Math. Z. 169, 223–236 (1979)

Mathematische Zeitschrift

Multiple Points of Immersions, and the Kahn-Priddy Theorem

Ulrich Koschorke

Fachbereich Mathematik der Gesamthochschule Siegen, Hölderlinstraße 3, D-5900 Siegen 21, Federal Republic of Germany

Introduction

Classical methods of Pontrjagin-Thom [17] and of Hirsch [7] allow us to identify the n-th stable homotopy group π_n^S of spheres with the bordism group of oriented n-manifolds smoothly immersed into $\mathbb{R}^{n+1}$. Thus an analysis of the selfintersections of such immersed manifolds leads to potentially interesting invariants in homotopy theory.

In this paper we study, as a first example, the homomorphism

$$\theta = \theta_n \colon \pi_n^S \to \mathbb{Z}_2$$

which assigns to every immersion $i: M^n \to \mathbb{R}^{n+1}$ the mod 2 number of its (generic) $(n+1)$-fold points. This homomorphism was also investigated recently by Freedman [5] who conjectured that θ_n is the stable Hopf invariant (i.e. θ_n is nontrivial precisely when $n=0, 1, 3$ or 7) and who checked this for $n \leqq 3$.

Our results strongly support Freedman's conjecture.

Theorem A. (i) *If* $\alpha \in \pi_a^S$, $\beta \in \pi_b^S$ *with* $a, b > 0$, *then*

$$\theta_{a+b}(\alpha \circ \beta) = 0.$$

(ii) *Given* $\alpha \in \pi_a^S$, $\beta \in \pi_b^S$ *and* $\gamma \in \pi_c^S$ *such that* $\alpha \circ \beta = \beta \circ \gamma = 0$, *consider the Toda bracket* $\langle \alpha, \beta, \gamma \rangle = \pm \langle \gamma, \beta, \alpha \rangle$ *as a subset of* $\pi_{a+b+c+1}^S$. *Assume* $0 < a \leqq b + c$. *If* $\theta_c(\gamma) = 0$ *or* $c+1$ *does not divide* $a+b+1$, *then*

$$\theta_{a+b+c+1}(\langle \alpha, \beta, \gamma \rangle) = \{0\}.$$

Theorem B. *If* n *is even and strictly positive, then* $\theta_n \equiv 0$.

An inspection of Toda's tables now gives the following result (use [18], pp. 189, 110, and interchange α and γ when necessary).

Corollary C. *If* $n \leqq 20$, $n \neq 0, 1, 3$ *or* 7, *then* $\theta_n \equiv 0$.

0025-5874/79/0169/0223/$02.80

On the other hand, θ is the unique $\mathbb{Z}_2$-valued epimorphism on the cyclic groups π_0^S, π_1^S and π_3^S ([5]; see also our § 2, especially example 2.4, where the case $n=7$ is included in the discussion).

We prove Theorem A by straightforward differential topology (in § 1). The proof of Theorem B, however, involves the Kahn-Priddy theorem [8] which states e.g. that a certain map

$$\lambda_*: \pi_n^S(P^q) \to \pi_n^S \qquad (0<n<q)$$

is an epimorphism of 2-primary components. In § 2 we show that λ_* can be interpreted as "twisted" multiplication with the generator $\eta \in \pi_1^S$. This leads to Theorem B, and, more generally, to a description of θ_n in terms of the selfintersection behavior of certain *un*oriented codimension-2 immersions in $\mathbb{R}^{n+1}$. As a side result we obtain:

Theorem D. *Every element in the 2-primary part of the framed bordism group of arbitrary dimension can be represented by the total space M of a circle bundle with structure group $\mathbb{Z}_2$ (acting by a reflection), together with a parallelisation (not just stable framing!) of M which is preserved by the (locally defined) S^1-action on M.*

Note that for every natural number r the nontrivial standard r-fold cover of each fiber determines an r-fold covering map of M over itself which preserves the parallelisation up to homotopy. Also, the fiberwise antipodal map on M allows us to establish the following variant of a theorem of Brown [3].

Corollary E. *Every closed framed manifold of dimension $n>0$ is framed bordant to a parallelised manifold which allows a fixed point free involution preserving the parallelisation.*

In [9] it was already shown that parallelised bordism of parallelised manifolds coincides with the corresponding (stably!) framed bordism; this is used here to handle the situation away from the prime 2.

Another immediate consequence of Theorem D is a result of Ray [14]: clearly M can be reframed (as the boundary of a disk bundle) to become framed bordant to zero.

The Kahn-Priddy theorem implies also that for every prime p there is homomorphism

$$\beta_*: \pi_n^S \to \pi_n^S(\mathrm{B}\Sigma_p)$$

which is injective on the p-primary part of π_n^S (see also [16]). In § 3, on the other hand, we construct many selfintersection invariants on π_n^S, among them the "basic" homomorphism

$$\psi_p: \pi_n^S \to \pi_n^S((B\Sigma_p)^{\phi_p})$$

where $(B\Sigma_p)^{\phi_p}$ denotes the Thom space of the canonical $(p-1)$-plane bundle ϕ_p over the classifying space of the symmetric group Σ_p. It is a very interesting question whether the Kahn-Priddy theorem can be rephrased in terms of selfintersections, and especially, whether ψ_p is injective on the p-component for every prime p. In particular, this would imply that oriented codimension-1 immersions into euclidean space are entirely determined, up to bordism, by their selfintersection behavior - a possible result which seems to be out of the reach of direct geometric methods. One can go one step further and ask: given $x \in \pi_n^S$ and a natural number q such that for all $p \geqq q$ $\psi_p(x)=0$, can x be represented by an immersion without q-tuple points? Also, here is an interesting geometric problem: how can one express arbitrary intersection invariants in terms of the basic ones?

A certain invariant $\psi_2^{(n)}$ defined in § 3 can be identified with the stable Hopf invariant. This leads to the following (equivalent) form of the Freedman's conjecture: If N is a closed (not necessarily orientable) manifold of even dimension $n=2m$, and a (generic) immersion $j: N \to \mathbb{R}^{n+2}$ allows a normal nowhere zero vectorfield, then the number of $(m+1)$-tuple points of j equals, mod 2, the Euler number of N (which is known to be even except possibly when $n=0,2$ or 6).

Finally, in the last section we study the behavior of the selfintersection invariants in low dimensions. This is done by means of explicitly constructed immersions. We get e.g. the following result which contains some folk theorems (cf. [2]) and many of the results of [5].

Theorem F. a) *For any selftransverse immersion i of a closed surface M into $\mathbb{R}^3$ the following integers are equal modulo 2:*

(i) *the Euler number of M,*

(ii) *the number of triple points of i,*

(iii) *the number of circles in $\mathbb{R}^3$ along which i intersects itself in such a way that one cannot distinguish globally a "first" sheet and a "second" sheet of $i(M)$,*

(iv) *the number of double points of any selftransverse immersion $M \to \mathbb{R}^4$ which is regularly homotopic to $M \xrightarrow{i} \mathbb{R}^3 \subset \mathbb{R}^4$.*

Clearly these numbers are all even when M is orientable.

b) *For any selftransverse immersion j of a closed 3-manifold M into $\mathbb{R}^4$ the following integers are equal modulo 2:*

(i) *the Euler number of the surface of double points of j,*

(ii) *the number of quadruple points of j,*

(iii) *the number of double points of any selftransverse immersion $M \to \mathbb{R}^6$ which is regularly homotopic to $M \xrightarrow{j} \mathbb{R}^4 \subset \mathbb{R}^6$.*

Moreover, there exists an oriented 3-manifold immersed into $\mathbb{R}^4$ such that all these numbers are odd.

It is a pleasure to thank Paul Schweitzer und Duane Randall for many interesting conversations; also, I would like to express my gratitude to the Mathematics Department of PUC in Rio de Janeiro, to the Brasilian CNP_q and to the German GMD for their hospitality and financial support during the time when this work developed.

Conventions. All manifolds, immersions, embeddings etc. will be smooth. Given an n-manifold M, the *tangent bundle* is denoted by TM and the *orientation bundle* $\Lambda^n TM$ by ξ_M. The *canonical line bundle over real projective space* P^∞ is written γ.

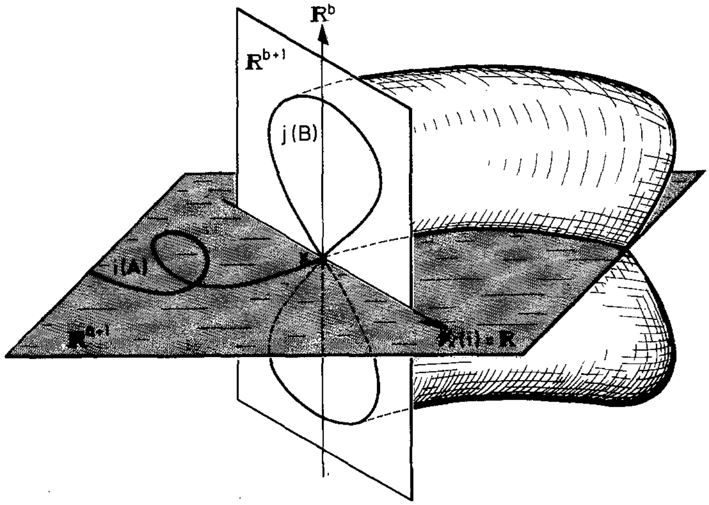


Fig. 1. The immersion $i*j$

§ 1. Proof of Theorem A

First let us specify the identification between oriented bordism of codimension-one immersions and the stable homotopy of spheres. Given an immersion $i: M^n \to \mathbb{R}^{n+1}$, its tangent map induces a monomorphism $i_*: TM \hookrightarrow M \times \mathbb{R}^{n+1} = \underline{\mathbb{R}}^{n+1}$ whose cokernel is isomorphic to the orientation bundle ξ_M of M. Thus if M is oriented, i_* can be extended to a stable parallelisation $TM \oplus \underline{\mathbb{R}} \cong \underline{\mathbb{R}}^{n+1}$ of M. We identify the resulting framed bordism class, or rather its image under the Pontrjagin-Thom construction, with the bordism class of i.

Next represent $\alpha \in \pi_a^S$, $\beta \in \pi_b^S$ by immersions $i: A \to \mathbb{R}^{a+1}$, $j: B \to \mathbb{R}^{b+1}$. Since A is oriented, the normal bundle $\nu(i)$ of i is trivial and i extends to an "immersed tubular neighborhood" map $h: A \times \mathbb{R}^{b+1} \to \mathbb{R}^{a+b+1}$. Then clearly $i*j$, defined by

$$i*j: A \times B \xrightarrow{\mathrm{Id}_A \times j} A \times \mathbb{R}^{b+1} \xrightarrow{h} \mathbb{R}^{a+b+1},$$

represents $\pm \alpha \circ \beta \in \pi_{a+b}^S$. (In other words, each normal slice of the immersion $A \to \mathbb{R}^{a+b} \subset \mathbb{R}^{a+b+1}$ can be identified with $\mathbb{R}^{b+1}$, and then contains a copy of $j(B)$, see Fig. 1.)

Now let a, b be strictly positive. Since $\alpha \circ \beta = \pm \beta \circ \alpha$, we may assume that $a \leqq b$ and hence $2a < a+b+1$. Then, after a small deformation, the composed immersion $A \to \mathbb{R}^{a+1} \subset \mathbb{R}^{a+b+1}$ has only transverse double points, and therefore none at all. Similarly, h can be deformed into an embedding h'. Since generically j has at most $(b+1)$-tuple points, so has $h' \circ (\mathrm{id}_A \times j)$. Therefore $\theta_{a+b}(\alpha \circ \beta) = 0$.

Finally let arbitrary elements $\alpha \in \pi_a^S$, $\beta \in \pi_b^S$ and $\gamma \in \pi_c^S$ be represented by codimension-one immersions i, j, k of manifolds A, B and C. If $\alpha \circ \beta = \beta \circ \gamma = 0$, then $i*j: A \times B \to \mathbb{R}^{a+b+1}$ and $j*k$: $B \times C \to \mathbb{R}^{b+c+1}$ extend to immersions $\ell_+: X \to \mathbb{R}^{a+b+1} \times [0, \infty)$ and $\ell_-: Y \to \mathbb{R}^{b+c+1} \times (-\infty, 0]$ respectively, where e.g.

X is a compact manifold with boundary $A \times B$ and $\ell_+ = (i * j) \times \mathrm{Id}$ on a collar $A \times B \times [0, 1) \subset X$. The two resulting bordisms

$$\ell_+ * k: X \times C \to \mathbb{R}^{a+b+c+1} \times [0, \infty)$$

and

$$i * \ell_- : A \times Y \to \mathbb{R}^{a+b+c+1} \times (-\infty, 0]$$

of $i * j * k$ fit together to give a smooth codimension-one immersion into $\mathbb{R}^{a+b+c+2}$. It is not hard to see that every element in $\langle \alpha, \beta, \gamma \rangle$ can be represented this way.

Now deform the composed immersions $A \to \mathbb{R}^{a+1} \subset \mathbb{R}^{a+b+c+1}$, $B \to \mathbb{R}^{b+1} \subset \mathbb{R}^{b+c+1}$ and $C \to \mathbb{R}^{c+1}$ to (framed) immersions i', j', k' with transverse selfintersections (i.e., wherever sheets $V_1, \dots V_r$ meet, V_r lies transverse to $V_1 \cap \dots \cap V_{r-1}$). Then $i' * j'$, $j' * k'$ and $i' * j' * k'$ have again transverse selfintersections provided the immersed tubular neighborhood used in the $*$-construction are shrunk sufficiently. Deform ℓ_+ and ℓ_- accordingly to achieve transversality.

Assume $0 < a \leqq b + c$. Then i' is an embedding and the highest order of selfintersection of $i' * \ell'_-$ is $b + c + 2 < a + b + c + 2$. If the codimension $c + 1$ of $\ell'_+ : X \to \mathbb{R}_+^{a+b+c+2}$ does not divide $a + b + c + 2$, the highest order of selfintersection of ℓ'_+ will be strictly smaller than $(a + b + c + 2)/(c + 1)$, and hence $\ell'_+ * k'$ has no $(a + b + c + 2)$-tuple point; indeed, the tubular neighborhood of each sheet of ℓ'_+ contributes at most $c + 1$ sheets to a selfintersection of $\ell'_+ * k'$. On the other hand, consider the case that $a + b + c + 2 = r(c + 1)$ for some $r \in \mathbb{Z}$. Then in the neighborhood of each r-tuple point x of ℓ'_+ the image of $\ell'_+ * k'$ looks like the union

$$\bigcup_{s=0}^{r-1} \mathbb{R}^{s(c+1)} \times k'(C) \times \mathbb{R}^{(r-s-1)(c+1)}$$

in $\mathbb{R}^{r(c+1)}$. Therefore the number of $(a + b + c + 2)$-tuple points of $\ell'_+ * k'$ near X is q^r provided k' has q $(c + 1)$-tuple points. If $\theta_c(\gamma) = 0$, then q and q^r are even, and hence again $\theta_{a+b+c+1}$ vanishes on $(\ell'_+ * k' \cup i' * \ell'_-) \sim (\ell_+ * k \cup i * \ell_-)$.

Remark 1.1. In [4] Cohen proved that every element in the 2-torsion of π_n^S, $n \neq 1, 3$ or 7, belongs to a higher Toda bracket. Can Theorem A be sharpened to prove the vanishing part of Freedman's conjecture?

§ 2. The Figure-8 Construction and the Kahn-Priddy Theorem

It is wellknown that the stable homotopy group $\pi_n^S(P^q)$, $n < q$, can be identified with *un*oriented bordism of immersions $i: N^{n-1} \to \mathbb{R}^n$. This can be seen along the lines of our geometric interpretation of π_n^S (however, note the different dimensions!). More details will be given below.

Starting from such an immersion i, we will now construct a new immersion 8_i. Each normal slice of the composed immersion $N^{n-1} \xrightarrow{i} \mathbb{R}^n \subset \mathbb{R}^{n+1}$ is diffeomorphic to a plane $v_x \times \mathbb{R}$ and hence contains a figure 8, centered at 0 and vertical (i.e. intersecting $\{0\} \times \mathbb{R}$ three times). Because of the invariance of the figure -8 under the $\mathbb{Z}_2$-action on v_x, this leads to a global immersion 8_i of the unit circle bundle of $v(i) \oplus \underline{\mathbb{R}}$ into $\mathbb{R}^{n+1}$ (compare with Fig. 1). Since $v(i)$ is canonically isomorphic to the orientation bundle $\xi_N = \Lambda^{n-1} TN$, we obtain an *oriented* codimension-one immersion into $\mathbb{R}^{n+1}$. With the previous identifications in mind, we have constructed a homomorphism

$$8_*: \pi_n^S(P^q) \to \pi_n^S.$$

Note that the obvious composed map $\pi_{n-1}^S \to \pi_n^S(P^q) \to \pi_n^S$ is just multiplication with the generator η of π_1^S.

On the other hand, each line in $\mathbb{R}^{q+1}$ defines a reflection and hence a "loop" $(\mathbb{R}^{q+1} \cup \infty, \infty) \to (\mathbb{R}^{q+1} \cup \infty, \infty)$ to which we add the identity loop (by loop addition). After a deformation around the basepoint $* = \mathbb{R} \cdot (1, 0, 0, \ldots, 0)$ of P^q we obtain a map $\ell: (P^q, *) \to (\Omega^{q+1} S^{q+1}$, constant) whose adjoint defines a homomorphism

$$\lambda_*: \pi_n^S(P^q) \to \pi_n^S.$$

Theorem 2.1[1]. *For $0 < n < q$ the homomorphism 8_* coincides with λ_* (and hence is an epimorphism of 2-primary components by* [8]*).*

Proof. We may assume q as large as we like. Thus, given $i: N^{n-1} \to \mathbb{R}^n$, pick an embedding $e: N \subset \mathbb{R}^{q+1}$; so (i, e): $N \to \mathbb{R}^n \times \mathbb{R}^{q+1}$ is again an embedding. Choosing an (immersed) tubular neighborhood $v(i) \to \mathbb{R}^n$ of i, we can identify $v(i) \times \mathbb{R}^{q+1}$ with a neighborhood of $(i, e)(N)$ in $\mathbb{R}^{n+q+1}$ in the obvious way.

Now let $\bar{P}^{q-1}$ denote the projective space of $\{0\} \times \mathbb{R}^q \subset \mathbb{R}^{q+1}$; note that the Thom space of the canonical line bundle $\bar{\gamma}$ over $\bar{P}^{q-1}$ is P^q. Thus a classifying map $N \to \bar{P}^{q-1}$ of $v(i)$ determines a map $v(i) \times \mathbb{R}^{q+1} \to \bar{\gamma} \times \mathbb{R}^{q+1}$ which extends to a map f from $S^{n+q+1} = \mathbb{R}^{n+q+1} \cup \{\infty\}$ to the Thom space of $\bar{\gamma} \times \mathbb{R}^{q+1}$, i.e. $P^q \wedge S^{q+1}$. This defines the element in $\pi_n^S(P^q)$ corresponding to the immersion i.

Composing with the adjoint $\ell^*: P^q \wedge S^{q+1} \to S^{q+1}$ we obtain the element $\lambda_*[i] = [\ell^* \circ f]$ in π_n^S which we will now describe by an oriented codimension-one immersion. We may assume that $0 \in \mathbb{R}^{q+1} \cup \{\infty\} = S^{q+1}$ is a regular value of $\ell^* \circ f$, and we have to study the framed submanifold $(\ell^* \circ f)^{-1}(0)$ in $v(i) \times \mathbb{R}^{q+1} \subset \mathbb{R}^{n+q+1}$.

Given $x \in N$, consider $\ell^* \circ f$, restricted to the slice $v_x(i) \times \mathbb{R}^{q+1}$ (normal to $(i, e)(N)$ at $(i, e)(x)$). For $y \in v_x(i)$ usually $\ell^* \circ f | \{y\} \times \mathbb{R}^{q+1}$ is a loop sum of the reflection with respect to the line $f(y) \in P^q$, and of the identity on $\mathbb{R}^{q+1}$; however, if y is very far from $0 \in v_x(i)$, i.e. if $f(y)$ is close to the base point of P^q, then $\ell^* \circ f | \{y\} \times \mathbb{R}^{q+1}$ is an intermediate stage of the homotopy between the loop sum $(-\mathrm{Id}_{\mathbb{R}} \times \mathrm{Id}_{\mathbb{R}^n}) + \mathrm{Id}_{\mathbb{R}^{n+1}}$ and the constant map into $\infty \in S^{q+1}$. It follows that $(\ell^* \circ f)^{-1}(0) \cap v_x(i) \times \mathbb{R}^{q+1}$ consists of two arcs parallel to the v_x-direction, joined together at their ends to form a circle. The normal framing is the obvious one except around the center of the "lower" arc; as one moves along this arc, the normal framing is given by reflections with respect to vectors which rotate from

[1] I am told that Gray obtained this result independently

$(1,0,\dots 0)\in\mathbb{R}^{q+1}$ through a unit vector v in the line $f((i.e)(x))\in\bar{P}^{q-1}$ towards $(-1,0,\dots 0)$.

Now twist the lower part of our embedded circle by 180° around the axis $\mathbb{R}(1,0,\dots 0)$ in the threedimensional subspace $v_x(i)\times\mathbb{R}\cdot(1,\dots 0)\times\mathbb{R}\cdot v$ of $v_x(i)\times\mathbb{R}^{q+1}$. (For precise details see the isotopy Theorem 5.2 in [10].) At the end of this isotopy the framing is the identity von $\{0\}\times\mathbb{R}^q$ everywhere, and the projection from $v_x(i)\times\mathbb{R}^{q+1}$ to $v_x(i)\times\mathbb{R}\times\{0\}$ takes our circle into the (vertical) figure-8. This construction is compatible with the $\mathbb{Z}_2$-action on $v_x(i)=\mathbb{R}\cdot v$, and hence can be performed simultaneously in all normal slices $v_x\times\mathbb{R}^{q+1}$, $x\in N$.

It follows that $(\ell^*\circ f)^{-1}(0)$ is the total space of the circle bundle of $v(i)\oplus\underline{\mathbb{R}}=\xi_N\oplus\underline{\mathbb{R}}$, and that its figure-8 immersion represents $\lambda_*[i]$. This proves Theorem 2.1.

At this point we digress to describe the homomorphism $\lambda_*\colon\pi_n^S(P^q)\to\pi_n^S$ in terms of the geometric interpretation of stable homotopy by bordism of framed manifolds with maps (which is perhaps better known than the immersion interpretation). Thus let $z\in\pi_n^S(P^q)$ correspond to the bordism class of a framed n-dimensional manifold L together with a map $f\colon L\to P^q$. Let $N\subset L$ be the $(n-1)$-dimensional zero set of a transverse section of $f^*(\gamma)$, where γ is the canonical line bundle over P^q. After a suitable deformation, the framing of L restricts to a trivialization of $TL|N$ which in turn is canonically isomorphic to $TN\oplus\xi_N$. Now let $p\colon M\to N$ be the unit circle bundle of $\xi_N\oplus\underline{\mathbb{R}}$. Then TM splits into a "horizontal" part (isomorphic to $p^*(TN)$) and the line bundle tangent to the fibers (which is isomorphic to $p^*(\xi_N)$). Thus we obtain a parallelization

$$TM\cong p^*(TN\oplus\xi_N)\cong\underline{\mathbb{R}}^n$$

which we can choose to be invariant under the action of S^1 on the fibers. (This action is only defined over oriented subsets of N; change of orientations corresponds to conjugation in S^1.) Then the framed bordism class of M, parallelized this way, corresponds to $\lambda_*(z)$. This is easy to see once one notices (and applies fiberwise) that the standard normal vector field of the figure-8 immersion in the (x,y)-plane never points in the negative y-direction; hence, up to homotopy, the resulting stable parallelization of the unit circle comes from the invariant vector field on S^1. Theorem D and its corollaries follow immediately.

Now we return to our study of immersions. Consider the homomorphism $\vartheta\colon\pi_n^S(P^\infty)\to\mathbb{Z}_2$ defined as follows. Given an element in $\pi_n^S(P^\infty)$, represent it by an immersion $i\colon N^{n-1}\hookrightarrow\mathbb{R}^n$. If $n+1$ is even, deform the composed immersion $N^{n-1}\xrightarrow{i}\mathbb{R}^n\subset\mathbb{R}^{n+1}$ into an immersion i' with transverse selfintersections, and let $\vartheta[i]$ denote the mod 2 number of the $\frac{n+1}{2}$-tuple points of i'.

If $n+1$ is odd, put $[i]=0$.

Proposition 2.2. *The following diagram commutes*

$$\begin{array}{ccc} \pi_n^S(P^\infty) & \xrightarrow{\;8_*\;} & \pi_n^S \\ & \underset{\vartheta}{\searrow} & \big\downarrow{\scriptstyle \theta_n} \\ & & \mathbb{Z}_2 \end{array}$$

In particular, if $n>1$, then ϑ vanishes on the image of the obvious homomorphism from π_{n-1}^S into $\pi_n^S(P^\infty)$ (which forgets orientations).

Proof. Each sheet of i' at a selfintersection of i' gives rise to at most two sheets of $8_{i'}$ at the corresponding selfintersection of $8_{i'}$. Thus, for $n+1$ even, the $((n+1)/2)$-tuple points of i' are precisely the $(n+1)$-tuple points of $8_{i'}$. If $n+1$ is odd, $8_{i'}$ has no $(n+1)$-tuple points. Since $\theta_n \circ 8_*[i]$ is the mod 2 number of $(n+1)$-tuple points of $8_{i'}$, the first statement in our proposition follows. In view of Theorem A, (i), the second statement is also clear, since $\pi_{n-1}^S \to \pi_n^S(P^\infty) \xrightarrow{8_*} \pi_n^S$ is multiplication with $\eta \in \pi_1^S$.

The results 2.1 and 2.2 clearly imply the theorem B of the introduction. Also, it follows easily from Proposition 2.2. and the examples in section 4 that θ_3 is onto.

The next result leads to a uniform approach to the exceptional dimensions $n=1, 3$ and 7.

Proposition 2.3. *Let λ be a line bundle over a closed manifold M of dimension n, let $p: \tilde{M} \to M$ be the corresponding double cover, and let $N \subset M$ be the $(n-1)$-dimensional zero set of a generic smooth section of $\lambda \otimes \xi_M$ $(\xi_M = \Lambda^n TM)$. Given an immersion $i: M \to \mathbb{R}^{n+1}$, the number of (generic) $(n+1)$-tuple points of $i \circ p: \tilde{M} \to \mathbb{R}^{n+1}$ equals,* mod 2, *the number of (generic) $\frac{n+1}{2}$-tuple points of the codimension-2 immersion $i|N: N \to \mathbb{R}^{n+1}$. (In particular, if $\lambda = \xi_M$, then $\theta_n[i \circ p] = 0$ trivially.)*

Proof. Note that $\xi_M \cong \nu(i)$, and interpret the section of $\lambda \otimes \xi_M = \mathrm{Hom}(\lambda, \nu(i))$ as a line bundle homomorphism $h: \lambda \to \nu(i)$ which is injective outside of N. Now $\tilde{M}$ is the boundary of the unit disk bundle $D(\lambda)$. Thus $h|D(\lambda)$ provides an immersed bordism between the immersions $h|\tilde{M} \to \nu(i)$ (whose only double points occur at N) and the figure-8 immersion centered at N and "vertical" with respect to the normal direction of N in M. Now compose with an immersion of $\nu(i)$ into $\mathbb{R}^{n+1}$ as tubular neighborhood of i. The resulting figure-8 immersion is clearly bordant to $i \circ p$. The arguments of the previous proof now apply to yield our proposition.

Example 2.4. For $n=1, 3$ or 7 the division algebra structure on $\mathbb{R}^{n+1}$ (complex, quaternionic or Cayley) leads to a parallelization of P^n and hence to an immersion $i: P^n \to \mathbb{R}^{n+1}$. The corresponding immersion $S^n \to \mathbb{R}^{n+1}$ generates π_n^S. Moreover, the canonical line bundle over P^n allows a section with zero set P^{n-1}. Thus $\theta_n: \pi_n^S \to \mathbb{Z}_2$ is the unique epimorphism iff $i|P^{n-1}: P^{n-1} \to \mathbb{R}^{n+1}$ has an odd number of generic $((n+1)/2)$-tuple points. This is satisfied for $n=1$ (trivial) and $n=3$ (see Theorem F and § 4). However, I do not know a proof for $n=7$.

§ 3. General Selfintersection Invariants

In this section we construct many invariants for elements in π_*^S along the lines of [11].

Given integers $k, r \geqq 1$, consider a framed immersion $i: M \to \mathbb{R}^{n+k}$ of a closed n-dimensional manifold M. After a small deformation, we may assume that i has only transverse selfintersections. Then the set

$$\tilde{M}(r)=\{(x_1,\ldots,x_r)\in(M)^r \mid i(x_1)=\ldots=i(x_r);\ x_s \neq x_t \text{ for } s\neq t\}$$

forms a closed $(n+k-kr)$-dimensional submanifold of $(M)^r=M\times\ldots\times M$ on which the symmetric group Σ_r acts freely. We call $M(r)=\tilde{M}(r)/\Sigma_r$ the *r-tuple point manifold* of i. $M(r)$ is naturally equipped with a classifying map $g\colon M(r)\to B\Sigma_r$ of the covering space $\tilde{M}(r)$ over $M(r)$, and with a stable vector bundle isomorphism $\bar{g}\colon T(M(r))\oplus g^*(k\phi_r)\cong\underline{\mathbb{R}}^n$ derived from the obvious framed immersion of $\tilde{M}(r)$ into $\mathbb{R}^{n+k}$; here ϕ_r denotes the $(r-1)$-dimensional vector bundle over $B\Sigma_r$ which corresponds to the standard Σ_r-action on the complement of the diagnonal in $\mathbb{R}^r$.

Note that the bordism group of framed codimension-k immersions into $\mathbb{R}^{n+k}$ can again be identified with π_n^S. Thus the construction above defines a homomorphisms from π_n^S into the bordism group of triples consisting of a closed $(n-k(r-1))$-manifold N, a continuous map $g\colon N\to B\Sigma_r$ and a stable isomorphism $\bar{g}\colon TN\oplus g^*(k\phi_r)\cong\underline{\mathbb{R}}^n$. This "*normal*" bordism group, denoted by $\Omega_{n-k(r-1)}(B\Sigma_r;k\phi_r)$ (cf. [9]), is canonically isomorphic, via the Pontrjagin-Thom construction, to the stable homotopy group $\pi_n^S((B\Sigma_r)^{k\phi_r})$ of the Thom space of $k\phi_r=\phi_r\oplus\ldots\oplus\phi_r$($k$ factors). Hence the "*codimension-k, r-tuple point homomorphism*" takes the following form

$$\psi_r^{(k)}\colon \pi_n^S\to\pi_n^S((\mathrm{B}\Sigma_r)^{k\phi_r}).$$

If $k=1$, we write simply ψ_r for $\psi_r^{(1)}$. If in addition r equals a prime p, we call $\psi_p=\psi_p^{(1)}$ the *basic p-primary selfintersection homomorphism.*

Note that $\psi_r^{(k)}$ is only interesting when $1<r\leqq\frac{n}{k}+1$; otherwise our homomorphism is the identity on π_n^S (for $r=1$) or it vanishes.

Recall from [9], § 9 that a 0-dimensional bordism group of a (reasonable) pathconnected space is $\mathbb{Z}$ or $\mathbb{Z}_2$ according to whether the coefficient bundle is orientable or not.

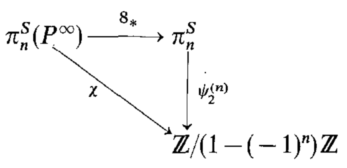
Example 3.1. For $n>0$ it follows from [10] that

$$\psi_2^{(n)}\colon \pi_n^S\to\Omega_0(P^\infty;n\gamma)=\begin{cases}\mathbb{Z}_2 & n \text{ odd}\\ \mathbb{Z} & n \text{ even}\end{cases}$$

coincides with the stable Hopf invariant (and hence, by [1], vanishes iff $n\neq 1,3,7$).

We now compute $\psi_2^{(n)}\circ 8_*$. Given $x\in\pi_n^S(P^\infty)$, represent x by an immersion $N^{n-1}\to\mathbb{R}^n$, and define $\chi(x)$ in $\mathbb{Z}_2$ or $\mathbb{Z}$ to be the (mod 2) Euler number of N.

Proposition 3.2. *The following diagram commutes for all* $n>0$



In particular, if $n \neq 1, 3$ or 7, then every closed $(n-1)$-manifold N which immerses into $\mathbb{R}^n$ must have an even Euler number.

(Thus the case $N = P^0$, P^2 or P^6 (see example 2.4.) is quite exceptional.)

Proof. We need to consider only the case when n is odd, since otherwise $\psi_2^{(n)} \circ 8_* \equiv \chi \equiv 0$. Given $i: N^{n-1} \to \mathbb{R}^n$, we obtain $\psi_2^{(n)} \circ 8_*[i]$ by counting the generic double points of $8_i': S(\xi_N \oplus \underline{\mathbb{R}}) \xrightarrow{8_i} \mathbb{R}^{n+1} \subset \mathbb{R}^{2n}$. For this purpose deform the immersion $N^{n-1} \to \mathbb{R}^n \subset \mathbb{R}^{2n}$ into an embedding, with tubular neighborhood $(\xi_N \oplus \underline{\mathbb{R}}) \oplus \underline{\mathbb{R}}^{n-1}$. Note that there is a rather obvious embedding e of $S(\xi_N \oplus \underline{\mathbb{R}})$ into $(\xi_N \oplus \underline{\mathbb{R}}) \oplus \xi_N$ which projects to the figure-8 immersion $S(\xi_N \oplus \underline{\mathbb{R}}) \to \xi_N \oplus \underline{\mathbb{R}}$. Now any generic bundle morphism $h: \xi_N \to \underline{\mathbb{R}}^{n-1}$ over N combines with e to give a generic immersion into $(\xi_N \oplus \underline{\mathbb{R}}) \oplus \underline{\mathbb{R}}^{n-1} \subset \mathbb{R}^{2n}$ which is homotopic to $8_i'$ and which has its double points precisely at the zeros of h. The mod 2 number of such zeroes equals

$$\begin{aligned} w_{n-1}(\mathbf{Hom}(\xi_N, \mathbb{R}^{n-1}))[N] &= w_{n-1}((n-1)\xi_N)[N] \\ &= w_1(N)^{n-1}[N] \\ &= w_{n-1}(N)[N] \end{aligned}$$

(since $TN \oplus \xi_N$ is trivial and therefore $w(N) = (1 + w_1(N))^{-1}$).
It follows that $\psi_2^{(n)} \circ 8_*[i] = \chi[i]$.

Example 3.3. By definition

$$\theta_n = \psi_{n+1}^{(1)} = \psi_{n+1}: \pi_n^S \to \Omega_0(B\Sigma_{n+1}; \phi_{n+1}) = \mathbb{Z}_2.$$

A priori, it is not clear why there should be a simple relation between θ_n and the stable Hopf invariant $\psi_2^{(n)}$. Note that by the results 2.1, 2.2 and 3.2, Freedman's conjecture is equivalent to the claim that

$$\vartheta \equiv \chi: \pi_n^S(P^\infty) \to \mathbb{Z}_2 \qquad \text{for all odd } n > 0.$$

Example 3.4. Let k be even and let $n = km$ be strictly positive. Then

$$\psi_{m+1}^{(k)}: \pi_n^S \to \Omega_0(B\Sigma_{m+1}; k\phi_{m+1}) = \mathbb{Z}$$

must obviously vanish. For $k = 2$ this proves again the second statement in Proposition 2.2.

Remark 3.5. Note that normal bordism groups into which our selfintersection homomorphisms $\psi_r^{(k)}$ map, can again be interpreted in terms of immersions. E.g.

$$\psi_2 = \psi_2^{(1)}: \pi_n^S \to \pi_n^S((B\Sigma_2)^\gamma) = \pi_n^S(P^\infty)$$

takes oriented codimension-1 immersions in $\mathbb{R}^{n+1}$ into unoriented codimension-1 immersions in $\mathbb{R}^n$.

Clearly our selfintersection invariants can be generalized to give also a whole family of homomorphisms defined on $\Omega_*(B\Sigma_r, k\phi_r)$ etc. So it should be interesting to iterate our construction. Especially the unoriented analogues of $\psi_r^{(k)}$, to be defined on $\pi_*^S(P^\infty)$, should be very important, as well as relations among their composites with ψ_2, 8_* and $\psi_*^{(*)}$.

§ 4. Low-Dimensional Examples

In this section we will study our selfintersection invariants $\psi_r^{(k)}$ on low-dimensional groups π_n^S by means of explicit immersions. We will assume $r>1$ throughout. Also we will use the techniques of [9], § 9 to compute normal bordism groups.

Clearly, for $n=1$ or 2 the only non-trivial selfintersection invariant is given by the isomorphism

$$\psi_2 \colon \pi_n^S \xrightarrow{\cong} \pi_n^S(P^\infty) \cong \Omega_{n-1}(P^\infty; \gamma) \cong \mathbb{Z}_2$$

whose inverse is 8_*. To see this, note that π_n^S is generated by the figure-8 immersion $j\colon S^1 \hookrightarrow \mathbb{R}^2$ or by its square $j * j$.

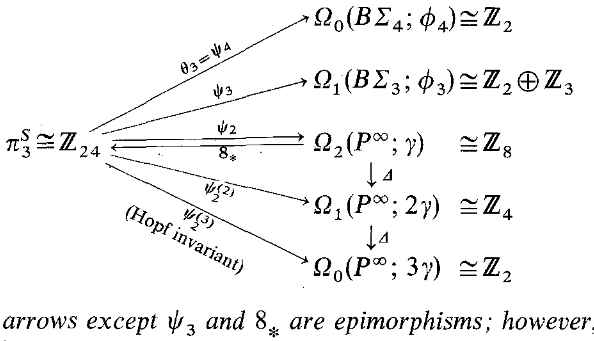
Thus let us investigate the case $n=3$. First recall that the double cover $p\colon S^\infty \to P^\infty$ gives rise to a long exact Gysin sequence (see [14], or also [9], § 9) of normal bordism groups

$$\begin{array}{l} \ldots \to \Omega_m(S^\infty; p^*(q\gamma) \xrightarrow{p_*} \Omega_m(P^\infty; q\gamma) \xrightarrow{\Delta} \Omega_{m-1}(P^\infty; (q+1)\gamma) \to \ldots \\ \qquad \| \wr \\ \qquad \pi_m^S \end{array} \tag{4.1}$$

Here $\Delta[N; g\colon N \to P^\infty; \bar{g}\colon TN \oplus g^*(q\gamma) \cong \underline{\mathbb{R}}^{m+q}]$ is represented by the zero set $Z \subset N$ of a transverse section of the line bundle $g^*(\gamma)$, together with the map $g|Z$ and the restricted trivialisation of $TZ \oplus (g|Z)^*((q+1)\gamma) = (TN \oplus g^*(q\gamma))|Z$.

The following result lists all nontrivial selfintersection invariants on π_3^S.

Proposition 4.2. *In the commutative diagram (up to signs)*



all arrows except ψ_3 and 8_* are epimorphisms; however, the 2-primary component of ψ_3 vanishes. Furthermore

$\psi_2 \circ 8_* = 3 \cdot$ *(identity on $\Omega_2(P^\infty; \gamma)$), and*

$8_* \circ \psi_2 = 3 \cdot$ *(projection of π_3^S onto its 2-primary part).*

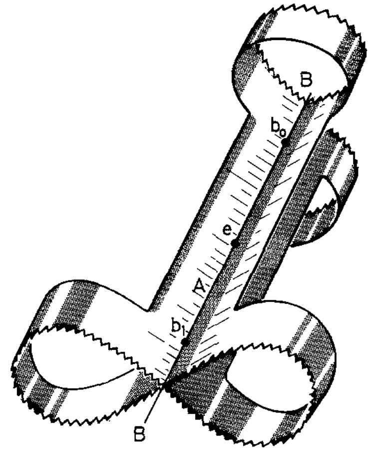


Fig. 2. The first step of the construction of i

The proof of this result and of Theorem F is based on the following immersion

$$i: P^2 \# (S^1 \times S^1) \to \mathbb{R}^3.$$

Decompose the unit circle $S \subset \mathbb{R}^2$ into a small arc A symmetric around $e=(1,0)$ and the remaining arc B. Identify a tubular neighborhood of S in $\mathbb{R}^3$ with $S \times D_\varepsilon$ in the obvious way ($D_\varepsilon = \varepsilon$-disk in $\mathbb{R}^2$). Consider the immersion $i_B: B \times S^1 \to B \times D_\varepsilon$ which maps $\{b\} \times S^1$ into the figure-8 in $\{b\} \times D_\varepsilon$, $b \in B$, in such a way that the vertical axis of the figure-8 rotates by 90° when b moves from b_0 to b_1 (b_0, b_1 being the endpoints of B).

Now use the two immersed rectangles $A \times \{(t, \pm t) \in D_\varepsilon \,|\, |t| \leqq \delta\}$, δ small, to join the first sheet of i_B at b_0 with the second sheet of i_B at b_1, and vice versa. We obtain an immersed surface whose boundary is a circle consisting of the four arcs $A \times \{(\pm\delta, \pm\delta)\}$, two horizontal arcs in $b_0 \times D_\varepsilon$ (namely the figure-8 minus the δ-neighborhood of $b_0 \times 0$) and likewise two vertical arcs in $b_1 \times D_\varepsilon$. The immersed surface can be extended "outwards" until the boundary circle is just $\{e\} \times \partial D_\varepsilon$, with a collar neighborhood lying in $\{e\} \times D_\varepsilon$. Join this in the obvious way (along $[e, b_0] \times \partial D_\varepsilon$) to the embedded disk $\{b_0\} \times D_\varepsilon$ and obtain a closed surface smoothly immersed into $\mathbb{R}^3$. It is not hard to see that its Euler number is -1, so that we get an immersion $i: P^2 \# (S^1 \times S^1) \to \mathbb{R}^3$ with transverse selfintersections.

The only triple point of i is b_0 where the disk $\{b_0\} \times D_\varepsilon$ meets the previous two sheets. Double points occur at S (no coherent global distinction between "first" and "second" sheet possible) and at the figure-8 in $\{b_0\} \times D$ (where the original surface ("first sheet") meets the second sheet $\{b_0\} \times D$). In $\mathbb{R}^4$ we can separate the first and second sheets by lifting them to different x_4-levels. However, selfintersections will occur even in $\mathbb{R}^4$ at points where to rôle of the sheets switches. Thus i, considered as an immersion into $\mathbb{R}^4$, can be deformed to an immersion i' with precisely one transverse double point.

This example shows that the mod 2 invariants listed in Theorem F, a) are nontrivial. Since they are additive bordism invariants, and since the unoriented bordism group of immersed surfaces $\pi_3^S(P^\infty)$ is cyclic of order 8 (see [20], p. 82, or [9], § 10), all these invariants give rise to the only epimorphism for $\mathbb{Z}_8$ to $\mathbb{Z}_2$, and therefore they must coincide.

Similarly, the invariants listed in Theorem F, b) define $\mathbb{Z}_2$-valued homomorphisms on the unoriented immersion group $\pi_4^S(P^\infty)=\mathbb{Z}_2$. Thus the proof of Theorem F is complete as soon as we have shown that these $\mathbb{Z}_2$-invariants are nontrivial on the figure-8 immersion in $8_{i'}$, belonging to i'. But this follows from the fact that the double point manifold of $8_{i'}$ consists of $P^2 \# S^1 \times S^1$ and of a torus.

This torus arises at the double point of i' in $\mathbb{R}^4$; since we have a product situation here, the resulting contribution to $\psi_2[8_{i'}]$ is the image of $\eta^2 \in \pi_2^S$, i.e. the only non-trivial element of order 2 in $\pi_3^S(P^\infty)$ (recall that $\eta^3=0$ in π_3^S). The contribution of $P^2 \# S^1 \times S^1$ is $-[i]$. Thus $[i]$ is a generator of $\pi_3^S(P^\infty)=\mathbb{Z}_8$ with $\psi_2 \circ 8_*[i]=3[i]$. The claims on $\psi_2 \circ 8_*$ and $8_* \circ \psi_2$ (and on ψ_4) in Proposition 4.2. follow. So does the claim on ψ_3, since the triple point manifold of $8_{i'}$ consists of two circles with constant maps into $B\Sigma_3$ and equal framing data.

Next use (4.1.) in the special cases $(m,q)=(2,1)$ and $(1,2)$ to get the exact sequences

$$\mathbb{Z}_2 \to \underset{\cong \mathbb{Z}_8}{\Omega_2(P^\infty, \gamma)} \xrightarrow{\Delta} \Omega_1(P^\infty; 2\gamma)$$

and

$$\mathbb{Z}_2 \to \Omega_1(P^\infty; 2\gamma) \xrightarrow{\Delta} \underset{\cong \mathbb{Z}_2}{\Omega_0(P^\infty; 3\gamma)};$$

they combine to show that $\Omega_1(P^\infty; 2\gamma) \cong \mathbb{Z}_4$, and that Δ is onto in both cases. Apply the techniques of the proof of Proposition 3.2. to the immersion 8_i to see that $\psi_2^{(2)}$ is onto. This implies also the commutativity up to sign which we claimed in 4.2.

Finally, we compute $\Omega_1(B\Sigma_3; \phi_3)$. According to [9], Theorem 9.3, there is an exact sequence

$$\mathbb{Z}_2 \xrightarrow{\delta_1} \Omega_1(B\Sigma_3; \phi_3) \to \bar{\Omega}_1(B\Sigma_3; \phi_3) \to 0,$$

where the auxiliary group $\bar{\Omega}_1(B\Sigma_3; \phi_3)$ can easily be identified with $\mathbb{Z}_3$ (cf. [9], 9.21). An element of order 3 or 6 in $\Omega_1(B\Sigma_3; \phi_3)$ can be obtained as follows. Take a loop generating $\pi_1(B\mathbb{Z}_3)$, project into $B\Sigma_3$, and add any trivialization of the Whitney sum of TS^1 and the (orientable) pullback of ϕ_3.

Also, δ_1 is known to be injective provided we can show that the second Stiefel-Whitney class $w_2(\phi_3)$ vanishes. Now consider the exact sequence

$$0 \to \mathbb{Z}_3 \subset \Sigma_3 \xrightarrow{\pi} \mathbb{Z}_2 \to 0$$

and the inclusion $\iota: \mathbb{Z}_2 = \Sigma_2 \subset \Sigma_3$. Theorems IV.7.1., IV.11.5. and XI.10.1. of [12] imply that $H^*(B\mathbb{Z}_2; \mathbb{Z}_2) \cong H^*(B\Sigma_3; \mathbb{Z}_2)$; this isomorphism is induced by the map $B\iota: B\mathbb{Z}_2 \to B\Sigma_3$, since $\pi \circ \iota$ os the identity on $\mathbb{Z}_2$. On the other hand, $(B\iota)^*(\phi_3)$ is stably isomorphic to a line bundle. Therefore $(B\iota)^*(w_2(\phi_3))$ and $w_2(\phi_3)$ have to vanish. This completes the proof of Proposition 4.2.

References

1. Adams, F.: On the non-existence of elements of Hopf invariant one. Ann. of Math. **72**, 20–104 (1960)
2. Banchoff, T.: Triple points and surgery of immersed surfaces. Proc. Amer. Math. Soc. **46**, 407–413 (1974)
3. Brown, E.: Framed manifolds with a fixed point free involution. Michigan Math. J. **23**, 257–260 (1976)
4. Cohen, J.: The decomposition of stable homotopy. Ann. of Math. **87**, 305–320 (1968)
5. Freedman, M.: Quadruple points of 3-manifolds in S^4. Comment. Math. Helv. **53**, 385–394 (1978)
6. Hilbert, D., Cohn-Vossen, S.: Geometry and the imagination. New York: Chelsea 1952
7. Hirsch, M.: Immersions of manifolds. Trans. Amer. Math. Soc. **93**, 242–276 (1959)
8. Kahn, D., Priddy, S.: Applications of the transfer to stable homotopy theory. Bull. Amer. Math. Soc. **78**, 981–987 (1972)
9. Koschorke, U.: A singularity approach to framefields and other vector bundle morphisms. Lecture Notes in Mathematics. Berlin-Heidelberg-New York: Springer (to appear)
10. Koschorke, U., Sanderson, B.: Geometric interpretations of the generalized Hopf invariant. Math. Scand. **41**, 199–217 (1977)
11. Koschorke, U., Sanderson, B.: Selfintersections and higher Hopf invariants. Topology **17**, 283–290 (1978)
12. MacLane, S.: Homology. Grundlehren der Mathematischen Wissenschaften 114. Berlin-Heidelberg-NewYork: Springer 1975
13. Milnor, J., Kervaire, M.: Bernoulli numbers, homotopy groups, and a theorem of Rohlin. In: Proceedings of the International Congress of Mathematicians (Edinburgh 1958), pp. 454–458. Cambridge: Cambridge University Press 1960
14. Ray, N.: A geometrical observation on the Arf invariant of a framed manifold. Bull. London Math. Soc. **4**, 163–164 (1972)
15. Salomonsen, M.A.: Bordism and geometric dimension. Math. Scand. **32**, 87–111 (1973)
16. Segal, G.: Operations in stable homotopy theory. London Mathematical Society Lecture Note Series **11**, 105–110. Cambridge: Cambridge University Press 1974
17. Thom, R.: Quelques propriétés globales des variétés différentiables. Comment. Math. Helv. **28**, 17–86 (1954)
18. Toda, H.: Composition methods in homotopy groups of spheres. Princeton, NewYork: Princeton University Press 1962
19. Toda, H., Saito, Y., Yokota, I.: Note on the generator of $\pi_7(SO(n))$. Mem. Coll. Sci. Univ. Kyoto Ser. A **30**, 227–230 (1957)
20. Whitehead, G.: Recent advances in homotopy theory. Regional Conference Series in Mathematics **5**. Providence, Rhode Island: American Mathematical Society 1970

Received August 23, 1978

Note Added in Proof. Brayton Gray informs me that there is considerable overlap between work of his and §2 (as well as theorem D).