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CUTTING, PASTING AND THE DOUBLES OF MANIFOLDS WITH BOUNDARY

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Dedicated to the memory of the late Professor Katsuo Kawakubo

0. Introduction

Cuttings and pastings of manifolds lead to the so-called SK-groups of manifolds. Various kinds of such groups are found in the literature [**1, 2, 4–7**]. In this paper we will investigate the relations between them.

The manifolds considered here are all in the unoriented category. Let P and P' be m-dimensional compact smooth manifolds with boundary, and $\varphi : \partial P \to \partial P'$ be a diffeomorphism. Pasting P and P' along the boundary by φ, we obtain a closed smooth manifold $P \cup_\varphi P'$. For another diffeomorphism $\psi : \partial P \to \partial P'$, we obtain another manifold $P \cup_\psi P'$. The two manifolds $P \cup_\varphi P'$ and $P \cup_\psi P'$ are said to be obtained from each other by *cutting and pasting* (*Schneiden und Kleben* in German). If a closed manifold M is obtained from a closed manifold M' by a finite sequence of cuttings and pastings, we say that M and M' are *SK-equivalent* to each other. This is an equivalence relation on $\mathcal{M}_m$, the set of m-dimensional closed smooth manifolds. Note that if M and M' are SK-equivalent then $\chi(M) = \chi(M')$ since

$$\chi(P \cup_\varphi P') = \chi(P) + \chi(P') - \chi(\partial P) = \chi(P \cup_\psi P'),$$

where χ denotes the Euler characteristic. The set of all equivalence classes of m-dimensional closed smooth manifolds, denoted by $\mathcal{M}_m/SK$, is a semigroup with the addition induced from a disjoint union of manifolds. The Grothendieck group of $\mathcal{M}_m/SK$ is called the *SK-group* of m-dimensional closed manifolds and is denoted by SK_m. This group has been introduced and observed in Karras *et al* [**4**].

As a generalization of this group, in Komiya [**5**] the author introduced the SK-group of pairs of manifolds. Let $m \geq n \geq 0$ be integers. Let (P, Q) be a pair of an m-dimensional compact smooth manifold P and an n-dimensional compact smooth submanifold Q of P with $\partial Q = Q \cap \partial P$. Let (P', Q') be another pair, and $\varphi : (\partial P, \partial Q) \to (\partial P', \partial Q')$ a diffeomorphism inducing a diffeomorphism

$\varphi|\partial Q : \partial Q \to \partial Q'$. Then we obtain a pair $(P \cup_\varphi P', Q \cup_{\varphi|\partial Q} Q')$ of an m-dimensional closed smooth manifold $P \cup_\varphi P'$ and an n-dimensional closed smooth submanifold $Q \cup_{\varphi|\partial Q} Q'$. If $\psi : (\partial P, \partial Q) \to (\partial P', \partial Q')$ is another diffeomorphism, then two pairs $(P \cup_\varphi P', Q \cup_{\varphi|\partial Q} Q')$ and $(P \cup_\psi P', Q \cup_{\psi|\partial Q} Q')$ are said to be obtained from each other by cutting and pasting. As in the absolute case, the cutting and pasting process induces an SK-equivalence relation on $\mathcal{M}_{m,n}$, the set of pairs (M, N) of m-dimensional closed smooth manifolds M and n-dimensional closed smooth submanifolds N of M. The set of equivalence classes denoted by $\mathcal{M}_{m,n}/SK$ is a semigroup, and we obtain the SK-group $SK_{m,n}$ of pairs as the Grothendieck group of $\mathcal{M}_{m,n}/SK$.

Koshikawa [**6**] also generalized SK_m to the SK-group $SK_m(\partial)$ of manifolds with boundary. Let (P, Q_1, Q_2) be a triple such that P is an m-dimensional compact smooth manifold with boundary $\partial P = Q_1 \cup Q_2$, Q_1 and Q_2 are $(m-1)$-dimensional compact smooth submanifolds of ∂P with $Q_1 \cap Q_2 = \partial Q_1 = \partial Q_2$. Let (P', Q'_1, Q'_2) be another triple, and $\varphi : Q_2 \to Q'_2$ a diffeomorphism. Then we obtain a compact smooth manifold $P \cup_\varphi P'$ with boundary $\partial(P \cup_\varphi P') = Q_1 \cup_{\varphi|\partial Q_1} Q'_1$ after an appropriate smoothing on the boundary. Given another diffeomorphism $\psi : Q_2 \to Q'_2$, we obtain a second manifold with boundary. In the same way as above, we define an SK-equivalence relation on $\mathcal{M}_m(\partial)$, the set of m-dimensional compact smooth manifolds with boundary, and obtain the SK-group $SK_m(\partial)$. In Koshikawa [**6**] the notation $SK_m^O(\text{pt}, \text{pt})$ is used instead of $SK_m(\partial)$.

We denote by $(M, \partial M)$ a compact manifold M with boundary. If $(M, \partial M) \in \mathcal{M}_m(\partial)$, then its double $DM = M \cup_{\text{id}} M \in \mathcal{M}_m$ is defined where id : $\partial M \to \partial M$ is the identity. Since ∂M is considered as an $(m-1)$-dimensional submanifold of DM, a pair $(DM, \partial M)$ is in $\mathcal{M}_{m,m-1}$. The first purpose of this paper is to study the relation between $\mathcal{M}_m(\partial)$ and $\mathcal{M}_{m,m-1}$ through the SK-groups and the construction of the double.

We also obtain the equivariant SK-group SK_m^G of m-dimensional closed smooth G-manifolds where G is a compact Lie group. See Karras *et al.* [**4**] and Kosniowski [**7**]. See also Hara [**1**] and Hara and Koshikawa [**2**] for the equivariant SK-group of compact smooth G-manifolds with boundary. Let $\mathcal{M}_m^G$ be the set of m-dimensional closed smooth G-manifolds. As in the non-equivariant case, we define an equivariant SK-equivalence relation on $\mathcal{M}_m^G$, and obtain a semigroup $\mathcal{M}_m^G/SK$. SK_m^G is the Grothendieck group of $\mathcal{M}_m^G/SK$. In this paper we only consider the case of $G = \mathbb{Z}_2$, the cyclic group of order two. For $(M, \partial M) \in \mathcal{M}_m(\partial)$, DM admits a smooth $\mathbb{Z}_2$-action which interchanges the two copies of M in DM. The second purpose of this paper is to study the relation between $\mathcal{M}_m(\partial)$ and $\mathcal{M}_m^{\mathbb{Z}_2}$.

Let $\mathcal{M}^{ev}_{m,m-1}$ be the subset of $\mathcal{M}_{m,m-1}$ consisting of $(M, N) \in \mathcal{M}_{m,m-1}$ with $\chi(M) \equiv \chi(N) \bmod 2$. Note that at least one of $\chi(M)$ and $\chi(N)$ is zero since an odd-dimensional closed manifold has zero as its Euler characteristic. We obtain the SK-group $SK^{ev}_{m,m-1}$ as the Grothendieck group of a semigroup $\mathcal{M}^{ev}_{m,m-1}/SK$. For $(M, \partial M) \in \mathcal{M}_m(\partial)$, we see $\chi(DM) \equiv \chi(\partial M) \bmod 2$ since $\chi(DM) = 2\chi(M) - \chi(\partial M)$. The construction of the double of $(M, \partial M) \in \mathcal{M}_m(\partial)$ leads to a homomorphism $D : SK_m(\partial) \to SK^{ev}_{m,m-1}$.

Let $\mathcal{M}^{\mathbb{Z}_2}_m(m-1)$ be the subset of $\mathcal{M}^{\mathbb{Z}_2}_m$ consisting of $M \in \mathcal{M}^{\mathbb{Z}_2}_m$ such that all components of the fixed point set $M^{\mathbb{Z}_2}$ of M are of dimension $m-1$ or empty. The SK-group $SK^{\mathbb{Z}_2}_m(m-1)$ is obtained as the Grothendieck group of $\mathcal{M}^{\mathbb{Z}_2}_m(m-1)/SK$. The construction of the double also leads to a homomorphism $D : SK_m(\partial) \to SK^{\mathbb{Z}_2}_m(m-1)$ since the $\mathbb{Z}_2$-action on DM has ∂M as its fixed point set.

For a compact $\mathbb{Z}_2$-manifold M we see $\chi(M) \equiv \chi(M^{\mathbb{Z}_2}) \bmod 2$ (see, for example, Kawakubo [**3**, Ch. 5]). There is a correspondence $\mathcal{M}^{\mathbb{Z}_2}_m(m-1) \to \mathcal{M}^{ev}_{m,m-1}$ which sends $M \in \mathcal{M}^{\mathbb{Z}_2}_m(m-1)$ to a pair $(M, M^{\mathbb{Z}_2}) \in \mathcal{M}^{ev}_{m,m-1}$, and this induces a homomorphism $F : SK^{\mathbb{Z}_2}_m(m-1) \to SK^{ev}_{m,m-1}$.

We will obtain the following result.

THEOREM. *The following diagram is commutative*

$$\begin{array}{ccc} & SK_m(\partial) & \\ {}^{D}\swarrow & & \searrow^{D} \\ SK^{\mathbb{Z}_2}_m(m-1) & \xrightarrow[F]{\quad\quad} & SK^{ev}_{m,m-1} \end{array}$$

and the three homomorphisms D, D and F are isomorphisms.

1. SK-group of pairs

There are correspondences

$$\mathcal{M}_m \longrightarrow \mathcal{M}_{m,n}, \quad M \longmapsto (M, \emptyset),$$
$$\mathcal{M}_{m,n} \longrightarrow \mathcal{M}_n, \quad (M, N) \longmapsto N.$$

These induce homomorphisms between SK-groups,

$$i : SK_m \to SK_{m,n} \quad \text{and} \quad j : SK_{m,n} \to SK_n.$$

From Komiya [**5**, Theorem 1.1] we have a split short exact sequence

$$0 \longrightarrow SK_m \xrightarrow{i} SK_{m,n} \xrightarrow{j} SK_n \longrightarrow 0.$$

Karras *et al.* [**4**, Theorem 1.3a] shows that if m is odd then $SK_m = 0$, and if m is even then SK_m is isomorphic to the group $\mathbb{Z}$ of integers by the isomorphism which sends $[M] \in SK_m$ to $\chi(M) \in \mathbb{Z}$, where $[M]$ denotes the class represented by $M \in \mathcal{M}_m$. When $n = m - 1$, these facts imply that $i : SK_m \to SK_{m,m-1}$ is an isomorphism if m is even, and that $j : SK_{m,m-1} \to SK_{m-1}$ is an isomorphism if m is odd.

To prove the theorem, we need to define a certain subgroup of SK_m. Let $\mathcal{M}_m^{ev}$ be the subset of $\mathcal{M}_m$ consisting of $M \in \mathcal{M}_m$ with $\chi(M) \equiv 0 \bmod 2$, and SK_m^{ev} be the Grothendieck group of $\mathcal{M}_m^{ev}/SK$. In the same way as above, there are homomorphisms $i : SK_m^{ev} \to SK_{m,n}^{ev}$ and $j : SK_{m,n}^{ev} \to SK_n^{ev}$, and we obtain the following.

PROPOSITION 1.

$$SK_{m,m-1}^{ev} \cong \begin{cases} SK_m^{ev}, & m \text{ is even}, \\ SK_{m-1}^{ev}, & m \text{ is odd}. \end{cases}$$

In fact, $i : SK_m^{ev} \to SK_{m,m-1}^{ev}$ is an isomorphism if m is even, and $j : SK_{m,m-1}^{ev} \to SK_{m-1}^{ev}$ is an isomorphism if m is odd.

2. SK-group of manifolds with boundary

The correspondences

$$\mathcal{M}_m \longrightarrow \mathcal{M}_m(\partial), \quad M \longmapsto (M, \emptyset),$$
$$\mathcal{M}_m(\partial) \longrightarrow \mathcal{M}_{m-1}^{ev}, \quad (M, \partial M) \longmapsto \partial M$$

induce homomorphisms

$$i_b : SK_m \to SK_m(\partial) \quad \text{and} \quad \partial : SK_m(\partial) \to SK_{m-1}^{ev}.$$

PROPOSITION 2.

$$SK_m(\partial) \cong \begin{cases} SK_m, & m \text{ is even}, \\ SK_{m-1}^{ev}, & m \text{ is odd}. \end{cases}$$

In fact, $i_b : SK_m \to SK_m(\partial)$ is an isomorphism if m is even, and $\partial : SK_m(\partial) \to SK_{m-1}^{ev}$ is an isomorphism if m is odd.

Proof. Koshikawa [**6**, Theorem 1.2] implies that $[M_1, \partial M_1] = [M_2, \partial M_2]$ in $SK_m(\partial)$ if and only if $\chi(M_1) = \chi(M_2)$. Using this fact, we first show that i_b is an isomorphism. Assume m is even. The construction of the double of $(M, \partial M) \in \mathcal{M}_m(\partial)$ induces a homomorphism $D : SK_m(\partial) \to SK_m$ whose image corresponds to $2\mathbb{Z}$ under the isomorphism $SK_m \cong \mathbb{Z}$. Hence there is a homomorphism $D' : SK_m(\partial) \to SK_m$ such that $2D' = D : SK_m(\partial) \to SK_m$. We easily see that $D' \circ i_b = \mathrm{id}$. To show $i_b \circ D' = \mathrm{id}$, for $(M, \partial M) \in \mathcal{M}_m(\partial)$ let $N \in \mathcal{M}_m$ be a manifold with $\chi(N) = \chi(DM)/2 (= \chi(M)$ since m is even). Then $i_b \circ D'([M, \partial M]) = [N, \emptyset] = [M, \partial M]$. Hence $i_b \circ D' = \mathrm{id}$.

We now turn to the proof for ∂ to be an isomorphism if m is odd. Consider the correspondence

$$\mathcal{M}^{ev}_{m-1} \longrightarrow \mathcal{M}_m(\partial), \quad N \longmapsto (I \times N', 2N'),$$

where $N' \in \mathcal{M}_{m-1}$ is a manifold with $\chi(N') = \chi(N)/2$, I is the unit interval $[0, 1]$ and $2N' = \partial(I \times N') = N' \cup N'$. This induces a homomorphism $q : SK^{ev}_{m-1} \to SK_m(\partial)$. We easily see $\partial \circ q = \mathrm{id}$. To show $q \circ \partial = \mathrm{id}$, for $(M, \partial M) \in \mathcal{M}_m(\partial)$ let $N'' \in \mathcal{M}_{m-1}$ be a manifold with $\chi(N'') = \chi(\partial M)/2$. Note that $\chi(\partial M) = 2\chi(M)$ if m is odd. We see $q \circ \partial([M, \partial M]) = [I \times N'', 2N''] = [M, \partial M]$. Hence $q \circ \partial = \mathrm{id}$. □

3. SK-group of $\mathbb{Z}_2$-manifolds

Consider the correspondence

$$\mathcal{M}^{ev}_m \longrightarrow \mathcal{M}^{\mathbb{Z}_2}_m(m-1), \quad M \longmapsto 2M'.$$

Here $M' \in \mathcal{M}_m$ is a manifold with $\chi(M') = \chi(M)/2$, and $2M'$ is given a free $\mathbb{Z}_2$-action which interchanges the components. This correspondence induces a homomorphism $i_e : SK^{ev}_m \to SK^{\mathbb{Z}_2}_m(m-1)$, and the correspondence

$$\mathcal{M}^{\mathbb{Z}_2}_m(m-1) \longrightarrow \mathcal{M}^{ev}_{m-1}, \quad M \longmapsto M^{\mathbb{Z}_2}$$

induces a homomorphism $\eta : SK^{\mathbb{Z}_2}_m(m-1) \to SK^{ev}_{m-1}$.

PROPOSITION 3.

$$SK^{\mathbb{Z}_2}_m(m-1) \cong \begin{cases} SK^{ev}_m, & m \text{ is even,} \\ SK^{ev}_{m-1}, & m \text{ is odd.} \end{cases}$$

In fact, $i_e : SK^{ev}_m \to SK^{\mathbb{Z}_2}_m(m-1)$ is an isomorphism if m is even, and $\eta : SK^{\mathbb{Z}_2}_m(m-1) \to SK^{ev}_{m-1}$ is an isomorphism if m is odd.

To prove Proposition 3, we need one more SK-group, $SK^{\mathbb{Z}_2}_m(\text{free})$. Let $\mathcal{M}^{\mathbb{Z}_2}_m(\text{free})$ be the subset of $\mathcal{M}^{\mathbb{Z}_2}_m$ consisting of $M \in \mathcal{M}^{\mathbb{Z}_2}_m$ such that the $\mathbb{Z}_2$-action on M is fixed point free, i.e. $M^{\mathbb{Z}_2} = \emptyset$. $\mathcal{M}^{\mathbb{Z}_2}_m(\text{free})$ is also a subset of $\mathcal{M}^{\mathbb{Z}_2}_m(m-1)$.

$SK_m^{\mathbb{Z}_2}(\text{free})$ is the Grothendieck group of $\mathcal{M}_m^{\mathbb{Z}_2}(\text{free})/SK$. There is a canonical homomorphism $SK_m^{\mathbb{Z}_2}(\text{free}) \to SK_m^{\mathbb{Z}_2}(m-1)$. As is shown in Kosniowski [**7**, 5.3], if m is even then $SK_m^{\mathbb{Z}_2}(\text{free})$ is isomorphic to $\mathbb{Z}$ by the isomorphism which sends $[M] \in SK_m^{\mathbb{Z}_2}(\text{free})$ to $\chi(M/\mathbb{Z}_2) \in \mathbb{Z}$ where $M/\mathbb{Z}_2$ is the orbit space of M, and if m is odd then $SK_m^{\mathbb{Z}_2}(\text{free}) = 0$.

Proof of Proposition 3. Assume m is even. For $M \in \mathcal{M}_m^{\mathbb{Z}_2}(m-1)$ let T be a $\mathbb{Z}_2$-invariant open tubular neighborhood of $M^{\mathbb{Z}_2}$ in M. Then the Euler characteristic of the double $D(M-T)$ of $M-T$ is divisible by 4, since $\chi(D(M-T)) = 2\chi(M-T)$ and $\chi(M-T)$ is even since the $\mathbb{Z}_2$-action on $M-T$ is free. Hence there is a manifold $M' \in \mathcal{M}_m^{ev}$ with $\chi(M') = \chi(D(M-T))/2$. The correspondence $M \mapsto M'$ induces a homomorphism $E : SK_m^{\mathbb{Z}_2}(m-1) \to SK_m^{ev}$. $E \circ i_e = \text{id}$ is easily shown.

To show $i_e \circ E = \text{id}$, we first recall some known results from Kosniowski [**7**]. In Kosniowski [**7**, 2.2], $SK_m^G[H; U]$ is defined as the SK-group of G-vector bundles of type $[H; U]$ over closed G-manifolds. When $G = \mathbb{Z}_2$ and $\sigma = [\mathbb{Z}_2; \tilde{\mathbb{R}}]$, there is a homomorphism $\nu : SK_m^{\mathbb{Z}_2}(m-1) \to SK_m^{\mathbb{Z}_2}[\sigma]$ which sends $[M] \in SK_m^{\mathbb{Z}_2}(m-1)$ to the SK-equivalence class of the normal bundle $\nu(M)$ of $M^{\mathbb{Z}_2}$ in M. $SK_m^{\mathbb{Z}_2}[\sigma]$ is isomorphic to SK_{m-1} by the isomorphism which sends a bundle to its base space (see Kosniowski [**7**, 5.3.4]). Hence we see for $M \in \mathcal{M}_m^{\mathbb{Z}_2}(m-1)$ that $[\nu(M)] = 0$ in $SK_m^{\mathbb{Z}_2}[\sigma](= 0$ since m is even). We also see that the bordism class of $\nu(M)$ in the bordism group of bundles is zero, because the line bundle associated to the double covering $M-T \to M-T/\mathbb{Z}_2$ gives a bordism between $\nu(M)$ and the zero, where T is a $\mathbb{Z}_2$-invariant open tubular neighborhood of $M^{\mathbb{Z}_2}$ in M. From Kosniowski [**7**, Corollary 2.7.2] we see $[M] = [N_1] - [N_2]$ in $SK_m^{\mathbb{Z}_2}(m-1)$ for some $N_1, N_2 \in \mathcal{M}_m^{\mathbb{Z}_2}(\text{free})$. Since $i_e \circ E([M]) = [2N_1'] - [2N_2']$ for some $N_i \in \mathcal{M}_m$ with $\chi(N_i') = \chi(N_i)/2$ for $i = 1, 2$, we must show $[2N_1'] - [2N_2'] = [N_1] - [N_2]$ in $SK_m^{\mathbb{Z}_2}(m-1)$. We easily see this equality in $SK_m^{\mathbb{Z}_2}(\text{free})$ and hence in $SK_m^{\mathbb{Z}_2}(m-1)$. This completes the proof for i_e to be an isomorphism.

We now turn to the proof for η. Assume m is odd. The inverse homomorphism $r : SK_{m-1}^{ev} \to SK_m^{\mathbb{Z}_2}(m-1)$ of η is constructed as follows. For $[N] \in SK_{m-1}^{ev}$ let $N' \in \mathcal{M}_{m-1}$ be a manifold with $\chi(N') = \chi(N)/2$. Consider a $\mathbb{Z}_2$-manifold $N' \times \tilde{S}^1 \in \mathcal{M}_m^{\mathbb{Z}_2}(m-1)$, where N' has a trivial $\mathbb{Z}_2$-action and $\tilde{S}^1$ is the one-dimensional sphere with a $\mathbb{Z}_2$-action whose fixed point set consists of two points. Define r to be the homomorphism which sends $[N] \in SK_{m-1}^{ev}$ to $[N' \times \tilde{S}^1] \in SK_m^{\mathbb{Z}_2}(m-1)$. Then $\eta \circ r = \text{id}$ is easily shown.

For $[M] \in SK_m^{\mathbb{Z}_2}(m-1)$, $r \circ \eta([M]) = [N'' \times \tilde{S}^1]$ where $N'' \in \mathcal{M}_{m-1}$ is a manifold with $\chi(N'') = \chi(M^{\mathbb{Z}_2})/2$. To show $r \circ \eta = \text{id}$, we must show that

$[M] = [N'' \times \tilde{S}^1]$ in $SK_m^{\mathbb{Z}_2}(m-1)$. Note that $\eta([M]) = \eta([N'' \times \tilde{S}^1])$. There is a commutative diagram

$$\begin{array}{ccc} SK_m^{\mathbb{Z}_2}(m-1) & \xrightarrow{\eta} & SK_{m-1}^{ev} \\ {\scriptstyle \nu}\downarrow & & \downarrow\cap \\ SK_m^{\mathbb{Z}_2}[\sigma] & \xrightarrow{\cong} & SK_{m-1}. \end{array}$$

From this we have $[\nu(M)] = [\nu(N'' \times \tilde{S}^1)]$ in $SK_m^{\mathbb{Z}_2}[\sigma]$, where $\nu(\ )$ denotes the normal bundle of the fixed point set. As in the proof for i_e, the bordism classes of $\nu(M)$ and $\nu(N'' \times \tilde{S}^1)$ are both zero in the bordism group of bundles. Hence Kosniowski [**7**, Corollary 2.7.2] implies that $[M] - [N'' \times \tilde{S}^1]$ is contained in the image of $SK_m^{\mathbb{Z}_2}(\text{free}) \to SK_m^{\mathbb{Z}_2}(m-1)$. Hence we have $[M] = [N'' \times \tilde{S}^1]$ since m is odd and $SK_m^{\mathbb{Z}_2}(\text{free}) = 0$. This completes the proof. □

4. Proof of the theorem

It is straightforward that the diagram in the theorem is commutative, and it is shown as follows that $D : SK_m(\partial) \to SK_{m,m-1}^{ev}$ is an isomorphism. There is a commutative diagram

$$\begin{array}{ccc} SK_m & \xrightarrow{i_b} & SK_m(\partial) \\ {\scriptstyle 2}\downarrow & & \downarrow{\scriptstyle D} \\ SK_m^{ev} & \xrightarrow{i} & SK_{m,m-1}^{ev}, \end{array}$$

where 2 is the homomorphism induced from the correspondence $M \mapsto 2M$. The homomorphism 2 is an isomorphism. When m is even, i_b and i are isomorphisms by Propositions 1 and 2. Hence D is an isomorphism if m is even. In a similar way, using Propositions 1, 2 and 3, we can show that the homomorphisms $D : SK_m(\partial) \to SK_{m,m-1}^{ev}$, $D : SK_m(\partial) \to SK_m^{\mathbb{Z}_2}(m-1)$ and $F : SK_m^{\mathbb{Z}_2}(m-1) \to SK_{m,m-1}^{ev}$ are isomorphisms for any m.

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