{cusps}\}$ are "trivial" in the following sense. If $v \in Y - \{Y \cap \{\text{cusps}\}\}$, then the stabilizer of v in $PGL_2(k[C])$ is a k-vector space.

Thus, with $\mathbb{Z}/p$-coefficients, $H_i(\Gamma_v, \mathbb{Z}/p) = 0$ for $i \geq 1$. It follows that for $i \geq 2$, the groups $H_i(PGL_2(k[C]), \mathbb{Z}/p)$ depend only on the cusps. Note that $H_1(Y, \mathbb{Z}/p)$ will contribute to the group $H_1(PGL_2(k[C]), \mathbb{Z}/p)$ (this is not an issue for elliptic curves since in this case Y is contractible).

The "nontrivial" part of $PGL_2(k[C])\backslash X$ is therefore a disjoint union of trees indexed by the Jacobian, $\text{Pic}^0(S)$. If $L \in \text{Pic}^0(S)$ satisfies $2L = 0$, the tree associated with L is a path $v_0, v_1, \ldots$ where $\Gamma_{v_0} \cong PGL_2(k)$ and Γ_{v_i} is the semidirect product of $k^\times$ with k^i (this is the same as the group $\Gamma_{c(p,i)}$ of Proposition 4.5.1). By the same arguments as above, this branch contributes a copy of $H_i(PGL_2(k), \mathbb{Z}/p)$ to $H_i(PGL_2(k[C]), \mathbb{Z}/p)$ for $i \geq 2$. If $L \in \text{Pic}^0(S)$ is not of order 2, then the branch associated to L consists of a path $v_1, v_2, \ldots$. Each stabilizer is the semidirect product of $k^\times$ with a k-vector space. Homologically, this is equivalent to $k^\times$. Moreover, the branch is adjacent to a vertex v_L and one has $v_L = v_{-L}$. Thus, we have a tree similar to that in Figure 4.3.

This contributes a copy of $H_i(k^\times, \mathbb{Z}/p)$ to $H_i(PGL_2(k[C]), \mathbb{Z}/p)$ for $i \geq 2$. Thus, we have the following result.

THEOREM 4.5.7. *Suppose the field k is algebraically closed. Then for all $i \geq 2$,*

$$H_i(PGL_2(k[C]), \mathbb{Z}/p) \cong \bigoplus_{\substack{L \in \mathrm{Pic}^0(S) \\ 2L=0}} H_i(PGL_2(k), \mathbb{Z}/p) \oplus \bigoplus_{\substack{L \in \mathrm{Pic}^0(S) \\ 2L \neq 0, L \equiv -L}} H_i(k^\times, \mathbb{Z}/p).$$

Note that this agrees with Theorem 4.5.2. Indeed, an elliptic curve E and its Jacobian coincide. The points on E corresponding to the $l \in k \cup \{\infty\}$ where $F(l, y) = 0$ has a unique solution are precisely those of order 2, and the identification of a point with its negative is built into the indexing of the $H_i(k^\times, \mathbb{Z}/p)$ factors in Theorem 4.5.2.

These homology decompositions have important corollaries; we discuss this in Chapter 5.

4.6. Groups of Higher Rank

The impressive array of results for rank one groups is something of an anomaly; once one passes to SL_n, PGL_n, $n \geq 3$, things become much more complicated. In this section we discuss what is known for groups of higher rank.

4.6.1. $SL_n(\mathbb{Z})$. It is possible, in principle, to compute the homology of $SL_n(\mathbb{Z})$ in a manner similar to that of $SL_2(\mathbb{Z})$. There is an obvious contractible space X upon which $SL_n(\mathbb{Z})$ acts, namely the space of positive definite quadratic forms on $\mathbb{R}^n$ (up to positive scalars). When $n = 2$, this space can be identified with the hyperbolic plane $\mathcal{H} \subset \mathbb{C}$ with the usual action of $SL_2(\mathbb{Z})$ by linear fractional transformations.

Unfortunately, the quotient $SL_n(\mathbb{Z})\backslash X$ is not compact. Compactifications exist, but it is still difficult (if not virtually impossible) to compute the cohomology in this way. C. Soulé [**113**] discovered a way to get around this in the case $n = 3$. The basic idea is to replace X by a contractible subspace X' such that $SL_3(\mathbb{Z})\backslash X'$ is compact (it is also contractible).

The space X' is defined as follows. The space X consists of symmetric positive definite 3×3 matrices (modulo scalars). Write $[h]$ for the class of a matrix $h = (h_{ij})$. The group $SL_3(\mathbb{Z})$ acts on X via $[h].g = [g^T h g]$. A fundamental domain for this action is the set D of "reduced" points. These are the matrices which are minimal in their own orbit with respect to the ordering $[h] < [h']$ if and only if the sequence of diagonal coefficients of h is less than

that of h' in the lexicographic order on $\mathbb{R}^3$. Let D' be the set of reduced points $[h]$ such that $h_{11} = h_{22} = h_{33}$. Then $X' = D'.SL_3(\mathbb{Z})$.

Soulé described the cells of $SL_3(\mathbb{Z})\backslash X'$ and completed the (formidable!) spectral sequence calculation to arrive at the following result.

THEOREM 4.6.1. *The integral cohomology of $SL_3(\mathbb{Z})$ is given by the following table (where $(a)G$ means a direct sum of a copies of G).*

n	$H^n(SL_3(\mathbb{Z}), \mathbb{Z})$
$12m+1$	$(6m)\mathbb{Z}/2$
$12m+2$	$(6m)\mathbb{Z}/2$
$12m+3$	$(6m+2)\mathbb{Z}/2$
$12m+4$	$(2)\mathbb{Z}/3 \oplus (2)\mathbb{Z}/4 \oplus (6m)\mathbb{Z}/2$
$12m+5$	$(6m+1)\mathbb{Z}/2$
$12m+6$	$(6m+4)\mathbb{Z}/2$
$12m+7$	$(6m+3)\mathbb{Z}/2$
$12m+8$	$(2)\mathbb{Z}/2 \oplus (2)\mathbb{Z}/4 \oplus (6m+1)\mathbb{Z}/2$
$12m+9$	$(6m+5)\mathbb{Z}/2$
$12m+10$	$(6m+5)\mathbb{Z}/2$
$12m+11$	$(6m+4)\mathbb{Z}/2$
$12m+12$	$(2)\mathbb{Z}/3 \oplus (2)\mathbb{Z}/4 \oplus (6m+5)\mathbb{Z}/2$

No such computation exists for $n \geq 4$; this technique is just too complicated.

Armed with Theorem 4.6.1, it should be possible to compute the cohomology of $SL_3(\mathbb{Z}[1/p])$ using a higher dimensional analogue of the technique of Section 4.1. One now needs to compute the cohomology of subgroups $\Gamma \subset SL_3(\mathbb{Z})$ which are parabolic modulo p. This has been carried out by Henn [**56**] in the case $p = 2$.

4.6.2. $SL_n(k[t])$ **and** $SL_n(k[t, t^{-1}])$. There is a generalization to higher rank of the Bruhat–Tits tree. If K is a field with valuation v, local ring $\mathcal{O}$, and residue field k, one defines lattices in K^n as before. Define a simplicial complex X as follows. The vertices are equivalence classes of lattices. A collection $\Lambda_0, \Lambda_1, \ldots, \Lambda_i$ forms an i-simplex if there exist representatives $L_0, L_1, \ldots, L_i$ such that

$$\pi L_i \subset L_0 \subset \cdots \subset L_i.$$

One checks that the maximal dimension of a cell is $n-1$. The complex X is contractible (see Brown's book [**22**] for a nice discussion of this).

In the case $K = \mathcal{L}$ = field of Laurent series over k, a fundamental domain for the action of $SL_n(\mathcal{L})$ on X is the $(n-1)$-simplex D with vertices

$v_1, v_2, \ldots, v_n$, where

$$v_i = [\mathcal{O} \oplus \cdots \oplus \mathcal{O} \oplus \underbrace{t\mathcal{O} \oplus \cdots \oplus t\mathcal{O}}_{i}].$$

The stabilizers of the v_i are conjugates of $SL_n(\mathcal{O})$ by diagonal matrices with $n-i$ 1's and i t's.

Since $k[t, t^{-1}]$ is dense in $\mathcal{L}$, the simplex D is also a fundamental domain for the action of $SL_n(k[t, t^{-1}])$ on X. The vertex stabilizers are conjugates of $SL_n(k[t])$. The stabilizer of a simplex is the intersection of its vertex stabilizers. Thus, the computation of $H_\bullet(SL_n(k[t, t^{-1}]), \mathbb{Z})$ reduces to the computation of the homology of groups $\Gamma_P \subseteq SL_n(k[t])$ of the form

$$1 \longrightarrow K \longrightarrow \Gamma_P \xrightarrow{t=0} P \longrightarrow 1$$

where $P \subseteq SL_n(k)$ is a standard parabolic subgroup containing the upper triangular matrices and K is the subgroup of $SL_n(k[t])$ consisting of matrices congruent to the identity modulo t (compare with Section 4.4). All simplex stabilizers are conjugates of the various Γ_P.

In Section 4.4, we showed that the map $\Gamma \to B(k)$ induces an isomorphism in integral homology. It should come as no surprise that this generalizes to higher dimensions.

THEOREM 4.6.2. *If the field k is infinite, then the split surjection $\Gamma_P \to P$ induces an isomorphism $H_\bullet(\Gamma_P, \mathbb{Z}) \to H_\bullet(P, \mathbb{Z})$.*

COROLLARY 4.6.3. *The inclusion $SL_n(k) \to SL_n(k[t])$ induces an isomorphism $H_\bullet(SL_n(k), \mathbb{Z}) \to H_\bullet(SL_n(k[t]), \mathbb{Z})$.*

PROOF. This is the special case $P = SL_n(k)$. □

PROOF OF THEOREM 4.6.2. The first step is to compute a fundamental domain for the action of $SL_n(k[t])$ on the Bruhat–Tits building Y associated to $k(t)$; this was done by Soulé [**114**]. It is a subcomplex $T \subset Y$ which looks like an infinite "wedge"; see Figure 4.4 for the case $n = 3$. The vertices of T are the classes $[t^{r_1}e_1, t^{r_2}e_2, \ldots, t^{r_{n-1}}e_{n-1}, e_n]$, where $e_1, e_2, \ldots, e_n$ is the standard basis of $k(t)^n$ and $r_1 \geq r_2 \geq \cdots \geq r_{n-1} \geq 0$.

The stabilizers have the form

$$\begin{pmatrix} L_1 & V_{12} & \cdots & V_{1m} \\ 0 & L_2 & & \vdots \\ 0 & 0 & \ddots & V_{m-1,m} \\ 0 & 0 & 0 & L_m \end{pmatrix}$$

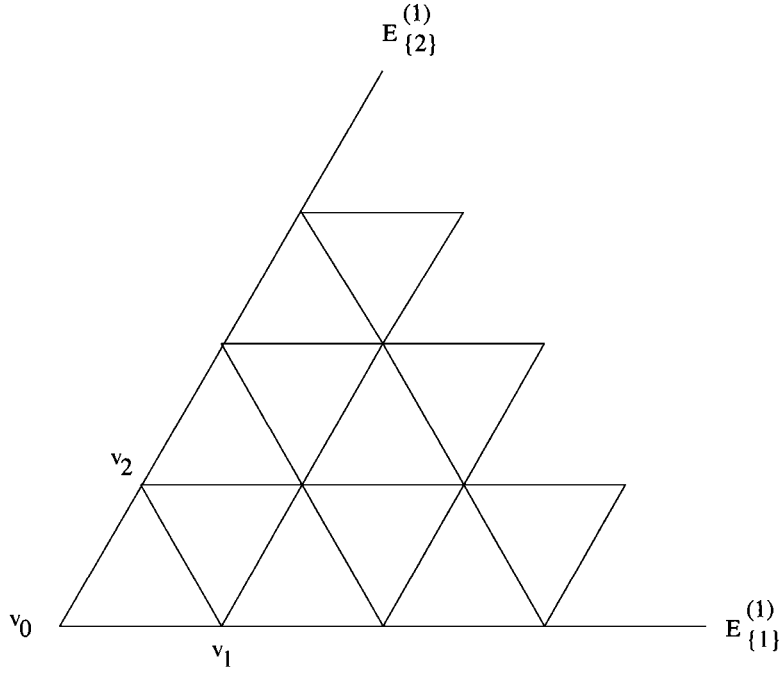


FIGURE 4.4. The fundamental domain T for $n = 3$

where $L_i \subseteq GL_{n_i}(k)$ and $V_{ij} \subset M_{n_i,n_j}(k[t])$. Homologically, these are equivalent to the reductive group

$$\begin{pmatrix} L_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & L_m \end{pmatrix}.$$

One then uses the spectral sequence

$$E^1_{p,q} = \bigoplus_{\sigma^{(p)} \subset T} H_q(\Gamma_\sigma) \Longrightarrow H_{p+q}(SL_n(k[t]))$$

to prove Corollary 4.6.3 as follows. Denote by v_0 the vertex $[e_1, \ldots, e_n]$ and by v_i the vertex $[te_1, \ldots, te_i, e_{i+1}, \ldots, e_n]$ for $i = 1, \ldots, n-1$. For a k element subset $I = \{i_1, \ldots, i_k\}$ of $\{1, 2, \ldots, n-1\}$, define $E_I^{(k)}$ to be the subcomplex of T which is the union of all rays with origin v_0 passing through the $(k-1)$-simplex $\langle v_{i_1}, \ldots, v_{i_k} \rangle$. If $I = \{1, \ldots, n-1\}$, then $E_I^{(n-1)} = T$. When we write $E_J^{(l)}$, the superscript denotes the cardinality of the set J.

Define a filtration $V^\bullet$ of T by setting $V^{(0)} = v_0$ and

$$V^{(k)} = \bigcup_I E_I^{(k)}, \qquad 1 \le k \le n-1,$$

where I ranges over the k element subsets of $\{1, 2, \ldots, n-1\}$. One checks easily that if $l \le k-1$ and if σ is an l-simplex in $E_I^{(k)}$, where $I = \{i_1, \ldots, i_k\}$, not lying entirely in any $E_J^{(k-1)}$, where $J \subset I$, then Γ_σ has the block form of the intersection $\Gamma_{v_{i_1}} \cap \cdots \cap \Gamma_{v_{i_k}}$. Thus, the coefficient system $\sigma \mapsto H_q(\Gamma_\sigma)$ is "locally constant" in the sense of Proposition A.2.7 and hence we see that the inclusion $v_0 \to T$ induces an isomorphism

$$H_\bullet(v_0, \mathcal{H}_q) \longrightarrow H_\bullet(T, \mathcal{H}_q)$$

for all $q \ge 0$. Thus, the E^2-term of the spectral sequence is

$$E^2_{pq} = \begin{cases} H_q(SL_n(k)) & p = 0 \\ 0 & p > 0. \end{cases}$$

The general case is proved by showing that a fundamental domain for the action of Γ_P on Y is the subcomplex

$$D_P = \bigcup_{s \in \Sigma_n / H} sT,$$

where $\Sigma_n \subset SL_n(k)$ is the symmetric group on n letters, and $H = \Sigma_n \cap P$. This is done in stages as follows. Let $K \subset SL_n(k[t])$ be the subgroup of matrices congruent to the identity modulo t. Then as in the case $n = 2$, the subcomplex

$$D' = \bigcup_{s \in SL_n(k)/B(k)} sT$$

is a fundamental domain for the action of K on Y. Then using the action of $B(k)$ on D', one shows that

$$D_B = \bigcup_{s \in \Sigma_n} sT$$

is a fundamental domain for the action of Γ_B on Y. Finally, since Γ_P contains a subgroup of permutation matrices, the action of P on D_B shows that D_P is the required fundamental domain for Γ_P.

Translating the filtration $V^\bullet$ of T gives a filtration $W_P^\bullet$ of D_P. Another application of Proposition A.2.7 shows that the map

$$H_\bullet(v_0, \mathcal{H}_q) \longrightarrow H_\bullet(D_P, \mathcal{H}_q)$$

is an isomorphism. Here, the coefficient system is $\sigma \mapsto H_q(\Gamma_\sigma)$ where Γ_σ is the stabilizer of σ in Γ_P. The proof is finished by noting that the stabilizer Γ_{v_0} is P. □

Theorem 4.6.2 provides a complete computation of the E^1–term of the spectral sequence for computing $H_\bullet(SL_n(k[t,t^{-1}]),\mathbb{Z})$:

$$E^1_{p,q} = \bigoplus_{\sigma\subset D} H_q(\Gamma_P).$$

The differential, d^1, is difficult to compute. However, one has the following results.

THEOREM 4.6.4. *If $n \geq 3$, then the E^2–term of the spectral sequence is*

$$E^2_{p,q} = \begin{cases} \mathbb{Z} & p=0, q=0 \\ 0 & p>0, q=0 \\ 0 & p\neq 1, q=1 \\ k^\times & p=1, q=1 \\ H_2(SL_n(k),\mathbb{Z}) & p=0, q=2. \end{cases}$$

PROOF. See [**67**]. The row $q=0$ of E^1 is the simplicial chain complex of the $(n-1)$-simplex D. Since D is contractible, the computation of $E^2_{\bullet,0}$ follows easily. The row $q=1$ of E^1 is similar to the bar construction; this makes it possible to write down a contracting homotopy. □

COROLLARY 4.6.5. *If $n \geq 3$, then*

$$\begin{aligned} H_2(SL_n(k[t,t^{-1}]),\mathbb{Z}) &= H_2(SL_n(k),\mathbb{Z}) \oplus k^\times \\ &= K_2(k) \oplus K_1(k) \\ &= K_2(k[t,t^{-1}]). \end{aligned}$$

PROOF. This follows directly from the spectral sequence calculation. The fact that $H_2(SL_n(k),\mathbb{Z}) = K_2(k)$ for $n \geq 3$ is standard (see, e.g. [**80**]). The last equality is the Fundamental Theorem of Algebraic K-theory. □

The other differentials, d^1, remain a mystery. One needs a description of the maps $H_\bullet(GL_n(k)) \to H_\bullet(GL_m(k))$ for $n < m$ in order to carry out the calculation.

4.6.3. Unstable Homotopy Invariance. Corollary 4.6.3 may be viewed as an unstable version of homotopy invariance in algebraic K-theory:

$$K_\bullet(R[t]) \cong K_\bullet(R)$$

for R regular. A natural question to ask is if the hypotheses in Corollary 4.6.3 can be weakened; that is, do we have $H_\bullet(SL_n(R[t]),\mathbb{Z}) \cong H_\bullet(SL_n(R),\mathbb{Z})$ for all regular R? The answer is certainly no: consider $R=\mathbb{F}_p$. Still, one might hope for a result in this direction provided the ring R has enough units. We discuss such a result now.

Suppose R is an integral domain with field of fractions Q. Denote by $E_2(R[t])$ the subgroup of $SL_2(R[t])$ generated by elementary matrices. In general this is a proper subgroup. Indeed, Krstić and McCool [**74**] have shown that if R is an integral domain which is not a field, then there is a surjective homomorphism

$$SL_2(R[t])/U_2(R[t]) \longrightarrow F,$$

where $U_2(R[t])$ is the normal subgroup generated by all unipotent matrices (we have $E_2 \subseteq U_2$) and F is a free group of countable rank.

Nagao's theorem asserts that there is a decomposition

$$SL_2(Q[t]) \cong SL_2(Q) *_{B(Q)} B(Q[t]).$$

By applying a result of Serre [**110**], p. 6, we obtain the following.

THEOREM 4.6.6. *If R is an integral domain, then*

$$E_2(R[t]) \cong E_2(R) *_{B(R)} B(R[t]).$$

This theorem has the following application. By a modification of Theorem 2.2.2, if A has many units (see Section 2.2), then $H_\bullet(B(A)) \cong H_\bullet(B(A[t]))$. Thus, using the same argument as in Theorem 4.3.1 we obtain the following result.

THEOREM 4.6.7. *If R is an integral domain with many units, then the inclusion $E_2(R) \to E_2(R[t])$ induces an isomorphism*

$$H_\bullet(E_2(R), \mathbb{Z}) \longrightarrow H_\bullet(E_2(R[t]), \mathbb{Z}).$$

Theorem 4.6.7 cannot hold in general for all n, however. Consider the ring $R = (k[x,y]/(y^2 - x^3 - x^2))_{(x,y)}$. If $H_\bullet(E_n(R), \mathbb{Z}) \cong H_\bullet(E_n(R[t]), \mathbb{Z})$ for all n, then we would have

$$H_2(E(R[t]), \mathbb{Z}) \cong H_2(E(R), \mathbb{Z});$$

i.e., $K_2(R[t]) \cong K_2(R)$. But Weibel [**132**] has shown that $K_2(R[t]) \neq K_2(R)$. Perhaps the added assumption that R be regular would suffice to prove the result for all $n \geq 3$.

Theorem 4.6.7 does hold for $n = 3$, however.

THEOREM 4.6.8. *If R is an integral domain with many units, then the inclusion $E_3(R) \to E_3(R[t])$ induces an isomorphism*

$$H_\bullet(E_3(R), \mathbb{Z}) \longrightarrow H_\bullet(E_3(R[t]), \mathbb{Z}).$$

PROOF. Let Q be the field of fractions of R and denote by T the fundamental domain for the action of $SL_3(Q[t])$ on the Bruhat–Tits building Y associated to $Q(t)$. Define a subcomplex U of Y by

$$U = \bigcup_{x \in E_3(R[t])} xT.$$

Clearly, T is a fundamental domain for the action of $E_3(R[t])$ on U. Using the argument in the proof of Theorem 4.6.2, we are finished provided the complex U is acyclic.

To prove this we need the following result of Soulé [**112**]. Suppose G is a group which acts on a simplicial complex Z with fundamental domain Z'. Denote by $\tilde{G}$ the amalgam of the stabilizers of the simplices of Z (i.e., view the complex Z' as a partially ordered set, and take the colimit of the stabilizers using this poset as the indexing category). Then there is an exact sequence

$$\pi_1(Z) \longrightarrow \tilde{G} \longrightarrow G \longrightarrow \pi_0(Z) \longrightarrow 0.$$

Since $E_3(R[t])$ is generated by the vertex stabilizers in T, the map $\tilde{E}_3(R[t]) \to E_3(R[t])$ is surjective; that is, U is connected. We claim that U is also simply connected. This finishes the proof in light of the long exact sequence

$$0 \longrightarrow H_2(U) \longrightarrow H_2(Y) \longrightarrow H_2(Y,U) \longrightarrow \cdots$$

and the fact that $H_2(Y) = 0$. The simple connectivity of U is proved as follows. The amalgam $\tilde{E}_3(R[t])$ is the colimit of the stabilizers of the simplices in T. Since Y is contractible, the group $SL_3(Q[t])$ is the amalgam of its vertex stabilizers. Thus we have an injective map of systems

$$\{E_3(R[t])_\sigma\}_{\sigma \in T} \hookrightarrow \{SL_3(Q[t])_\sigma\}_{\sigma \in T},$$

and consequently an inclusion

$$\tilde{E}_3(R[t]) \hookrightarrow SL_3(Q[t]).$$

Moreover, the image of $\tilde{E}_3(R[t])$ in $SL_3(Q[t])$ is clearly the subgroup generated by the $E_3(R[t])_\sigma$; that is, $\tilde{E}_3(R[t]) \xrightarrow{\cong} E_3(R[t])$. Thus, the map $\pi_1(U) \to \tilde{E}_3(R[t])$ is the zero map.

Soulé's theorem also asserts that the map $\pi_1(Z) \to \tilde{G}$ is injective if (1) Z' is simply connected, and (2) for each $g \in G$, the complex $gZ' \cap Z'$ is connected (or empty). Since this is clearly the case for $E_3(R[t])$, the map $\pi_1(U) \to \tilde{E}_3(R[t])$ is injective. Thus, $\pi_1(U) = 0$. □

REMARK 4.6.9. The above argument works for any n to show that the complex U is connected and simply connected. Moreover, we always have $H_{n-1}(U) = 0$. Thus, for larger n and the ring R mentioned above, there must be some nontrivial 2-cycle in U.

Unstable homotopy invariance is an important step in attacking the Friedlander–Milnor Conjecture. We discuss this in Chapter 5

In [**73**], we used Theorem 4.6.6 to study the homology of $SL_2(\mathbb{Z}[t])$. Note that Theorem 4.6.7 does not apply to $E_2(\mathbb{Z}[t])$. However, since $B(\mathbb{Z}) = \mathbb{Z}/2 \times \mathbb{Z}$ and $B(\mathbb{Z}[t]) = \mathbb{Z}/2 \times \mathbb{Z}[t]$, the Künneth formula implies that $H_i(B(\mathbb{Z}[t]))$ contains a copy of $H_i(t\mathbb{Z}[t]) = \bigwedge^i_{\mathbb{Z}} t\mathbb{Z}[t]$ as a summand. Since $H_i(\mathbb{Z}) = 0$ for

$i > 1$ and since the map $H_1(\mathbb{Z}) \to H_1(\mathbb{Z}[t])$ is the usual inclusion, we see that $H_i(E_2(\mathbb{Z}[t]), \mathbb{Z})$ also contains $\bigwedge^i_{\mathbb{Z}} t\mathbb{Z}[t]$ as a direct summand.

Now consider the group $SL_2(\mathbb{F}_p[t])$ for $p = 2, 3$. Nagao's Theorem applies here and we have $B(\mathbb{F}_p[t]) = \mathbb{F}_p^\times \times \mathbb{F}_p[t]$. Again using the Künneth formula and results about the homology of abelian groups [**21**], p. 123, we see that $H_i(SL_2(\mathbb{F}_p[t]), \mathbb{Z})$ contains a copy of $\bigwedge^i_{\mathbb{Z}} t\mathbb{F}_p[t]$. Moreover, it is clear that under the obvious map

$$(\varphi_p)_* : H_i(E_2(\mathbb{Z}[t]), \mathbb{Z}) \longrightarrow H_i(SL_2(\mathbb{F}_p[t]), \mathbb{Z}),$$

the element $t^{l_1} \wedge \cdots \wedge t^{l_i} \in \bigwedge^i_{\mathbb{Z}} t\mathbb{Z}[t]$ maps to $t^{l_1} \wedge \cdots \wedge t^{l_i} \in \bigwedge^i_{\mathbb{Z}} t\mathbb{F}_p[t]$. Now consider the map $j : E_2(\mathbb{Z}[t]) \to SL_2(\mathbb{Z}[t])$ and the elements $x_{\underline{l}} = j_*(t^{l_1} \wedge \cdots \wedge t^{l_i})$. Via the commutative diagram

$$\begin{array}{ccc} E_2(\mathbb{Z}[t]) & \longrightarrow & SL_2(\mathbb{Z}[t]) \\ & \searrow & \downarrow \\ & & SL_2(\mathbb{F}_p[t]) \end{array}$$

we see that $x_{\underline{l}} \neq 0$ and if $x_{\underline{m}}$ is another such element, then $x_{\underline{l}}$ and $x_{\underline{m}}$ are independent. Thus we have the following.

THEOREM 4.6.10. *For all $i \geq 1$, the group $H_i(SL_2(\mathbb{Z}[t]), \mathbb{Z})$ is not finitely generated.* □

Moreover, if we consider the group $H_1(SL_2(\mathbb{Z}[t]), \mathbb{Z})$, we see that the situation is worse than was known previously. Write

$$H_1(SL_2(\mathbb{Z}[t]), \mathbb{Z}) = H_1(SL_2(\mathbb{Z}), \mathbb{Z}) \oplus H_1(F, \mathbb{Z}) \oplus X$$

where X is some abelian group. Since $E_2(\mathbb{Z}[t])$ maps to 1 in F, we have $H_1(E_2(\mathbb{Z}[t]), \mathbb{Z}) \mapsto X$. Thus, the group X is also not finitely generated.

4.6.4. Twisted Coefficients. Another natural question about unstable homotopy invariance is the following. Suppose char $k = 0$ and V is a rational $SL_n(k)$-module. The group $SL_n(k[t])$ acts on V via evaluation at $t = 0$. What can be said about the cohomology $H^\bullet(SL_n(k[t]), V)$?

In [**68**], we studied the group H^1. Denote by B the upper triangular subgroup of $SL_n(k)$, by K the subgroup of $SL_n(k[t])$ consisting of matrices congruent to I modulo t, and by C the upper triangular subgroup of K.

THEOREM 4.6.11. *Suppose* char $k = 0$ *and let V be a finite dimensional rational $SL_n(k)$-module. Then*

$$H^1(SL_n(k[t]), V) = \begin{cases} H^1(SL_n(k), V) & V \neq \mathrm{Ad} \\ H^1(SL_n(k), V) \oplus F^\infty & V = \mathrm{Ad}, n = 2 \\ H^1(SL_n(k), V) \oplus F & V = \mathrm{Ad}, n \geq 3. \end{cases}$$

The case $n = 2$ may be proved using the long exact sequence associated to the amalgamated free product decomposition of $SL_2(k[t])$:

$$\to H^1(SL_2(k[t]), V) \to H^1(SL_2(k), V) \oplus H^1(B(k[t]), V) \to H^1(B, V).$$

To compute $H^1(B(k[t]), V)$ we use the Hochschild–Serre spectral sequence associated to the split extension

$$1 \longrightarrow C \longrightarrow B(k[t]) \longrightarrow B \longrightarrow 1.$$

This yields a split exact sequence

$$0 \to H^0(B, H^1(C, V)) \to H^1(B(k[t]), V) \to H^1(B, V) \longrightarrow 0.$$

Since C acts trivially on V, $H^1(C, V) = \mathrm{Hom}(H_1(C), V)$. Thus, we have $H^0(B, H^1(C, V)) = \mathrm{Hom}_B(C, V)$ (since C is abelian). The action of B on C is given by

$$\begin{pmatrix} a & b \\ 0 & 1/a \end{pmatrix} : \begin{pmatrix} 1 & tp(t) \\ 0 & 1 \end{pmatrix} \mapsto \begin{pmatrix} 1 & a^2 tp(t) \\ 0 & 1 \end{pmatrix}.$$

If V is an irreducible SL_2-module, then B acts on V with weight n for some integer n. Hence if $n \neq 2$, $\mathrm{Hom}_B(C, V) = 0$. If $n = 2$ (i.e., $V = \mathrm{Ad}$), then $\mathrm{Hom}_B(C, V) = \mathrm{Hom}_k(tk[t], k)$.

If $n \geq 3$, then one uses the spectral sequence associated to the action of $SL_n(k[t])$ on the Bruhat–Tits building associated to $k(t)^n$. The bottom row of the spectral sequence satisfies $E^2_{p,0} = 0$ for $p > 0$ and hence $H^1(SL_n(k[t]), V) = E^2_{0,1}$. A careful analysis of the structure of various groups of the form $\mathrm{Hom}_P(H_1(C_P), V)$, where P is a parabolic subgroup of $SL_n(k)$ and C_P is an associated subgroup of K, yields the final result that

$$E^2_{0,1} \cong \mathrm{Hom}_B(H_1(C), V).$$

This group is easily seen to vanish if $V \neq \mathrm{Ad}$. It is a one-dimensional k-vector space if $V = \mathrm{Ad}$.

Nothing is known about the higher cohomology groups.

Exercises

1. Prove that if $\mathbb{Z}/n \to \mathbb{Z}/(mn)$ is the inclusion $1 \mapsto m$, then the map $H_{2i-1}(\mathbb{Z}/n) \to H_{2i-1}(\mathbb{Z}/(mn))$ is the same inclusion.
2. Show that $H_{2i-1}(SL_2(\mathbb{Z}))$ is cyclic of order 12.
3. Complete the proof of Proposition 4.1.2.
4. Show that there are two orbits for the action of $SL_2(K)$ on the Bruhat–Tits tree X; the representatives are $\mathcal{O} \oplus \mathcal{O}$ and $\mathcal{O} \oplus \pi\mathcal{O}$. Show also that there is a single edge orbit.
5. Show that $H_1(SL_2(\mathbb{F}_2[t]), \mathbb{Z})$ contains a copy of an infinite dimensional $\mathbb{F}_2$-vector space, while $H_1(SL_2(\mathbb{F}_2), \mathbb{Z})$ is finite.

6. Let k be a field and let p be a prime distinct from the characteristic of k. Prove that the inclusion $SL_2(k) \to SL_2(k[t])$ induces an isomorphism in mod p homology.
7. Let Γ be the subgroup of $SL_2(k[t])$ consisting of matrices which are upper triangular modulo t. Prove that the map $\Gamma \to B(k)$ induces an isomorphism in mod p homology for p not equal to the characteristic of k.
8. Consider the complex D defined in the proof of Theorem 4.4.1. Show that the vertices of D are inequivalent modulo the action of K. Show also that every edge of X is equivalent to an edge of D modulo K.
9. Prove that if k is of characteristic zero, then

$$H_p(B(k), H_q(K)) = 0, \qquad q \geq 1$$

(see the proof of Theorem 4.4.1).
10. Use the action of $B(k)$ on D to show that the path D' defined in the proof of Theorem 4.4.1 is a fundamental domain for the action of Γ on X.
11. Prove the assertions in Remark 4.5.3.
12. Prove that the maximal dimension of a simplex in the Bruhat–Tits building associated to K^n is $n-1$.
13. Prove that the stabilizers in $SL_n(k[t])$ of the simplices in the fundamental domain T (described in Theorem 4.6.2) have the form indicated.
14. Prove that the inclusion of the reductive part of the stabilizers in the previous exercise into the full stabilizer induces a homology isomorphism. (Hint: use induction on $m =$ the number of diagonal blocks.)
15. Calculate the differential d^1 in the row $q = 1$ of the spectral sequence of Theorem 4.6.4. (Hint: it looks similar to the bar construction in Appendix A).
16. Prove that for all $i \geq 1$, the group $H_i(B(\mathbb{Z}[t]))$ contains a copy of $H_i(t\mathbb{Z}[t])$ as a summand.
17. Prove that for all $i \geq 1$, the group $H_i(SL_2(\mathbb{F}_p[t]), \mathbb{Z})$ contains a copy of $\bigwedge^i_{\mathbb{Z}} t\mathbb{F}_p[t]$ (provided $p = 2, 3$). Show that under the obvious map

$$H_i(E_2(\mathbb{Z}[t]), \mathbb{Z}) \longrightarrow H_i(SL_2(\mathbb{F}_p[t]), \mathbb{Z})$$

the element $t^{l_1} \wedge \cdots \wedge t^{l_i}$ maps to $t^{l_1} \wedge \cdots \wedge t^{l_i}$.
18. Show that $\mathrm{Hom}_B(C, \mathrm{Ad}) = \mathrm{Hom}_k(tk[t], k)$.
19. Prove that if $n \geq 3$, then the group $\mathrm{Hom}_B(H_1(C), V)$ vanishes if $V \neq \mathrm{Ad}$ and is one-dimensional if $V = \mathrm{Ad}$.

Chapter 5

The Friedlander–Milnor Conjecture

Aside from Quillen's conjecture (Section 1.2), the most important unsolved problem in the study of the homology of linear groups is E. Friedlander's generalized isomorphism conjecture [**41**]. This is usually called the Friedlander–Milnor conjecture because of Milnor's study of the special case of Lie groups [**81**].

Let G be a reductive algebraic group over an algebraically closed field k and let p be a prime distinct from char k. Consider the simplicial classifying scheme BG_k and the simplicial set $BG(k)$ ($G(k)$ denotes the discrete group of k-rational points of G). The cohomology of $BG(k)$ is simply the cohomology of the discrete group $G(k)$. The conjecture asserts that there is a natural map

$$H^\bullet_{\text{ét}}(BG_k, \mathbb{Z}/p) \longrightarrow H^\bullet(BG(k), \mathbb{Z}/p)$$

which is an isomorphism, where $H^\bullet_{\text{ét}}$ denotes the étale cohomology of the simplicial scheme BG_k (see Appendix C).

In the case $k = \mathbb{C}$, we have $H^\bullet_{\text{ét}}(BG_{\mathbb{C}}, \mathbb{Z}/p) \cong H^\bullet(BG, \mathbb{Z}/p)$, where BG is the classifying space of the complex lie group G defined in Appendix B. Since a great deal is known about $H^\bullet_{\text{ét}}(BG_k, \mathbb{Z}/p)$ (see, for example, [**42**]), this conjecture would give detailed information about $H^\bullet(G(k), \mathbb{Z}/p)$.

In this chapter we describe the conjecture in detail along with all known results. In contrast with Quillen's conjecture, there are no known counterexamples.

5.1. Lie Groups

While the generalized isomorphism conjecture assumes that the ground field is algebraically closed, we may say something about the case $k = \mathbb{R}$ as well. Let G be a Lie group and let G^δ be the group G with the discrete topology.

CONJECTURE 5.1.1. *The natural map $BG^\delta \to BG$ defined in Appendix B induces isomorphisms on homology and cohomology with $\mathbb{Z}/p$ coefficients.*

To begin our attack on this conjecture, we first prove a few lemmas. If G is a topological group, denote by $\overline{G}$ the homotopy fiber of the map $G^\delta \to G$; that is, if $\mathcal{P}(G)$ denotes the space of paths in G, then

$$\overline{G} = \{(g, f) \in G^\delta \times \mathcal{P}(G) : f(0) = e \in G^\delta, f(1) = g\}.$$

If G is locally contractible, so that the identity component G_0 has a universal covering group U, then the homomorphisms $U \to G_0 \to G$ induce isomorphisms $\overline{U} \to \overline{G}_0 \to \overline{G}$. So the homology groups of $B\overline{G}$ depend only on the universal covering group of G. If G is a Lie group, it follows that $H_\bullet(B\overline{G})$ depends only upon the Lie algebra $\mathfrak{g}$.

LEMMA 5.1.2. *Conjecture* 5.1.1 *is true for a connected lie group G if and only if the space $B\overline{G}$ has the mod p homology of a point for every prime p. Moreover, if H is locally isomorphic to G, then if Conjecture* 5.1.1 *holds for G, it also holds for H.*

PROOF. Consider the homology spectral sequence

$$E^2_{p,q} = H_p(BG, H_q(B\overline{G}, \mathbb{Z}/p)) \Longrightarrow H_{p+q}(BG^\delta, \mathbb{Z}/p).$$

If $H_q(B\overline{G}, \mathbb{Z}/p) = 0$ for $q > 0$, then

$$H_p(BG^\delta, \mathbb{Z}/p) = E^\infty_{p,0} = E^2_{p,0} = H_p(BG, \mathbb{Z}/p).$$

Conversely, suppose $H_\bullet(BG^\delta, \mathbb{Z}/p) \cong H_\bullet(BG, \mathbb{Z}/p)$. Note that since $\pi_0(G) = \{1\}$, the space BG is simply connected. We proceed by induction on q to show that $H_q(B\overline{G}, \mathbb{Z}/p) = 0$. Suppose $q = 1$ and consider the map

$$d^2 : H_2(BG, \mathbb{Z}/p) \longrightarrow H_0(BG, H_1(B\overline{G}, \mathbb{Z}/p)).$$

Since $H_2(BG^\delta, \mathbb{Z}/p) \cong H_2(BG, \mathbb{Z}/p)$ by hypothesis, we see that d^2 is the zero map. It follows that we have a short exact sequence

$$0 \to H_0(BG, H_1(B\overline{G}, \mathbb{Z}/p)) \to H_1(BG^\delta, \mathbb{Z}/p) \xrightarrow{\cong} H_1(BG, \mathbb{Z}/p) \to 0.$$

Since BG is simply connected, we have

$$H_0(BG, H_1(B\overline{G}, \mathbb{Z}/p)) = H_1(B\overline{G}, \mathbb{Z}/p) = 0.$$

Now assume that $H_i(B\overline{G}, \mathbb{Z}/p) = 0$ for $i < q$. Consider the map

$$d^{q+1} : H_{q+1}(BG, \mathbb{Z}/p) \longrightarrow H_0(BG, H_q(B\overline{G}, \mathbb{Z}/p)).$$

By hypothesis, this map is trivial and by the inductive hypothesis we have an exact sequence

$$0 \to H_0(BG, H_q(B\overline{G}, \mathbb{Z}/p)) \to H_q(BG^\delta, \mathbb{Z}/p) \xrightarrow{\cong} H_q(BG, \mathbb{Z}/p) \to 0.$$

It follows that $H_q(B\overline{G}, \mathbb{Z}/p) = 0$.

The second assertion is proved by considering the spectral sequence associated to the fibration $B\overline{G} \longrightarrow BH^\delta \longrightarrow BH$. □

LEMMA 5.1.3. *Let Γ be a discrete uniquely divisible group. Then $B\Gamma$ has the mod p homology of a point.*

PROOF. By definition, Γ is a rational vector space. Thus, for $i > 0$, we have

$$H_i(B\Gamma, \mathbb{Z}/p) = \bigwedge^i_{\mathbb{Z}/p}(\Gamma \otimes \mathbb{Z}/p) = 0.$$

□

THEOREM 5.1.4. *If the component of the identity of G is solvable, then Conjecture* 5.1.1 *holds for G.*

PROOF. Proceed by induction on the dimension of G. Lemma 5.1.2 allows us to assume that G is simply connected. If $\dim_{\mathbb{R}} G = 1$, then $G \cong \mathbb{R}$. Since $B\mathbb{R}$ is contractible, and $H_i(B\mathbb{R}^\delta, \mathbb{Z}/p) = 0$ for $i > 0$ by Lemma 5.1.3, we see that the conjecture holds for G. Now if G has real dimension greater than 1, choose a homomorphism $G \to \mathbb{R}$ with kernel N. Then we have a fibration

$$B\overline{N} \longrightarrow B\overline{G} \longrightarrow B\overline{\mathbb{R}}.$$

By induction, we may assume that $B\overline{N}$ has the mod p homology of a point. The associated spectral sequence may then be used to show that $H_i(B\overline{G}, \mathbb{Z}/p) = 0$ for $i > 0$. □

Theorem 5.1.4 allows us to reduce Conjecture 5.1.1 to the case of simple groups. Indeed, if G is a Lie group, the Lie algebra $\mathfrak{g}$ has a maximal solvable ideal $\mathfrak{n}$. The quotient $\mathfrak{g}/\mathfrak{n}$ splits as a direct product of simple Lie algebras $\mathfrak{s}_i$. Let S_i be the corresponding simple Lie groups. Then we have a fibration

$$B\overline{N} \longrightarrow B\overline{G} \longrightarrow \prod B\overline{S}_i.$$

Since $B\overline{N}$ has the mod p homology of a point by Theorem 5.1.4, we see that $B\overline{G}$ does if and only if each $B\overline{S}_i$ does.

For an arbitrary Lie group G, not much more can be said. We do have the following result.

THEOREM 5.1.5. *Assume G has finitely many components. Then the canonical map*

$$\eta_* : H_i(BG^\delta, \mathbb{Z}/p) \longrightarrow H_i(BG, \mathbb{Z}/p)$$

is a split surjection (and hence the map η^* on cohomology is a split injection).

PROOF. We use the notion of Becker-Gottlieb transfer [**6**]; we shall not define it explicitly. Let $\pi : E \to B$ be a smooth fiber bundle with a closed manifold F as fiber. Then there is a transfer map $\mathrm{tr} : H_i(B) \to H_i(E)$ such that the composition

$$H_i(B) \xrightarrow{\mathrm{tr}} H_i(E) \xrightarrow{\pi_*} H_i(B)$$

is multiplication by the Euler characteristic $\chi(F)$.

Now let G be a Lie group and choose a maximal compact subgroup K. It is a standard fact that G/K is contractible so that the map $i : BK \to BG$ is a

homotopy equivalence. Let N be the normalizer of a maximal torus in K. The manifold K/N has Euler characteristic 1 [58]. Considering the transfer map associated to the fibration $K/N \to BN \xrightarrow{\pi_*} BK$, we see that the map

$$\pi_* : H_i(BN) \to H_i(BK)$$

is a split surjection. By Theorem 5.1.4, the map

$$H_\bullet(BN^\delta, \mathbb{Z}/p) \longrightarrow H_\bullet(BN, \mathbb{Z}/p)$$

is an isomorphism. The proof is finished by noting that the diagram

$$\begin{array}{ccc} H_i(BN^\delta, \mathbb{Z}/p) & \longrightarrow & H_i(BG^\delta, \mathbb{Z}/p) \\ \cong\downarrow & & \downarrow\eta_* \\ H_i(BN, \mathbb{Z}/p) & \xrightarrow{i_*\circ\pi_*} & H_i(BG, \mathbb{Z}/p) \end{array}$$

commutes. □

COROLLARY 5.1.6. *Every element of order n in $H_i(BG, \mathbb{Z})$ lifts to an element of order n in $H_i(BG^\delta, \mathbb{Z})$.*

PROOF. Consider the exact sequence

$$0 \longrightarrow \mathbb{Z} \xrightarrow{n} \mathbb{Z} \longrightarrow \mathbb{Z}/n \longrightarrow 0$$

and the commutative diagram of associated long exact sequences

$$\begin{array}{ccccc} H_{i+1}(BG^\delta, \mathbb{Z}/n) & \longrightarrow & H_i(BG^\delta, \mathbb{Z}) & \xrightarrow{n} & H_i(BG^\delta, \mathbb{Z}) \\ \downarrow\eta_* & & \downarrow\eta_* & & \downarrow\eta_* \\ H_{i+1}(BG, \mathbb{Z}/n) & \longrightarrow & H_i(BG, \mathbb{Z}) & \xrightarrow{n} & H_i(BG, \mathbb{Z}). \end{array}$$

Since $\eta_* : H_{i+1}(BG^\delta, \mathbb{Z}/n) \to H_{i+1}(BG, \mathbb{Z}/n)$ is surjective, the result follows. □

Rational Coefficients. A natural question to study is the behavior of the map

$$H_i(BG^\delta, \mathbb{Q}) \longrightarrow H_i(BG, \mathbb{Q}).$$

In this case, a great deal can be said.

THEOREM 5.1.7. *If G is compact, then the canonical map*

$$H_i(BG^\delta, \mathbb{Q}) \longrightarrow H_i(BG, \mathbb{Q})$$

is zero for $i > 0$.

PROOF. The proof makes use of the Chern–Weil homomorphism

$$\theta : \mathrm{Inv}_G(\mathbb{R}[\mathfrak{g}']) \longrightarrow H^\bullet(BG, \mathbb{R})$$

associated to the Lie group G and its Lie algebra $\mathfrak{g}$. The algebra $\mathrm{Inv}_G(\mathbb{R}[\mathfrak{g}'])$ is the graded algebra of real-valued polynomial functions on the vector space $\mathfrak{g}$ which are invariant under the adjoint action of G. The map θ is defined as follows. Let $f : \mathfrak{g} \to \mathbb{R}$ be an invariant polynomial which is homogeneous of degree n. Let M be a smooth manifold with a smooth principal G-bundle (see Appendix B) $E \to M$ and let ω be a smooth G-invariant connection on E (i.e., ω is a 1-form with values in $\mathfrak{g}$). Let Ω be the curvature 2-form of ω. Then Ω^n is a $2n$-form on E and it gives rise to a closed $2n$-form $f(\Omega)$ whose cohomology class lies in $H^{2n}(M, \mathbb{R})$. This class corresponds to the required class $\theta(f) \in H^{2n}(BG, \mathbb{R})$ under the canonical map

$$H^{2n}(BG, \mathbb{R}) \longrightarrow H^{2n}(M, \mathbb{R})$$

(for further details, see [**116**]). The Chern–Weil theorem [**18**] asserts that if G is compact, then θ is an isomorphism. In particular, BG has only even-dimensional cohomology with real (and hence with rational) coefficients.

Now, any homology class x in $H_{2n}(BG^\delta, \mathbb{Q})$ can be realized as the image of a homology class from some smooth open manifold M which is mapped into BG^δ. To prove that its image in $H_{2n}(BG, \mathbb{Q})$ is zero, we evaluate on an arbitrary real cohomology class in $H^\bullet(BG, \mathbb{R})$. If $n > 0$, choose a homogeneous polynomial $f \in \mathrm{Inv}_G(\mathbb{R}[\mathfrak{g}'])$ of degree n and consider the class $(f(\Omega))$ of the induced bundle over M. This induced bundle has curvature zero and hence the value of x, when evaluated on $(f(\Omega))$, is zero. □

There is a complex version of the Chern–Weil theorem. If G is a semi-simple complex Lie group, then the analogous map

$$\mathrm{Inv}_G(\mathbb{C}[\mathfrak{g}']) \longrightarrow H^\bullet(BG, \mathbb{C})$$

is an isomorphism [**81**]. A similar argument now proves the following.

THEOREM 5.1.8. *If G is a complex semi-simple group with finitely many components, then the natural map*

$$H_i(BG^\delta, \mathbb{Q}) \longrightarrow H_i(BG, \mathbb{Q})$$

is zero for $i > 0$. □

5.2. Groups over Algebraically Closed Fields

There is a natural generalization of Conjecture 5.1.1 for algebraic groups over fields. This generalized isomorphism conjecture is due to E. Friedlander [**41**].

Let S be a simplicial set and let k be an algebraically closed field. Define a simplicial scheme $S \otimes \operatorname{spec}(k)$ by

$$(S \otimes \operatorname{spec}(k))_n = \coprod_{S_n} \operatorname{spec}(k)$$

with simplicial structure induced from S. This scheme has the property that

$$H^\bullet_{\text{ét}}(S \otimes \operatorname{spec}(k), \mathbb{Z}/n) \cong H^\bullet(S, \mathbb{Z}/n).$$

Now let Γ be a discrete group and let G_k be an algebraic group over k. If we have a homomorphism

$$\rho : \Gamma \longrightarrow G(k)$$

from Γ to the discrete group $G(k)$ of k-rational points of G_k, then we obtain an induced map

$$\rho : \Gamma_k = \Gamma \otimes \operatorname{spec}(k) \longrightarrow G_k$$

of group schemes over k. In turn, this induces a map

$$B\rho : B\Gamma_k \longrightarrow BG_k$$

of simplicial schemes. Thus, we have an induced map

$$B\rho^* : H^\bullet_{\text{ét}}(BG_k, \mathbb{Z}/n) \longrightarrow H^\bullet(B\Gamma_k, \mathbb{Z}/n) \cong H^\bullet(B\Gamma, \mathbb{Z}/n).$$

CONJECTURE 5.2.1. *Let k be an algebraically closed field and let p be a prime distinct from* char k. *Let G_k be an algebraic group over k. Then the natural map of group schemes $G(k)_k \longrightarrow G_k$ induces an isomorphism*

$$H^\bullet_{\text{ét}}(BG_k, \mathbb{Z}/p) \longrightarrow H^\bullet(BG(k), \mathbb{Z}/p).$$

As in the case for Lie groups, we can reduce to the case of reductive groups.

PROPOSITION 5.2.2. *Let G_k be a connected algebraic group over k and let U_k denote the unipotent radical of G_k. Then Conjecture* 5.2.1 *holds for G_k if and only if it holds for the reductive group G_k/U_k.*

PROOF. The discrete group $U(k)$ is a successive extension of k_0-vector spaces (where k_0 denotes the prime subfield of k). It follows that $U(k)$ is acyclic for cohomology with $\mathbb{Z}/p$-coefficients, $p \neq$ char k. Thus, the natural map

$$H^\bullet(BG/U(k), \mathbb{Z}/p) \longrightarrow H^\bullet(BG(k), \mathbb{Z}/p)$$

is an isomorphism. On the scheme side, since $G_k \longrightarrow G_k/U_k$ is an affine bundle, the map $BG_k \longrightarrow BG_k/U_k$ induces an isomorphism in étale cohomology. Since the map $H^\bullet_{\text{ét}}(BG_k, \mathbb{Z}/p) \longrightarrow H^\bullet(BG(k), \mathbb{Z}/p)$ is natural, the result follows. □

PROPOSITION 5.2.3. *Let $G_{\mathbb{C}}$ be an algebraic group over $\mathbb{C}$. Then there is a natural commutative diagram*

$$\begin{array}{ccc} H^{\bullet}_{\text{ét}}(BG_{\mathbb{C}}, \mathbb{Z}/p) & \longrightarrow & H^{\bullet}(BG(\mathbb{C})^{top}, \mathbb{Z}/p) \\ & \searrow \quad \swarrow & \\ & H^{\bullet}(BG(\mathbb{C})^{\delta}, \mathbb{Z}/p) & \end{array}$$

where the two diagonal maps are induced by the maps $G(\mathbb{C})_{\mathbb{C}} \to G_{\mathbb{C}}$ and id : $G(\mathbb{C})^{\delta} \to G(\mathbb{C})^{top}$. *Thus, Conjecture* 5.2.1 *holds for $G_{\mathbb{C}}$ if and only if Conjecture* 5.1.1 *holds for the complex Lie group $G(\mathbb{C})^{top}$.*

PROOF. This is obvious from the definitions. □

In the proof of Proposition 5.2.2, we showed that Conjecture 5.2.1 holds for unipotent groups. This is also true for tori and normalizers of maximal tori (just as in the complex case). We will prove this in Section 5.3, but we will use this fact before then.

We now prove Conjecture 5.2.1 for the case $k = \overline{\mathbb{F}}_p$. We first need the following result.

THEOREM 5.2.4. *Let G be a reductive complex Lie group and let l, p be primes with $l \neq p$. Then there exists a map $BG(\overline{\mathbb{F}}_p) \longrightarrow BG$ which induces an isomorphism*

$$H_{\bullet}(BG(\overline{\mathbb{F}}_p), \mathbb{Z}/l) \longrightarrow H_{\bullet}(BG, \mathbb{Z}/l)$$

where $G(\overline{\mathbb{F}}_p)$ is the discrete group of $\overline{\mathbb{F}}_p$-rational points of a Chevalley integral group scheme associated to G.

PROOF. The proof is rather technical and relies on some facts from étale homotopy theory [**40**]. Let $G_{\mathbb{Z}}$ be an integral Chevalley group scheme associated to G and let $G_{\overline{\mathbb{F}}_p} = G_{\mathbb{Z}} \otimes \overline{\mathbb{F}}_p$. Choose an embedding of the Witt vectors of $\overline{\mathbb{F}}_p$ into $\mathbb{C}$. Then for any pth power $q = p^d$ we have a homotopy commutative square ([**40**], Theorem 12.2)

$$\begin{array}{ccc} BG(\mathbb{F}_q) & \longrightarrow & (\mathbb{Z}/l)_{\infty}(BG) \\ {\scriptstyle D_q}\downarrow & & \downarrow{\scriptstyle \Delta} \\ (\mathbb{Z}/l)_{\infty}(BG) & \xrightarrow{1 \times \phi^q} & (\mathbb{Z}/l)_{\infty}(BG^{\times 2}) \end{array} \tag{5.1}$$

such that some map on the homotopy fibers $\text{fib}(D_q) \to \text{fib}(\Delta) \cong (\mathbb{Z}/l)_{\infty}(BG)$ induces a homology isomorphism with $\mathbb{Z}/l$-coefficients. Here, $(\mathbb{Z}/l)_{\infty}(BG)$ is the Bousfield–Kan l-completion of the singular complex of BG [**20**] (the relevant property of this completion is that in this case the map $BG \to (\mathbb{Z}/l)_{\infty}(BG)$ induces a mod l homology isomorphism); ϕ^q is associated to the geometric

Frobenius map $\phi^q : G_{\overline{\mathbb{F}}_p} \to G_{\overline{\mathbb{F}}_p}$; and Δ is induced by the diagonal map $G \to G^{\times 2}$. The proof of this fact uses the Lang isomorphism, which asserts the following. Let $\psi : G_k \to G_k$ be a surjective endomorphism such that the group of ψ-invariant k-rational points, $H = G_k(k)^\psi$, is finite. Then the map

$$1/\psi : G_k \longrightarrow G_k$$

defined by sending a k-rational point g to $g\psi(g)^{-1}$ is a principal H-fibration; that is,

$$1/\psi : G_k/H \xrightarrow{\cong} G_k.$$

The above diagram is obtained by interpreting the Lang isomorphism in this context. Note that the Frobenius map satisfies the condition of the theorem.

Denote by $i : BG(\mathbb{F}_q) \to BG(\mathbb{F}_{q'})$ the map induced by an inclusion $G(\mathbb{F}_q) \to G(\mathbb{F}_{q'})$. Then as a corollary to the above statement (see Corollary 12.4 of [40]), we have, for any $q' = q^e = p^{de}$, a natural map of fibration sequences

$$\begin{array}{ccccc} \mathrm{fib}(D_q) & \longrightarrow & BG(\mathbb{F}_q) & \xrightarrow{D_q} & (\mathbb{Z}/l)_\infty(BG) \\ \downarrow{\scriptstyle j} & & \downarrow{\scriptstyle i} & & \downarrow{\scriptstyle \mathrm{id}} \\ \mathrm{fib}(D_{q'}) & \longrightarrow & BG(\mathbb{F}_{q'}) & \xrightarrow{D_{q'}} & (\mathbb{Z}/l)_\infty(BG) \end{array} \tag{5.2}$$

such that $j^* : H^\bullet(\mathrm{fib}(D_{q'}), \mathbb{Z}/l) \to H^\bullet(\mathrm{fib}(D_q), \mathbb{Z}/l)$ can be identified with the map

$$\theta^* : H^\bullet_{\text{ét}}(G_{\overline{\mathbb{F}}_p}, \mathbb{Z}/l) \longrightarrow H^\bullet_{\text{ét}}(G_{\overline{\mathbb{F}}_p}, \mathbb{Z}/l)$$

induced by $\theta = \mu \circ (1 \times \phi^q \times \cdots \times \phi^{q'/q}) : G_{\overline{\mathbb{F}}_p} \to G_{\overline{\mathbb{F}}_p}$, where $\mu : (G_{\overline{\mathbb{F}}_p})^e \to G_{\overline{\mathbb{F}}_p}$ is the product map. Associated to each sequence in (5.2), we have a spectral sequence. Taking direct limits we obtain a spectral sequence with

$$E^2_{s,t} = H_s(BG, \varinjlim_q H_t(\mathrm{fib}(D_q), \mathbb{Z}/l)) \Longrightarrow H_{s+t}(BG(\overline{\mathbb{F}}_p), \mathbb{Z}/l).$$

We now show that $E^2_{s,t} = 0$ for $t \neq 0$.

LEMMA 5.2.5. *For any $q = p^d$ there exists some $q' = p^e$ such that the endomorphism on the $\mathbb{Z}/l$-dual Hopf algebra of $H^\bullet_{\text{ét}}(G_{\overline{\mathbb{F}}_p}, \mathbb{Z}/l)$*

$$j_* : H^\bullet_{\text{ét}}(G_{\overline{\mathbb{F}}_p}, \mathbb{Z}/l)^\# \cong H_\bullet(\mathrm{fib}(D_q), \mathbb{Z}/l) \to H_\bullet(\mathrm{fib}(D_{q'}), \mathbb{Z}/l)$$

induced by $j : \mathrm{fib}(D_q) \to \mathrm{fib}(D_{q'})$ satisfies

1. *if $x \in H^\bullet_{\text{ét}}(G_{\overline{\mathbb{F}}_p}, \mathbb{Z}/l)^\#$ is primitive, then $j_*(x) = 0$*
2. *if $x \in H^\bullet_{\text{ét}}(G_{\overline{\mathbb{F}}_p}, \mathbb{Z}/l)^\#$ is such that $j_*(x) \neq 0$ while $j_*(y) = 0$ for all y with homological degree less than the degree of x, then $j_*(x)$ is primitive.*

PROOF. Identify j_* with the dual of the map θ^* defined above. Then if x is primitive, we have

$$j_*(x) = x + \phi_*^q(x) + \cdots + \phi_*^q \circ \phi_*^q \circ \cdots \circ \phi_*^q(x)$$

where ϕ_*^q is the dual of ϕ^{q*}. If the order of ϕ_*^q as an automorphism of the finite-dimensional Hopf algebra $H_{\text{ét}}^\bullet(G_{\overline{\mathbb{F}}_p}, \mathbb{Z}/l)^\#$ is m, then $j_*(x) = 0$ provided that $q' = q^e$ is such that $e = lm$. To prove the second assertion, assume that x satisfies the stated conditions and let $e = lm$. Denote by Δ the comultiplication in the Hopf algebra $H_{\text{ét}}^\bullet(G_{\overline{\mathbb{F}}_p}, \mathbb{Z}/l)$. We have $\Delta_*(x) = 1 \otimes x + x \otimes 1 + \sum x_i \otimes x_j$ in $H_{\text{ét}}^\bullet(G_{\overline{\mathbb{F}}_p} \times G_{\overline{\mathbb{F}}_p}, \mathbb{Z}/l)^\#$. Then

$$\begin{aligned} \Delta_*(j_*(x)) &= j_*(\Delta_*(x)) \\ &= 1 \otimes j_*(x) + j_*(x) \otimes 1 + \sum j_*(x_i) \otimes j_*(x_j) \\ &= 1 \otimes j_*(x) + j_*(x) \otimes 1 \end{aligned}$$

so that $j_*(x)$ is primitive. □

Lemma 5.2.5 implies that for each $q = p^d$ there is some $q' = q^e$ such that

$$j_* : \tilde{H}_\bullet(\text{fib}(D_q), \mathbb{Z}/l) \longrightarrow \tilde{H}_\bullet(\text{fib}(D_{q'}), \mathbb{Z}/l)$$

is the zero map. Hence, we have $E_{s,t}^2 = 0$ for $t \neq 0$ and therefore the map

$$\begin{aligned} (\varinjlim D_q)_* : H_\bullet(BG(\overline{\mathbb{F}}_p), \mathbb{Z}/l) &= \varinjlim H_\bullet(BG(\mathbb{F}_q), \mathbb{Z}/l) \\ &\to H_\bullet((\mathbb{Z}/l)_\infty(BG), \mathbb{Z}/l) \\ &= H_\bullet(BG, \mathbb{Z}/l) \end{aligned}$$

is an isomorphism. Since $\tilde{H}_\bullet(BG(\overline{\mathbb{F}}_p), \mathbb{Q}) = \tilde{H}_\bullet(BG(\overline{\mathbb{F}}_p), \mathbb{Z}/p) = 0$, we can find a unique lift [**120**]

$$\begin{array}{ccc} & & BG \\ & \nearrow & \downarrow \\ BG(\overline{\mathbb{F}}_p) & \xrightarrow{\varinjlim D_q} & (\mathbb{Z}/l)_\infty(BG) \end{array}$$

which gives the required isomorphism

$$H_\bullet(BG(\overline{\mathbb{F}}_p), \mathbb{Z}/l) \longrightarrow H_\bullet(BG, \mathbb{Z}/l).$$

This completes the proof of Theorem 5.2.4. □

COROLLARY 5.2.6. *Let p be a prime and let $G_{\overline{\mathbb{F}}_p}$ be an algebraic group over $\overline{\mathbb{F}}_p$. Then Conjecture* 5.2.1 *holds for $G_{\overline{\mathbb{F}}_p}$.*

PROOF. We may assume that $G_{\overline{\mathbb{F}}_p}$ is reductive and that $G_{\overline{\mathbb{F}}_p} = G_{\mathbb{Z}} \otimes \overline{\mathbb{F}}_p$. The map

$$\varinjlim D_q : BG(\overline{\mathbb{F}}_p) \longrightarrow (\mathbb{Z}/l)_\infty(BG(\mathbb{C})^{\mathrm{top}})$$

is induced by the map $G(\overline{\mathbb{F}}_p)_{\overline{\mathbb{F}}_p} \longrightarrow G_{\overline{\mathbb{F}}_p}$ and a choice of embedding of the Witt vectors of $\overline{\mathbb{F}}_p$ into $\mathbb{C}$ induces an isomorphism

$$H^\bullet_{\text{ét}}(BG(\overline{\mathbb{F}}_p), \mathbb{Z}/l) \cong H^\bullet_{\text{ét}}(BG_{\mathbb{C}}, \mathbb{Z}/l) \cong H^\bullet(BG(\mathbb{C})^{\mathrm{top}}, \mathbb{Z}/l).$$

(The first isomorphism is implicit in the statement of Theorem 5.2.4 and is due to Friedlander and Parshall [42]. They proved that if k is a separably closed field with ring of Witt vectors R, then the base change maps

$$G_k \longrightarrow G_R \longleftarrow G_K \longrightarrow G_{\mathbb{C}}$$

associated with embeddings $R \to K \leftarrow \mathbb{C}$ (K algebraically closed) induce isomorphisms in mod l étale cohomology.) Consider the commutative diagram

$$\begin{array}{ccc} H^\bullet_{\text{ét}}(BG_{\overline{\mathbb{F}}_p}, \mathbb{Z}/l) & \longrightarrow & H^\bullet(BG(\mathbb{C})^{\mathrm{top}}, \mathbb{Z}/l) \\ & \searrow & \downarrow \\ & & H^\bullet(BG(\overline{\mathbb{F}}_p), \mathbb{Z}/l). \end{array}$$

By Theorem 5.2.4, the right arrow is an isomorphism. The result follows. □

Finite fields revisited. Diagram (5.1) allows one to compute the cohomology of the finite group $G(\mathbb{F}_q)$ via the associated Eilenberg–Moore spectral sequence. This was noted by Quillen in his ICM address [97], and discovered independently by Friedlander [39]. This spectral sequence has the form

$$E_2^{s,\bullet} = \mathrm{Tor}_{-s}^{H^\bullet((\mathbb{Z}/l)_\infty(BG)\times(\mathbb{Z}/l)_\infty(BG))}(H^\bullet((\mathbb{Z}/l)_\infty(BG)), H^\bullet((\mathbb{Z}/l)_\infty(BG))) \Longrightarrow H^\bullet(BG(\mathbb{F}_q), \mathbb{Z}/l).$$

Note that this holds for the unstable group $BG(\mathbb{F}_q)$. The computation given in Chapter 1 involved this spectral sequence for the stable space BU followed by a difficult restriction argument to the unstable case.

For example, if $G = GL_n$, then we have

$$\begin{aligned} H^\bullet((\mathbb{Z}/l)_\infty(BG), \mathbb{Z}/l) &= H^\bullet(BG, \mathbb{Z}/l) \\ &= H^\bullet(BU_n, \mathbb{Z}/l) \\ &= H^\bullet(G(n, \infty), \mathbb{Z}/l) \\ &= \mathbb{Z}/l[c_1, c_2, \ldots, c_n], \quad \deg c_i = 2i. \end{aligned}$$

The spectral sequence then has the form

$$\mathrm{Tor}_{-s}^{\mathbb{Z}/l[c_1,\ldots,c_n,c'_1,\ldots,c'_n]}(\mathbb{Z}/l[c_1, \ldots, c_n], \mathbb{Z}/l[c'_1, \ldots, c'_n]) \Longrightarrow H^\bullet(BG(\mathbb{F}_q)).$$

The E_2-term is easily calculated (see Chapter 1), giving the result.

We now return to our study of Conjecture 5.2.1. If $H \subseteq G$ is an inclusion of discrete groups and if A is a G-module, then $x \in H^\bullet(BH, A)$ is *G-invariant* if the images of x under the maps

$$H^\bullet(BH, A) \xrightarrow{\text{res}} H^\bullet(B(H \cap gHg^{-1}), A)$$

and

$$H^\bullet(BH, A) \xrightarrow{c(g)^*} H^\bullet(B(gHg^{-1}), A) \xrightarrow{\text{res}} H^\bullet(B(H \cap gHg^{-1}), A)$$

coincide, where $c(g)^*$ is the map induced by conjugation (see Appendix A). Denote the set of G-invariant elements by $H^\bullet(BH, A)^S$. For example, if G is finite with p-Sylow subgroup H, then the p-primary part of $H^\bullet(BG, A)$ is detected on H; that is,

$$H^n(BG, A)_{(p)} \xrightarrow{\cong} H^n(BH, A)^S$$

for $n > 0$.

PROPOSITION 5.2.7. *Let $G_{\overline{\mathbb{F}}_p}$ be a connected linear algebraic group over $\overline{\mathbb{F}}_p$ and let $N_{\overline{\mathbb{F}}_p} \subset G_{\overline{\mathbb{F}}_p}$ be the normalizer of a maximal torus in $G_{\overline{\mathbb{F}}_p}$. Then the restriction map*

$$H^\bullet(BG(\overline{\mathbb{F}}_p), \mathbb{Z}/l) \longrightarrow H^\bullet(BN(\overline{\mathbb{F}}_p), \mathbb{Z}/l)$$

is injective with image equal to the set of G-invariant elements.

PROOF. Let $G_{\mathbb{Z}}$ be an associated reductive group scheme over $\mathbb{Z}$. The group $N(\mathbb{F}_q)$ of $\mathbb{F}_q$-rational points of $N_{\mathbb{Z}}$ contains an l-Sylow subgroup of $G(\mathbb{F}_q)$ [**117**]. Thus the restriction maps

$$H^\bullet(BG(\mathbb{F}_q), \mathbb{Z}/l) \longrightarrow H^\bullet(BN(\mathbb{F}_q), \mathbb{Z}/l)^S$$

are isomorphisms for any $q = p^d$. Now pass to the limit. □

THEOREM 5.2.8. *Let G_k be a connected linear algebraic group over k and let $N_k \subset G_k$ be the normalizer of a maximal torus. Then the composition*

$$H^\bullet_{\text{ét}}(BG_k, \mathbb{Z}/l) \longrightarrow H^\bullet_{\text{ét}}(BN_k, \mathbb{Z}/l) \xrightarrow{\cong} H^\bullet(BN(k), \mathbb{Z}/l)$$

is an injection with image the stable elements with respect to the embedding $N(k) \subset G(k)$.

PROOF. The proof that Conjecture 5.2.1 holds for N_k will be postponed until Section 5.3. Let $R \subset \mathbb{C}$ be the strict henselization of the local ring $\mathbb{Z}_{(p)}$. Then R has residue field $\overline{\mathbb{F}}_p$ and field of fractions contained in $\overline{\mathbb{Q}}$. By Hensel's lemma, the kernel of the surjection $R^\times \to \overline{\mathbb{F}}_p^\times$ is uniquely l-divisible, as are the cokernels of the injections $R^\times \to \overline{\mathbb{Q}}^\times$ and $\overline{\mathbb{Q}}^\times \to \mathbb{C}^\times$. As a consequence, we have isomorphisms (with $\mathbb{Z}/l$-coefficients)

$$H^\bullet(BN(\overline{\mathbb{F}}_p)) \xrightarrow{\cong} H^\bullet(BN(R)) \xleftarrow{\cong} H^\bullet(BN(\overline{\mathbb{Q}})) \xleftarrow{\cong} H^\bullet(BN(\mathbb{C})^\delta).$$

Hence the map

$$H^\bullet(BN(\overline{\mathbb{F}}_p), \mathbb{Z}/l)^S \longrightarrow H^\bullet(BN(R), \mathbb{Z}/l)^S$$

is injective. To see that it is surjective, let $x \in H^\bullet(BN(R), \mathbb{Z}/l)$ be stable for $g \in G(R)$. Denote by $\overline{g}$ the image of g in $G(\overline{\mathbb{F}}_p)$. Note that

$$H^\bullet(BN(\overline{\mathbb{F}}_p) \cap N(\overline{\mathbb{F}}_p)^{\overline{g}}, \mathbb{Z}/l) \cong H^\bullet(BN(R) \cap N(R)^g, \mathbb{Z}/l)$$

so that the element in $H^\bullet(BN(\overline{\mathbb{F}}_p), \mathbb{Z}/l)$ corresponding to x is stable for $\overline{g}$. Since R is Henselian, the map $G(R) \to G(\overline{\mathbb{F}}_p)$ is surjective, so we have an isomorphism

$$H^\bullet(BN(\overline{\mathbb{F}}_p), \mathbb{Z}/l)^S \xrightarrow{\cong} H^\bullet(BN(R), \mathbb{Z}/l)^S.$$

Now consider the commutative diagram

$$\begin{array}{ccccc}
H^\bullet(BN(\overline{\mathbb{F}}_p), \mathbb{Z}/l)^S & \xrightarrow{\cong} & H^\bullet(BN(R), \mathbb{Z}/l)^S & \longleftarrow & H^\bullet(BN(\overline{\mathbb{Q}}), \mathbb{Z}/l)^S \\
\uparrow & & \uparrow & & \uparrow \\
H^\bullet_{\text{ét}}(BG_{\overline{\mathbb{F}}_p}, \mathbb{Z}/l) & \xleftarrow{\cong} & H^\bullet_{\text{ét}}(BG_R, \mathbb{Z}/l) & \xrightarrow{\cong} & H^\bullet_{\text{ét}}(BG_{\overline{\mathbb{Q}}}, \mathbb{Z}/l).
\end{array}$$

The lower horizontal maps are induced by base change and hence are isomorphisms. By Corollary 5.2.6 and Proposition 5.2.7, we have an isomorphism

$$H^\bullet_{\text{ét}}(BG_{\overline{\mathbb{F}}_p}, \mathbb{Z}/l) \xrightarrow{\cong} H^\bullet(BN(\overline{\mathbb{F}}_p), \mathbb{Z}/l)^S$$

and hence all the vertical arrows are isomorphisms. In particular,

$$H^\bullet_{\text{ét}}(BG_{\overline{\mathbb{Q}}}, \mathbb{Z}/l) \xrightarrow{\cong} H^\bullet(BN(\overline{\mathbb{Q}}), \mathbb{Z}/l)^S.$$

Now if k is any algebraically closed field, let $k_0 \subseteq k$ be the algebraic closure of the prime field. Then since $k^\times/k_0^\times$ is uniquely l-divisible, we have an injection

$$H^\bullet(BN(k), \mathbb{Z}/l)^S \longrightarrow H^\bullet(BN(k_0), \mathbb{Z}/l)^S.$$

Consider the commutative diagram

$$\begin{array}{ccc}
H^\bullet_{\text{ét}}(BG_{k_0}, \mathbb{Z}/l) & \xrightarrow{\cong} & H^\bullet_{\text{ét}}(BG_k, \mathbb{Z}/l) \\
\downarrow \cong & & \downarrow \\
H^\bullet_{\text{ét}}(BN(k_0), \mathbb{Z}/l)^S & \longleftarrow & H^\bullet(BN(k), \mathbb{Z}/l)^S.
\end{array}$$

The top map is an isomorphism by base change. Since the bottom map is injective, we obtain the desired isomorphism

$$H^\bullet_{\text{ét}}(BG_k, \mathbb{Z}/l) \xrightarrow{\cong} H^\bullet(BN(k), \mathbb{Z}/l)^S$$

for any k. □

COROLLARY 5.2.9. *The map*

$$H^\bullet_{\text{ét}}(BG_k, \mathbb{Z}/l) \longrightarrow H^\bullet(BG(k), \mathbb{Z}/l)$$

is split injective and the map

$$H^\bullet(BG(k), \mathbb{Z}/l) \longrightarrow H^\bullet(BN(k), \mathbb{Z}/l)^S$$

is split surjective.

PROOF. Consider the commutative diagram

$$\begin{array}{ccc} H^\bullet_{\text{ét}}(BG_k, \mathbb{Z}/l) & \longrightarrow & H^\bullet(BG(k), \mathbb{Z}/l) \\ & \searrow & \downarrow \\ & & H^\bullet(BN(k), \mathbb{Z}/l)^S. \end{array}$$

The left map is an isomorphism. □

COROLLARY 5.2.10. *Conjecture* 5.2.1 *holds for G_k if and only if the map*

$$H^\bullet(BG(k), \mathbb{Z}/l) \longrightarrow H^\bullet(BN(k), \mathbb{Z}/l)$$

is injective. □

Conjecture 5.2.1 admits an equivalent formulation in terms of finite subgroups.

CONJECTURE 5.2.11. *Let k be an algebraically closed field, let G_k be an algebraic group over k, and let l be a prime distinct from the characteristic of k. Then for any nonzero $x \in H^\bullet(BG(k), \mathbb{Z}/l)$, there exists a finite subgroup $H \subset G(k)$ such that x restricts nontrivially to $H^\bullet(BH, \mathbb{Z}/l)$.*

THEOREM 5.2.12. *Conjecture* 5.2.1 *holds for G_k if and only if Conjecture* 5.2.11 *does.*

PROOF. Suppose G_k satisfies 5.2.11. We show that the map

$$H^\bullet(BG(k), \mathbb{Z}/l) \to H^\bullet(BN(k), \mathbb{Z}/l)$$

is injective. Let $x \in H^\bullet(BG(k), \mathbb{Z}/l)$ be nonzero and choose a finite subgroup $H \subset G(k)$ such that x restricts nontrivially to $H^\bullet(BH, \mathbb{Z}/l)$. We may replace H by an l-Sylow subgroup and hence assume that H is an l-group. Such a group consists entirely of semi-simple elements and hence is conjugate to a subgroup of $N(k)$. The restriction of x to $H^\bullet(BN(k), \mathbb{Z}/l)$ is therefore nontrivial.

Conversely, if G_k satisfies 5.2.1, then the map

$$H^\bullet(BG(k), \mathbb{Z}/l) \longrightarrow H^\bullet(BN(k), \mathbb{Z}/l)$$

is injective. If char $k = p > 0$, choose an embedding $\overline{\mathbb{F}}_p \subset k$. This induces isomorphisms

$$H^\bullet(BN(k), \mathbb{Z}/l) \xrightarrow{\cong} H^\bullet(BN(\overline{\mathbb{F}}_p), \mathbb{Z}/l) \xrightarrow{\cong} \varprojlim H^\bullet(BN(\mathbb{F}_q), \mathbb{Z}/l)$$

and hence the mod l cohomology of $G(k)$ is detected on the finite subgroups $N(\mathbb{F}_q) \subset G(k)$. If k has characteristic zero, choose a prime p which does not divide the order of the Weyl group W of G_k. Choose an embedding of the strict henselization R of $\mathbb{Z}_{(p)}$ into k. This gives rise to maps $\overline{\mathbb{F}}_p \leftarrow R \rightarrow k$. These induce isomorphisms

$$H^\bullet(BN(\overline{\mathbb{F}}_p), \mathbb{Z}/l) \cong H^\bullet(BN(R), \mathbb{Z}/l) \cong H^\bullet(BN(k), \mathbb{Z}/l).$$

Since the map $R^\times \rightarrow \overline{\mathbb{F}}_p^\times$ has a splitting with uniquely $|W|$-divisible cokernel, we get a W-equivariant map $T(\overline{\mathbb{F}}_p) \rightarrow T(R)$ inducing the inverse to the reduction map

$$H^\bullet(BW, T(R)) \xrightarrow{\cong} H^\bullet(BW, T(\overline{\mathbb{F}}_p)).$$

In particular, the reduction map $N(R) \rightarrow N(\overline{\mathbb{F}}_p)$ (which is a map of extensions of W) admits a splitting which induces an isomorphism

$$H^\bullet(BN(R), \mathbb{Z}/l) \xrightarrow{\cong} H^\bullet(BN(\overline{\mathbb{F}}_p), \mathbb{Z}/l).$$

The composite map $N(\overline{\mathbb{F}}_p) \rightarrow N(R) \rightarrow N(k) \rightarrow G(k)$ detects the mod l cohomology of $G(k)$. Since the mod l cohomology of $N(\overline{\mathbb{F}}_p)$ is detected by its finite subgroups $N(\mathbb{F}_q)$, the result follows. □

COROLLARY 5.2.13. *Let $k = \bigcup k_\alpha$, where each k_α is algebraically closed. Then G_k satisfies Conjecture* 5.2.1 *if and only if each G_{k_α} does.*

PROOF. Suppose G_k satisfies 5.2.1. For each α, $k = \bigcup A_\beta$, where A_β is a finitely generated k_α-algebra. Each A_β admits a k_α-algebra map $A_\beta \rightarrow k_\alpha$. It follows that the map

$$H_\bullet(BG(k_\alpha), \mathbb{Z}/l) \longrightarrow H_\bullet(BG(k), \mathbb{Z}/l)$$

is injective, and hence

$$H^\bullet(BG(k), \mathbb{Z}/l) \longrightarrow H^\bullet(BG(k_\alpha), \mathbb{Z}/l)$$

is surjective. Consider the diagram

$$\begin{array}{ccc} H^\bullet(BG(k), \mathbb{Z}/l) & \longrightarrow & H^\bullet(BN(k), \mathbb{Z}/l) \\ \downarrow & & \downarrow \cong \\ H^\bullet(BG(k_\alpha), \mathbb{Z}/l) & \longrightarrow & H^\bullet(BN(k_\alpha), \mathbb{Z}/l). \end{array}$$

It follows that the bottom map is injective.

The converse is easy and is left to the reader. □

As a consequence we see that we need only find a "large enough" field of each characteristic to prove Conjecture 5.2.1. Indeed, let k be an algebraically closed field of infinite transcendence degree over the prime subfield and let L be algebraically closed with $\mathrm{char}(L) = \mathrm{char}(k)$. Assume that 5.2.1 holds for G_k. Write $L = \bigcup L_\alpha$ where each L_α is algebraically closed and of finite transcendence degree over the prime subfield. Then each L_α admits an embedding into k and hence each G_{L_α} satisfies Conjecture 5.2.1. Since L is the union of the L_α, we see that G_L does also.

There is another reformulation of Conjecture 5.2.1, due to Jardine [**62**], which involves ultraproducts of simplicial sets. Recall that an *ultrafilter* $\mathcal{U}$ on a set I is a collection of subsets of I such that (1) $\mathcal{U}$ does not contain the empty set; (2) if $X \subset Y$ and $X \in \mathcal{U}$, then $Y \in \mathcal{U}$; (3) $\mathcal{U}$ is closed under finite intersections; and (4) if $X \cup Y \in \mathcal{U}$, then $X \in \mathcal{U}$ or $Y \in \mathcal{U}$. Given an ultrafilter $\mathcal{U}$, the corresponding *ultraproduct* of a collection of sets $\{F_i\}_{i \in I}$ is the set

$$\left(\prod_{i \in I} F_i\right) / \sim$$

where the equivalence relation $\sim$ is defined by $(a_i) = (b_i)$ if there is a $Z \in \mathcal{U}$ with $a_i = b_i$ for all $i \in Z$.

Now assume that the sets F_i are fields and let R be their ordinary product. Let $\mathfrak{p}$ be a prime ideal of R. If $X \subset I$, define an element $e_X = (e_{X,i}) \in R$ by

$$e_{X,i} = \begin{cases} 0 & \text{if } i \in X \\ 1 & \text{if } i \notin X. \end{cases}$$

Let $\mathcal{U}(\mathfrak{p})$ be the set of subsets $X \subset I$ such that $e_X \in \mathfrak{p}$. One checks easily that $\mathcal{U}(\mathfrak{p})$ is an ultrafilter. Conversely, every ultrafilter $\mathcal{U}$ corresponds to some prime ideal $\mathfrak{p}(\mathcal{U})$ of R. In particular, the kernel $\mathfrak{m}_i$ of the projection

$$\mathrm{pr}_i : R \longrightarrow F_i$$

is a maximal ideal of R which corresponds to the ultrafilter $\mathcal{U}_i = \mathcal{U}(\mathfrak{m}_i)$ that is generated by the element $e_{\{i\}}$.

Denote by j the canonical morphism of schemes

$$j : \coprod_{i \in I} \mathrm{spec}(F_i) \longrightarrow \mathrm{spec}(R).$$

If Y is a collection of sets indexed by I, then there is a sheaf j_*Y on $\mathrm{spec}(R)$ such that the ultraproduct of the sets Y_i corresponding to $\mathcal{U}$ is the stalk $j_*Y_{\mathfrak{p}(\mathcal{U})}$ of j_*Y at the point $\mathfrak{p}(\mathcal{U})$. We will denote the ultraproduct by $j_*(Y)_{\mathcal{U}}$.

CONJECTURE 5.2.14. *Let I be the set of prime numbers and let $P : J \longrightarrow I$ be a function, where J is some infinite set. Let $BSL_n(\overline{\mathbb{F}}_P)$ denote the sheaf on the discrete space $\mathcal{P}(J)$ of all subsets of J corresponding to the family of*

simplicial sets $\{BSL_n(\overline{\mathbb{F}}_{P(j)})\}$, *and let* $BSL_n(\overline{\mathbb{F}}_P)^\wedge_l$ *be the sheaf associated to the family of* l*-completions* $\{BSL_n(\overline{\mathbb{F}}_P(j))^\wedge_l\}$. *Let* $\mathcal{U}$ *be an ultrafilter on the set* J *such that the set* $\{j \in J : P(j) = l\}$ *is not a member of* $\mathcal{U}$. *Then the induced map*

$$j_*(BSL_n(\overline{\mathbb{F}}_P))_{\mathcal{U}} \longrightarrow j_*(BSL_n(\overline{\mathbb{F}}_P)^\wedge_l)_{\mathcal{U}}$$

is a mod l *homology isomorphism.*

According to Jardine ([**62**], Theorem 17), this conjecture is equivalent to Conjecture 5.2.1 for SL_n. Moreover, one expects that this construction works equally well for any simple simply connected algebraic group. Also, this can be reduced to a statement about ultraproducts of classifying spaces for finite special linear groups. The reader is referred to [**62**] for further details.

5.3. Rigidity

The study of Conjecture 5.2.1 is closely related to the concept of rigidity, which we now explain. Let V be a contravariant functor on an appropriate category of schemes over a field k (e.g., the category of smooth affine schemes) with values in the category of torsion abelian groups.

DEFINITION 5.3.1. The functor V is rigid if for every connected variety X over k and closed points $x, y : \mathrm{spec}(k) \longrightarrow X$, the corresponding maps

$$x^*, y^* : V(X) \longrightarrow V(\mathrm{spec}(k)) = V(k)$$

coincide.

To produce examples of rigid functors, we need the following notions. We say that V is *homotopy invariant* if $V(X \times \mathbb{A}^1_k) = V(X)$ for any scheme X. The functor V *admits transfers* if for any finite flat morphism $X \longrightarrow Y$, there is a homomorphism $\mathrm{tr}_{X/Y} : V(X) \longrightarrow V(Y)$ satisfying the usual properties (e.g., the composition

$$V(Y) \longrightarrow V(X) \xrightarrow{\mathrm{tr}} V(Y)$$

is multiplication by the degree of the map).

THEOREM 5.3.2. *Suppose* V *is a homotopy invariant functor with transfers. Then* V *is rigid.*

PROOF. It suffices to consider the case where X is a smooth affine curve. Let $\mathrm{Div}(X)$ be the group of divisors on X and let $\mathrm{Div}^0(X)$ be the subgroup of principal divisors. Consider the pairing

$$\mathrm{Div}(X) \times V(X) \longrightarrow V(k)$$

defined by $(x, u) \mapsto x^*(u)$. We show that this vanishes on $\mathrm{Div}^0(X) \times V(X)$. Let $\overline{X}$ be a smooth projective model for X and set $X_\infty = \overline{X} - X$. Let f be

a rational function on $\overline{X}$ which is equal to 1 on X_∞. Consider the principal divisor (f). Let X_0 be the curve obtained from X by deleting the points where f equals 1. Then f defines a covering $X_0 \longrightarrow \mathbb{A}^1_k = \mathbb{P}^1_k - \{1\}$. The properties of transfer imply that the image of $((f), u)$ in $V(k)$ coincides with the image of $((0-\infty), \mathrm{tr}_{X_0/\mathbb{A}^1}(u|_{X_0}))$; the latter is zero by homotopy invariance. Thus, the pairing factors through $\mathrm{Pic}^0(\overline{X}, X_\infty) \otimes V(X)$. Since $\mathrm{Pic}^0(\overline{X}, X_\infty)$ is divisible and $V(X)$ is torsion, we see that the pairing vanishes on $\mathrm{Div}^0(X) \times V(X)$; that is, $x^*(u) = y^*(u)$ for any $x, y : \mathrm{spec}(k) \longrightarrow X$. □

As an easy corollary, one sees that if V commutes with limits:

$$V(\mathrm{spec}(\varinjlim A_i)) = \varinjlim V(\mathrm{spec}(A_i)),$$

then $V(k) = V(k_0)$ for any extension $k_0 \longrightarrow k$ of algebraically closed fields [**125**]. To see this, note that if X_0 is a connected variety over k_0 and if $x, y : \mathrm{spec}(k) \to X_0$ are any two k_0-points, then $x^* = y^*$. In turn, this implies that the image of the map $V(X_0) \to V(k)$ corresponding to $x : \mathrm{spec}(k) \to X_0$ lies in the image of $V(k_0)$: choose a rational point $\mathrm{spec}(k_0) \to X_0$ and apply the above remark to x and $y : \mathrm{spec}(k) \to \mathrm{spec}(k_0) \to X_0$. Now write k as the limit of the finitely generated k_0-subalgebras of k. Then $V(k) = \varinjlim V(\mathrm{spec}(A))$. The homomorphism $V(\mathrm{spec}(A)) \to V(k)$ is induced by a k_0-point $\mathrm{spec}(k) \to \mathrm{spec}(A)$ corresponding to the embedding of A in k. The image of this map is contained in the image of $V(k_0)$ and hence the map $V(k_0) \to V(k)$ is surjective. The injectivity of this map is clear since each algebra A is split over k_0.

Since the K-groups are homotopy invariant, admit transfers and commute with limits, we obtain the following result.

COROLLARY 5.3.3. *Let $k_0 \longrightarrow k$ be an extension of algebraically closed fields. Then for any n, the natural maps $K_i(k_0)/n \longrightarrow K_i(k)/n$, ${}_nK_i(k_0) \longrightarrow {}_nK_i(k)$, and $K_i(k_0, \mathbb{Z}/n) \longrightarrow K_i(k, \mathbb{Z}/n)$ are isomorphisms.* □

COROLLARY 5.3.4. *Let k be an algebraically closed field of positive characteristic p. Then $K_i(k)$ is divisible for $i \geq 1$. The torsion subgroup is trivial for i even and is isomorphic to $\bigoplus_{l \neq p} \mathbb{Q}_l/\mathbb{Z}_l$ if i is odd.*

PROOF. The follows from the computation 1.1.12 for $k_0 = \overline{\mathbb{F}}_p$. □

Corollary 5.3.3 begs the question: are the homology functors

$$H_i(G(-), \mathbb{Z}/n),$$

where G is an affine group scheme over k, rigid on the category of smooth k-algebras? This question has not been settled.

CONJECTURE 5.3.5. *Let G be a group over an algebraically closed field k and let p be a prime distinct from the characteristic of k. Let X be a smooth*

affine curve over k and let x, y be closed points on X. Then the corresponding specialization homomorphisms

$$s_x, s_y : H_\bullet(G(k[X]), \mathbb{Z}/p) \longrightarrow H_\bullet(G(k), \mathbb{Z}/p)$$

coincide.

Note that 5.3.5 holds for the affine line when $G = GL_n, SL_n, PGL_n$: By Corollary 4.6.3, the inclusion $G(k) \longrightarrow G(k[t])$ induces an isomorphism on integral homology and this map is split by evaluation at any $x \in \mathbb{A}^1$. More generally, it holds for $G = PGL_2$, $X = C - \{x\}$, where C is a smooth projective curve over k and x is a closed point on C. This follows from Theorem 4.5.7 by an argument similar to the proof of the rigidity theorem in [**126**].

Consider the functors $H_i(E_2(-), \mathbb{Z})$ on the category of smooth algebras, where k is an infinite field. In Section 4.6.3 we showed that these satisfy

$$H_i(E_2(R[t]), \mathbb{Z}) \cong H_i(E_2(R), \mathbb{Z})$$

whenever R has many units. If R is a smooth k-algebra, then

$$k[\mathrm{spec}(R) \times \mathbb{A}^1_k] = R[t];$$

thus, the functors $H_i(E_2(-), \mathbb{Z})$ are homotopy invariant. If they also admit transfers, then Theorem 5.3.2 would imply that the functors $H_i(E_2(-), \mathbb{Z}/n)$ are rigid for $(n, \mathrm{char}(k)) = 1$. Unfortunately, there appears to be no way to equip these functors with transfer maps.

Conjecture 5.3.5 is implied by the following stronger statement.

CONJECTURE 5.3.6. *Let G be an algebraic group over k and let p be a prime distinct from the characteristic of k. Let X be a smooth affine curve over k and denote by $\mathcal{O}_x^h$ the henselization of the local ring $\mathcal{O}_x$, $x \in X$. Then the natural map*

$$H_\bullet(G(k), \mathbb{Z}/p) \longrightarrow H_\bullet(G(\mathcal{O}_x^h), \mathbb{Z}/p)$$

is an isomorphism.

This conjecture (and hence Conjecture 5.3.5) holds in a few special cases. We discuss this in Section 5.5.

As promised, we have the following result.

PROPOSITION 5.3.7. *Conjecture* 5.3.6 *implies Conjecture* 5.3.5.

PROOF. Let x, y be closed points on the smooth affine curve X. Note that

$$H_\bullet(G(\mathcal{O}_x^h), \mathbb{Z}/p) = \varinjlim_{Y/X \text{ étale}} H_\bullet(G(k[Y]), \mathbb{Z}/p).$$

Let $\alpha \in H_\bullet(G(k[X]), \mathbb{Z}/p)$ and denote by $\alpha(x)$ the image of α under the specialization map s_x. Since $H_\bullet(G(k), \mathbb{Z}/p)$ is a summand of $H_\bullet(G(k[X]), \mathbb{Z}/p)$, we can view $\alpha(x)$ in the latter. Consider the class $\alpha - \alpha(x)$. This specializes to 0 under s_x and by assumption maps to 0 in $H_\bullet(G(\mathcal{O}_x^h), \mathbb{Z}/p)$. Thus there

exists a curve Y which is étale over X such that $\alpha - \alpha(x)$ maps to 0 in $H_\bullet(G(k[Y]), \mathbb{Z}/p)$. Denote by α_Y the image of α in $H_\bullet(G(k[Y]), \mathbb{Z}/p)$. Then $\alpha_Y = \alpha(x)$, which is a constant class. This implies that the specialization of α_Y is the same at all points of Y and hence there is an affine open neighborhood of $x \in X$ on which the specialization of α is constant. Similarly, we can find an open neighborhood of y on which the specialization of α is constant. Since these two neighborhoods intersect nontrivially, we have $\alpha(x) = \alpha(y)$. □

On the surface, rigidity has no apparent connection to Conjecture 5.2.1. The next result provides the link.

PROPOSITION 5.3.8. *Let k be an algebraically closed field and denote by K the algebraic closure of $k(T)$. Then if rigidity holds, the natural map*

$$i_* : H_\bullet(G(k), \mathbb{Z}/p) \longrightarrow H_\bullet(G(K), \mathbb{Z}/p)$$

is an isomorphism for $p \neq char(k)$.

PROOF. Injectivity does not require rigidity. We have $K = \varinjlim k[C]$, where C ranges over the smooth affine curves over k, and each map

$$H_\bullet(G(k), \mathbb{Z}/p) \longrightarrow H_\bullet(G(k[C]), \mathbb{Z}/p)$$

is split injective. It follows that i_* is injective.

To prove surjectivity, note that each class in $H_\bullet(G(K), \mathbb{Z}/p)$ comes from some $\alpha \in H_\bullet(G(k[C]), \mathbb{Z}/p)$, where C is a smooth affine curve over k. If $x, y :$ spec$(K) \longrightarrow C$ are two points of the k-variety C, then the corresponding specialization maps

$$s_x, s_y : H_\bullet(G(k[C]), \mathbb{Z}/p) \longrightarrow H_\bullet(G(K), \mathbb{Z}/p)$$

coincide (see the discussion following Theorem 5.3.2). It follows that the image of s_x is contained in the image of $H_\bullet(G(k), \mathbb{Z}/p)$. Indeed, choose a rational point spec$(k) \to C$ and let $y :$ spec$(K) \to C$ be the composition

$$\text{spec}(K) \longrightarrow \text{spec}(k) \longrightarrow C.$$

Then the image of s_y is clearly contained in the image of $H_\bullet(G(k), \mathbb{Z}/p)$. The homomorphism

$$H_\bullet(G(k[C]), \mathbb{Z}/p) \longrightarrow H_\bullet(G(K), \mathbb{Z}/p)$$

is induced by a map spec$(K) \to C$ of schemes over k. In view of the above statements, the image of this map is contained in the image of $H_\bullet(G(k), \mathbb{Z}/p)$. It follows that i_* is surjective. □

REMARK 5.3.9. Consider the functors $H_i(E_2(-), \mathbb{Z}/p)$. If we knew these were rigid, then the conclusion of Proposition 5.3.8 would hold for $G = SL_2$.

Indeed, we would have an isomorphism

$$H_\bullet(E_2(k), \mathbb{Z}/p) \longrightarrow H_\bullet(E_2(K), \mathbb{Z}/p),$$

and $E_2 = SL_2$ for fields.

COROLLARY 5.3.10. *If rigidity holds and k has positive characteristic, then Conjecture* 5.2.1 *holds for G.*

PROOF. Note that if k is an algebraically closed field of characteristic $p > 0$, then k is a union of transcendental extensions of $\overline{\mathbb{F}}_p$. It follows that the natural map

$$H_\bullet(G(\overline{\mathbb{F}}_p), \mathbb{Z}/l) \longrightarrow H_\bullet(G(k), \mathbb{Z}/l)$$

is an isomorphism. Consider the commutative diagram

$$\begin{array}{ccc} H^\bullet_{\text{ét}}(BG_k, \mathbb{Z}/l) & \longrightarrow & H^\bullet(BG(k), \mathbb{Z}/l) \\ \uparrow & & \downarrow \\ H^\bullet_{\text{ét}}(BG_{\overline{\mathbb{F}}_p}, \mathbb{Z}/l) & \longrightarrow & H^\bullet(BG(\overline{\mathbb{F}}_p), \mathbb{Z}/l). \end{array}$$

The left vertical arrow is an isomorphism by base change and the bottom arrow is an isomorphism by Theorem 5.2.6. □

REMARK 5.3.11. If we could prove 5.2.1 for a field of characteristic zero (e.g., $\overline{\mathbb{Q}}$), then Corollary 5.3.10 would apply in that case as well.

To summarize, we have the following. If, for each smooth affine curve X over k and closed points $x, y \in X$, the map

$$H_\bullet(G(k), \mathbb{Z}/p) \longrightarrow H_\bullet(G(\mathcal{O}^h_x), \mathbb{Z}/p)$$

is an isomorphism, then the specialization maps

$$s_x, s_y : H_\bullet(G(k[X]), \mathbb{Z}/p) \longrightarrow H_\bullet(G(k), \mathbb{Z}/p)$$

coincide, and hence the canonical map

$$H^\bullet_{\text{ét}}(BG_k, \mathbb{Z}/p) \longrightarrow H^\bullet(BG(k), \mathbb{Z}/p)$$

is an isomorphism.

This allows us to prove more cases of Conjecture 5.2.1.

THEOREM 5.3.12 ([**61**]). *Suppose G is one of the following groups over k: a unipotent group, a torus, a solvable group, or the normalizer of a maximal torus in a connected algebraic group. Then Conjecture* 5.2.1 *holds for G.*

PROOF. Let X be a smooth affine curve over k and let x be a closed point on X. If G is the additive group, then $G(\mathcal{O}^h_x)$ is the abelian group underlying $\mathcal{O}^h_x$; since p is invertible in k, this group is uniquely p-divisible. Thus,

$$H_i(G(k), \mathbb{Z}/p) = H_i(G(\mathcal{O}^h_x), \mathbb{Z}/p) = 0, \quad i > 0.$$

An arbitrary connected unipotent group U has a normal connected codimension one unipotent subgroup U'; the quotient is the additive group. Any map $\mathrm{spec}(\mathcal{O}_x^h) \longrightarrow U/U'$ lifts to U. Arguing by induction and using the short exact sequence

$$1 \longrightarrow U'(\mathcal{O}_x^h) \longrightarrow U(\mathcal{O}_x^h) \longrightarrow \mathbb{G}_a(\mathcal{O}_x^h) \longrightarrow 0,$$

we see that

$$H_i(U(\mathcal{O}_x^h), \mathbb{Z}/p) = 0, \quad i > 0.$$

The same is true for $U(k)$; thus, 5.2.1 holds for any unipotent group.

If G is the multiplicative group, then $G(\mathcal{O}_x^h) = (\mathcal{O}_x^h)^\times$ is p-divisible and contains the group $\mathbb{Z}/p^\infty$ of p-primary roots of unity as its p-torsion subgroup. It follows that the induced map

$$H_\bullet(k^\times, \mathbb{Z}/p) \longrightarrow H_\bullet((\mathcal{O}_x^h)^\times, \mathbb{Z}/p)$$

is an isomorphism. Using the Künneth formula, we obtain the result for any torus.

Now, if G is solvable, we have a split exact sequence

$$1 \longrightarrow U \longrightarrow G \longrightarrow T \longrightarrow 1,$$

where T is a torus and U is unipotent. The associated spectral sequence calculation shows that

$$H_\bullet(G(k), \mathbb{Z}/p) \cong H_\bullet(G(\mathcal{O}_x^h), \mathbb{Z}/p).$$

Finally, let G be the normalizer of a maximal torus T in a connected algebraic group. The quotient $W = G/T$ is a finite discrete group; hence $W(k) = W(\mathcal{O}_x^h)$. We have a commutative diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & T(k) & \longrightarrow & G(k) & \longrightarrow & W(k) & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \| & & \\ 1 & \longrightarrow & T(\mathcal{O}_x^h) & \longrightarrow & G(\mathcal{O}_x^h) & \longrightarrow & W(\mathcal{O}_x^h) & \longrightarrow & 1. \end{array}$$

Comparing the spectral sequences, we see that the map

$$H_\bullet(G(k), \mathbb{Z}/p) \longrightarrow H_\bullet(G(\mathcal{O}_x^h), \mathbb{Z}/p)$$

is an isomorphism. □

There is one more important case which we now mention. Consider the infinite general linear group GL. We have the following result of Gabber [**43**] and Gillet–Thomason [**47**].

THEOREM 5.3.13. *Let X be a smooth variety over a field k (not necessarily algebraically closed) and let $x \in X$ be a closed point. Then if p is prime to* $\mathrm{char}(k)$, *the natural map*

$$K_\bullet(\mathcal{O}_x^h, \mathbb{Z}/p) \longrightarrow K_\bullet(k, \mathbb{Z}/p)$$

is an isomorphism. □

COROLLARY 5.3.14. *Under the same hypotheses as* 5.3.13, *the natural map*

$$H_\bullet(GL(\mathcal{O}_x^h), \mathbb{Z}/p) \longrightarrow H_\bullet(GL(k), \mathbb{Z}/p)$$

is an isomorphism.

PROOF. This is true more generally. Suppose $f : X \longrightarrow Y$ is a map of connected H-spaces and m is an integer such that both the kernel and cokernel of the map $\pi_1(X) \longrightarrow \pi_1(Y)$ are uniquely m-divisible. Then $\pi_\bullet(X, \mathbb{Z}/m) \longrightarrow \pi_\bullet(Y, \mathbb{Z}/m)$ is an isomorphism if and only if $H_\bullet(X, \mathbb{Z}/m) \longrightarrow H_\bullet(Y, \mathbb{Z}/m)$ is an isomorphism ([**127**], Proposition 1.5). In this case, taking $X = BGL(\mathcal{O}_x^h)^+$ and $Y = BGL(k)^+$, Theorem 5.3.13 implies the result. □

REMARK 5.3.15. 1. The analogue of Corollary 5.3.14 holds for the group SL since $BE(R)^+$ is the universal cover of $BGL(R)^+$.

2. The analogue of Corollary 5.3.14 is known for the infinite orthogonal group O and the infinite symplectic group Sp by the work of M. Karoubi [**65**].

COROLLARY 5.3.16. *If k has positive characteristic p, then the natural map*

$$H^\bullet_{\text{ét}}(BGL_k, \mathbb{Z}/l) \longrightarrow H^\bullet(BGL(k), \mathbb{Z}/l)$$

is an isomorphism.

PROOF. By the previous corollary, the map

$$H^\bullet(BGL(k), \mathbb{Z}/l) \longrightarrow H^\bullet(BGL(\overline{\mathbb{F}}_p), \mathbb{Z}/l)$$

is an isomorphism. To extend étale cohomology to direct limits of schemes, we view a scheme as a sheaf and work in the sheaf category (see [**61**]). Then $BGL_k = \varinjlim BGL_{n,k}$ and there is an isomorphism

$$H^\bullet_{\text{ét}}(BGL_k, \mathbb{Z}/l) \longrightarrow \varprojlim H^\bullet_{\text{ét}}(BGL_{n,k}, \mathbb{Z}/l)$$

([**61**], Section 4). Consider the commutative diagram

$$\begin{array}{ccc} H^\bullet_{\text{ét}}(BGL_k, \mathbb{Z}/l) & \longrightarrow & H^\bullet(BGL(k), \mathbb{Z}/l) \\ \uparrow \cong & & \downarrow \cong \\ H^\bullet_{\text{ét}}(BGL_{\overline{\mathbb{F}}_p}, \mathbb{Z}/l) & \longrightarrow & H^\bullet(BGL(\overline{\mathbb{F}}_p), \mathbb{Z}/l). \end{array}$$

The bottom arrow is an isomorphism by Theorem 5.2.6. □

REMARK 5.3.17. Jardine's proof of this result ([**61**], Theorem 4.1) works in any characteristic. It is based upon the theory of simplicial étale sheaves on the category of smooth schemes over k. A thorough discussion of this would take us too far afield, but we can say what is going on in general. Corollary 5.3.14 implies that the homology sheaves $H_\bullet(BGL, \mathbb{Z}/p)$ are constant. Comparing the universal coefficient spectral sequences for discrete cohomology and étale cohomology yields the isomorphism $H^\bullet_{\text{ét}}(BGL_k, \mathbb{Z}/p) \to H^\bullet(BGL(k), \mathbb{Z}/p)$. The

same argument works for any algebraic group G, so the analogue of Corollary 5.3.14 for G implies the Friedlander–Milnor conjecture for G. (See [**61**] for further details.)

The proof of Corollary 5.3.16 would work in characteristic zero if we knew the result for $\mathbb{C}$ (or $\overline{\mathbb{Q}}$). In the next section, we show that there is an isomorphism $H^\bullet(BGL(\mathbb{C})^{\mathrm{top}}, \mathbb{Z}/p) \cong H^\bullet(BGL(\mathbb{C}), \mathbb{Z}/p)$.

5.4. Stable Results

The results in this section are due to A. Suslin [**127**]. Let k be a field and consider the simplicial scheme $BGL_{n,k}$. The ith space $X_{n,i}$ is the i-fold product

$$X_{n,i} = \underbrace{GL_{n,k} \times \cdots \times GL_{n,k}}_{i},$$

and the jth face map $p_j^i : X_{n,i} \longrightarrow X_{n,i-1}$ is defined by

$$p_j^i(g_1, \ldots, g_i) = \begin{cases} (g_2, \ldots, g_i) & j = 0 \\ (g_1, \ldots, g_j g_{j+1}, \ldots, g_i) & 1 \leq j \leq i-1 \\ (g_1, \ldots, g_{i-1}) & j = i. \end{cases}$$

Consider the k-rational point $\mathrm{spec}(k) \to X_{n,i}$ given by $x \mapsto (1, 1, \ldots, 1)$, where $1 \in GL_{n,k}$ is the unit element. Let $X_{n,i}^h$ be the henselization of $X_{n,i}$ at this closed point and denote its coordinate algebra by $\mathcal{O}_{n,i}^h$. The ring $\mathcal{O}_{n,i}^h$ has maximal ideal $\mathfrak{m}_{n,i}^h$.

Since the maps p_j^i preserve the unit section, we have induced morphisms $X_{n,i}^h \longrightarrow X_{n,i-1}^h$. Denote by $(p_j^i)^*$ the maps

$$\mathcal{O}_{n,i-1}^h \longrightarrow \mathcal{O}_{n,i}^h,$$

$$GL(\mathcal{O}_{n,i-1}^h, \mathfrak{m}_{n,i-1}^h) \longrightarrow GL(\mathcal{O}_{n,i}^h, \mathfrak{m}_{n,i}^h),$$

and

$$\tilde{C}_\bullet(GL(\mathcal{O}_{n,i-1}^h, \mathfrak{m}_{n,i-1}^h), \mathbb{Z}/p) \longrightarrow \tilde{C}_\bullet(GL(\mathcal{O}_{n,i}^h, \mathfrak{m}_{n,i}^h), \mathbb{Z}/p),$$

where $\tilde{C}(G, \mathbb{Z}/p)$ denotes the reduced standard complex of G with coefficients in $\mathbb{Z}/p$ (i.e., $\tilde{C}_0(G, \mathbb{Z}/p) = 0$).

We have morphisms of schemes over $\mathrm{spec}(k)$:

$$X_{n,i}^h \longrightarrow X_{n,i} \xrightarrow{\mathrm{pr}_m} GL_n$$

preserving the unit section. These give rise to canonical matrices

$$\alpha_m \in GL_n(\mathcal{O}_{n,i}^h, \mathfrak{m}_{n,i}^h)$$

(i.e., $\mathcal{O}_{n,i}^h$-valued points of GL_n).

Denote the chain $[\alpha_1, \ldots, \alpha_i] \otimes 1 \in \tilde{C}_i(GL(\mathcal{O}_{n,i}^h, \mathfrak{m}_{n,i}^h), \mathbb{Z}/p)$ by $u_{n,i}$ (set $u_{n,0} = 0$).

LEMMA 5.4.1. *There exist chains* $c_{n,i} \in C_{i+1}(GL(\mathcal{O}^h_{n,i}, \mathfrak{m}^h_{n,i}), \mathbb{Z}/p)$ *such that*

$$d(c_{n,i}) = u_{n,i} - \sum_{j=0}^{i}(-1)^j (p^i_j)^*(c_{n,i-1}).$$

PROOF. Let $c_{n,0} = 0$ and assume that $c_{n,0}, \ldots, c_{n,i-1}$ have been constructed. Note that $d(u_{n,i}) = \sum_{j=0}^{i}(-1)^j (p^i_j)^*(u_{n,i-1})$. Then

$$\begin{aligned} d\left(u_{n,i} - \sum_{j=0}^{i}(-1)^j (p^i_j)^*(c_{n,i-1})\right) = & \\ \sum_{j=0}^{i}(-1)^j (p^i_j)^* \left(\sum_{k=0}^{i-1}(-1)^k (p^{i-1}_j)^*(c_{n,i-2})\right) & \\ = \sum_{j=0}^{i}\sum_{k=0}^{i-1}(-1)^{j+k}(p^{i-1}_k p^i_j)^*(c_{n,i-2}) & \\ = 0. & \end{aligned}$$

This shows that $u_{n,i} - \sum_{j=0}^{i}(-1)^j (p^i_j)^*(c_{n,i-1})$ is a cycle and hence a boundary since $\tilde{H}_i(GL(\mathcal{O}^h_{n,i}, \mathfrak{m}^h_{n,i}), \mathbb{Z}/p) = 0$ by Corollary 5.3.14. This gives us the required chain $c_{n,i}$. □

Lemma 5.4.1 leads to a computation of the K-theory of any algebraically closed field. The idea is to prove that if k is an algebraically closed field of positive characteristic l, and if E is the algebraic closure of the quotient field of the ring of Witt vectors of k, then there is an isomorphism $K_*(k, \mathbb{Z}/p) \cong K_*(E, \mathbb{Z}/p)$. Thus the groups $K_*(F, \mathbb{Z}/m)$ for $(m, \mathrm{char}(F)) = 1$ do not even depend on the characteristic of the algebraically closed field F. Thus, the computation of Corollary 5.3.4 is valid for any F.

Lemma 5.4.1 also plays a role in the computation of the $\mathbb{Z}/p$ homology of $BGL(k)^{\mathrm{top}}$ for $k = \mathbb{R}, \mathbb{C}$. We have the map

$$H_*(BGL(k), \mathbb{Z}/p) \longrightarrow H_*(BGL(k)^{\mathrm{top}}, \mathbb{Z}/p)$$

obtained by taking the limit of the maps

$$H_*(BGL_n(k), \mathbb{Z}/p) \longrightarrow H_*(BGL_n(k)^{\mathrm{top}}, \mathbb{Z}/p).$$

The space $BGL(\mathbb{C})^{\mathrm{top}}$ has the homotopy type of BU and $BGL(\mathbb{R})^{\mathrm{top}}$ has that of BO (see Appendix B). The mod p (co)homology of these spaces is well understood.

THEOREM 5.4.2. *Let* $k = \mathbb{R}$ *or* $\mathbb{C}$. *The canonical map*

$$H_*(BGL(k), \mathbb{Z}/p) \longrightarrow H_*(BGL(k)^{\mathrm{top}}, \mathbb{Z}/p)$$

is an isomorphism.

We first prove a few lemmas. Let G be a discrete group. The space BG is the geometric realization of a certain simplicial set (see Appendix B). If G is a Lie group with finitely many components, fix a left invariant metric on G and denote by G_ε the ε-ball with center at $1 \in G$. Let BG_ε be the geometric realization of the simplicial set whose p-simplices are p-tuples $[g_1, \dots, g_p]$ of elements of G such that

$$G_\varepsilon \cap g_1 G_\varepsilon \cap \cdots \cap g_1 \cdots g_p G_\varepsilon \neq \emptyset$$

with the usual face and degeneracy maps.

LEMMA 5.4.3. *If $\varepsilon > 0$ is small enough, then the sequence $BG_\varepsilon \longrightarrow BG^\delta \longrightarrow BG^{\mathrm{top}}$ is a fibration up to homotopy (i.e., BG_ε is homotopy equivalent to the space $B\overline{G}$ of Section 5.1).*

PROOF. Let $G^{\mathrm{top}} \longrightarrow EG^{\mathrm{top}} \longrightarrow BG^{\mathrm{top}}$ be the universal principal G^{top}-bundle. For a space X, denote by $\mathrm{Sin}(X)$ its singular simplicial set (the homology of the associated simplicial abelian group is the singular homology of X). We obtain a commutative diagram of fibrations

$$\begin{array}{ccccc} |\mathrm{Sin}(G^{\mathrm{top}})| & \longrightarrow & |\mathrm{Sin}(EG^{\mathrm{top}})| & \longrightarrow & |\mathrm{Sin}(BG^{\mathrm{top}})| \\ \downarrow & & \downarrow & & \downarrow \\ G^{\mathrm{top}} & \longrightarrow & EG^{\mathrm{top}} & \longrightarrow & BG^{\mathrm{top}} \end{array}$$

where the vertical arrows are homotopy equivalences. The discrete group G acts on $|\mathrm{Sin}(EG^{\mathrm{top}})|$ freely and hence we obtain a new fibration

$$|\mathrm{Sin}(G^{\mathrm{top}})|/G \longrightarrow |\mathrm{Sin}(EG^{\mathrm{top}})|/G \longrightarrow |\mathrm{Sin}(BG^{\mathrm{top}})|.$$

Since $|\mathrm{Sin}(EG^{\mathrm{top}})|$ is contractible, the space $|\mathrm{Sin}(EG^{\mathrm{top}})|/G$ is homotopy equivalent to BG. Thus, we have a fibration up to homotopy

$$|\mathrm{Sin}(G^{\mathrm{top}})|/G \longrightarrow BG \longrightarrow BG^{\mathrm{top}}.$$

Suppose that ε is small. Then G_ε is geodesically convex and hence every nonempty intersection $g_0 G_\varepsilon \cap \cdots \cap g_p G_\varepsilon$ is contractible. Denote by $X_\bullet$ the simplicial space with

$$X_p = \coprod_{g_0, \dots, g_p} g_0 G_\varepsilon \cap \cdots \cap g_p G_\varepsilon.$$

Let $Y_{\bullet\bullet}$ be the bisimplicial set with $Y_{pq} = \mathrm{Sin}_p(X_q)$ and denote by $(\mathrm{Sin}(G^{\mathrm{top}}))_\varepsilon$ the subobject of $\mathrm{Sin}(G^{\mathrm{top}})$ consisting of singular simplices lying in some gG_ε. Let E_ε be the simplicial set with p-simplices the $(p+1)$-tuples $\langle g_0, \dots, g_p \rangle$ with

$g_0 G_\varepsilon \cap \cdots \cap g_p G_\varepsilon \neq \emptyset$. Consider the evident maps of bisimplicial sets

$$\begin{array}{ccc} Y_{\bullet\bullet} & \xrightarrow{\psi} & E_\varepsilon \\ \varphi \downarrow & & \\ (\mathrm{Sin}(G^{\mathrm{top}}))_\varepsilon & & \end{array}$$

where we view $(\mathrm{Sin}(G^{\mathrm{top}}))_\varepsilon$ (resp. E_ε) as a bisimplicial set trivial in the q (resp. p) direction. One checks easily that φ and ψ are homotopy equivalences. Moreover, the embedding $(\mathrm{Sin}(G^{\mathrm{top}}))_\varepsilon \longrightarrow \mathrm{Sin}(G^{\mathrm{top}})$ is also a homotopy equivalence. Thus, we have homotopy equivalences

$$|\mathrm{Sin}(G^{\mathrm{top}})| \xleftarrow{\simeq} |Y_{\bullet\bullet}| \xrightarrow{\simeq} |E_\varepsilon|.$$

The discrete group G acts freely on all of these and the above maps are G-equivariant. Thus, we get a homotopy equivalence

$$BG_\varepsilon = |E_\varepsilon|/G \simeq |\mathrm{Sin}(G^{\mathrm{top}})|/G$$

and it is clear that the diagram

$$\begin{array}{ccc} BG_\varepsilon & \xrightarrow{\simeq} & |\mathrm{Sin}(G^{\mathrm{top}})|/G \\ & \searrow & \downarrow \\ & & BG \end{array}$$

commutes up to homotopy. □

LEMMA 5.4.4. *Let $k = \mathbb{R}$ or $\mathbb{C}$. If ε is small enough, then the map $BGL_n(k)_\varepsilon \longrightarrow BGL(k)$ induces the zero map on $\tilde{H}_\bullet(-, \mathbb{Z}/p)$.*

PROOF. We use the construction of Lemma 5.4.1. Denote by $\mathcal{O}_{n,i}^{\mathrm{cont}}$ the ring of germs of continuous functions $GL_n(k)^i \longrightarrow k$ defined in a neighborhood of the identity. The group $GL_r(\mathcal{O}_{n,i}^{\mathrm{cont}})$ may be identified with the group of germs of continuous maps $GL_n(k)^i \longrightarrow GL_r(k)$ defined in a neighborhood of 1. Thus, every chain $c \in C_q(GL_r(\mathcal{O}_{n,i}^{\mathrm{cont}}), \mathbb{Z}/p)$ defines a continuous map from some neighborhood of $1 \in GL_n(k)^i$ to $C_q(GL_r(k), \mathbb{Z}/p)$. Now, the ring $\mathcal{O}_{n,i}^{\mathrm{cont}}$ is henselian, so we obtain a unique map $\mathcal{O}_{n,i}^h \longrightarrow \mathcal{O}_{n,i}^{\mathrm{cont}}$. Denote by $c_{n,i}^{\mathrm{cont}} \in \tilde{C}_{i+1}(GL(\mathcal{O}_{n,i}^{\mathrm{cont}}), \mathbb{Z}/p)$ the image of the chain $c_{n,i}$ constructed in 5.4.1. Let $N > 0$ and find $\varepsilon > 0$ so that the chains $c_{n,i}^{\mathrm{cont}}$ are defined on $GL_n(k)_\varepsilon \times \cdots \times GL_n(k)_\varepsilon$ for $0 \leq i \leq N$. Then we get maps

$$s_i : C_i(BGL_n(k)_\varepsilon, \mathbb{Z}/p) \longrightarrow C_{i+1}(GL(k), \mathbb{Z}/p) = C_{i+1}(BGL(k), \mathbb{Z}/p)$$

for $0 \leq i \leq N$. It is clear from 5.4.1 that s_i is a null-homotopy (for $i \leq N$) for the embedding

$$\tilde{C}_\bullet(BGL_n(k)_\varepsilon, \mathbb{Z}/p) \longrightarrow \tilde{C}_\bullet(BGL(k), \mathbb{Z}/p) = \tilde{C}_\bullet(GL(k), \mathbb{Z}/p).$$

The result follows. □

COROLLARY 5.4.5. *If $\varepsilon > 0$ is small, then the embedding*

$$BSL_n(k)_\varepsilon \longrightarrow BSL(k)$$

induces the zero map on $\tilde{H}_\bullet(-, \mathbb{Z}/p)$.

PROOF. This follows since for any field k, the map

$$H_\bullet(SL(k)) \longrightarrow H_\bullet(GL(k))$$

is split injective. □

COROLLARY 5.4.6. *If $\varepsilon > 0$ is small, then*

$$\tilde{H}_i(BSL_n(k)_\varepsilon, \mathbb{Z}/p) = 0$$

for $0 \leq i \leq (n-1)/2$.

PROOF. Consider the spectral sequence associated to the fibration

$$BSL_n(k)_\varepsilon \longrightarrow BSL_n(k) \longrightarrow BSL_n(k)^{\mathrm{top}}.$$

By Theorem 5.1.5, the map

$$H_\bullet(BSL_n(k), \mathbb{Z}/p) \longrightarrow H_\bullet(BSL_n(k)^{\mathrm{top}}, \mathbb{Z}/p)$$

is (split) surjective. Let i_0 be the least positive integer for which the group $H_{i_0}(BSL_n(k)_\varepsilon, \mathbb{Z}/p) \neq 0$. Then the spectral sequence shows that we have a short exact sequence (with $\mathbb{Z}/p$-coefficients)

$$0 \longrightarrow H_{i_0}(BSL_n(k)_\varepsilon) \longrightarrow H_{i_0}(BSL_n(k)) \longrightarrow H_{i_0}(BSL_n(k)^{\mathrm{top}}) \longrightarrow 0.$$

Now, the homology $H_i(SL_n)$ stabilizes at $i = (n-1)/2$ (Theorem 2.1.7). So if $i \leq (n-1)/2$, we see that

$$H_i(BSL_n(k)_\varepsilon, \mathbb{Z}/p) \longrightarrow H_i(BSL_n(k), \mathbb{Z}/p)$$

is the zero map. Thus, $i_0 > (n-1)/2$. □

PROOF OF THEOREM 5.4.2. By Lemma 5.4.6, we see that the map

$$BSL(k) \longrightarrow BSL(k)^{\mathrm{top}}$$

induces an isomorphism on homology with $\mathbb{Z}/p$-coefficients. Since both spaces are simply connected, the map extends to $BSL(k)^+ \longrightarrow BSL(k)^{\mathrm{top}}$ and this

map is an isomorphism on homotopy with $\mathbb{Z}/p$–coefficients. Consider the commutative diagram of fibrations

$$\begin{array}{ccccc} BSL(k)^+ & \longrightarrow & BGL(k)^+ & \longrightarrow & Bk^\times \\ \downarrow & & \downarrow & & \downarrow \\ BSL(k)^{\mathrm{top}} & \longrightarrow & BGL(k)^{\mathrm{top}} & \longrightarrow & B(k^\times)^{\mathrm{top}}. \end{array}$$

The outside arrows induce isomorphisms on homology and homotopy with finite coefficients. Hence, so does the middle arrow. □

COROLLARY 5.4.7. *The map*

$$H_i(BGL_n(k), \mathbb{Z}/p) \longrightarrow H_i(BGL_n(k)^{\mathrm{top}}, \mathbb{Z}/p)$$

is an isomorphism for $i \leq n$.

PROOF. This follows from Theorem 2.3.1. □

Aside from this last low-dimensional result, we can say a few more things.

5.5. H_1, H_2, and H_3

Recall from Section 5.1 that if G is a Lie group, then Conjecture 5.1.1 holds for G if and only if the space $B\overline{G}$ has the mod p homology of a point. This is true if and only if $H_i(B\overline{G}, \mathbb{Z})$ is uniquely divisible for all $i > 0$.

PROPOSITION 5.5.1. *Let G be a connected semi-simple Lie group. Then $H_1(B\overline{G}, \mathbb{Z}) = 0$ and there is a short exact sequence*

$$0 \longrightarrow H_2(B\overline{G}, \mathbb{Z}) \longrightarrow H_2(BG^\delta, \mathbb{Z}) \longrightarrow H_2(BG, \mathbb{Z}) \longrightarrow 0.$$

PROOF. For the computation of $H_1(B\overline{G})$, we may assume that G is simply connected. This implies that $H_2(BG, \mathbb{Z}) = 0$. Since G is perfect, $H_1(BG^\delta, \mathbb{Z}) = 0$. The spectral sequence associated to the fibration

$$B\overline{G} \longrightarrow BG^\delta \longrightarrow BG$$

shows that $H_1(B\overline{G}, \mathbb{Z}) = 0$.

For any connected Lie group, results of Borel [13] imply that the group $H_3(BG, \mathbb{Z})$ is finite (since the rational cohomology of BG is a polynomial algebra on even dimensional generators). By Corollary 5.1.6, we see that $H_3(BG^\delta, \mathbb{Z})$ surjects onto $H_3(BG, \mathbb{Z})$. If in addition G is semi-simple, then $H_1(B\overline{G}, \mathbb{Z}) = 0$ and the spectral sequence gives us the required exact sequence. □

THEOREM 5.5.2. *If G is a Chevalley group over the real or complex numbers, then Conjecture* 5.1.1 *holds for H_2.*

PROOF. By the work of Sah and Wagoner [**108**], the group $H_2(B\overline{G},\mathbb{Z})$ is isomorphic to $K_2(\mathbb{C})$ in the complex case and to $K_2(\mathbb{C})^-$ in the real case. Since $K_2(\mathbb{C})$ is uniquely divisible, we see that $H_2(B\overline{G},\mathbb{Z}/p)=0$. □

The results of Section 3.2 can be used to prove Conjecture 5.1.1 for $H_3(SL_2(\mathbb{C}),\mathbb{Z}/p)$ and $H_3(GL_2(\mathbb{C}),\mathbb{Z}/p)$ (the former implies the latter). Let F be an algebraically closed field of characteristic zero. We have an exact sequence

$$0 \longrightarrow \mathbb{Q}/\mathbb{Z} \longrightarrow H_3(SL_2(F),\mathbb{Z}) \longrightarrow \mathfrak{p}(F)$$

(here, $\mathbb{Q}/\mathbb{Z}$ is the group of roots of unity in F). Since $\mathfrak{p}(F)$ is uniquely divisible, $H_3(SL_2(F),\mathbb{Z})$ is the sum of a uniquely divisible group and $\mathbb{Q}/\mathbb{Z}$. If $F=\mathbb{C}$, then $H_3(SL_2(\mathbb{C})^\delta,\mathbb{Z}/p)=0$. Since this group surjects onto the group $H_3(SL_2(\mathbb{C})^{\mathrm{top}},\mathbb{Z}/p)$, this implies the result. With a bit more work, one can prove the same result for $H_3(SL_2(\mathbb{R}),\mathbb{Z}/p)$ [**107**].

In positive characteristic, we can prove Conjecture 5.2.1 for $H^i(GL_n)$ for $i\le 3$. As we saw in Section 5.3, it suffices to show that the natural map

$$H_i(GL_n(k),\mathbb{Z}/p) \longrightarrow H_i(GL_n(\mathcal{O}_x^h),\mathbb{Z}/p)$$

is an isomorphism when X is a smooth affine curve over k. Theorems 2.3.1 and 5.3.13 show that this holds for $i\le n$. In Section 3.3, we showed that if F is an infinite field and if A is a hensel local F-algebra, then the natural map of Bloch groups

$$B(F)\otimes\mathbb{Z}/p \longrightarrow B(A)\otimes\mathbb{Z}/p$$

is an isomorphism for $p\neq \mathrm{char}(F)$. Moreover, there is an exact sequence

$$H_3(GM_2(R)) \longrightarrow H_3(GL_2(R)) \longrightarrow B(R) \longrightarrow 0$$

for any local F-algebra R.

PROPOSITION 5.5.3. *Let $\mathcal{O}_x^h$ be the henselization of the local ring $\mathcal{O}_x$, where x is a closed point on the curve X. Then the natural map*

$$H_3(GL_2(k))\otimes\mathbb{Z}/p \longrightarrow H_3(GL_2(\mathcal{O}_x^h))\otimes\mathbb{Z}/p$$

is an isomorphism.

PROOF. Consider the commutative diagram

$$\begin{array}{ccccc}
H_3(GM_2(k))\otimes\mathbb{Z}/p & \longrightarrow & H_3(GL_2(k))\otimes\mathbb{Z}/p & \longrightarrow & B(k)\otimes\mathbb{Z}/p \\
\downarrow & & \downarrow & & \downarrow \\
H_3(GM_2(\mathcal{O}_x^h))\otimes\mathbb{Z}/p & \longrightarrow & H_3(GL_2(\mathcal{O}_x^h))\otimes\mathbb{Z}/p & \longrightarrow & B(\mathcal{O}_x^h)\otimes\mathbb{Z}/p.
\end{array}$$

The left vertical map is an isomorphism since we have an isomorphism

$$H_\bullet(GM_2(k), \mathbb{Z}/p) \longrightarrow H_\bullet(GM_2(\mathcal{O}_x^h), \mathbb{Z}/p)$$

(see the proof of Theorem 5.3.12). The result now follows from the Five Lemma. □

COROLLARY 5.5.4. *The natural map*

$$H_3(GL_2(k), \mathbb{Z}/p) \longrightarrow H_3(GL_2(\mathcal{O}_x^h), \mathbb{Z}/p))$$

is an isomorphism.

PROOF. Consider the commutative diagram of universal coefficient sequences

$$\begin{array}{ccccc} H_3(GL_2(k)) \otimes \mathbb{Z}/p & \longrightarrow & H_3(GL_2(k), \mathbb{Z}/p) & \longrightarrow & {}_pH_2(GL_2(k)) \\ \downarrow & & \downarrow & & \downarrow \\ H_3(GL_2(\mathcal{O}_x^h)) \otimes \mathbb{Z}/p & \longrightarrow & H_3(GL_2(\mathcal{O}_x^h), \mathbb{Z}/p) & \longrightarrow & {}_pH_2(GL_2(\mathcal{O}_x^h)). \end{array}$$

The last map is an isomorphism by stability and the corresponding statement for $H_2(GL)$. The result now follows from the Five Lemma. □

COROLLARY 5.5.5. *If k has positive characteristic l, then the map*

$$H^3_{\text{ét}}(BGL_{n,k}, \mathbb{Z}/p) \longrightarrow H^3(BGL_n(k), \mathbb{Z}/p)$$

is an isomorphism.

PROOF. Stability implies the result for $n \geq 3$. The case $n = 2$ now follows from the previous corollary. □

Exercises

1. Let G be a locally contractible topological group, G_0 its identity component, and U its universal covering group. Prove that the homomorphisms $U \to G_0 \to G$ induce isomorphisms $\overline{U} \to \overline{G}_0 \to \overline{G}$.
2. Prove the second assertion of Lemma 5.1.2: if H is locally isomorphic to G, then if Conjecture 5.1.1 holds for G, it also holds for H.
3. Prove that if N is the normalizer of a maximal torus T in a Lie group G, then the identity component of N is T.
4. Prove that if R is the strict henselization of the local ring $\mathbb{Z}_{(p)}$, then there is an isomorphism

$$H^\bullet(BN(\overline{\mathbb{F}}_p) \cap N(\overline{\mathbb{F}}_p)^g, \mathbb{Z}/l) \cong H^\bullet(BN(R) \cap N(R)^g, \mathbb{Z}/l).$$

5. Complete the proof of Corollary 5.2.13.
6. Show that $\mathcal{U}(\mathfrak{p})$ is an ultrafilter and that every ultrafilter on I corresponds to some prime ideal of $R = \prod_{i \in I} F_i$.

7. Let k be an algebraically closed field and let p be a prime not equal to the characteristic of k. Let X be a smooth affine curve over k and let x be closed point on X. Let $\mathcal{O}_x^h$ be the henselization of the local ring $\mathcal{O}_x$. Prove that the homomorphism
$$H_\bullet(k^\times, \mathbb{Z}/p) \longrightarrow H_\bullet((\mathcal{O}_x^h)^\times, \mathbb{Z}/p)$$
is an isomorphism.
8. Prove that Corollary 5.3.14 holds for the infinite special linear group SL.
9. Prove that $\tilde{H}_i(GL(\mathcal{O}_{n,i}^h, \mathfrak{m}_{n,i}^h), \mathbb{Z}/p) = 0$ (you may assume that the action of $GL(k)$ on this homology is trivial).
10. Prove that the map φ in Lemma 5.4.3 is a homotopy equivalence as follows. It suffices to show that for each p, the map $Y_{p,\bullet} \to (\mathrm{Sin}(G^{\mathrm{top}}))_{\varepsilon,p}$ is a homotopy equivalence. Use the definitions of these spaces and note that the fiber of this map over a point t is the simplicial set whose p-simplices are $(p+1)$-tuples of elements of $G_t = \{g \in G : t \in \mathrm{Sin}_p(gG_\varepsilon)\}$. This set is contractible.
11. Prove that the map ψ in Lemma 5.4.3 is a homotopy equivalence by showing that the fiber of $Y_{\bullet,q} \to (E_\varepsilon)_q$ over $\langle g_0, \ldots, g_q \rangle$ is $\mathrm{Sin}(g_0 G_\varepsilon \cap \cdots \cap g_q G_\varepsilon)$.
12. Show that the map s_i defined in the proof of Lemma 5.4.4 is a null-homotopy.
13. Prove that for any field k, the map
$$H_\bullet(SL(k), \mathbb{Z}) \longrightarrow H_\bullet(GL(k), \mathbb{Z})$$
is split injective.

Appendix A

Homology of Discrete Groups

In this appendix we summarize basic facts about group (co)homology. We assume a basic familiarity with the homology of topological spaces and homological algebra. Good references for most of this material are the books of K. Brown [**21**] and L. Evens [**37**].

A.1. Basic Concepts

A.1.1. The definition. Let G be a group and choose a presentation of G:

$$1 \longrightarrow R \longrightarrow F \longrightarrow G \longrightarrow 1$$

where R and F are free groups. Construct a CW-complex BG as follows. Take a point x and for each generator of G (*i.e.*, for each element of F) attach a 1-cell to x. One now has a bouquet of circles, $X^{(1)}$, with $\pi_1(X^{(1)}) = F$. Each element of R is a word in the generators of F and hence corresponds to a path γ in $X^{(1)}$. For each such element, attach a 2-cell e^2 via a map $f : \partial e^2 \to \gamma$. This yields a space $X^{(2)}$ with $\pi_1(X^{(2)}) = F/R = G$ (each path corresponding to an element of R is now nullhomotopic). Now, attach a 3-cell for each generator of $\pi_2(X^{(2)})$ to obtain a space $X^{(3)}$ with $\pi_1(X^{(3)}) = G$ and $\pi_2(X^{(3)}) = 0$. Continue this process, adding i-cells to obtain a space $X^{(i)}$ at each stage with $\pi_1(X^{(i)}) = G$ and $\pi_j(X^{(i)}) = 0$ for $1 < j < i$. Now define

$$BG = \bigcup_i X^{(i)}.$$

Clearly, $\pi_1(BG) = G$ and $\pi_j(BG) = 0$ for $j > 1$. This construction is covariantly functorial in G and one checks easily that BG is the unique space, up to homotopy equivalence, satisfying $\pi_1 = G$ and $\pi_j = 0$ for $j > 1$.

EXAMPLE A.1.1. If $G = \mathbb{Z}$, then a presentation for G is

$$0 \longrightarrow 0 \longrightarrow \mathbb{Z} \xrightarrow{\text{id}} \mathbb{Z} \longrightarrow 0.$$

Thus, to build BG, we take a point and attach a 1-cell. This gives $X^{(1)} = S^1$. Since $\pi_i(S^1) = 0$ for $i > 1$, the process stops and hence $BG = S^1$.

EXAMPLE A.1.2. Consider $G = \mathbb{Z}/2$. A presentation is

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\times 2} \mathbb{Z} \longrightarrow \mathbb{Z}/2 \longrightarrow 0.$$

Thus, $X^{(1)} = S^1$ and we attach a 2-cell e^2 to S^1 via the map $\partial e^2 = S^1 \xrightarrow{\times 2} S^1$. The resulting space $X^{(2)}$ is the real projective plane $\mathbb{RP}^2$. Now, $\pi_2(\mathbb{RP}^2) = \pi_2(S^2) = \mathbb{Z}$ and the generator $f : S^2 \to \mathbb{RP}^2$ is the double cover. Hence,

$$X^{(3)} = \mathbb{RP}^2 \cup_f e^3 = \mathbb{RP}^3.$$

Continuing this process, we see that at each stage we attach an i-cell via the double covering map so that $X^{(i)} = \mathbb{RP}^i$. Hence, $B\mathbb{Z}/2 = \mathbb{RP}^\infty$.

DEFINITION A.1.3. The homology of the group G with coefficients in the trivial module A is

$$H_\bullet(G, A) = H_\bullet(BG, A).$$

The cohomology is defined similarly as

$$H^\bullet(G, A) = H^\bullet(BG, A).$$

The above computations of the homotopy types of $B\mathbb{Z}$ and $B\mathbb{Z}/2$ give the following homology groups:

$$H_i(\mathbb{Z}, \mathbb{Z}) = H_i(S^1, \mathbb{Z}) = \begin{cases} \mathbb{Z} & i = 0, 1 \\ 0 & i > 1 \end{cases}$$

$$H_i(\mathbb{Z}/2, \mathbb{Z}) = H_i(\mathbb{RP}^\infty, \mathbb{Z}) = \begin{cases} \mathbb{Z} & i = 0 \\ \mathbb{Z}/2 & i \text{ odd} \\ 0 & i \text{ even.} \end{cases}$$

Unfortunately, it is usually impossible to obtain such simple models for BG. Moreover, we would like to be able to compute homology with coefficients in nontrivial G-modules M. We carry this out via the following device.

Let $\mathbb{Z}G$ be the group ring of G over $\mathbb{Z}$ and let M be a (left) $\mathbb{Z}G$-module. A *projective resolution* of M over $\mathbb{Z}G$ is an exact sequence of $\mathbb{Z}G$-modules

$$\cdots \to P_i \to P_{i-1} \to \cdots \to P_1 \to P_0 \to M \to 0$$

where each P_j is a projective G-module. Such resolutions exist for any M.

Now, let G be a group and choose a resolution

$$P_\bullet \longrightarrow \mathbb{Z}$$

of the trivial module $\mathbb{Z}$. If M is a G-module, we define the homology groups of G with coefficients in M to be

$$H_i(G, M) = H_i(P_\bullet \otimes_{\mathbb{Z}G} M).$$

(In the sequel, we shall abbreviate $\otimes_{\mathbb{Z}G}$ by $\otimes_G$.) The cohomology groups are defined similarly:

$$H^\bullet(G, M) = H^\bullet(\mathrm{Hom}_{\mathbb{Z}G}(P_\bullet, M)).$$

That this is well-defined is a consequence of the following.

PROPOSITION A.1.4. *Let $P_\bullet \to \mathbb{Z}$ and $Q_\bullet \to \mathbb{Z}$ be two projective resolutions over $\mathbb{Z}G$. Then there is a $\mathbb{Z}G$-linear chain map $f_\bullet : P_\bullet \to Q_\bullet$ such that $f_\bullet$ is a homotopy equivalence of chain complexes which is unique up to a unique chain homotopy equivalence.* □

To see that our new definition agrees with the previous definition using BG, consider the (contractible!) universal cover $\tilde{X} \to BG$. The group G acts on $\tilde{X}$ as the group of deck transformations and hence the cellular chain complex $C_\bullet(\tilde{X})$ is a chain complex of G-modules. Moreover, since $\tilde{X}$ is contractible, the augmented complex $C_\bullet(\tilde{X}) \to \mathbb{Z}$ is a (free) resolution of $\mathbb{Z}$ over $\mathbb{Z}G$. Each $C_i(\tilde{X})$ is a free G-module with one basis element for each G-orbit of i-cells.

This new definition allows us to use ad hoc resolutions to compute homology. For example, let $G = \langle t : t^n = 1 \rangle$ be the cyclic group of order n with generator t (written multiplicatively). Denote by Δ the endomorphism of $\mathbb{Z}G$ given by multiplication by $1 + t + \cdots + t^{n-1}$. Consider the sequence

$$\cdots \longrightarrow \mathbb{Z}G \xrightarrow{t-1} \mathbb{Z}G \xrightarrow{\Delta} \mathbb{Z}G \xrightarrow{t-1} \mathbb{Z}G \xrightarrow{\varepsilon} \mathbb{Z} \longrightarrow 0.$$

This sequence is exact since $\Delta(t-1) = t^n - 1 = 0$. This gives the following result.

PROPOSITION A.1.5. *The integral homology of the cyclic group $\mathbb{Z}/n$ is*

$$H_i(\mathbb{Z}/n, \mathbb{Z}) = \begin{cases} \mathbb{Z} & i = 0 \\ \mathbb{Z}/n & i \text{ odd} \\ 0 & i \text{ even.} \end{cases}$$

PROOF. Apply $- \otimes_G \mathbb{Z}$ to the above resolution:

$$\cdots \xrightarrow{\Delta} \mathbb{Z} \xrightarrow{t-1} \mathbb{Z} \xrightarrow{\Delta} \mathbb{Z} \xrightarrow{t-1} \mathbb{Z}.$$

Since t acts trivially on $\mathbb{Z}$, this complex has the form

$$\cdots \xrightarrow{n} \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{n} \mathbb{Z} \xrightarrow{0} \mathbb{Z}.$$

The homology of this complex is easily computed. □

The standard resolution. This resolution is obtained from the "simplex" spanned by G; *i.e.*, we build a space X with vertices the elements of G and simplices the finite subsets of G. This space is clearly contractible. The corresponding free resolution $F_\bullet = C_\bullet(X)$ is explicitly given as follows. The module F_n is the free abelian group with basis all $(n+1)$-tuples $(g_0, g_1, \ldots, g_n)$. The

G-action is given by $g(g_0, \ldots, g_n) = (gg_0, \ldots, gg_n)$ and the boundary map ∂ is defined as

$$\partial(g_0, \ldots, g_n) = \sum_{i=0}^{n} (-1)^i (g_0, \ldots, \hat{g}_i, \ldots, g_n).$$

A basis for the free $\mathbb{Z}G$-module F_n consists of those $(n+1)$-tuples whose first element is 1. Write such a tuple as $(1, g_1, g_1g_2, \ldots, g_1g_2 \cdots g_n)$ and introduce the *bar notation*

$$[g_1|g_2|\cdots|g_n] = (1, g_1, g_1g_2, \ldots, g_1g_2 \cdots g_n).$$

(If $n = 0$, there is only one such element, denoted [ ].) In terms of this basis, the map ∂ is given as $\partial = \sum_{i=0}^{n} (-1)^i d_i$, where

$$d_i[g_1|\cdots|g_n] = \begin{cases} g_1[g_2|\cdots|g_n] & i = 0 \\ [g_1|\cdots|g_{i-1}|g_ig_{i+1}|g_{i+2}|\cdots|g_n] & 0 < i < n \\ [g_1|\cdots|g_{n-1}] & i = n. \end{cases}$$

In low dimensions, the bar resolution has the form

$$F_2 \xrightarrow{\partial_2} F_1 \xrightarrow{\partial_1} \mathbb{Z}G \xrightarrow{\varepsilon} \mathbb{Z} \longrightarrow 0,$$

where $\varepsilon(1) = 1$, $\partial_1([g]) = g[\ ] - [\ ] = g - 1$, and $\partial_2([g|h]) = g[h] - [gh] + [g]$.

PROPOSITION A.1.6. *Let M be a G-module. Then $H_0(G, M)$ is the module M_G of coinvariants; that is,*

$$M_G = M/\langle gm - m : g \in G, m \in M \rangle.$$

PROOF. Apply $- \otimes_G M$ to the standard resolution:

$$\xrightarrow{\partial_2} F_1 \otimes_G M \xrightarrow{\partial_1} \mathbb{Z}G \otimes_G M.$$

We have $H_0(G, M) = \operatorname{coker}(\partial_1)$. The formula for ∂_1 is given by

$$\partial_1([g] \otimes m) = (g - 1) \otimes m$$

and under the isomorphism $\mathbb{Z}G \otimes_G M \to M$, we have $(g-1) \otimes m \mapsto gm - m$. □

REMARK A.1.7. Let M and N be G-modules. Then there is a canonical isomorphism $M \otimes_G N \cong (M \otimes N)_G$, where G acts diagonally on $M \otimes N$. This gives us a second definition of $H_\bullet(G, M)$. If $F_\bullet$ is a projective resolution of $\mathbb{Z}$, then

$$H_\bullet(G, M) = H_\bullet((F_\bullet \otimes M)_G).$$

PROPOSITION A.1.8. *For any group G, $H_1(G, \mathbb{Z}) = G_{ab}$, the abelianization of G.*

PROOF. Applying $- \otimes_G \mathbb{Z}$ to the standard resolution, we obtain

$$\bigoplus_{[g|h]} \mathbb{Z} \xrightarrow{\partial_2} \bigoplus_{[g]} \mathbb{Z} \xrightarrow{\partial_1} \mathbb{Z}.$$

The map ∂_1 maps the element $[g] \otimes n$ to $gn - n$. But since $\mathbb{Z}$ is a trivial G-module, $gn - n = n - n = 0$. Thus, $H_1(G, \mathbb{Z}) = \operatorname{coker}(\partial_2)$ and the map ∂_2 is

$$\begin{aligned} \partial_2([g|h]) &= g[h] - [gh] + [g] \\ &= [h] - [gh] + [g]. \end{aligned}$$

Therefore,

$$H_1(G, \mathbb{Z}) = \bigoplus_{[g]} \mathbb{Z} / \langle [g] + [h] = [gh] \rangle,$$

which is G_{ab}, as claimed. □

REMARK A.1.9. This also follows from the fact that the first homology group of the space BG is the abelianization of $\pi_1(BG) = G$.

The definition of $H_\bullet(G)$ is clearly functorial.

Relative Homology. Let G be a group, M a G-module, and $H \subseteq G$ a subgroup. We define the relative homology groups $H_\bullet(G, H; M)$ as follows. Consider the sequence of complexes

$$0 \longrightarrow C_\bullet(H) \otimes_H M \longrightarrow C_\bullet(G) \otimes_G M \longrightarrow C_\bullet(G, H; M) \longrightarrow 0,$$

where $C_\bullet(G, H; M) = C_\bullet(G) \otimes_G M / C_\bullet(H) \otimes_H M$. We claim that this is exact; that is, we claim that the map

$$C_\bullet(H) \otimes_H M \longrightarrow C_\bullet(G) \otimes_G M$$

is injective. To see this, note that this map is none other than the inclusion

$$C_\bullet(BH; \tilde{M}) \longrightarrow C_\bullet(BG; \tilde{M}),$$

where $\tilde{M}$ is the coefficient system on BG given by M. A less topological proof is obtained by noting that the map in question is the composition

$$C_\bullet(H) \otimes_H M \longrightarrow C_\bullet(G) \otimes_H M \longrightarrow C_\bullet(G) \otimes_G M.$$

It therefore suffices to treat the special case

$$\mathbb{Z}H \otimes_H M \longrightarrow \mathbb{Z}G \otimes_H M \longrightarrow \mathbb{Z}G \otimes_G M$$

where the composition is clearly an isomorphism.

DEFINITION A.1.10. The relative homology groups $H_\bullet(G, H; M)$ are defined as

$$H_\bullet(G, H; M) = H_\bullet(C_\bullet(G, H; M)).$$

Note that there is a long exact sequence

$$\cdots \to H_i(H;M) \to H_i(G;M) \to H_i(G,H;M) \to H_{i-1}(H;M) \to \cdots .$$

Center kills. Let $g \in G$ and consider the map of pairs $c(g) : (G,M) \to (G,M)$ given by

$$c(g) : (h \mapsto ghg^{-1}, m \mapsto gm).$$

This induces a map $c(g)_* : H_\bullet(G,M) \to H_\bullet(G,M)$.

PROPOSITION A.1.11. *The map $c(g)_*$ is the identity.*

PROOF. Consider the map $h_i : C_i(G,M) \to C_{i+1}(G,M)$ given by

$$h_i([g_1|\cdots|g_i] \otimes m) = \sum_{j=0}^{i} (-1)^j [g_1|\cdots|g_j|g^{-1}|g^{-1}g_{j+1}g|\cdots|g^{-1}g_ig] \otimes m.$$

Then $h_\bullet$ is a homotopy from $c(g)_*$ to the identity. □

COROLLARY A.1.12 ("center kills"). *Let z be a central element of G and let M be a G-module. Then the endomorphism of M defined by $m \mapsto zm$ induces the identity map on $H_\bullet(G,M)$.*

PROOF. Conjugation by z is the identity on G. □

EXAMPLE A.1.13. Center kills is very useful in computing homology. Consider the group $GL_n(\mathbb{Z})$ and let $\mathbb{Z}^n$ have its usual $GL_n(\mathbb{Z})$ action. Then $-I$ acts as multiplication by -1 on $\mathbb{Z}^n$ and hence acts the same way on $H_\bullet(GL_n(\mathbb{Z}),\mathbb{Z}^n)$. But since $-I$ is central, the induced map on homology is the identity. Thus, $\mathrm{id}_* = -\mathrm{id}_*$; that is, $H_\bullet(GL_n(\mathbb{Z}),\mathbb{Z}^n)$ is all 2-torsion.

A.1.2. Induced Modules. Let G be a group with subgroup H. The ring $\mathbb{Z}G$ is easily seen to be a free $\mathbb{Z}H$-module with basis a set of coset representatives of $H\backslash G$. Let M be an H-module. We define the induced module as

$$\mathrm{Ind}_H^G M = \mathbb{Z}G \otimes_{\mathbb{Z}H} M.$$

This is a G-module via $g : g' \otimes m \mapsto gg' \otimes m$. Clearly, we have a decomposition

$$\mathrm{Ind}_H^G M = \bigoplus_{g \in G/H} gM.$$

This characterizes modules of the form $\mathrm{Ind}_H^G M$. Let N be a G-module whose underlying abelian group is a sum $\bigoplus_{i \in I} M_i$. Suppose the G-action transitively permutes the summands.

PROPOSITION A.1.14. *Let N be a G-module as above. Let M be one of the summands M_i and let $H \subset G$ be the stabilizer of i in G. Then M is an H-module and $N \cong \mathrm{Ind}_H^G M$.* □

COROLLARY A.1.15. *Let $N = \bigoplus_{i\in I} M_i$ be a G-module. Assume that the G-action permutes the summands according to some action of G on I. Let G_i be the stabilizer of i and let E be a set of representatives for I modulo G. Then M_i is a G_i-module and $N \cong \bigoplus_{i\in E} \mathrm{Ind}_{G_i}^G M_i$.* □

EXAMPLE A.1.16. Let X be a complex on which G acts. The G-module $C_n(X)$ is a direct sum of copies of $\mathbb{Z}$, one for each n-cell of X, and G permutes the summands. Let Σ_n be a set of representatives for the G-orbits of n-cells, and let G_σ be the stabilizer of σ. Then

$$C_n(X) = \bigoplus_{\sigma\in\Sigma_n} \mathrm{Ind}_{G_\sigma}^G \mathbb{Z}_\sigma,$$

where $\mathbb{Z}_\sigma$ is the G_σ-module associated to σ; that is, if $g \in G_\sigma$, g acts on $\mathbb{Z}_\sigma$ as $+1$ if g preserves the orientation of σ and by -1 if g reverses orientation.

PROPOSITION A.1.17 (Shapiro's Lemma). *If $H \subseteq G$ and M is an H-module, then $H_\bullet(H, M) \cong H_\bullet(G, \mathrm{Ind}_H^G M)$.*

PROOF. Let $F_\bullet \to \mathbb{Z}$ be a projective resolution over $\mathbb{Z}G$. Since $\mathbb{Z}G$ is a free $\mathbb{Z}H$-module, $F_\bullet$ is also a resolution over $\mathbb{Z}H$. Thus

$$H_\bullet(H, M) = H_\bullet(F_\bullet \otimes_H M).$$

But $F_\bullet \otimes_H M \cong F_\bullet \otimes_G (\mathbb{Z}G \otimes_H M) \cong F_\bullet \otimes_G (\mathrm{Ind}_H^G M)$. □

A.1.3. Transfer. Let G be a group and let H be a subgroup of finite index. The inclusion $i : H \to G$ induces a map $i_* : H_\bullet(H, M) \to H_\bullet(G, M)$. Since $(G : H) < \infty$, we can construct a map

$$\mathrm{tr}_H^G : H_\bullet(G, M) \longrightarrow H_\bullet(H, M)$$

as follows. Denote by X the space BG and by $\tilde{X}$ the covering of X corresponding to the subgroup H. If σ is a cell of X, then there is a cell $\tilde{\sigma}$ of $\tilde{X}$ lying above σ for each coset representative of G/H. Define a map

$$C_n(X, \tilde{M}) \longrightarrow C_n(\tilde{X}, \tilde{M}),$$

where $\tilde{M}$ is the coefficient system on X associated to M, by

$$\sigma \otimes m \mapsto \sum_{G/H} \tilde{\sigma} \otimes m.$$

The induced map on homology is tr_H^G. Since the composition

$$C_n(X, \tilde{M}) \longrightarrow C_n(\tilde{X}, \tilde{M}) \longrightarrow C_n(X, \tilde{M})$$

is the map

$$\sigma \otimes m \mapsto \sum_{G/H} \tilde{\sigma} \otimes m \mapsto \sum_{G/H} \sigma \otimes m,$$

we see that $i_* \circ \mathrm{tr}_H^G = (G : H)\mathrm{id}$.

PROPOSITION A.1.18. *If G is finite, then $H_n(G,M)$ is annihilated by $|G|$ for all $n>0$. If $|G|$ is invertible in M, then $H_n(G,M)=0$ for all $n>0$.*

PROOF. Let $H=\{1\}\subset G$. Then $i_*\circ \mathrm{tr}_H^G = |G|\mathrm{id}$. But $H_n(H,M)=0$ for $n>0$. This proves the first assertion. The second follows from the first since if $|G|$ is invertible in M, then it is also invertible in $H_\bullet(G,M)$. □

A.1.4. Abelian groups. We state the following propositions about the homology of abelian groups.

THEOREM A.1.19. *Assume that k is a principal ideal domain.*

1. *There is a map $\psi:\bigwedge^\bullet(G\otimes k)\to H_\bullet(G,k)$ which is injective for every abelian group G and split injective if G is finitely generated.*
2. *Suppose that every prime p such that G has p-torsion is invertible in k. Then ψ is an isomorphism.*
3. *If k has characteristic zero, then ψ is an isomorphism in dimension 2.*

PROOF. This is all obvious if G is cyclic: $\bigwedge^i(G\otimes k)=0$ for $i>1$ and $H_i(G,k)=0$ for $i>1$ if $|G|$ is invertible in k. The case of G finitely generated now follows easily via the Künneth formula and induction since G is a finite direct product of cyclic groups. The general case follows from the fact that $G=\varinjlim G_\alpha$, where G_α ranges over the finitely generated subgroups of G. □

THEOREM A.1.20. *Let G be an abelian group. Then there is a natural isomorphism*

$$\bigwedge^\bullet(G\otimes\mathbb{Z}/p)\otimes\Gamma({}_pG)\longrightarrow H_\bullet(G,\mathbb{Z}/p)$$

where Γ is a divided power algebra and ${}_pG$ denotes the p-torsion subgroup of G.

PROOF. See [21], p. 126. □

A.2. Spectral Sequences

A.2.1. Basic definitions. Spectral sequences are a generalization of the long exact homology sequence

$$\cdots\to H_i(C'_\bullet)\to H_i(C_\bullet)\to H_i(C_\bullet/C'_\bullet)\to H_{i-1}(C'_\bullet)\to\cdots$$

associated to a short exact sequence

$$0\longrightarrow C'_\bullet\longrightarrow C_\bullet\longrightarrow C_\bullet/C'_\bullet\longrightarrow 0$$

of chain complexes. Suppose we are given a complex $C_\bullet$ and an increasing sequence of subcomplexes $\{F_pC_\bullet\}_{p\in\mathbb{Z}}$. Assume the filtration is dimensionwise finite; that is, $\{F_pC_n\}_{p\in\mathbb{Z}}$ is a filtration of finite length for each n. There is an induced filtration on the homology $H_\bullet(C_\bullet)$ given by

$$F_pH_\bullet(C_\bullet)=\mathrm{im}(H_\bullet(F_pC_\bullet)\longrightarrow H_\bullet(C_\bullet)).$$

We have the associated graded module

$$\mathrm{gr}H_\bullet(C_\bullet) = \bigoplus_p F_pH_\bullet(C_\bullet)/F_{p-1}H_\bullet(C_\bullet).$$

The spectral sequence associated to the filtered complex $C_\bullet$ is a sequence $\{E^r\}_{r\geq 0}$ of "successive approximations" to $\mathrm{gr}H_\bullet(C_\bullet)$ with E^1 consisting of the groups $H_\bullet(F_pC_\bullet/F_{p-1}C_\bullet)$. More precisely, E^r is a bigraded module equipped with *differentials*

$$d^r_{p,q} : E^r_{p,q} \longrightarrow E^r_{p-r,q+r-1}$$

such that E^{r+1} is the homology of E^r:

$$E^{r+1}_{p,q} = \ker(d^r_{p,q})/\mathrm{im}(d^r_{p+r,q-r+1}).$$

Note that if $E^r_{p,q} = 0$ for some r, then for $r' \geq r$, $E^{r'}_{p,q} = 0$. Since the filtration is assumed dimensionwise finite, the module $E^r_{p,q}$, for fixed p, q, stabilizes at some point $r = r(p,q)$. We define $E^\infty_{p,q}$ to be this stable module: $E^\infty_{p,q} = E^{r(p,q)}_{p,q}$. We say that the spectral sequence *converges* to $H_\bullet(C_\bullet)$.

This is best illustrated with an example. Suppose we have a first quadrant double complex of modules:

$$\begin{array}{ccccccc}
\downarrow & & \downarrow & & \downarrow & & \\
C_{0,2} & \longleftarrow & C_{1,2} & \longleftarrow & C_{2,2} & \longleftarrow & \\
\downarrow & & \downarrow & & \downarrow & & \\
C_{0,1} & \longleftarrow & C_{1,1} & \longleftarrow & C_{2,1} & \longleftarrow & \\
{\scriptstyle d^v}\downarrow & & \downarrow & & \downarrow & & \\
C_{0,0} & \underset{d^h}{\longleftarrow} & C_{1,0} & \longleftarrow & C_{2,0} & \longleftarrow &
\end{array}$$

We have $d^h : C_{p,q} \to C_{p-1,q}$ and $d^v : C_{p,q} \to C_{p,q-1}$, $d^hd^v + d^vd^h = d^hd^h = d^vd^v = 0$. This is the E^0-term of a spectral sequence with $d^0 = d^v$. Thus $E^1_{p,q} = H_q(C_{p,\bullet}, d^v)$. Since $d^hd^v + d^vd^h = 0$, the horizontal map d^h induces a map

$$d^1 = (d^h)_* : E^1_{p,q} \longrightarrow E^1_{p-1,q}.$$

Taking homology again, we obtain

$$E^2_{p,q} = H_p(E^1_{\bullet,q}, d^1) = H^h_pH^v_q(C_{\bullet,\bullet});$$

that is, the E^2-term is obtained by first taking the vertical homology of the double complex and then taking the horizontal homology of the resulting complex.

The map d^2 is easily described (see [77], Appendix D, for a good discussion).

Now, since $C_{p,q} = 0$ for $p < 0$ or $q < 0$, we have $E^r_{p,q} = 0$ for $p < 0$ or $q < 0$ for all r. As a result, for fixed p, q, there exists $r = r(p,q)$ such that the differentials d^r starting and ending at $E^r_{p,q}$ are zero. Thus, $E^r_{p,q} = E^{r+1}_{p,q} = \cdots = E^\infty_{p,q}$ for $r \geq r(p,q)$.

Consider the total complex $\mathrm{Tot}C_{\bullet,\bullet}$ defined by

$$\mathrm{Tot}_n C_{\bullet,\bullet} = \bigoplus_{p+q=n} C_{p,q}.$$

This complex has a differential induced by d^h and d^v. There is a canonical filtration on $\mathrm{Tot}C_{\bullet,\bullet}$:

$$F_i \mathrm{Tot}_n C_{\bullet,\bullet} = \bigoplus_{p \leq i} C_{p,n-p}.$$

In other words, we take only those modules in columns 0 through i. This induces a filtration on $H_\bullet(\mathrm{Tot}C_{\bullet,\bullet})$:

$$0 \subset F_0 \subset F_1 \subset \cdots \subset F_n = H_n(\mathrm{Tot}C_{\bullet,\bullet}).$$

THEOREM A.2.1. *For all $p, q \geq 0$,*

$$E^\infty_{p,q} = F_p H_{p+q}(\mathrm{Tot}C_{\bullet,\bullet}) / F_{p-1} H_{p+q}(\mathrm{Tot}C_{\bullet,\bullet}).$$

In this case we see that the spectral sequence converges to the homology of $\mathrm{Tot}C_{\bullet,\bullet}$ and write

$$E^1_{p,q} \Longrightarrow H_{p+q}(\mathrm{Tot}C_{\bullet,\bullet}).$$

Note that we could have filtered the complex by rows:

$$F_i \mathrm{Tot}_n C_{\bullet,\bullet} = \bigoplus_{q \leq i} C_{n-q,q}.$$

The associated spectral sequence is then obtained by first taking horizontal homology and then taking vertical homology:

$$E^2_{p,q} = H^v_p H^h_q(C_{\bullet,\bullet}).$$

This sequence also converges to $H_\bullet(\mathrm{Tot}C_{\bullet,\bullet})$, but gives a different filtration and hence a different E^∞-term.

Example: Chain complexes of coefficients. Let G be a group and M a G-module. The homology $H_\bullet(G, M)$ is defined as $H_\bullet(F_\bullet \otimes_G M)$, where $F_\bullet$ is a projective resolution of $\mathbb{Z}$. If $C_\bullet$ is a nonnegative chain complex of G-modules, we set

$$H_\bullet(G, C_\bullet) = H_\bullet(F_\bullet \otimes_G C_\bullet).$$

If $C_\bullet$ consists of a single module M in dimension zero, then $H_\bullet(G, C_\bullet) = H_\bullet(G, M)$.

Now, $F_\bullet \otimes_G C_\bullet$ is the total complex of the double complex $(F_p \otimes_G C_q)$. Thus, we have two spectral sequences converging to $H_\bullet(G, C_\bullet)$. The first sequence has

$$E^1_{p,q} = H_q(F_p \otimes_G C_\bullet) = F_p \otimes_G H_q(C_\bullet)$$

since $F_p \otimes_G -$ is an exact functor (F_p is projective). Computing the E^2-term, we have

$$E^2_{p,q} = H_p(G, H_q(C_\bullet)). \tag{A.1}$$

PROPOSITION A.2.2. *If $C_\bullet \simeq C'_\bullet$, then $H_\bullet(G, C_\bullet) = H_\bullet(G, C'_\bullet)$.*

PROOF. Since $C_\bullet \simeq C'_\bullet$, $H_\bullet(C_\bullet) = H_\bullet(C'_\bullet)$. It follows that

$$E^2_{p,q}(C_\bullet) = H_p(G, H_q(C_\bullet)) = H_p(G, H_q(C'_\bullet)) = E^2_{p,q}(C'_\bullet),$$

and hence the E^∞-terms are isomorphic. □

The second spectral sequence has

$$E^1_{p,q} = H_q(F_\bullet \otimes_G C_p) = H_q(G, C_p). \tag{A.2}$$

The E^2-term is then the pth homology group of the complex $H_q(G, C_\bullet)$.

Suppose for example that each C_p is a free $\mathbb{Z}G$-module, or more generally an $H_\bullet$-acyclic G-module (that is, $H_q(G, C_p) = 0$ for $q > 0$; for example C_p projective or induced). Then $E^1_{p,q} = 0$ for $q > 0$ and $E^1_{p,0} = (C_p)_G$. Thus, in this case the second spectral sequence collapses to give an isomorphism

$$H_\bullet(G, C_\bullet) = H_\bullet((C_\bullet)_G).$$

Now, using the first spectral sequence A.1, we see that

$$E^2_{p,q} = H_p(G, H_q(C_\bullet)) \Longrightarrow H_{p+q}((C_\bullet)_G). \tag{A.3}$$

This is a typical argument using spectral sequences. One uses one spectral sequence to identify a computable E^1- or E^2-term, and the other spectral sequence to identify the abutment.

A.2.2. Two important examples. Let

$$1 \longrightarrow H \longrightarrow G \longrightarrow Q \longrightarrow 1$$

be a group extension. Then we have the following result, due to Hochschild and Serre.

THEOREM A.2.3. *For any G-module M, there is a spectral sequence of the form*

$$E^2_{p,q} = H_p(Q, H_q(H, M)) \Longrightarrow H_{p+q}(G, M).$$

PROOF. Let $F_\bullet \to \mathbb{Z}$ be a projective resolution over $\mathbb{Z}G$. Then $F_\bullet \otimes_G M$ can be computed by first factoring out the H-action, then factoring out the Q-action:

$$F_\bullet \otimes_G M = ((F_\bullet \otimes M)_H)_Q = (F_\bullet \otimes_H M)_Q.$$

Writing $C_\bullet = F_\bullet \otimes_H M$, we have

$$H_\bullet(G, M) = H_\bullet((C_\bullet)_Q).$$

Moreover, we have an isomorphism of Q-modules

$$H_\bullet(H, M) = H_\bullet(C_\bullet).$$

Now, we must show that the Q-modules $C_p = F_p \otimes_H M$ are $H_\bullet$- acyclic. Since we can take $F_\bullet$ to be the standard resolution, it suffices to show that $\mathbb{Z}G \otimes_H M$ is acyclic. But this latter module is an induced module $\mathbb{Z}Q \otimes A$ (see [21], p. 69). Thus, using spectral sequence A.3, we have

$$E^2_{p,q} = H_p(Q, H_q(C_\bullet)) \Longrightarrow H_{p+q}((C_\bullet)_Q).$$

But the E^2-terms are isomorphic to $H_p(Q, H_q(H, M))$. □

EXAMPLE A.2.4. Let G be the group of 3×3 upper triangular matrices over $\mathbb{Z}$ with 1's on the diagonal (the Heisenberg group). Then we have an extension

$$0 \longrightarrow \mathbb{Z} \longrightarrow G \longrightarrow \mathbb{Z} \oplus \mathbb{Z} \longrightarrow 0$$

where $G \to \mathbb{Z} \oplus \mathbb{Z}$ is the map

$$\begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} \mapsto (a, c).$$

The kernel of this map is central; thus, $H_q(\mathbb{Z})$ is a trivial $\mathbb{Z} \oplus \mathbb{Z}$-module. The Hochschild–Serre spectral sequence takes the following form:

$$E^2_{p,q} = H_p(\mathbb{Z} \oplus \mathbb{Z}, H_q(\mathbb{Z})) \Longrightarrow H_{p+q}(G).$$

Since $H_q(\mathbb{Z}) = 0$ for $q > 1$, the spectral sequence is concentrated on the lines $q = 0$ and $q = 1$:

$$E^2 = \begin{array}{|cccc} 0 & 0 & 0 & 0 \\ \mathbb{Z} & \mathbb{Z} \oplus \mathbb{Z} & \mathbb{Z} & 0 \\ \mathbb{Z} & \mathbb{Z} \oplus \mathbb{Z} & \mathbb{Z} & 0 \end{array}$$

The only nontrivial differential is $d^2 : E^2_{2,0} \to E^2_{0,1}$. We claim that this map is an isomorphism. Note that $H_2(\mathbb{Z} \oplus \mathbb{Z}) = \bigwedge^2(\mathbb{Z} \oplus \mathbb{Z}) = \mathbb{Z}$ generated by $(1,0) \wedge (0,1)$. We claim that to compute d^2, we lift $(1,0)$ and $(0,1)$ to G and

compute the commutator of the two elements. (The interested reader can check this.) The obvious lifts are

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

and their commutator is

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

so that $d^2 : (1,0) \wedge (0,1) \mapsto 1 \in H_1(\mathbb{Z})$. Thus,

$$H_i(G) = \begin{cases} \mathbb{Z} & i = 0, 3 \\ \mathbb{Z} \oplus \mathbb{Z} & i = 1, 2 \\ 0 & i > 3. \end{cases}$$

REMARK A.2.5. One could also compute the homology of G by noting that BG is a circle bundle over the torus with Euler class 1. The Gysin sequence (which is really just a special case of a spectral sequence) then gives the same result.

Equivariant homology. Suppose $C_\bullet(X)$ is the cellular chain complex of a G-complex X. The homology groups $H_\bullet(G, C_\bullet(X))$ are denoted $H_\bullet^G(X)$ and called the equivariant homology groups of (G, X). We can perform this construction with any G-module M:

$$H_\bullet^G(X, M) = H_\bullet(G, C_\bullet(X) \otimes M)$$

where G acts diagonally on $C_\bullet(X) \otimes M$.

Note that $H_\bullet^G(\mathrm{pt}, M) = H_\bullet(G, M)$. Since any G-complex X admits a map to a point, there is a canonical map

$$H_\bullet^G(X, M) \longrightarrow H_\bullet(G, M).$$

Consider the two spectral sequences associated to $H_\bullet(G, C_\bullet(X) \otimes M)$. The first spectral sequence satisfies

$$E^2_{p,q} = H_p(G, H_q(X, M)) \Longrightarrow H^G_{p+q}(X, M).$$

PROPOSITION A.2.6. *If X is acyclic, then the canonical map*

$$H_\bullet^G(X, M) \longrightarrow H_\bullet(G, M)$$

is an isomorphism.

PROOF. In this case, $H_q(X, M) = 0$ for all $q > 0$ so that the spectral sequence is concentrated on the line $q = 0$. Thus

$$E^{\infty}_{p,0} = E^2_{p,0} = H_p(G, M) = H^G_p(X, M).$$

□

The second spectral sequence provides an important computational tool. Let Σ_p be a set of representatives for the G-action on $C_p(X)$. We have, for each σ, the orientation module $\mathbb{Z}_\sigma$ mentioned above. Let $M_\sigma = \mathbb{Z}_\sigma \otimes M$. Then we have an isomorphism of G-modules

$$C_p(X, M) = \bigoplus_{\sigma \in \Sigma_p} \mathrm{Ind}^G_{G_\sigma} M_\sigma.$$

By Shapiro's Lemma, we have

$$H_q(G, C_p(X, M)) \cong \bigoplus_{\sigma \in \Sigma_p} H_q(G_\sigma, M_\sigma).$$

The second spectral sequence A.2 then takes the form

$$E^1_{p,q} = \bigoplus_{\sigma \in \Sigma_p} H_q(G_\sigma, M_\sigma) \Longrightarrow H^G_{p+q}(X, M),$$

and, if X is acyclic, then

$$E^1_{p,q} = \bigoplus_{\sigma \in \Sigma_p} H_q(G_\sigma, M_\sigma) \Longrightarrow H_{p+q}(G, M). \tag{A.4}$$

The d^1 map is easily seen to be induced by the boundary map in X; that is,

$$d^1|_{H_q(G_\sigma, M_\sigma)} : H_q(G_\sigma, M_\sigma) \mapsto \bigoplus_{\tau \subset \sigma} H_q(G_\tau, M_\tau)$$

is the direct sum of the maps induced by the inclusions $G_\sigma \to G_\tau$, where τ ranges over the faces of σ.

In more compact terms, we have

$$E^1_{\bullet,q} = C_\bullet(X/G, \mathcal{H}_q)$$

where $\mathcal{H}_q$ is the coefficient system $\sigma \mapsto H_q(G_\sigma, M_\sigma)$. The E^2-term is then

$$E^2_{p,q} = H_p(X/G, \mathcal{H}_q).$$

This point of view is used frequently in Chapter 4, where the following result is used repeatedly (see [**113**], Lemma 6, or [**67**], Lemma 3.3).

PROPOSITION A.2.7. *Suppose $F^{(0)} \subset F^{(1)} \subset \cdots \subset F^{(k)} = X$ is a filtration of the simplicial complex X such that each $F^{(i)}$ and each component of $F^{(i)} - F^{(i-1)}$ is contractible. Let $\mathcal{M}$ be a coefficient system on X such that the restriction of $\mathcal{M}$ to each component of $F^{(i)} - F^{(i-1)}$ is constant. Then the inclusion $F^{(0)} \to X$ induces an isomorphism*

$$H_\bullet(F^{(0)}, \mathcal{M}) \longrightarrow H_\bullet(X, \mathcal{M}).$$

PROOF. The filtration of X yields a filtration of $C_\bullet(X, \mathcal{M})$. This gives a spectral sequence converging to $H_\bullet(X, \mathcal{M})$ with E^1-term having ith column

$$H_\bullet(F^{(i)}, F^{(i-1)}; \mathcal{M}).$$

Consider the relative chain complex $C_\bullet(F^{(i)}, F^{(i-1)}; \mathcal{M})$. By hypothesis, this chain complex is a direct sum of chain complexes with constant coefficients. Since each $F^{(i)}$ is contractible, it follows that

$$H_\bullet(F^{(i)}, F^{(i-1)}; \mathcal{M}) = 0, \quad i \geq 1.$$

Thus, only the 0th column $H_\bullet(F^{(0)}, \mathcal{M})$ is (potentially) nonzero. □

The Solomon–Tits Theorem. In Chapter 3, we used the following fact. Let k be a field and let S be the partially ordered set of proper subspaces of k^n, ordered by inclusion. Let T be the geometric realization of S.

THEOREM A.2.8. *T is homotopy equivalent to a wedge of $(n-2)$-spheres.*

PROOF. We proceed by induction on n, beginning at $n = 2$. The only proper subspaces of k^2 are lines. It follows that T is simply a collection of points; that is, it is a wedge of 0-spheres.

Now assume that $n \geq 3$. Let ℓ be a fixed line in k^n. Let Y denote the set of hyperplanes in k^n such that $H + \ell = k^n$. Denote by S_0 the complement $S - Y$ and let $T_0 \subset T$ be the geometric realization of S_0. We claim that T_0 is contractible. Define a poset map $f : S_0 \to S_0$ by $A \mapsto A + \ell$. This map is well-defined since $A + \ell$ is a proper subspace for $A \in S_0$. Note that $f(A) \geq A$ for all $A \in S_0$.

LEMMA A.2.9. *Let P_1 and P_2 be posets with $P_1 \subseteq P_2$. Let $f : P_1 \to P_2$ be a map with $f(s) \geq s$ for all $s \in P_1$. Then $|P_1| \simeq |f(P_1)|$.*

PROOF. We need a homotopy $F : |P_1| \times [0,1] \to |P_2|$ with $F_0 = |i|$ and $F_1 = |f|$ (here, i is the inclusion of P_1 in P_2). Triangulate $|P_1| \times [0,1]$ in the usual way. The condition that $f(s) \geq s$ implies that $|i| \times \{0\} \coprod |f| \times \{1\}$ extends to all of $|P_1| \times [0,1]$. □

Now, the image of the map f defined above is the set of subspaces of k^n which contain the line ℓ. The geometric realization of this is clearly homotopy

equivalent to the vertex $\ell \in T_0$ (it is a minimal element of S_0). By the lemma, we have $T_0 \simeq f(T_0) \simeq \{\ell\}$; that is, T_0 is contractible.

Now, if H is an element of Y, its link is, by definition, the set of subspaces of k^n properly contained in H. The geometric realization of this link is isomorphic to the space T in dimension $n-1$ (H is an $(n-1)$-dimensional k-vector space). By the induction hypothesis, this link has the homotopy type of a wedge of $(n-3)$-spheres.

To complete the proof, note that T is obtained from T_0 by attaching the links of all the hyperplanes in k^n. Since T_0 is contractible, we see that T is homotopy equivalent to a wedge of $(n-2)$ spheres (i.e., when we contract T_0, we are getting the wedge of the suspensions of the links of the hyperplanes). □

Appendix B

Classifying Spaces and K-theory

This appendix gives a brief introduction to the general theory of classifying spaces and Quillen's definition of higher algebraic K-theory. Good references for this material are the books of Husemoller [**59**] and Srinivas [**118**].

B.1. Classifying Spaces

Let F be a functor on some category of topological spaces (such as the category of finite CW-complexes). A classifying space for F is a space B together with an equivalence of functors

$$F \simeq [-, B];$$

that is, the set $F(X)$ is the same as the set of homotopy classes of maps $X \to B$. In this section we find classifying spaces for principal G-bundles and for vector bundles of a given rank. This is a consequence of the more general notion of classifying space for small categories.

B.1.1. Principal Bundles. Let G be a group. A *principal G-bundle* consists of a locally trivial fibration

$$p : E \longrightarrow B$$

with fiber G and a right G-action

$$E \times G \longrightarrow E.$$

Two such bundles are *equivalent* if there is a homeomorphism $f : E_1 \to E_2$ such that the diagram

$$\begin{array}{ccc} E_1 & \xrightarrow{f} & E_2 \\ {\scriptstyle p_1}\downarrow & & \downarrow{\scriptstyle p_2} \\ B & \xrightarrow{\mathrm{id}} & B \end{array}$$

commutes.

DEFINITION B.1.1. A classifying space for G is a space BG with a principal G-bundle

$$p : EG \longrightarrow BG,$$

where EG is contractible, which is universal in the following sense: if $q : E \to B$ is any principal G- bundle, then there is a continuous map $B \to BG$, unique up to homotopy, such that E is the fiber product

$$\begin{array}{ccc} E & \longrightarrow & EG \\ \downarrow & & \downarrow \\ B & \longrightarrow & BG. \end{array}$$

EXAMPLE B.1.2. Let $U(n)$ be the group of unitary $n \times n$ complex matrices. Denote by $S(n)$ the Stiefel manifold of unitary n-frames in $\mathbb{C}^\infty$, and by $G(n)$ the Grassmann manifold of n-planes in $\mathbb{C}^\infty$. The space $S(n)$ is contractible [**119**] and there is a fibration

$$\begin{array}{ccc} U(n) & \longrightarrow & S(n) \\ & & \downarrow \pi \\ & & G(n) \end{array}$$

where π sends a frame to the plane it spans. A proof that this $U(n)$-bundle is universal may be found in [**59**], p. 83. Thus, we have $BU(n) = G(n)$. Moreover, since the inclusion $U(n) \to GL_n(\mathbb{C})$ is a homotopy equivalence, we also have $BGL_n(\mathbb{C}) = G(n)$. Similarly, if $O(n)$ denotes the group of orthogonal $n \times n$ real matrices, then $BO(n)$ is the Grassmannian of n-planes in $\mathbb{R}^\infty$.

REMARK B.1.3. Interestingly enough, the same space $G(n)$ also classifies rank n complex vector bundles in the sense that if $E \xrightarrow{p} X$ is such a bundle, there is a bundle map (unique up to homotopy)

$$\begin{array}{ccc} E & \longrightarrow & \gamma^n \\ \downarrow & & \downarrow \\ X & \longrightarrow & G(n) \end{array}$$

where γ^n is the *canonical bundle* whose fiber over $V \in G(n)$ is V (that is, $\gamma^n \subset G(n) \times \mathbb{C}^\infty$ is the set $\{(V, v) : v \in V\}$). Similarly, rank n real vector bundles are classified by the real Grassmann manifold $BO(n)$ (see [**83**], p. 61).

B.1.2. The Classifying Space of a Small Category. Let $\mathcal{C}$ be a small category (that is, the class of objects of $\mathcal{C}$ forms a set). Define a simplicial set $\mathcal{NC}$, the *nerve* of $\mathcal{C}$ as follows. An n-simplex is a diagram in $\mathcal{C}$ of the form

$$A_0 \xrightarrow{f_1} A_1 \xrightarrow{f_2} \cdots \xrightarrow{f_{n-1}} A_{n-1} \xrightarrow{f_n} A_n.$$

The ith face map applied to this simplex is

$$A_0 \xrightarrow{f_1} A_1 \xrightarrow{f_2} \cdots \xrightarrow{f_{i-1}} A_{i-1} \xrightarrow{f_{i+1} \circ f_i} A_{i+1} \xrightarrow{f_{i+2}} \cdots \xrightarrow{f_n} A_n,$$

and the ith degeneracy is

$$A_0 \xrightarrow{f_1} A_1 \xrightarrow{f_2} \cdots \xrightarrow{f_i} A_i \xrightarrow{\mathrm{id}} A_i \xrightarrow{f_{i+1}} \cdots \xrightarrow{f_n} A_n.$$

DEFINITION B.1.4. The classifying space $B\mathcal{C}$ of $\mathcal{C}$ is the geometric realization of the simplicial set $\mathcal{N}\mathcal{C}$ (see [**118**], Chapter 3).

Let G be a topological group (perhaps with the discrete topology). Denote by $\underline{G}$ the category with a single object e with morphism set equal to G. Composition of morphisms is given by the group operation. The n-simplices of $\mathcal{N}\underline{G}$ are n-tuples $(g_1, \ldots, g_n)$ of elements of G with ith face map

$$(g_1, \ldots, g_n) \mapsto (g_1, \ldots, g_i g_{i+1}, \ldots, g_n).$$

Consider the classifying space $B\underline{G}$.

THEOREM B.1.5 ([**109**]). *If G is an absolute neighborhood retract (*e.g.*, G discrete or G a Lie group), then $B\underline{G}$ is a classifying space for G on the category of paracompact spaces.* □

THEOREM B.1.6. *Let G be a discrete group. Then the space $B\underline{G}$ is an Eilenberg–MacLane space $K(G,1)$, and hence is homotopy equivalent to the space BG defined in Appendix A.*

PROOF. Let $\underline{\underline{G}}$ be the category with object set G and morphism set $G \times G$; that is, if $g_1, g_2 \in G$, then $\mathrm{Hom}_{\underline{\underline{G}}}(g_1, g_2) = (g_1, g_2)$. There is a functor $\underline{\underline{G}} \to \underline{G}$ defined by

$$g \mapsto e; \qquad (g_1, g_2) \mapsto g_2 g_1^{-1} \in \mathrm{Hom}_{\underline{G}}(e, e).$$

The group G acts on $\underline{\underline{G}}$ by $g.h = hg^{-1}$, $g.(g_1, g_2) = (g_1 g^{-1}, g_2 g^{-1})$. This action is free and hence G acts freely on $B\underline{\underline{G}}$. The induced map $B\underline{\underline{G}} \to B\underline{G}$ is G-equivariant for the trivial action on $B\underline{G}$ and so $B\underline{\underline{G}}$ is a covering space of $B\underline{G}$ with fiber G. Since $\underline{\underline{G}}$ has an initial object (any object is initial), the space $B\underline{\underline{G}}$ is contractible. Thus, $\pi_1(B\underline{G}) = G$ and all the higher homotopy groups vanish. Moreover, since $B\underline{\underline{G}} \to B\underline{G}$ is a principal G-bundle, there is a map $BG \to B\underline{G}$ which is an isomorphism on homotopy and hence is a homotopy equivalence. □

REMARK B.1.7. The standard resolution of $\mathbb{Z}$ over $\mathbb{Z}G$ (see Appendix A) is the chain complex associated to the simplicial abelian group obtained by applying the free abelian group functor dimensionwise to the simplicial set $\mathcal{N}\underline{\underline{G}}$. The "simplex" X mentioned in Appendix A is the space $B\underline{\underline{G}}$.

Now, if G is a topological group, denote by G^δ the group G viewed as a discrete group. The identity map $G^\delta \to G$ is continuous (it is *not* continuous in the other direction) and hence induces a map $BG^\delta \to BG$ of classifying spaces. Moreover, any homomorphism $G \to H$ induces a map $BG \to BH$ (this was not obvious for the construction of BG given in Appendix A).

B.2. K-theory

B.2.1. Topological K-theory. Topological K-theory is a generalized cohomology theory on the category of CW-complexes. Let F denote $\mathbb{R}$ or $\mathbb{C}$ and denote by $\mathrm{Vect}_F(X)$ the set of isomorphism classes of F-vector bundles on X. This set is a semiring with addition given by Whitney sum of bundles and multiplication given by tensor product. The trivial bundle of rank zero is the additive identity and the trivial line bundle is the multiplicative identity. Choose a basepoint $x \in X$. There is a semiring map $\mathrm{rk} : \mathrm{Vect}_F(X) \to \mathbb{Z}$ defined by

$$\mathrm{rk}(E) = \dim_F(E_x)$$

where E_x is the fiber over x. The rank of a vector bundle is constant on each connected component of X so that if X is connected, this function is independent of the choice of basepoint.

Recall that if R is a semiring, there exists a ring S called the *ring completion* and a map $R \to S$ which satisfies the obvious universal property.

DEFINITION B.2.1. The ring $K_F^0(X)$ is defined to be the ring completion of $\mathrm{Vect}_F(X)$.

The functor K_F^0 is contravariant in X. Indeed, if $f : X \to Y$ is continuous, then we define $f^* : K_F^0(Y) \to K_F^0(X)$ by

$$f^*([E]) = [f^*(E)],$$

where for a bundle E, $[E]$ denotes the isomorphism class of E and $f^*(E)$ denotes the pullback of E. The map rk induces a ring homomorphism $\mathrm{rk} : K_F^0(X) \to \mathbb{Z}$.

DEFINITION B.2.2. The reduced K-theory of X is

$$\tilde{K}_F^0(X) = \ker(\mathrm{rk} : K_F^0(X) \longrightarrow \mathbb{Z}).$$

The elements of $\tilde{K}_F^0(X)$ may be characterized as follows. Call two vector bundles E, E' over X *stably equivalent* if there are trivial bundles η, η' with $E \oplus \eta \cong E' \oplus \eta'$. This is clearly an equivalence relation on $\mathrm{Vect}_F(X)$.

THEOREM B.2.3 ([59], p. 105). *Let X be a space such that for each vector bundle E over X there exists a bundle E' with $E \oplus E'$ trivial. Then the elements of $\tilde{K}_F^0(X)$ are in one-to-one correspondence with stable equivalence classes in $\mathrm{Vect}_F(X)$.* □

The functor $\tilde{K}_F^0$ has a classifying space defined as follows. For each n, consider the Grassmannian $G(n, F^{2n})$ of n-planes in F^{2n} and denote by B_F the union $\bigcup_{n \geq 1} G(n, F^{2n})$. If $F = \mathbb{R}$, this is usually denoted BO and if $F = \mathbb{C}$ it is denoted BU.

THEOREM B.2.4 ([**59**], p. 107). *Let X be a finite connected CW-complex. Then there is a natural isomorphism*

$$\tilde{K}_F^0(X) \cong [X, B_F].$$

Thus, giving a stable equivalence class of an F-vector bundle on X is the same as giving a homotopy class of maps $X \to B_F$.

REMARK B.2.5. The notation K_F^0 indicates that there are functors K_F^i. This is indeed the case, but we shall not describe them here.

B.2.2. Algebraic K-theory. The definition of the group $K_0(R)$ for a ring R is classical and dates back to the work of Grothendieck on the Riemann–Roch problem. The definition is quite easy to state, yet the group $K_0(R)$ is often very difficult to compute.

Let R be a ring. The set of isomorphism classes of finitely generated projective R-modules forms a monoid with addition given by direct sum and identity element the trivial module. If R is commutative, then we have a multiplication given by tensor product with identity the rank one free module R. We define $K_0(R)$ to be the completion of this monoid (semiring if R is commutative).

EXAMPLE B.2.6. If R is a field, then finitely generated projective modules are free and are determined up to isomorphism by rank. Thus, $K_0(R) = \mathbb{Z}$. If R is the ring of integers in a number field, then $K_0(R) = \mathbb{Z} \oplus \mathrm{Cl}(R)$, where $\mathrm{Cl}(R)$ is the ideal class group of R.

Loosely speaking, projective modules over R correspond to locally free sheaves (= vector bundles) on Spec R. This is the intuitive idea behind the following result, due to Swan [**121**].

THEOREM B.2.7. *Let $F = \mathbb{R}$ or $\mathbb{C}$, let X be a finite connected CW-complex, and let $R = C^F(X)$ be the ring of continuous F-valued functions on X. If E is a vector bundle on X, let*

$$\Gamma(X, E) = \{s : X \longrightarrow E : p \circ s = id_X\}$$

be the set of continuous sections of E. Then $\Gamma(X, E)$ is a finitely generated projective R-module. Moreover, the map $E \mapsto \Gamma(X, E)$ induces an isomorphism of categories from the category $Vect_F(X)$ to the category of finitely generated projective R-modules and hence induces an isomorphism $K_F^0(X) \to K_0(R)$. □

The groups $K_1(R)$ and $K_2(R)$ are also classical. Denote by $GL(R)$ the infinite general linear group and by $E(R)$ the subgroup generated by elementary matrices. By the Whitehead lemma, the group $E(R)$ is a perfect normal subgroup equal to the commutator subgroup of $GL(R)$. The group $K_1(R)$ is defined as $GL(R)/E(R)$, the abelianization of $GL(R)$. Milnor defined the

group $K_2(R)$ to be $H_2(E(R), \mathbb{Z})$. When R is a field, we have the following presentation of $K_2(R)$, due to Matsumoto (see, e.g. [**80**]).

THEOREM B.2.8. *Let F be a field. Then the group $K_2(F)$ has a presentation with generators $\{x, y\}$, where $x, y \in F^\times$, and relations*

1. $\{x, 1-x\} = 1$ *for* $x \neq 0, 1$,
2. $\{x_1 x_2, y\} = \{x_1, y\}\{x_2, y\}$,
3. $\{x, y_1 y_2\} = \{x, y_1\}\{x, y_2\}$. □

Furthermore, there are relative groups $K_i(R, I)$, for $i = 0, 1$, defined for an ideal $I \subset R$ such that there is an exact sequence

$$K_2(R) \to K_2(R/I) \to K_1(R, I) \to K_1(R) \to K_1(R/I)$$
$$\to K_0(R, I) \to K_0(R) \to K_0(R/I) \to 0.$$

Until the work of Quillen, no one knew how to define higher K-groups K_i which extend this sequence to the left.

Quillen gave two definitions of algebraic K-theory: the +-construction and the Q-construction. A hard theorem asserts that these two definitions give the same result. While K-groups can be defined for schemes, we shall concentrate on the definition for rings using the +- construction.

Let R be a ring and consider the classifying space $BGL(R)$ of the discrete group $GL(R)$. Quillen defines a space $BGL(R)^+$ and then defines

$$K_i(R) = \pi_i(\mathbf{K}(R)),$$

where $\mathbf{K}(R) = BGL(R)^+ \times K_0(R)$. We describe this construction in the proof of the following result.

THEOREM B.2.9. *Let X be a connected CW-complex with basepoint x_0 and let π be a perfect normal subgroup of $\pi_1 = \pi_1(X, x_0)$. Then there exists a CW-complex X^+, which is obtained from X by attaching only 2-cells and 3-cells such that*

1. *The map $\pi_1(X, x_0) \to \pi_1(X^+, x_0)$ is the quotient map $\pi_1 \to \pi_1/\pi$;*
2. *For any π_1/π-module M, we have $H_\bullet(X^+, X; M)$, where M is viewed as a local coefficient system on X^+.*

Moreover, X^+ is unique up to homotopy equivalence.

PROOF. Let $\{\gamma_i\}_{i \in I}$ be a set of generators for π and let $g_i : (S^1, *) \to (X, x_0)$ be a representing map for γ_i. Since the kernel of the Hurewicz map $\pi_1 \to H_1(X, \mathbb{Z})$ is the commutator subgroup $[\pi_1, \pi_1]$, each map g_i is trivial on homology. Let Y be the complex obtained by attaching a 2-cell e_i^2 for each $i \in I$ via the maps $g_i : \partial e_i^2 = S^1 \to X$. The inclusion $X \to Y$ clearly satisfies 1 above.

Let $\tilde{X} \to \tilde{Y}$ be the covering spaces with covering group π_1/π so that $\tilde{Y}$ is the universal cover of Y and $\pi_1(\tilde{X}) = \pi$. Since π is perfect, $H_1(\tilde{X}, \mathbb{Z}) = \pi_{\mathbf{ab}} = 0$. The relative homology $H_\bullet(Y, X; \mathbb{Z})$ is concentrated in degree 2, where it is the free abelian group on the $[e_i^2]$. Similarly, $H_\bullet(\tilde{Y}, \tilde{X}; \mathbb{Z})$ is concentrated in degree 2 where it is the free $\mathbb{Z}[\pi_1/\pi]$-module on the $[e_i^2]$. Since the connecting map $\partial : H_2(\tilde{Y}, \tilde{X}; \mathbb{Z}) \to H_1(\tilde{X}, \mathbb{Z}) = 0$ is trivial, $H_\bullet(\tilde{Y}, \mathbb{Z})$ differs from $H_\bullet(\tilde{X}, \mathbb{Z})$ by adding $\bigoplus_{i \in I} \mathbb{Z}[\pi_1/\pi][e_i^2]$ in degree 2. Since $\tilde{Y}$ is simply connected, $[e_i^2]$ is in the image of the Hurewicz map $\pi_2(\tilde{Y}) \to H_2(\tilde{Y}, \mathbb{Z})$ and similarly for Y (by pushing down). Let $h_i : (S^2, *) \to (Y, x_0)$ be an element mapping to $[e_i^2]$ via the Hurewicz map. Attach a 3-cell e_i^3 to Y via h_i for each $i \in I$ to obtain a space X^+. Property 1 is clear. Property 2 is easy to check. The uniqueness of X^+ is left to the reader. □

Now, since $BGL(R)^+$ is path connected, we have $\pi_0(\mathbf{K}(R)) = K_0(R)$ and

$$\pi_1(\mathbf{K}(R)) = \pi_1(BGL(R)^+) = GL(R)/E(R) = K_1(R).$$

Moreover, it is not difficult to see that $K_2(R)$ agrees with Milnor's definition.

Let $I \subset R$ be an ideal. If we define $\mathbf{K}(R, I)$ to be the fiber of the map $\mathbf{K}(R) \to \mathbf{K}(R/I)$, then we obtain a long exact sequence of K-groups

$$\cdots \longrightarrow K_{i+1}(R/I) \longrightarrow K_i(R, I) \longrightarrow K_i(R) \longrightarrow K_i(R/I) \longrightarrow \cdots .$$

Moreover, the definition of $K_i(R, I)$ agrees with the classical definition for $i = 0, 1$.

We also have the following result, due to Quillen, which is known as the fundamental theorem of algebraic K-theory.

THEOREM B.2.10. *Let R be a regular ring. Then there are natural isomorphisms*

$$K_i(R[t]) \cong K_i(R)$$

and

$$K_i(R[t, t^{-1}]) \cong K_i(R) \oplus K_{i-1}(R).$$

By construction, for any ring R there is a Hurewicz map

$$h_i : K_i(R) \longrightarrow H_i(GL(R), \mathbb{Z}).$$

This provides the motivation for studying the homology of linear groups over R. If we tensor the above groups with $\mathbb{Q}$, the resulting map is injective with image equal to the primitive elements. Thus, computations of $H_\bullet(GL(R), \mathbb{Z})$ can give information about $K_\bullet(R)$.

Appendix C
Étale Cohomology

This appendix provides a quick summary, mostly without proofs, of the basics of étale cohomology. A good reference for this material is J. Milne's book [**79**].

C.1. Étale Morphisms and Henselian Rings

C.1.1. Étale morphisms.

DEFINITION C.1.1. A morphism of schemes $f : X \longrightarrow Y$ is affine if $f^{-1}(U)$ is an open affine subset of X for every open affine $U \subseteq Y$. If, in addition, $\Gamma(f^{-1}(U), \mathcal{O}_X)$ is a finite $\Gamma(U, \mathcal{O}_Y)$-algebra for each U, then we say that f is finite.

PROPOSITION C.1.2. *(a) A closed immersion is finite.*
(b) The composite of two finite morphisms is finite.
(c) Any base change of a finite morphism is finite.

PROOF. It suffices to consider opens in some affine covering of the target. These statements then translate into statements about rings which are obvious. For example, (b) boils down to showing that a finite extension of a finite extension is finite. Statement (c) asserts that if $f : X \longrightarrow Y$ is finite and if $Z \longrightarrow Y$ is any morphism, then the induced morphism $X \times_Y Z \longrightarrow Z$ is also finite:

$$\begin{array}{ccc} X \times_Y Z & \longrightarrow & X \\ \downarrow & & \downarrow f \\ Z & \longrightarrow & Y. \end{array}$$

This reduces to an obvious statement about tensor products. □

PROPOSITION C.1.3. *Any finite morphism $f : X \longrightarrow Y$ is proper; that is, it is separated, of finite-type, and universally closed.*

PROOF. This may be found in [**79**], p.4. Recall that f is separated if the diagonal morphism

$$\Delta : X \longrightarrow X \times_Y X$$

is a closed immersion. It is of finite-type if there exists an open affine covering of Y by subsets $V_i = \mathrm{spec}(B_i)$ such that for each i, $f^{-1}(V_i)$ can be covered by

a finite number of affine subsets $U_{ij} = \operatorname{spec}(A_{ij})$, where each A_{ij} is a finitely generated B_i-algebra. The map f is universally closed if it is closed and if any base change of f is closed. □

Finite morphisms over $\operatorname{spec}(k)$, where k is a field, have a particularly nice description.

PROPOSITION C.1.4. *Let $f : X \longrightarrow \operatorname{spec}(k)$ be a morphism of finite-type. The following are equivalent.*

1. *X is affine and $\Gamma(X, \mathcal{O}_X)$ is an Artin ring;*
2. *X is finite and discrete as a topological space;*
3. *X is discrete;*
4. *X is finite.* □

DEFINITION C.1.5. A morphism $f : X \longrightarrow Y$ is quasi-finite if it is of finite-type and has finite fibers. Similarly, an A-algebra B is quasi-finite if it is of finite-type and if $B \otimes_A k(\mathfrak{p})$ is a finite $k(\mathfrak{p})$-algebra for all prime ideals $\mathfrak{p} \subset A$ (here, $k(\mathfrak{p})$ is the fraction field of the domain $A/\mathfrak{p}$).

Clearly, any immersion is quasi-finite, as the composition of two quasi-finite morphisms. Also, any base change of a quasi-finite morphism is quasi-finite.

We know that finite morphisms are proper. Conversely, proper morphisms which are quasi-finite are finite. This is a consequence of Zariski's main theorem [**79**], p. 6.

Recall that a homomorphism of rings $f : A \longrightarrow B$ is *flat* if B is a flat A-module (via f); that is, the functor $- \otimes_A B$ is exact. Flatness is preserved by localization.

DEFINITION C.1.6. A morphism $f : X \longrightarrow Y$ of schemes is flat if for all $x \in X$, the induced map $\mathcal{O}_{Y,f(x)} \longrightarrow \mathcal{O}_X, x$ is flat. Equivalently, f is flat if for any pair of open affines $U \subseteq X, V \subseteq Y$, with $f(U) \subseteq V$, the map $\Gamma(V, \mathcal{O}_Y) \longrightarrow \Gamma(U, \mathcal{O}_X)$ is flat.

As one might expect, open immersions are flat, as is the composition of two flat morphisms. A base change of a flat morphism is flat.

EXAMPLE C.1.7. If A is any ring, then $A[x_1, \ldots, x_n]$ is a free A-module; thus, $\mathbb{A}^n_A$ is flat over $\operatorname{spec}(A)$. More generally, let $Z \subset \mathbb{A}^n_A$ be a hypersurface; that is, Z is the zero set of a single nonzero polynomial P. Then Z is flat over $\operatorname{spec}(A)$ if and only if the ideal generated by the coefficients of P is A. This may be restated by saying that Z is flat if and only if its closed fibers over $\operatorname{spec}(A)$ have the same dimension. Thus, flatness is the algebraic analogue of a continuously varying family.

DEFINITION C.1.8. A flat morphism $f : A \longrightarrow B$ is faithfully flat if $B \otimes_A M$ is nonzero for any nonzero A-module M. Such a map is necessarily injective (take $M = (a)$, where $a \in A$). Note that the associated map $f^* : \text{spec}(B) \longrightarrow \text{spec}(A)$ is surjective. A morphism $f : X \longrightarrow Y$ of schemes is faithfully flat if it is flat and surjective.

DEFINITION C.1.9. Let k be a field with algebraic closure $\overline{k}$. A k-algebra A is separable if $\overline{A} = A \otimes_k \overline{k}$ has zero Jacobson radical; that is, the intersection of the maximal ideals is zero.

DEFINITION C.1.10. A morphism $f : Y \longrightarrow X$ that is locally of finite-type is unramified at $y \in Y$ if $\mathcal{O}_{Y,y}/\mathfrak{m}_x \mathcal{O}_{Y,y}$ is a finite separable field extension of $k(x) = \mathcal{O}_{X,x}/\mathfrak{m}_x \mathcal{O}_{X,x}$, where $x = f(y)$. In terms of rings, a homomorphism $f : A \longrightarrow B$ of finite type is unramified at $\mathfrak{q} \in \text{spec}(B)$ if and only if $\mathfrak{p} = f^{-1}(\mathfrak{q})$ generates the maximal ideal in $B_\mathfrak{q}$ and $k(\mathfrak{q})$ is a finite separable field extension of $k(\mathfrak{p})$. A morphism $f : Y \longrightarrow X$ is unramified if it is unramified at each $y \in Y$.

REMARK C.1.11. Any closed immersion is unramified.

DEFINITION C.1.12. A morphism $f : Y \longrightarrow X$ is étale if it is flat and unramified (hence also locally of finite-type).

Clearly, any open immersion is étale, as is the composition of two étale morphisms. Étale morphisms are also preserved by base change.

Étale morphisms are the algebraic analogue of local isomorphisms. Let A be a ring and consider the ring $A[t_1, \ldots, t_n]$. Let $P_1, \ldots, P_n$ be polynomials and let $B = A[t_1, \ldots, t_n]/(P_1, \ldots, P_n)$. Then B is étale over A if and only if the image of $\det(\partial P_i/\partial t_j)$ in B is a unit. If $Y = \text{spec}(B)$ and $X = \text{spec}(A)$ are analytic manifolds, then this criterion means that the induced maps on tangent spaces are isomorphisms. This is made precise by the following result.

PROPOSITION C.1.13. *A morphism $f : Y \longrightarrow X$ is étale if and only if for every $y \in Y$, there exist open affine neighborhoods $V = \text{spec}(C)$ of y and $U = \text{spec}(A)$ of $x = f(y)$ such that for some polynomials $P_1, \ldots, P_n$,*

$$C = A[t_1, \ldots, t_n]/(P_1, \ldots, P_n)$$

and $\det(\partial P_i/\partial t_j)$ is a unit in C.

PROOF. See [**79**], Corollary 3.16, p. 27. □

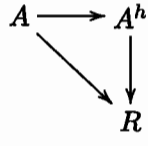
C.1.2. Henselian rings. Let A be a local ring with maximal ideal $\mathfrak{m}$ and residue field k. Denote the maps $A \to k$ and $A[t] \to k[t]$ by $a \mapsto \overline{a}$ and $f \mapsto \overline{f}$. Two polynomials $f, g \in B[t]$ (where B is any ring) are *strictly coprime* if the ideal generated by f, g is $B[t]$.

DEFINITION C.1.14. A local ring A is Henselian if the following condition holds: If f is a monic polynomial in $A[t]$ such that $\overline{f}$ factors as $\overline{f} = g_0 h_0$ with g_0 and h_0 monic and coprime, then f factors as $f = gh$ where g and h are monic and $\overline{g} = g_0, \overline{h} = h_0$.

PROPOSITION C.1.15. *Any complete local ring is Henselian.*

PROOF. See [**79**], p.35. □

Let A be a local ring and denote by $\hat{A}$ its $\mathfrak{m}$-adic completion. Since A is a subring of $\hat{A}$, we see that A is a subring of a Henselian ring. The smallest such ring is called the Henselization of A; denote this ring by A^h. The ring A^h satisfies the obvious universal mapping property; namely, if $j : A \to R$ is a homomorphism where R is Henselian, then there is a unique map $A^h \to R$ such that the diagram



commutes. This characterizes A^h uniquely, provided it exists.

DEFINITION C.1.16. Let A be a local ring. An étale neighborhood of A is a pair $(B, \mathfrak{q})$ where B is an étale A-algebra and $\mathfrak{q}$ is a prime ideal lying over $\mathfrak{m}$ such that the induced map $k \to k(\mathfrak{q})$ is an isomorphism.

One checks easily that the étale neighborhoods of A with connected spectra form a filtered direct system. Let the ring $(A^h, \mathfrak{m}^h)$ be the limit of this system:

$$(A^h, \mathfrak{m}^h) = \varinjlim (B, \mathfrak{q}).$$

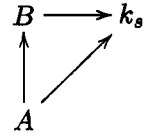
It is not hard to check that A^h is local with $A^h/\mathfrak{m}^h = k$, and that A^h is the Henselization of A.

EXAMPLE C.1.17. Let k be a field, and let A be the localization of $k[t_1, \ldots, t_n]$ at $(t_1, \ldots, t_n)$. Then the Henselization of A is the ring of power series $P \in k[[t_1, \ldots, t_n]]$ that are algebraic over A.

DEFINITION C.1.18. Let X be a scheme and let $x \in X$. An étale neighborhood of x is a pair (Y, y), where Y is an étale X-scheme and y is a point of Y mapping to x such that $k(x) = k(y)$. The connected étale neighborhoods of x form a filtered system and $\varinjlim \Gamma(Y, \mathcal{O}_Y) = \mathcal{O}^h_{X,x}$.

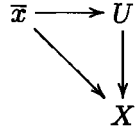
Note that by definition a Henselian ring A has no finite étale extensions with trivial residue field extensions except those of the form $A \to A^r$ for some r. Thus, if the residue field of A is separably algebraically closed, then A has no finite étale extensions. Such a ring is called *strictly Henselian*. Every local

ring A has a strict Henselization A^{sh} satisfying the obvious universal mapping property. It can be constructed as follows. Fix a separable closure k_s of k. Then $A^{sh} = \varinjlim B$, where the limit runs over all diagrams



with $A \to B$ étale. If $A = k$ is a field, then A^{sh} is any separable closure of k.

Let $\overline{x} \to X$ be a geometric point of a scheme X. An étale neighborhood of $\overline{x}$ is a commutative diagram



with $U \to X$ étale. Then $\mathcal{O}^{sh}_{X,\overline{x}} = \varinjlim \Gamma(U, \mathcal{O}_U)$, where the limit is taken over all étale neighborhoods of $\overline{x}$. This is the analogue for the étale topology of the local ring for the Zariski topology. Indeed, the two definitions are the same: take the direct limit over "open" sets containing $\overline{x}$.

C.2. Étale Cohomology

C.2.1. Sheaves. We assume that the reader is familiar with the concept of a sheaf on a topological space. It is possible to define "topologies" on the category of schemes which are more general than the standard Zariski topology. With this generalized notion of "open covering", we can extend the formal properties of sheaves to obtain sheaves for these topologies.

Let E be a class of morphisms of schemes satisfying

1. All isomorphisms are in E;
2. The composite of two morphisms in E is in E;
3. Any base change of a morphism in E is in E.

The full subcategory of Sch/X (the category of schemes over a fixed base scheme X) whose structure morphism is in E will be denoted by E/X. There three obvious examples:

1. $E = (\mathrm{Zar})$ consists of all open immersions;
2. $E = (\text{ét})$ consists of all étale morphisms of finite-type;
3. $E = (\mathrm{fl})$ consists of all flat morphisms locally of finite-type.

Fix a base scheme X, a class E of morphisms, and a full subcategory C/X of Sch/X that is closed under fiber products and is such that for any $Y \to X$ in C/X and any E-morphism $U \to Y$, the composite $U \to X$ is in C/X. An E-covering of $Y \in C/X$ is a family $(U_i \xrightarrow{g_i} Y)_{i\in I}$ of E-morphisms such that $Y = \bigcup g_i(U_i)$. The class of all such coverings is the E-topology on C/X. The category C/X with the E-topology is called the E-site and is denoted by $(C/X)_E$ or X_E.

A *presheaf* P on a site $(C/X)_E$ is a contravariant functor $C/X \to \mathrm{Ab}$, where Ab is the category of abelian groups. A presheaf P is a *sheaf* if it satisfies

1. If $s \in P(U)$, and $(U_i \to U)_{i\in I}$ is a covering of U such that $\mathrm{res}_{U_i,U}(s) = 0$ for all i, then $s = 0$;
2. If $(U_i \to U)_{i\in I}$ is a covering and the family $(s_i)_{i\in I}$, $s_i \in P(U_i)$ is such that
$$\mathrm{res}_{U_i\times_U U_j,U_i}(s_i) = \mathrm{res}_{U_i\times_U U_j,U_j}(s_j)$$
for all $i, j \in I$, then there exists $s \in P(U)$ such that $\mathrm{res}_{U_i,U}(s) = s_i$ for all $i \in I$.

Note that these axioms are the usual sheaf axioms in the case $E = (\mathrm{Zar})$; indeed, the fiber product $U_i \times_U U_j$ is just the intersection $U_i \cap U_j$.

The usual properties of sheaves on a topological space generalize to this setting. For further details, see Chapter II of [**79**].

C.2.2. Cohomology. Let $\mathcal{A}$ be an abelian category. An object I of $\mathcal{A}$ is *injective* if the functor
$$M \mapsto \mathrm{Hom}_{\mathcal{A}}(M, I)$$
is exact. The category $\mathcal{A}$ has *enough injectives* if for every M in $\mathcal{A}$ there is a monomorphism $M \to I$, with I injective. If $\mathcal{A}$ has enough injectives and if $f : \mathcal{A} \to \mathcal{B}$ is a left exact functor into another abelian category, then there are functors $R^i f : \mathcal{A} \to \mathcal{B}$ satisfying

1. $R^0 f = f$;
2. $R^i f(I) = 0$ if I is injective and $i > 0$;
3. For any exact sequence $0 \to M' \to M \to M'' \to 0$ in $\mathcal{A}$, there are morphisms $\partial^i : R^i f(M'') \to R^{i+1} f(M')$, $i \geq 0$ such that the sequence
$$\cdots \to R^i f(M) \to R^i f(M'') \xrightarrow{\partial^i} R^{i+1} f(M') \to R^{i+1} f(M) \to \cdots$$
is exact. Moreover, this construction is functorial.

The derived functors are defined as follows. If $M \in \mathcal{A}$, choose an injective resolution
$$0 \longrightarrow M \longrightarrow I_0 \longrightarrow I_1 \longrightarrow \cdots$$

(that is, each I_j is an injective object of $\mathcal{A}$). Such resolutions are essentially unique. Consider the complex

$$C : 0 \longrightarrow f(I_0) \longrightarrow f(I_1) \longrightarrow \cdots .$$

The objects $R^i f(M)$ are the cohomology objects of this complex:

$$R^i f(M) = H^i(C).$$

PROPOSITION C.2.1. *Let X be a scheme and let X_E be the E-site on X. Denote by $\mathcal{S}(X_E)$ the category of sheaves of abelian groups on X_E. Then $\mathcal{S}(X_E)$ has enough injectives.*

PROOF. See [79], p. 83. □

DEFINITION C.2.2. (a) The global sections functor

$$\Gamma(X, -) : \mathcal{S}(X_E) \longrightarrow \mathrm{Ab}$$

with $\Gamma(X, F) = F(X)$ is left exact and its derived functors are written

$$R^i\Gamma(X, -) = H^i(X, -) = H^i(X_E, -).$$

The group $H^i(X_E, F)$ is called the ith cohomology group of X_E with values in F.

(b) For any $U \to X$ in C/X, the derived functors of $F \mapsto F(U)$ are written $H^i(U, F)$.

(c) For any fixed sheaf F_0 on X_E, the functor

$$F \mapsto \mathrm{Hom}_{\mathcal{S}}(F_0, F)$$

is left exact. Its derived functors are written $\mathrm{Ext}^i_{\mathcal{S}}(F_0, -)$.

REMARK C.2.3. 1. $H^i(X_E, F)$ is a contravariant functor on X_E; that is, if $\pi^* : \mathcal{S}(X_E) \to \mathcal{S}(X'_{E'})$ is exact, then the maps $H^i(X_E, F) \to H^i(X'_{E'}, \pi^* F)$ are induced by the obvious map $H^0(X, F) \to H^0(X', \pi^* F)$, where $\pi : X' \to X$.
2. There is an isomorphism of functors $\Gamma(X, -) \cong \mathrm{Hom}(\mathbb{Z}, -)$, where $\mathbb{Z}$ is the constant sheaf on X_E. It follows that $H^i(X, -) \cong \mathrm{Ext}^i(\mathbb{Z}, -)$.

EXAMPLE C.2.4. Let $X = \mathrm{spec}(k)$, where k is a field, and consider the étale site on X. Let k_s be a separable closure of k and let G be the Galois group of k_s over k. Then there is an isomorphism

$$\mathcal{S}(X_E) \cong G - \mathrm{mod}$$

of categories, where G-mod is the category of continuous G-modules. If the sheaf F corresponds to the module M, then $\Gamma(X, F) = M^G$ and

$$H^i(X, F) = H^i(G, M) = H^i(k, M),$$

where the groups on the right are the Galois cohomology groups of k with coefficients in M. These are defined to be $\varinjlim H^i(G(k'/k), M^{k'})$, where k'/k is a finite galois extension of k (note that $k_s = \bigcup k'$).

EXAMPLE C.2.5. (Hilbert's Theorem 90) The canonical map

$$H^1(X_{\mathrm{Zar}}, \mathcal{O}_X^\times) \longrightarrow H^1(X_{\text{ét}}, \mathbb{G}_m)$$

is an isomorphism, where $\mathbb{G}_m$ is the sheaf of multiplicative groups. It follows that

$$H^1(X_{\text{ét}}, \mathbb{G}_m) = \mathrm{Pic}(X).$$

The functors $H^i_{\text{ét}}(X, -) = H^i(X_{\text{ét}}, -)$ satisfy many of the properties of a cohomology theory, such as excision.

The cohomology groups $H^i(X_E, -)$ agree, in most cases of interest, with the Čech cohomology groups defined in the usual manner via coverings of X. This is the case, for example, if F is a quasi-coherent sheaf of $\mathcal{O}_X$-modules in the Zariski topology, assuming X is separated. The same is true for étale cohomology if the following condition holds: X is quasi-compact and every finite subset of X is contained in an affine open subset (e.g., X quasi-projective over an affine scheme). As usual, there is a spectral sequence relating Čech cohomology to derived functor cohomology. For full details, see [**79**], Chapter III, Section 2.

Given two topologies on C/X, one would like to know the relationship between the two cohomology theories. We present one such result, which is used in Chapter 5.

THEOREM C.2.6. *Let X be a smooth scheme over $\mathbb{C}$. Then for any finite abelian group M, we have an isomorphism*

$$H^i(X(\mathbb{C}), M) \cong H^i(X_{\text{ét}}, M).$$

REMARK C.2.7. This is absolutely not true with integral coefficients. For example, if X is a smooth complete curve over $\mathbb{C}$ of genus g, then

$$H^1(X(\mathbb{C}), \mathbb{Z}) = \mathbb{Z}^{2g} = \mathrm{Hom}(\pi_1(X), \mathbb{Z}).$$

For a scheme X, there is an algebraic fundamental group $\pi_1^{\mathrm{alg}}(X)$ which classifies étale coverings of X ([**79**], Chapter I). We have

$$H^1(X_{\text{ét}}, \mathbb{Z}) = \mathrm{Hom}_{\mathrm{cont}}(\pi_1^{\mathrm{alg}}(X), \mathbb{Z}) = 0$$

in this case since for a complex variety, the group $\pi_1^{\mathrm{alg}}(X)$ is the profinite completion of the topological fundamental group, and hence has only finite discrete quotients. However, we do have

$$\begin{aligned} H^1(X(\mathbb{C}), \mathbb{Z}/n) &= (\mathbb{Z}/n)^{2g} \\ &= \mathrm{Hom}_{\mathrm{cont}}(\pi_1^{\mathrm{alg}}(X), \mathbb{Z}/n) \\ &= H^1_{\text{ét}}(X, \mathbb{Z}/n). \end{aligned}$$

Let l be a prime number. A sheaf F is *torsion* (resp. l-torsion) if, for all quasi-compact U, $F(U)$ is torsion (resp. l-torsion).

PROPER BASE CHANGE THEOREM C.2.8 ([79], p.224). Let $k \subset K$ be separably closed fields and let X be a scheme which is proper over k. Let X_K be the scheme $X \times_k K$ and let F be a torsion sheaf on $X_{\text{ét}}$. Then the natural map

$$H^i(X, F) \longrightarrow H^i(X_K, F|_{X_K})$$

is an isomorphism for $i \geq 0$. □

C.3. Simplicial Schemes

Most of the above theory can be extended to simplicial schemes. A thorough discussion of this may be found in [40]. The étale site of a simplicial scheme $X_\bullet$, denoted $\text{Et}(X_\bullet)$, is the category whose objects are étale maps $U \to X_n$ for some $n \geq 0$, and whose maps are commutative diagrams

$$\begin{array}{ccc} U & \longrightarrow & V \\ \downarrow & & \downarrow \\ X_n & \longrightarrow & X_m \end{array}$$

where the bottom arrow is a structure map of $X_\bullet$. A covering of an object $U \to X_n$ is a family of étale maps $(U_i \xrightarrow{g_i} U)_{i \in I}$ over X_n with $U = \bigcup g_i(U_i)$. Note that this definition does not imply that there is a map of simplicial schemes $U_\bullet \longrightarrow X_\bullet$ given in dimension n by $U \to X_n$.

A presheaf on $\text{Et}(X_\bullet)$ is a contravariant functor on $\text{Et}(X_\bullet)$; that is, to each object $U \to X_n$ we associate a set (or group, ring, etc.) $P(U)$. The presheaf F is a sheaf if the usual axioms hold. The category of sheaves of abelian groups on $\text{Et}(X_\bullet)$ has enough injectives.

DEFINITION C.3.1. Let $X_\bullet$ be a simplicial scheme. For $i \geq 0$, the cohomology group functor

$$H^i(X_\bullet, -)$$

is the ith derived functor of the functor sending a sheaf F to the abelian group given as the kernel of the map

$$d_0^* - d_1^* : F(X_0) \longrightarrow F(X_1).$$

Equivalently, $H^i(X_\bullet, -) = \text{Ext}^i_{S(X_\bullet)}(\mathbb{Z}, -)$, where $\mathbb{Z}$ is the constant sheaf.

As usual, there is a definition of Čech cohomology via coverings. This theory agrees with the above definition if, for example, each X_n is quasi-projective over a noetherian ring. The reader is referred to Chapter 3 of [40] for further details. Also, the cohomology of a simplicial scheme over $\mathbb{C}$ agrees with that of the corresponding simplicial manifold with coefficients in a torsion sheaf. Moreover, the proper base change theorem holds in this more general context as well.

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