Topology Vol. 17, pp. 283–290
© Pergamon Press Ltd., 1978. Printed in Great Britain

0040-9383/78/0901-0283/$02.00/0

SELF-INTERSECTIONS AND HIGHER HOPF INVARIANTS

ULRICH KOSCHORKE and BRIAN SANDERSON

(Received 6 April 1977)

INTRODUCTION

IN THIS paper we show how the well known models for loop spaces of Boardman and Vogt[3], James[5], May[9], and Segal[10], can be viewed in a natural way as "Thom spaces for immersions". Thus homotopy classes of maps into these models correspond to bordism classes of immersed manifolds with certain extra structures. By considering the multiple points of such immersions we obtain operations in homotopy theory. Special cases are the generalised and higher Hopf invariants of James[6], the Hopf ladder of Boardman and Steer[2], and the cohomotopy operations of Snaith[12], and Segal[11].

In §1 we establish the connection between models of loop spaces and structured immersions. In §2 we describe the process of "taking k-tuple points" in what might be termed an "external Hopf invariant", and set out its properties in Theorem 2.2. We then get the Segal and Snaith operations by composing with a suitable "forgetful" function. A similar procedure is followed in §3 where the James operations are described.

We are grateful to the referee for many helpful comments.

§1. IMMERSIONS AND COMBINATORIAL MODELS OF LOOP SPACES

We work in the category of compactly generated Hausdorff spaces[13] with non-degenerate base points, denoted $*$. A frequent example of such a space will be the Thom space $T(\xi)$ of a vector bundle ξ. We also let ξ denote the total space of the vector bundle. The n-fold suspension S^nX, of a space X is $X \wedge (I^n/\partial I^n)$, and $X^{[k]}$(resp. $\Lambda^k X$) denotes the k-fold Cartesian (resp. smash) product of X with itself. Then, for example, $\Lambda^k T(\xi) = T(\xi^{[k]})$. Define an element of $\tilde{X}_{m,k}$ (resp. $X_{m,k}$) to be an ordered (resp. unordered) subset of $\mathbf{R}^m$, $0 \leq m \leq \infty$, of k elements each with a label in X. Thus, $\tilde{X}_{m,k}$ is an open subspace of $(\mathbf{R}^m \times X)^{[k]}$, and $X_{m,k}$ is its quotient under the action of the symmetric group Σ_k. In particular, $\tilde{X}_{m,1} \cong X_{m,1} \cong \mathbf{R}^m \times X$. If C denotes a one-point space, note that the "configuration spaces"

$$\tilde{C}_{m,k} \cong \{(r_1, \ldots, r_k) | r_i \in \mathbf{R}^m,\ r_i \neq r_j \text{ if } i \neq j\}$$

and $C_{m,k}$ have natural smooth structures.

Forgetting points labelled $*$, we get the topological quotient $C_m(X) = (\coprod_{k\geq 0} X_{m,k})/\sim$. See [10] for details. A point of $C_m(X)$ is uniquely represented by a function $\varphi: A \to X$, where A is a finite subset of $\mathbf{R}^m$ and $\varphi(A) \subset X \setminus \{*\}$. In particular, $C_m(S^0) \cong \coprod_{k\geq 0} C_{m,k}$ is the space of finite subsets of $\mathbf{R}^m$. $C_m(X)$ is filtered by the subspaces $F_kC_m(X) = \{\varphi \in C_m(X)\colon |\mathrm{dom}\varphi| \leq k\}$.

Finally define a space $C_m{}^k(X)$ as follows. Consider finite sets Φ of functions with distinct domains, so that if $\varphi \in \Phi$ then $\varphi\colon A \to X$, where $A \subset \mathbf{R}^m$ and $|A| = k$. We identify two such sets Φ and Φ' if $\Phi' \subset \Phi$ and for each $\varphi \in \Phi \setminus \Phi'$ there is some $a \in \mathrm{dom}\varphi$ so that $\varphi(a) = *$. The resulting set $C_m{}^k(X)$ of equivalence classes is topologised as a quotient of a subspace of $\coprod_{k\geq 0} (X_{m,k})^s/\Sigma_s$. Note that a point in $C_m{}^k(X)$ is uniquely represented by a Φ consisting of functions into $X \setminus \{*\}$; we will usually represent elements of $C_m{}^k(X)$ in this way, and it will be convenient to identify $C_m{}^k(X)$ with this set of unique representatives. Note also that $C_m{}^1(X) \cong C_m(X)$.

Now let V be a manifold without boundary (all manifolds in this paper will be assumed smooth), and let ξ be a vector bundle over some space B. We will consider "decompressible" immersions into V, with normal bundles modelled on ξ or its "twisted power" $\xi_{m,k}$ over $B_{m,k}$. More precisely let $(M,g,\bar{g})$ be a triple determined by a commutative diagram,

$$\begin{array}{ccccc} & & \nu & \xrightarrow{\bar{g}} & \xi_{m,k} \\ & & \downarrow & & \downarrow \\ V & \xleftarrow{g_1} & M & \longrightarrow & B_{m,k} \\ & & & \overset{g_2}{\searrow} & \downarrow \\ & & & & C_{m,k} \end{array}$$

where M is a closed manifold, unlabelled maps are the obvious ones, $\bar{g}$ is a bundle map with domain the normal bundle ν of the immersion $g_1 : M \to V$, and $g = (g_1, g_2) : M \to V \times C_{m,k}$ is an embedding. Define $\mathcal{J}_m{}^k(V,\xi)$ to be the bordism set of such triples (bordisms over $V \times I$). We will be mainly concerned with the case $k = 1$, i.e. with bordism of embeddings in $V \times \mathbf{R}^m$ which project to immersions in V, with normal bundles expressed as pullbacks of ξ. We will denote the resulting set $\mathcal{J}_m{}^1(V,\xi)$ simply by $\mathcal{J}_m(V, \xi)$.

Examples. (i) If ξ is the trivial bundle ϵ^n over a one point space B of dimension $n > 0$, then $\mathcal{J}_m(V, \xi)$ is the set of bordism classes of embeddings g: $M \to V \times \mathbf{R}^m$ which project to codim. n framed immersions in V. (ii) If $\xi = \epsilon^0$ (over a one point *space*), $m = \infty$, then $\mathcal{J}_m(V, \xi)$ corresponds to the set of isomorphism classes of finite coverings of the closed components of V. (iii) If $\xi = \gamma^n$ is the universal n-vector bundle and $m = 0$ (resp. $m = \infty$), then $\mathcal{J}_m(V, \xi)$ may be identified with bordism classes of codim. n embeddings (resp. immersions) in V.

Let [,] denote based homotopy classes, and let V_c denote the one-point compactification of V.

THEOREM 1.1. *There is a bijection*

$$\beta : \mathcal{J}_m{}^k(V,\xi) \to [V_c, C_m{}^k(T(\xi))].$$

Proof. Let $[M,g,\bar{g}] \in \mathcal{J}_m{}^k(V,\xi)$. Without loss of generality assume g_1 extended to an immersion $g_1' : \nu \to V$ so that (g_1', g_2'): $\nu \to V \times C_{m,k}$ is an embedding, where g_2' is the composition $v \to \xi_{m,k} \to C_{m,k}$, and further assume that for each $v \in V$ $(g_1')^{-1}(v)$ is finite. Define f: $V_c \to C_m{}^k(T(\xi))$ as follows. We have $\xi_{m,k} \subset T(\xi)_{m,k}$, and we let $f(v)$ be the set of functions $\{\bar{g}(x) \in T(\xi)_{m,k}$: $g_1'(x) = v\}$. Now set $\beta[M,g,\bar{g}] = [f]$. To describe an inverse α for β, begin with f: $V_c \to C_m{}^k(T(\xi))$, and consider $\hat{V} \subset V \times C_{m,k}$ defined by $\hat{V} = \{(v, \mathrm{dom}\varphi)$: $\varphi \in f(v)\}$. Then $\hat{V}$ is a manifold with manifold structure pulled back from V. Define $\hat{f} : \hat{V} \to \xi_{m,k}$ by $\hat{f}(v, \mathrm{dom}\varphi) = \varphi$. After a suitable approximation, we may assume that $\hat{V}$ is a smooth submanifold of $V \times C_{m,k}$ and that $\hat{f}$ is transverse to the zero section $B_{m,k}$ of $\xi_{m,k}$. Let $M = \hat{f}^{-1} B_{m,k}$, let g be the inclusion $M \subset \hat{V} \subset V \times C_{m,k}$, and let $\bar{g}$: $\nu \to \xi_{m,k}$ be the "restriction" of f, where ν is the normal bundle of M in $\hat{V}$, which may be identified with the normal bundle of g_1. Then set $\alpha[f] = [M,g,\bar{g}]$. As in the classical Pontrjagin–Thom construction it is not hard to see that α and β are well defined and inverse of one another.

Let $D_m(X)$ be the space of finite sets of disjoint "little cubes" in $\mathbf{R}^m$, labelled in X. There is a map q: $D_m(X) \to \Omega^m S^m(X)$ which is a weak homotopy equivalence provided X is connected, [10, p. 15]. We show in the appendix that the map α: $D_m(X) \to C_m(X)$, which assigns to a set of cubes their centres, is a homotopy equivalence. Composing q with a homotopy inverse for α gives a map i: $C_m(X) \to \Omega^m S^m(X)$. On the other hand, there is a natural map ι: $\mathcal{J}_m(V, \xi) \to \mathcal{J}_0(V + \mathbf{R}^m, \xi \oplus \epsilon^m)$ which forgets that g_1 is an immersion.

THEOREM 1.2. *The following diagram commutes, and in case* dim $\xi > 0$ *each function is a bijection.*

$$\begin{array}{ccc} \mathcal{I}_m(V,\xi) & \xrightarrow{\iota} & \mathcal{I}_0(V\times\mathbf{R}^m, \xi\oplus\epsilon^m) \\ \downarrow \beta & & \downarrow \beta \\ [V_c, C_m(T(\xi))] & \xrightarrow{i_*} & [V_c, \Omega^m S^m(T(\xi))] \cong [S^m V_c, S^m(T(\xi))] \end{array}$$

Proof. $f \in i_*\beta[M,g,\bar{g}]$, where $f: S^m V_c \to S^m T(\xi)$, may be described as follows. Let $g_1': \nu \to V$ and $g_2': \nu \to \mathbf{R}^m$ be as in the proof of 1.1, and let $\bar{g}_1$ be the composition $\nu \to \xi_{m,1} \to \xi$. Further assume a tubular neighbourhood of $(g_1' \times g_2')\ \nu$ in $V \times \mathbf{R}^m$ is trivialised in the obvious way. Then f is the Pontrjagin–Thom construction $S^m V_c \to T(\nu\oplus\epsilon^m)$ composed with $S^m T(\bar{g})$: $S^m T(\nu) \to S^m T(\xi)$. Clearly f also represents $\beta\iota[M,g,\bar{g}]$. Finally, recall that β is an isomorphism by 1.1, and i_* is an isomorphism if X is connected (i.e. dim$\xi > 0$).

Remark 1.3. We have used results from homotopy theory to show that, up to bordism, embeddings can be compressed into immersions in a unique way, i.e. ι is bijective (in particular in the notation of [7] $\mathcal{I}_r^{n,m} \cong \mathcal{I}_r^{n+m,0}$, $n > 0$). This geometric result is not at all obvious in general. However, for large m the bijectivity of ι (and hence of i_*) follows directly from immersion theory. Thus we have a simple geometric proof for the well known weak equivalence of $(\coprod_{s\geq 0} (E\Sigma_s \times X^s)/\Sigma_s)/\sim$ with $\Omega^\infty S^\infty X$ (see e.g. 4.10 and 5.4 of [1]) at least if X is a (connected) Thom space. In fact, by using the theory of [4] chapter VIII, one could extend the geometric proof to cover the case when X is a connected CW-complex.

Theorem 1.1 may be sharpened (in case $m > 0$) by introducing a monoid structure in the relevant sets. A choice of (smooth) embedding j: $\mathbf{R}^m \amalg \mathbf{R}^m \to \mathbf{R}^m$ determines an obvious *addition* $C_m{}^k(X) \times C_m{}^k(X) \to C_m{}^k(X)$. The corresponding monoid structure on $\mathcal{I}_m{}^k(V,\xi)$ may be very crudely described as 'disjoint union' of immersions.

In the next section we will also need a *multiplication* $C_m{}^k(X) \times C_m{}^l(X) \to C_m^{k+l}(X)$. This is defined by $(\Phi_0, \Phi_1) \mapsto \Phi = \{(\varphi_0 \amalg \varphi_1)j^{-1}: \ \varphi_i \in \varphi_i\}$. The corresponding multiplication $\mathcal{I}_m{}^k(V,\xi) \times \mathcal{I}_m{}^l(V,\xi) \to \mathcal{I}_m^{k+l}(V,\xi)$ may be described as follows:

$$([M',g',\bar{g}'], [M'',g'',\bar{g}'']) \mapsto [M,g,\bar{g}],$$

where we assume g_1' is transverse to g_1'', and therefore

$$M = \{(x,y) \in M' \times M'': g_1'(x) = g_1''(y)\}$$

is a submanifold of $M' \times M''$, and g_1' (or g_1'') determines an immersion g_1: $M \to V$. Then for the normal bundle of g_1 we have $\nu \cong \nu' \times \nu''|$, and $(g,\bar{g})$ may be constructed, using j, from the diagram

$$\begin{array}{ccccccc} \nu & \to & \nu' \times \nu'' & \to & \xi_{m,k} \times \xi_{m,l} & \to & \xi_{m,k+l} \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ M & \to & M' \times M'' & \to & B_{m,k} \times B_{m,l} & \to & B_{m,k+l} \end{array} \qquad (1.4)$$

§2. MULTIPLE POINTS OF IMMERSIONS AND COHOMOTOPY OPERATIONS

In this section we will define operations

$$\psi_m{}^k: \mathcal{I}_m(V,\xi) \to \mathcal{I}_m{}^k(V,\xi), \quad k = 1,2,3,\ldots,$$

which arise from an analysis of self-intersections of immersions. We then compose with a function $\mathcal{I}_m{}^k(V,\xi) \to \mathcal{I}_\infty(V,\xi_{m,k})$ to get an operation $\theta_m{}^k$: $\mathcal{I}_m(V,\xi) \to \mathcal{I}_\infty(V,\xi_{m,k})$, which is the geometric interpretation of a map $C_m(X) \to C_\infty(F_k C_m(X)/F_{k-1}C_m(X))$ when X is a Thom space, and which is briefly described as follows. To an immersion g_1: $M \to V$ with normal bundle classified by ξ we associate an immersion $g_1{}^k$: $M(k) \to V$, of the k-tuple point manifold of g_1, with normal bundle classified by $\xi_{m,k}$, and the definition is completed by choosing an embedding of $M(k)$ in R^∞. The details follow.

Consider a class $[M,g,\bar{g}] \in \mathscr{I}_m(V,\xi)$. Thus we have an immersion g_1: $M \to V$ decompressed by g_2 to an embedding in $V \times \mathbf{R}^m$ and equipped with a bundle map $\bar{g}_1$: $\nu \to \xi$ over a map $M \to B$.

By [8] we may assume that all multiple self-intersections of g_1 are transverse. We say simply that g_1 is *self-transverse*. In particular given $k > 1$ the set

$\tilde{M}(k) = \{(x_1, \ldots x_k) \in M^{[k]}$: $g_1(x_1) = \ldots = g_1(x_k)$, and $x_i \neq x_j$ if $i \neq j\}$ forms a closed submanifold of $M^{[k]}$ on which the symmetric group Σ_k acts freely. We call $M(k) = \tilde{M}(k)/\Sigma_k$ the *k-tuple point manifold.* Let $g_1{}^k$: $M(k) \to V$ be the immersion defined by $g_1{}^k$ $[x_1, \ldots, x_k] = g_1(x_1) = \ldots = g_1(x_k)$. The immersion $\tilde{g}_1{}^k$: $\tilde{M}(k) \to M(k) \to V$ has normal bundle $\tilde{\nu}(k)$ say. There is the diagram of bundle maps

$$\begin{array}{ccccc} \tilde{\nu}(k) & \to & \nu \times \cdots \times \nu & \to & \xi \times \cdots \times \xi \\ \downarrow & & \downarrow & & \downarrow \\ \tilde{M}(k) & \to & M \times \cdots \times M & \to & B \times \cdots \times B, \end{array}$$

and we have $\tilde{g}_2{}^k$: $\tilde{\nu}(k) \to \tilde{M}(k) \to \tilde{C}_{m,k}$, where $\tilde{M}(k) \to \tilde{C}_{m,k}$ is given by $(x_1, \ldots, x_k) \to (g_2(x_1), \ldots, g_2(x_k))$. Putting together the bundle map $\tilde{\nu}(k) \to \xi^{[k]}$ and $\tilde{g}_2{}^k$, and factoring by the action of Σ_k gives $\bar{g}^k$: $\nu(g_1{}^k) \to \xi_{m,k}$. Set

$$\psi_m{}^k[M,g,\bar{g}] = [M(k),g^k,\bar{g}^k].$$

Remark 2.1. Define ψ: $C_m(X) \to C_m{}^k(X)$ by $\varphi' \in \psi(\varphi)$ whenever φ' is a restriction of φ to a subset of k elements. Then with $X = T(\xi)$ ψ induces $\psi_m{}^k$.

Let S: $\mathscr{I}_m{}^k(V,\xi) \to \mathscr{I}^k_{m+1}(V,\xi)$ be the 'suspension' induced by the inclusion $\mathbf{R}^m \subset \mathbf{R}^{m+1}$.

THEOREM 2.2. *The family of operations*

$$\psi_m{}^k: \mathscr{I}_m(V,\xi) \to \mathscr{I}_m{}^k(V,\xi) \quad k = 1,2, \ldots,$$

satisfies the following properties:

(a) (*identity*) $\psi_m{}^1 = id$.

(b) (*normalisation*) *if* $y \in \mathscr{I}_m(V,\xi)$ *can be represented by* $(M,g,\bar{g})$ *where* g_1 *is an embedding, then* $\psi_m{}^k(y) = 0$ *for* $k > 1$.

(c) (*Cartan formula*).

$$S\,\psi_m{}^k(y + y') = S\psi_m{}^k(y) + S(\psi_m^{k-1}(y) \cdot \psi_m{}^1(y')) + \ldots + S\psi_m{}^k(y').$$

Proof. (a) and (b) are immediate from the definition of $\psi_m{}^k$. For the Cartan formula, let $y = [M,g,\bar{g}], y' = [M',g',\bar{g}']$, and assume that the immersion $g_1 \amalg g_1'$: $M \amalg M' \to V$ is self-transverse. The k-tuple point manifold then has a component $M(i,j)$ coming from the intersection of $g_1(i)$: $M(i) \to V$ with $g_1'(j)$: $M'(j) \to V$ for each i,j, so that $i + j = k$. These components may be linked in $V \times C_{m,k}$, but not in $V \times C_{m+1,k}$; just push them out first to different x_{m+1} – levels and then into appropriate x_1 – regions in $\mathbf{R}^{m+1}$.

Define a function $\kappa_m{}^k$: $\mathscr{I}_m{}^k(V, \xi) \to \mathscr{I}_\infty(V, \xi_{m,k})$ by forgetting f_1: specifically, $\kappa_m{}^k[M,g,\bar{g}] = [M,g',\bar{g}']$, where $g' = (g_1,g_2')$, $\bar{g}' = (\bar{g},g_2')$: $\nu \to \xi_{m,k} \times \mathbf{R}^\infty$, and g_2': $M \to \mathbf{R}^\infty$ is any embedding. $\kappa_m{}^k$ is compatible with addition and with the multiplication

$$\mathscr{I}_\infty(V,\xi_{m,k}) \times \mathscr{I}_\infty(V,\xi_{m,l}) \to \mathscr{I}_\infty(V,\xi_{m,k+l}),$$

defined from transverse intersection along the lines of (1.4).

The composed operations

$$\theta_m{}^k = \kappa_m{}^k \cdot \psi_m{}^k: \mathscr{I}_m(V,\xi) \to \mathscr{I}_\infty(V,\xi_{m,k}), \quad k = 1,2, \ldots,$$

satisfy analogous properties to these given for ψ in Theorem 2.2; no suspension S is needed in the Cartan formula here, since previously S was used only to correct g_2.

Remark 2.3. There is a map

$$\kappa: C_m{}^k(X) \to C_\infty(T(\xi_{m,k})) = C_\infty(F_kC_m/F_{k-1}(C_m(X))$$

corresponding to κ_m^k. Indeed, κ is induced by the map $X_{m,k} \to \mathbf{R}^\infty \times F_k(C(X)/F_{k-1}(C_m(X)),\ \varphi \to (e(\mathrm{dom}\varphi, p(\varphi))$, where $e\colon C_{m,k} \to \mathbf{R}^\infty$ is any embedding and $p\colon X_{m,k} \to F_kC_m(X)/F_{k-1}C_m(X)$ is the obvious quotient map. The composition

$$\theta\colon C_m(X) \to C_m{}^k(X) \to C_\infty(F_kC_m(X)/F_{k-1}C_m(X)),$$

corresponding to the operation $\theta_m{}^k$, can be described as follows. Given $\varphi \in C_m(X)$, $\theta(\varphi)$ has domain $\{e(A) \in \mathbf{R}^\infty | A \subset \mathrm{dom}\,\varphi,\ |A| = k\}$, and $\theta(\varphi)\,(e(A)) = p(\varphi|A)$. θ is adjoint (after using the equivalence of D_∞ with $\Omega^\infty S^\infty$ and with C_∞) to the stable map defined by Snaith[12].

By taking $\xi = \epsilon^0$ over a point, we recover the operations (on covering spaces embedded in $V \times \mathbf{R}^m$) of Segal[11, p. 108], induced by $\theta\colon C_m(S^0) \to C_\infty(C^+_{m,k})$; moreover our Cartan formula corresponds to formula (b) in Theorem 4 of [11].

If V is Euclidean space and ξ is a trivial bundle over a point, then the operation $\theta_m{}^2$ is just the "double-point Hopf invariant" $\mathcal{J}H_m$ of [7]. In the meta-stable range, it corresponds to the generalised Hopf invariant of Whitehead and James, as was shown in [7].

§4. THE JAMES–HOPF INVARIANT

In this section we compose a modification $\tilde{\psi}_1{}^k$ of $\psi_1{}^k$ with a forgetful map κ^k to get an operation $\theta^k\colon \mathcal{J}_1(V,\xi) \to \mathcal{J}_1(V,\xi^{[k]})$. We then show that θ^k corresponds to the higher James–Hopf invariant[6]. We follow the conventions of [2].

θ^k is briefly described by, $\theta^k[M,g,\bar{g}] = [M(k),g(k),\bar{g}(k)]$, where $g(k)_1$ is the immersion of the k-tuple point manifold $M(k)$ and $g(k)_2$ is chosen to preserve the lexicographic order on $g(k)_1^{-1}(v)$, for each $v \in V$. Details follow.

An element of $C_{1,k}$ is uniquely represented by $(t_1,t_2,\ldots,t_k)$ where $t_1 > t_2 > \ldots > t_k$, therefore $C_{1,k}$ may be ordered (lexicographically from the left). Now let $\tilde{C}_1{}^k(X)$ be the space of finite sequences of functions $\Phi = (\varphi_1,\varphi_2,\ldots,\varphi_s)$, so that in particular $C_1{}^k(X)$ is a quotient of $\tilde{C}_1{}^k(X)$. Then $\psi\colon C_1(X) \to C_1{}^k(X)$ factors through $\tilde{C}_1{}^k(X)$; use the order on $C_{1,k}$ to order functions by their domains. Correspondingly we have $\tilde{\psi}_1{}^k\colon \mathcal{J}_1(V,\xi) \to \tilde{\mathcal{J}}_1{}^k(V,\xi)$, where an element of $\tilde{\mathcal{J}}_1{}^k(V,\xi)$ is represented by $(M,g,\bar{g},o)$ where $[M,g,\bar{g}] \in \mathcal{J}_1{}^k(V,\xi)$ and for each $v \in V$ o_v is a total order on $g_1^{-1}(v)$ depending continuously on v, i.e. for any two paths c, c' in M with $g_1c = g_1c'$ we have: $c(1) < c'(1)$ if $c(0) < c'(0)$.

Note that the order on $\mathbf{R}^1$ induces a homeomorphism $Y_{1,k} \cong C_{1,k} \times Y^{[k]}$ for any Y. We can now describe $\kappa^k\colon \tilde{\mathcal{J}}_1{}^k(V,\xi) \to \mathcal{J}_1(V,\xi^{[k]})$. Corresponding to $[M,g,\bar{g},o]$ there is a diagram

$$\begin{array}{ccccc} & & \nu & \longrightarrow & C_{1,k} \times \xi^{[k]} \\ & & \downarrow & & \downarrow \\ V & \xleftarrow{g_1} & M & \xrightarrow{(g_2,g_3)} & C_{1,k} \times B^{[k]}. \end{array}$$

To get $\kappa^k[M,g,\bar{g},o]$, replace g_2 by any map $g_2'\colon M \to \mathbf{R}$ which satisfies the following condition. For each $v \in V$ the order on $g_1^{-1}(v)$ induced by the embedding $g_2'|\colon g_1^{-1}(v) \to \mathbf{R}$ coincides with the order O_v. We must show that such a g_2' exists. We will do this shortly. First recall the infinite reduced product Y_∞ of [5]. A point of Y_∞ is uniquely represented by a word $y_1 + \ldots + y_k$, where no $y_i = *$. The nondegeneracy of the base point means that there is a map $u\colon Y \to \mathbf{R}_+{}^1$, with $u^{-1}(0) = *$ and a homotopy $h_t\colon Y \to Y$ so that $h_0 = id.$ and $h_1(u^{-1}[0,1)) = *$. Then there is a cononical map $w\colon Y_\infty \to \Omega SY$ whose adjoint $\hat{w}\colon Y_\infty \times I \to Y \wedge (I/\partial I)$ is defined by

$$\hat{w}(y_1 + \ldots + y_k,t) = [y_i,(t\sigma_k - \sigma_{i-1})/u(y_i)]$$

for $\sigma_{i-1}/\sigma_k \leq t \leq \sigma_i/\sigma_k$, where $\sigma_j = u(y_1) + \ldots + u(y_j)$ for $1 \leq j \leq k$, and $\sigma_0 = 0$.

Recall that we have maps $q\colon D_1(Y) \to \Omega SY$ and $\alpha\colon D_1(Y) \rightarrowtail C_1(Y)$. Let $D_m'(Y)$ be the modification of $D_m(Y)$ in which cubes are allowed to meet in their boundaries. Then the inclusion $D_m(Y) \subset D_m'(Y)$ is a homotopy equivalence, and α, q extend to α', q' respectively.

LEMMA 3.1. *There is a homotopy* w_s: $Y_\infty \to \Omega SY$ *of* $w = w_0$, *and a map* l': $Y_\infty \to D_1'(Y)$ *so that* $q'\ l' = w_1$.

Proof. Define l': $Y_\infty \to D_1'(Y)$ by $l'(y_1 + \ldots + y_k) = [c_1, \ldots c_k, y_1', \ldots, y_k']$, where the cube c_i has centre $\sigma_i/\sigma_k - u(y_i)/2\sigma_k$ and diameter $u(y_i)/\sigma_k$, and

$$y_i' = \begin{cases} y_i & \text{if } u(y_i) = 1 \\ h(y_i, 2(1-u(y_i))) & \text{if } 1/2 \leq u(y_i) \leq 1 \\ * & \text{if } 0 \leq u(y_i) \leq 1/2. \end{cases}$$

Let π_1, π_2 be projections of $Y \times (I/\partial I)$ on the first and second factors respectively. Define $\hat{w}_s$: $Y_\infty \times I \to Y \wedge (I/\partial I)$ as follows;

$$\pi_2 \hat{w}_s = \pi_2 \hat{w} \quad \text{and for } \sigma_{i-1}/\sigma_k \leq t \leq \sigma_i/\sigma_k;$$

$$\pi_1 \hat{w}_s(y_1 + \ldots + y_k, t) = \begin{cases} y_i & s \leq u(y_i) \leq 1 \\ h(y_i, 2(s - u(y_i))/s & s/2 \leq u(y_i) \leq s \\ * & 0 \leq u(y_i) \leq s/2. \end{cases}$$

Now let π: $\tilde{C}_1{}^k(X) \to (\Lambda^k X)_\infty$ be given by

$$\pi(\Phi) = \sum_{i=1}^{s} \varphi_i(t_i{}^1) \wedge \ldots \wedge \varphi_i(t_i{}^k),$$

where $\Phi = (\varphi_1, \ldots, \varphi_s)$ and dom $\varphi_i = (t_i{}^1, \ldots, t_i{}^k)$ with indexing agreeing with the given order in Φ and the usual order in $\mathbf{R}^1$ respectively.

Define $\kappa = \alpha' l' \pi$: $\tilde{C}_1{}^k(X) \to C_1(\Lambda^k X)$. One can now see that the g_2', required for the definition of κ^k, exists by working around the following diagram

$$\begin{array}{ccc} \tilde{\mathcal{J}}_1{}^k(V,\xi) & \overset{\kappa^k}{- - - - - - \to} & \mathcal{J}_1(V,\xi^{[k]}) \\ \cong\downarrow \tilde{\beta} & & \cong\downarrow \beta \\ [V_c, \tilde{C}_1{}^k(T(\xi))] & \xrightarrow{\kappa_*} & [V_c, C_1(\Lambda^k T(\xi))]. \end{array}$$

THEOREM 3.2. *Let* $\theta^k = \kappa^k.\psi^k$ *then the diagram*

$$\begin{array}{ccc} \mathcal{J}_1(V,\xi) & \xrightarrow{\theta^k} & \mathcal{J}_1(V,\xi^{[k]}) \\ \cong\downarrow \beta\cdot\iota & & \cong\downarrow \beta\cdot\iota \\ [SV_c, ST(\xi)] & \xrightarrow{\gamma_k} & [SV_c, S(\Lambda^k T(\xi))] \end{array}$$

commutes for $k = 1,2,3,\ldots$, *where* γ_k *is the James–Hopf invariant, and* dim $\xi > 0$.

Proof. Recall that γ_k is defined by commutativity in the following diagram

$$\begin{array}{ccc} [SV_c, SX] & \xrightarrow{\gamma_k} & [SV_c, S\Lambda^k X] \\ \cong & & \cong \\ [V_c, \Omega SX] & & [V_c, \Omega S\Lambda^k X] \\ \uparrow w* & & \uparrow w_* \\ [V_c, X_\infty] & \xrightarrow[g_*]{} & [V_c, (\Lambda^k X)_\infty] \end{array}$$

where g is the combinatorial extension given by

$$g(x_1 + \ldots + x_s) = \sum_\sigma x_{\sigma(1)} \wedge \ldots \wedge x_{\sigma(k)}$$

the sum is ordered lexicographically and σ runs over all order preserving injections $\{1, \ldots, k\} \to \{1, \ldots, s\}$. The proof is completed by a diagram chase, using 1.2 and 3.1, and by checking commutativity in the following diagram,

$$\begin{array}{ccc} [V_c, C_1(X)] & \xrightarrow{\psi_*} & [V_c, \tilde{C}_1^{\,k}(X)] \\ \uparrow (\alpha' l')_* & & \downarrow \pi_* \\ [V_c, X_\infty] & \xrightarrow[g^*]{} & [V_c, (\Lambda^k X)_\infty]. \end{array}$$

Remark 3.3. There is the addition in $\mathcal{J}_1(V, \xi^{[k]})$, given by "stacking the immersions on top of each other," and a multiplication

$$\mathcal{J}_1(V, \xi^{[k]}) \times \mathcal{J}_1(V, \xi^{[l]}) \to \mathcal{J}_1(V, \xi^{[k+l]})$$

given by transverse intersection and coming from the obvious map $(\Lambda^k X)_\infty \times (\Lambda^l X)_\infty \to (\Lambda^{k+l} X)_\infty \cdot \kappa^k$ commutes with these operations and one easily sees that 2.2 holds for θ^k in place of $\psi_1^{\,k}$. In particular we recover the Hopf ladders

$$\lambda_k = S^{k-1}\gamma_k \quad k = 1,2,3, \ldots \quad \text{of [2].}$$

Note that the characteristic axioms (a), (b), (c) in [2, p. 185] correspond to the properties (a), (b), (c) in our 2.2.

Finally, there is a geometric interpretation of λ_k in [2] (Theorem 6.8). This corresponds to the composition of $\psi_1^{\,k}$: $\mathcal{J}_1(V, \xi) \to \mathcal{J}_1^{\,k}(V, \xi)$ with the isomorphism $\mathcal{J}_1^{\,k}(V; \xi) \cong \mathcal{J}_k(V, \xi^{[k]})$, induced by mapping $\{t_1, \ldots, t_k\} \in C_{1,k}$ (where $t_1 > t_2 > \ldots > t_k$) to $(t_1, \ldots, t_k) \in C_{k,1} = \mathbf{R}^k$.

REFERENCES

1. M. G. Barratt and P. J. Eccles: Γ^+-structures-I: A free group functor for stable homotopy theory, *Topology* **13** (1974), 25–45.
2. J. M. Boardman and B. Steer: On Hopf invariants, *Comment. math. Helvet.* **42** (1967), 180–221.
3. J. M. Boardman and R. M. Vogt: Homotopy-everything H-spaces, *Bull. Am. math. Soc.* **74** (1968), 1117–1122.
4. S. Buoncristiano, C. P. Rourke and B. J. Sanderson: A geometric approach to homology theory, *London Math. Soc. Lecture Note series* 18 (1976).
5. I. M. James: Reduced product spaces, *Ann. Math.* **62** (1955), 170–197.
6. I. M. James: On the suspension Triad, *Ann. Math.* **63** (1956), 191–247.
7. U. Koschorke and B. J. Sanderson: Geometric interpretations of the generalised Hopf invariant (to appear in *Math. Scand.*).
8. R. K. Lashof and S. Smale: Self-intersections of immersed manifolds, *J. Math. Mech.* **8** (1959), 143–157.
9. J. P. May: The geometry of iterated loop spaces, *Springer Lecture Notes in Mathematics* 271. Springer, Berlin (1972).
10. G. Segal: Configuration spaces and iterated loop spaces, *Invent. Math.* (1973), 213–221.
11. G. Segal: Operations in stable homotopy theory, new developments in topology, *London Math. Soc. lecture notes* series **11** (1974), 105–110.
12. V. P. Snaith: A stable decomposition of $\Omega^n S^n X$, *J. Lond. math. Soc.* **7** (1974), 577–583.
13. N. E. Steenrod: A convenient category of topological spaces, *Mich. Math. J.* **14** (1967), 133–152.

University of Bonn
University of Warwick

APPENDIX

Recall that a point of $D_m(X)$ (see p. 5) is uniquely represented by

$$[\tilde{c}_1, \ldots, \tilde{c}_k, x_1, \ldots, x_k] \in (\tilde{D}_{m,k} \times X^{(k)})/\Sigma_k,$$

where each $\tilde{c}_i$ is a 'little cube' in $\mathbf{R}^m$; see [8] for details.

Proposition. *If X is a compactly generated Hausdorff space with a non-degenerate base point, then the 'centre' map $\alpha: D_m(X) \to C_m(X)$ is a homotopy equivalence.*

Proof. Choose $u: X \to [0, 1]$ so that $u^{-1}(0) = *$, and h: $X \times I \to X$ so that $h_1 u^{-1}[0, 1) = *$ and $h_0 = id$. If $[c_1, \ldots, c_k] \in C_{m,k}$ let $\varphi \cdot \tilde{c}_i$ be a little cube with centre c_i and diameter φ. Define $\beta: C_m(X) \to D_m(X)$ by

$$\beta[c_1, \ldots, c_k, x_1, \ldots, x_k] = [\varphi \cdot \tilde{c}_1, \ldots, \varphi \cdot \tilde{c}_k, v(x_1), \ldots, v(x_k)],$$

where $\varphi = \min\{1, 1/4|c_i - c_j|/w(x_i)\colon 1 \leq i, j \leq k\}$, $|\cdot|$ is the product norm on $\mathbf{R}^m$, $w(x_i) = \min\{u(x_i), 1/2\}$ and $v(x_i)$

is given by

$$v(x_i) = \begin{cases} x_i & \text{if } u(x_i) = 1 \\ h(x_i, 2(1 - u(x_i)) & \text{if } 1/2 \leq u(x_i) \leq 1 \\ * & \text{if } 0 \leq u(x_i) \leq 1/2. \end{cases}$$

It is easy to check that β is well defined and that $\alpha\beta \simeq 1$ and $\beta\alpha \simeq 1$.