Math. Proc. Camb. Phil. Soc. (1989), **106**, 531
Printed in Great Britain

A Seifert-matrix interpretation of Cappell and Shaneson's approach to link cobordisms

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(*Received* 11 *August* 1988; *revised* 21 *February* 1989)

Introduction

We classified the set of F_m-cobordism classes of F_m-links by their Seifert matrices in [**5**]. On the other hand Cappell and Shaneson identified them with essentially a quotient group of their homology surgery obstruction group [**2**]. In this paper, we will find a description of their surgery obstruction in terms of a Seifert matrix. In relation to Ledimet's recent results [**7**], we hope this might provide some clue to whether F_m-cobordism or boundary cobordism is stronger than ordinary link cobordism. It also seems to be an interesting algebraic question to find an algorithm for obtaining a Seifert matrix from their surgery obstruction.

It has been known that all formulations of the knot cobordism group are equivalent (see [**1**, **4**, **8**, **9**, **10**, and **11**]). In [**11**], the equivalence between the Seifert matrix description and the Γ-group description for the knot cobordism group was established in an abstract setting, and in [**10**], the same equivalence was obtained by explicitly converting a given Γ-group obstruction to the corresponding Seifert matrix.

1. *Construction of normal maps*

A link in S^{n+2} is an embedded oriented submanifold that is homeomorphic to m copies of S^n and a link cobordism between two given links in S^{n+2} is a properly embedded oriented submanifold in $S^{n+2}\times[0,1]$ that is homeomorphic to $S^n\times[0,1]$ and intersects $S^{n+2}\times 0$ and $S^{n+2}\times 1$ at the given links. Let F_m be the free group on m generators. A pair (L,θ) is an *F_m-link* if L is a link in S^{n+2} and $\theta:\pi_1(X)\rightarrow F_m$ is an epimorphism sending a set of meridians to a fixed set of generators of F_m where X is an exterior of L in S^{n+2}, that is, the complement of an open tubular neighbourhood of L in S^{n+2}. By the transversality argument, the epimorphism θ gives a Seifert surface, i.e. disjoint oriented submanifolds bounded by L. Conversely, the existence of a Seifert surface for L produces such an epimorphism by the Thom–Pontryagin construction. Two such epimorphisms for the same link differ by a generator-conjugating automorphism of F_m (see [**2**, **5**]). A pair $(\mathbf{L},\Theta)$ is an *F_m-cobordism* between (L_0,θ_0) and (L_1,θ_1) if $\mathbf{L}$ is a link cobordism between links L_0 and L_1 and Θ: $\pi_1\mathbf{X}\rightarrow F_m$ is an epimorphism extending θ_0 and θ_1 up to inner automorphisms (see [**2**]) where $\mathbf{X}$ is an exterior of $\mathbf{L}$ in $S^{n+1}\times[0,1]$. $C_n(F_m)$ denotes the set of F_m-cobordism classes of F_m-links. $C_n(F_m)$ is an abelian group under the connected sum and the group $\mathfrak{U}_m$ of all generator-conjugating automorphisms of F_m acts on $C_n(F_m)$ by compositions on the epimorphisms.

For our purposes we need naturally a new normal map involving Seifert surfaces.

We begin by constructing the target of our normal map. Let

V_* be the boundary connected sum of m copies of $D^{n+2}\times S^1$;

T_* be the disjoint union of m copies of $D^{n+1}\times S^1$;

X_* be the interior connected sum of m copies of $D^{n+1}\times S^1$;

Y_* be the disjoint union of m copies of $S^n\times S^1$.

Then they are related by $\partial V_* = T_* \cup_{Y_*} X_*$. Thus (V_*, T_*, X_*) is a simple Poincaré triad because $\mathrm{wh}(F_m)=0$. In fact V_* can serve as a model of an exterior of a trivial disk link: $m(D^{n+1}, S^n) \hookrightarrow (D^{n+3}, S^{n+2})$.

Let (L,θ) be an F_m-link in S^{n+2} and let $\alpha_\partial : \partial X \to Y_*$ be the obvious homeomorphism. According to [2], proposition 2·2, α_∂ extends to a map $\alpha : X \to X_*$ that induces θ on fundamental groups and that induces isomorphisms on homology groups with integer coefficients. Let

$$Z_* = X_* \underset{Y_*}{\cup} m(S^n\times D^2) \quad \text{and} \quad U_* = V_* \underset{T_*}{\cup} m(D^{n+1}\times D^2).$$

Then $(U_*, Z_*) = (D^{n+3}, S^{n+2})$. By putting the tubular neighbourhood of L back to X, α extends to a map $\bar\alpha : S^{n+2} \to Z_*$. We note that α_∂, α and $\bar\alpha$ all have degree 1. Let L_* be the trivial link given by $m(S^n\times 0)$ of $m(S^n\times D^2)$ inside Z_*. Then $\bar\alpha^{-1}(L_*) = L$. Let D_* be the disjoint union of m copies of D^{n+1} in Z_* such that

(1) $\partial D_* = L_*$;

(2) if $e: X_* \to \bigvee^m S^1$ is the map given by the Thom–Pontryagin construction of the framed submanifold $D_* \cap X_*$ of X_*, then $e_* : \pi_1(X_*) \to F_m$ sends the ith generator to the ith generator for each $i = 1, \ldots, m$;

(3) $\bar\alpha$ is transverse to D_*.

$\bar\alpha : S^{n+2} \to Z_*$ extends to a map $\tilde\alpha : D^{n+3} \to U_*$ by taking cones. By a small homotopy of $\tilde\alpha$ relative to the boundary, $\tilde\alpha$ becomes transverse to $m(D^{n+1}\times 0)$ in U_*. Also we may assume that $\tilde\alpha^{-1}(m(D^{n+1}\times D^2))$ is a tubular neighbourhood of $\tilde\alpha^{-1}(m(D^{n+1}\times 0))$ whose intersection with the ambient sphere S^{n+2} is the tubular neighbourhood of L. Let $\tilde b$ be a bundle map covering $\tilde\alpha$, of trivial bundles so that $\tilde b|_{\tilde\alpha^{-1}(m(D^{n+1}\times D^2))}$ preserves the tubular neighbourhood structure, or is a SO(2)-bundle map. Let

$$V = D^{n+3}\backslash \tilde\alpha^{-1}(m(D^{n+1}\times \mathrm{int} D^2)) \quad \text{and} \quad T = \tilde\alpha^{-1}(m(D^{n+1}\times \partial D^2)).$$

We now define a normal map of triads

$$(f,b) : (V,T,X) \to (V_*, T_*, X_*)$$

by the restriction of $(\tilde\alpha, \tilde b)$. We call this normal map (f,b) of triads a Seifert normal map for an F_m-link (L,θ).

There is a nicer way, which justifies the name of the normal map, to look at the triad (V,T,X). We had D_* in Z_* such that $\bar\alpha^{-1}(D_*)$ is a Seifert surface of (L,θ). Let Δ_* be the disjoint union of m copies of D^{n+2} in U_* such that

(1) $\partial\Delta_* = D_* \cup_{L_*} m(D^{n+1}\times 0)$ where $m(D^{n+1}\times 0) \subset m(D^{n+1}\times D^2) \subset U_*$;

(2) Δ_* lies on U_* parallel to the radial direction of the cone on which we have extended $\bar\alpha$ to $\tilde\alpha$;

(3) Δ_* does not contain the cone point of U_*.

Then $\tilde\alpha$ is automatically transverse to Δ_*. It is easy to see that $\tilde\alpha^{-1}(m(D^{n+1}\times 0))$ is the Seifert surface $\bar\alpha^{-1}(D_*)$ pushed into the D^{n+3} except the boundary which is the link L. In fact the trace of the isotopy pushing $\bar\alpha^{-1}(D_*)$ into the D^{n+3} to place it on

$\tilde{\alpha}^{-1}(m(D^{n+1}\times 0))$ can be identified with $\tilde{\alpha}^{-1}(\Delta_*)$ and looks like the quotient

$$(\bar{\alpha}^{-1}(D_*)\times I)/(\bar{\alpha}^{-1}(D_*)\times \partial I).$$

Thus T is a part of boundary of a tubular neighbourhood of the Seifert surface pushed in, and V is an exterior of the Seifert surface pushed into D^{n+3}.

2. *Surgery obstructions*

Given an F_m-link (L,θ), let $(f,b):(V,T,X)\to(V_*,T_*,X_*)$ be a Seifert normal map. Then $f|_X$ induces isomorphisms of integral homology groups and $f|_{\partial X}$ is a homotopy equivalence. Let Φ_m be the diagram

$$\begin{array}{ccc} \mathbb{Z}[\pi(T_*)] & \xrightarrow{\text{id}} & \mathbb{Z}[\pi(T_*)] \\ {\scriptstyle i_*}\downarrow & & \downarrow{\scriptstyle a} \\ \mathbb{Z}[\pi_1(V_*)] & \xrightarrow[a]{} & \mathbb{Z} \end{array}$$

where rings are integral group rings of fundamental groupoids and a denotes the augmentation maps and i_* is induced by the inclusion: $T_*\to V_*$. Thus Φ_m is simply the integral group ring of the diagram

$$\begin{array}{ccc} \mathbb{Z}\cup\cdots\cup\mathbb{Z} & \longrightarrow & \mathbb{Z}\cup\cdots\cup\mathbb{Z} \\ {\scriptstyle i_*}\downarrow & & \downarrow \\ F_m & \longrightarrow & \{e\}. \end{array}$$

According to [1] (see also [2]), one can define a surgery obstruction $\sigma(f,b)$ as an element of $\Gamma_{n+3}(\Phi_m)$. Since $\text{wh}(F_m)=0$, we are omitting the s or h superscript. This is the obstruction to finding a normal map

$$(f',b'):(V',T',X)\to(V_*,T_*,X_*)$$

normally cobordant to (f,b) relative X such that f' induces isomorphisms of integral homology groups and $f'|_{T'}$ is a homotopy equivalence.

$\tau(m,n)$ denotes the map taking the surgery obstruction $\sigma(f,b)$ of a Seifert normal map (f,b) for a given F_m-link (L,θ).

PROPOSITION 1. *The map $\tau(m,n):C_n(F_m)\to\Gamma_{n+3}(\Phi_m)$ is well defined for $n\geqslant 2$.*

Proof. Let $(f_i,b_i):(V_i,T_i,X_i)\to(V_*,T_*,X_*)$ be Seifert normal maps for F_m-links (L_i,θ_i) for $i=0,1$. Then we want to show that if (L_0,θ_0) is F_m-cobordant to (L_1,θ_1) then $\sigma(f_0,b_0)=\sigma(f_1,b_1)$ in $\Gamma_{n+3}(\Phi_m)$.

Let $\mathbf{L}$ be an F_m-cobordism of (L_0,θ_0) to (L_1,θ_1) in $S^{n+2}\times I$ and $T(\mathbf{L})$ a tubular neighbourhood of $\mathbf{L}$ meeting $S^{n+2}\times i$ at the given tubular neighbourhood of L_i for $i=0,1$. Let $\mathbf{X}$ be the closure of $S^{n+2}\times I\backslash T(\mathbf{L})$. According to [2], proposition 2·3, there are maps

$$A:\mathbf{X}\to X_* \quad\text{and}\quad \beta:(X_*,Y_*)\to(X_*,Y_*)$$

such that

(1) A is an extension of the obvious homotopy equivalence

$$A_\partial:\partial\mathbf{X}\backslash\text{int}\,(X_0\cup X_1)\to Y_*;$$

(2) $A|_{X_1} = \alpha_1$ and $A|_{X_0} = \beta \circ \alpha_0$ where $\alpha_i : (X_i, Y_i) \to (X_*, Y_*)$ are the maps which arise from the construction of the Seifert normal maps;

(3) $\beta|_{Y_*}$ is the identity and β induces the identity on $\pi_1 X_*$ and β is a (simple) homotopy equivalence;

(4) A induces isomorphisms on integral homology groups.

By the construction of β (see the proof of 2·3 in [2]), β is actually the identity outside the $(n+2)$-cell of X_* which can be assumed to have no intersection with D_*. Thus the triad (V_0, T_0, X_0) can be assumed to be also obtained from $\beta \circ \alpha_0$. Let $\hat{\beta}$ be any bundle map whose domain is the target bundle of b_0. Let

$$(\beta \circ f_0, \hat{\beta} \circ b_0) : (V_0, T_0, X_0) \to (V_*, T_*, X_*)$$

be another Seifert normal map for (L_0, θ_0) obtained from using $\beta \circ \alpha_0 : X_0 \to X_*$ and the bundle map $\hat{\beta} \circ b_0$ covering it. Then we have $\sigma(\beta \circ f_0, \hat{\beta} \circ b_0) = \sigma(f_0, b_0)$ by the functorial property of this obstruction since β induces the identity on the fundamental group (see the proof of proposition 3·2 in [2] and theorem 3·1 and the following discussion in [1]).

We construct a map $\bar{A} : S^{n+2} \times I \to Z_*$ by putting $T(\mathbf{L})$ back to $\mathbf{X}$, that is, by glueing the obvious homotopy equivalence: $T(\mathbf{L}) \to m(S^n \times D^2)$ together with $A : \mathbf{X} \to X_*$ along $\partial \mathbf{X} \setminus \operatorname{int}(X_0 \cup X_1)$. Let $\bar{A}_t$ be the restriction $\bar{A}|_{S^{n+2} \times t} : S^{n+2} \times t \to Z_*$ for $t \in I$. For each $t \in I$, $\bar{A}_t$ extends to a map $\tilde{A}_t : D^{n+3} \times t \to U_*$ by taking cones and $\tilde{A} : D^{n+3} \times I \to U_*$ is defined by $\tilde{A}|_{D^{n+3} \times t} = \tilde{A}_t$. Let $\tilde{B}$ be a bundle map over $\tilde{A}$ of trivial bundles extending $\hat{\beta} \circ \tilde{b}_0$ and $\tilde{b}_1$ which cover $\widetilde{\beta \circ \alpha_0}$ and $\tilde{\alpha}_1$. By a small homotopy of $\tilde{A}$ relative to the boundary, we can assume that

(1) $\tilde{A}$ is transverse to $m(D^{n+1} \times D^2)$ in D_*;

(2) $\tilde{B}|_{\tilde{A}^{-1}(m(D^{n+1} \times D^2))}$ is a SO(2)-bundle map.

We put

$$\mathbf{V} = D^{n+3} \times I \setminus \tilde{A}^{-1}(m(D^{n+1} \times \operatorname{int} D^2))$$

and

$$\mathbf{T} = \tilde{A}^{-1}(m(D^{n+1} \times \partial D^2)).$$

Let $(F, B) : (\mathbf{V}, \mathbf{T}, \mathbf{X}) \to (V_*, T_*, X_*)$ be the normal map given by the restriction of $(\tilde{A}, \tilde{B})$. Then it is now clear by the construction of $(\tilde{A}, \tilde{B})$ that (F, B) is a normal bordism between normal maps $(\beta \circ f_0, \hat{\beta} \circ b_0)$ and (f_1, b_1). We glue the restricted normal map

$$(F, B)|_{\mathbf{X}} : (\mathbf{X}, \partial \mathbf{X} \setminus \operatorname{int}(X_0 \cup X_1)) \to (X_*, Y_*)$$

together with $(f_1, b_1) : (V_1, T_1, X_1) \to (V_*, T_*, X_*)$ along X_1, and denote the resulting normal map by (f_1', b_1'). But one may consider

$$(F, B)|_{\mathbf{X}} : (\mathbf{X}, X_0, X_1, \partial \mathbf{X} \setminus \operatorname{int}(X_0 \cup X_1)) \to (X_* \times I, X_* \times 0, X_* \times 1, Y \times I)$$

as a Cappell–Shaneson complementary normal cobordism. Since $F|_{\mathbf{X}} = A$ induces isomorphisms on integral homology groups and $F|_{\partial \mathbf{X} \setminus \operatorname{int}(X_0 \cup X_1)}$ is a homotopy equivalence, $\sigma((F, B)|_{\mathbf{X}}) = 0$ in $\Gamma_{n+3}(\Phi_m)$. We have

$$\sigma(f_1, b_1) = \sigma(f_1', b_1') \quad \text{in } \Gamma_{n+3}(\Phi_m)$$

by the additivity of obstructions.

We define the product normal map

$$(\beta \circ f_0, \hat{\beta} \circ b_0) \times I : (V_0, T_0, X_0) \times I \to (V_*, T_*, X_*)$$

by composing the projection: $(V_0, T_0, X_0) \times I \to (V_0, T_0, X_0)$ with $(\beta \circ f_0, \hat{\beta} \circ b_0)$, and we glue $(\beta \circ f_0, \hat{\beta} \circ b_0) \times I$ and (F, B) together along (V_0, T_0, X_0) to get a new normal map (F', B'). Then it is easy to see that the normal maps $(\beta \circ f_0, \hat{\beta} \circ b_0)$ and (f_1', b_1') are normally cobordant relative X_0 by the normal map (F', B'). Thus we have

$$\sigma(\beta \circ f_0, \hat{\beta} \circ b_0) = \sigma(f_1', b_1') \quad \text{in } \Gamma_{n+3}(\Phi_m).$$

Recall the map $i_* : \mathbb{Z} \cup \ldots \cup \mathbb{Z} \to F_m$ induced by the inclusion: $T_* \to V_*$. The natural map: $L_{n+3}(i_*) \to \Gamma_{n+3}(\Phi_m)$ is injective (see [**2**]) and so we identify $L_{n+3}(i_*)$ with its image. According to theorem 4·1 of [**2**], the assignment of the surgery obstruction $\sigma(G, C)$ of the complementary normal cobordism (G, C) to an F_m-link of dimension n induces an isomorphism

$$\rho(m, n) : C_n(F_m) \to \Gamma_{n+3}(\Phi_m)/L_{n+3}(i_*), \quad \text{for } n \geqslant 2, n \neq 3.$$

When $n = 3$, $\rho(m, n)$ is an isomorphism onto a subgroup of $\Gamma_{n+3}(\Phi_m)/L_{n+3}(i_*)$ of index 2^m.

Proposition 1 says that the surgery obstruction given by $\tau(m, n)$ does not have the ambiguity of $L_{n+3}(i_*)$ (compare [**2**], proposition 3·2).

PROPOSITION 2. *For* $n \geqslant 2$, $\tau(m, n) = \rho(m, n)$ modulo $L_{n+3}(i_*)$.

Proof. Let (L, θ) be an F_m-link and $(f, b) : (V, T, X) \to (V_*, T_*, X_*)$ be its Seifert normal map for (L, θ). Let

$$(G, C) : (W, \partial_- W, \partial_+ W, \partial_0 W) \to (X_* \times I, X_* \times 0, X_* \times 1, Y_* \times I)$$

be a Cappell–Shaneson complementary normal cobordism for (L, θ). Then $\partial_- W = X_0$ is an exterior of the trivial link and $\partial_+ W = X$. Now we must show that $\sigma(f, b) - \sigma(G, C)$ is an element of $L_{n+3}(i_*)$.

By the construction of (f, b) and the construction of (G, C) (see [**2**], lemma 3·1), we may glue $-(G, C)$ and (f, b) together along X to get a new normal map of triads

$$(f', b') : (V \underset{X}{\cup} W, T \underset{\partial X}{\cup} \partial_0 W, X_0) \to (V_* \underset{X_* \times 1}{\cup} X_* \times I, T_* \underset{Y_* \times 1}{\cup} Y_* \times I, X_*)$$

where $-(G, C)$ is the upside-down normal map of (G, C). But the target of the new normal map is obviously (V_*, T_*, X_*) again. By the additivity of the surgery obstruction,

$$\sigma(f', b') = \sigma(f, b) - \sigma(G, C).$$

Since $(f', b')|_{X_0}$ is already a homotopy equivalence, $\sigma(f', b')$ is an element of $L_{n+3}(i_*)$.

According to proposition 5·2 and the following discussion in [**2**], the injection from $L_{n+3}(i_*)$ to $\Gamma_{n+3}(\Phi_m)$ becomes an isomorphism for n even, while $L_{n+3}(i_*) = 0$ for n odd. Thus $\tau(m, n) = \rho(m, n)$ for all odd $n \geqslant 3$.

Let $\mathfrak{F}_m : \mathbb{Z}[F_m] \to \mathbb{Z}$ be the augmentation map. Since the natural map from $L_{n+3}(F_m)$ to $\Gamma_{n+3}(\mathfrak{F}_m)$ is injective for n odd, we identify $L_{n+3}(F_m)$ with its image (see [**1**]). Define $\tilde{\Gamma}_{n+3}(\mathfrak{F}_m) = \Gamma_{n+3}(\mathfrak{F}_m)/L_{n+3}(F_m)$. The kernel of the natural map from $\Gamma_{n+3}(\mathfrak{F}_m)$ to $\Gamma_{n+3}(\Phi_m)$ contains $L_{n+3}(F_m)$ for n odd. In fact $L_{n+3}(F_m)$ is equal to the image $(L_{n+3}(\mathbb{Z} \cup \ldots \cup \mathbb{Z}) \to \Gamma_{n+3}(\mathfrak{F}_m))$ and

$$L_{n+3}(\mathbb{Z} \cup \ldots \cup \mathbb{Z}) \to \Gamma_{n+3}(\mathfrak{F}_m) \to \Gamma_{n+3}(\Phi_m) \to 0$$

is exact (see [**2**]). So there is an induced map: $\tilde{\Gamma}_{n+3}(\mathfrak{F}_m) \to \Gamma_{n+3}(\Phi_m)$. According to [**2**], theorem 6·2, for $n = 2k-1 \geqslant 5$ there is a split short exact sequence

$$0 \to \tilde{\Gamma}_{n+3}(\mathfrak{F}_m) \xrightarrow{\gamma} C_n(F_m) \xrightarrow{\Delta} mP_{n+1} \to 0$$

where Δ is given by the signature or the Arf invariant of the components viewed as knots and γ is the composite of the induced map $\tilde{\Gamma}_{n+3}(\mathfrak{F}_m) \to \Gamma_{n+3}(\Phi_m)$ with $\rho(m,n)^{-1}$. When $n = 3$, we replace Δ by $\frac{1}{2}\Delta$ and we can still have such a split short exact sequence. Furthermore this sequence respects the natural action of $\mathfrak{U}_m$ with the trivial action on mP_{n+1}.

We recall some results from [5]. $G(m,\epsilon)$ was the set of cobördism classes of Seifert matrix of type (m,ϵ). Then the maps taking Seifert matrices

$$\phi(m,k): C_{2k-1}(F_m) \to G(m,(-1)^k) \quad \text{for } k \geqslant 3$$

$$\phi(m,2): C_3(F_m) \to G^0(m,+1)$$

are isomorphisms. Let $K(m,(-1)^k)$ be the kernel of the (split) surjective map

$$\Delta: G(m,(-1)^k) \to mP_{2k}$$

taking the signature or the Arf invariant of diagonal blocks. Then we have split exact sequences

$$0 \to K(m,(-1)^k) \xrightarrow{r} C_{2k-1}(F_m) \xrightarrow{\Delta} mP_{2k} \to 0 \quad (k \geqslant 3),$$

or

$$0 \to K(m,+1) \xrightarrow{r} C_3(F_m) \xrightarrow{\frac{1}{2}\Delta} mp_4 \to 0$$

where r is $\phi^{-1}(m,k)$, that is, the realization of Seifert matrices (see [5], theorem 3·4). This sequence is also equivariant under the action of $\mathfrak{U}_m$ with the trivial action on mP_{2k} (see [5]).

Comparing this sequence with Cappell–Shaneson's short exact sequence, we must have an isomorphism

$$\psi(m,\epsilon): K(m,\epsilon) \to \tilde{\Gamma}_{2k+2}(\mathfrak{F}_m), \quad \epsilon = (-1)^k$$

preserving $\mathfrak{U}_m$-actions and $\psi(m,\epsilon)$ is given by $\gamma^{-1} \circ r$.

3. *Computations of surgery obstructions*

Now we try to find the map $\psi(m,\epsilon)$ explicitly. A main tool will be the fact that when we define γ^{-1}, we can use $\tau(m,2k-1)$ instead of $\rho(m,2k-1)$ because they are the same map. Since $\psi(m,\epsilon)$ depends only on k modulo 2, we will assume that $k \geqslant 3$.

By the definition of $K(m,\epsilon)$, any class of $K(m,\epsilon)$ has a representative Seifert matrix $A = (A_{ij})$ of type (m,ϵ) with the property that for each $i = 1, \ldots, m$

$$\text{when } \epsilon = 1, \quad A_{ii} + A_{ii}^T = \begin{pmatrix} 0 & I_i \\ I_i & * \end{pmatrix}$$

(*) $$\text{when } \epsilon = -1, \quad A_{ii} - A_{ii}^T = \begin{pmatrix} 0 & I_i \\ -I_i & 0 \end{pmatrix}$$

and a_{ss}^i is even for $s = 1, \ldots, l_i$, where A_{ij} are $(2l_i \times 2l_j)$-matrices and a^i_{ss} is the (s,s) entry of A_{ii} and I_i are $(l_i \times l_i)$ identity matrices for $i,j = 1,\ldots,m$.

We will fix $A = (A_{ij})$ and I_i's in the form given as above in the rest of this paper, and we understand that $\epsilon = (-1)^k$.

Let $M = M_1 \cup \ldots \cup M_m$ be a Seifert surface in S^{2k+1} realizing the Seifert matrix A of type (m,ϵ) (see [5], theorem 3·4). Then M is a $2k$-dimensional handle body with m 0-handles and $2l_i$ k-handles on each ith 0-handle. So M is an ordered disjoint union of m $(k-1)$-connected submanifolds $M_1,\ldots,M_m$ in S^{2k+1}. The Seifert pairing

$$\sigma: H_k M \times H_k M \to \mathbb{Z}$$

is represented by the matrix A on the basis of $H_k M$ given by the k-handles of M. ∂M is a simple m-link of dimension $2k-1$ with the splitting map id_{F_m}. We denote this F_m-link by (L_A, id).

Let $(f,b):(V,T,X) \to (V_*,T_*,X_*)$ be a Seifert normal map for (L_A,id). Then the surgery obstruction $\sigma(f,b)$ in $\Gamma_{2k+2}(\Phi_m)$ is the image of $[A]$ under $\tau(m,2k-1) \circ \Phi(m,2k-1)^{-1}$. Since $[A] \in K(m,\epsilon)$, we have $\Delta(\sigma(f,b)) = 0$ in mP_{2k}. We are looking for an element ξ in $\tilde{\Gamma}_{2k+2}(\mathfrak{F}_m)$ such that $\sigma(f,b)$ is the image of ξ under the injection $\tilde{\Gamma}_{2k+2}(\mathfrak{F}_m) \to \Gamma_{2k+2}(\Phi_m)$. Since $\sigma((f,b)|_T) = \Delta(\sigma(f,b)) = 0$ in $mP_{2k} \cong L_{2k+1}(\mathbb{Z} \cup \ldots \cup \mathbb{Z})$, the normal map $(f,b)|_T$ is normally cobordant, relative to the boundary, to a homotopy equivalence. This normal cobordism can be realized as $N \times S^1$ where N is a trace of framed interior surgeries which make M into mD^{2k}.

We recall that T is the boundary of a tubular neighbourhood of M pushed into the D^{2k+2}. We attach $N \times S^1$ to V along T and denote the result by W, and extend the normal map $(f,b):(V,T,X) \to (V_*,T_*,X_*)$ to a normal map

$$(f',b'):(W, m(D^{2k} \times S^1), X) \to (V_*,T_*,X_*)$$

(see for example [12], chapter 1). Then the surgery obstruction $\sigma(f',b')$ is an element of $\Gamma_{2k+2}(\mathfrak{F})$ because $f'|_{\partial W}$ induces at least isomorphisms on integral homology groups. The quotient of $\sigma(f',b')$ modulo $L_{2k+2}(F_m)$ is the element ξ in $\tilde{\Gamma}_{2k+2}(\mathfrak{F})$ which we were looking for (see the proof of 7·1 in [2]).

For $i = 1,\ldots,m$ let $\{\alpha^i_1,\ldots,\alpha^i_{2l_i}\}$ be the basis of $H_k M_i$ given by k-handles of M_i such that the Seifert pairing $\sigma: H_k M \times H_k M \to \mathbb{Z}$ with respect to this basis of $H_k M$ gives the Seifert matrix A. It follows from the property (*) of the Seifert matrix A that we can kill $\alpha^i_1,\ldots,\alpha^i_{l_i}$ by framed surgeries on M_i. Let N_i be the trace of these surgeries on M_i, that is,

$$N_i = M_i \times I \cup h^i_1 \cup \ldots \cup h^i_{l_i}$$

where h^i_j are $(k+1)$-handles with the attaching spheres α^i_j. Then

$$\partial N_i = M_i \cup \partial M_i \times I \cup D^{2k}.$$

We take the ordered disjoint union $N = N_1 \cup \ldots \cup N_m$.

By using Van Kampen's theorem and the 1-connectedness of the M_i's one can check that the map $f: V \to V_*$ induces an isomorphism on fundamental groups and so does the map $f': W \to V_*$. Let $\tilde{V}$ and $\tilde{W}$ be the universal covers of V and W, respectively. Then integral homology groups $H_*\tilde{V}$ and $H_*\tilde{W}$ are $\mathbb{Z}[F_m]$-modules.

$\tilde{V}$ can be constructed as follows. Let $S = S_1 \cup \ldots \cup S_m$ be the trace of the isotopy pushing the Seifert surface $M = M_1 \cup \ldots \cup M_m$ into V; more precisely $S = \bar{\alpha}^{-1}(\Delta_*) \cap V$ (see §1). We note that S is homeomorphic to $M \times I$. If we cut V open along S, we have a manifold B which is homeomorphic to D^{2k+2}. Two copies S^+, S^- of S are parts of ∂B. We take a copy $B(w)$ of B for each $w \in F_m$. Let $x_1,\ldots,x_m$ be generators of F_m. We identify $S_i^+(w)$ of $B(w)$ with $S_i^-(x_i w)$ of $B(x_i w)$ for all $w \in F_m$ and $i = 1, \ldots, m$. Then the result is $\tilde{V}$.

We consider

$$B_{\text{odd}} = \bigcup_{l(w)=\text{odd}} B(w) \quad \text{and} \quad B_{\text{even}} = \bigcup_{l(w)=\text{even}} B(w)$$

where l is the length of (freely reduced) words of F_m. Then we have in $\tilde{V}$ that $B_{\text{odd}} \cap B_{\text{even}} = \bigcup_{w \in F_m} S(w)$. Let $\Lambda_m = \mathbb{Z}[F_m]$. The Meyer–Vietoris sequence of the couple $(B_{\text{odd}}, B_{\text{even}})$ is given as

$$\ldots \to H_i B \bigotimes_{\mathbf{Z}} \Lambda_m \to H_i \tilde{V} \to H_{i-1} S \bigotimes_{\mathbf{Z}} \Lambda_m \to \ldots.$$

The sequence is one of Λ_m-modules. Since B is homeomorphic to D^{2k+2} and S is homeomorphic to $M \times I$, we have that $\tilde{H}_i \tilde{V} = 0$ for $i \neq k+1$ and $H_{k+1} \tilde{V}$ is isomorphic to the free Λ_m-module over $2(l_1 + \ldots + l_m)$ generators. We choose a canonical basis

$$\{\hat{\alpha}_j^i\}_{i=1,\ldots,m \text{ and } j=1,\ldots,2l_i}$$

of $H_{k+1} \tilde{V}$ as follows. For $i = 1, \ldots, m$ and $j = 1, \ldots, l_i$ the cycle which is in $S_i^+(e) = S_i^-(x_i)$ and represents the homology class α_j^i of $H_k M_i \cong H_k S$ bounds a chain $c_+ \alpha_j^i$ in $B(x_i)$ and bounds a chain $c_- \alpha_j^i$ in $B(e)$, then we let $\hat{\alpha}_j^i$ be the homology class represented by the cycle $c_- \alpha_j^i \cup c_+ \alpha_j^i$ in V where e is the identity element of F_m and we give the chains orientations so that $\partial c_+ \alpha_j^i = \partial c_- \alpha_j^i$ with orientations. At this point we calculate the intersection pairings λ among these generators of $H_{k+1} \tilde{V}$, which will be used later. For $i, j = 1, \ldots, m$ and $r = 1, \ldots, l_i$ and $j = 1, \ldots, l_j$

$$\lambda(\hat{\alpha}_r^i, \hat{\alpha}_s^j) = \sum_{w \in F_m} (\hat{\alpha}_r^i . w(\hat{\alpha}_s^j))\, w.$$

But it is easy to see that $\hat{\alpha}_r^i . w(\hat{\alpha}_s^i) = 0$ unless $w = e, x_i, x_i^{-1}$. We may assume that the cycles $\hat{\alpha}_r^i$ and $\hat{\alpha}_s^i$ do not intersect each other on $S_i^+(e)$ by a small homotopy of $\hat{\alpha}_s^i$; this can be done, for example, by a small translation on $S_i^+(e) \approx M \times I$ in the I-direction. In the following computation, lk is the linking pairing in the ambient sphere S^{2k+1} and i_+ and $i_- : M \to X$ are small translations off M along the positive and negative direction of M.

$$\begin{aligned}
\hat{\alpha}_r^i . \hat{\alpha}_s^i &= c_- \alpha_r^i . c_- \alpha_s^i + c_+ \alpha_r^i . c_+ \alpha_s^i = \mathrm{lk}\,(\alpha_r^i, i_+ \alpha_s^i) + \mathrm{lk}\,(\alpha_r^i, i_- \alpha_s^i) \\
&= \mathrm{lk}\,(\alpha_r^i, i_+ \alpha_s^i) - \epsilon\, \mathrm{lk}\,(\alpha_s^i, i_+ \alpha_r^i). \\
\hat{\alpha}_r^i . x_i(\hat{\alpha}_s^i) &= c_+ \alpha_r^i . x_i(c_- \alpha_s^i) = c_+ \alpha_r^i . i_-(-c_+ \alpha_s^i) \\
&= \mathrm{lk}\,(\alpha_r^i, i_-(-\alpha_s^i)) = \epsilon\, \mathrm{lk}\,(\alpha_s^i, i_+ \alpha_r^i). \\
\hat{\alpha}_r^i . x_i^{-1}(\hat{\alpha}_s^i) &= c_- \alpha_r^i . x_i^{-1}(c_+ \alpha_s^i) = c_- \alpha_r^i . i_+(-c_- \alpha_s^i) \\
&= \mathrm{lk}\,(\alpha_r^i, i_+(-\alpha_s^i)) = -\mathrm{lk}\,(\alpha_r^i, i_+ \alpha_s^i).
\end{aligned}$$

For $i \neq j$, it is easy to see that $\hat{\alpha}_r^i . w(\hat{\alpha}_s^i) = 0$ unless $w = e, x_i, x_j^{-1}, x_i x_j^{-1}$.

$$\begin{aligned}
\hat{\alpha}_r^i . \hat{\alpha}_s^j &= c_- \alpha_r^i . c_- \alpha_s^j = \mathrm{lk}\,(\alpha_r^i, i_+ \alpha_s^j). \\
\hat{\alpha}_r^i . x_i(\hat{\alpha}_s^j) &= c_+ \alpha_r^i . x_i(c_- \alpha_s^j) = c_+ \alpha_r^i . i_-(-c_+ \alpha_s^j) \\
&= -\mathrm{lk}\,(\alpha_r^i, i_- \alpha_s^j) = -\mathrm{lk}\,(\alpha_r^i, i_+ \alpha_s^j)
\end{aligned}$$

(because α_r^i, α_s^j lie on distinct surfaces).

$$\hat{\alpha}^i_r . x_j^{-1}(\hat{\alpha}^j_s) = c_-\alpha^i_r . x_j^{-1}(c_+\alpha^j_s) = c_-\alpha^i_r . i_+(-c_-\alpha^j_s) = -\operatorname{lk}(\alpha^i_r, i_+\alpha^j_s).$$

$$\hat{\alpha}^i_r . x_i x_j^{-1}(\hat{\alpha}^j_s) = c_+\alpha^i_r . x_i x_j^{-1}(c_+\alpha^j_s) = c_+\alpha^i_r . c_+\alpha^j_s = \operatorname{lk}(\alpha^i_r, i_+\alpha^j_s).$$

We recall that the Seifert pairing

$$\sigma : H_k M \times H_k M \to \mathbb{Z}$$

is given by $\sigma(\alpha, \beta) = \operatorname{lk}(\alpha, i_+\beta)$ and gives the matrix $A = (A_{ij})$ on the basis $\{\alpha^i_r\}$. Then the intersection pairing λ on $\tilde{V}$ is represented by a matrix $\hat{A} = (\hat{A}_{ij})$ on the basis $\{\hat{\alpha}^i_r\}$ where

$$\hat{A}_{ii} = (1-x_i^{-1})A_{ii} - \epsilon(1-x_i)A^T_{ii}$$

and

$$\hat{A}_{ij} = (1-x_i)(1-x_j^{-1})A_{ij} \quad \text{for} \quad i \neq j.$$

Further the quadratic function μ associated to the intersection pairing λ is then given by

$$\mu(\hat{\alpha}^i_r) \equiv (1-x_i^{-1})\,\sigma(\alpha^i_r, \alpha^i_r) \mod \Omega(-\epsilon)$$

where $\Omega(-\epsilon) = \{p + \epsilon p^* \mid p \in \Lambda_m\}$ is a subgroup of Λ_m.

We now construct $\tilde{W}$ from $\tilde{V}$. We recall that $M = M_1 \cup \ldots \cup M_m$ is a Seifert surface of (L_A, id), and $N = N_1 \cup \ldots \cup N_m$ is the trace of surgeries on M, and $W = V \cup_T (N \times S^1)$, $T = M \times S^1$, $\partial V = X \cup_{\partial X} T$, and X is an exterior of L_A and so $\pi_1 X \cong F_m$. Let $F_m/\langle x_i\rangle$ be the set of cosets of the cyclic group $\langle x_i \rangle$ for $i = 1, \ldots, m$ and let $\tilde{X}$ be the universal cover of X. Then

$$\partial\tilde{V} = \tilde{X} \cup \left(\bigcup_{i=1}^m \left(\bigcup_{w \in F_m/\langle x_i\rangle} (M_i \times \mathbb{R}^1)_w\right)\right).$$

We attach $\bigcup_{w \in F_m/\langle x_i\rangle}(N_i \times \mathbb{R}^1)_w$ to $\tilde{V}$ along $\bigcup_{w \in F_m/\langle x_i\rangle}(M_i \times \mathbb{R}^1)_w$ for all $i = 1, \ldots, m$. Then what we get is $\tilde{W}$.

Using the Meyer–Vietoris sequence, we have a sequence of Λ_m-modules

$$0 \to H_{k+1}\tilde{V} \xrightarrow{j_*} H_{k+1}\tilde{W} \xrightarrow{\partial} \bigoplus_{i=1}^m (H_k M_i \otimes_{\mathbb{Z}} \Lambda_m/(1-x_i)\Lambda_m)$$

$$\xrightarrow{i_*} \bigoplus_{i=1}^m (H_k N_i \otimes_{\mathbb{Z}} \Lambda_m/(1-x_i)\Lambda_m) \to H_k\tilde{W} \to 0.$$

We now recall that for $i = 1, \ldots, m$

$$H_* M_i \cong \begin{cases} \mathbb{Z}\langle \alpha^i_1, \ldots, \alpha^i_{2l_i}\rangle & \text{for} \quad * = k \\ 0 & \text{otherwise} \end{cases}$$

$$H_* N_i \cong \begin{cases} \mathbb{Z}\langle \alpha^i_{l_i+1}, \ldots, \alpha^i_{2l_i}\rangle & \text{for} \quad * = k \\ 0 & \text{otherwise} \end{cases}$$

$$H_* \tilde{V} \cong \begin{cases} \Lambda_m\langle \hat{\alpha}^i_r : i = 1, \ldots, m, r = 1, \ldots, 2l_i\rangle & \text{for} \quad * = k+1 \\ 0 & \text{otherwise.} \end{cases}$$

By the construction of N, the map i_* in the above sequence is just a projection. Thus $H_k\tilde{W} = 0$. Moreover we can identify $\ker i_*$ with

$$\bigoplus_{i=1}^m (\Lambda_m/(1-x_i)\Lambda_m\langle \alpha^i_r \otimes 1 : r = 1, \ldots, l_i\rangle).$$

We will show that $H_{k+1}\tilde{W}$ is a free Λ_m-module by determining the extensions of the given bases of $H_{k+1}\tilde{V}$ and $\ker i_*$ as follows. For $i = 1, \ldots, m$ and $r = 1, \ldots, l_i$, the cycle which is in $S_i^+(e)$ and represents α_r^i, now bounds a chain $c\alpha_r^i$ in a copy of $N_i \times \mathbb{R}^1$ in $\tilde{W}$. We let

$$\tilde{\alpha}_r^i = c_- \alpha_r^i \cup c\alpha_r^i;$$

then $\partial(\tilde{\alpha}_r^i) = \alpha_r^i \otimes 1$. Since $-x_i \tilde{\alpha}_r^i = -((-c_+ \alpha_r^i) \cup (c\alpha_r^i)) = c_+ \alpha_r^i \cup (-c\alpha_r^i)$, we have $j_*(\hat{\alpha}_r^i) = (1-x_i)\tilde{\alpha}_r^i$. For $i = 1, \ldots, m$ and $r = l_i+1, \ldots, 2l_i$, we let

$$\tilde{\alpha}_r^i = j_*(\hat{\alpha}_r^i).$$

Thus with respect to these bases, j_* is represented by a matrix J which is a block sum of matrices

$$J_i = \begin{pmatrix} (1-x_i) I_i & 0 \\ 0 & I_i \end{pmatrix} \quad \text{for} \quad i = 1, \ldots, m.$$

We recall the normal map $(f', b'):(W, \partial W) \to (V_*, T_*, \cup_{Y_*} X_*)$. $f'|_{\partial W}$ is an integral homology equivalence and we now know that f' is $(k+1)$-connected. Let λ and μ be the intersection pairing and the associated quadratic function on $\tilde{W}$. Then the surgery obstruction (f', b') is the triple $(H_{k+1} \tilde{W}, \lambda, \mu)$ as an element of $\Gamma_{2k+2}(\mathfrak{F}_m)$. Thus the isomorphism $\psi: K(m, \epsilon) \to \tilde{\Gamma}_{2k+2}(\mathfrak{F}_m)$ sends a cobordism class of a Seifert matrix A to a triple $(H_{k+1} \tilde{W}, \lambda, \mu)$ modulo $L_{2k+2}(F_m)$.

We now describe $\sigma(f', b') = (H_{k+1} \tilde{W}, \lambda, \mu)$ in terms of the Seifert matrix A. We had a canonical basis $\{\tilde{\alpha}_r^i\}$ of $H_{k+1} \tilde{W}$. For $i = 1, \ldots, m$ and $r, s = 1, \ldots, l_i$, we have $\tilde{\alpha}_r^i . w(\tilde{\alpha}_s^i) = 0$ unless $w = e$. As we did before, we may assume that the cycles $\tilde{\alpha}_r^i, \tilde{\alpha}_s^i$ do not intersect each other on $S_i^+(e)$ by translating $\tilde{\alpha}_s^i$ on $S_i^+(e)$ by i_+. Thus

$$\tilde{\alpha}_r^i . \tilde{\alpha}_s^i = (c_- \alpha_r^i . c_- \alpha_s^i) + (c\alpha_r^i . c\alpha_s^i) = \operatorname{lk}(\alpha_r^i, i_+ \alpha_s^i).$$

Similarly we can compute $\lambda(\tilde{\alpha}_r^i, \tilde{\alpha}_s^j)$ for all i, j, r, s. However by using the map $j_*: H_{k+1} \tilde{V} \to H_{k+1} \tilde{W}$, we can more easily obtain λ, μ on $\tilde{W}$ from the result about λ, μ on $\tilde{V}$. In another words we have

$$\lambda_W(\tilde{\alpha}_r^i, \tilde{\alpha}_s^j) = \lambda_V(j_*^{-1}(\tilde{\alpha}_r^i), j_*^{-1}(\tilde{\alpha}_s^j))$$

and

$$\mu_W(\tilde{\alpha}_r^i) = \mu_V(j_*^{-1}(\tilde{\alpha}_r^i))$$

where the notation is self-explanatory. We note that j_*^{-1} makes sense because of the conditions $(*)$ on A. Moreover j_*^{-1} is given by the matrix J^{-1} on our choices of bases, where J^{-1} is the block sum of the matrices J_i^{-1}

$$\begin{pmatrix} (1-x_i)^{-1} I_i & 0 \\ 0 & I_i \end{pmatrix} \quad \text{for} \quad i = 1, \ldots, m.$$

Thus λ_W is represented by a matrix $\tilde{A} = (\tilde{A}_{ij})$ given by

$$\tilde{A}_{ii} = J_i^{-1} \hat{A}_{ii} J_i^{*-1}$$

and

$$\tilde{A}_{ij} = J_i^{-1} \hat{A}_{ij} J_j^{*-1} \quad \text{for} \quad i \neq j,$$

where $*$ stands for the involution on the ring Λ_m and so

$$J_i^{*-1} = \begin{pmatrix} (1-x_i^{-1})^{-1} I_i & 0 \\ 0 & I_i \end{pmatrix} \quad \text{for} \quad i = 1, \ldots, m.$$

The quadratic function μ_W is given when $\epsilon = 1$ by

$$\mu_W(\tilde{\alpha}_r^i) \equiv \begin{cases} 0 & \text{for} \quad r = 1, \ldots, l_i \\ (1-x_i^{-1})\, \sigma(\alpha_r^i, \alpha_r^i) & \text{for} \quad r = l_i+1, \ldots, 2l_i \end{cases}$$

and when $\epsilon = -1$ by

$$\mu_W(\tilde{\alpha}_r^i) \equiv \begin{cases} \tfrac{1}{2}\sigma(\alpha_r^i, \alpha_r^i) & \text{for } \ r = 1, \ldots, l_i \\ (1-x_i^{-1})\,\sigma(\alpha_r^i, \alpha_r^i) & \text{for } \ r = l_i+1, \ldots, 2l_i \end{cases}$$

modulo $\Omega(-\epsilon) = \{p + \epsilon p^* \mid p \in \Omega_m\}$.

For the case $\epsilon = -1$, we note that

$$\left(\frac{1}{1-x_i}\right)^* (1-x_i^{-1}) \left(\frac{1}{1-x_i}\right)$$

$$\equiv (1-x_i^{-1})^* \frac{(1-x_i^{-1})(1-x_i)}{2} (1-x_i^{-1}) \left(\frac{1}{1-x_i}\right) \equiv \tfrac{1}{2} \quad \operatorname{mod} \Omega(+).$$

μ_W can be defined for all elements in $H_{k+1}\tilde{W}$ by using the formulae

$$\mu_W(\alpha+\beta) \equiv \mu_W(\alpha) + \mu_W(\beta) + \lambda_W(\alpha, \beta),$$

$$\mu_W(p\alpha) \equiv p\mu_W(\alpha)p^* \quad \text{for } \alpha, \beta \in H_{k+1}\tilde{W} \text{ and } p \in \Lambda_m.$$

4. *Main results*

The following theorem reviews our long journey.

THEOREM 3. *The isomorphism $\psi(m, \epsilon): K(m, \epsilon) \to \tilde{\Gamma}_{2k+2}(\mathfrak{F}_m)$, $\epsilon = (-1)^k$ is given by the composition*

$$\begin{array}{lll}
K(m,\epsilon) & A & \\
\downarrow & \downarrow & \\
C_{2k-1}(F_m) & (L_A, \mathrm{id}) & \\
\downarrow & \downarrow & \\
\Gamma_{2k+2}(\Phi_m) & \sigma(f,b) & (f,b):(V,T,X) \to (V_*, T_*, X_*) \\
\downarrow & \downarrow & \\
\Gamma_{2k+2}(\mathfrak{F}_m) & \sigma(f',b') & (f',b'):(W,\partial W) \to (V_*, T_* \cup X_*) \\
\downarrow & \downarrow & \\
\tilde{\Gamma}_{2k+2}(\mathfrak{F}_m) & \sigma(f',b') & \operatorname{mod} L_{2k+2}(F_m)
\end{array}$$

where $\sigma(f', b') = (\Lambda_m\langle\{\tilde{\alpha}_r^i\}\rangle, \lambda, \mu)$ and λ is represented on the basis $\{\tilde{\alpha}_r^i\}$ by $\tilde{A} = (\tilde{A}_{ij})$ given by

$$\tilde{A}_{ii} = \begin{pmatrix} (1-x_i)^{-1}I_i & \\ & I_i \end{pmatrix} A_{ii} \begin{pmatrix} I_i & \\ & (1-x_i^{-1})I_i \end{pmatrix} - \epsilon \begin{pmatrix} I_i & \\ & (1-x_i)I_i \end{pmatrix} A_{ii}^T \begin{pmatrix} (1-x_i^{-1})^{-1}I_i & \\ & I_i \end{pmatrix}$$

$$\tilde{A}_{ij} = \begin{pmatrix} I_i & \\ & (1-x_i)I_i \end{pmatrix} A_{ij} \begin{pmatrix} I_j & \\ & (I-x_j^{-1})I_j \end{pmatrix} \quad \textit{for} \quad i \neq j$$

and μ is given on the basis by

$$\mu(\tilde{\alpha}_r^i) \equiv \begin{cases} \tfrac{1}{4}(1-\epsilon)\,\sigma(\alpha_r^i, \alpha_r^i) & \operatorname{mod} \Omega(-\epsilon) \quad \textit{for } 1 \leqslant r \leqslant l_i \\ (1-x_i^{-1})\,\sigma(\alpha_r^i, \alpha_r^i) & \operatorname{mod} \Omega(-\epsilon) \quad \textit{for } l_i+1 \leqslant r \leqslant 2l_i. \end{cases}$$

THEOREM 4. *A is S-equivalent to the empty matrix if and only if $\tilde{A}$ is a non-singular matrix over Λ_m.*

Proof. It follows from [**6**], theorem 1·6·1 that A is a Seifert matrix of the trivial link if and only if A is S-equivalent to the empty Seifert matrix. We use the notation that we have been using in this section.

The associated normal map to the Seifert matrix A

$$(f', b'): (W, \partial W) \to (V_*, T_* \cup X_*)$$

has the property that $f'|_{\partial W}$ is a homotopy equivalence because X is now an exterior of the trivial link. Thus the surgery obstruction $\sigma(f', b')$ is an element of $L_{2k+2}(F_m)$.

Conversely, we suppose that $\sigma(f', b')$ is in $L_{2k+2}(F_m)$. Since X is an exterior of a simple link, $f'|_X$ is k-connected and hence $f'|_{\partial W}$ is k-connected. Since f' itself is $(k+1)$-connected, we have an exact sequence

$$0 \to H_{k+1}\partial\tilde{W} \to H_{k+1}\tilde{W} \to H_{k+1}(\tilde{W}, \partial\tilde{W}) \to H_k\partial\tilde{W} \to 0.$$

But the middle map in the sequence is the adjoint of the intersection pairing which is non-singular over Λ_m. Thus

$$H_k\partial\tilde{W} = H_{k+1}\partial\tilde{W} = 0,$$

that is, $f'|_{\partial W}$ is a homotopy equivalence. Thus the link L_A is trivial by the unlinking theorem (see [**3**]).

We recall the action of $\mathfrak{U}_m$ on $G(m, \epsilon)$ and $\Gamma_{2k+2}(\mathfrak{F}_m)$.

THEOREM 5. *The map $\psi(m, \epsilon)$ in Theorem 3 is equivariant under the action of $\mathfrak{U}_m$.*

Proof. The realization map $r: G(m, \epsilon) \to C_{2k-1}(F_m)$ sending A to (L_A, id), preserves the action, that is, $(L_{\alpha.A}, \mathrm{id})$ is F_m-cobordant to (L_A, α). In fact they are F_m-ambient isotopic (see [**6**], theorem 1·6·2). Moreover the Cappell–Shaneson map

$$\rho(m, 2k-1): C_{2k-1}(F_m) \to \Gamma_{2k+2}(\Phi_m)$$

preserves the action. The other maps in Theorem 3 arise in Γ-groups and so preserve the action.

Remark. We may prove Theorem 5 directly as follows. We recall from Theorem 3 that the intersection pairing of $\psi(m, \epsilon)(A)$ is given by the matrix $\tilde{A}$. Let α_{ij} be a generator of $\mathfrak{U}_m$ sending x_i to $x_j x_i x_j^{-1}$ and fixing other generators of F_m (see [**5**], lemma 2·4). Let $\alpha_{ij}.A = B$. The matrix B is given in [**5**]. Let $\tilde{B}$ be the intersection matrix of $\psi(m, \epsilon)(B)$. We fix some notation:

$$U_j = \begin{pmatrix} (1-x_j)^{-1}I_j & & & \\ & I_j & & \\ & & (1-x_j)^{-1}I_{2i} & \\ & & & I_{2i} \end{pmatrix}, \quad V_j = \begin{pmatrix} I_j & & & \\ & (1-x_j)I_j & & \\ & & I_{2i} & \\ & & & (1-x_j)I_{2i} \end{pmatrix}$$

$$V = \begin{pmatrix} I_i & \\ & (1-x_i)I_i \end{pmatrix}, \qquad V_s = \begin{pmatrix} I_s & \\ & (1-x_s)I_s \end{pmatrix} \quad \text{for } s \neq j.$$

We denote the involuted transpose of a matrix U over Λ_m by U^*. Then $\tilde{B} = (\tilde{B}_{st})$ is given by

$$C_{jj} = U_j B_{jj} V_j^* - \epsilon V_j B_{jj}^T U_j^* = \begin{pmatrix} \tilde{A}_{jj} & 0 & VA_{ji} \\ 0 & 0 & x_j^{-1} S_i \\ A_{ij} V^* & -x_j S_i & 0 \end{pmatrix},$$

$$C_{ji} = V_j B_{ji} V_i^* = \begin{pmatrix} 0 \\ -S_i V_i^* \\ 0 \end{pmatrix}, \quad C_{jt} = V_j B_{jt} V_t^* = \begin{pmatrix} \tilde{A}_j \\ 0 \\ (1-x_j) A_{it} V_t^* \end{pmatrix},$$

$$C_{ij} = V_i B_{ij} V_j^* = (0 \quad V_i S_i \quad 0),$$

$$C_{sj} = V_s B_{sj} V_j^* = (\tilde{A}_{sj} \quad 0 \quad V_s(1-x_j^{-1}) A_{si}),$$

$$\tilde{B}_{st} = \tilde{A}_{st}, \quad \text{for } s, t \neq i, j.$$

Let $W = (W_{st})$ be a non-singular matrix over Λ_m given by

$$W_{jj} = \begin{pmatrix} I_{2j} & -Vx_j A_{ji} S_i^{-1} & 0 \\ 0 & x_j I_{2i} & 0 \\ 0 & 0 & I_{2i} \end{pmatrix},$$

$$W_{ii} = x_j I_{2i}, \quad W_{ij} = (0 \quad 0 \quad x_j V_i x_j^{-1}),$$

$$W_{ss} = I_{2s}, \quad W_{sj} = (0 \quad V_s(1-x_j) A_{si} S_i^{-1} \quad 0), \quad \text{for } s \neq i, j$$

$$W_{st} = 0, \quad \text{otherwise.}$$

Then $W\tilde{B}W^* = (\tilde{C}_{st})$ is given by

$$\tilde{C}_{jj} = \begin{pmatrix} \tilde{A}_{jj} & 0 & 0 \\ 0 & 0 & S_i \\ 0 & -S_i & 0 \end{pmatrix}, \quad \tilde{C}_{ji} = \begin{pmatrix} x_j \tilde{A}_{ji} x_j^{-1} \\ 0 \\ 0 \end{pmatrix}, \quad \tilde{C}_{jt} = \begin{pmatrix} \tilde{A}_{jt} \\ 0 \\ 0 \end{pmatrix},$$

$$\tilde{C}_{ij} = (x_j \tilde{A}_{ij} x_j^{-1} \quad 0 \quad 0), \qquad \tilde{C}_{ii} = x_j \tilde{A}_{ii} x_j^{-1}, \qquad \tilde{C}_{it} = x_j V_i x_j^{-1} A_{it} V_s^*,$$

$$\tilde{C}_{sj} = (\tilde{A}_{sj} \quad 0 \quad 0), \qquad \tilde{C}_{si} = V_s A_{si} x_j V_i^* x_j^{-1}, \quad \tilde{C}_{st} = \tilde{A}_{st}, \quad \text{for } s, t \neq i, j.$$

We recall the $\mathfrak{U}_m$-action on $\Gamma_{2k+2}(\mathfrak{F}_m)$. The matrix $\alpha_{jj}.\tilde{A}$ is the matrix obtained from $\tilde{A}$ by substituting x_i by $x_j x_i x_j^{-1}$ in the entries of $\tilde{A}$. Then it is easy to see that $\tilde{C} = (\tilde{A}_{st})$ is congruent to

$$(\alpha_{ij}.\tilde{A}) \oplus \begin{pmatrix} 0 & S_i \\ -S_i & 0 \end{pmatrix}.$$

Since S_i is an ϵ-symmetric non-singular matrix over $\mathbb{Z}$, we have proved that $\tilde{B} = \widetilde{\alpha_{ij}.A}$ and $\alpha_{ij}.\tilde{A}$ represent the same intersection pairing in $\Gamma_{2k+2}(\mathfrak{F}_m)$. One can easily check by using the matrix W that the quadratic function of $\psi(m,\epsilon)(\alpha_{ij}.A)$ has value zero on the submodule giving the intersection pairing

$$\begin{pmatrix} 0 & S_i \\ -S_i & 0 \end{pmatrix}$$

and agrees with that of $\alpha_{ij}.(\psi(m,\epsilon)(A))$ on the orthogonal complement of the submodule. This proves that $\psi(m,\epsilon)(\alpha_{ij}.A) = \alpha_{ij}.(\psi(m,\epsilon)(A))$ in $\Gamma_{2k+2}(\mathfrak{F}_m)$.

5. *Examples*

Finally we give few examples to illustrate Theorem 3. Cappell and Shaneson gave an example of infinitely many boundary links that are not cobordant to split links in terms of their surgery obstruction (see the proof of theorem 1 in [2]). The cobordism mentioned here is the usual link cobordism without the F_m-structure restriction. The abelianization of Γ-group obstruction was shown to be invariant under this cobordism in these circumstances with mild restrictions. By computing signatures they showed that the intersection forms of their example are non-trivial and that infinitely many of them are distinct. In fact the individual components of these links are all trivial and so it was sufficient to prove that they are not cobordant to split links.

For any integer N, let A be the Seifert matrix of the boundary link of two components in S^{2k+1} given by

$$A = \begin{pmatrix} 0 & 1 & 1 & N \\ 0 & 0 & 1 & N \\ -\epsilon & -\epsilon & 0 & 1 \\ -\epsilon N & -\epsilon N & 0 & 0 \end{pmatrix} \quad \text{for } \epsilon = (-1)^k.$$

For $N = 1$ and $k = 1$, this link is obtained by taking Whitehead doubles on both components of the Hopf link. Let x, y be the generators of F_m for $m = 2$. Then the conversion through Theorem 3 produces the following λ-forms exactly as given in [2]:

$$\lambda = \begin{pmatrix} x+x^{-1}-2 & 1 \\ 1 & N(y+y^{-1}-2) \end{pmatrix} \quad \text{when } \epsilon = -1,$$

$$\lambda = \begin{pmatrix} x-x^{-1} & 1 \\ -1 & N(y^{-1}-y) \end{pmatrix} \quad \text{when } \epsilon = 1.$$

In [5], an example of a boundary link was given to demonstrate that the $\mathfrak{U}_m$ actions on boundary links are non-trivial. The three-component link defined by the Seifert matrix C as given on p. 679 in [5] is not F_m-cobordant to the trivial link but any invariants coming from the abelianization will vanish due to its construction. In fact its λ-form is given as follows:

$$\lambda = \begin{pmatrix} \bar{y} - x\bar{y}x^{-1} & -\epsilon\overline{yz} - 1 \\ \overline{zy} + \epsilon & -\epsilon\overline{zyz} - \bar{z} \end{pmatrix}$$

where $\bar{w} = -\epsilon w + w^{-1} + \epsilon - 1$.

This research was partly supported by a Korea Science and Engineering Foundation Grant.

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