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The Karoubi Tower and *K*-Theory Invariants of Hermitian Forms

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Abstract. By the use of the Karoubi Tower diagram we generalize the classical invariants of quadratic forms. Similar to Quillen's higher K-theory generalization of the classical K-theory groups, these invariants are an extension of the classical invariants by the use of homotopy theory. The iterated forgetful maps in the Karoubi Tower are *KR* valued and yield a generalization of the standard (rank, discriminant and total Hasse–Witt) invariants of quadratic forms in two directions. First, we get invariants of all degrees. Second, these invariants are defined for every Hermitian ring. They yield and generalize the Clifford invariant in the case of a field of characteristic different from 2, or in the case of an arithmetic Dedekind domain containing $\frac{1}{2}$.

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1. Introduction

Quadratic forms over a field R have been extensively studied, in particular their classification by the use of invariants. The classical invariants of non-singular symmetric bilinear forms over a field are rank, discriminant, Hasse–Witt and signature invariants [12, 16]. Invariants rank, rank(mod 2), discriminant and Hasse–Witt are K_0, $K_0/2K_0$, K_1/K_1^2 and $K_2/2K_2$ valued, respectively. We write K_i for the ith K-theory group K_iR. These invariants assemble into the Clifford invariant, which is a homomorphism from the Witt ring $W(R)$ into the Brauer–Wall group $BW(R)$. In some special cases, for example in the case of a finite field F_{p^i}, in the case of p-adics $\widehat{\mathbb{Q}}_p$, or in the case of algebraic number fields, the Clifford invariant (together with rank and signatures) classify the quadratic forms (see [16]). Recall that when $\frac{1}{2} \in R$ the category of quadratic forms and the category of Hermitian forms over $(R, id, 1)$ are equivalent.

The classical invariants of quadratic forms over a field were generalized in many ways. In particular, one desires to have a general theory of quadratic forms over any commutative ring (see [2, 8]). For another example of generalized classical invariants see Giffen's K_2 valued Hasse–Witt invariants for (α, ϵ)-reflexive forms [4]. In [11], Milnor defines 'Stiefel–Whitney invariants' of quadratic forms over a field. These invariants are defined by the use of Milnor's K-theory groups and in low degrees Stiefel–Whitney invariants yield the classical invariants of quadratic forms over a field.

Using the Karoubi Tower, we give a construction of invariants of Hermitian forms over any Hermitian ring. Our construction resembles Quillen's generalization of the classical K-theory groups. There is an obvious analogy in defining the classical K-theory groups and the classical invariants of quadratic forms. Quillen's K-theory groups are the homotopy groups of a certain space, whose lower homotopy groups realize the classical K-theory groups. Similarly, our invariants are the maps between the homotopy groups of certain spaces into $\pi_*(KR)$, where in lower dimensions these maps realize the classical invariants. Our invariants (using all degrees) yield the classification of forms over fields and rings for which the inclusion of the K-theory fixed set into the K-theory homotopy fixed set induces an isomorphism on π_0 (compare [5]). Low degree cases of these invariants generalize the classical invariants of quadratic modules over a field and over (arithmetic) Dedekind domains ([2, 8]). In particular, our invariants yield and generalize the Clifford invariant in the case of a field (char $\neq 2$) or in the case of an arithmetic Dedekind domain containing $\frac{1}{2}$.

Let (R, α, ϵ) be a Hermitian ring, i.e. R is a ring with unit element 1, α is an anti-involution and ϵ is a central element, such that $\alpha(\epsilon)\epsilon = 1$. Let $\mathcal{P}(R)$ be the category of finitely generated projective left R-modules and let $\mathcal{H}(R, \alpha, \epsilon)$ be the category of Hermitian modules, that is the category of finitely generated projective left R-modules with an (R, α, ϵ)-Hermitian form [9]. Let KR and $K\ \mathrm{Herm}(R, \alpha, \epsilon)$ be the associated K-theory spaces.

1.1. THE KAROUBI TOWER

For a Hermitian ring (R, α, ϵ) we constructed [9] the Karoubi Tower diagram:

$$\begin{array}{ccccccccccc}
 & \Omega^n K V_R^n & & \Omega K V_R & & KR & & KU_R & & KU_R^n & \\
 & \downarrow{\scriptstyle \Omega^n H_{V_R^n}} & & \downarrow{\scriptstyle \Omega H_{V_R}} & & \downarrow{\scriptstyle H_R} & & \downarrow{\scriptstyle H_{U_R}} & & \downarrow{\scriptstyle H_{U_R^n}} & \\
\cdots \to & \Omega^n K H^{(n)} & \to \cdots \to & \Omega K H^{(1)} & \to & K H^{(0)} & \to & K H^{(-1)} & \to \cdots \to & K H^{(-n)} & \to \cdots \\
 & \downarrow{\scriptstyle \Omega^n F_{V_R^n}} & & \downarrow{\scriptstyle \Omega F_{V_R}} & & \downarrow{\scriptstyle F_R} & & \downarrow{\scriptstyle F_{U_R}} & & \downarrow{\scriptstyle F_{U_R^n}} & \\
 & \Omega^n K V_R^n & & \Omega K V_R & & KR & & KU_R & & KU_R^n &
\end{array}$$

where $V_R^0 = R = U_R^0$, $KH^{(n)} = K\mathrm{Herm}(V_R^n, \alpha, \epsilon)$ and $KH^{(-n)} = K\mathrm{Herm}(U_R^n, \alpha, \epsilon)$.

1.2. INVARIANTS OF A HERMITIAN RING

Karoubi Tower generalizes the classical invariants of quadratic forms. The maps $\pi_0(\Omega^n F_{V_R^n})$ are $K_n R$ valued homomorphisms, which yield invariants for every Hermitian ring and for every $n \geqslant 0$. These generalizations simplify and under some assumptions agree with the generalizations of Bass and others.

THEOREM. *In the case of a local ring R the map $\pi_0(F_R)$ is the classical rank map. In the case of a Dedekind domain R which contains $\frac{1}{2}$, the map $\pi_0(F_R)$ yields the rank map* [8]. *In the case of an arithmetic Dedekind domain (or in the case of a field) containing $\frac{1}{2}$, the map $\pi_1(F_{V_R})$ yields the discriminant map and the map $\pi_2(F_{V_R^2})$ yields the total Hasse–Witt invariant.*

2. Classical Invariants

If G is a group, we write ${}_nG$ to denote the subgroup of elements of order n in G and G/n to denote the group G/nG or G/G^n, respectively, for an additive or multiplicative group. The following is a brief dictionary of the classical theory of invariants of quadratic forms over fields:

- Rank: Rank is the dimension of the free module underlying a quadratic form [12].
- Discriminant: Let (M, q) be a (non-singular) quadratic module of rank n over a field R. The discriminant of (M, q), denoted by $d(M, q)$ is the element $(-1)^{n(n-1)/2}\det(q) \in R^*/2$, where $\det(q)$ is the determinant of the matrix associated to q. The discriminant depends only on the Witt class in $W(R)$. If $I \leqslant W(R)$ is the fundamental ideal (the ideal of classes of even rank) in $W(R)$, then d defines a homomorphism $d\colon I \to R^*/2$ with the kernel I^2, where R^* is the multiplicative Abelian group of invertible elements in R [12].
- Hasse–Witt invariant: Let R be a field and B a multiplicative Abelian group of exponent 2. A symbol $\varphi\colon R^* \times R^* \to B$ is a bimultiplicative function satisfying $\varphi(x, 1-x) = 1$ for every $x \neq 1$ in R^*. The universal symbol is a symbol $\varphi\colon R^* \times R^* \to B$, such that for any other symbol $\varphi'\colon R^* \times R^* \to B'$ there exists a unique homomorphism $f\colon B \to B'$, such that $\varphi' = f \circ \varphi$. There exists a universal symbol (Steinberg symbol) $\varphi\colon R^* \times R^* \to K_2/2 \sim I^2/I^3 \sim {}_2\mathrm{Br}(R)$, where $\mathrm{Br}(R)$ is the Brauer group [11, 16(12.11)]. Under the determinant isomorphism $K_1 \sim R^*$ the Steinberg symbol φ is given by $(x, y) \to [x \cup y] \in K_2/2$, where $K_1 \times K_1 \xrightarrow{\cup} K_2$ is the standard map. Let φ be the Steinberg symbol. Any quadratic module (M, q) admits an orthogonal decomposition $(M, q) = \langle x_1\rangle \oplus \cdots \oplus \langle x_n\rangle$ and one defines the total Hasse invariant $H_\varphi(M, q)$ by $H_\varphi(M, q) = \prod_{i \leqslant j} \varphi(x_i, x_j)$. $H_\varphi(M, q)$ does not depend on the choice of the orthogonal decomposition. Furthermore, quadratic modules of the same rank and belonging to the same class in $W(R)$ have the same total Hasse invariant. For any $X \in I$ one can choose a quadratic module (M, q) whose class in $W(R)$ is X such that $\mathrm{rank}(M, q) \equiv 0 (\mathrm{mod}\ 8)$. The total Hasse–Witt invariant of $X \in I$ is $h_\varphi(X) \stackrel{\mathrm{def}}{=} H_\varphi(M, q)$. The restriction of the total Hasse–Witt invariant to I^2 is a homomorphism $I^2 \to {}_2\mathrm{Br}(R)$, with the kernel I^3.

There exists a general theory of the classical invariants of quadratic forms over a commutative ring (see [2, 8]).

Let R be a Dedekind domain and assume that $2 \in R^*$. The Clifford invariant is a homomorphism $W(R) \xrightarrow{\text{Cliff}} BW(R)$, where $W(R)$ is the Witt ring and $BW(R)$ is the ‘Brauer–Wall group’ (see [2, 8]). The image of the Clifford invariant is called the ‘Hasse–Wall group’ and is denoted by $HW(R)$. It follows from the study of $BW(R)$ (see [8, 10]) that there exists a filtration of $BW(R)$:

$$1 \hookrightarrow BW_2(R) \hookrightarrow BW_1(R) \hookrightarrow BW_0(R) \hookrightarrow BW(R)$$

with the following properties:

$$BW_2(R) \sim \mathrm{Br}(R), \qquad \frac{BW_1(R)}{BW_2(R)} \sim \frac{R^*}{R^{*2}},$$

$$\frac{BW_0(R)}{BW_1(R)} \sim {}_2\mathrm{Pic}(R), \qquad \frac{BW(R)}{BW_0(R)} \sim \mathbb{Z}_2,$$

where $\mathrm{Br}(R)$ is the Brauer group of R and $\mathrm{Pic}(R)$ is the Picard group of R. Furthermore, $BW_2(R) \cap HW(R) \sim {}_2\mathrm{Br}(R)$. The group $BW(R)/BW_2(R)$ is known as the graded quadratic group of R and is denoted by $\mathrm{QU}(R)$. Note also that if R is a field (char $\neq 2$), then ${}_2\mathrm{Br}(R) \sim K_2/2$ [11, 16 page 89]), $R^*/2 \sim K_1/2$, $\mathrm{Pic}(R) \sim 1$ and $\mathbb{Z}_2 \sim K_0/2$. The Clifford homomorphism and the above filtration of $BW(R)$ induce the filtration of $W(R)$:

$$1 \hookrightarrow \overline{W_3}(R) \hookrightarrow \overline{W_2}(R) \hookrightarrow \overline{W_1}(R) \hookrightarrow \overline{W_0}(R) \hookrightarrow W(R)$$

where $\overline{W_3}(R) = \mathrm{Ker}(\mathrm{Cliff})$ and $\overline{W_i}(R) = \mathrm{Cliff}^{-1}(BW_i(R))$ for $i = 0, 1, 2$. In the case of a field (char $\neq 2$) this filtration coincides with the filtration of the powers of the fundamental ideal, i.e. $I^i \sim \overline{W_i}(R)$ for $i = 1, 2, 3$, where I is the fundamental ideal in the Witt ring $W(R)$. Also $\overline{W_1}(R) = \overline{W_0}(R)$.

3. Generalized Invariants

Let (R, α, ϵ) be a Hermitian ring. Apply π_0 to the Karoubi Tower (see Section 1.1 and [9 (1.5)]). To simplify the notation we write the associate diagram of Abelian groups in the following way:

$$\begin{tikzcd}
 & K_n \arrow[d, "h_n"] & K_{n-1} \arrow[d, "h_{n-1}"] & & K_1 \arrow[d, "h_1"] & K_0 \arrow[d, "h_0"] & \\
\cdots \arrow[r, "i_{n+1}"] & H_n \arrow[r, "i_n"] \arrow[d, "f_n"] & H_{n-1} \arrow[r, "i_{n-1}"] \arrow[d, "f_{n-1}"] & \cdots \arrow[r, "i_2"] & H_1 \arrow[r, "i_1"] \arrow[d, "f_1"] & H_0 \arrow[r, "i_0"] \arrow[d, "f_0"] & W \\
 & K_n & K_{n-1} & & K_1 & K_0 &
\end{tikzcd} \tag{KT}$$

where $K_i = \pi_i(KR)$, $H_i = \pi_i(K\mathrm{Herm}(V_R^i, \alpha, \epsilon))$ and W is the cokernel of h_0. Recall, the diagram from [9(2.3)] and its properties. The map $H_0 \xrightarrow{f_0} K_0$ factors as $H_0 \to \pi_0 F_{\mathbb{Z}_2}({}_aS_+^n, KR) \to K_0$. Let Γ_n be the homomorphism $H_0 \xrightarrow{\Gamma_n} \pi_0 F_{\mathbb{Z}_2}({}_aS_+^n, KR)$.

DEFINITION 1. The homomorphisms Γ_n, for $n = 0, 1, \ldots, \infty$ are called the invariants of a Hermitian ring (R, α, ϵ).

Consider the following diagram

$$\begin{array}{ccccccc}
K_0 & \xrightarrow{h_0} & H_0 & \longrightarrow & W & \longrightarrow & 0 \\
\downarrow{\scriptstyle id} & & \downarrow{\scriptstyle \Gamma_n} & & \downarrow{\scriptstyle w_n} & & \\
K_0 & \xrightarrow{\Gamma_n \circ h_0} & \pi_0 F_{\mathbb{Z}_2}({}_a S^n_+, KR) & \longrightarrow & W_n & \longrightarrow & 0
\end{array}$$

where W and W_n are the cokernels of h_0 and $\Gamma_n \circ h_0$, respectively and w_n is the homomorphism (invariant) induced by the invariant Γ_n. The rest of this chapter is mostly devoted to the following theorem:

THEOREM 2. *Let $(R, id, 1)$ be a Hermitian ring.*

(a) *If R is a field of characteristic different from 2, then the homomorphism w_2 is equivalent to the Clifford invariant and the group W_2 is isomorphic to the Hasse–Wall group $HW(R)$.*

(b) *If R is an arithmetic Dedekind domain containing $\frac{1}{2}$ then the Clifford invariant $W \to BW$ factors as $W \overset{\omega_2}{\to} W_2 \to BW$.*

For any Hermitian ring (R, α, ϵ), let I_n be the image of the map $\alpha_n = i_0 \circ i_1 \circ \cdots \circ i_n$ (see diagram (KT) above). The invariants Γ_n (or simply homomorphisms f_n from (KT)) yield homomorphisms $\varphi_n \colon I_n \to f_n(H_n)/f_n(\mathrm{Ker}\alpha_n) \sim I_n/I_{n+1}$ for every $n \geqslant 0$.

Remark 3. Let I_∞ be the inverse limit of the tower of groups $\cdots \hookrightarrow I_n \hookrightarrow \cdots \hookrightarrow I_1 \hookrightarrow W$. Invariants φ_n, for $n = 0, 1, 2, \ldots$ classify W upto the subgroup I_∞. In particular, if $I_\infty = 0$, then the invariants φ_n classify W.

PROPOSITION 4. *The map $W \xrightarrow{w_n} W_n$ factors as $W \to W/I_{n+1} \hookrightarrow W_n$, where the first map is the quotient map, the second is an inclusion, $W \subset \pi_0(K\mathrm{Herm}(U_R, \alpha, \epsilon))$ and $W_n \subset \pi_0 F_{\mathbb{Z}_2}({}_a S^{n+1}_+, KU_R)$. If KR is -1 connected, then $W = \pi_0(K\,\mathrm{Herm}\,(U_R, \alpha, \epsilon))$ and $W_n = \pi_0 F_{\mathbb{Z}_2}({}_a S^{n+1}_+, KU_R)$.*

Proof. The first statement is clear since the two sequences $H_n \to W \to W_n$ and $H_n \to W \to W/I_n \to 0$ are exact. $W \subset \pi_0(K\mathrm{Herm}(U_R, \alpha, \epsilon))$, $W_n \subset \pi_0 F_{Z_2}({}_a S^{n+1}_+, KU_R)$ ($W = \pi_0(K\,\mathrm{Herm}(U_R, \alpha, \epsilon))$ and $W_n = \pi_0 F_{Z_2}({}_a S^{n+1}_+, KU_R)$, if *KR* is -1 connected) by the exact sequences following from [9 (2.3)] □

Remark. The homomorphisms φ_n are I_n/I_{n+1} valued and in general I_n/I_{n+1} is isomorphic to a subgroup of the quotient group of $\widehat{H}^n(\mathbb{Z}_2, K_n)$. Namely $f_n h_n$ is the map $1+(-1)^n\tau$, where τ is the involution on K_n and $\mathrm{Im}(h_n) \subset \mathrm{Ker}(\alpha_n)$. Thus

$$\begin{aligned}\widehat{H}^n(\mathbb{Z}_2, K_n) &= \frac{\mathrm{Ker}(1-(-1)^n\tau)}{\mathrm{Im}(1+(-1)^n\tau)}\\ &= \frac{\mathrm{Ker}(1-(-1)^n\tau)}{f_n((\mathrm{Im}(h_n))} \xrightarrow{\text{quotient}}\\ &\xrightarrow{\text{quotient}} \frac{\mathrm{Ker}(1-(-1)^n\tau)}{f_n((\mathrm{Ker}(\alpha_n))} \supset \frac{f_n(H_n)}{f_n((\mathrm{Ker}(\alpha_n))} \sim I_n/I_{n+1}.\end{aligned}$$

The inclusion $\mathrm{Ker}(1-(-1)^n\tau) \supset f_n(H_n)$ follows from the exact sequences in the Karoubi Tower.

THEOREM 2(a)

(i) *Let $(R, id, 1)$ be a Hermitian field and let the characteristic of R be different from 2. The group W_2 is isomorphic to the Hasse–Wall group $HW(R)$, the homomorphism $W \xrightarrow{w_2} W_2$ is equivalent to the Clifford invariant and $I_i = I^i$ for $i = 1, 2, 3$. (Therefore, $W_1 \sim W/I_2 \sim BW(R)/BW_2(R)$, $W_0 \sim W/I_1 \sim BW(R)/BW_1(R)$, $\varphi_2\colon I_2 \to I_2/I_3 \sim K_2/2 \sim_2 \mathrm{Br}(R)$ is the 'total Hasse–Witt' invariant, $\varphi_1\colon I_1 \to I_1/I_2 \sim K_1/2 \sim R^*/2$ is equivalent to the discriminant homomorphism, $w_0 \equiv \varphi_0\colon W \to W/I_1 \sim K_0/2 \sim \mathbb{Z}_2$ is the* rank(mod 2) *invariant and Γ_0 is the rank map.)*

(ii) *Let $(R, id, 1)$ be a Hermitian ring and assume that R is an arithmetic Dedekind domain with $2 \in R^*$. The group W_1 is isomorphic to the quotient group $BW(R)/BW_2(R) \sim QU(R)$, the homomorphism $W \xrightarrow{w_1} W_1$ is equivalent to the composition $W(R) \xrightarrow{\text{Cliff}} BW(R) \to BW(R)/BW_2(R)$, $\overline{W_1}(R) = I_1$ and $\overline{W_2}(R) = I_2$. (Therefore, $W_0 \sim W/I_1 \sim BW(R)/BW_1(R)$, $\varphi_1\colon I_1 \to I_1/I_2 \sim K_1/2 \sim R^*/2$ is the discriminant homomorphism, $w_0 \equiv \varphi_0\colon W \to W/I_1 \sim K_0/2 \sim \mathbb{Z}_2 \oplus \mathrm{Pic}(R)/2$ composed with the projection $\mathbb{Z}_2 \oplus \mathrm{Pic}(R)/2 \to \mathbb{Z}_2$ is the* rank(mod 2) *invariant and Γ_0 composed with the projection $K_0 \sim \mathbb{Z} \oplus \mathrm{Pic}(R) \to \mathbb{Z}$ is the standard* rank *map.)*

(iii) *Let $(R, id, 1)$ be a Hermitian ring and assume that R is a Dedekind domain with $2 \in R^*$. The group W_0 is isomorphic to the quotient group $BW(R)/BW_1(R)$, the homomorphism $W \xrightarrow{w_0} W_0$ is equivalent to the composition $W(R) \xrightarrow{\text{Cliff}} BW(R) \to BW(R)/BW_1(R)$ and $I_1 \subset \overline{W_0}(R)$. (Therefore, $w_0 \equiv \varphi_0\colon W \to W/I_1 \sim K_0/2 \sim \mathbb{Z}_2 \oplus \mathrm{Pic}(R)/2$ composed with the projection $\mathbb{Z}_2 \oplus \mathrm{Pic}(R)/2 \to \mathbb{Z}_2$ is the* rank(mod 2) *invariant and Γ_0 composed with the projection $K_0 \sim \mathbb{Z} \oplus \mathrm{Pic}(R) \to \mathbb{Z}$ is the standard rank map.)*

Proof. A detailed proof can be found in [10]. Compare also [7(4.2.3)], which turns out to be equivalent to the statement (i) above. □

Now let F be a number field, that is some finite extension of rationals $\mathbb{Q}$. Let R be such a ring that $\mathcal{O} \subset R \subset F$, where $\mathcal{O}$ is the ring of (algebraic) integers in F.

Such a ring R is called an arithmetic Dedekind domain. Let $\mathbb{P}$ be the set of non-zero prime ideals $\mathcal{P} \subset R$. Let $\mathsf{f}_{\mathcal{P}} = R/\mathcal{P}$, that is $\mathsf{f}_{\mathcal{P}}$ is a finite field of characteristic p, where $\mathcal{P}$ is over the ideal (p). Let $\widehat{F}_{\mathcal{P}}$ be the $\mathcal{P}$-adic completion of the field F and $\widehat{\mathcal{O}}_{\mathcal{P}}$ the completion of $\mathcal{O}_{\mathcal{P}}$ in $\widehat{F}_{\mathcal{P}}$, where $\mathcal{O}_{\mathcal{P}}$ is the ring of integers in F localized at $\mathcal{P}$. By assumption $\frac{1}{2} \in R$ and so all the fields $\mathsf{f}_{\mathcal{P}}$, for $\mathcal{P} \in \mathbb{P}$, have odd characteristic. Let $\mathcal{P} \in \mathbb{P}$ be an ideal over (p) and let $0 \to K_2(\widehat{\mathcal{O}}_{\mathcal{P}}) \to K_2(\widehat{F}_{\mathcal{P}}) \to K_1(\mathsf{f}_{\mathcal{P}}) \to 0$ be the 'localization sequence' (see for example [15(5.3.28)]).

LEMMA 5. *The map $K_2(\widehat{F}_{\mathcal{P}}) \to K_1(\mathsf{f}_{\mathcal{P}})$ from the exact sequence above induces an isomorphism $K_2(\widehat{F}_{\mathcal{P}})/2 \xrightarrow{\sim} K_1(\mathsf{f}_{\mathcal{P}})/2 \sim \mathbb{Z}_2$.*

Proof. Recall that $K_2(\widehat{F}_{\mathcal{P}}) \sim \mu(\widehat{F}_{\mathcal{P}}) \oplus H$, where $\mu(\widehat{F}_{\mathcal{P}})$ is the group of roots of unity in $\widehat{F}_{\mathcal{P}}$ and H is a divisible and torsion free group (see [6(10)]). One gets the exact 'torsion sequence' $0 \to \operatorname{tor}(K_2(\widehat{\mathcal{O}}_{\mathcal{P}})) \to \mu(\widehat{F}_{\mathcal{P}}) \to K_1(\mathsf{f}_{\mathcal{P}}) \to 0$. The group $\operatorname{tor}(K_2(\widehat{\mathcal{O}}_{\mathcal{P}}))$ is the kernel of the surjective map $\mu(\widehat{F}_{\mathcal{P}}) \to \mathsf{f}_{\mathcal{P}}{}^*$ and is isomorphic to the p-primary subgroup ${}_{p^i}\mu(\widehat{F}_{\mathcal{P}})$ of $\mu(\widehat{F}_{\mathcal{P}})$ (see [17(14.10)] or [13(5.8 Cor.)]). Since $p \neq 2$ one concludes that $\mu(\widehat{F}_{\mathcal{P}})/2 \xrightarrow{\sim} K_1(\mathsf{f}_{\mathcal{P}})/2 \sim \mathbb{Z}_2$. Since $K_2(\widehat{F}_{\mathcal{P}})/2 \sim \mu(\widehat{F}_{\mathcal{P}})/2$ we are done. $\square$

Recall the Karoubi Tower (KT) and especially the following relations:

$$\frac{I_2(R)}{I_3(R)} \sim \frac{f_2(H_2(R))}{f_2(\mathrm{Ker}\alpha_2(R))} \subset \frac{K_2(R)}{f_2(\mathrm{Ker}\alpha_2(R))},$$

$$h_2(K_2(R)) \subset \mathrm{Ker}\alpha_2(R), \qquad h_2(K_2(F)) = \mathrm{Ker}\alpha_2(F),$$

and

$$K_2(R)^2 \subset f_2(\mathrm{Ker}\alpha_2(R)) \subset K_2(R).$$

Let $E = f_2(\mathrm{Ker}\alpha_2(R))$.

LEMMA 6. *There exists a natural surjection $K_2(R)/E \to {}_2\mathrm{Br}(R)$.*

Proof. The map $K_2(R) \to K_2(F)$ induces the map $K_2(R)/2 \to K_2(F)/2$. Because of the functoriality and since $h_2(K_2(F)) = \mathrm{Ker}\alpha_2(F)$ the map $K_2(R)/2 \to K_2(F)/2$ factors as $K_2(R)/2 \xrightarrow{j} K_2(R)/E \xrightarrow{k} K_2(F)/2$, where j is surjective and thus $\mathrm{Im}(k) = \mathrm{Im}(kj)$. Look at the following commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & K_2(R) \arrow[r] & K_2(F) \arrow[r] \arrow[d, leftrightarrow, "id"] & \bigoplus_{\mathcal{P}\in\mathbb{P}} K_1(\mathsf{f}_{\mathcal{P}}) \arrow[r] & 0 \\
& & K_2(F) \arrow[r] \arrow[d] & \bigoplus_{\mathcal{P}\in\mathbb{P}} K_2(\widehat{F}_{\mathcal{P}}) \arrow[u, twoheadrightarrow] \arrow[d] & \\
0 \arrow[r] & \mathrm{Br}(R) \arrow[r] & \mathrm{Br}(F) \arrow[r] & \bigoplus_{\mathcal{P}\in\mathbb{P}} \mathrm{Br}(\widehat{F}_{\mathcal{P}}). &
\end{tikzcd}$$

The top and the bottom sequences are the 'localization exact sequences' for K-theory [15(5.3.28)] and for the Brauer group [14(6.35)], respectively. The bottom two vertical arrows are the maps which induce the isomorphisms $K_2(F)/2 \xrightarrow{\sim} {}_2\mathrm{Br}(F)$ and $\bigoplus_{\mathcal{P}\in\mathbb{P}} K_2(\widehat{F}_{\mathcal{P}})/2 \xrightarrow{\sim} \bigoplus_{\mathcal{P}\in\mathbb{P}} {}_2\mathrm{Br}(\widehat{F}_{\mathcal{P}})$ [11, 16 page 89]. Using Lemma 5 we get the following diagram:

$$\begin{tikzcd}
& K_2(R)/E \arrow[r] \arrow[d, dashed] & K_2(F)/2 \arrow[r] \arrow[d, "\sim"] & \bigoplus_{\mathcal{P}\in\mathbb{P}} K_2(\widehat{F}_{\mathcal{P}})/2 \arrow[r] \arrow[d, "\sim"] & 0 \\
0 \arrow[r] & {}_2\mathrm{Br}(R) \arrow[r] & {}_2\mathrm{Br}(F) \arrow[r] & \bigoplus_{\mathcal{P}\in\mathbb{P}} {}_2\mathrm{Br}(\widehat{F}_{\mathcal{P}}) &
\end{tikzcd}$$

where the two horizontal sequences are exact and the two isomorphisms induce the desired surjection. □

Let $(R, id, 1)$ be a Hermitian ring and assume that R is an arithmetic Dedekind domain containing $\frac{1}{2}$. All the groups which appear in the following theorem are associated to the ring R.

THEOREM 2(b) *The Clifford invariant $W \to BW$ factors as $W \xrightarrow{\omega_2} W_2 \to BW$. (Therefore $I_3 \subset \overline{W}_3$ and there exist natural surjections $K_2/2 \to {}_2\mathrm{Br}$ and $I_2/I_3 \to {}_2\mathrm{Br}$.)*

Proof. The following diagram is self-explaining:

$$\begin{tikzcd}
O \arrow[r] & I_2 = \overline{W}_2 \arrow[r] \arrow[d, dashed] \arrow[ddd, bend right=60, "\varphi_2"'] & W \arrow[r] \arrow[d, dashed, "\mathrm{Cliff}"] \arrow[ddd, bend right=60, "\omega_2"'] & W_1 & \\
O \arrow[r] & {}_2Br \arrow[r] & HW \arrow[r] & W_1 \arrow[r] \arrow[u, "id"'] \arrow[dd, leftrightarrow, "id"] & O \\
& K_2/E \arrow[u, dashed] & & & \\
O \arrow[r] & I_2/I_3 \arrow[r] \arrow[u] & W_2 \arrow[r] \arrow[uu, dashed] & W_1 &
\end{tikzcd}$$

The existence of the map $K_2/E \to {}_2\mathrm{Br}$ is essential in defining the map $W_2 \to HW$. □

EXAMPLE. In the case of a finite field $(F_q, 1, 1)$ the computations of the Karoubi Tower [9(4), 10(3)] imply that $I_2 = I^2 = \{0\}$. (If q is even, then $I = \{0\}$.) Therefore, the quadratic forms over a finite field are determined by dimension and discriminant (see Remark 3).

EXAMPLE. If in the case of a field $(R, 1, 1)$ the group I^3, where I is a fundamental ideal, is torsion free, then the quadratic forms over R are classified by dimension, discriminant, Hasse–Witt and signature invariants. If $I^3 = \{0\}$, then the quadratic forms are classified by dimension, discriminant and Hasse–Witt invariants. Namely, the kernel of the total signature homomorphism equals to the torsion subgroup of W (see [12] and compare Remark 3).

Finally, in the case when KR is -1 connected (see [9(1.8)]), the invariants Γ_n, or more precisely the induced invariants w_n, can be described by the use of equivariant Postnikov tower of the infinite loop space KR. Recall the map

$$\begin{tikzcd} H_*(\mathbb{Z}_2, KR) \arrow[r] \arrow[rr, bend right, "\mathcal{N}"'] & K\,\mathrm{Herm}(R, \alpha, \epsilon) \arrow[r] & H^*(\mathbb{Z}_2, KR) \end{tikzcd}$$

from the remark at the end of chapter 3 in [9]. The homotopy cofibre of $H_*(\mathbb{Z}_2, KR) \to K\,\mathrm{Herm}(R, \alpha, \epsilon)$ is homotopy equivalent to the homotopy direct limit of the Karoubi Tower and its 0th homotopy group is isomorphic to W. The homotopy cofibre of the norm map $\mathcal{N}\colon H_*(\mathbb{Z}_2, KR) \to H^*(\mathbb{Z}_2, KR)$ is the Tate spectrum $\widehat{H}(\mathbb{Z}_2, KR)$. Thus, we get the map $W \to \pi_0\widehat{H}(\mathbb{Z}_2, KR)$. There exists the map $\pi_0\widehat{H}(\mathbb{Z}_2, KR) \to \pi_0\widehat{H}(\mathbb{Z}_2, KR\langle 0, \dots, n\rangle)$, induced by the Postnikov tower projection.

THEOREM 7. If KR is -1 connected, then the composition homomorphism $W \to \pi_0\widehat{H}(\mathbb{Z}_2, KR) \to \pi_0\widehat{H}(\mathbb{Z}_2, KR\langle 0, \dots, n-1\rangle)$ is equivalent to the invariant w_n.

Proof. See [10]. □

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