A CERTAIN TYPE OF IRREGULAR ALGEBRAIC SURFACES*

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By a *complex* 4-*manifold* we shall mean a complex manifold of *topological* dimension 4. For any complex 4-manifold M we let $c_\nu(M)$ denote the ν^{th} Chern class of M. By the Hirzebruch index theorem, the index of M is equal to

$$\tau(M) = \frac{1}{3}[c_1^2(M) - 2c_2(M)].$$

A. Van de Ven has pointed out in connection with his recent results(1) that there are not many known examples of compact *connected* complex 4-manifolds with positive indices. The purpose of this note is to exhibit a series of compact connected complex 4-manfolds $M_{n,m}$, $n, m = 2, 3, 4, \cdots$, with positive indices. Each complex 4-manifold $M_{n,m}$ is an irregular algebraic surface having a structure of locally non-trivial complex analytic family of non-singular algebraic curves of genus $m(2n-1)$ whose base space is a compact Riemann surface. The complex 4-manifold $M_{n,m}$ serves as an example of differentiable fibre bundle such that the index of the bundle space is *not* equal to the product of the indices of the fibre and of the base space(2).

Let R_0 be a compact Riemann surface of genus $n \geqq 2$. The surface R_0 is topologically a "sphere with n handles". Letting γ_0 be a "meridian" circle on a handle of R_0, we construct a 2-sheeted unramified covering surface R of R_0 with a covering map $\lambda: R \to R_0$ such that $R - \lambda^{-1}(\gamma_0)$ consists of two

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(1) See A. Van de Ven: On the Chern numbers of certain complex and almost complex manifolds, Proc. Nat. Acad. Sci., U. S. A., Vol. 55 (1966), pp. 1624–1627.

(2) See S. S. Chern, F. Hirzebruch and J.-P. Serre: On the index of a fibered manifold, Proc. Am. Math. Soc., Vol. 8 (1957), pp. 587–596.

connected components each of which is homeomorphic to $R_0 - \gamma_0$. The genus of the Riemann surface R is $g = 2n - 1$. Writing $\lambda^{-1}\lambda(t) = \{t, t^*\}$ for any point $t \in R$, we define a involution $\iota : t \to t^*$ of R which has no fixed point. The fundamental group $\pi_1(R)$ of R is generated by $2g$ elements $a_1, a_2, \cdots, a_{2g}$ satisfying the relation

$$a_1 a_2 a_1^{-1} a_2^{-1} a_3 a_4 a_3^{-1} \cdots\cdots a_{2g-1} a_{2g} a_{2g-1}^{-1} a_{2g}^{-1} = 1 .$$

Let $Z = \{\varepsilon\}$ denote a cyclic group of order $m \geqq 2$ generated by an element ε. We take $2g$ copies $Z_i, i = 1, 2, \cdots, 2g$, of Z and form their direct product

$$Z^{2g} = Z_1 \times Z_2 \times \cdots \times Z_i \times \cdots \times Z_{2g} .$$

We let ρ denote the homomorphism of $\pi_1(R)$ onto Z^{2g} determined by the assignment

$$\rho : a_i \to 1_1 \times 1_2 \times \cdots \times 1_{i-1} \times \varepsilon_i \times 1_{i+1} \times \cdots \times 1_{2g} ,$$

where 1 indicates the unit of Z. We then define S to be the m^{2g}-sheeted unramified covering surface of R which corresponds to ρ in the sense that the covering map f of S onto R maps the fundamental group $\pi_1(S)$ of S onto the kernel of ρ:

$$(1) \qquad 1 \to f(\pi_1(S)) \to \pi_1(R) \xrightarrow{\rho} Z^{2g} \to 1 .$$

The genus of the Riemann surface S is $h = m^{2g}(g-1) + 1$. Let f^* be the holomorphic map: $u \to f(u)^* = \iota f(u)$. We form a cartesian product $W = R \times S$. We define Γ and Γ^* to be, respectively, the graphs of the holomorphic maps f and f^*, i.e.,

$$\Gamma = \{(f(u), u) \mid u \in S\} \subset W,$$

$$\Gamma^* = \{(f^*(u), u) \mid u \in S\} \subset W,$$

and let

$$W'' = W - \Gamma - \Gamma^* .$$

In order to construct an m-sheeted covering manifold of W with branch loci Γ and Γ^*, we examine the homology group $H_1(W'', Z)$ of W'' of 1-cycles with compact supports. We indicate by the symbol $\sim$ the homology with coefficients in the integers Z. We choose a point $u_0 \in S$ and identify R with $R \times u_0$. Let D be a small circular disk of center t_0 on R with positive orientation and let $\gamma = \partial D$ be the boundary of D.

Lemma. *The homology class of the circle γ generates a cyclic subgroup $\{\gamma\}$ of order m of $H_1(W'', Z)$ and*

$$H_1(W'', Z) \cong H_1(R, Z) \oplus H_1(S, Z) \oplus \{\gamma\}. \tag{2}$$

Proof. First we show that $m\gamma \sim 0$ in W''. We represent each generator a_i by a simple closed differentiable curve α_i on R with a parametric representation: $x \to \alpha_i(x)$, $0 \leqq x \leqq 1$, starting and ending at $t_0 : \alpha_i(0) = \alpha_i(1) = t_0$. We may assume that $\alpha_1 \cap \alpha_2 = t_0$, that α_1 and α_2 intersect transversally and that $\alpha_1^* \cap \alpha_2$ is empty, where $\alpha_1^* = \iota\alpha_1$. Since $\rho(a_1^m) = 1$, there exists on S a simple closed differentiable curve β with a parametric representation: $x \to \beta(x)$, $0 \leqq x \leqq 1$, $\beta(0) = \beta(1)$, such that $f(\beta(x)) = \alpha_1(mx)$. The cartesian product $\alpha_2 \times \beta$ is a differentiable submanifold of $W = R \times S$ which is homeomorphic to a torus. The graph Γ intersects $\alpha_2 \times \beta$ at m points: $p_k = (\alpha_2(0), \beta(k/m)), k = 0, 1, 2, \cdots, m-1$. Choosing an appropriate orientation for $\alpha_2 \times \beta$, we infer readily that the intersection multiplicity of Γ and $\alpha_2 \times \beta$ at the points p_k are equal to 1:

$$I_{p_k}(\Gamma, \alpha_2 \times \beta) = 1. \tag{3}$$

The graph Γ^* of f^* does not intersect $\alpha_2 \times \beta$, since $\alpha_1^* = f^*(\beta)$ does not meet α_2. Letting γ_k denote the boundary of a small positively oriented circular disk of center p_k on $\alpha_2 \times \beta$, we infer from (3) that

$$\gamma_k \sim \gamma, \qquad \text{in } W'',$$

while

$$\sum_{k=1}^{m} \gamma_k \sim 0, \qquad \text{on } \alpha_2 \times \beta - \cup p_k \subset W''.$$

It follows that $m\gamma \sim 0$ in W''.

To prove that $k\gamma \nsim 0$ in W'' for $k \not\equiv 0\,(m)$, we suppose that $k\gamma \sim 0$ in W'' and derive the congruence: $k \equiv 0\,(m)$. We choose simple closed differentiable curves $\beta_j, j = 1, 2, \cdots, 2h$, on S which generate the fundamental group $\pi_1(S)$ such that the curves $f(\beta_j)$ and $f^*(\beta_j)$ intersect the curves α_i transversally. The cartesian product $\alpha_i \times \beta_j$ is a differentiable submanifold of $W = R \times S$ which is homeomorphic to a torus. The intersection $\Gamma \cap \alpha_i \times \beta_j$ consists of a finite number of points $p = (t, u)$ determined by the condition that $u \in \beta_j$, $t = f(u) \in \alpha_i$, and, for each intersection $p = (t, u)$, the equality

$$I_p(\Gamma, \alpha_i \times \beta_j) = I_t(\alpha_i, f(\beta_j)) \tag{4}$$

holds, where $I_t(\alpha_i, f(\beta_j))$ denotes the intersection multiplicity at t of the curves α_i and $f(\beta_j)$ on R, provided that the choice of the orientation of $\alpha_i \times \beta_j$ is appropriate. For any pair of 2-cycles B and C on a compact complex 4-manifold, we denote the total intersection multiplicity of B and C by (BC) or (B, C). By (1) we have a homology

$$f(\beta_j) \sim m \sum_{r=1}^{2g} n_r \alpha_r, \qquad \text{on } R,$$

where the coefficients n_r are integers. It follows that the total intersection multiplicity of α_i and $f(\beta_j)$ is divisible by m. Hence we infer from (4) that

$$I(\Gamma, \alpha_i \times \beta_j) \equiv 0\,(m). \tag{5}$$

Similarly we obtain

$$I(\Gamma^*, \alpha_i \times \beta_j) \equiv 0\,(m). \tag{6}$$

There exists a 2-chain $A'' \subset W''$ such that $\partial A'' = k\gamma$. Since $\gamma = \partial D$, the difference $kD - A''$ is a 2-cycle in W. We have a homology

$$kD - A'' \sim rR + sS + \sum_{i,j} n_{ij} \alpha_i \times \beta_j, \qquad \text{in } W,$$

where $R = R \times u_0$, $S = t_0 \times S$, and the coefficients r, s, n_{ij} are integers. Hence we get

$$k = (\Gamma, kD - A'') = r(\Gamma R) + s(\Gamma S) + \sum_{i,j} n_{ij}(\Gamma, \alpha_i \times \beta_j),$$

$$0 = (\Gamma^*, kD - A'') = r(\Gamma^* R) + s(\Gamma^* S) + \sum_{i,j} n_{ij}(\Gamma^*, \alpha_i \times \beta_j).$$

Combining these equalities with (5) and (6), we obtain the congruence: $k \equiv 0(m)$, since $(\Gamma R) = (\Gamma^* R)$ and $(\Gamma S) = (\Gamma^* S)$. Thus we see that γ generates a cyclic subgroup $\{\gamma\}$ of order m of $H_1(W'', \boldsymbol{Z})$.

$\gamma^* = \iota\gamma$ is a circle on $R = R \times u_0$ of the center $t_0^* = \Gamma^* \cap R$. Take an arbitrary 1-cycle ζ in W''. We have a homology

$$\zeta \sim \sum m_i\alpha_i + \sum n_j\beta_j, \qquad \text{in } W,$$

where the coefficients m_i and n_j are integers. Hence we get

$$\zeta \sim \sum m_i\alpha_i + \sum n_j\beta_j + b\gamma + c\gamma^*, \qquad \text{in } W'',$$

where b and c are integers, while

$$\gamma^* \sim -\gamma, \qquad \text{on } R - t_0 - t_0^* \subset W''.$$

It follows that

$$\zeta \sim \sum m_i\alpha_i + \sum n_j\beta_j + k\gamma \qquad \text{in } W'',$$

where k is an integer. Thus we obtain the isomorphism (2), q.e.d.

The isomorphism (2) determines a homomorphism: $H_1(W'', \boldsymbol{Z}) \to \{\gamma\}$ in an obvious manner and there is a canonical homomorphism: $\pi_1(W'') \to H_1(W'', \boldsymbol{Z})$. Composing these two homomorphisms we obtain a homomorphism

$$\eta : \pi_1(W'') \to \{\gamma\}.$$

We let M'' be the m-sheeted unramified covering manifold of W'' corresponding to the homomorphism η in the sense that the covering map μ'' of M'' onto W'' maps $\pi_1(M'')$ onto the kernel of η. We infer readily that M'' can be extended to an m-sheeted ramified covering manifold M of W with branch loci Γ and

Γ^*. We define $M_{n,m}$ to be the covering manifold M of W thus obtained. The compact connected complex 4-manifold $M_{n,m}$ is an irregular algebraic surface. The covering map μ of $M_{n,m}$ onto W is an extension of μ''. The inverse images $\Delta = \mu^{-1}(\Gamma)$ and $\Delta^* = \mu^{-1}(\Gamma^*)$ are complex submanifolds of $M_{n,m}$ and

$$M_{n,m} = M'' \cup \Delta \cup \Delta^* .$$

Moreover μ maps Δ and Δ^* biholomorphically onto Γ and Γ^*, respectively.

We define Ψ to be the composite $P_S \circ \mu$ of the covering map μ and the projection $P_S : W = R \times S \to S$. The holomorphic map $\Psi : M_{n,m} \to S$ determines a structure of complex analytic family on $M_{n,m}$ of which the base space is S. For any point $u \in S$, the fibre $C_u = \Psi^{-1}(u)$ is a compact Riemann surface which is an m-sheeted cyclic covering surface of R with two branch points $f(u)$ and $f^*(u)$. Hence we infer that, for any point $v \in S$, the number of those fibres C_u which are conformally equivalent to C_v is finite. Thus the complex analytic family $M_{n,m}$ is locally non-trivial. Moreover the infinitesimal deformation $\partial C_u / \partial u$ does not vanish at any point $u \in S$. To verify this we cover R by a finite number of small disks U_i, $i = 1, 2, 3, \cdots$. Let z_i denote a local coordinate defined on U_i and let $V_i = \mu^{-1}(U_i \times u)$. Note that C_u is covered by the open subsets V_i. Assuming that $f(u) \in U_1$, $f^*(u) \in U_2$, we introduce local coordinates

$$\begin{cases} w_1 = [z_1 - f(u)]^{1/m}, & \text{on } V_1, \\ w_2 = [z_2 - f^*(u)]^{1/m}, & \text{on } V_2, \\ w_i = z_i, & \text{on } V_i,\ i \geqq 3. \end{cases}$$

In terms of these local coordinates we calculate the 1-cocycle $\{\theta_{ik}\}$ composed of holomorphic vector fields θ_{ik} on $V_i \cap V_k$ which represents the infinitesimal deformation $\partial C_u / \partial u$. We obtain

$$\theta_{k1} = -\theta_{1k} = -\frac{\partial}{\partial u}[z_1 - f(u)]^{1/m} \frac{d}{dw_1} = \frac{f'(u)}{m w_1^{m-1}} \frac{d}{dw_1}, \quad \text{for} \quad k \geqq 3,$$

and similarly

$$\theta_{k2} = -\theta_{2k} = \frac{f^{*\prime}(u)}{mw_2^{m-1}} \frac{d}{dw_2} \qquad \text{for } k \geqq 3,$$

$$\theta_{ik} = 0, \qquad \text{for } i,\ k \geqq 3,$$

where $f'(u) \neq 0$, $f^{*\prime}(u) \neq 0$. Suppose that $\{\theta_{ik}\}$ is cobounded by a 0-cochain $\{\theta_i\}$ composed of holomorphic vector fields θ_i on V_i, i.e., $\theta_{ik} = \theta_k - \theta_i$ on $V_i \cap V_k$. Then the formula

$$\theta = \begin{cases} \theta_1 - \dfrac{f'(u)}{mw_1^{m-1}} \dfrac{d}{dw_1}, & \text{on } V_1, \\ \theta_2 - \dfrac{f^{*\prime}(u)}{mw_2^{m-1}} \dfrac{d}{dw_2}, & \text{on } V_2, \\ \theta_k, & \text{on } V_k,\ k \geqq 3, \end{cases}$$

defines a meromorphic vector field θ on C_u with two poles of order $m-1$. This contradicts that the genus of C_u is equal to $mg \geqq 3m$.

Now we calculate the Chern numbers of $M_{n,m}$. For any compact Riemann surface C, we denote by $\pi(C)$ the genus of C. Note that

$$\pi(\Delta) = \pi(\Delta^*) = \pi(\Gamma) = \pi(\Gamma^*) = h = m^{2g}(g-1) + 1.$$

We have

$$c_2(M_{n,m}) = mc_2(W) - (m-1)[2 - 2\pi(\Gamma) + 2 - 2\pi(\Gamma^*)]$$

$$= 4m(g-1)(h-1) + 4(m-1)(h-1) = 4(h-1)(mg-1).$$

Let ϕ and ψ denote, respectively, holomorphic 1-forms on R and on S. The divisors of ϕ and ψ are

$$(\phi) = \sum_{i=1}^{2g-2} t_i, \qquad (\psi) = \sum_{k=1}^{2h-2} u_k.$$

The holomorphic 2-form $\phi \wedge \psi$ on $W = R \times S$ induces a holomorphic 2-form Φ on $M_{n,m}$. We infer readily that the divisor of Φ is

$$K = (\Phi) = \sum_{i=1}^{2g-2} B_i + \sum_{k=1}^{2h-2} C_k + (m-1)\Delta + (m-1)\Delta^*,$$

where

$$B_i = \mu^{-1}(t_i \times S), \qquad C_k = \mu^{-1}(R \times u_k) = \Psi^{-1}(u_k).$$

It is clear that

$$(B_i C_k) = m, \qquad (B_i \Delta) = (B_i \Delta^*) = m^{2g}, \qquad (C_k \Delta) = (C_k \Delta^*) = 1.$$

We have

$$2\pi(\Delta) - 2 = (K\Delta) + (\Delta\Delta) = \sum_i (B_i \Delta) + \sum_k (C_k \Delta) + m(\Delta\Delta).$$
$$= m^{2g}(2g-2) + (2h-2) + m(\Delta\Delta).$$

Thus we get

$$(\Delta\Delta) = (\Delta^*\Delta^*) = -2m^{2g-1}(g-1).$$

We have

$$c_1^2(M_{m,n}) = (KK) = 2\sum_i \sum_k (B_i C_k) + 2(m-1)\sum_i (B_i, \Delta + \Delta^*)$$
$$+ 2(m-1)\sum_k (C_k, \Delta + \Delta^*) + (m-1)^2(\Delta\Delta) + (m-1)^2(\Delta^*\Delta^*)$$
$$= 8m(g-1)(h-1) + 8(m-1)m^{2g}(g-1) + 8(m-1)(h-1)$$
$$- 4(m-1)^2 m^{2g-1}(g-1) = 4(h-1)(2mg + m - 2 - 1/m).$$

Thus we obtain the following formulae:

$$c_1^2(M_{n,m}) = 8m^{4n-3}(n-1)(4m^2 n - m^2 - 2m - 1),$$

$$c_2(M_{n,m}) = 8m^{4n-3}(n-1)(2m^2 n - m^2 - m),$$

$$\tau(M_{n,m}) = 8m^{4n-4}(n-1) \cdot m(m^2-1)/3.$$

The index $\tau(M_{n,m})$ is obviously positive. We remark that

$$\lim_{n \to +\infty} \frac{c_1^2(M_{n,m})}{c_2(M_{n,m})} = 2,$$

$$\lim_{m \to +\infty} \frac{c_1^2(M_{n,m})}{c_2(M_{n,m})} = 2 + \frac{1}{2n-1}.$$

The complex 4-manifold $M_{n,m}$ is the bundle space of a differentiable fibre bundle of which the fibre and the base space are both compact differentiable surfaces. Since the indices of compact surfaces are defined to be zero, the index $\tau(M_{n,m})$ is not equal to the product of of the indices of the fibre and of the base space.

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