[4] S. S. Chang, K. K. Tan, H. W. J. Lee, C. K. Chan, On the convergece of implicit iteration process with error for a finite family of asymptotically nonexpansive mappings, J. Math. Anal. Appl. **313** (2006), 273–283.
[5] A. Genel, J. Lindenstrass, An example concerning fixed points, Israel J. Math. **22** (1975), 81–86.
[6] K. Goebel, W. A. Kirk, A fixed point theorem for asymptotically nonexpansive mappings, Proc. Am. Math. Soc. **35** (1972), 171–174.
[7] J. Gornicki, Weak convergence theorems for asymptotically nonexpansive mappings in uniformly convex Banach spaces, Comment. Math. Univ. Carolin. **301** (1989), 249–252.
[8] T. H. Kim and H. K. Xu, Convergence of the modified Mann's iteration method for asymptotically strict pseudo-contractions. Nonlinear Anal. in press, available online 1 March 2007, doi:10.1016/j.na. 2007.02.029.
[9] W. R. Mann, Mean value methods in iteration, Proc. Amer. Math. Soc. **4** (1953), 506–510.
[10] G. Marino, H. K. Xu, Weak and strong convergence theorems for strict pseudo-contractions in Hilbert space, J. Math. Anal. Appl. **329** (2007), 336–346.
[11] L. Qihou, Convergence theorems of the sequence of iterates for asymptotically demicontractive and hemicontractive mappings, Nonliear Anal. **26** (1996), 1835–1842.
[12] S. Reich, Weak convergence theorems for nonexpansive mappings in Banach spaces, J. Math. Anal. Appl. **67** (1979), 274–276.
[13] S. Reich, Strong convergence theorems for resolvents of accretive operators in Banach spaces, J. Math. Anal. Appl. **75** (1980), 287–292.
[14] Arif Rafiq, On Mann iteration in Hilbert spaces, doi:10.1016/j.na. 2006.03.012.
[15] J. Schu, Weak and strong convergence of fixed points of asymptotically nonexpansive mappings, Bull. Austral. Math. Soc. **43** (1991), 153–159.
[16] K. K. Tan, K. K. Xu, Approximating fixed points of nonexpansive mappings by the Ishikawa iteration process, J. Math. Anal. Appl. **178** (2) (1993), 301–308.
[17] H. K. Xu and R. G. Ori, An implicit iteration process for nonexpansive mappings, Numer. Funct. Anal. and Optim. **22** (2001), 767–773.
[18] Y. Y. Zhou, S. S. Chang, Convergence of implicit iterative process for a finite of asymptotically nonexpansive mappings in Banach spaces, Numer. Funct. Anal. Optim. **23** (2002), 911–921.

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UNIVERSAL WU CLASSES, UNIVERSAL STIEFEL-WHITNEY CLASSES AND AN ENDOMORPHISM D_k OF $H^*(BO; Z_2)$

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Abstract: Let BO be the space which classifies stable real vector bundles. The mod 2 cohomology ring $H^*(BO; Z_2)$ is the polynomial algebra over Z_2 generated by the universal Stiefel-Whitney classes $w_1, w_2, w_3, \ldots$. In this paper, we define an endomorphism D_k $(k \geq 1)$ of $H^*(BO; Z_2)$, and determine $D_k v_i$ for the i-dimensional universal Wu class v_i. As applications, we prove that, in the polynomial $v_k = v_k(w_1, w_2, w_3, \ldots, w_k)$, w_k appears with coefficient 1 if and only if k is a power of 2, and obtain detailed formulas on the Wu class and the Stiefel-Whitney class.

1. Introduction

For a closed smooth manifold M, let $v_i(M)$ $(\in H^i(M; Z_2))$ denote the i-dimensional Wu class of M and let $w_i(M)$ $(\in H^i(M; Z_2))$ denote the i-dimensional Stiefel-Whitney class of M. Then the following equalities hold (cf. [3, Theorem 11.14] and [4, p. 245]).

$$(1.1) \qquad w_0(M) = v_0(M) = 1, \qquad w_i(M) = \sum_{0 \leq j \leq i} Sq^j v_{i-j}(M) \quad \text{if } i \geq 1,$$

where Sq^j is the Steenrod squaring operation (cf. [5, Chapter 1]). Thus the Stiefel-Whitney class $w_i(M)$ is determined by the Wu classes $v_j(M)$ $(0 \leq j \leq i)$. Conversely, the Wu class $v_i(M)$ is determined by the Stiefel-Whitney classes $w_j(M)$ $(0 \leq j \leq i)$ inductively. Using (1.1) and the properties of the Steenrod squaring operations, we have $v_0(M) = w_0(M) = 1$, $v_1(M) = w_1(M)$, $v_2(M) = w_2(M) + w_1(M)^2$, $v_3(M) = w_2(M)w_1(M), \ldots$.

Since $v_i(M) = 0$ for $i > (\dim M)/2$ (cf. [7, Section 1]), we have $w_3(M) = 0$, $w_2(M)w_1(M) = 0$ and $w_1(M)^3 = 0$ if $\dim M = 3$. Thus all Stiefel-Whitney numbers of a 3-dimensional closed smooth manifold M are zero (cf. [3, Problem 11-D]). So M can be realized as the boundary of some smooth compact manifold of dimension 4 (cf. [3, Theorem 4.10]).

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Let BO be the space which classifies stable real vector bundles. Then its mod 2 cohomology ring $H^*(BO; Z_2)$ is the polynomial algebra $Z_2[w_1, w_2, w_3, \dots]$ over Z_2 on the universal Stiefel-Whitney classes w_i $(\in H^i(BO; Z_2))$ for $i \geq 1$ (cf. [3, Theorem 7.1]). In the way parallel to (1.1), the i-dimensional universal Wu class v_i $(\in H^i(BO; Z_2))$ is defined by the equalities:

$$(1.2) \qquad w_0 = v_0 = 1, \qquad w_i = \sum\nolimits_{0 \leq j \leq i} Sq^j v_{i-j} \quad \text{if } i \geq 1$$

(cf. [1, (1.2)], [8], [9, (1.1)]). Let $f : M \to BO$ be the classifying map for the stable tangent bundle of M. Then $f^*(w_i) = w_i(M)$ and $f^*(v_i) = v_i(M)$ for $i \geq 0$ (cf. [9, Section 1]). In this paper, we study the problem to express v_i as a polynomial of $w_i, w_{i-1}, \dots, w_1$.

As for the endomorphisms of the cohomology ring $H^*(BO; Z_2) = Z_2[w_1, w_2, w_3, \dots]$, J. Milnor defined in [2, p. 225] the doubling homomorphism $d : H^*(BO; Z_2) \to H^*(BO; Z_2)$ by $d(w_i) = w_{2i}$. We define, for any positive integer k, another endomorphism

$$D_k : H^*(BO; Z_2) \to H^*(BO; Z_2)$$

as follows.

If $x = w_{i(1)} w_{i(2)} \cdots w_{i(\ell)}$, where, for any n with $1 \leq n \leq \ell$, $i(n)$ is a positive integer and $w_{i(n)}$ is the $i(n)$-dimensional universal Stiefel-Whitney class, then

$$D_k x = w_{i(1)-k} w_{i(2)} \cdots w_{i(\ell)} + w_{i(1)} w_{i(2)-k} \cdots w_{i(\ell)} + \cdots + w_{i(1)} w_{i(2)} \cdots w_{i(\ell)-k},$$

where $w_j = 0$ if $j < 0$, and $D_k x = 0$ if $\dim x \leq 0$.

If $x = \sum_i x_i$, where x_i is a monomial, then

$$D_k x = \sum\nolimits_i D_k x_i.$$

In the following, let $v_j = 0$ if $j < 0$. Then we prove

THEOREM 1. *For any i-dimensional universal Wu class v_i,*

$$D_k v_i = v_{i-k} \quad \text{if } k = 2^a \ (a \geq 0),$$
$$= 0 \quad \text{otherwise.}$$

As an application of Theorem 1, we have

COROLLARY 2. *In the polynomial $v_k = v_k(w_1, w_2, \dots, w_k)$, w_k appears with coefficient 1 if and only if $k = 2^a$ $(a \geq 0)$.*

Let $n = 2^a + b$, where a and b are integers with $a \geq 1$ and $2^a > b \geq 0$. According to Lemma 6.3, we may write

$$v_n = \sum\nolimits_{[n/2]+1 \leq j \leq 2^a} w_j P_{j-1} + R_{[n/2]},$$

where $[x]$ is the integer m with $m \leq x < m + 1$, and where P_k and R_k denote some polynomials of $w_k, w_{k-1}, \dots, w_1$. Using Theorem 1, we determine P_{j-1} completely in Theorem 6.4.

If $a \geq 2$, then, according to Lemma 7.2, we may write

$$v_n = \sum\nolimits_{[n/3]+1 \leq j \leq 2^a} w_j P_{j-1} + \sum\nolimits_{[n/3]+1 \leq j \leq 2^{a-1}+[b/2]} w_j^2 Q_{j-1} + R_{[n/3]},$$

where P_k, Q_k and R_k denote some polynomials of $w_k, w_{k-1}, \dots, w_1$. Using Theorem 1, we determine P_{j-1} completely in Theorems 7.1 and 8.1.

In the next paper, we will study Q_{j-1}.

R. E. Stong proved in [6, Lemma, p. 315] that v_{2^a} is indecomposable. Corollary 2 and Theorem 6.4 improve the result.

This paper is arranged as follows. In Section 2 we recall the binomial coefficients and the Wu formula. In Section 3 we study some properties of the endomorphism D_k. In Section 4 we establish a formula on $D_k Sq^i x$ for $x \in H^*(BO; Z_2)$. In Section 5, using the formula in Section 4 we give a proof of Theorem 1, and using Theorem 1 we have Corollary 2. As other applications of Theorem 1, we prove Theorem 6.3 in Section 6, Theorems 7.1 and 8.1 which are detailed formulas on Wu classes and Stiefel-Whitney classes in Sections 7 and 8.

2. Binomial coefficients and the Wu formula

Let r and s be integers with $r \geq s \geq 0$ and let $C_{r,s}$ denote the binomial coefficient $r!/(s!(r-s)!)$. Then the following is well-known (cf. [5, Lemma 2.6, p. 5]).

LEMMA 2.1. *Let $a = \sum_{0 \leq i \leq m} a(i) 2^i$ and $b = \sum_{0 \leq i \leq m} b(i) 2^i$ be 2-adic expansions, where $0 \leq a(i)$, $b(i) \leq 1$ for $0 \leq i \leq m$. Then*

$$C_{b,a} \equiv \prod\nolimits_{0 \leq i \leq m} C_{b(i),a(i)} \mod 2.$$

For any integers r and s, let $C_{r,s}$ denote the usual binomial coefficient whenever $r \geq s \geq 0$, let $C_{-1,0} = 1$ and let $C_{r,s} = 0$ otherwise. Then the following is easily verified.

LEMMA 2.2. *For any integers r and s such that $(r,s) \neq (-1,0), (0,1)$,*

$$C_{r,s} = C_{r-1,s} + C_{r-1,s-1}.$$

For non-negative integers ℓ, m and n, define an integer $A(\ell,m,n)$ as follows.

$$A(\ell,m,n)=\sum\nolimits_{0\le s\le \ell}\mathrm{C}_{\ell-s,s}\mathrm{C}_{m+s,n-s}.$$

Then we have

LEMMA 2.3. *Let ℓ, m and n be non-negative integers. Then*

$$A(\ell,m,n)\equiv \mathrm{C}_{\ell+m+1,n}+\mathrm{C}_{m,n-\ell-1} \mod 2.$$

In particular, if $n\le \ell$, then

$$A(\ell,m,n)=\sum\nolimits_{0\le s\le n}\mathrm{C}_{\ell-s,s}\mathrm{C}_{m+s,n-s}\equiv \mathrm{C}_{\ell+m+1,n} \mod 2.$$

PROOF. We prove the former congruence by induction on (ℓ,m,n). If $\ell=0$, then we have, for any non-negative integers m and n,

$$\begin{aligned}A(0,m,n)&=\mathrm{C}_{m,n}\equiv \mathrm{C}_{m,n}+\mathrm{C}_{m,n-1}+\mathrm{C}_{m,n-1} \mod 2\\ &=\mathrm{C}_{m+1,n}+\mathrm{C}_{m,n-1} \qquad \text{by Lemma 2.2.}\end{aligned}$$

Hence the former congruence holds for $(\ell,m,n)=(0,m,n)$. If $\ell=1$, then we have, for any non-negative integers m and n,

$$\begin{aligned}A(1,m,n)&=\mathrm{C}_{1,0}\mathrm{C}_{m,n}+\mathrm{C}_{0,1}\mathrm{C}_{m+1,n-1}=\mathrm{C}_{m,n}\\ &\equiv \mathrm{C}_{m,n}+\mathrm{C}_{m,n-1}+\mathrm{C}_{m,n-1}+\mathrm{C}_{m,n-2}+\mathrm{C}_{m,n-2} \mod 2\\ &=\mathrm{C}_{m+1,n}+\mathrm{C}_{m+1,n-1}+\mathrm{C}_{m,n-2} \qquad \text{by Lemma 2.2}\\ &=\mathrm{C}_{m+2,n}+\mathrm{C}_{m,n-2} \qquad \text{by Lemma 2.2.}\end{aligned}$$

Hence the former congruence holds for $(\ell,m,n)=(1,m,n)$. If $n=0$, then we have, for any non-negative integers ℓ and n,

$$A(\ell,m,0)=\mathrm{C}_{\ell,0}\mathrm{C}_{m,0}=1=\mathrm{C}_{\ell+m+1,0}+\mathrm{C}_{m,-\ell-1}.$$

Hence the former congruence holds for $(\ell,m,n)=(\ell,m,0)$.

Assume that $s\ge 2$ and that the former congruence holds for any (ℓ,m,n) where $\ell\le s-1$ and $n\ge 1$. Then we have

$$\begin{aligned}A(s,m,n)&=\sum\nolimits_{0\le i\le s}\mathrm{C}_{s-i,i}\mathrm{C}_{m+i,n-i}\\ &=\sum\nolimits_{0\le i\le s}(\mathrm{C}_{s-i-1,i}+\mathrm{C}_{s-i-1,i-1})\mathrm{C}_{m+i,n-i} \qquad \text{by Lemma 2.2}\\ &=\sum\nolimits_{0\le i\le s-1}\mathrm{C}_{s-i-1,i}\mathrm{C}_{m+i,n-i}+\sum\nolimits_{1\le i\le s-1}\mathrm{C}_{s-i-1,i-1}\mathrm{C}_{m+i,n-i}\end{aligned}$$

$$\begin{aligned}&=A(s-1,m,n)+A(s-2,m+1,n-1)\\ &\equiv \mathrm{C}_{s+m,n}+\mathrm{C}_{m,n-s}+\mathrm{C}_{s+m,n-1}+\mathrm{C}_{m+1,n-s} \mod 2\\ &\qquad \text{by the inductive assumption}\\ &=\mathrm{C}_{s+m,n}+\mathrm{C}_{m,n-s}+\mathrm{C}_{s+m,n-1}+\mathrm{C}_{m,n-s}+\mathrm{C}_{m,n-s-1} \qquad \text{by Lemma 2.2}\\ &\equiv \mathrm{C}_{s+m,n}+\mathrm{C}_{s+m,n-1}+\mathrm{C}_{m,n-s-1} \mod 2\\ &=\mathrm{C}_{s+m+1,n}+\mathrm{C}_{m,n-s-1} \qquad \text{by Lemma 2.2.}\end{aligned}$$

Hence the former congruence holds for $(\ell,m,n)=(s,m,n)$.

So the former congruence holds for any non-negative integers ℓ, m and n. The latter congruence is clear since $\mathrm{C}_{m+s,n-s}=0$ for $n<s$ and $\mathrm{C}_{m,n-\ell-1}=0$ for $n\le \ell$. □

The following Wu formula is useful for our proofs.

THEOREM 2.4 (Wu) (cf. [4, p. 245], [8], [9, (1.3)]). *Let i and j be any integers. Then, for the universal Stiefel-Whitney class w_j,*

$$Sq^i w_j=\sum\nolimits_{0\le t\le i}\mathrm{C}_{j-i-1+t,t}w_{i-t}w_{j+t},$$

where Sq^i is the zero homomorphism for $i\le -1$ and $w_j=0$ for $j\le -1$. (Here $\mathrm{C}_{r,s}$ denotes the Z_2 reduction of the binomial coefficient $\mathrm{C}_{r,s}$.)

Using (1.2) and Theorem 2.4, we have

EXAMPLE 2.5. $v_0=w_0=1$, $v_1=w_1$, $v_2=w_2+w_1^2$, $v_3=w_2w_1$,

$$v_4=w_4+w_3w_1+w_2^2+w_1^4, \qquad v_5=w_4w_1+w_3w_1^2+w_2^2w_1+w_2w_1^3,$$

$$v_6=w_4w_2+w_4w_1^2+w_3^2+w_3w_2w_1+w_3w_1^3+w_2^2w_1^2,$$

$$v_7=w_4w_2w_1+w_3^2w_1+w_3w_2w_1^2.$$

3. Some properties of D_k

In this section we study some fundamental properties of D_k.

LEMMA 3.1. *For any elements $x, y\in H^*(BO;Z_2)$,*

$$D_k(xy)=(D_kx)y+x(D_ky).$$

In particular, $D_k(x^2)=0$.

PROOF. If $x, y \in \{0,1\} \cup \{\text{monomials}\}$, the equality holds clearly.

If $x = \sum_i x_i$ and $y = \sum_j y_j$, where $x_i, y_j \in \{0,1\} \cup \{\text{monomials}\}$, then we have

$$D_k(xy) = \sum_{i,j} D_k(x_i y_j) = \sum_{i,j} \{(D_k x_i) y_j + x_i (D_k y_j)\}$$

$$= \left(\sum_i D_k x_i\right)\left(\sum_j y_j\right) + \left(\sum_i x_i\right)\left(\sum_j D_k y_j\right) = (D_k x) y + x (D_k y). \quad \square$$

LEMMA 3.2. *For the universal Stiefel-Whitney class* w_j,

$$D_1 Sq^i w_j = Sq^i w_{j-1}.$$

PROOF. If $i > j$, then $D_1 Sq^i w_j = D_1 0 = 0$. If $i = j$, then $D_1 Sq^i w_i = D_1 w_i^2 = 0$ by Lemma 3.1. On the other hand, $Sq^i w_{j-1} = 0$ for $i \geq j$. Hence the equality holds for $i \geq j$.

Suppose that $j > i$. Then we have

$$D_1 Sq^i w_j = \sum_{0 \leq m \leq i} C_{j-i-1+m,m} D_1 (w_{i-m} w_{j+m}) \quad \text{by Theorem 2.4}$$

$$= \sum_{0 \leq m \leq i} C_{j-i-1+m,m} (w_{i-m-1} w_{j+m} + w_{i-m} w_{j+m-1})$$

$$= \sum_{1 \leq m \leq i} (C_{j-i+m-2,m-1} + C_{j-i+m-1,m}) w_{i-m} w_{j+m-1} + w_i w_{j-1}$$

$$= \sum_{1 \leq m \leq i} C_{j-i+m-2,m} w_{i-m} w_{j+m-1} + w_i w_{j-1} \quad \text{by Lemma 2.2.}$$

Now, if $j \geq i+2$, then

$$D_1 Sq^i w_j = \sum_{0 \leq m \leq i} C_{j-i+m-2,m} w_{i-m} w_{j+m-1} = Sq^i w_{j-1} \quad \text{by Theorem 2.4,}$$

and if $j = i+1$, then

$$D_1 Sq^i w_{i+1} = \sum_{1 \leq m \leq i} C_{m-1,m} w_{i-m} w_{i+m} + w_i^2 = w_i^2 = Sq^i w_i. \quad \square$$

LEMMA 3.3. *Let t be any positive integer. Then, for the universal Stiefel-Whitney class* w_j,

$$D_{2t+1} Sq^i w_j = D_{t+1} D_t Sq^i w_j + D_1 Sq^{i-t} w_{j-t}.$$

PROOF. Note that by Lemma 3.1

$$D_{t+1} D_t (w_{i-m} w_{j+m}) + D_1 (w_{i-m-t} w_{j+m-t}) = D_{2t+1} (w_{i-m} w_{j+m}).$$

Then we have

$$D_{2t+1} Sq^i w_j = \sum_{0 \leq m \leq i} C_{j-i-1+m,m} D_{2t+1} (w_{i-m} w_{j+m}) \quad \text{by Theorem 2.4}$$

$$= \sum_{0 \leq m \leq i} C_{j-i-1+m,m} \{D_{t+1} D_t (w_{i-m} w_{j+m}) + D_1 (w_{i-m-t} w_{j+m-t})\}$$

$$= D_{t+1} D_t \sum_{0 \leq m \leq i} C_{j-i-1+m,m} (w_{i-m} w_{j+m})$$

$$+ D_1 \sum_{0 \leq m \leq i-t} C_{j-i-1+m,m} (w_{i-m-t} w_{j+m-t})$$

$$= D_{t+1} D_t Sq^i w_j + D_1 Sq^{i-t} w_{j-t} \quad \text{by Theorem 2.4.} \quad \square$$

LEMMA 3.4. *Let t be any positive integer. Then, for any element* $x \in H^*(BO; Z_2)$,

$$D_{2t} x = D_t D_t x.$$

PROOF. It suffices to prove the equality when x is a monomial of the universal Stiefel-Whitney classes, namely it suffices to prove the equality

$$D_{2t}(w_{i(1)} w_{i(2)} \cdots w_{i(\ell)}) = D_t D_t (w_{i(1)} w_{i(2)} \cdots w_{i(\ell)}),$$

where $w_{i(n)}$ $(1 \leq n \leq \ell)$ is the $i(n)$-dimensional universal Stiefel-Whitney class.

We prove the equality above by induction on ℓ. If $\ell = 1$, then $D_t D_t (w_{i(1)}) = D_t w_{i(1)-t} = w_{i(1)-2t} = D_{2t} w_{i(1)}$.

Assume that the equality holds for $\ell = s$. Then we have

$$D_t D_t (w_{i(1)} \cdots w_{i(s)} w_{i(s+1)})$$

$$= D_t \{D_t (w_{i(1)} \cdots w_{i(s)}) w_{i(s+1)} + (w_{i(1)} \cdots w_{i(s)}) w_{i(s+1)-t}\} \quad \text{by Lemma 3.1}$$

$$= D_t D_t (w_{i(1)} \cdots w_{i(s)}) w_{i(s+1)} + D_t (w_{i(1)} \cdots w_{i(s)}) w_{i(s+1)-t}$$

$$+ D_t (w_{i(1)} \cdots w_{i(s)}) w_{i(s+1)-t} + w_{i(1)} \cdots w_{i(s)} w_{i(s+1)-2t} \quad \text{by Lemma 3.1}$$

$$= D_{2t} (w_{i(1)} \cdots w_{i(s)}) w_{i(s+1)} + w_{i(1)} \cdots w_{i(s)} w_{i(s+1)-2t}$$

by the inductive assumption

$$= D_{2t} (w_{i(1)} \cdots w_{i(s)} w_{i(s+1)}) \quad \text{by Lemma 3.1.}$$

Hence the equality holds for $\ell = s+1$. $\square$

THEOREM 3.5. *Let k be any power of 2. Then, for the universal Stiefel-Whitney class* w_j,

$$D_k Sq^i w_j = Sq^i w_{j-k}.$$

PROOF. Let $k = 2^r$ $(r \geq 0)$. For fixed i, we prove the equality by induction on (j, r). By Lemma 3.2, the equality holds for $(j, r) = (j, 0)$, where j is any integer.

Assume that the equality holds for (j, r), where j is any integer and $k = s = 2^r$ (≥ 1). Then we have

$$\begin{aligned} D_{2s}Sq^i w_j &= D_s D_s Sq^i w_j && \text{by Lemma 3.4} \\ &= D_s Sq^i w_{j-s} && \text{by the inductive assumption} \\ &= Sq^i w_{j-2s} && \text{by the inductive assumption.} \end{aligned}$$

Hence the equality holds for $(j, r+1)$, where j is any integer and $k = 2s = 2^{r+1}$. □

4. A formula on $D_k Sq^i x$

THEOREM 4.1. *Let k be any positive integer. Then, for the universal Stiefel-Whitney class w_j,*

$$D_k Sq^i w_j = \sum\nolimits_{0 \le s \le k-1} C_{k-s-1,s} Sq^{i-s} w_{j-k+s}.$$

PROOF. We prove the equality by induction on (i, j, k). If $(i, j, k) = (i, j, 1)$, the equality holds for any i and j by Lemma 3.2.

Assume that $n \ge 2$ and that the equality holds for (i, j, k), where i and j are any integers and $k \le n-1$. Then we intend to prove the equality for (i, j, n).

If $n = 2t$ $(t \ge 1)$, then we have

$$\begin{aligned} D_{2t}Sq^i w_j &= D_t D_t Sq^i w_j \qquad \text{by Lemma 3.4} \\ &= \sum\nolimits_{0 \le s \le t-1} C_{t-s-1,s} D_t Sq^{i-s} w_{j-t+s} \qquad \text{by the inductive assumption} \\ &= \sum\nolimits_{0 \le s \le t-1} \sum\nolimits_{0 \le r \le t-1} C_{t-s-1,s} C_{t-r-1,r} Sq^{i-(s+r)} w_{j-2t+(s+r)} \\ &\qquad \text{by the inductive assumption} \\ &= \sum\nolimits_{0 \le m \le 2t-2} B_m Sq^{i-m} w_{j-2t+m}, \end{aligned}$$

where $B_m = \sum_{s+r=m} C_{t-s-1,s} C_{t-r-1,r}$. Here, if m is odd, $B_m \equiv 0 \bmod 2$, and if m is even, $B_m \equiv C_{t-m/2-1,m/2} \equiv C_{2t-m-2,m} \equiv C_{2t-m-1,m}$ by Lemma 2.1. We therefore have

$$D_{2t}Sq^i w_j = \sum\nolimits_{0 \le m \le t-1} C_{2t-2m-1,2m} Sq^{i-2m} w_{j-2t+2m}.$$

On the other hand,

$$\sum\nolimits_{0 \le s \le 2t-1} C_{2t-s-1,s} Sq^{i-s} w_{j-2t+s} = \sum\nolimits_{0 \le m \le t-1} C_{2t-2m-1,2m} Sq^{i-2m} w_{j-2t+2m}$$

since $C_{2t-s-1,s}$ is even by Lemma 2.1 if s is odd. Hence we obtain the desired equality for $(i, j, n) = (i, j, 2t)$.

If $n = 2t+1$ $(t \ge 1)$, then we have

$$\begin{aligned} D_{2t+1}Sq^i w_j &= D_{t+1} D_t Sq^i w_j + Sq^{i-t} w_{j-t-1} \qquad \text{by Lemmas 3.3 and 3.2} \\ &= \sum\nolimits_{0 \le s \le t-1} C_{t-s-1,s} D_{t+1} Sq^{i-s} w_{j-t+s} + Sq^{i-t} w_{j-t-1} \\ &\qquad \text{by the inductive assumption} \\ &= \sum\nolimits_{0 \le s \le t-1} \sum\nolimits_{0 \le r \le t} C_{t-s-1,s} C_{t-r,r} Sq^{i-(s+r)} w_{j-2t+(s+r)-1} + Sq^{i-t} w_{j-t-1} \\ &\qquad \text{by the inductive assumption} \\ &= \sum\nolimits_{0 \le m \le 2t-1} C_m Sq^{i-m} w_{j-2t+m-1} + Sq^{i-t} w_{j-t-1}, \end{aligned}$$

where $C_m = \sum_{s+r=m} C_{t-s-1,s} C_{t-r,r}$.

Case 1. $0 \le m \le t-1$.

$$\begin{aligned} C_m &= \sum\nolimits_{0 \le s \le m} C_{t-s-1,s} C_{t-m+s,m-s} = \sum\nolimits_{0 \le s \le t-1} C_{t-s-1,s} C_{t-m+s,m-s} \\ &= A(t-1, t-m, m) \equiv C_{2t-m,m} \bmod 2 \\ &\qquad \text{by the latter part of Lemma 2.3.} \end{aligned}$$

Case 2. $t \le m \le 2t-1$. If $C_{t-s-1,s} C_{t-r,r}$ is odd, then $t-s-1 \ge s$ and $t-r \ge r$, and so $s+r \le t-1$. But this is impossible since $s+r = m \ge t$. Hence C_m is even.

Combining Cases 1 and 2, we have

$$\begin{aligned} D_{2t+1}Sq^i w_j &= \sum\nolimits_{0 \le m \le t-1} C_{2t-m,m} Sq^{i-m} w_{j-2t+m-1} + Sq^{i-t} w_{j-t-1} \\ &= \sum\nolimits_{0 \le s \le t} C_{2t-s,s} Sq^{i-s} w_{j-2t+s-1} \\ &= \sum\nolimits_{0 \le s \le 2t} C_{2t-s,s} Sq^{i-s} w_{j-2t+s-1}. \end{aligned}$$

The last equality holds, because if $C_{2t-s,s}$ is odd for $0 \le s \le 2t$, then $2t - s \ge s$, namely $0 \le s \le t$. Thus we obtain the desired equality for $(i, j, n) = (i, j, 2t+1)$.

Hence the equality holds for any (i, j, k). □

REMARK. If k is a power of 2, then $C_{k-s-1,s}$ is even for $1 \le s \le k-1$ by Lemma 2.1. Hence Theorem 4.1 is a generalization of Theorem 3.5.

THEOREM 4.2. *Let k be any positive integer. Then, for any element $x \in H^*(BO; Z_2)$,*

$$D_k Sq^i x = \sum\nolimits_{0 \le s \le k-1} C_{k-s-1,s} Sq^{i-s} D_{k-s} x.$$

PROOF. If $\dim x \leq 0$, then each side of the equality is 0.

For fixed k and $x = w_{j(1)}w_{j(2)}\cdots w_{j(\ell)}$, where $w_{j(m)}$ $(1 \leq m \leq \ell)$ is the universal Stiefel-Whitney class, we prove the equality by induction on (i, ℓ). If $(i, \ell) = (i, 1)$, then the equality holds for any integer i by Theorem 4.1 since $D_{k-s}w_{j(1)} = w_{j(1)-k+s}$.

Assume that $n \geq 1$ and that the equality holds for $(i, \ell) = (i, n)$, namely for $x = \alpha = w_{j(1)}w_{j(2)}\cdots w_{j(n)}$, where i is any integer. Set $\beta = w_{j(n+1)}$. Then we have

$$D_kSq^i(\alpha\beta) = D_k\sum_{0\leq p\leq i}(Sq^p\alpha)(Sq^{i-p}\beta)$$

$$= \sum_{0\leq p\leq i}(D_kSq^p\alpha)(Sq^{i-p}\beta) + \sum_{0\leq p\leq i}(Sq^p\alpha)(D_kSq^{i-p}\beta)$$

by Lemma 3.1

$$= \sum_{0\leq p\leq i}\Big\{\sum_{0\leq s\leq k-1} C_{k-s-1,s}(Sq^{p-s}D_{k-s}\alpha)(Sq^{i-p}\beta)\Big\}$$

$$+ \sum_{0\leq p\leq i}\Big\{\sum_{0\leq s\leq k-1} C_{k-s-1,s}(Sq^p\alpha)(Sq^{i-p-s}D_{k-s}\beta)\Big\}$$

by the inductive assumption

$$= \sum_{0\leq s\leq k-1} C_{k-s-1,s}Sq^{i-s}\{(D_{k-s}\alpha)\beta\}$$

$$+ \sum_{0\leq s\leq k-1} C_{k-s-1,s}Sq^{i-s}\{\alpha(D_{k-s}\beta)\}$$

$$= \sum_{0\leq s\leq k-1} C_{k-s-1,s}Sq^{i-s}\{(D_{k-s}\alpha)\beta + \alpha(D_{k-s}\beta)\}$$

$$= \sum_{0\leq s\leq k-1} C_{k-s-1,s}Sq^{i-s}D_{k-s}(\alpha\beta) \quad \text{by Lemma 3.1.}$$

Hence the equality holds for $x = \alpha\beta$, namely for $(i, \ell) = (i, n+1)$, where i is any integer. Thus the equality holds for any monomials.

For $x = \sum_j x_j$, where x_j is a monomial, we have

$$D_kSq^ix = \sum_j D_kSq^ix_j$$

$$= \sum_j\sum_{0\leq s\leq k-1} C_{k-s-1,s}Sq^{i-s}D_{k-s}x_j$$

$$= \sum_{0\leq s\leq k-1} C_{k-s-1,s}Sq^{i-s}D_{k-s}\sum_j x_j$$

$$= \sum_{0\leq s\leq k-1} C_{k-s-1,s}Sq^{i-s}D_{k-s}x. \qquad \square$$

5. Proofs of Theorem 1 and Corollary 2

First, we prepare two lemmas.

LEMMA 5.1. *Let k be any power of 2. Then, for any element $x \in H^*(BO; Z_2)$,*

$$D_kSq^ix = Sq^iD_kx.$$

PROOF. If k is a power of 2, $C_{k-s-1,s}$ is even for any s with $1 \leq s \leq k-1$ by Lemma 2.1. Hence the result follows from Theorem 4.2. $\square$

LEMMA 5.2. *Let k be any power of 2. Then, for any integer $i \geq k$,*

$$D_kv_i = v_{i-k}.$$

PROOF. We prove the equality by induction on i for any fixed k, where $i \geq k$. If $i = k$, then we have

$$D_kv_k = D_k\Big(w_k + \sum_{1\leq j\leq k} Sq^jv_{k-j}\Big) \quad \text{by (1.2)}$$

$$= w_0 + \sum_{1\leq j\leq k} Sq^jD_kv_{k-j} \quad \text{by Lemma 5.1.}$$

Here $D_kv_{k-j} = 0$ for $1 \leq j \leq k$. Hence

$$D_kv_k = w_0 = v_0 \quad \text{by (1.2).}$$

Assume that $j \geq k+1$ and that $D_kv_i = v_{i-k}$ holds for any i, where $k \leq i \leq j-1$. Then we intend to prove the equality for $i = j$.

Now, we have

$$D_kv_j = D_k\Big(w_j + \sum_{1\leq s\leq j} Sq^sv_{j-s}\Big) \quad \text{by (1.2)}$$

$$= w_{j-k} + \sum_{1\leq s\leq j-k} Sq^sD_kv_{j-s} \quad \text{by Lemma 5.1}$$

$$= w_{j-k} + \sum_{1\leq s\leq j-k} Sq^sv_{j-s-k} \quad \text{by the inductive assumption}$$

$$= v_{j-k} \quad \text{by (1.2).}$$

Hence the equality holds for $i = j$.

We therefore have the equality $D_kv_i = v_{i-k}$ for any $i \geq k$, provided k is a power of 2. $\square$

PROOF OF THEOREM 1. If $i < k$, $D_kv_i = 0 = v_{i-k}$. Hence it suffices to consider the case where $i \geq k$. We prove the result by induction on (i, k), where $i \geq k$.

Step 1. If $k = 1$, the result holds for any integer $i \geq 1$ by Lemma 5.2.

Step 2. Assume that $m \geq n \geq 2$ and that the result holds for any (i,k), where $k \leq n-1$ and $i \geq k$, and, in addition, for any (i,n), where $n \leq i \leq m-1$ if $n < m$. Then we intend to prove the result for $(i,k) = (m,n)$.

Now, we have

$$D_n v_m = D_n\Big(w_m + \sum\nolimits_{1\leq t\leq m} Sq^t v_{m-t}\Big) \quad \text{by (1.2)}$$

$$= w_{m-n} + \sum\nolimits_{1\leq t\leq m} D_n Sq^t v_{m-t}$$

$$= w_{m-n} + \sum\nolimits_{1\leq t\leq m}\sum\nolimits_{0\leq s\leq n-1} C_{n-s-1,s} Sq^{t-s} D_{n-s} v_{m-t}$$

by Theorem 4.2

$$= w_{m-n} + \sum\nolimits_{0\leq s\leq n-1} C_{n-s-1,s}\sum\nolimits_{1\leq t\leq m} Sq^{t-s} D_{n-s} v_{m-t}. \qquad (*)$$

If $n = 2^p$, then $C_{n-s-1,s}$ is even for $1 \leq s \leq n-1$. Hence we have

$$D_n v_m = w_{m-n} + \sum\nolimits_{1\leq t\leq m} Sq^t D_n v_{m-t}$$

$$= w_{m-n} + \sum\nolimits_{1\leq t\leq m-n} Sq^t v_{m-t-n} \quad \text{by the inductive assumption}$$

$$= v_{m-n} \quad \text{by (1.2)}.$$

If $n = 2^p + q$ $(2^p > q \geq 1)$, then, unless $n-s$ is a power of 2,

$$D_{n-s} v_{m-t} = 0 \quad \text{for } 0 \leq s \leq n-1 \text{ by the inductive assumption.}$$

Hence, in the equality $(*)$, it suffices to consider the case where $n-s = 2^a$, $0 \leq s \leq n-1$ and $0 \leq s \leq n-s-1$. These imply $2^a \leq n \leq 2^{a+1}-1$, and so $a = p$. Then we have

$$D_{2^p} v_{m-t} = v_{m-t-2^p} \quad \text{by the inductive assumption.}$$

Since $C_{2^a-1,n-2^a}$ $(= C_{2^p-1,q})$ is odd by Lemma 2.1, we have

$$D_n v_m = w_{m-n} + C_{2^p-1,q}\sum\nolimits_{1\leq t\leq m} Sq^{t-q} v_{m-2^p-t} \quad \text{by } (*)$$

$$= w_{m-n} + \sum\nolimits_{1\leq t\leq m} Sq^{t-q} v_{m-2^p-t}$$

$$= w_{m-n} + \sum\nolimits_{0\leq j\leq m-n} Sq^j v_{m-n-j}$$

$$= w_{m-n} + w_{m-n} \quad \text{by (1.2)}$$

$$= 0.$$

Hence the result holds for $(i,k) = (m,n)$.

Combining Step 1 with Step 2, we obtain the result for

$$(1,1),(2,1),\ldots,(i,1),\ldots;(2,2),(3,2),\ldots,(i,2),\ldots;$$

$$\ldots;(n,n),(n+1,n),\ldots,(i,n),\ldots;\ldots$$

namely for all (i,k) with $i \geq k \geq 1$. □

PROOF OF COROLLARY 2. By (1.2) and Theorem 2.4 we can write

$$v_k = cw_k + \sum\nolimits_i x_i,$$

where $c \in Z_2$ $(= H^0(BO;Z_2))$, $x_i = w_{i(1)}w_{i(2)}\cdots w_{i(\ell)}$ and $1 \leq i(m) \leq k-1$ for every m with $1 \leq m \leq \ell$. Then $D_k v_k = c$ since $D_k w_k = w_0$ $(=1)$ and $D_k x_i = 0$.

On the other hand, by Theorem 1, $D_k v_k = v_0$ $(=1)$ if and only if k is a power of 2. Hence $c = 1$ if and only if k is a power of 2. □

6. A formula on v_n for $n = 2^a + b$ $(2^a > b \geq 0)$

Let $\bar{w}_i$ $(\in H^i(BO;Z_2))$ stand for the i-dimensional dual universal Stiefel-Whitney class, namely let $\bar{w}_i$ satisfy

$$(w_0 + w_1 + w_2 + \cdots)(\bar{w}_0 + \bar{w}_1 + \bar{w}_2 + \cdots) = 1 \quad (\in H^*(BO;Z_2))$$

(cf. [3, p. 40]). We prepare three lemmas.

LEMMA 6.1. *If $i \geq k \geq 1$, then, for the dual universal Stiefel-Whitney class $\bar{w}_i \in H^i(BO;Z_2)$,*

$$D_k \bar{w}_i = \bar{w}_{i-k}.$$

PROOF. We prove the equality by induction on (i,k), where $i \geq k \geq 1$. If $(i,k) = (k,k)$, then we have

$$D_k \bar{w}_k = D_k\Big(\sum\nolimits_{1\leq s\leq k-1} w_s\bar{w}_{k-s} + w_k\Big) = \sum\nolimits_{1\leq s\leq k-1} D_k(w_s\bar{w}_{k-s}) + w_0$$

$$= \sum\nolimits_{1\leq s\leq k-1}\{(D_k w_s)\bar{w}_{k-s} + w_s(D_k\bar{w}_{k-s})\} + w_0 \quad \text{by Lemma 3.1}$$

$$= w_0 = 1 = \bar{w}_0.$$

Assume that $j \geq k+1$ and that $D_k\bar{w}_i = \bar{w}_{i-k}$ for any (i,k), where $1 \leq k \leq i \leq j-1$. Then we have

$$D_k\bar{w}_j = D_k\Big(\sum\nolimits_{1\leq s\leq j-1} w_s\bar{w}_{j-s} + w_j\Big) = \sum\nolimits_{1\leq s\leq j-1} D_k(w_s\bar{w}_{j-s}) + w_{j-k}$$

$$= \sum\nolimits_{1\leq s\leq j-1}\{(D_k w_s)\bar{w}_{j-s} + w_s(D_k\bar{w}_{j-s})\} + w_{j-k} \quad \text{by Lemma 3.1}$$

$$= \sum_{1\le s\le k-1}(D_k w_s)\bar{w}_{j-s} + \sum_{k\le s\le j-1}(D_k w_s)\bar{w}_{j-s}$$
$$+ \sum_{1\le s\le j-k} w_s(D_k\bar{w}_{j-s}) + \sum_{j-k+1\le s\le j-1} w_s(D_k\bar{w}_{j-s}) + w_{j-k}.$$

Here

$$\sum_{1\le s\le k-1}(D_k w_s)\bar{w}_{j-s} = 0,$$
$$\sum_{k\le s\le j-1}(D_k w_s)\bar{w}_{j-s} = \sum_{k\le s\le j-1} w_{s-k}\bar{w}_{j-s}$$
$$= w_0\bar{w}_{j-k} + \sum_{k+1\le s\le j-1} w_{s-k}\bar{w}_{j-s},$$
$$\sum_{1\le s\le j-k} w_s(D_k\bar{w}_{j-s}) = \sum_{1\le s\le j-k} w_s\bar{w}_{j-k-s}$$

by the inductive assumption

$$= \sum_{k+1\le s\le j-1} w_{s-k}\bar{w}_{j-s} + w_{j-k}\bar{w}_0,$$
$$\sum_{j-k+1\le s\le j-1} w_s(D_k\bar{w}_{j-s}) = 0.$$

Thus we obtain $D_k\bar{w}_j = w_0\bar{w}_{j-s} + w_{j-k}\bar{w}_0 + w_{j-k} = \bar{w}_{j-s}$. Hence the equality holds for $(i,k) = (j,k)$.

We therefore have the equality for any (i,k) with $i \ge k \ge 1$. □

In the following, let P_k and R_k denote polynomials of $w_k, w_{k-1}, \dots, w_1$ $(k \ge 1)$ and $P_0, R_0 \in Z_2$ $(= H^0(BO; Z_2))$.

LEMMA 6.2. *For any positive integer n, write $n = 2^a + b$ $(2^a > b \ge 0)$. Then we have*

$$v_n = w_{2^a} v_b + R_{2^a-1}.$$

PROOF. If $n = 1$, that is, $a = 0$ and $b = 0$, then the equality holds for $R_0 = 0$. If $n = 2^a$ $(a \ge 1)$, that is, $b = 0$, then we may put $v_n = P_0 w_n + R_{n-1}$ by (1.2) and Theorem 2.4. Using Lemma 3.1 repeatedly, we have $D_n R_{n-1} = 0$. Hence, by Theorem 1, $v_0 = D_n v_n = D_n(P_0 w_n + R_{n-1}) = P_0 w_0 = P_0$ as desired.

If $n = 2^a + b$ $(2^a > b > 0)$, then we may put

$$v_n = \sum_{0\le i\le b} w_{n-i}P_i + R_{2^a-1}$$

by (1.2) and Theorem 2.4. Using Lemma 3.1 repeatedly, we have $D_n(w_{n-i}P_i) = 0$ for $1 \le i \le b$ and $D_n R_{2^a-1} = 0$. Then, by Theorem 1, $0 = D_n v_n = P_0 w_0 = P_0$.

Assume that $0 < j < b$ and that $P_i = 0$ for any i with $0 \le i \le j-1$. Then

$$v_n = \sum_{j\le i\le b} w_{n-i}P_i + R_{2^a-1}.$$

Using Lemma 3.1 repeatedly, we have $D_{n-j}(\sum_{j\le i\le b} w_{n-i}P_i) = w_0 P_j$ and $D_{n-j}R_{2^a-1} = 0$. Hence, by Theorem 1, $0 = D_{n-j}v_n = w_0 P_j = P_j$. We therefore have $P_i = 0$ for any i with $0 \le i < b$ by induction on i, and so $v_n = w_{2^a}P_b + R_{2^a-1}$. Then $v_b = D_{2^a}v_n = (D_{2^a}w_{2^a})P_b = P_b$ by Theorem 1 and Lemma 3.1. Thus we have obtained the result. □

LEMMA 6.3. *Let $n = 2^a + b$, where a and b are integers with $a \ge 1$ and $2^a > b \ge 0$. Then we have*

$$v_n = \sum_{[n/2]+1\le j\le 2^a} w_j P_{j-1} + R_{[n/2]},$$

where P_k and R_k are some polynomials of $w_k, w_{k-1}, \dots, w_1$, and $P_{2^a-1} = v_b$. Furthermore, P_{j-1} may be replaced by P_{n-j}.

PROOF. By Lemma 6.2, $v_n = w_{2^a}v_b + R_{2^a-1}$, where R_k is a polynomial of $w_k, w_{k-1}, \dots, w_1$. If, for $j \le [n/2]$, $w_j P_{j-1}$ exists in the polynomial R_{2^a-1}, then clearly $w_j P_{j-1}$ is a polynomial of $w_{[n/2]}, w_{[n/2]-1}, \dots, w_1$. If $w_j^2 P_j$ exists in R_{2^a-1}, then $\dim P_j = n - 2j \ge 0$, and so $j \le [n/2]$. Hence $w_j^2 P_j$ is a polynomial of $w_{[n/2]}, w_{[n/2]-1}, \dots, w_1$. Thus we obtain the former part.

If $w_j P_{j-1}$ is in the sum of the lemma, then $\dim P_{j-1} = n - j$. Hence the latter part is clear. (Note that $n - j \le j - 1$ for $[n/2] + 1 \le j$.) □

THEOREM 6.4. *Let $n = 2^a + b$, where a and b are integers with $a \ge 1$ and $2^a > b \ge 0$. Then we have*

$$v_n = \sum_{0\le i\le 2^{a-1}-[b/2]-1} w_{2^a-i}\bar{w}_i v_b + R_{[n/2]}.$$

In particular, $v_{2^a} = w_{2^a} + \sum_{1\le i\le 2^{a-1}-1} w_{2^a-i}\bar{w}_i + R_{2^{a-1}}$. Here R_k denotes some polynomial of $w_k, w_{k-1}, \dots, w_1$.

PROOF. According to Example 2.5, we have the former part for $2 \le n \le 7$ as follows.

If $n = 2$, that is, $a = 1$ and $b = 0$, then $v_2 = w_2 + w_1^2 = w_2\bar{w}_0 v_0 + w_1^2$.

If $n = 3$, that is, $a = 1$ and $b = 1$, then $v_3 = w_2 w_1 = w_2\bar{w}_0 v_1$.

If $n = 4$, that is, $a = 2$ and $b = 0$, then $v_4 = w_4 + w_3 w_1 + w_2^2 + w_1^4 = (w_4\bar{w}_0 + w_3\bar{w}_1)v_0 + w_2^2 + w_1^4$.

If $n = 5$, that is, $a = 2$ and $b = 1$, then $v_5 = w_4 w_1 + w_3 w_1^2 + w_2^2 w_1 + w_2 w_1^3 = (w_4\bar{w}_0 + w_3\bar{w}_1)v_1 + w_2^2 w_1 + w_2 w_1^3$.

If $n = 6$, that is, $a = 2$ and $b = 2$, then $v_6 = w_4 w_2 + w_4 w_1^2 + w_3^2 + w_3 w_2 w_1 + w_3 w_1^3 + w_2^2 w_1^2 = w_4\bar{w}_0 v_2 + w_3^2 + w_3 w_2 w_1 + w_3 w_1^3 + w_2^2 w_1^2$.

If $n=7$, that is, $a=2$ and $b=3$, then $v_7 = w_4w_2w_1 + w_3^2w_1 + w_3w_2w_1^2 = w_4\bar{w}_0v_3 + w_3^2w_1 + w_3w_2w_1^2$.

Let $a \geq 3$. Then we may put

$$v_n = w_{2^a}v_b + \sum\nolimits_{1\leq i\leq 2^{a-1}-[b/2]-1} w_{2^a-i}P_{b+i} + R_{[n/2]}$$

for $n=2^a+b$ $(2^a > b \geq 0)$ by Lemma 6.3. We intend to prove that $P_{b+i} = \bar{w}_i v_b$ by induction on i. Now, we have

$$0 = D_{2^a-1}v_n = w_1v_b + (D_{2^a-1}w_{2^a-1})P_{b+1} = \bar{w}_1v_b + P_{b+1}$$

by Theorem 1 and Lemma 3.1. Hence $P_{b+1} = \bar{w}_1v_b$.

Assume that $2 \leq j \leq 2^{a-1}-[b/2]-1$ and that $P_{b+i} = \bar{w}_iv_b$ for any i with $1 \leq i \leq j-1$. Then, using Theorem 1, we have

$$0 = D_{2^a-j}v_n = w_jv_b + \sum\nolimits_{1\leq i\leq 2^{a-1}-[b/2]-1}\{(D_{2^a-j}w_{2^a-i})P_{b+i}\}$$

$$= w_jv_b + \sum\nolimits_{1\leq i\leq j} w_{j-i}P_{b+i} = w_jv_b + \sum\nolimits_{1\leq i\leq j-1} w_{j-i}P_{b+i} + P_{b+j}$$

$$= w_jv_b + \sum\nolimits_{1\leq i\leq j-1} w_{j-i}\bar{w}_iv_b + P_{b+j} \qquad \text{by the inductive assumption}$$

$$= (w_j + \sum\nolimits_{1\leq i\leq j-1} w_{j-i}\bar{w}_i)v_b + P_{b+j} = \bar{w}_jv_b + P_{b+j}.$$

Hence $P_{b+j} = \bar{w}_jv_b$. We therefore obtain that $P_{b+i} = \bar{w}_iv_b$ for any i with $1 \leq i \leq 2^{a-1}-[b/2]-1$.

Thus we have completed the proof of the former part. The latter part follows immediately from the former part. □

Corollary 2 follows also from Theorem 6.4.

Using (1.2), Example 2.5 and Theorem 6.4, we have

EXAMPLE 6.5.

$$v_8 = (w_8 + w_7\bar{w}_1 + w_6\bar{w}_2 + w_5\bar{w}_3)v_0 + R_4$$

$$= w_8 + w_7w_1 + w_6(w_2 + w_1^2) + w_5(w_3 + w_1^3) + R_4,$$

$$v_9 = (w_8 + w_7\bar{w}_1 + w_6\bar{w}_2 + w_5\bar{w}_3)v_1 + R_4$$

$$= w_8w_1 + w_7w_1^2 + w_6(w_2w_1 + w_1^3) + w_5(w_3w_1 + w_1^4) + R_4,$$

$$v_{10} = (w_8 + w_7\bar{w}_1 + w_6\bar{w}_2)v_2 + R_5$$

$$= w_8(w_2 + w_1^2) + w_7(w_2w_1 + w_1^3) + w_6(w_2^2 + w_1^4) + R_5.$$

7. A formula on v_n for $n = 2^a + b$ $(2^a > b \geq 2^{a-1})$

In this section we prove

THEOREM 7.1. *Let $n = 2^a + b$, where a and b are integers with $a \geq 2$ and $2^a > b \geq 2^{a-1}$. Then we have*

$$v_n = \sum\nolimits_{0\leq i\leq 2^a-[n/3]-1} w_{2^a-i}\bar{w}_iv_b$$

$$+ \sum\nolimits_{2^{a-1}-[b/2]\leq i\leq 2^a-[n/3]-1} w_{2^a-i}^2Q_{2^a-i-1} + R_{[n/3]},$$

where Q_k and R_k are polynomials of $w_k, w_{k-1}, \ldots, w_1$.

Before proving Theorem 7.1, we prepare a lemma.

LEMMA 7.2. *Let $n = 2^a + b$, where a and b are integers with $a \geq 2$ and $2^a > b \geq 0$. Then we have*

$$v_n = w_{2^a}v_b + \sum\nolimits_{[n/3]+1\leq j\leq 2^a-1} w_jP_{j-1} + \sum\nolimits_{[n/3]+1\leq j\leq [n/2]} w_j^2Q_{j-1} + R_{[n/3]},$$

where P_k, Q_k and R_k are polynomials of $w_k, w_{k-1}, \ldots, w_1$.

PROOF. By Lemma 6.2, $v_n = w_{2^a}v_b + R_{2^a-1}$, where R_k is a polynomial of $w_k, w_{k-1}, \ldots, w_1$. If $j \leq [n/3]$, then both w_jP_{j-1} and $w_j^2Q_{j-1}$ are polynomials of $w_{[n/3]}, w_{[n/3]-1}, \ldots, w_1$. If $w_j^2Q_{j-1}$ exists in the sum above, then $\dim Q_{j-1} = n - 2j \geq 0$, and so $j \leq [n/2]$. If $w_j^3Q_j$ exists in the sum above, then $\dim Q_j = n - 3j \geq 0$, and so $j \leq [n/3]$. Hence $w_j^3Q_j$ is a polynomial of $w_{[n/3]}$, $w_{[n/3]-1}, \ldots, w_1$. □

PROOF OF THEOREM 7.1. By Lemma 7.2, we have

$$v_n = \sum\nolimits_{0\leq i\leq 2^a-[n/3]-1} w_{2^a-i}P_{2^a-i-1}$$

$$+ \sum\nolimits_{2^{a-1}-[b/2]\leq i\leq 2^a-[n/3]-1} w_{2^a-i}^2Q_{2^a-i-1} + R_{[n/3]},$$

where P_k, Q_k and R_k are polynomials of $w_k, w_{k-1}, \ldots, w_1$ and $P_{2^a-1} = v_b$.

Let r be any integer with $1 \leq r \leq 2^a - [n/3] - 1$ and assume that the equality $P_{2^a-i-1} = \bar{w}_iv_b$ holds for any integer i with $0 \leq i \leq r-1$. Then we intend to prove that the equality $P_{2^a-r-1} = \bar{w}_rv_b$ holds.

Applying D_{2^a-r} to the equality of Lemma 7.2, we have, by Lemma 3.1,

$$D_{2^a-r}v_n = \sum\nolimits_{0\leq i\leq r} w_{r-i}P_{2^a-i-1} + \sum\nolimits_{0\leq i\leq 2^a-[n/3]-1} w_{2^a-i}D_{2^a-r}P_{2^a-i-1}$$

$$+ \sum\nolimits_{2^{a-1}-[b/2]\leq i\leq 2^a-[n/3]-1} w_{2^a-i}^2D_{2^a-r}Q_{2^a-i-1} + D_{2^a-r}R_{[n/3]}.$$

Now, $2^a - 1 \geq 2^a - r \geq [n/3] + 1 \geq 2^{a-1} + 1$. (The last inequality follows from the inequality $b \geq 2^{a-1}$.) Hence $2^a - r$ is not a power of 2, and so $D_{2^a-r}v_n = 0$ by Theorem 1. Furthermore, $D_{2^a-r}R_{[n/3]} = 0$ clearly.

If $i > r-1$, then $D_{2^a-r}P_{2^a-i-1} = 0$ and $D_{2^a-r}Q_{2^a-i-1} = 0$ by Lemma 3.1. Moreover, if $2^{a-1} - [b/2] \leq i \leq r$, then $D_{2^a-r}Q_{2^a-i-1} = 0$ since $2^a - r > \dim Q_{2^a-i-1} = 2^a + b - 2(2^a - i)$.

Thus we have, by Lemma 3.1 and the assumption,

$$\begin{aligned} 0 &= \sum\nolimits_{0 \leq i \leq r} w_{r-i}P_{2^a-i-1} + \sum\nolimits_{0 \leq i \leq r-1} w_{2^a-i}D_{2^a-r}P_{2^a-i-1} \\ &= \sum\nolimits_{0 \leq i \leq r-1} D_{2^a-r}(w_{2^a-i}P_{2^a-i-1}) + P_{2^a-r-1} \\ &= \sum\nolimits_{0 \leq i \leq r-1} D_{2^a-r}(w_{2^a-i}\overline{w}_i v_b) + P_{2^a-r-1} \\ &= \sum\nolimits_{0 \leq i \leq r-1} \{w_{r-i}\overline{w}_i v_b + w_{2^a-i}(D_{2^a-r}\overline{w}_i)v_b + w_{2^a-i}\overline{w}_i D_{2^a-r}v_b\} + P_{2^a-r-1}. \end{aligned}$$

Here, $D_{2^a-r}\overline{w}_i = 0$ since $2^a - r > i$ from $b \geq 2^{a-1}$, and $D_{2^a-r}v_b = 0$ since $2^a - 1 \geq 2^a - r \geq 2^{a-1} + 1$. Therefore $P_{2^a-r-1} = (\sum_{0 \leq i \leq r-1} w_{r-i}\overline{w}_i)v_b = \overline{w}_r v_b$. □

Using (1.2), Example 2.5 and Theorem 7.1, we have

EXAMPLE 7.3.

$$\begin{aligned} v_6 &= w_4w_2 + w_4w_1^2 + w_3^2 + w_3w_2w_1 + w_3w_1^3 + w_2^2w_1^2 \\ &= (w_4 + w_3\overline{w}_1)v_2 + w_3^2Q_2 + R_2, \\ v_7 &= w_4w_2w_1 + w_3^2w_1 + w_3w_2w_1^2 = (w_4 + w_3\overline{w}_1)v_3 + w_3^2Q_2 + R_2, \\ v_{12} &= (w_8 + w_7\overline{w}_1 + w_6\overline{w}_2 + w_5\overline{w}_3)v_4 + w_6^2Q_5 + w_5^2Q_4 + R_4, \\ v_{13} &= (w_8 + w_7\overline{w}_1 + w_6\overline{w}_2 + w_5\overline{w}_3)v_5 + w_6^2Q_5 + w_5^2Q_4 + R_4, \\ v_{14} &= (w_8 + w_7\overline{w}_1 + w_6\overline{w}_2 + w_5\overline{w}_3)v_6 + w_7^2Q_6 + w_6^2Q_5 + w_5^2Q_4 + R_4, \\ v_{15} &= (w_8 + w_7\overline{w}_1 + w_6\overline{w}_2)v_7 + w_7^2Q_6 + w_6^2Q_5 + R_5. \end{aligned}$$

8. A formula on v_n for $n = 2^a + b$ $(2^{a-1} > b \geq 0)$

For $X \in H^*(BO; Z_2)$ and for a positive integer k, the symbol $\langle X; k \rangle$ denotes the sum of all monomials of $w_k, w_{k-1}, \ldots, w_1$ which appear in X, namely

$$\langle X; k \rangle = \sum\nolimits_{i(j) \leq k\ (1 \leq j \leq n)} w_{i(1)}w_{i(2)} \cdots w_{i(n)}$$

for $X = \sum_{i(j) \leq k\ (1 \leq j \leq n)} w_{i(1)}w_{i(2)} \cdots w_{i(n)} + \sum_{j>k} w_jY_j$, where Y_j is a sum of monomials of $w_j, w_{j-1}, \ldots, w_1$ in $H^*(BO; Z_2)$.

THEOREM 8.1. *Let $n = 2^a + b$, where a and b are integers with $a \geq 2$ and $2^{a-1} > b \geq 0$. Then we have*

$$\begin{aligned} v_n = &\sum\nolimits_{0 \leq i \leq 2^{a-1}-1} w_{2^a-i}\overline{w}_i v_b \\ &+ \sum\nolimits_{2^{a-1} \leq i \leq 2^a-[n/3]-1} w_{2^a-i}P_{2^a-i-1} \\ &+ \sum\nolimits_{2^{a-1}-[b/2] \leq i \leq 2^a-[n/3]-1} w_{2^a-i}^2 Q_{2^a-i-1} + R_{[n/3]}, \end{aligned}$$

where P_k, Q_k and R_k are polynomials of $w_k, w_{k-1}, \ldots, w_1$.

Furthermore, for $2^{a-1} \leq i \leq 2^a - [n/3] - 1$, we have

$$P_{2^a-i-1} = \langle \overline{w}_i v_b + \overline{w}_{i-2^{a-1}}v_{2^{a-1}+b}; 2^a - i - 1 \rangle.$$

PROOF. The former part: If we replace, in the proof of Theorem 7.1, $1 \leq r \leq 2^a - [n/3] - 1$ by $1 \leq r \leq 2^{a-1} - 1$ and $2^a - 1 \geq 2^a - r \geq [n/3] + 1 \geq 2^{a-1} + 1$ by $2^a - 1 \geq 2^a - r \geq 2^{a-1} + 1$, then we have the desired equality by Lemma 7.2.

The latter part: We prove the equality of the latter part by induction on i.

Step 1. We prove the equality for $i = 2^{a-1}$. Applying $D_{2^{a-1}}$ to the equality of the former part, we have, by Theorem 1 and Lemma 3.1,

$$\begin{aligned} v_{2^{a-1}+b} = &\sum\nolimits_{0 \leq i \leq 2^{a-1}-1} w_{2^{a-1}-i}\overline{w}_i v_b + \sum\nolimits_{0 \leq i \leq 2^{a-1}-1} w_{2^a-i}D_{2^{a-1}}(\overline{w}_i v_b) \\ &+ \sum\nolimits_{2^{a-1} \leq i \leq 2^a-[n/3]-1} (w_{2^{a-1}-i}P_{2^a-i-1} + w_{2^a-i}D_{2^{a-1}}P_{2^a-i-1}) \\ &+ \sum\nolimits_{2^{a-1}-[b/2] \leq i \leq 2^a-[n/3]-1} w_{2^a-i}^2 D_{2^{a-1}}Q_{2^a-i-1} + D_{2^{a-1}}R_{[n/3]}. \end{aligned}$$

Here, by Lemma 3.1, $D_{2^{a-1}}P_{2^a-i-1} = 0$ for $i \geq 2^{a-1}$ since $2^{a-1} > 2^a - i - 1$, and $D_{2^{a-1}}Q_{2^a-i-1} = 0$ for $i \leq 2^a - [n/3] - 1$ since $\dim Q_{2^a-i-1} = n - 2(2^a - i) < 2^{a-1}$. Moreover, by Lemma 3.1, $D_{2^{a-1}}(\overline{w}_i v_b) = 0$ for $i < 2^{a-1}$ since $b < 2^{a-1}$, and $D_{2^{a-1}}R_{[n/3]} = 0$ by Lemma 3.1. Hence $v_{2^{a-1}+b} = \sum_{0 \leq i \leq 2^{a-1}-1} w_{2^{a-1}-i}\overline{w}_i v_b + P_{2^{a-1}-1}$. We therefore have

$$\begin{aligned} P_{2^{a-1}-1} &= \left\langle v_{2^{a-1}+b} + \sum\nolimits_{0 \leq i \leq 2^{a-1}-1} w_{2^{a-1}-i}\overline{w}_i v_b; 2^{a-1} - 1 \right\rangle \\ &= \langle v_{2^{a-1}+b} + \overline{w}_{2^{a-1}}v_b; 2^{a-1} - 1 \rangle. \end{aligned}$$

Step 2. If $a = 2$, then $b = 0$ or 1, and so $n = 4$ or 5. In case $n = 4$, by Example 2.5,

$$v_4 = w_4 + w_3w_1 + w_2^2 + w_1^4 = w_4 + w_3\overline{w}_1 + w_2P_1 + w_2^2Q_1 + R_1,$$

where $P_1 = \langle \overline{w}_2 + v_2; 1\rangle = \langle (w_2 + w_1^2) + (w_2 + w_1^2); 1\rangle = 0$. In case $n = 5$, by Example 2.5,

$$v_5 = w_4 w_1 + w_3 w_1^2 + w_2 w_1^3 + w_2^2 w_1 = w_4 v_1 + w_3 \overline{w}_1 v_1 + w_2 P_1 + w_2^2 Q_1 + R_1,$$

where $P_1 = \langle \overline{w}_2 v_1 + v_3; 1\rangle = \langle (w_2 + w_1^2) w_1 + w_2 w_1; 1\rangle = w_1^3$. Hence the result holds for $a = 2$.

Let $a \geq 3$. Assume that $2^{a-1} < r \leq 2^a - [n/3] - 1$ and that the equality

$$P_{2^a-i-1} = \langle \overline{w}_i v_b + \overline{w}_{i-2^{a-1}} v_{2^{a-1}+b}; 2^a - i - 1\rangle$$

holds for any i with $2^{a-1} \leq i \leq r-1$. We intend to prove that the equality holds for $i = r$.

Since $2^{a-2} + 1 \leq [n/3] + 1 \leq 2^a - r \leq 2^{a-1} - 1$, $2^a - r$ is not a power of 2. Hence $D_{2^a-r} v_n = 0$ and $D_{2^a-r} v_b = 0$ by Theorem 1. Furthermore, $D_{2^a-r} R_{[n/3]} = 0$ by Lemma 3.1. So applying D_{2^a-r} to the equality of the former part, we have

$$\begin{aligned} 0 &= D_{2^a-r} \sum\nolimits_{0 \leq i \leq 2^{a-1}-1} w_{2^a-i} \overline{w}_i v_b \\ &\quad + D_{2^a-r} \sum\nolimits_{2^{a-1} \leq i \leq 2^a-[n/3]-1} w_{2^a-i} P_{2^a-i-1} \\ &\quad + D_{2^a-r} \sum\nolimits_{2^{a-1}-[b/2] \leq i \leq 2^a-[n/3]-1} w_{2^a-i}^2 Q_{2^a-i-1} \\ &= ① + ② + ③, \end{aligned}$$

where, by Lemma 3.1,

$$\begin{aligned} ① &= \sum\nolimits_{0 \leq i \leq 2^{a-1}-1} \{w_{r-i} \overline{w}_i v_b + w_{2^a-i} (D_{2^a-r} \overline{w}_i) v_b\}, \\ ② &= \sum\nolimits_{2^{a-1} \leq i \leq r-1} (w_{r-i} P_{2^a-i-1} + w_{2^a-i} D_{2^a-r} P_{2^a-i-1}) \\ &\quad + \sum\nolimits_{r \leq i \leq 2^a-[n/3]-1} (w_{r-i} P_{2^a-i-1} + w_{2^a-i} D_{2^a-r} P_{2^a-i-1}) \\ &= \sum\nolimits_{2^{a-1} \leq i \leq r-1} (w_{r-i} P_{2^a-i-1} + w_{2^a-i} D_{2^a-r} P_{2^a-i-1}) + P_{2^a-r-1}, \\ ③ &= \sum\nolimits_{2^{a-1}-[b/2] \leq i \leq 2^a-[n/3]-1} w_{2^a-i}^2 D_{2^a-r} Q_{2^a-i-1}. \end{aligned}$$

Now, using the fact that $2^a - i > 2^a - r - 1$ in ① and ②, we have

$$\begin{aligned} \langle ①; 2^a - r - 1\rangle &= \left\langle \sum\nolimits_{0 \leq i \leq 2^{a-1}-1} w_{r-i} \overline{w}_i v_b; 2^a - r - 1 \right\rangle, \\ \langle ②; 2^a - r - 1\rangle &= \left\langle \sum\nolimits_{2^{a-1} \leq i \leq r-1} w_{r-i} P_{2^a-i-1}; 2^a - r - 1 \right\rangle + P_{2^a-r-1}. \end{aligned}$$

If $2^a - i \leq 2^a - r - 1$, then $D_{2^a-r} Q_{2^a-i-1} = 0$. Hence $\langle ③; 2^a - r - 1\rangle = 0$. Using the inductive assumption and the fact that $2^a - i - 1 \geq r - i$ and $2^a - i - 1 \geq 2^a - r - 1$ for $2^{a-1} \leq i \leq r-1$, we have

$$\begin{aligned} &\langle ②; 2^a - r - 1\rangle \\ &= \left\langle \sum\nolimits_{2^{a-1} \leq i \leq r-1} w_{r-i} \langle \overline{w}_i v_b + \overline{w}_{i-2^{a-1}} v_{2^{a-1}+b}; 2^a - i - 1\rangle; 2^a - r - 1 \right\rangle + P_{2^a-r-1} \\ &= \sum\nolimits_{2^{a-1} \leq i \leq r-1} \langle w_{r-i} \overline{w}_i v_b + w_{r-i} \overline{w}_{i-2^{a-1}} v_{2^{a-1}+b}; 2^a - r - 1\rangle + P_{2^a-r-1}. \end{aligned}$$

We therefore obtain

$$\begin{aligned} P_{2^a-r-1} &= \left\langle \left(\sum\nolimits_{0 \leq i \leq 2^{a-1}-1} w_{r-i} \overline{w}_i + \sum\nolimits_{2^{a-1} \leq i \leq r-1} w_{r-i} \overline{w}_i \right) v_b \right. \\ &\quad \left. + \left(\sum\nolimits_{2^{a-1} \leq i \leq r-1} w_{r-i} \overline{w}_{i-2^{a-1}} \right) v_{2^{a-1}+b}; 2^a - r - 1 \right\rangle \\ &= \langle \overline{w}_r v_b + \overline{w}_{r-2^{a-1}} v_{2^{a-1}+b}; 2^a - r - 1\rangle. \qquad \square \end{aligned}$$

Using (1.2), Example 2.5 and Theorem 8.1, we have

EXAMPLE 8.2.

$$\begin{aligned} v_8 &= w_8 + w_7 \overline{w}_1 + w_6 \overline{w}_2 + w_5 \overline{w}_3 + w_4 P_3 + w_3 P_2 + w_4^2 Q_3 + w_3^2 Q_2 + R_2, \\ P_3 &= \langle \overline{w}_4 + v_4; 3\rangle \\ &= \langle (w_4 + w_2^2 + w_2 w_1^2 + w_1^4) + (w_4 + w_3 w_1 + w_2^2 + w_1^4); 3\rangle = w_3 w_1 + w_2 w_1^2, \\ P_2 &= \langle \overline{w}_5 + \overline{w}_1 v_4; 2\rangle \\ &= \langle (w_5 + w_3 w_1^2 + w_2^2 w_1 + w_1^5) + w_1 (w_4 + w_3 w_1 + w_2^2 + w_1^4); 2\rangle = 0, \\ v_9 &= (w_8 + w_7 \overline{w}_1 + w_6 \overline{w}_2 + w_5 \overline{w}_3) v_1 + w_4 P_3 + w_4^2 Q_3 + R_3, \\ P_3 &= \langle \overline{w}_4 v_1 + v_5; 3\rangle \\ &= \langle (w_4 + w_2^2 + w_2 w_1^2 + w_1^4) w_1 + (w_4 w_1 + w_3 w_1^2 + w_2^2 w_1 + w_2 w_1^3); 3\rangle \\ &= w_3 w_1^2 + w_1^5, \\ v_{10} &= (w_8 + w_7 \overline{w}_1 + w_6 \overline{w}_2 + w_5 \overline{w}_3) v_2 + w_4 P_3 + w_5^2 Q_4 + w_4^2 Q_3 + R_3, \\ P_3 &= \langle \overline{w}_4 v_2 + v_6; 3\rangle \\ &= \langle (w_4 + w_2^2 + w_2 w_1^2 + w_1^4)(w_2 + w_1^2) \\ &\quad + (w_4 w_2 + w_4 w_1^2 + w_3^2 + w_3 w_2 w_1 + w_3 w_1^3 + w_2^2 w_1^2); 3\rangle \\ &= w_3^2 + w_3 w_2 w_1 + w_3 w_1^3 + w_2^3 + w_2^2 w_1^2 + w_1^6, \end{aligned}$$

$$v_{11} = (w_8 + w_7\bar{w}_1 + w_6\bar{w}_2 + w_5\bar{w}_3)v_3 + w_4P_3 + w_5^2Q_4 + w_4^2Q_3 + R_3,$$

$$P_3 = \langle \bar{w}_4 v_3 + v_7; 3\rangle$$

$$= \langle (w_4 + w_2^2 + w_2w_1^2 + w_1^4)w_2w_1 + (w_4w_2w_1 + w_3^2w_1 + w_3w_2w_1^2); 3\rangle$$

$$= w_3^2w_1 + w_3w_2w_1^2 + w_2^3w_1 + w_2^2w_1^3 + w_2w_1^5.$$

References

[1] H. Ichikawa and T. Yoshida, Some monomials in the universal Wu classes, Hiroshima Math. J. **20** (1990), 127–136.

[2] J. W. Milnor, On the Stiefel-Whitney numbers of complex manifolds and of spin manifolds, Topology **3** (1965), 223–230.

[3] J. W. Milnor and J. D. Stasheff, Characteristic Classes, Ann. of Math. Stud. **76**, Princeton Univ. Press, Princeton, 1974.

[4] H. Osborn, Vector Bundles, Vol. 1, Foundations and Stiefel-Whitney Classes, Academic Press, New York-London, 1982.

[5] N. E. Steenrod and D. B. A. Epstein, Cohomology Operations, Ann. of Math. Stud. **50**, Princeton Univ. Press, Princeton, 1962.

[6] R. E. Stong, Notes on Cobordism Theory, Math. Notes, Princeton Univ. Press, Princeton, 1968.

[7] R. E. Stong and T. Yoshida, Wu classes, Proc. Amer. Math. Soc. **100** (1987), 352–354.

[8] W. T. Wu, Les *i*-carrés dans une variété grassmannienne, C. R. Acad. Sci. Paris **230** (1950), 918–920.

[9] T. Yoshida, The polynomials on w_1, w_2 and w_3 in the universal Wu classes, Hiroshima Math. J. **26** (1996), 189–207.

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SLIGHTLY GENERALIZED CONTINUOUS FUNCTIONS

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Abstract: A new class of functions, slightly generalized continuous functions, is introduced. Basic properties of slightly generalized continuous functions are studied. The class of slightly generalized functions properly includes the class of slightly continuous functions and generalized continuous functions. Also, by using slightly generalized continuous functions, some properties of domain/range of functions are characterized.

1. Introduction and preliminaries

Slightly continuity were introduced by Jain [8] in 1980 and next have been developed by Singal and Jain [16]. Balachandran et al. [2] introduced the notion of generalized continuous functions and investigated some of their basic properties. The same authors [12] continued the study generalized continuous functions and defined the concepts of generalized homeomorphism and *gc*-homeomorphism. Next, Cueva [5, 6] obtained further results on g-continuous and g-closed (g-open) functions which defined by Malgan [13]. On the other hand, Nour [15] and Baker [1] introduced the some weak forms of slightly continuity which are called slightly semi-continuity and slightly precontinuity, respectively. In this paper, we first defined slightly generalized continuous functions and show that the class of slightly generalized continuous functions properly includes the classes of slightly continuous functions and generalized continuous functions. Second, we obtain some results on g-closed sets and investigate basic properties of slightly generalized continuous functions concerning composition and restriction. Finally, we study of the behavior of some separation axioms, related properties and GO-compactness, GO-connectedness under slightly generalized continuous functions. Relationships between slightly generalized continuous functions and GO-connected spaces are investigated. In particularly, it is shown that slightly generalized continuous image of a GO-connected space is connected.

Throughout this paper, spaces (X, τ) and (Y, σ) (or simply X and Y) always mean topological spaces on which no seperation axioms are assumed

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