A GEOMETRICAL PROOF OF A THEOREM OF KAHN AND PRIDDY

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0. *Introduction*

The following is known as the Kahn–Priddy Theorem (see [3]).

(0.1) THEOREM. *Let $t:\Omega_n^{\mathrm{fr}}(B\mathbb{Z}_p)\to\Omega_n^{\mathrm{fr}}$ be the transfer map defined on bordism class representatives by $t(\tilde{M}\downarrow M)=\tilde{M}$, where $\tilde{M}$ carries the induced tangential framing. Then $t:\Omega_n^{\mathrm{fr}}(B\mathbb{Z}_p)\to p\text{-tors}\,\Omega_n^{\mathrm{fr}}$ is a split surjection for any prime p.*

For odd primes, it is more usual to state the theorem with $B\Sigma_p$; however, a simple application of the transfer of the covering $B\mathbb{Z}_p\to B\Sigma_p$ shows that Σ_p and $\mathbb{Z}_p$ are interchangeable.

Assuming that Ω_n^{fr} is finite, an equivalent formulation (see [6]) is as follows.

(0.2) THEOREM. *The map $t:\Omega_n^{\mathrm{fr}}(B\mathbb{Z}_p)\to\Omega_n^{\mathrm{fr}}$ is a split surjection for $n\geqslant 1$.*

We shall give a proof of the above assertion using framed manifolds.

We shall employ the following notation. For $p=2$, let $\mathbb{R}^{n,k}$ denote $\mathbb{R}^{n+k}$ with the negating involution on the first n co-ordinates. Let $S^{n,k-1}$ be the unit sphere in $\mathbb{R}^{n,k}$. Let $\varepsilon^{n,k}$ be the trivial $\mathbb{Z}_2$-bundle with fibre $\mathbb{R}^{n,k}$ over an arbitrary $\mathbb{Z}_2$-space.

For p odd and $q=\frac{1}{2}(p-1)$, let $\mathbb{C}^{qn,k}$ denote $\mathbb{C}^{qn}\times\mathbb{R}^k$ where $\mathbb{Z}_p$ acts by multiplying the q complex subspaces $\mathbb{C}^n$ by $\omega,\omega^2,\ldots,\omega^q$ respectively $\big(\omega=\exp(2\pi i/p)\big)$. Let $S^{2qn,k-1}$ be the unit sphere in $\mathbb{C}^{qn,k}$. Let $\varepsilon^{2qn,k}$ be the trivial $\mathbb{Z}_p$-bundle with fibre $\mathbb{C}^{qn,k}$ over an arbitrary $\mathbb{Z}_p$-space.

If M_1 and M_2 are framed bordant, we shall write $M_1\sim M_2$, and use $[M_1]$ to stand for their common framed bordism class.

Our aim is to produce a splitting ψ for t in (0.2).

1. *The case when $p=2$*

Let M with tangential framing α represent $x\in\Omega_n^{\mathrm{fr}}$. Consider the manifold $W=M\times M\times[-1,1]$, with involution T given by $T(m,n,u)=(n,m,-u)$. The fixed point set of T is the diagonal $\Delta=\{(m,m,0)\}$.

Now $\Delta=M$ has normal bundle $tM\oplus\varepsilon^{1,0}$ (where t denotes the tangent bundle), which is equivariantly trivialised by $\alpha:tM\oplus\varepsilon^{1,0}\to\mathbb{R}^{n+1,0}$. We can thus perform a Pontrjagin–Thom collapse to construct an equivariant map

$$p_M: W \longrightarrow M^{t\oplus\varepsilon} \longrightarrow S^{n+1,0}.$$

Observe that p_M is transverse to $0\in S^{n+1,0}=\mathbb{R}^{n+1,0}\cup\{\infty\}$, and that $p_M^{-1}(0)=\Delta$.

Received 18 January, 1980.

[J. LONDON MATH. SOC. (2), 24 (1981), 329–334]

(1.1) LEMMA. *The map p_M is equivariantly homotopic to a map q_M which agrees with p_M on a tubular neighbourhood $N(\Delta)$, and which is transverse to ∞.*

Proof. Let $W' = W - \mathring{N}(\Delta)$, and $X = S^{n+1,0} - \mathring{N}(0)$. Then W' is a free $\mathbb{Z}_2$-manifold with boundary, and $p_M : W' \to X$ is an equivariant map such that $p_M(\partial W') \subset \partial X$.

In these circumstances, equivariant transversality is possible by a simple consequence of Thom theory (see [5; Lemma (5.1)]), and the homotopy takes place away from boundaries since $\infty \notin \partial X$.

Thus we may construct $q_M : W' \to X$, equivariant and transverse to ∞, and agreeing with p_M on W'. To conclude, we paste back in $N(\Delta)$ and $N(0)$.

We now write $q_M^{-1}(\infty) \subset W$ as X_M. This has a free $\mathbb{Z}_2$ action by very construction, and as we shall see in (1.3) carries a compatible framing. In fact, according to the homotopy chosen in (1.1), it is only well defined up to free equivariant framed bordism.

Furthermore, if we choose $M \sim M'$ via a framed bordism B, we can apply the above construction to all of B and obtain a suitable bordism $X_M \sim X_{M'}$.

So it is permissible to define $\psi(x)$ to be $[X_M \downarrow X_M/\mathbb{Z}_2]$.

(1.2) LEMMA. *If $n \geqslant 1$, then ψ so defined is a homomorphism satisfying $t\psi = 1$.*

Proof. In $S^{n+1,0}$, choose an interval I connecting 0 and ∞. Now q_M is transverse to the end points, and so by relative (but non-equivariant!) transversality can be made transverse to all of I so long as $n > 0$. Then $q_M^{-1}(I)$ is a framed bordism in W from X_M to $\Delta = M$. So $t\psi = 1$.

Now let $M' = M_1 \coprod M_2$, so that

$$W = (M_1 \times M_1 \coprod M_2 \times M_2 \coprod \mathbb{Z}_2 \times M_1 \times M_2) \times I.$$

Then Δ lies within the first two components, and hence so does the framed bordism above. Thus $X_{M'} = X_{M_1} \coprod X_{M_2}$, and ψ is a homomorphism.

It remains to establish the following.

(1.3) LEMMA. *We may frame X_M by a framing which is equivariantly equivalent to a $\mathbb{Z}_2$ invariant framing; so $[X_M \downarrow X_M/\mathbb{Z}_2] \in \Omega_n^{\mathrm{fr}}(B\mathbb{Z}_2)$.*

Proof. It suffices to produce an equivariant isomorphism

$$tX_M \oplus \varepsilon^{0,s} \cong \varepsilon^{0,n+s} \qquad (s \text{ large})$$

from the data that $tX_M \oplus \varepsilon^{n+1,0} \cong tW|X_M$.

So consider $W \times 0 \subset W \times [-1, 1] = W_1$, where W_1 carries T extended by the identity. Then

$$tW \times \varepsilon^{0,1} \cong t(M \times I) \times t(M \times I)$$

where the latter carries the switch involution. Then applying the framing α we see that $tW \oplus \varepsilon^{0,1} \cong \varepsilon^{n+1,n+1}$, and so

$$tX_M \oplus \varepsilon^{n+1,1} \cong \varepsilon^{n+1,n+1}. \tag{1.4}$$

Also, there is the classifying map $f: X_M \to S^{n+2,-1}$ for T restricted to X_M. Since $2^a\eta$ (where η is the Hopf bundle over $\mathbb{R}P^{n+1}$) can be trivialised by Clifford algebra for suitably large a, we see that over $S^{n+2,-1}$ there is an isomorphism $\varepsilon^{b,0} \cong \varepsilon^{0,b}$ whenever $b \equiv 0 \pmod{2^a}$. So pulling back along f, the same is true over X_M. Hence

$$\begin{aligned} tX_M \oplus \varepsilon^{0,b+1} &\cong tX_M \oplus \varepsilon^{b,1} \cong \varepsilon^{b,n+1} \quad \text{by (1.4)} \\ &\cong \varepsilon^{0,n+1+b} \end{aligned}$$

as sought.

Note that several choices have been made above: presumably distinct choices will yield different versions of ψ.

Combining (1.2) and (1.3), we have our proof of (0.2). In turn we therefore have a simple geometric proof of the following result of Thom, first (and recently) proven geometrically by S. Buoncristiano using a more complicated argument.

(1.4) PROPOSITION. *Any framed manifold is an unoriented boundary.*

Proof. From above, $M \sim X_M$. Now attach to X_M the total space of the associated line bundle.

2. *The case when p is odd*

We now outline the construction of ψ for odd primes. The idea is similar to the case when $p = 2$, but simpler in one interesting aspect.

Let $W = M \times \ldots \times M$ with p factors, and let $\mathbb{Z}_p$ act by cyclic permutation. The fixed point set is the diagonal $M = \Delta \subset W$, which has normal bundle $(p-1)tM$. Furthermore, $\mathbb{Z}_p$ acts on $(p-1)tM \cong q(tM \otimes \mathbb{C})$ by multiplying successive factors by $\omega, \omega^2, \ldots, \omega^q$.

Now recall (for example from [1]) how the framing α of $tM \oplus \varepsilon$ gives rise to the 'Gauss map' $\nu(\alpha): tM \to tS^n$. Using this, together with a Pontrjagin–Thom collapse, we may construct an equivariant map

$$r'_M : W \longrightarrow M^{(p-1)t} = M^{q(t\otimes\mathbb{C})} \xrightarrow[\nu(\alpha)]{} (S^n)^{q(t\otimes\mathbb{C})}.$$

But $tS^n \otimes \mathbb{C}$ admits a trivialisation as a $U(n)$ bundle, which can be combined with r'_M to produce the equivariant map

$$r_M : W \longrightarrow S^{(p-1)n,0} = \mathbb{C}^{qn,0} \cup \{\infty\}.$$

Note that r_M is transverse to 0, and that $r_M^{-1}(0) = \Delta$.

The perceptive reader will observe that we have introduced $\nu(\alpha)$ merely as a device to trivialise $(p-1)tM$ systematically for all p and α.

Since $W - \mathring{N}(\Delta)$ is a free $\mathbb{Z}_p$-space, we can proceed as for $p = 2$ and assume that r_M is equivariantly transverse to ∞, and transverse to an interval $[0, \infty]$ in $S^{(p-1)n,0}$. Then $Y_M = r_M^{-1}(\infty)$ is a free $\mathbb{Z}_p$-manifold framed bordant to Δ. To show that it has a framing invariant under the $\mathbb{Z}_p$ action, we must prove the analogue of (1.3). This depends on three observations:

(i) $tY_M \oplus \varepsilon^{(p-1)n,0} \cong tW|Y_M$,

(ii) $tW \oplus \varepsilon^{p-1,1} \cong \varepsilon^{(p-1)(n+1),n+1}$ via α,

(iii) $\varepsilon^{(p-1)m,0} \cong \varepsilon^{0,(p-1)m}$ over Y_M for $m \equiv 0 \pmod{p^a}$.

The third of these isomorphisms follows since the regular complex line bundle over a lens space has order p^a for some suitably large a (for example, see [2]).

The analogue of (1.2) follows at once, and so $\psi : \Omega_n^{\mathrm{fr}} \to \Omega_n^{\mathrm{fr}}(B\mathbb{Z}_p)$ is constructed. It is not unique.

The first point to note is that the above proof does not apply directly to Σ_p, since Σ_p does not act freely on $W - \mathring{N}(\Delta)$.

Secondly, and unlike the case when $p = 2$, Y_M is actually embedded in M^p rather than in $M^2 \times I$. This is the basis of §3.

3. *An immersed version for* $p = 2$

Returning to $p = 2$ and the data of §1, we consider the Gauss map $\nu(\alpha) : M \to S^n$ and assume that it is transverse to $0 \in S^n$. Thus if $\nu(\alpha)$ has degree d,

$$\nu(\alpha)^{-1}(0) = \{p_1, p_2, \ldots, p_d\}, \quad \text{where each } p_i \in M .$$

(Note that, if $d = 0$ then M is parallelised by α and §1 applies to give $X_M \subset M \times M$.)

Writing $\sigma'(\alpha)$ for the map $\nu(\alpha)^2 : M \times M \to S^n \times S^n$, where T acts on both the range and the domain, we see that $\sigma'(\alpha)$ is equivariant.

Now let $A \subset S^n \times S^n$ denote the immersed submanifold consisting of the axes $S^n \vee S^n$. Then $\sigma'(\alpha)$ is equivariantly transverse to A, whose inverse image consists of $2d$ copies of M, immersed with trivial normal bundle in $M \times M$ and having d^2 double points. The involution T interchanges the two sets of d copies of M.

Of course, as a map of diagonals $M \to S^n$, the two maps $\sigma'(\alpha)$ and $\nu(\alpha)$ coincide. But $\nu(\alpha)$ extends by definition to a bundle map $tM \to tS^n$. Since these are the normal bundles of the respective diagonals, we may adjust $\sigma'(\alpha)$ by a homotopy to a map $\sigma(\alpha)$, which as a map of neighbourhoods

$$\sigma(\alpha) : N(\Delta) \longrightarrow N(\Delta)$$

is transverse to $\Delta = S^n$, with $\sigma(\alpha)^{-1}(S^n) = \Delta = M$.

Now regard the restriction

$$\sigma(\alpha) : W' \longrightarrow S^n \times S^n ,$$

where $W' = M \times M - \mathring{N}(\Delta)$. Then as in §1, we may apply equivariant transversality and assume that $\sigma(\alpha)$ is so transverse to Δ, obtaining $\sigma(\alpha)^{-1}(\Delta) = Z_M \coprod \Delta$, where Z_M carries a free involution.

However, we may do more. There is an immersed and stably framed bordism B in $S^n \times S^n$ between A and Δ. Making $\sigma(\alpha)$ further transverse to B, we deduce that $\sigma(\alpha)^{-1}(A) \sim Z_M \coprod \Delta$. But $\sigma(\alpha) \simeq \sigma'(\alpha)$, and so $2dM \sim Z_M \coprod \Delta$. Thus

$J_M = 2dM - Z_M$ is a framed manifold with free involution, bordant to M and immersible in $M \times M$.

So J_M is a suitable candidate for a representative of $\psi(x)$. The verification of the details concerning the action of the involution on the framing is similar to the calculations in §1. It transpires that the immersion of $2dM$ has normal bundle isomorphic to $\varepsilon^{n,0}$. We leave the remaining computation to the interested reader.

4. *An application*

We conclude by showing how the methods of §1 can be extended to give a wholly geometric proof of the following.

(4.1) PROPOSITION. *Suppose that the n-stem Ω_n^{fr} is finitely generated. Then it is finite* ($n \geqslant 1$).

Proof. Let x be any element of Ω_n^{fr} which does not have finite order. Using our constructions, we have

$$\psi(x) = x_1 + y_1 \qquad \text{where } x_1 \in \Omega_n^{\mathrm{fr}},\ y_1 \in \tilde{\Omega}_n^{\mathrm{fr}}(B\mathbb{Z}_p).$$

Therefore

$$x = t\psi(x) = px_1 + t(y_1);$$

therefore

$$x = p^m x_m + t(y_m)$$

by iteration. Applying (4.2) below, we deduce that $p^n x = p^{m+n} x_m$ for $m = 1, 2, \ldots$. So $p^n x$ does not have finite order, and yet it is divisible by arbitrarily high powers of p. So $p^n x = 0$, whence $x = 0$.

So it remains to give a geometric demonstration of (4.2) (note that the choice of p is irrelevant). To this end, it is convenient to consider signed, or virtual, coverings of a framed manifold N. These arise from standard coverings by negating the first framing vector (and hence the orientation) of the total space. Examples of 'degree' $-p$ are $-pN \downarrow N$ and $-\pi : -\tilde{N} \downarrow N$ where $\pi : \tilde{N} \downarrow N$ is a given p-fold covering.

Such coverings, along with standard ones, may be added over a common base by taking disjoint union of total spaces. In particular, if we form $\pi - \pi : (\tilde{N} - \tilde{N}) \downarrow N$, there is an obvious framed bordism of the total space to zero, expressed as $\pi - \pi \sim \varnothing \downarrow N$.

(4.2) LEMMA. *Suppose that y lies in the reduced group $\tilde{\Omega}_n^{\mathrm{fr}}(B\mathbb{Z}_p)$, and is represented by $\tilde{N} \downarrow N$ with N connected. Then $p^n\tilde{N} \sim 0$ in Ω_n^{fr}.*

Proof. Consider the virtual covering $\rho : (\tilde{N} - pN) \downarrow N$, and its cartesian square $\rho^2 : (\tilde{N} - pN)^2 \downarrow N^2$. We wish to restrict ρ^2 to the diagonal $N \xrightarrow[\delta]{} N^2$.

Well, $\tilde{N}^2 \downarrow N^2$ restricts to $p\tilde{N} \downarrow N$, whilst both $(\tilde{N} \times N) \downarrow N^2$ and $(N \times \tilde{N}) \downarrow N^2$ restrict to $\tilde{N} \downarrow N$. Hence

$$\delta^* \rho^2 \sim (p^2 N - p\tilde{N}) \downarrow N.$$

Continuing by induction, we deduce that

$$\delta^*\rho^k \sim (-1)^k(p^k N - p^{k-1}\tilde{N}) \downarrow N .$$

Furthermore, on the j-th subproduct $N_j = N^{j-1} \times * \times N^{k-j} \overset{}{\underset{i}{\hookrightarrow}} N^k$,

$$i^*\rho^k = (p-p)(\tilde{N}-pN)^{k-1} \downarrow N^{k-1} \sim \varnothing \downarrow N^{k-1} .$$

So, if we choose $k > n$ in order that $\delta : N \to N^k$ is homotopic to a map which factors through $\bigcup_{j=1}^{k} N_j$ $\left(\text{for } \bigwedge_k N \text{ is } (k-1)\text{-connected}\right)$, we have that

$$\delta^*\rho^k \sim \varnothing \downarrow N .$$

Hence

$$(-1)^k(p^k N - p^{k-1}\tilde{N}) \sim 0 .$$

But $N \sim 0$ by choice, and so $p^{k-1}\tilde{N} \sim 0$ for all $k > n$.

Thus (see §0) our proof of (0.1) is entirely geometrical, modulo only that Ω_n^{fr} is finitely generated.

Two interesting problems are as follows:

(i) describe X_M and Y_M of §§1, 2 intrinsically in terms of M and α;

(ii) relate our ψ to the Hopf invariants of [4].

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