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ISOTOPIC CLASSIFICATION OF ODD-DIMENSIONAL SIMPLE LINKS OF CODIMENSION TWO

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V. L. KOBEL'SKIĬ

ABSTRACT. This paper gives a PL-isotopy classification of odd-dimensional simple links of dimension $\geqslant 5$ in terms of their Seifert matrices.

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This paper gives an isotopy classification of odd-dimensional (of dimension $\geqslant 5$) simple links in terms of their Seifert matrices. A similar ambient-isotopy classification of simple links was obtained by Liang [1], following the ambient-isotopy classification of simple knots carried out by Levine [5]. In contrast to ambient isotopy, isotopy of knots is an uninteresting equivalence relation: any knot is isotopic to the trivial knot.

The formulation in this paper is given for two-component links, but everything generalizes easily to the case of a larger number of components.

The main results of the paper are formulated in the Introduction (§1.6); §§2 and 3 are devoted to the proofs, and in a supplement (§4) some isotopy invariants of links connected with cobordism invariants of matrices are considered.

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§1. Introduction

1.1. Links. By an n-dimensional *link* is meant a piecewise-linear locally flat oriented submanifold L of the sphere S^{n+2}, homeomorphic to the disjoint sum $S_1^n \amalg S_2^n$ of two n-dimensional spheres. Denote by K_1 and K_2 the connected components of L. It is well known that each submanifold K_i bounds a piecewise-linear locally flat compact oriented submanifold V_i of S^{n+2}, called a *Seifert surface of the knot* K_i. If there exist nonintersecting Seifert surfaces V_1 and V_2 of the components K_1 and K_2, then L is called a *boundary link*, and the manifold $V = V_1 \cup V_2$ is called a *Seifert surface of the link* L. A link of dimension $2q-1$ is called *simple* if it has a Seifert surface consisting of $(q-1)$-connected components.

1.2. Seifert matrices of links. Let L be a boundary link of dimension $2q-1$, and let V be a Seifert surface of L. We denote by TV a regular neighborhood of the surface V in S^{2q+1}. The manifold TV is homeomorphic to the product $V \times J$, where $J = [-1, 1]$, and where the surface V, lying in TV, is carried to $V \times \{0\}$ by this homeomorphism. Putting

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$V_+ = V \times \{1\}$, let i: $V \to V_+$ be the homeomorphism defined by translation $V \times \{0\} \to V \times \{1\}$ in the product $V \times J$, and let j: $V_+ \to S^{2q+1} \setminus V$ be the inclusion. We define a form

$$l: H_q(V; \mathbf{Z})/\mathrm{Tors} \times H_q(V; \mathbf{Z})/\mathrm{Tors} \to \mathbf{Z}$$

as follows: if $x, y \in H_q(V; \mathbf{Z})/\mathrm{Tors}$, then $l(x, y) = lk((j \circ i)_* x, y)$, where

$$lk: H_q(S^{2q+1} \setminus V; \mathbf{Z})/\mathrm{Tors} \times H_q(V; \mathbf{Z})/\mathrm{Tors} \to \mathbf{Z}$$

is the linking coefficient in the sphere S^{2q+1}.

If V_1 and V_2 are Seifert surfaces of the components of the link L, then in the basis of the group $H_q(V; \mathbf{Z})/\mathrm{Tors}$, consisting of bases of $H_q(V_1; \mathbf{Z})/\mathrm{Tors}$ and $H_q(V_2; \mathbf{Z})/\mathrm{Tors}$, the matrix of the form l can be written as

$$M = \left(\begin{array}{c|c} M_1 & P \\ \hline -\varepsilon P' & M_2 \end{array} \right),$$

where M_i is the matrix of the restriction of l to $H_q(V_i; \mathbf{Z})/\mathrm{Tors}$, called a *Seifert matrix of the knot* K_i, $\varepsilon = (-1)^q$, and the prime denotes transposition (see [1]). The matrix M of the form l, equipped with such a decomposition, will be called a *Seifert matrix of the link L*. Since M_i is a Seifert matrix of the component K_i, the matrix $M_i + \varepsilon M_i'$ is unimodular (see [7]). This is obviously equivalent to unimodularity of the matrix $M + \varepsilon M'$.

1.3. L-matrices. Let $\varepsilon = \pm 1$. By an *L-matrix* is meant a square matrix M, equipped with a decomposition of the form

$$M = \left(\begin{array}{c|c} M_1 & P \\ \hline -\varepsilon P' & M_2 \end{array} \right),$$

where M_1 and M_2 are square matrices such that $M + \varepsilon M'$ is unimodular. We denote the number ε in the definition of M by $\varepsilon(M)$.

THEOREM 1.3.1 (LIANG [1]). *For any L-matrix M and any integer* $n \geq 1$, *there exists a simple link of dimension* $4n + 2 + \varepsilon(M)$, *having M as its Seifert matrix.*

1.4. I-equivalence of L-matrices. Let

$$M = \left(\begin{array}{c|c} M_1 & P \\ \hline -\varepsilon P' & M_2 \end{array} \right) \quad \text{and} \quad N = \left(\begin{array}{c|c} N_1 & Q \\ \hline -\varepsilon Q' & N_2 \end{array} \right)$$

be L-matrices, where the matrices M_1 and N_1 have size $m_1 \times m_1$, and M_2 and N_2 have size $m_2 \times m_2$. The L-matrices M and N are called *l-congruent* if there exist unimodular B_1 of size $m_1 \times m_1$ and B_2 of size $m_2 \times m_2$ such that

$$N = \begin{pmatrix} B_1 & 0 \\ 0 & B_2 \end{pmatrix} M \begin{pmatrix} B_1' & 0 \\ 0 & B_2' \end{pmatrix}.$$

We shall say that the L-matrix M^1 is an *I-enlargement* of the L matrix

$$M = \left(\begin{array}{c|c} M_1 & P \\ \hline -\varepsilon P' & M_2 \end{array} \right)$$

if

$$M^1 = \left(\begin{array}{cccc|c} 0 & X & 0 & 0 & 0 \\ Y & Z & Q_3 & Q_1 & Q_2 \\ 0 & -\varepsilon Q_3' & M_3 & 0 & 0 \\ 0 & -\varepsilon Q_1' & 0 & M_1 & P \\ \hline 0 & -\varepsilon Q_2' & 0 & -\varepsilon P' & M_2 \end{array}\right)$$

or

$$M^1 = \left(\begin{array}{c|cccc} M_1 & P & 0 & 0 & Q_1 \\ \hline -\varepsilon P' & M_2 & 0 & 0 & Q_2 \\ 0 & 0 & M_3 & 0 & Q_3 \\ 0 & 0 & 0 & 0 & X \\ -\varepsilon Q_3' & -\varepsilon Q_2' & -\varepsilon Q_3' & Y & Z \end{array}\right),$$

where X, Y, Z and M_3 are square matrices. It is clear that the matrices $X + \varepsilon Y'$ and $M_3 + \varepsilon M_3'$ are unimodular. The matrix $\left(\begin{smallmatrix} 0 & X \\ Y & Z \end{smallmatrix}\right)$ will be called the *nontrivial part* of the I-enlargement. The L-matrix M will, in its turn, be called an *I-reduction* of the L-matrix M^1. We shall say that L-matrices M_0^1 and M_1^1 are *I-equivalent* if they can be connected by a chain of I-enlargements, I-reductions and l-congruences.

If in the definition of I-equivalence we restrict ourselves to the special form of I-enlargements and I-reductions in which the matrix M_3 has size zero and the nontrivial part is equal to $\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$ or $\left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)$, we obtain the definition of *l-equivalence* of L-matrices (see [1]).

REMARK. It is easy to show that any I-enlargement of an L-matrix M is l-congruent to an I-enlargement of the L-matrix M for which $Z + \varepsilon Z' = 0$ and the matrix $X + \varepsilon Y'$ is the identity.

1.5. Isotopy of links. Links L_0 and L_1 of dimension n are called *isotopic* if there exists a piecewise-linear embedding

$$F\colon (S_1^n \amalg S_2^n) \times I \to S^{n+2} \times I$$

(not necessarily locally flat) such that

(1) $F((S_1^n \amalg S_2^n) \times \{\alpha\}) \subset S^{n+2} \times \{\alpha\}$ for all $\alpha \in I$, and

(2) $F((S_1^n \amalg S_2^n) \times \{i\}) = L_i$, where $i = 0, 1$.

1.6. Formulation of the main results.

THEOREM 1.6.1. *If odd-dimensional boundary links are isotopic, then their Seifert matrices are I-equivalent.*

THEOREM 1.6.2. *If the Seifert matrices of simple odd-dimensional links of dimension* $\geqslant 5$ *are I-equivalent, then these links are isotopic.*

These assertions, taken together with Theorem 1.3.1, yield an isotopy classification of simple odd-dimensional links of dimension $\geqslant 5$ in terms of their Seifert matrices.

§2. Proof of Theorem 1.6.1

2.1. Local isotopy and Rolfsen's theorem. Let L_0 and L_1 be links of dimension n, and suppose that there exists in the ambient sphere S^{n+2} a piecewise-linear ball D of dimension $n + 2$ such that

(1) L_i and ∂D intersect transversally,

(2) the paris $(D, D \cap L_i)$ are proper ball pairs, and

(3) outside D the links L_0 and L_1 coincide.

We shall then say that the links L_0 and L_1 are *locally isotopic* (see [3]), or that *they can be connected by a local isotopy in the ball D*. It is clear that locally isotopic links are isotopic.

Links L_0 and L_1 of dimension n are called *ambient-isotopic* if there exists a piecewise-linear homeomorphism $F: S^{n+2} \times I \to S^{n+2} \times I$ such that

(1) $F(S^{n+2} \times \{\alpha\}) \subset S^{n+2} \times \{\alpha\}$ for all $\alpha \in I$,

(2) $F|_{S^{n+2} \times \{0\}} = \mathrm{id}$, and

(3) $F(L_0 \times \{1\}) = L_1$.

Obviously, ambient-isotopic links are isotopic. All these concepts are connected by the following theorem.

THEOREM 2.1 (ROLFSEN [2]). *If L_0 and L_1 are isotopic links, then L_1 can be obtained from L_0 by a finite sequence of local and ambient isotopies; moreover, two local isotopies, one for each component, are sufficient.*

In view of this theorem and the l-equivalence of Seifert matrices of ambient-isotopic links (see [1]), to prove Theorem 1.6.1 it suffices to prove the following lemma.

2.2. LEMMA. *Seifert matrices of locally isotopic links are I-equivalent.*

Let L_0 and L_1 be locally isotopic links of dimension $2q - 1$, and let D be the ball of this local isotopy. It is easy to show that there exist Seifert surfaces V_0 and V_1 of the links L_0 and L_1 satisfying the following conditions:

(1) V_0 and V_1 coincide outside D,

(2) V_i and ∂D intersect transversally, and

(3) $V_i \cap \partial D$ is connected.

It is clear that in such a case the surface $W = V_0 \cap \partial D = V_1 \cap \partial D$ is a connected Seifert surface of the knot $K = L_0 \cap \partial D = L_1 \cap \partial D$ in the sphere ∂D. Let U be the connected component of V_0 intersecting D, and put $W_1 = U \cap D$ and $W_2 = \overline{U \setminus W_1}$, so that $U = W_1 \cup_W W_2$.

We first prove Lemma 2.2 for local isotopies such that the group $H_{q-1}(W; \mathbf{Z})$ has no torsion. We denote $H_i(X; \mathbf{Z})/\mathrm{Tors}$ by $\overline{H}_i(X)$.

2.3. A special case: $\mathrm{Tors}\, H_{q-1}(W; \mathbf{Z}) = 0$. Let G be a finitely generated abelian group and H a subgroup. We denote by $S(H, G)$ the smallest pure subgroup of G containing H. It is clear that $\mathrm{rank}\, S(H, G) = \mathrm{rank}\, H$.

Let $\overline{\mathrm{in}}_i$: $\overline{H}_q(W) \to \overline{H}_q(W_i)$, for $i = 1, 2$, be the homomorphisms induced by the inclusion homomorphisms in_i: $H_q(W; \mathbf{Z}) \to H_q(W_i, \mathbf{Z})$. We put $G_i = S(\mathrm{Im}\, \overline{\mathrm{in}}_i, \overline{H}_q(W_i))$. It is clear that the group $B_i = \overline{H}_q(W_i)/G_i$ has no torsion, so that $\overline{H}_q(W_i) = B_i \oplus G_i$. We consider a segment of the exact sequence of the triad (U, W_1, W_2):

$$\cdots \to H_q(W; \mathbf{Z}) \to H_q(W_1; \mathbf{Z}) \oplus H_q(W_2; \mathbf{Z}) \xrightarrow{\rho} H_q(U; \mathbf{Z}) \xrightarrow{\partial} H_{q-1}(W; \mathbf{Z}) \to \cdots$$

and a segment of the induced sequence:

$$\cdots \to \overline{H}_q(W) \to \overline{H}_q(W_1) \oplus \overline{H}_q(W_2) \xrightarrow{\bar\rho} \overline{H}_q(U) \xrightarrow{\bar\partial} \overline{H}_{q-1}(W) \to \cdots.$$

Since $\mathrm{Tors}\, H_{q-1}(W; \mathbf{Z}) = 0$, the induced sequence is exact at the term $\overline{H}_q(U)$, and therefore $\overline{H}_q(U) = \mathrm{Im}\, \bar\rho \oplus G$, where $G \simeq \mathrm{Im}\, \bar\partial$.

Put $\bar{B}_i = S(\bar{\rho}(B_i), \bar{H}_q(U))$. Let $\overline{\text{in}}\colon \bar{H}_q(W) \to \bar{H}_q(U)$ be the homomorphism induced by the inclusion homomorphism in: $H_q(W; \mathbf{Z}) \to H_q(U; \mathbf{Z})$, and put $T = S(\operatorname{Im}\overline{\text{in}}, \bar{H}_q(U))$. It is easy to see that $\operatorname{Im}\bar{\rho} = \bar{B}_1 \oplus \bar{B}_2 \oplus T$, and so $\bar{H}_q(U) = \bar{B}_1 \oplus \bar{B}_2 \oplus T \oplus G$.

Let $\tau\colon \bar{H}_q(U) \times H_q(U) \to \mathbf{Z}$ be the intersection index in the manifold U; in a basis of $\bar{H}_q(U)$, consisting of bases for the subgroups $\bar{B}_1$, $\bar{B}_2$, T and G, the matrix of the form τ obviously can be written as

$$A = \begin{pmatrix} X_1 & 0 & 0 & P_1 \\ 0 & X_2 & 0 & P_2 \\ 0 & 0 & 0 & S_1 \\ \varepsilon P_1' & \varepsilon P_2' & \varepsilon S_1' & S_2 \end{pmatrix},$$

where $\varepsilon = (-1)^q$ and the matrix S_1 is unimodular. Adding to the generators of the subgroups $\bar{B}_1$ and $\bar{B}_2$ elements of the subgroup T, we make the matrices P_1 and P_2 vanish. These elements, obtained as a result of such a modification of the generators of $\bar{B}_1$ and $\bar{B}_2$, generate subgroups C_1 and C_2 of $\bar{H}_q(U)$, which is represented in the form of a direct sum $\bar{H}_q(U) = C_1 \oplus C_2 \oplus T \oplus G$.

We turn now to the Seifert from constructed on the manifold U:

$$l_U\colon \bar{H}_q(U) \times \bar{H}_q(U) \to \mathbf{Z}.$$

Let N be the matrix of the form l_U, in a basis of $\bar{H}_q(U)$ consisting of bases of the subgroups C_1, C_2, T and G. As is easily seen, the subgroup T is orthogonal to the subgroup $C_1 \oplus C_2 \oplus T$ with respect to l_U, and the subgroups C_1 and C_2 are also orthogonal. Taking into account that

$$N + \varepsilon N' = \begin{pmatrix} X_1 & 0 & 0 & 0 \\ 0 & X_2 & 0 & 0 \\ 0 & 0 & 0 & S_1 \\ 0 & 0 & \varepsilon S_1' & S_2 \end{pmatrix},$$

we obtain the following form for the matrix N:

$$N = \begin{pmatrix} M_1 & 0 & 0 & Q_1 \\ 0 & M_2 & 0 & Q_2 \\ 0 & 0 & 0 & X \\ -\varepsilon Q_1' & -\varepsilon Q_2' & Y & Z \end{pmatrix};$$

thus the Seifert matrix N_0 of the link L_0 has the form

$$N_0 = \left(\begin{array}{cccc|c} M_1 & 0 & 0 & Q_1 & 0 \\ 0 & M_2 & 0 & Q_2 & P \\ 0 & 0 & 0 & X & 0 \\ -\varepsilon Q_1' & -\varepsilon Q_2' & Y & Z & Q_3 \\ \hline 0 & -\varepsilon P' & 0 & -\varepsilon Q_3' & M_3 \end{array}\right),$$

i.e., it is an I-enlargement of the L-matrix

$$M = \left(\begin{array}{c|c} M_2 & P \\ \hline -\varepsilon P' & M_3 \end{array}\right).$$

Analogous reasoning, carried out for the Seifert surface V_1 of the link L_1, shows that its Seifert matrix N_1 is also an I-enlargement of M. Lemma 2.2 is proved for the case $\operatorname{Tors} H_{q-1}(W; \mathbf{Z}) = 0$.

Now let $H_{q-1}(W; \mathbf{Z})$ be arbitrary; we restrict consideration to the case $q \geqslant 4$. The general proof for all q is essentially analogous to that carried out above, only more cumbersome; in the case $q \geqslant 4$ we shall reduce the proof of Lemma 2.2 to the special case examined above, utilizing the concept of a cross-section of a Seifert surface of a knot (see [4]).

2.4. Cross-sections of Seifert surfaces of knots. Let K be a knot of dimension $2q - 2$, W its Seifert surface, and suppose that in the ambient sphere S^{2q} there exists a piecewise-linear $2q$-dimensional ball B such that

(1) ∂B intersects K and W transversally,

(2) the pair $(B, B \cap K)$ is a proper ball pair, and

(3) the pairs $(W \cap B, W \cap \partial B)$ and $(W \cap \overline{(S^{2q} \setminus B)}, W \cap \partial B)$ are $(q-1)$-connected.

Then the surface $W \cap \partial B$ is called a *cross-section of the Seifert surface W of the knot K.*

LEMMA 2.4 (KEARTON [4]). *A Seifert surface of a $(2q-2)$-dimensional knot for $q \geqslant 4$ has a cross-section.*

Let

$$\mathrm{in}_1 \colon H_{q-1}(W \cap \partial B; \mathbf{Z}) \to H_{q-1}(W \cap B; \mathbf{Z})$$

and

$$\mathrm{in}_2 \colon H_{q-1}(W \cap \partial B; \mathbf{Z}) \to H_{q-1}\left(W \cap \overline{(S^{2q} B)}\,; \mathbf{Z}\right)$$

be inclusion homomorphisms; we shall say that the surface $W \cap \partial B$ is a *regular cross-section of the surface W* if $\operatorname{Ker} \mathrm{in}_1 = \operatorname{Ker} \mathrm{in}_2$. It is easy to show that if W has a regular cross-section, then $\operatorname{Tors} H_{q-1}(W; \mathbf{Z}) = 0$.

2.5. Conclusion of the proof of Lemma 2.2: reduction to the case $\operatorname{Tors} H_{q-1}(W; \mathbf{Z}) = 0$. We shall represent the given local isotopy in the form of a composition of three local isotopies satisfying the conditions of §2.3. We adopt the notation of §2.2.

Let Z_0 and Z_1 be the principal components of the links L_0 and L_1, such that $Z_i \cap D = L_i \cap D$, and let $W' = W \cap \partial B$ be a cross-section of the surface W. Put $U_1 = W \cap B$ and $U_2 = \overline{W \setminus U_1}$, and let $S = K \cap \partial B$ and $C = K \cap B$.

Thicken the ball B to a cylinder $B \times J$, where $J = [-1, 1]$, in the ambient sphere S^{2q+1}, so that $\partial D \cap (B \times J) = B = B \times \{0\}$. The links L_0 and L_1 coincide near ∂D, so that $L_0 \cap (B \times J) = L_1 \cap (B \times J)$. We can suppose that $L_i \cap (B \times J) = Z_i \cap (B \times J)$. The link L_0 intersects the cylinder $B \times J$ in the disk $C \times J$, and intersects $\partial(B \times J)$ in the sphere Q, where

$$Q = (C \times \{-1\}) \cup_{(S \times \{-1\})} (S \times J) \cup_{(S \times \{1\})} (C \times \{1\}),$$

and the Seifert surface F of the knot Q in the sphere $\partial(B \times J)$:

$$F = (U_1 \times \{-1\}) \cup_{(W' \times \{-1\})} (W' \times J) \cup_{(W' \times \{1\})} (U_1 \times \{1\})$$

has the surface W' as a regular cross-section. We subject the link L_0 to a local isotopy, transforming (within the ball $B \times J$) the disk $C \times J$ to a disk which is symmetric, with respect to $\partial(B \times J)$, to the disk $\overline{Z_0 \setminus (C \times J)}$; the resulting link is denoted by L_0'. Since F has a regular cross-section, $\operatorname{Tors} H_{q-1}(F; \mathbf{Z}) = 0$, and for the above local isotopy Lemma 2.2 is valid.

We subject the link L_1 to an analogous local isotopy (which transforms the disk $C \times J$ to a disk symmetric, with respect to $\partial(B \times J)$, to the disk $\overline{Z_1 \backslash (C \times J)}$); the resulting link is denoted by L_1'. The links L_0' and L_1' are connected by a local isotopy in D, and the Seifert surface $F' = U_2 \cup_{W'} U_2$ of the knot $K' = L_0' \cap \partial D = L_1' \cap \partial D$ in ∂D has a regular cross-section W', so that for this local isotopy, and consequently also for the given local isotopy, the assertion of Lemma 2.2 is valid, which completes the proof of Theorem 1.6.1.

§3. Proof of Theorem 1.6.2.

3.1. Scheme of the proof. Let the L-matrix N_1 be an I-enlargement of the L-matrix N_0. For each $q \geqslant 1$ we construct simple $(4q + 2 + \varepsilon(N_0))$-dimensional links L_0 and L_1, connected by a local isotopy in a ball D, whose Seifert matrices are the L-matrices N_0 and N_1. From this, in view of the results of [1], Theorem 1.6.2 will be proved.

First, using the nontrivial part of the given I-enlargement, we construct a Seifert surface W of the knot $L_i \cap \partial D$ in ∂D; and then we will construct Seifert surfaces of the desired links L_0 and L_1, intersecting ∂D in the surface W.

3.2. Construction of the surface W. It obviously suffices to restrict attention to one of the two forms of an I-enlargement, since one form is obtained from the other by renumbering the components of links. Consider the L-matrix

$$N_0 = \left(\begin{array}{c|c} M_1 & P \\ \hline -\varepsilon P' & M_2 \end{array}\right)$$

and its I-enlargement

$$N_1 = \left(\begin{array}{cccc|c} 0 & X & 0 & 0 & 0 \\ Y & Z & Q_3 & Q_1 & Q_2 \\ 0 & -\varepsilon Q_3' & M_3 & 0 & 0 \\ 0 & -\varepsilon Q_1' & 0 & M_1 & P \\ \hline 0 & -\varepsilon Q_2' & 0 & -\varepsilon P' & M_2 \end{array}\right).$$

Let X, Y and Z be $m \times m$ matrices. As we remarked in §1.4, we can assume that $Z + \varepsilon Z' = 0$ and that the matrix $X + \varepsilon Y'$ is the identity. Let $4q + 2 + \varepsilon(N_0) = 2n - 1$ (such an n can always be found, since $\varepsilon(N_0) = \pm 1$).

In $\mathbf{R}^{2n+2}$ with coordinates $x_1, \ldots, x_{2n+2}$ we consider the unit sphere S^{2n+1}. Denote by D the hemisphere in S^{2n+1} given by the inequality $x_1 \geqslant 0$. The hemisphere D will be the ball of our local isotopy. Put $S_0 = \partial D$, let S_1 be the sphere defined in S^{2n+1} by $x_1 = x_2 = 0$, and let D_1 be the ball defined in S^{2n+1} by $x_1 = x_2 = 0$ and $x_3 \geqslant 0$. Put $S_2 = \partial D_1$, and let $\Omega_1, \ldots, \Omega_m$ be disjoint smooth $(2n - 2)$-dimensional balls in S_2. To D_1 in S_1 we attach m handles $h_1^n, \ldots, h_m^n$ of index n, so that the attaching sphere ξ_i of the handle h_i^n lies in Ω_i. For the resulting manifold D_1', generators $[\alpha_1], \ldots, [\alpha_m]$ of the group $H_n(D_1'; \mathbf{Z})$ are realized by disjoint smoothly embedded spheres $\alpha_1, \ldots, \alpha_m$ of dimension n. Let $\gamma_1, \ldots, \gamma_m$ be a family of n-dimensional smooth balls in S_2 such that $\gamma_i \in \Omega_i$ and $\partial\gamma_i = \xi_i$, and let $\tau_1, \ldots, \tau_m$ be a family of smooth disjoint $(n - 2)$-dimensional spheres in S_2 such that τ_i intersects only γ_i, in exactly one of its interior points. To D_1' in S_0 we attach m handles $h_1^{n-1}, \ldots, h_m^{n-1}$ of index $n - 1$, using $\tau_1, \ldots, \tau_m$ as attaching spheres, obtaining a manifold we denote by W'. The handles $h_1^{n-1}, \ldots, h_m^{n-1}$ yield generators $[\beta_1], \ldots, [\beta_m]$ of $H_{n-1}(W'; \mathbf{Z})$, realized by disjoint smoothly embedded spheres $\beta_1, \ldots, \beta_m$ of dimension $n - 1$, and the incidence number of the classes $[\alpha_i]$ and $[\beta_j]$ in W' is equal to δ_j^i. It is easy to see that the boundary of W' is a $(2n - 2)$-dimensional sphere.

Finally, modifying, if necessary, the embeddings of the handles $h_1^{n-1},\dots,h_m^{n-1}$ without changing their attaching spheres, we obtain a submanifold W of S_0 such that $l_W([\alpha_i],[\beta_j]) = x_{ij}$, where $l_W\colon H_n(W;\mathbf{Z})\times H_{n-1}(W;\mathbf{Z})\to\mathbf{Z}$ is the Seifert pairing constructed for the submanifold W of S_0, and x_{ij} is an element of the matrix X.

3.3. Construction of the link L_1. Let $J=[-\varepsilon_0,\varepsilon_0]$, where ε_0 is a sufficiently small positive number. Thicken the sphere S_0 in S^{2n+1} to a strip $R(-\varepsilon_0,\varepsilon_0)$, defined by $-\varepsilon_0\leqslant x_1\leqslant\varepsilon_0$, correspondingly thicken the manifold W to a cylinder $W\times J$ in $R(-\varepsilon_0,\varepsilon_0)$, and let W_1 and W_2 be the upper and lower bases of $W\times J$. Generators of $H_{n-1}(W_i;\mathbf{Z})$ and $H_n(W_i;\mathbf{Z})$ are realized by smoothly embedded spheres, which we denote by the corresponding spheres in W supplied with a superscript, for example α_i^1, β_j^2 etc.

Attach to $W\times J$ in $R(\varepsilon_0,2\varepsilon_0)$, defined in S^{2n+1} be the inequalities $\varepsilon_0\leqslant x_1\leqslant 2\varepsilon_0$, m handles $H_{11}^n,\dots,H_{m1}^n$ of index n, using $\beta_1^1,\dots,\beta_m^1$ as attaching spheres. Attach also to $W\times J$ in $R(-2\varepsilon_0,-\varepsilon_0)$, defined in S^{2n+1} by $-2\varepsilon_0\leqslant x_1\leqslant-\varepsilon_0$, m handles $H_{12}^n,\dots,H_{m2}^n$ of index n, using $\beta_1^2,\dots,\beta_m^2$ as attaching spheres. It is clear that such attachings are realizable. Denote the resulting manifold by T:

$$T=(W\times J)\cup H_{11}^n\cup\cdots\cup H_{m1}^n\cup H_{12}^n\cup\cdots\cup H_{m2}^n.$$

A simple argument shows that the boundary of T is a $(2n-1)$-dimensional sphere. The manifold T is $(n-1)$-connected, and $H_n(T;\mathbf{Z})=\mathbf{Z}\oplus\cdots\oplus\mathbf{Z}$ ($2m$ summands). As a basis for $H_n(T;\mathbf{Z})$ we can take the classes $[\alpha_1],\dots,[\alpha_m]$ and $[\delta_1],\dots,[\delta_m]$, where the notation $[\alpha_i]$ is preserved for the image of the classes $[\alpha_i]$ under the inclusion homomorphism in: $H_n(W;\mathbf{Z})\to H_n(T;\mathbf{Z})$, and a representative δ_i of the class $[\delta_i]$ is obtained, by gluing, from the middle disks of the handles H_{i1}^n and H_{i2}^n and the collar $\beta_i\times J$. For such a basis the Seifert matrix of the knot ∂T will obviously take the form $\left(\begin{smallmatrix}0&X\\Y&F\end{smallmatrix}\right)$, where F is an $m\times m$ matrix. We may suppose that $F+\varepsilon F'=0$ (this can be attained by modifying the generators $[\delta_i]$). Finally, we may suppose that $F=\mathbf{Z}$ (this equality can be attained by modifying the embeddings of the handles $H_{12}^n,\dots,H_{m2}^n$ without changing their attaching spheres). Thus the Seifert matrix of the knot ∂T is equal to $\left(\begin{smallmatrix}0&X\\Y&Z\end{smallmatrix}\right)$.

We construct in the ball $D(-3\varepsilon_0)$, defined in S^{2n+1} by $x_1\leqslant-3\varepsilon_0$, a simple $(2n-1)$-dimensional link L with Seifert matrix

$$\left(\begin{array}{c|c} M_1 & P\\ \hline -\varepsilon P' & M_2\end{array}\right),$$

which can be done in view of Theorem 1.3.1. Denote by B_1 and B_2 the components of L, and let A_1 and A_2 be $(n-1)$-connected disjoint Seifert surfaces of the components. The manifold A_i can be represented as a $2n$-dimensional disk to which are attached m_i handles $g_{i1},\dots,g_{im_i}$ of index n. Analogously [7], we transform A_1 into a submanifold A_1' of $D(-\varepsilon_0)$, by modifying the embeddings of the handles $g_{11},\dots,g_{1m_1}$ without changing their attaching spheres, in order that the link consisting of the components of ∂T and $\partial A_1'$ have Seifert matrix N_2, where

$$N_2=\left(\begin{array}{cc|c} 0 & X & 0\\ Y & Z & Q_1\\ \hline 0 & -\varepsilon Q_1' & M_1\end{array}\right).$$

Using Q_2, we transform A_2 into a submanifold A'_2 of $D(-\varepsilon_0)$ in order that the link consisting of the components of ∂T and $\partial A'_2$ have Seifert matrix N_3, where

$$N_3 = \left(\begin{array}{cc|c} 0 & X & 0 \\ Y & Z & Q_2 \\ \hline 0 & -\varepsilon Q'_2 & M_2 \end{array}\right).$$

We construct in the ball $D(3\varepsilon_0)$, defined in S^{2n+1} by $x_i \geqslant 3\varepsilon_0$, a simple knot B_3 with Seifert matrix M_3, and let A_3 be an $(n-1)$-connected Seifert surface of B_3. Using the matrix Q_3, we transform A_3 into a submanifold A'_3 of $D(\varepsilon_0)$, in order that the link consisting of the components of ∂T and $\partial A'_3$ have Seifert matrix N_4, where

$$N_4 = \left(\begin{array}{cc|c} 0 & X & 0 \\ Y & Z & Q_3 \\ \hline 0 & -\varepsilon Q'_3 & M_3 \end{array}\right).$$

Denote by L_1 the link consisting of the components of $\partial(T\#_\partial A'_3\#_\partial A'_1)$ and $\partial A'_2$, where the symbol $\#_\partial$ denotes connected sum along the boundary. It is clear that the link L_1 is simple, and that its Seifert matrix is equal to N_1.

3.4. Construction of the link L_0. Put

$$L_0 \cap \overline{(S^{2n+1}\setminus D)} = L_1 \cap \overline{(S^{2n+1}\setminus D)}.$$

For the construction of the part of L_0 lying in D, to the cylinder $W\times J$ in the strip $R(\varepsilon_0, 2\varepsilon_0)$ we glue m handles $\varphi_1^{n+1},\dots,\varphi_m^{n+1}$ of index $n+1$, using $\alpha_1^1,\dots,\alpha_m^1$ as attaching spheres. From the construction of W it is clear that such an attaching is possible. Let $S=(W\times J)\cup\varphi_1^{n+1}\cup\cdots\cup\varphi_m^{n+1}$, and then put $L_0\cap D=\partial S\cap D$. It is easy to see that the submanifold L_0 so defined is a simple link with the Seifert matrix

$$N_0 = \left(\begin{array}{c|c} M_1 & P \\ \hline -\varepsilon P' & M_2 \end{array}\right).$$

Moreover, by construction, the links L_0 and L_1 are connected by a local isotopy in the ball D. Theorem 1.6.2 is proved.

§4. Supplement: some isotopy invariants of links

4.1. Cobordism of matrices. Matrices M_1 and M_2 are called *cobordant* if their block difference

$$\begin{pmatrix} M_1 & 0 \\ 0 & -M_2 \end{pmatrix}$$

is congruent to a matrix of the form $\left(\begin{smallmatrix} 0 & A \\ B & C \end{smallmatrix}\right)$, where A and B are square matrices of the same size. Let $\varepsilon=\pm1$; we shall say that a matrix M *satisfies property* ε if the matrix $M+\varepsilon M'$ is unimodular.

As Levine showed, cobordism classes of matrices satisfying property ε form an abelian group with respect to block addition, and the cobordism class of a matrix M is completely determined by invariants $\varepsilon_\lambda(M)$, $\sigma_\lambda^k(M)$ and $\mu_\lambda^k(M)$ (see [6] and [8]).

4.2. Enlargements of Seifert matrices of links. Let a boundary link L have Seifert matrix

$$\left(\begin{array}{c|c} M_1 & P \\ \hline -\varepsilon P' & M_2 \end{array}\right).$$

We call the matrix $\overline{M}$ an *enlargement of the Seifert matrix of the link*, where

$$\overline{M} = \begin{pmatrix} M_1 & P & 0 & 0 \\ -\varepsilon P' & M_2 & 0 & 0 \\ 0 & 0 & -M_1 & 0 \\ 0 & 0 & 0 & -M_2 \end{pmatrix}.$$

From Theorem 1.6.1 easily follows

ASSERTION 4.2.1. *Enlargements of Seifert matrices of isotopic links are cobordant.*

This assertion can also be obtained differently, making use of the following theorem.

THEOREM 4.2.2 (ROLFSEN [2]). *Isotopic links are cobordant if and only if their corresponding components are cobordant.*

From Assertion 4.2.1 there plainly follows the isotopy invariance of the family $\{\varepsilon_\lambda(\overline{M}), \sigma_\lambda^k(\overline{M}), \mu_\lambda^k(\overline{M})\}$, where $\overline{M}$ is an enlargement of the Seifert matrix of a link.

4.3. EXAMPLES. *A link not cobordant to a split link*; *nonisotopic links not distinguished by Rolfsen's invariants.* As O. Ya. Viro has told me, the signature of a symmetrized enlargement of the Seifert matrix of a link provides an elementary way to disprove the erroneous theorem of Gutierrez [9], in which it was asserted that every link of codimension 2 is cobordant to a split link (i.e. to a link whose components can be separated by disjoint embedded balls). In fact, it is obviously a cobordism invariant, and is equal to zero for split links. On the other hand, for a $(4q-3)$-dimensional link L with Seifert matrix M, where

$$M = \left(\begin{array}{cc|cc} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \hline 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{array}\right),$$

it is equal to 2 by an easy computation.

Other counterexamples to the formulation of Gutierrez were found by Cappell and Shaneson [10] and Kawauchi [11].

Moreover, the link L just mentioned and the link $-L$ provide an example of nonisotopic links that are not distinguished by localized Alexander invariants (see [3]) but (obviously) distinguished by the invariant $\sigma(\overline{M}+\overline{M}')$.

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