CHARACTERISTIC CLASSES OF VECTOR BUNDLES ON A REAL ALGEBRAIC VARIETY

1992 Math. USSR Izv. 39 703

(http://iopscience.iop.org/0025-5726/39/1/A02)

Изв. Акад. Наук СССР
Сер. Матем. Том 55 (1991), № 4

Math. USSR Izvestiya
Vol. 39 (1992), No. 1

CHARACTERISTIC CLASSES OF VECTOR BUNDLES ON A REAL ALGEBRAIC VARIETY

UDC 513.6+517.6

V. A. KRASNOV

ABSTRACT. For a vector bundle E on a real algebraic variety X, the author studies the connections between the characteristic classes

$$c_k(E(\mathbf{C})) \in H^{2k}(X(\mathbf{C}), Z), \qquad w_k(E(\mathbf{R})) \in H^k(X(\mathbf{R}), \mathbf{F}_2).$$

It is proved that for M-varieties the equality $w_k(E(\mathbf{R}) = 0$ implies the congruence $c_k(E(\mathbf{C})) \equiv 0 \bmod 2$. Sufficient conditions are found also for the converse to hold; this requires the construction of new characteristic classes $cw_k(E(\mathbf{C})) \in H^{2k}(X(\mathbf{C}); G, \mathbf{z}(k))$. The results are applied to study the topology of $X(\mathbf{R})$.

INTRODUCTION

Let X be a nonsingular real projective algebraic variety, and E a vector bundle on X. Then the following two topological vector bundles are defined: the complex bundle $E(\mathbf{C})$ on $X(\mathbf{C})$ and the real bundle $E(\mathbf{R})$ on $X(\mathbf{R})$, where $E(\mathbf{C})$ and $X(\mathbf{C})$ are the sets of complex points, and $E(\mathbf{R})$ and $X(\mathbf{R})$ the sets of real points. Our principal problem consists in looking for connections between the characteristic classes

$$c_k(E(\mathbf{C})) \in H^{2k}(X(\mathbf{C}), \mathbf{Z}), \qquad w_k(E(\mathbf{R})) \in H^k(X(\mathbf{R}), \mathbf{F}_2).$$

Let us give examples of such connections. We start with rather obvious ones. The best known is

Assertion 00. *In* $H^{2k}(X(\mathbf{R}), \mathbf{F}_2)$,

$$c_k(E(\mathbf{C}))|_{X(\mathbf{R})} \mod 2 = (w_k(E(\mathbf{R})))^2.$$

This means in particular that the equality $w_k(E(\mathbf{R})) = 0$ implies the congruence $c_k(E)(\mathbf{C}))|_{X(\mathbf{R})} \equiv 0 \bmod 2$. Subsequent examples will show that under certain appropriate conditions the equality $w_k(E(\mathbf{R})) = 0$ implies the congruence $c_k(E(\mathbf{C})) = 0$ mod 2 on all of $X(\mathbf{C})$.

Assertion 01. *Suppose* $\dim X = n$. *Then*

$$(c_n(E(\mathbf{C})), [X(\mathbf{C})]) \mod 2 = (w_n(E(\mathbf{R})), [X(\mathbf{R})]),$$

where $[X(\mathbf{C})] \in H_{2n}(X(\mathbf{C}), \mathbf{Z})$ *and* $[X(\mathbf{R})] \in H_n(X(\mathbf{R}), \mathbf{F}_2)$ *are the fundamental homology classes.*

This results from the following fact: if the class $c_n(E(\mathbf{C}))$ is given by the zero-cycle

$$\pm x_1 \pm \cdots \pm x_p \pm (x_1' + x_1'') \pm \cdots \pm (x_q' + x_q''),$$

where $x_1, \dots, x_p \in X(\mathbf{R})$ and the pairs x_r', x_r'' are complex conjugate, then the class $w_n(E(\mathbf{R}))$ is given by the zero-cycle $x_1 + \cdots + x_p$. As a consequence we have

1991 *Mathematics Subject Classification.* Primary 14P25; Secondary 14F05, 14J60, 55R40.

Corollary 02. *Suppose* $\dim X = n$. *Then the equality* $w_n(E(\mathbf{R})) = 0$ *implies the congruence* $c_n(E(\mathbf{C})) \equiv 0 \bmod 2$; *and if* $X(\mathbf{R})$ *consists of a single component, then the converse holds.*

Assertion 01 can be generalized as follows.

Assertion 03. *Let* $f\colon Y \to X$ *be a mapping of a nonsingular* k*-dimensional real algebraic variety. Then*

$$(f^* c_k(E(\mathbf{C}), [Y(\mathbf{C})]) \mod 2 = (f^* w_k(E(\mathbf{R})), [Y(\mathbf{R})]).$$

Before stating the corollaries of this assertion, we recall that by $A_k(X)$ is meant the group of rational equivalence classes of k-cycles; then certain homomorphisms

$$\mathrm{d}_{\mathbf{C}}\colon A_k(X) \to H_{2k}(X(\mathbf{C}), \mathbf{Z}), \qquad \mathrm{cl}_{\mathbf{C}}\colon A_k(X) \to H_{2k}(X(\mathbf{C}), \mathbf{F}_2)$$

are defined.

Corollary 04. *Suppose the homomorphism* $\mathrm{cl}_{\mathbf{C}}\colon A_k(X) \to H_{2k}(X(\mathbf{C}), \mathbf{F}_2)$ *is epi. Then the equality* $w_k(E(\mathbf{R})) = 0$ *implies the congruence* $c_k(E(\mathbf{C})) \equiv 0 \bmod 2$.

This follows from Assertion 03, using resolution of singularities for cycles on X.

Corollary 05. *Suppose* X *is an* n*-dimensional nonsingular complete intersection. Then for* $2k > n$ *the equality* $w_k(E(\mathbf{R})) = 0$ *implies the congruence* $c_k(E(\mathbf{C})) \equiv 0 \bmod 2$ (*when* X *is of odd degree this holds also for* $0 < 2k < n$).

We show now that the following holds.

Proposition 06. *Let* X *be an* M*-variety. Then the equality* $w_1(E(\mathbf{R})) = 0$ *implies the congruence* $c_1(E(\mathbf{C})) \equiv 0 \bmod 2$.

It suffices to prove this for a line bundle; in the contrary case we replace E by $\det E$. For a line bundle, consider a continuous mapping $f\colon X(\mathbf{C}) \to \mathbf{P}^N(\mathbf{C})$, commuting with complex conjugation, such that the bundle $E(\mathbf{C})$ is topologically isomorphic with the bundle $f^* V(\mathbf{C})$, where $V(\mathbf{C})$ is the universal bundle on $\mathbf{P}^N(\mathbf{C})$ and the isomorphism preserves the real structures on $E(\mathbf{C})$ and $f^* V(\mathbf{C})$. Then

$$c_1(E(\mathbf{C})) = f^* c_1(V(\mathbf{C})), \qquad w_1(E(\mathbf{R})) = f^* w_1(V(\mathbf{R})). \tag{1}$$

If $w_1(E(\mathbf{R})) = 0$, then from (1) we obtain the equality

$$\sum_q \dim \operatorname{Ker}[H^q(\mathbf{P}^N(\mathbf{R}), \mathbf{F}_2) \to H^q(X(\mathbf{R}), \mathbf{F}_2)] = N. \tag{2}$$

On the other hand, for the mapping f we have the Harnack-Thom inequality

$$\begin{aligned} &\sum_q \dim \operatorname{Ker}[H^q(\mathbf{P}^N(\mathbf{R}), \mathbf{F}_2) \to H^q(X(\mathbf{R}), \mathbf{F}_2)] \\ &\quad \le \sum_q \dim \operatorname{Ker}[H^q(\mathbf{P}^N(\mathbf{C}), \mathbf{F}_2) \to H^q(X(\mathbf{C}), \mathbf{F}_2)] \end{aligned} \tag{3}$$

(see [1]), which is satisfied for any continuous mapping. From (1)–(3) we obtain the congruence $c_1(E(\mathbf{C})) \equiv 0 \bmod 2$.

Proposition 06 implies the following interesting fact.

Corollary 07. *Let* X *be an* M*-surface, and let all the components of* $X(\mathbf{R})$ *be orientable. Then the Euler characteristic satisfies the congruence* $\chi(X(\mathbf{R})) \equiv 0 \bmod 16$, *and the homology class* $[X(\mathbf{R})]$ *in* $H_2(X(\mathbf{C}), \mathbf{F}_2)$ *is equal to zero.*

The proof will be given in §4.5.

We note now that if the method of the proof of Proposition 06 is applied to a mapping $f\colon X(\mathbf{C}) \to \mathrm{Gr}_m(\mathbf{C}^N)$, where $m = \operatorname{rk} E$ and $f^* V(\mathbf{C})$ is topologically isomorphic to $E(\mathbf{C})$, one arrives at the following result.

Proposition 08. *Let X be an M-variety. Then the equalities*

$$w_1(E(\mathbf{R})) = 0, \ldots, w_m(E(\mathbf{R})) = 0$$

imply collectively the congruences

$$c_1(E(\mathbf{C})) \equiv 0 \mod 2, \ldots, c_m(E(\mathbf{C})) \equiv 0 \mod 2.$$

We prove in this paper the following generalization of Proposition 06.

Theorem 09. *Let X be an M-variety. Then the equality $w_k(E(\mathbf{R})) = 0$ implies the congruence $c_k(E(\mathbf{C})) \equiv 0 \bmod 2$.*

We also prove

Theorem 010. *Let X be an M-variety. Then the equality $v_k(X(\mathbf{R})) = 0$ implies the equality $v_{2k}(X(\mathbf{C})) = 0$, where $v_r(\cdot)$ is the Wu class of the variety.*

This implies

Corollary 011. *Let X be a $2k$-dimensional M-variety, with $v_k(X(\mathbf{R})) = 0$. Then the Euler characteristic satisfies the congruence $\chi(X(\mathbf{R})) \equiv 0 \bmod 8$, and the homology class $[X(\mathbf{R})]$ in $H_{2k}(X(\mathbf{C}), \mathbf{F}_2)$ is equal to zero.*

The proof of this corollary is given in §4.5. For the proof of Theorem 09 we need to construct certain new characteristic classes (for line bundles they have already come up in [2]). The precise definition of these classes will be given in the main part of the paper; for the moment, we simply explain where they reside.

Let $\mathbf{Z}(k)$ be the constant sheaf on $X(\mathbf{C})$ with stalk $(2\pi i)^k\mathbf{Z} \subset \mathbf{C}$. Let $\tau : X(\mathbf{C}) \to X(\mathbf{C})$ be the involution of complex conjugation; then τ acts on $\mathbf{Z}(k)$ also concordantly with the action on $X(\mathbf{C})$, by complex conjugation. Let $G = \{e, \tau\}$ be the group of order 2; then $\mathbf{Z}(k)$ becomes a G-sheaf, and the Galois-Grothendieck cohomology $H^{2k}(X(\mathbf{C}); G, \mathbf{Z}(k))$ is defined (see [3]), for which there exist canonical homomorphisms

$$\alpha\colon H^{2k}(X(\mathbf{C}); G, \mathbf{Z}(k)) \to H^{2k}(X(\mathbf{C}), \mathbf{Z}),$$
$$\beta\colon H^{2k}(X(C); G, \mathbf{Z}(k)) \to H^{k}(X(\mathbf{R}), \mathbf{F}_2).$$

We shall construct characteristic classes

$$cw_k(E(\mathbf{C}), \tau) \in H^{2k}(X(\mathbf{C}); G, \mathbf{Z}(k)),$$

which we call *mixed*, since they satisfy the equalities

$$\alpha(cw_k(E(\mathbf{C}), \tau)) = c_k(E(\mathbf{C})), \qquad \beta(cw_k(E(\mathbf{C}), \tau)) = w_k(E(\mathbf{R})).$$

Indeed, we have the following commutative diagram:

$$\begin{tikzcd}
 & c_k(E) \arrow[d, phantom, "\rotatebox{-90}{$\in$}"] & \\
 & A^k(X) \arrow[dl, "\mathrm{cl}_{\mathbf{C}}"'] \arrow[dr, "\mathrm{cl}_{\mathbf{R}}"] \arrow[dd, "\mathrm{cl}"] & \\
c_k(E(\mathbf{C})) \in H^{2k}(X(\mathbf{C}), \mathbf{Z}) & & H^k(X(\mathbf{R}), \mathbf{F}_2) \ni w_k(E(\mathbf{R})) \\
 & H^{2k}(X(\mathbf{C}); G, \mathbf{Z}(k)) \arrow[ul, "\alpha"] \arrow[ur, "\beta"'] & \\
 & cw_k(E(\mathbf{C}), \tau) \arrow[u, phantom, "\rotatebox{90}{$\in$}"] &
\end{tikzcd} \tag{4}$$

Here $A^k(X)$ is the group of rational equivalence classes of cycles of codimension k. The homomorphism $\mathrm{cl}_{\mathbf{C}}$ is the composite of the homomorphism $\mathrm{cl}_{\mathbf{C}}: A^k(X) \to H_{2n-2k}(X(\mathbf{C}), \mathbf{Z})$, defined by taking the complex points of the cycle, with the Poincaré duality

$$H_{2n-2k}(X(\mathbf{C}), \mathbf{Z}) \xrightarrow{\sim} H^{2k}(X(\mathbf{C}), \mathbf{Z}).$$

The homomorphism $\mathrm{cl}_{\mathbf{R}}$ is constructed similarly, but its definition involves certain subtleties (see §2.2). The homomorphism cl is defined in §2.1. Under all the homomorphisms cl, $\mathrm{cl}_{\mathbf{C}}$, $\mathrm{cl}_{\mathbf{R}}$, α, and β, characteristic classes go into characteristic classes. A good part of the paper is concerned with the validation of diagram (4).

Let it be noted, also, that these mixed characteristic classes are being defined for Real (capital letter!) bundles on a topological space with involution. We recall that a *Real bundle* on such a topological space X means a complex vector bundle E on X provided with an antilinear involution (of the real structure) $\tau: E \to E$ that commutes with the involution on X (see [4]).

We have also made use of the mixed characteristic classes toward the solution of the converse problem: to find sufficient conditions that the congruence $c_k(E(\mathbf{C})) \equiv 0 \bmod 2$ imply the equality $w_k(E(\mathbf{R})) = 0$. In this direction there already exists the following previous result (see [5] and [6]).

Proposition 012. *Suppose* $H^1(X(\mathbf{C}), \mathbf{Z}) = 0$ *and the group* $H^2(X(\mathbf{C}), \mathbf{Z})$ *has no elements of order* 2. *Then the congruence* $c_1(E(\mathbf{C})) \equiv 0 \bmod 2$ *implies the equality* $w_1(E(\mathbf{R})) = 0$.

We re-prove this proposition by using the mixed characteristic classes, and attempt a generalization for the classes $c_k(E(\mathbf{C}))$ with $k > 1$. In particular, we prove

Theorem 013. *Suppose* X *is a* GM-*variety such that* $H^{2q-1}(X(\mathbf{C}), \mathbf{Z}) = 0$ *for* $1 \leq q \leq k$, $H^{2q}(X(\mathbf{C}), \mathbf{Z})$ *has no elements of order* 2 *for* $1 \leq q \leq k$, *and the homomorphism* $\mathrm{cl}_{\mathbf{C}}: A^q(X) \to H^{2q}(X(\mathbf{C}), \mathbf{Z})$ *is epi for* $q < k$. *Then the congruence* $c_k(E(\mathbf{C})) \equiv 0 \bmod 2$ *implies the equality* $w_k(E(\mathbf{R})) = 0$.

Finally, let us observe that by a nonsingular real algebraic variety we mean in this paper a scheme X over the field $\mathbf{R}$ such that $X \otimes_{\mathbf{R}} \mathbb{C}$ is a nonsingular irreducible variety. A subvariety of a variety X is a reduced and irreducible closed subscheme of X.

§1. Line bundles

This section is introductory. It will assist in defining the mixed characteristic classes of an arbitrary complex vector bundle with real structure.

1.1. **Picard groups.** Let X be a topological space with involution $\tau: X \to X$, i.e., a real topological space in the terminology of [4]. We consider only compact spaces. The Picard group $\operatorname{Pic} X$ is by definition the group of complex line bundles on X (isomorphic bundles being identified); $\operatorname{Pic}(X, \tau)$ is the group of Real (capital letter!) line bundles on X, i.e., the group of complex line bundles with real structure; and $\operatorname{Pic}^{\mathbf{R}} X^\tau$ is the group of real (lower-case!) line bundles on X^τ.

Let $\mathscr{O}_X$ be the sheaf of germs of continuous complex-valued functions on X, and $\mathscr{O}_X^*$ the sheaf of germs of invertible complex-valued functions. Then $\operatorname{Pic} X = H^1(X, \mathscr{O}_X^*)$. Similarly, if $\mathscr{A}_{X^\tau}$ and $\mathscr{A}_{X^\tau}^*$ are the sheaves of germs of real-valued functions, then $\operatorname{Pic}^{\mathbf{R}} X^\tau = H^1(X^\tau, \mathscr{A}_{X^\tau}^*)$. A corresponding description applies to the group $\operatorname{Pic}(X, \tau)$; for this, we observe that on the sheaves $\mathscr{O}_X$ and $\mathscr{O}_X^*$ there exists a canonical real structure given by the formula $\theta(\varphi) = \overline{\varphi \circ \tau}$, where φ is the germ of a function and the bar signifies complex conjugation. We shall now demonstrate the following fact.

Proposition 1.1.1. *There exist a canonical isomorphism*

$$\operatorname{Pic}(X, \tau) = H^1(X; G, \mathscr{O}_X^*),$$

where $G = \{e, \tau\}$ is the group of order 2 and $H^1(X; G, \mathscr{O}_X^*)$ is Galois-Grothendieck cohomology.

Proof. We first describe the cohomology group $H^1(X; G, \mathscr{O}_X^*)$ by means of coverings (see [3]). Let $\mathbf{U} = \{u_i\}$ be a covering of X, with τ acting on the index set I so that $\tau(U_i) = U_{\tau(i)}$ for every $i \in I$. In this case the covering $\mathbf{U}$ is called a *G-covering.* It will be called a covering without fixed points if τ acts on I without fixed points. Supposing, then, that the covering $\mathbf{U} = \{u_i\}$ is without fixed points, consider the cochain complex $\varphi \in C^k(\mathbf{U}, \mathscr{O}_X^*)$. It is acted on by the involution τ in the following way: if $\varphi \in C^k(\mathbf{U}, \mathscr{O}_X^*)$ and $\varphi = \{\varphi_{i_0} \cdots_{i_k}\}$, then

$$[\tau(\varphi)]_{i_0, \dots, i_k} = \theta(\varphi_{\tau(i_0), \dots, \tau(i_k)}).$$

Put

$$H^k(\mathbf{U}; G, \mathscr{O}_X^*) = H^k(C^*(\mathbf{U}, \mathscr{O}_X^*)^G);$$

then

$$H^k(X; G, \mathscr{O}_X^*) = \varinjlim H^k(\mathbf{U}; G, \mathscr{O}_X^*).$$

We now show how to construct, for a given cocycle $\varphi \in Z^1(\mathbf{U}, \mathscr{O}_X^*)^G$, a complex line bundle with real structure. Put

$$\widetilde{E} = \coprod_i U_i \times \mathbf{C}$$

and define an involution $\theta\colon \widetilde{E} \to \widetilde{E}$ as follows: if $(x, z) \in U_i \times \mathbf{C}$, then $\theta(x, z) = (\tau(x), \bar{z}) \in U_{\tau(i)} \times \mathbf{C}$. The cocycle φ determines an equivalence relation on $\widetilde{E}$ that is preserved under the action of θ. Indeed, if $x \in U_i \cap U_j$, then, by definition,

$$U_i \times \mathbf{C} \ni (x, z) \sim (x, \varphi_{ij}(x) \cdot z) \in U_j \times \mathbf{C}.$$

But

$$\begin{aligned}\theta(x, z) = (\tau(x), \bar{z}) \in U_{\tau(i)} \times \mathbf{C}, \ \theta(x, \varphi_{ij}(x) \cdot z) &= (\tau(x), \overline{\varphi_{ij}(x)} \cdot \bar{z}) \\ &= (\tau(x), \varphi_{ij}(\tau(x)) \cdot \bar{z}) \in U_{\tau(j)} \times \mathbf{C},\end{aligned}$$

i.e., $\theta(x, z) \sim \theta(x, \varphi_{ij}(x) \cdot z)$. Factoring $\widetilde{E}$, we obtain the complex line bundle

$$E = \widetilde{E}/{\sim} = \left(\coprod_i U_i \times \mathbf{C}\right) / \sim$$

s with real structure θ. This permits construction of a homomorphism $H^1(X; G, \mathscr{O}_X^*) \to \operatorname{Pic}(X, \tau)$, which is then easily verified to be an isomorphism; and the proposition is proved.

1.2. **The characteristic classes.** First of all, we observe that there exist canonical homomorphisms

$$\alpha\colon \operatorname{Pic}(X, \tau) \to \operatorname{Pic} X, \qquad \beta\colon \operatorname{Pic}(X, \tau) \to \operatorname{Pic}^{\mathbf{R}} X^\tau;$$

the first "forgets" the real structure, and the second is given by the formula $E \mapsto E^\theta$. For any $E \in \operatorname{Pic}(X, \tau)$ there are defined the characteristic classes $c_1(\alpha)(E)) \in H^2(X, \mathbf{Z})$ and $w_1(\beta(E)) \in H^1(X^\tau, \mathbf{F}_2)$. We shall now define a characteristic class

$cw_1(E) \in H^2(X; G, \mathbf{Z}(1))$ which is a "mix" of the classes $c_1(\alpha(E))$ and $w_1(\beta(E))$; here $\mathbf{Z}(1) = (2\pi i) \cdot \mathbf{Z} \subset \mathbf{C}$ is a constant sheaf on X, on which the involution τ acts by complex conjugation, i.e., by multiplication by -1. The new characteristic class is defined by means of the exponential exact sequence of G-sheaves

$$0 \to \mathbf{Z}(1) \to \mathscr{O}_X \overset{\exp}{\to} \mathscr{O}_X^* \to 1. \tag{1}$$

This exact sequence (1) determines a coboundary homomorphism

$$\delta : H^1(X; G, \mathscr{O}^*) \to H^2(X; G, \mathbf{Z}(1)),$$

and we denote the composite homomorphism

$$\operatorname{Pic}(X, \tau) \overset{\sim}{\to} H^1(X; G, \mathscr{O}^*) \overset{\delta}{\to} H^2(X; G, \mathbf{Z}(1))$$

by cw_1. Observe that in fact cw_1 is an isomorphism, since

$$H^1(X; G, \mathscr{O}) = H^2(X; G, \mathscr{O}) = 0.$$

We now define two more homomorphisms:

$$\alpha : H^2(X; G, \mathbf{Z}(1)) \to H^2(X, \mathbf{Z}), \qquad \beta : H^2(X; G, \mathbf{Z}(1)) \to H^1(X^\tau, \mathbf{F}_2).$$

The mapping α is the composite of projection onto $\mathrm{II}_\infty^{0,2}(X; G, \mathbf{Z}(1))$ and the inclusion $\mathrm{II}_\infty^{0,2}(X; G, \mathbf{Z}(1)) \subset H^2(X, \mathbf{Z}(1)) = H^2(X, \mathbf{Z})$. Before defining β, let us look at the spectral sequence

$$\mathrm{I}_2^{p,q}(X^\tau; G, \mathbf{Z}(1)) = H^p(X^\tau, \mathscr{H}^q(\mathbf{Z}(1))) \Rightarrow H^{p+q}(X^\tau; G, \mathbf{Z}(1)).$$

Since

$$\mathscr{H}^q(\mathbf{Z}(1)) = \begin{cases} 0 & \text{for } q \text{ even}, \\ \mathbf{F}_2 & \text{for } q \text{ odd}, \end{cases}$$

we have a canonical isomorphism

$$H^2(X^\tau; G, \mathbf{Z}(1)) \overset{\sim}{\to} H^1(X^\tau, \mathbf{F}_2).$$

The composite

$$H^2(X; G, \mathbf{Z}(1)) \to H^2(X^\tau; G, \mathbf{Z}(1)) \overset{\sim}{\to} H^1(X^\tau, \mathbf{F}_2)$$

is what we denote by β.

Proposition 1.2.1. *The following equalities hold:*

$$\alpha(cw_1(E)) = c_1(\alpha(E)), \qquad \beta(cw_1(E)) = w_1(\beta(E)). \tag{2}$$

We omit the verification of these equalities, since it is given in [2].

We also state the following obvious fact.

Proposition 1.2.2. *Let $f: X \to Y$ be an equivariant mapping of spaces with involution, and E a complex line bundle on Y with real structure; then*

$$cw_1(f^*E) = f^*(cw_1(E)). \tag{3}$$

Remark 1.2.3. It will be seen later that equalities (2) and (3) determine the mixed characteristic classes $cw_1(E)$ uniquely. For the moment we compute the Galois-Grothendieck cohomology group in which these classes lie, for the general case. Let E be a complex vector bundle on X with real structure, and suppose E splits into line bundles with real structure; that is, $E = E_1 \oplus \cdots \oplus E_m$. Then we must have

$$(1 + cw_1(E) + \cdots + cw_m(E)) = (1 + cw_1(E_1)) \cup \cdots \cup (1 + cw_1(E_m)),$$

and therefore

$$cw_k(E) \in H^{2k}(X; G, \mathbf{Z}(k)),$$

where $\mathbf{Z}(k) = \mathbf{Z}(1) \otimes \cdots \otimes \mathbf{Z}(1) = (2\pi i)^k \mathbf{Z} \subset \mathbf{C}$; that is, $\mathbf{Z}(k)$ is a constant sheaf with stalk $\mathbf{Z}$, on which the action of the involution is multiplication by -1 for odd k and trivial for even k.

1.3. **Divisors.** In this section X is an n-dimensional compact complex variety, $\tau: X \to X$ an antiholomorphic involution, $\operatorname{Pic} X$ and $\operatorname{Pic}(X, \tau)$ groups of complex-analytic line bundles, and $\mathscr{O}_X$ and $\mathscr{O}_X^*$ sheaves of germs of holomorphic functions. Let $\mathscr{D}iv(X)$ be the group of divisors on X Then τ acts on $\mathscr{D}iv(X)$, since if ψ is a hypersurface in X, so is $\tau(X)$; so that we have an involution

$$\tau^*: \mathscr{D}iv(X) \to \mathscr{D}iv(X).$$

Let D be a divisor invariant with respect to τ^*. Then on the sheaf $\mathscr{O}(D)$ there exists a canonical real structure; namely, put $\theta(\varphi) = \overline{\varphi \circ \tau}$, where φ is the germ of a meromorphic function such that $(\varphi) + D \geq 0$, and the bar means complex conjugation. The corresponding line bundle will be denoted by $L(D)$. Consequently, we have a homomorphism

$$L: \mathscr{D}iv(X)^{\tau*} \to \operatorname{Pic}(X, \tau),$$

and composition with the homomorphism

$$cw_1: \operatorname{Pic}(X, \tau) \to H^2(X; G, \mathbf{Z}(1))$$

gives a homomorphism

$$\mathrm{cl}: \mathscr{D}iv(X)^{\tau*} \to H^2(X; G, \mathbf{Z}(1)).$$

We want to describe the homomorphism cl geometrically. Suppose first that D is an irreducible hypersurface, invariant with respect to τ, and $[D] \in H_{2n-2}(D, \mathbf{Z})$ is the fundamental homology class. Consider the Poincaré-Lefschetz isomorphism

$$H_{2n-2}(D, \mathbf{Z}) \xrightarrow{\sim} H^2(X, X \backslash D; \mathbf{Z}),$$

and denote the image of $[D]$ under this isomorphism by $[D]^*$. Observe now that

$$H^2(X, X\backslash D; G, \mathbf{Z}(1)) = H^2(X, X\backslash D; \mathbf{Z}),$$

as follows from the spectral sequence

$$\mathrm{II}_2^{p,q}(\cdot; G, \mathbf{Z}(1)) = H^p(G, H^q(\cdot, \mathbf{Z}(1))) \Rightarrow H^{p+q}(\cdot; G, \mathbf{Z}(1));$$

so $[D]^*$ can be regarded as a cohomology class in $H^2(X, X\backslash D; G, \mathbf{Z}(1))$. Denote its image in $H^2(X; G, \mathbf{Z}(1))$ also by $[D]^*$. Suppose next that $D = Y_1 + Y_2$, where Y_1 and Y_2 are irreducible hypersurfaces such that $\tau(Y_1) = Y_2$. Putting $[D] = [Y_1] + [Y_2]$ and repeating the preceding argument, we obtain again a cohomology class $[D]^* \in H^2(X; G, \mathbf{Z}(1))$. Since the divisors of these two forms generate the group $\mathscr{D}iv(X)^{\tau^*}$, there is a homomorphism

$$[\ ]^*: \mathscr{D}iv(X)^{\tau^*} \to H^2(X; G, \mathbf{Z}(1)).$$

Proposition 1.3.1.

$$\mathrm{cl}(D) = [D]^*.$$

Proof. It suffices to verify this equality for divisors of the two forms just described. From the exact sequence of G-sheaves

$$0 \to \mathbf{Z}(1) \to \mathscr{O} \to \mathscr{O}^* \to 1$$

we obtain the commutative diagram

$$
\text{(1)}\qquad \begin{array}{ccccc}
L(D) \in & H^1(X, X\backslash D; G, \mathcal{O}^*) & \xrightarrow{\varepsilon} & H^2(X, X\backslash D; G, \mathbf{Z}(1)) & \ni [D]^* \\
\updownarrow & \downarrow & & \curvearrowright & \updownarrow \\
L(D) \in & H^1(X, X\backslash D; \mathcal{O}^*) & \xrightarrow{\delta} & H^2(X, X\backslash D; \mathbf{Z}) & \ni [D]^*
\end{array}
$$

Observe that the bundle $L(D)$ does indeed determine an element of the Picard group $H^1(X, X\backslash D; G, \mathcal{O}^*)$, as is easily seen from the definition of Čech cohomology. For the lower row in (1) we have

$$\text{(2)}\qquad \delta(L(D)) = [D]^*,$$

so that this equality holds also in the upper row, since the homomorphism

$$H^2(X, X\backslash D; G, \mathbf{Z}(1)) \to H^2(X, X\backslash D; \mathbf{Z})$$

is an inclusion. If now we regard $L(D)$ as an element of $H^1(X; G, \mathcal{O}^*)$, and $[D]^*$ as an element of $H^2(X; G, \mathbf{Z}(1))$, then (2) holds also in this case. This proves the proposition.

Remark 1.3.2. Suppose $D = k_1Y_1 + \cdots + k_mY_m$, where $Y_1, \ldots, Y_m$ are nonsingular irreducible hypersurfaces invariant with respect to τ, and put $D^\tau = k_1Y_1^\tau + \cdots + k_mY_m^\tau$; then D^τ is a divisor on X^τ, and a cohomology class $[D^\tau]^* \in H^1(X^\tau, \mathbf{F}_2)$ is defined. It is easily verified that $w_1(L(D^\tau)) = [D^\tau]^*$ (see [6]), and this means that $\beta([D]^*) = [D^\tau]^*$.

We note also that the manner of construction of the mapping $[\ ]^*: \mathcal{D}iv(X)^{\tau^*} \to H^2(X; G, \mathbf{Z}(1))$ has been borrowed from [12] and [13], with modification taking account of the real structure.

§2. Real algebraic cycles

In this section X is a nonsingular real projective algebraic variety, and we define and study certain mappings

$$\mathrm{cl}: A^k(X) \to H^{2k}(X(\mathbf{C}); G, \mathbf{Z}(k)), \qquad \mathrm{cl}_{\mathbf{C}}: A^k(X) \to H^{2k}(X(\mathbf{C}), \mathbf{Z}),$$
$$\mathrm{cl}_{\mathbf{R}}: A^k(X) \to H^k(X(\mathbf{R}), \mathbf{F}_2).$$

2.1. **The mappings cl and cl$_{\mathbf{C}}$.** If $Z^k(X)$ is the group of cycles of codimension k, we can define a mapping $\mathrm{cl}: \mathbf{Z}^k(X) \to H^{2k}(X(\mathbf{C}); G, \mathbf{Z}(k))$ by the equality $\mathrm{cl}(Y) = [Y]^*$, where the cohomology class $[Y]^*$ in $H^{2k}(X(\mathbf{C}); G, \mathbf{Z} \in k))$ is defined in the same way as for divisors in §1.3. Similarly, $\mathrm{cl}_{\mathbf{C}}(Y) = [Y(\mathbf{C})]^* \in H^{2k}(X(\mathbf{C}), \mathbf{Z})$ (see [13]). Denoting by α the canonical mapping

$$H^{2k}(X(\mathbf{C}); G, \mathbf{Z}(k)) \to H^{2k}(X(\mathbf{C}), \mathbf{Z}(k)) = H^{2k}(X(\mathbf{C}), \mathbf{Z}),$$

we obtain the equality $\alpha([Y]^*) = [Y(\mathbf{C})]^*$; i.e., $\alpha(\mathrm{cl}(Y)) = \mathrm{cl}_{\mathbf{C}}(Y)$.

Proposition 2.1.1. *If two cycles Y' and Y'' are rationally equivalent, then* $\mathrm{cl}(Y') = \mathrm{cl}(Y'')$.

To prove this, we first need the following fact.

Lemma 2.1.2. *Let X and Y be two topological spaces with involution, and $f, g: X \to Y$ equivariant mappings that are equivariantly homotopic. Then the cohomology group homomorphisms*

$$f^*, g^*: H^n(Y; G, \mathbf{Z}(k)) \to H^n(X; G, \mathbf{Z}(k))$$

coincide.

Proof. We show first that the mapping $i_t\colon X \to X \times I$, $i_t(x) = (x, t)$, induces an isomorphism

$$(1) \qquad i_t^*\colon H^n(X \times I; G, \mathbf{Z}(k)) \to H^n(X; G, \mathbf{Z}(k))$$

that is independent of t. Indeed, since

$$i_t^*\colon H^q(X \times I, \mathbf{Z}(k)) \to H^q(X, \mathbf{Z}(k))$$

is an isomorphism for every q, it follows from the spectral sequence

$$\mathrm{II}_2^{p,q}(\cdot\,; G, \mathbf{Z}(k)) = H^p(G, H^q(\cdot\,, \mathbf{Z}(k))) \Rightarrow H^{p+q}(\cdot\,; G, \mathbf{Z}(k))$$

that also (1) is an isomorphism. Furthermore, since the composite of the mappings

$$X \xrightarrow{i_t} X \times I \xrightarrow{\mathrm{pr}} X$$

coincides with id, the isomorphism $i_t^* \circ \mathrm{pr}^*$ is independent of t, and therefore so is i_t^*. Now let $F\colon X \times I \to Y$ be an equivariant homotopy from f to g. Then $f^* = i_0^* \circ F^*$ and $g^* = i_1^* \circ F^*$, and therefore $f^* = g^*$. This proves the lemma.

Proof of Proposition 2.1.1. Let $V \subset X \times \mathbf{P}^1$ be a subvariety of codimension k that projects dominantly onto $\mathbf{P}^1$. It determines a cohomology class

$$[V]^* \in H^{2k}(X(\mathbf{C}) \times \mathbf{P}^1(\mathbf{C}); G, \mathbf{Z}(k)).$$

If $t \in \mathbf{P}^1(\mathbf{R})$, we denote by V_t the cycle in $X = X \times \{t\}$ equal to $V \cap (X \times \{t\})$; then

$$(2) \qquad [V_t]^* = i_t^*[V]^*.$$

This equality follows from the commutativity of the diagram

$$\begin{array}{l}
[V]^* \in H^{2k}(X(\mathbf{C}) \times \mathbf{P}^1(\mathbf{C}), X(\mathbf{C}) \times \mathbf{P}^1(\mathbf{C})\backslash V(\mathbf{C}); G, \mathbf{Z}(k)) \xrightarrow{i_t^*} \\
\quad \updownarrow \qquad\qquad\qquad \curvearrowright\alpha \\
[V(\mathbf{C})]^* \in H^{2k}(X(\mathbf{C}) \times \mathbf{P}^1(\mathbf{C}), X(\mathbf{C}) \times \mathbf{P}^1(\mathbf{C})\backslash V(\mathbf{C}); \mathbf{Z}(k)) \xrightarrow{i_t^*} \\
\qquad\qquad\qquad\qquad \xrightarrow{i_t^*} H^{2k}(X(\mathbf{C}), X(\mathbf{C})\backslash V_t(\mathbf{C}); G, \mathbf{Z}(k)) \ni [V_t]^* \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad \curvearrowright a \qquad\qquad \updownarrow \\
\qquad\qquad\qquad\qquad \xrightarrow{i_t^*} H^{2k}(X(\mathbf{C}), X(\mathbf{C})\backslash V_t(\mathbf{C}); \mathbf{Z}(k)) \ni [V_t(\mathbf{C})]^*
\end{array}$$

and the equality $[V_t(\mathbf{C})]^* = i_t^*[V(\mathbf{C})]^*$. From (2) and the lemma, we conclude that the cohomology class $[V_t]^* \in H^{2k}(X(\mathbf{C}); G, \mathbf{Z}(k))$ is independent of t. This proves the proposition.

Thus, there is defined a mapping

$$\mathrm{cl}\colon A^k(X) \to H^{2k}(X(\mathbf{C}); G, \mathbf{Z}(k)).$$

We prove

Proposition 2.1.3. *The mapping* cl *preserves products.*

Proof. Let Y and Z be two subvarieties, of codimension k and l, with proper intersection. Then the cohomology class

$$[(Y \cdot Z)(\mathbf{C})]^* \in H^{2k+2l}(X(\mathbf{C}), X(\mathbf{C})\backslash Y(\mathbf{C}) \cap Z(\mathbf{C}); \mathbf{Z})$$

is the product of the classes

$$[Y(\mathbf{C})]^* \in H^{2k}(X(\mathbf{C}), X(\mathbf{C})\backslash Y(\mathbf{C}); \mathbf{Z}),$$
$$[Z(\mathbf{C})]^* \in H^{2l}(X(\mathbf{C}), X(\mathbf{C})\backslash Z(\mathbf{C}); \mathbf{Z}).$$

It follows that the class

$$[Y \cdot Z]^* \in H^{2k+2l}(X(\mathbf{C}), X(\mathbf{C})\backslash Y(\mathbf{C}) \cap Z(\mathbf{C}); G, \mathbf{Z}(k+l))$$

is the product of the classes

$$[Y]^* \in H^{2k}(X(\mathbf{C}), X(\mathbf{C})\backslash Y(\mathbf{C}); G, \mathbf{Z}(k)),$$
$$[Z]^* \in H^{2l}(X(\mathbf{C}), X(\mathbf{C})\backslash Z(\mathbf{C}); G, \mathbf{Z}(l)),$$

so that this relation also holds for the classes

$$[Y \cdot Z]^* \in H^{2k+2l}(X(\mathbf{C}); G, \mathbf{Z}(k+l)), \qquad [Y]^* \in H^{2k}(X(\mathbf{C}); G, \mathbf{Z}(k)),$$
$$[Z]^* \in H^{2l}(X(\mathbf{C}); G, \mathbf{Z}(l)).$$

This proves the proposition.

We now want to study the composite of the homomorphisms

$$A^k(X) \to H^{2k}(X(\mathbf{C}); G, \mathbf{Z}(k)) \to H^{2k}(X(\mathbf{R}); G, \mathbf{Z}(k))$$

For this we shall need first to define the fundamental homology class of the algebraic set $Y(\mathbf{R})$, where Y is a subvariety of X, and also to study the cohomology $H^{2k}(X(\mathbf{R}); G, \mathbf{Z}(k))$.

2.2. **The fundamental homology class and the mapping $\mathrm{cl}_{\mathbf{R}}$.** Let Y be a subvariety of X of dimension m, such that the set $Y(\mathbf{C})$ is irreducible and $\dim Y(\mathbf{R}) = m$. We shall define a homology class $[Y(\mathbf{R})] \in H_m(Y(\mathbf{R}), \mathbf{F}_2)$. For this we take a resolution of singularities

$$\begin{array}{ccc} \pi\colon Y' & \to & Y \\ \cup & & \cup \\ Z' & \to & Z \end{array}$$

where Z is the singularities of Y and $Z' = \pi^{-1}(Z)$, and consider the following commutative diagram of exact homology sequences with coefficients in $\mathbf{F}_2$:

$$\begin{array}{ccccccc} 0 & \to & H_m(Y'(\mathbf{R})) & \to & H_m(Y'(\mathbf{R}), Z'(\mathbf{R})) & \to & \cdots \\ & & \downarrow & & \downarrow \wr & & \\ 0 & \to & H_m(Y(\mathbf{R})) & \to & H_m(Y(\mathbf{R}), Z(\mathbf{R})) & \to & \cdots . \end{array}$$

The fundamental homology class $[Y'(\mathbf{R})] \in H_m(Y'(\mathbf{R}))$ is defined; its image in $H_m(Y(\mathbf{R}))$ gives the fundamental homology class $[Y(\mathbf{R})]$. Observe that if we take a triangulation of $Y(\mathbf{R})$, then $[Y(\mathbf{R})]$ is the sum of all the m-simplices; by [7], a triangulation of $Y(\mathbf{R})$ exists. We now define a mapping

$$\mathrm{cl}_{\mathbf{R}}\colon Z^k(X) \to H^k(X(\mathbf{R}), \mathbf{F}_2).$$

Let Y be a subvariety as above, with $m + k = \dim X$. Consider the sequence of homomorphisms

$$H_m(Y(\mathbf{R}), \mathbf{F}_2) \xrightarrow{\sim} H^k(X(\mathbf{R}), X(\mathbf{R})\backslash Y(\mathbf{R}); \mathbf{F}_2) \to H^k(X(\mathbf{R}), \mathbf{F}_2),$$

where the first is the Poincaré-Lefschetz isomorphism; the image of $[Y(\mathbf{R})]$ in both $H^k(X(\mathbf{R}), X(\mathbf{R})\backslash Y(\mathbf{R}); \mathbf{F}_2)$ and $H^k(X(\mathbf{R}), \mathbf{F}_2)$ will be denoted by $[Y(\mathbf{R})]^*$. In this case we put $\mathrm{cl}_{\mathbf{R}}(Y) = [Y(\mathbf{R})]^*$. In all other cases, $\dim_{\mathbf{R}} Y(\mathbf{R}) < \dim_{\mathbf{C}} Y(\mathbf{C})$, and we put $\mathrm{cl}_{\mathbf{R}}(Y) = 0$. This gives a homomorphism

$$\mathrm{cl}_{\mathbf{R}}\colon Z^k(X) \to H^k(X(\mathbf{R}), \mathbf{F}_2),$$

where the images of rationally equivalent cycles coincide; as a result, we have a homomorphism

$$\mathrm{cl}_{\mathbf{R}}\colon A^k(X) \to H^k(X(\mathbf{R}), \mathbf{F}_2).$$

This homomorphism is connected in a certain fashion with the homomorphism $\mathrm{cl}\colon A^k(X) \to H^{2k}(X(\mathbf{C}); G, \mathbf{Z}(k))$. To understand this connection, consider the homomorphism

$$\beta'\colon H^{2k}(X(\mathbf{C}); G, \mathbf{Z}(k)) \to H^{2k}(X(\mathbf{R}); G, \mathbf{F}_2)$$

which is the composite of the homomorphisms

$$H^{2k}(X(\mathbf{C}); G, \mathbf{Z}(k)) \to H^{2k}(X(\mathbf{C}); G, \mathbf{F}_2) \to H^{2k}(X(\mathbf{R}); G, \mathbf{F}_2),$$

of which the first is induced by the homomorphism of G-sheaves $\mathbf{Z}(k) \to \mathbf{F}_2$, $(2\pi i)^k p \mapsto p \bmod 2$, and the second is the restriction. Since

$$H^{2k}(X(\mathbf{R}); G, \mathbf{F}_2) = \bigoplus_{q=0}^{2k} H^q(X(\mathbf{R}), \mathbf{F}_2)$$

(see [1]), projection onto $H^k(X(\mathbf{R}), \mathbf{F}_2)$ defines also a homomorphism

$$\beta\colon H^{2k}(X(\mathbf{C}); G, \mathbf{Z}(k)) \to H^k(X(\mathbf{R}), \mathbf{F}_2).$$

Theorem 2.2.1. *The image of the composite of the homomorphisms*

$$A^k(X) \xrightarrow{\mathrm{cl}} H^{2k}(X(\mathbf{C}); G, \mathbf{Z}(k)) \xrightarrow{\beta'} H^{2k}(X(\mathbf{R}); G, \mathbf{F}_2)$$

lies in $H^k(X(\mathbf{R}), \mathbf{F}_2)$, *and* $\beta \circ \mathrm{cl} = \mathrm{cl}_{\mathbf{R}}$.

The proof will be given in §2.4; for the moment, we consider only the cases $k = 0, 1$.

Observe that

$$A^0(X) = \mathbf{Z}\cdot[X], \qquad H^0(X(\mathbf{C}); G, \mathbf{Z}) = H^0(X(\mathbf{C}), \mathbf{Z}) = \mathbf{Z}\cdot[X(\mathbf{C})]^*,$$
$$H^0(X(\mathbf{R}); G, \mathbf{F}_2) = H^0(X(\mathbf{R}), \mathbf{F}_2) = \mathbf{F}_2\cdot[X_1]^* + \cdots + \mathbf{F}_2\cdot[X_m]^*,$$

where $X_1, \ldots, X_m$ are the components of $X(\mathbf{R})$; furthermore,

$$\mathrm{cl}([X]) = [X(\mathbf{C})]^*, \qquad \beta([X(\mathbf{C})]^*) = [X_1]^* + \cdots + [X_m]^*,$$
$$\mathrm{cl}_{\mathbf{R}}([X]) = [X(\mathbf{R})]^*.$$

These equalities imply the theorem for the case $k = 0$. The case $k = 1$ has already been worked out (see Remark 1.3.2), since each divisor is linearly equivalent to a difference to a difference $Y' - Y''$, where Y' and Y'' are nonsingular hypersurfaces.

2.3. **Functorial properties of the mapping cl.** Suppose we have a triple $Z \subset Y \subset X$, where X and Y are nonsingular real algebraic varieties, while the variety Z can be singular. Then cohomology classes

$$[Y]^* \in H^{2k}(X(\mathbf{C}), X(\mathbf{C})\backslash Y(\mathbf{C}); G, \mathbf{Z}(k)),$$
$$[Z]^*_Y \in H^{2d}(Y(\mathbf{C}), Y(\mathbf{C})\backslash Z(\mathbf{C}); G, \mathbf{Z}(d)),$$
$$[Z]^*_X \in H^{2k+2d}(X(\mathbf{C}), X(\mathbf{C})\backslash Z(\mathbf{C}); G, \mathbf{Z}(k+d)),$$

are defined, where $k = \operatorname{codim}_X Y$ and $d = \operatorname{codim}_Y Z$. In this case we can define a product

$$(1) \qquad [Z]^*_Y \cdot [Y]^* \in H^{2k+2d}(X(\mathbf{C}), X(\mathbf{C})\backslash Z(\mathbf{C}); G, \mathbf{Z}(k+d))$$

as follows. Let $\pi: U \to Y(\mathbf{C})$ be a tubular neighborhood of $Y(\mathbf{C})$ in $X(\mathbf{C})$, where π is a real projection, i.e., commutes with τ, and let $V = \pi^{-1}(Z(\mathbf{C}))$. Then we have an isomorphism

$$\pi^*: H^{2d}(Y(\mathbf{C}), Y(\mathbf{C})\backslash Z(\mathbf{C}); G, \mathbf{Z}(d) \xrightarrow{\sim} H^{2d}(U, U\backslash V; G, \mathbf{Z}(d)),$$

since

$$\pi^*: H^q(Y(\mathbf{C}), Y(\mathbf{C})\backslash Z(\mathbf{C}); \mathbf{Z}) \to H^q(U, U\backslash V; \mathbf{Z})$$

is an isomorphism for every q. Consequently, we can regard $[Z]^*_Y$ as an element of the group $H^{2d}(U, U\backslash V; G, \mathbf{Z}(d))$, and the product (1) is therefore defined.

Proposition 2.3.1.

$$[Z]^*_Y \cdot [Y]^* = [Z]^*_X. \tag{2}$$

Proof. Equality (2) holds for the cohomology classes

$$\begin{gathered}
[Y(\mathbf{C})]^* \in H^{2k}(X(\mathbf{C}), X(\mathbf{C})\backslash Y(\mathbf{C}); \mathbf{Z}(k)), \\
[Z(\mathbf{C})]^*_{Y(\mathbf{C})} \in H^{2d}(Y(\mathbf{C}), Y(\mathbf{C})\backslash Z(\mathbf{C}); \mathbf{Z}(d)), \\
[Z(\mathbf{C})]^*_{X(\mathbf{C})} \in H^{2k+2d}(X(\mathbf{C}), X(\mathbf{C})\backslash Z(\mathbf{C}); \mathbf{Z}(k+d)).
\end{gathered}$$

Then the inclusions

$$\begin{gathered}
H^{2k}(X(\mathbf{C}), X(\mathbf{C})\backslash Y(\mathbf{C}); G, \mathbf{Z}(k)) \subset H^{2k}(X(\mathbf{C}), X(\mathbf{C})\backslash Y(\mathbf{C}); \mathbf{Z}(k)), \\
H^{2d}(Y(\mathbf{C}), Y(\mathbf{C})\backslash Z(\mathbf{C}); G, \mathbf{Z}(d)) \subset H^{2d}(Y(\mathbf{C}), Y(\mathbf{C})\backslash Z(\mathbf{C}); \mathbf{Z}(d)), \\
H^{2k+2d}(X(\mathbf{C}), X(\mathbf{C})\backslash Z(\mathbf{C}); G, \mathbf{Z}(k+d)) \subset H^{2k+2d}(X(\mathbf{C}), X(\mathbf{C})\backslash Z(\mathbf{C}); \mathbf{Z}(k+d))
\end{gathered}$$

imply that it also holds for the original classes. This proves the proposition.

Remark 2.3.2. If we regard the cohomology classes $[Z]^*_Y$ and $[Z]^*_X$ as elements of the groups $H^{2d}(Y(\mathbf{C}); G, \mathbf{Z}(d))$ and $H^{2k+2d}(X(\mathbf{C}); G, \mathbf{Z}(k+d))$, equality (2) remains true.

Now let $f: X \to Y$ be a mapping of nonsingular projective real algebraic varieties. Then there are homomorphisms

$$f^*: A^k(Y) \to A^k(X), \qquad f^*: H^{2k}(Y(\mathbf{C}); G, \mathbf{Z}(k)) \to H^{2k}(X(\mathbf{C}); G, \mathbf{Z}(k)).$$

Proposition 2.3.3.

$$f^* \circ \mathrm{cl} = \mathrm{cl} \circ f^*.$$

Proof. Identify X with the graph $\Gamma_f \subset X \times Y$; we obtain a commutative diagram

$$\begin{array}{ccccccc}
A^k(Y) & \xrightarrow{p^*} & A^k(X \times Y) & \xrightarrow{\cap \Gamma_f} & A^k(\Gamma_f) & = & A^k(X) \\
\mathrm{cl}\downarrow & & \mathrm{cl}\downarrow & & \mathrm{cl}\downarrow & & \mathrm{cl}\downarrow \\
H^{2k}(Y(\mathbf{C}); G, \mathbf{Z}(k)) & \xrightarrow{p^*} & H^{2k}(X(\mathbf{C}) \times Y(\mathbf{C}); G, \mathbf{Z}(k) & \longrightarrow & H^{2k}(\Gamma_f(\mathbf{C}); G, \mathbf{Z}(k)) & = & H^{2k}(X(\mathbf{C}); G, \mathbf{Z}(k))
\end{array}$$

where $p: X \times Y \to Y$ is the projection and

$$H^{2k}(X(\mathbf{C}) \times Y(\mathbf{C}); G, \mathbf{Z}(k)) \to H^{2k}(\Gamma_f(\mathbf{C}); G, \mathbf{Z}(k))$$

is the restriction homomorphism. It remains only to observe that the composite of the homomorphisms in the upper row of the diagram is by definition equal to the homomorphism $f^*: A^k(Y) \to A^k(X)$, while the composite in the lower row is equal to the homomorphism $f^*\colon H^{2k}(Y(\mathbf{C})\,;\,G\,,\mathbf{Z}(k)) \to H^{2k}(X(\mathbf{C})\,;\,G\,,\mathbf{Z}(k))$. This proves the proposition.

2.4. **Proof of the theorem on $\mathrm{cl}_{\mathbf{R}}$.** We carry out an induction on the dimension of X. The theorem holds for curves; assuming it holds for $\dim X \leq n$, we show it holds for $\dim X = n+1$. Thus, let Y be a subvariety of X of codimension k; we must show that the image of $\mathrm{cl}([Y])$ under the homomorphism

$$H^{2k}(X(\mathbf{C})\,;\,G\,,\mathbf{Z}(k)) \to H^{2k}(X(\mathbf{R})\,;\,G\,,\mathbf{F}_2) = \bigoplus_{q=0}^{2k} H^q(X(\mathbf{R})\,,\mathbf{F}_2)$$

lies in $H^k(X(\mathbf{R})\,,\mathbf{F}_2)$ and is equal to $\mathrm{cl}_{\mathbf{R}}([Y])$. We show first that Y can be supposed nonsingular. Consider a sequence of monoidal transformations with nonsingular centers

$$X_m \xrightarrow{\sigma_m} \cdots \xrightarrow{\sigma_2} X_1 \xrightarrow{\sigma_1} X_0 = X\,,$$

resolving the singularities of Y. Denote by $Y_i \subset X_i$ the inverse image of $Y_{i-1} \subset X_{i-1}$ under the mapping σ_i, where $Y_0 = Y$; then $Y_m \subset X_m$ is a nonsingular subvariety. We prove that if the theorem holds for the cycle $[Y_i] \in A^k(X_i)$, then it holds for the cycle $[Y_{i-1}] \in A^k(X_{i-1})$. Let Z_{i-1} be the center of the monoidal transformation $\sigma_i\colon X_i \to X_{i-1}$, and put $W_i = \sigma_i^{-1}(Z_{i-1})$; then W_i is a nonsingular hypersurface in X_i. Observe that

$$\sigma_i^*([Y_{i-1}]) = [Y_i] + \Delta\,, \tag{1}$$

where $\Delta \in A^k(X_i)$ is a cycle that lies on W_i. We verify first of all that the theorem holds for the cycle Δ. From Proposition 2.3.1 we have

$$\mathrm{cl}^{X_i}(\Delta) = \mathrm{cl}^{W_i}(\Delta) \cdot \mathrm{cl}([W_i])\,, \tag{2}$$

where $\mathrm{cl}([W_i]) = [W_i]^* \in H^2(X_i(\mathbf{C})\,, X_i(\mathbf{C})\backslash W_i(\mathbf{C})\,;\,G\,,\mathbf{Z}(1))$, which under the homomorphism

$$H^2(X_i(\mathbf{C})\,, X_i(\mathbf{C})\backslash W_i(\mathbf{C})\,;\,G\,,\mathbf{Z}(1)) \to H^2(X_i(\mathbf{R})\,, X_i(\mathbf{R})\backslash W_i(\mathbf{R})\,;\,G\,,\mathbf{F}_2)$$

goes into $\mathrm{cl}_{\mathbf{R}}([W_i])$. By the induction assumption, the cohomology class $\mathrm{cl}^{W_i}(\Delta) \in H^{2k-2}(W_i(\mathbf{C})\,;\,G\,,\mathbf{Z}(k-1))$ goes under the homomorphism

$$H^{2k-2}(W_i(\mathbf{C})\,;\,G\,,\mathbf{Z}(k-1)) \to H^{2k-2}(W_i(\mathbf{R})\,;\,G\,,\mathbf{F}_2)$$

into the class $\mathrm{cl}_{\mathbf{R}}^{W_i}(\Delta) \in H^{k-1}(W_i(\mathbf{R})\,,\mathbf{F}_2)$. Since (2) holds also for $\mathrm{cl}_{\mathbb{R}}$, we find that under the homomorphism

$$H^{2k}(X_i(\mathbf{C})\,;\,G\,,\mathbf{Z}(k)) \to H^{2k}(X_i(\mathbf{R})\,;\,G\,,\mathbf{F}_2)$$

the class $\mathrm{cl}^{X_i}(\Delta)$ goes into the class $\mathrm{cl}_{\mathbf{R}}^{X_i}(\Delta)$, as claimed. Thus, the theorem holds for the cycles $[Y_i]$ and Δ, and so by (1) it holds for the cycle $\sigma_i^*([Y_{i-1}]) \in A^k(X_i)$. Consider now the commutative diagram

$$\begin{array}{ccccc}
[Y_{i-1}] \in A^k(X_{i-1}) & \xrightarrow{\mathrm{cl}} & H^{2k}(X_{i-1}(\mathbf{C})\,;\,G\,,\mathbf{Z}(k)) & \to \\
\sigma_i^* \downarrow & & \sigma_i^* n \downarrow & \\
\sigma_i^*([Y_{i-1}]) \in A^k(X_i) & \xrightarrow{\mathrm{cl}} & H^{2k}(X_i(\mathbf{C})\,;\,G\,,\mathbf{Z}(k)) & \to
\end{array}$$

$$\begin{array}{ccccc}
\to & H^{2k}(X_{i-1}(\mathbf{R})\,;\,G\,,\mathbf{F}_2) & = \bigoplus_{q=0}^{2k} & H^q(X_{i-1}(\mathbf{R})\,,\mathbf{F}_2) \\
& \sigma_i^* \downarrow & & \sigma_i^* \downarrow \\
\to & H^{2k}(X_i(\mathbf{R})\,;\,G\,,\mathbf{F}_2) & = \bigoplus_{q=0}^{2k} & H^q(X_i(\mathbf{R})\,,\mathbf{F}_2)
\end{array}$$

The homomorphisms $\sigma_i^*\colon H^q(X_{i-1}(\mathbf{R}), \mathbf{F}_2) \to H^q(X_i(\mathbf{R}), \mathbf{F}_2)$ are inclusions; in addition, the diagram

$$\begin{array}{ccc} [Y_{i-1}] \in A^k(X_{i-1}) & \xrightarrow{\mathrm{cl}_{\mathbf{R}}} & H^k(X_{i-1}(\mathbf{R}), \mathbf{F}_2) \\ \sigma_i^* \downarrow & & \sigma_i^* \downarrow \\ \sigma_i^*([Y_{i-1}]) \in A^k(X_i) & \xrightarrow{\mathrm{cl}_{\mathbf{R}}} & H^k(X_i(\mathbf{R}), \mathbf{F}_2) \end{array}$$

is commutative; consequently, under the homomorphism

$$H^{2k}(X_{i-1}(\mathbf{C}); G, \mathbf{Z}(k)) \to H^{2k}(X_{i-1}(\mathbf{R}); G, \mathbf{F}_2)$$

the class $\mathrm{cl}([Y_{i-1}])$ maps into the class $\mathrm{cl}_{\mathbf{R}}([Y_{i-1}])$, as claimed. Thus, we can suppose the subvariety Y to be nonsingular. A monoidal transformation of X with center Y gives us a commutative diagram

$$\begin{array}{ccc} \widetilde{Y} & \subset & \widetilde{X} \\ \downarrow & & \sigma \downarrow \\ Y & \subset & X \end{array}$$

where $\widetilde{Y} = \sigma^{-1}(X)$. Consider the cycle $\sigma^*([Y]) \in A^k(\widetilde{Y})$. Since it lies on $\widetilde{Y}$, the class $\mathrm{cl}^{\widetilde{X}}(\sigma^*[Y])$ goes under the homomorphism

$$H^{2k}(\widetilde{X}(\mathbf{C}); G, \mathbf{Z}(k)) \to H^{2k}(\widetilde{X}(\mathbf{R}); G, \mathbf{F}_2)$$

into the class $\mathrm{cl}_{\mathbf{R}}^{\widetilde{X}}(\sigma^*[Y])$ (this is proved in the same way as for the cycle Δ above); consequently, the class $\mathrm{cl}^X([Y])$ goes under the homomorphism

$$H^{2k}(X(\mathbf{C}); G, \mathbf{Z}(k)) \to H^{2k}(X(\mathbf{R}); G, \mathbf{F}_2)$$

into the class $\mathrm{cl}_{\mathbf{R}}^X([Y])$, as claimed. This completes the proof of the theorem.

2.5. **General observations.** To sum up, we have constructed the following commutative diagram:

$$\begin{tikzcd} & A^k(X) \arrow[r, "\mathrm{cl}_R"] \arrow[dl, "\mathrm{cl}_{\mathbf{C}}"'] \arrow[d, "\mathrm{cl}"] & H^k(X(\mathbf{R}), F_2) \arrow[d, phantom, "\curvearrowright"] \\ H^{2k}(X(\mathbf{C}), \mathbf{Z}) & H^{2k}(X(\mathbf{C}); G, \mathbf{Z}(k)) \arrow[l, "\alpha"'] \arrow[r, "\beta'"] & H^{2k}(X(\mathbf{R}); G, \mathbf{F}_2) \end{tikzcd}$$

The homomorphism in it preserve multiplication, and it depends contravariantly on X.

Let E be a vector bundle on X. Since the Chern characteristic classes $c_k(E) \in A^k(X)$, are present, we can put

$$cw_k(E) = \mathrm{cl}(c_k(E)) \in H^{2k}(X(\mathbf{C}); G, \mathbf{Z}(k)).$$

This defines the mixed characteristic classes for real algebraic bundles, but we want to define them also in the topological situations. Various approaches exist for dealing with this problem. We shall solve it by means of the universal bundle, and also show that Grothendieck's method can be applied. For real algebraic bundles we prove that these definitions coincide with the definition in terms of algebraic cycles. These matters are taken up in the next section.

§3. The mixed characteristic classes

We first make some remarks concerning a special class of varieties.

3.1. **Varieties with cellular decomposition and Grassmannians.** Let X be a nonsingular projective real algebraic variety with cellular decomposition; i.e., X has a filtration

$$X = X_n \supset X_{n-1} \supset \cdots \supset X_0 \supset X_{-1} = \varnothing$$

by closed subschemes such that each difference $X_i \backslash X_{i-1}$ is a union of schemes U_{ij} isomorphic to affine spaces $\mathbf{A}^{n_{ij}}$. If V_{ij} is the closure of U_{ij}, then the classes $\{V_{ij}\}$ determine a basis for $A^*(X)$, and the homomorphism

$$\mathrm{cl}_{\mathbf{C}}\colon A^*(X) \to H^*(X(\mathbf{C}), \mathbf{Z})$$

is an isomorphism; this means that the composite of the mappings

$$A^k(X) \xrightarrow{\mathrm{cl}} H^{2k}(X(\mathbf{C}); G, \mathbf{Z}(k)) \xrightarrow{\alpha} H^{2k}(X(\mathbf{C}), \mathbf{Z})$$

is an isomorphism for every k. Put $\alpha^{-1} = \mathrm{cl} \circ (\mathrm{cl}_{\mathbf{C}})^{-1}$; then the homomorphism

$$\alpha^{-1}\colon H^{2k}(X(\mathbf{C}), \mathbf{Z}) \to H^{2k}(X(\mathbf{C}); G, \mathbf{Z}(k))$$

is an inclusion for every k.

As an example of a variety with cellular decomposition, we take the Grassmann variety Gr_m^n $(m \le n)$. We denote by $\mathrm{Gr}_m(\mathbf{C})$ the topological space $\bigcup_{n \ge m} \mathrm{Gr}_m^n(\mathbf{C})$, with involution $\tau\colon \mathrm{Gr}_m(\mathbf{C}) \to \mathrm{Gr}_m(\mathbf{C})$ by complex conjugation. The inclusions

$$\alpha_n^{-1}\colon H^{2k}(\mathrm{Gr}_m^n(\mathbf{C}), \mathbf{Z}) \to H^{2k}(\mathrm{Gr}_m^n(\mathbf{C}); G, \mathbf{Z}(k))$$

induce inclusions

$$\alpha^{-1}\colon H^{2k}(\mathrm{Gr}_m(\mathbf{C}), \mathbf{Z}) \to H^{2k}(\mathrm{Gr}_m(\mathbf{C}); G, \mathbf{Z}(k)),$$

since the restriction homomorphisms

$$H^{2k}(\mathrm{Gr}_m^N(\mathbf{C}), \mathbf{Z}) \to H^{2k}(\mathrm{Gr}_m^n(\mathbf{C}), \mathbf{Z}),$$
$$H^{2k}(\mathrm{Gr}_m^N(\mathbf{C}); G, \mathbf{Z}(k)) \to H^{2k}(\mathrm{Gr}_m^n(\mathbf{C}); G, \mathbf{Z}(k))$$

are isomorphisms for $N \gg n \gg k$. Let V be the universal bundle on $\mathrm{Gr}_m(\mathbf{C})$, and put

$$\mathrm{cw}_k(V) = \alpha^{-1}(c_k(V)) \in H^{2k}(\mathrm{Gr}_m(\mathbf{C}); G, \mathbf{Z}(k)). \tag{1}$$

Observe that then

$$\beta(\mathrm{cw}_k(V)) = w_k(V^\tau) \in H^k(\mathrm{Gr}_m(\mathbf{R}), \mathbf{F}_2), \tag{2}$$

where the homomorphism

$$\beta\colon H^{2k}(\mathrm{Gr}_m(\mathbf{C}); G, \mathbf{Z}(k)) \to H^k(\mathrm{Gr}_m(\mathbf{R}), \mathbf{F}_2)$$

is the composite of the homomorphisms

$$\begin{aligned} &H^{2k}(\mathrm{Gr}_m(\mathbf{C}); G, \mathbf{Z}(k)) \to H^{2k}(\mathrm{Gr}_m(\mathbf{C}); G, \mathbf{F}_2) \\ &\quad \to H^{2k}(\mathrm{Gr}_m(\mathbf{R}); G, \mathbf{F}_2) = \bigoplus_{q=0}^{2k} H^q(\mathrm{Gr}_m(\mathbf{R}), \mathbf{F}_2) \to H^k(\mathrm{Gr}_m(\mathbf{R}), \mathbf{F}_2). \end{aligned}$$

Equality (2) results from the following considerations. Let $\sigma_{1,\dots,1}$ be the Schubert cell in $\mathrm{Gr}_m^n(\mathbf{C})$, and V_n the restriction of V to $\mathrm{Gr}_m^n(\mathbf{C})$; then

$$c_k(V_n) = (-1)^k[\sigma_{1,\dots,1}]^*, \qquad w_k(V_n^\tau) = [(\sigma_{1,\dots,1})^\tau]^*,$$

whence

$$w_k(V_n^\tau) = \beta_n(\alpha_n^{-1}(c_k(V_n)))$$

by Theorem 2.2.1. If we now take the limit as $n \to +\infty$, we obtain (2).

3.2. **Definition of the mixed characteristic classes and their properties.** Let X be a compact topological space with involution $\tau: X \to X$, and $E \to X$ an m-dimensional complex bundle with real structure. Then there exists an equivariant mapping $f: X \to \mathrm{Gr}_m(\mathbf{C})$ such that $f^* V \approx E$; this is proved in the same way as the corresponding fact for bundles without real structure (see [8]). So by definition we put

$$\mathrm{cw}_k(E) = f^* \,\mathrm{cw}_k(V). \tag{1}$$

We must verify that this definition is legitimate. Let $g: X \to \mathrm{Gr}_m(\mathbf{C})$ be a second equivariant mapping such that $g^*(V) \approx E$. Then f and g are equivariantly homotopic; this is proved in the same way as the corresponding fact for bundles without real structure (see [8]). It remains only to apply Lemma 2.1.2.

Theorem 3.2.1. 1) *If E is a complex vector bundle with real structure, then*

$$\alpha(\mathrm{cw}_k(E)) = c_k(E), \qquad \beta(\mathrm{cw}_k(E)) = w_k(E^\tau).$$

2) *If $h: X \to Y$ is an equivariant mapping of topological spaces with involution, and F a complex vector bundle with real structure on Y, then*

$$\mathrm{cw}_k(h^* F) = h^* \,\mathrm{cw}_k(F).$$

3) *If E and F are complex vector bundles with real structure on X, then*

$$\mathrm{cw}(E \oplus F) = \mathrm{cw}(E) \cup \mathrm{cw}(F),$$

where $\mathrm{cw} = 1 + \mathrm{cw}_1 + \mathrm{cw}_2 + \cdots$ is the total mixed characteristic class.

4) *If E is a line bundle, then $\mathrm{cw}_1(E)$ is the characteristic class of* §1.2.

Proof. Part 1) need be verified only for the universal bundle, and for the latter it follows from equalities (1) and (2) of §3.1. Part 2) follows immediately from the definition of mixed classes in this section. To prove part 3), consider the canonical mapping

$$N : \mathrm{Gr}_m(\mathbf{C}) \times \mathrm{Gr}_n(\mathbf{C}) \to \mathrm{Gr}_{m+n}(\mathbf{C}),$$

where m and n are the dimensions of E and F. Let h be the composite of the mappings

$$X \xrightarrow{f \times g} \mathrm{Gr}_m(\mathbf{C}) \times \mathrm{Gr}_n(\mathbf{C}) \xrightarrow{N} \mathrm{Gr}_{m+n}(\mathbf{C}),$$

where f and g are mappings of X into $\mathrm{Gr}_m(\mathbf{C})$ and $\mathrm{Gr}_n(\mathbf{C})$ such that $f^* V^m \approx E$ and $g^* V^n \approx F$. Then it is easily verified that

$$h^*(V^{m+n}) \approx E \oplus F.$$

Consequently,

$$\mathrm{cw}(E \oplus F) = h^*(\mathrm{cw}(V^{m+n})) = ((f \times g)^* \circ N^*)(\mathrm{cw}(V^{m+n})).$$

Furthermore, since $N^*(c(V^{m+n})) = c(V^m) \otimes c(V^n)$, we obtain from the definition of $\mathrm{cw}(V)$ for the universal bundle (see (1) in §3.1) the corresponding equality

$$N^*(\mathrm{cw}(V^{m+n})) = \mathrm{cw}(V^m) \otimes \mathrm{cw}(V^n),$$

Hence

$$\begin{aligned}\mathrm{cw}(E \oplus F) &= (f \times g)^*(\mathrm{cw}(V^m) \otimes \mathrm{cw}(V^n)) \\ &= f^*(\mathrm{cw}(V^m)) \cup g^*(\mathrm{cw}(V^n)) = \mathrm{cw}(E) \cup \mathrm{cw}(F).\end{aligned}$$

Part 4) need be verified only for the universal line bundle on $\mathbf{P}^N(\mathbf{C})$. But on a projective space cw_1 is uniquely determined by the equality $\alpha(\mathrm{cw}_1) = c_1$, since the homomorphism $\alpha\colon H^2(\mathbf{P}^N(\mathbf{C}); G, \mathbf{Z}(1)) \to H^2(\mathbf{P}^N(\mathbf{C}), \mathbf{Z})$ is an isomorphism; and the characteristic class of §1.2 satisfies this equality. This completes the proof of the theorem.

The mixed characteristic classes can be defined also by Grothendieck's method using projectivization of the bundle; but to do this we must first compute the Galois-Grothendieck cohomology for the projectivization.

3.3. **Projectivization of a vector bundle.** In this subsection X is a compact topological space with involution $\tau : X \to X$; $\pi\colon E \to X$ is an m-dimensional complex vector bundle with real structure $\theta\colon E \to E$; and $\pi\colon P(E) \to X$ is the projectivization of $\pi\colon E \to X$. Note that the involution $\theta\colon E \to E$ induces an involution $\theta\colon P(E) \to P(E)$. We write $\mathbf{Z}_\pm$ for the constant G-sheaves on $P(E)$ with stalk $\mathbf{Z}$, on which the involution θ is equal to $\pm\,\mathrm{id}$.

Proposition 3.3.1. *The Leray spectral sequences for the Galois-Grothendieck cohomology*

$$E_{2,\pm}^{p,q} = H^p(X; G, R^q\pi_*(\mathbf{Z}_\pm)) \Rightarrow H^{p+q}(P(E); G, \mathbf{Z}_\pm)$$

are degenerate.

Proof. We can assume that X either is connected or consists of two connected components X' and X'' such that $\tau(X') = X''$; in any other case it can be represented as a disjoint union of such spaces. Since $R^q\pi_*(\mathbf{Z}) = 0$ for q odd, we have

$$d_{r,\pm}^{p,q} = 0 \quad \text{for } q \text{ odd and any } p \text{ and } r;$$
$$d_{r,\pm}^{p,q} = 0 \quad \text{for } r \text{ even and any } p \text{ and } q.$$

We prove that the remaining differentials are also zero. First we show that $d_{r,-}^{0,2} = 0$ for all r.

If $X^\tau \neq \varnothing$, put $Y = \{x\}$, where x is a fixed point in X^τ; otherwise put $Y = \{x, \tau(x)\}$, where $x \in X$. Denote by E_Y the restriction of E to Y; observe that the spectral sequence

$$E_{2,-}^{p,q}(Y) = H^p(Y; G, R^q\pi_*(\mathbf{Z}_-)) \Rightarrow H^{p+q}(P(E_Y); G, \mathbf{Z}_-)$$

is degenerate, with $E_{r,-}^{p,q}(Y) = 0$ for $p \neq 0$.

Consider the commutative diagram

$$\begin{array}{ccc}
t \in H^2(P(E); G, \mathbf{Z}_-) & \longrightarrow & H^2(P(E_Y); G, \mathbf{Z}_-) = \mathbf{Z}\cdot t_Y \\
\downarrow & & \downarrow \\
t^{0,2} \in E_{\infty,-}^{0,2} & \longrightarrow & E_{\infty,-}^{0,2}(Y) = \mathbf{Z}\cdot t_Y^{0,2}
\end{array}$$

where the horizontal homomorphisms are induced by the inclusion $P(E_Y) \subset P(E)$; also, $t = \mathrm{cw}_1(L)$ and $t_Y = \mathrm{cw}(L_Y)$, with $L = O(-1)$ and $L_Y = O_Y(-1)$ the canonical line bundles; and $t^{0,2}$ and $t_Y^{0,2}$ are the projections of t and t_Y. Since $t^{0,2}$ goes into $t_Y^{0,2} \neq \varnothing$, the homomorphism $E_{\infty,-}^{0,2} \to E_{\infty,-}^{0,2}(Y)$ is an epimorphism. On the other hand, the homomorphism $E_{2,-}^{0,2} \to E_{2,-}^{0,2}(Y)$ is an isomorphism. Therefore $d_{r,-}^{0,2} = 0$ for every r.

Now consider the sum of the spectral sequences:

$$E_2^{p,q} = H^p(X; G, R^q\pi_*(\mathbf{Z}_+ \oplus \mathbf{Z}_-)) \Rightarrow H^{p+q}(P(E); G, \mathbf{Z}_+ \oplus \mathbf{Z}_-).$$

Multiplication by $t^{0,2}$ gives an isomorphism

$$\begin{array}{ccc}
E_3^{p,0} & \xrightarrow{\sim} & E_3^{p,2} \\
\| & & \| \\
E_2^{p,0} & & E_2^{p,2}
\end{array}$$

commuting with the differentials. Since $d_3^{p,0} = 0$, it follows from the commutative diagram

$$\begin{array}{ccc} E_3^{p,0} & \xrightarrow{\sim} & E_3^{p,2} \\ {\scriptstyle d_3^{p,0}}\downarrow & & {\scriptstyle d_3^{p,2}}\downarrow \\ E_3^{p+3,-2} & \longrightarrow & E_3^{p+3,0} \end{array}$$

that $d_3^{p,2} = 0$. Considering next the isomorphisms

$$E_3^{p,2} \xrightarrow{\sim} E_3^{p,4} \xrightarrow{\sim} E_3^{p,6} \xrightarrow{\sim} \dots ,$$

we find that $d_3^{p,2q} = 0$ for all p and q. Continuing similarly, we find that $d_{2r+1}^{p,2q} = 0$ for all p, q, and r. This proves the proposition.

Corollary 3.3.2. (i) *The homomorphism* $\pi^*: H^q(X; G, \mathbf{Z}_\pm) \to H^q(P(E); G, \mathbf{Z}_\pm)$ *is an inclusion for every* q.

(ii) *The homomorphism* $t^p : H^q(X, G, \mathbf{Z}_+ \oplus \mathbf{Z}_-) \to (H^{q+2p}(P(E); G, \mathbf{Z}_+ \oplus \mathbf{Z}_-)$ *is an inclusion for* $0 \le p \le m-1$.

(iii)

$$\begin{aligned} H^q(P(E); G, \mathbf{Z}_+ \oplus \mathbf{Z}_-) = H^q(X; G, \mathbf{Z}_+ \oplus \mathbf{Z}_-) & \\ \oplus\, t \smile H^{q-2}(X; G, \mathbf{Z}_+ \oplus \mathbf{Z}_-) & \\ \oplus \dots \oplus t^{m-1} \smile H^{q-2m+2}(X; G, \mathbf{Z}_+ \oplus \mathbf{Z}_-). & \end{aligned}$$

Proof. Part (i) follows immediately from the proposition;

(ii) follows from the fact that the homomorphisms

$$t^p \smile : E_2^{q,0} \to E_2^{q,2p}$$

are isomorphisms for $0 \le p \le m-1$; and (iii) follows from (ii) and the equality

$$\bigoplus_{r+s=q} E_2^{r,s} = E_2^{q,0} \oplus t \smile E_2^{q-2,0} \oplus \dots \oplus t^{m-1} E_2^{q-2m+2,0}.$$

This proves the corollary.

3.4. **Definition of the mixed characteristic classes by Grothendieck's method.** We retain the notation of §3.3. Since there exist canonical isomorphisms $\mathbf{Z}(2q) = \mathbf{Z}_+$ and $\mathbf{Z}(2q-1) = \mathbf{Z}_-$, we have from Corollary 3.3.2 (iii) a decomposition

$$t^m = \sum_{k=1}^{m} t^{m-k} \smile a_k , \tag{1}$$

where $a_k \in H^{2k}(X; G, \mathbf{Z}(k))$.

Proposition 3.4.1.

$$\mathrm{cw}_k(E) = (-1)^{k+1} a_k , \tag{2}$$

where the a_k *are the cohomology classes in the decomposition* (1).

Proof. Consider the tautological exact sequence of complex vector bundles with real structures on $P(E)$:

$$0 \to L \to \pi^* E \to F \to 0.$$

It determines a decomposition

$$\pi^* E = L \oplus F. \tag{3}$$

Since $\operatorname{rk} F = m-1$, we have $\mathrm{cw}_k(F) = 0$ for $k \geq m$. This follows from the fact that $\mathrm{cw}_k(V^{m-1}) = 0$ for $k \geq m$, where V^{m-1} is the universal bundle. Similarly, $\mathrm{cw}_k(L) = 0$ for $k \geq 2$. From the decomposition (3) we have

$$\pi^* \mathrm{cw}(E) = (1+t) \smile \mathrm{cw}(F).$$

Consequently,

$$\mathrm{cw}(F) = (1+t)^{-1} \smile \pi^* \mathrm{cw}(E) = (1 - t + t^2 - \cdots) \smile \pi^* \mathrm{cw}(E),$$

and therefore

$$0 = \mathrm{cw}_m(F) = (-1)^m t^m + (-1)^{m-1} t^{m-1} \smile \pi^* \mathrm{cw}_1(E) + \cdots ,$$

i.e.,

$$t^m = -\sum_{k=1}^{m} (-1)^k t^{m-k} \smile \pi^* \mathrm{cw}_k(E).$$

Applying part (iii) of the corollary, we arrive at (2). This proves the proposition.

3.5. **The mixed characteristic classes of an algebraic bundle.** In this subsection X is a nonsingular projective real algebraic variety, and E a vector bundle on X. We have then the characteristic classes

$$c_k(E) \in A^k(X), \qquad c_k(E(\mathbf{C})) \in H^{2k}(X(\mathbf{C}), \mathbf{Z}), \qquad w_k(E(\mathbf{R})) \in H^k(X(\mathbf{R}), \mathbf{F}_2),$$

as well as the mixed characteristic classes $\mathrm{cw}_k(E(\mathbf{C})) \in H^{2k}(X(\mathbf{C}); G, \mathbf{Z}(k))$. Observe that $\mathrm{cl}_{\mathbf{C}}(c_k(E)) = c_k(E(\mathbf{C}))$. The next proposition generalizes this equality.

Proposition 3.5.1. *The following equalities hold*:

(1) $$\mathrm{cl}(c_k(E)) = \mathrm{cw}_K(E(\mathbf{C})), \ \mathrm{cl}_{\mathbf{R}}(c_k(E)) = w_k(E(\mathbf{R})).$$

Proof. For line bundles, these equalities have been verified in §1.3. For a bundle of rank m, consider the sequence of projectivizations

$$X_{m-1} \xrightarrow{\pi_{m-1}} X_{m-2} \xrightarrow{\pi_{m-2}} \cdots \xrightarrow{\pi_2} X_1 \xrightarrow{\pi_1} X,$$

where $X_1 = P(E)$ and $X_2 = P(F)$, with F the cokernel in the exact sequence $0 \to L \to \pi_1^* E \to F \to 0$, etc. Then for the bundle $\pi^*(E)$, $\pi = \pi_{m-1} \circ \cdots \circ \pi_2 \circ \pi_1$, we have a filtration

$$E = E_0 \supset E_1 \supset \cdots \supset E_{m-1} \supset E_m = 0,$$

where the factors $E_i/E_{i+1} = F_i$ are line bundles. Consequently, we obtain for the total characteristic classes the equalities

$$\begin{aligned} c(\pi^* E) &= c(F_0) \cdots c(F_{m-1}), \\ \mathrm{cw}(\pi^* E(\mathbf{C})) &= \mathrm{cw}(F_0(\mathbf{C}) \smile \cdots \smile \mathrm{cw}(F_{m-1}(\mathbf{C})), \\ w(\pi^* E(\mathbf{R})) &= w(F_0(\mathbf{R})) \smile \cdots \smile w(F_{m-1}(\mathbf{R})), \end{aligned}$$

and therefore

(2) $$\mathrm{cl}(c(*\pi^* E)) = \mathrm{cw}(\pi^* E(\mathbf{C})), \qquad \mathrm{cl}_{\mathbf{R}}(c(\pi^* E)) = w(\pi^* E(\mathbf{R})).$$

Since the homomorphisms

$$\pi^*\colon A^k(X) \to A^k(X_{m-1}), \qquad \pi^*\colon H^{2k}(X(\mathbf{C}); G, \mathbf{Z}(k)) \to H^{2k}(X_{m-1}(\mathbf{C}); G, \mathbf{Z}(k)),$$
$$\pi^*\colon H^k(X(\mathbf{R}), \mathbf{F}_2) \to H^k(X_{m-1}(\mathbf{R}), \mathbf{F}_2)$$

are inclusions, the equalities (2) imply (1). This proves the proposition.

§4. Applications of the mixed characteristic classes

4.1. Orientability of bundles.

Proposition 4.1.1. *Let X be a topological space with involution $\tau\colon X \to X$ such that $H^1(G, H^1(X, \mathbf{F}_2)) = 0$, and E a complex vector bundle on X with real structure. Then the congruence $c_1(E) \equiv 0 \bmod 2$ implies orientability of E^τ.*

Proof. Consider the commutative diagram

$$\begin{array}{ccccccc} H^2(X, \mathbf{Z}) & \xleftarrow{\alpha} & H^2(X; G, \mathbf{Z}(1)) & \xrightarrow{\beta} & H^1(X^\tau, \mathbf{F}_2) & & \\ \downarrow & & \downarrow & & \downarrow & & \\ H^2(X, \mathbf{F}_2) & \xleftarrow{\tilde\alpha} & H^2(X; G, \mathbf{F}_2) & \xrightarrow{\tilde\beta'} & H^2(X^\tau; G, \mathbf{F}_2) & = & \bigoplus_{q=0}^{2} H^q(X^\tau, \mathbf{F}_2). \end{array}$$

We must show that $w_1(E^\tau) = \beta(\mathrm{cw}_1(E)) = 0$. Let $\widetilde{\mathrm{cw}}_1(E)$ be the image of $\mathrm{cw}_1(E)$ under the homomorphism

$$H^2(X; G, \mathbf{Z}(1)) \to H^2(X; G, \mathbf{F}_2),$$

induced by the canonical G-sheaf homomorphism $\mathbf{Z}(1) \to \mathbf{F}_2$; then the congruence $c_1(E) \equiv 0 \bmod 2$ implies that

$$(1) \qquad \widetilde{\mathrm{cw}}_1(E) \in \operatorname{Ker}[H^2(X; G, \mathbf{F}_2) \xrightarrow{\tilde\alpha} H^2(X, \mathbf{F}_2)].$$

On the other hand, the condition $H^1(G, H^1(X, \mathbf{F}_2)) = 0$ and the spectral sequence

$$\mathrm{II}_2^{p,q} = H^p(G, H^q(X, \mathbf{F}_2)) \Rightarrow H^{p+q}(X; G, \mathbf{F}_2)$$

imply the equality

$$(2) \qquad \operatorname{Ker}[H^2(X; G, \mathbf{F}_2) \xrightarrow{\tilde\alpha} H^2(X, \mathbf{F}_2)] = F^0H^2(X; G, \mathbf{F}_2),$$

where $F^0H^2(X; G, \mathbf{F}_2)$ is the corresponding term of the filtration obtained from the spectral sequence. From (1) and (2) it follows that

$$\tilde\beta'(\widetilde{\mathrm{cw}}_1(E)) \in F^0H^2(X^\tau; G, \mathbf{F}_2) = H^0(X^\tau, \mathbf{F}_2);$$

but at the same time we have the relation

$$\tilde\beta'(\widetilde{\mathrm{cw}}_1(E)) = w_1(E^\tau) \in H^1(X^\tau, \mathbf{F}_2).$$

Therefore $w_1(E^\tau) = 0$. This proves the proposition.

Corollary 4.1.2. *Let X be a real algebraic variety such that $H^1(G, H^1(X(\mathbf{C}), \mathbf{F}_2)) = 0$, and E a vector bundle on X. Then the congruence $c_1(E(\mathbf{C})) \equiv 0 \bmod 2$ implies orientability of $E(\mathbf{R})$.*

Remark 4.1.3. A proposition close to 4.1.1 was proved in [6] by somewhat different means. Proposition 012, it is obvious, follows from Corollary 4.1.2, since in the absence of elements of order 2 in $H^2(X(\mathbf{C}), \mathbf{Z})$ we have the equality

$$\begin{aligned} \dim_{\mathbf{F}_2} H^1(G, H^1(X(\mathbf{C}), \mathbf{F}_2)) = {} & \dim_{\mathbf{F}_2} H^1(G, H^1(X(\mathbf{C}), \mathbf{Z})) \\ & + \dim_{\mathbf{F}_2} H^2(G, H^1(X(\mathbf{C}), \mathbf{Z})) \end{aligned}$$

(see [1]).

Remark 4.1.4. For curves, the condition $H^1(G, H^1(X(\mathbf{C}), \mathbf{F}_2)) = 0$ means that $X(\mathbf{R})$ consists of a single component (see [1]). The results of [1] also imply the following fact.

Assertion. *If X is a real algebraic* GM **Z**-*variety, $X(\mathbf{R})$ consists of a single component, and the group $H^2(X(\mathbf{C}), Z)$ has no elements of order 2, then*

$$H^1(G, H^1(X(\mathbf{C}), \mathbf{F}_2)) = 0.$$

Indeed, under these assumptions the set $A(\mathbf{R})$ of real points of the Albanese variety consists of a single component; and therefore

$$\dim H^1(G, H^1(X(\mathbf{C}), \mathbf{Z})) = \dim H^2(G, H^1(X(\mathbf{C}), \mathbf{Z})) = 0.$$

Remark 4.1.5. In addition to the condition on $X(\mathbf{C})$, $H^1(G, H^1(X(\mathbf{C}), \mathbf{F}_2)) = 0$, we can require that X be an M-variety; then from Proposition 06 we have

$$E(\mathbf{R}) \text{ is orientable } \Leftrightarrow c_1(E(\mathbf{C})) \equiv 0 \mod 2.$$

In particular:

Assertion. *Let $X = \bigcap H_i$ be a nonsingular complete intersection in $\mathbf{P}^N$, with* $\dim X \geq 2$ *and X an M-variety. Then*

$$X(\mathbf{R}) \textit{ is orientable } \Leftrightarrow \textit{ the number } \sum_i \deg H_i - N - 1 \textit{ is even.}$$

4.2. **Sufficient conditions for $c_k(E(\mathbf{C}))$ to be even.**

Lemma 4.2.1. *Let X be a real algebraic* GM-*variety, with the involution τ^* acting trivially on $H^q(X(\mathbf{C}), \mathbf{F}_2)$. Then the homomorphism*

$$\tilde{\beta}' \colon H^q(X(\mathbf{C}); G, \mathbf{F}_2) \to H^q(X(\mathbf{R}); G, \mathbf{F}_2)$$

is an inclusion.

Proof. Let $\dim X = n$; then for $N > 2n$ the homomorphism

$$\tilde{\beta}' \colon H^N(X(\mathbf{C}); G, \mathbf{F}_2) \to H^N(X(\mathbf{R}); G, \mathbf{F}_2)$$

is an isomorphism (see [1]). Consider the spectral sequence

$$\mathrm{II}_2^{p,q}(\cdot; G, \mathbf{F}_2) = H^p(G, H^q(\cdot, \mathbf{F}_2)) \Rightarrow H^{p+q}(\cdot; G, \mathbf{F}_2)$$

and the corresponding filtration $F^pH^{N-q}(\cdot; G, \mathbf{F}_2)$ $(q < N)$. Then

$$F^0H^{N-q}(X(\mathbf{C}); G, \mathbf{F}_2) = H^0(X(\mathbf{C}), \mathbf{F}_2) = \mathbf{F}_2,$$
$$F^0H^{N-q}(X(\mathbf{R}); G, \mathbf{F}_2) = H^0(X(\mathbf{R}), \mathbf{F}_2).$$

Denote by ω the generator of $H^0(X(\mathbf{C}), \mathbf{F}_2)$, and by $\tilde{\omega}_1, \dots, \tilde{\omega}_m$ the generators of $H^0(X(\mathbf{R}), \mathbf{F}_2)$ corresponding to the components of $X(\mathbf{R})$. Put $\tilde{\omega} = \tilde{\omega}_1 + \cdots + \tilde{\omega}_m$. Then $\tilde{\beta}'(\omega) = \tilde{\omega}$. Now consider the commutative diagram

$$\begin{array}{ccc} H^q(X(\mathbf{C}); G, \mathbf{F}_2) & \xrightarrow{\tilde{\beta}'} & H^q(X(\mathbf{R}); G, \mathbf{F}_2) \\ \big\downarrow{\scriptstyle \smile\omega} & & \big\downarrow{\scriptstyle \smile\tilde{\omega}} \\ H^N(X(\mathbf{C}); G, \mathbf{F}_2) & \xrightarrow{\tilde{\beta}'} & H^N(X(\mathbf{R}); G, \mathbf{F}_2) \end{array} \tag{1}$$

where $N = 2n + 1$, the elements ω and $\tilde{\omega}$ being regarded as belonging to the groups $F^0H^{N-q}(X(\mathbf{C}); G, \mathbf{F}_2)$ and $F^0H^{N-q}(X(\mathbf{R}); G, \mathbf{F}_2)$. As already observed, the homomorphism

$$\tilde{\beta}' \colon H^N(X(\mathbf{C}); G, \mathbf{F}_2) \to H^N(X(\mathbf{R}); G, \mathbf{F}_2)$$

is an isomorphism. The homomorphism

$$\smile \omega\colon H^q(X(\mathbf{C}); G, \mathbf{F}_2) \to H^N(X(\mathbf{C}); G, \mathbf{F}_2)$$

is mono; this follows from the lemma's hypothesis, since then the homomorphism

$$\smile \omega\colon \mathrm{II}_2^{p,q-p}(X(\mathbf{C}); G, \mathbf{F}_2) \to H^{p+N-q,q-p}(X(\mathbf{C}); G, \mathbf{F}_2)$$

is mono for all p. The homomorphism

$$\begin{array}{ccc} \smile \tilde{\omega}\colon H^q(X(\mathbf{R}); G, \mathbf{F}_2) & \to & H^N(X(\mathbf{R}); G, \mathbf{F}_2) \\ \| & & \| \\ \bigoplus_{p=0}^{q} H^p(X(\mathbf{R}), \mathbf{F}_2) & & \bigoplus_{p=0}^{N} H^p(X(\mathbf{R}), \mathbf{F}_2) \end{array}$$

is also mono. Consequently, we conclude from diagram (1) that

$$\tilde{\beta}'\colon H^q(X(\mathbf{C}); G, \mathbf{F}_2) \to H^q(X(\mathbf{R}); G, \mathbf{F}_2)$$

is mono. This proves the lemma.

Theorem 4.2.2. *Let X be a real algebraic* GM*-variety, let the involution τ^* act on $H^{2k}(X(\mathbf{C}), \mathbf{F}_2)$ trivially, and let E be a vector bundle on X. Then the equality $w_k(E(\mathbf{R})) = 0$ implies the congruence $c_k(E(\mathbf{C})) = 0 \bmod 2$.*

Proof. In the commutative diagram

$$\begin{array}{ccccc} H^{2k}(X(\mathbf{C}), \mathbf{Z}) & \xleftarrow{\alpha} & H^{2k}(X(\mathbf{C}); G, \mathbf{Z}(k)) & \xrightarrow{\beta'} & H^{2k}(X(\mathbf{R}); G, \mathbf{F}_2) = \bigoplus_{q=0}^{2k} H^q(X(\mathbf{R}), \mathbf{F}_2) \\ \downarrow & & \downarrow & & \downarrow \wr \\ H^{2k}(X(\mathbf{C}), \mathbf{F}_2) & \xleftarrow{\tilde{\alpha}} & H^{2k}(X(\mathbf{C}); G, \mathbf{F}_2) & \xrightarrow{\tilde{\beta}'} & H^{2k}(X(\mathbf{R}); G, \mathbf{F}_2) \end{array}$$

we have

$$\beta'(\mathrm{cw}_k(E(\mathbf{C}))) = w_k(E(\mathbf{R})) \in H^k(X(\mathbf{R}), \mathbf{F}_2),$$
$$\alpha(\mathrm{cw}_k(E(\mathbf{C}))) = c_k(E(\mathbf{C})) \in H^{2k}(X(\mathbf{C}), \mathbf{Z}).$$

Denote by $\widetilde{\mathrm{cw}}_k(E(\mathbf{C}))$ and $\tilde{c}_k(E(\mathbf{C}))$ the reductions mod 2 of the characteristic classes $\mathrm{cw}_k(E(\mathbf{C}))$ and $c_k(E(\mathbf{C}))$. We must prove that $\tilde{c}_k(E(\mathbf{C})) = 0$. But from the diagram it follows that

$$\tilde{c}_k(E(\mathbf{C})) = \tilde{\alpha}(\mathrm{cw}_k(E(\mathbf{C}))), \qquad \tilde{\beta}'(\widetilde{\mathrm{cw}}_k(E(\mathbf{C}))) = w_k(E(\mathbf{R})) = 0,$$

and it remains only to apply Lemma 4.2.1. This proves the theorem.

A similar proof gives

Theorem 4.2.3. *Let X be a nonsingular projective real algebraic* GM*-variety, with the involution τ^* acting trivially on $H^{2k}(X(\mathbf{C}), \mathbf{Z})$; and suppose $y \in A^k(X)$. Then the equality $\mathrm{cl}_{\mathbf{R}}(y) = 0$ implies the congruence $\mathrm{cl}_{\mathbf{C}}(y) \equiv 0 \bmod 2$.*

4.3. Sufficient conditions for the equality $w_k(E(\mathbf{R})) = 0$.

Proposition 4.3.1. *Let X be a nonsingular projective real algebraic variety such that the homomorphism $\mathrm{cl}_{\mathbf{C}}\colon A^k(X) \to H^{2k}(X(\mathbf{C}), \mathbf{Z})$ is an isomorphism. Then the congruence $\mathrm{cl}_{\mathbf{C}}(y) \equiv 0 \bmod 2$ implies the equality $\mathrm{cl}_{\mathbf{R}}(y) = 0$.*

Proof. If $\mathrm{cl}_{\mathbf{C}}(y) \equiv \bmod 2$, then $y = 2z$, $z \in A^k(X)$. Therefore $\mathrm{cl}_{\mathbf{R}}(y) = 2\,\mathrm{cl}_{\mathbf{R}}(z) = 0$, since $\mathrm{cl}_{\mathbf{R}}(z) \in H^k(X(\mathbf{R}), \mathbf{F}_2)$ has order 2. This proves the proposition.

Proposition 4.3.2. *Let X be a topological space with involution $\tau\colon X \to X$ such that*

$$\beta'(F^{2k-1}H^{2k}(X; G, \mathbf{Z}(k))) \subset F^{k-1}H^{2k}(X^\tau; G, \mathbf{F}_2),$$

and E a complex vector bundle with real structure. Then the congruence $c_k(E) \equiv 0$ mod 2 in $\mathrm{II}_\infty^{0,2k}(X; G, \mathbf{Z}(k)) \subset H^{2k}(X, \mathbf{Z})$ implies the equality $w_k(E^\tau) = 0$.

Proof. Consider the commutative diagram

$$\begin{array}{ccccc}
H^{2k}(X, \mathbf{Z}) \xleftarrow{\alpha} H^{2k}(X; G, \mathbf{Z}(k)) & \xrightarrow{\beta'} & H^{2k}(X^\tau; G, \mathbf{F}_2) & \supset & H^k(X^\tau, \mathbf{F}_2) \\
\cup & & & & \| \\
\mathrm{II}_\infty^{0,2k}(X; G, \mathbf{Z}(k)) = H^{2k}(X; G, \mathbf{Z}(k))/F^{2k-1} & \xrightarrow{\tilde{\beta}'} & H^{2k}(X^\tau; G, \mathbf{F}_2)/F^{k-1} & \supset & H^k(X^\tau, \mathbf{F}_2)
\end{array}$$

Let $\widetilde{\mathrm{cw}}_k(E)$ be the image of $\mathrm{cw}_k(E)$ under the homomorphism

$$H^{2k}(X; G, \mathbf{Z}(k)) \to H^{2k}(X; G, \mathbf{Z}(k))/F^{k-1}$$

Then $\widetilde{\mathrm{cw}}_k(E) = c_k(E) \in \mathrm{II}_\infty^{0,2k}(X; G, \mathbf{Z}(k))$, and $c_k(E) \equiv 0 \bmod 2$. Therefore $\tilde{\beta}'(\widetilde{\mathrm{cw}}_k(E)) = 0$. On the other hand, $\tilde{\beta}'(\widetilde{\mathrm{cw}}_k(E)) = \beta'(\mathrm{cw}_k(E)) = w_k(E^\tau)$. This proves the proposition.

A similar proof gives

Proposition 4.3.3. *Let X be a nonsingular projective real algebraic variety such that*

$$\beta'(F^{2k-1}H^{2k}(X(\mathbf{C}); G, \mathbf{Z}(k))) \subset F^{k-1}H^{2k}(X(\mathbf{R}); G, \mathbf{F}_2);$$

and suppose $y \in A^k(X)$. If $\mathrm{cl}_{\mathbf{C}}(y) \in \mathrm{II}_\infty^{0,2k}(X(\mathbf{C}); G, \mathbf{Z}(k))$ is even, then $\mathrm{cl}_{\mathbf{R}}(y) = 0$

Theorem 4.3.4. *Let X be a nonsingular projective real algebraic variety such that $H^{2q-1}(X(\mathbf{C}), \mathbf{Z}) = 0$ for $1 \le q \le k$, $H^{2q}(X(\mathbf{C}), \mathbf{Z})$ has no elements of order 2 for $q < k$, and the homomorphisms $\mathrm{cl}_{\mathbf{C}}\colon A^q(X) \to H^{2q}(X(\mathbf{C}), \mathbf{Z})$ are epimorphisms for $0 < q < k$. Then*

$$\beta'(F^{2k-1}H^{2k}(X(\mathbf{C}); G, \mathbf{Z}(k))) \subset F^{k-2}H^{2k}(X(\mathbf{R}); G, \mathbf{F}_2).$$

We preface the proof of this with some lemmas.

Lemma 4.3.5. *Let X be the variety of Theorem 4.3.4. Then the differentials $d_{r,\pm}^{p,q}$ in the spectral sequences*

$$\mathrm{II}_{2,\pm}^{p,q} = H^p(G, H^q(X(\mathbf{C}), \mathbf{Z}_\pm)) \Rightarrow H^{p+q}(X(\mathbf{C}); G, \mathbf{Z}_\pm)$$

are zero for $q < 2k$ and any p and r.

Proof. Since $H^{2q-1}(X(\mathbf{C}), \mathbf{Z}) = 0$ for $1 \le q \le k$, we have $d_{r,\pm}^{p,q} = 0$ for q odd ($q < 2k$). The fact that the homomorphism $\mathrm{cl}_{\mathbf{C}}\colon A^q(X) \to H^{2q}(X(\mathbf{C}), \mathbf{Z})$ is epi implies that the involution τ^* on $H^{2q}(X(\mathbf{C}), \mathbf{Z})$ is equal to $(-1)^q$; hence $H^p(G, H^{2q}(X(\mathbf{C}), \mathbf{Z}_\pm)) = 0$ ($q < k$) in all cases except the following: p even, q even, sign $+$; p even, q odd, sign $-$; p odd, q even, sign $-$; p odd, q odd, sign $+$. Consequently, $d_{r,\pm}^{p,2q} = 0$ in all cases except possibly those listed. Next, since $\mathrm{cl}_{\mathbf{C}}\colon A^q(X) \to H^{2q}(X(\mathbf{C}), \mathbf{Z})$ ($q < k$) is epi, the composite

$$A^q(X) \xrightarrow{\mathrm{cl}} H^{2q}(X(\mathbf{C}); G, \mathbf{Z}(q)) \xrightarrow{\alpha} H^{2q}(X(\mathbf{C}), \mathbf{Z})$$

is epi. This means in particular that, additionally, $d_{r,\pm}^{0,2q} = 0$ in the following cases: q even, sign $+$; q odd, sign $-$. We prove now that all the remaining differentials are also equal to zero for $q < k$. Since $d_{2,\pm}^{p,2q} = 0$, we prove first that $d_{3,\pm}^{p,2q} = 0$ ($q < k$). Let ω be the generator of the group

$$H^1(G, H^0(X(\mathbf{C}), \mathbf{Z}(1)) \subset H^1(X(\mathbf{C}); G, \mathbf{Z}(1))$$

and consider the commutative diagram

$$\begin{array}{ccc} H^0(G, H^{2q}(X(\mathbf{C}), \mathbf{Z}(q))) & \xrightarrow{d_3^{0,2q}} & H^3(G, H^{2q-2}(X(\mathbf{C}), \mathbf{Z}(q))) \\ {\scriptstyle \cup\omega^p}\downarrow & & {\scriptstyle \cup\omega^p}\downarrow \\ H^p(G, H^{2q}(X(\mathbf{C}), \mathbf{Z}(p+q))) & \xrightarrow{d_3^{p,2q}} & H^{p+3}(G, H^{2q-2}(X(\mathbf{C}), Z(p+q))) \end{array}$$

Since the multiplications of ω^p are epimorphisms, it follows from the equality $d^{0,2q}_{3,\pm} = 0$ that $d^{p,2q}_{3,\pm} = 0$ $(q < k)$. A similar proof gives $d^{p,2q}_{r,\pm} = 0$ for $r = 4, 5, \dots$. This completes the proof of the lemma.

Lemma 4.3.6. *Let X be the variety of Theorem 4.3.4, with $X(\mathbf{R}) \neq \varnothing$. Then*

$$\begin{aligned} F^{2q-1}H^{2q}(X(\mathbf{C}); G, \mathbf{Z}(q)) &= F^{2q-2}H^{2q}(X(\mathbf{C}); G, \mathbf{Z}(q)) \\ &= F^{2q-3}H^{2q}(X(\mathbf{C}); G, \mathbf{Z}(q)) \\ &= F^{2q-4}H^{2q}(X(\mathbf{C}); G, \mathbf{Z}(q)) \\ &= \Omega \smile H^{2q-4}(X(\mathbf{C}); G, \mathbf{Z}(q-2)) \qquad (q < k), \end{aligned}$$

where Ω is the generator of the group

$$H^4(G, H^0(X(\mathbf{C}), \mathbf{Z}(2))) \subset H^4(X(\mathbf{C}); G, \mathbf{Z}(2)).$$

Proof. The first three equalities follow from the equalities

$$H^{2q-1}(X(\mathbf{C}), \mathbf{Z}) = H^1(G, H^{2q-2}(X(\mathbf{C}), \mathbf{Z}(q))) = H^{2q-3}(X(\mathbf{C}), \mathbf{Z}) = 0,$$

and the last from Lemma 4.3.5 and the fact that the homomorphisms

$$\smile \Omega : H^r(G, H^s(X(\mathbf{C}), \mathbf{Z}(q-2))) \to H^{r+4}(G, H^s(X(\mathbf{C}), \mathbf{Z}(q))), \; r + s = 2q - 4,$$

are epi. This proves the lemma.

Lemma 4.3.7. *Let X be the variety of Theorem 4.3.4. Then*

$$H^{2q}(X(\mathbf{C}); G, \mathbf{Z}(q)) = F^{2q-1}H^{2q}(X(\mathbf{C}); G, \mathbf{Z}(q)) + \mathrm{cl}(A^q(X)) \qquad (q < k).$$

The proof follows from the equalities

$$F^{2q-1}H^{2q}(X(\mathbf{C}); G, \mathbf{Z}(q)) = \mathrm{Ker}[\alpha : H^{2q}(X(\mathbf{C}); G, \mathbf{Z}(q)) \to H^{2q}(X(\mathbf{C}), \mathbf{Z})],$$
$$\alpha(\mathrm{cl}(A^q(X))) = \mathrm{cl}_{\mathbf{C}}(A^q(X)) = H^{2q}(X(\mathbf{C}), \mathbf{Z}).$$

Proof of Theorem 4.3.4. If $X(\mathbf{R}) = \varnothing$, then $H^k(X(\mathbf{R}); G, \mathbf{F}_2) = 0$, and the theorem is of course valid. Suppose, then, that $X(\mathbf{R}) \neq \varnothing$. From Lemmas 4.3.5–4.3.7 we have the decomposition

$$F^{2k-1}H^{2k}(X(\mathbf{C}); G, \mathbf{Z}(k)) = \Omega \smile \mathrm{cl}(A^{k-2}(X)) + \Omega^2 \smile \mathrm{cl}(A^{k-4}(X)) + \cdots.$$

Since $\Omega \in F^0H^4(X(\mathbf{C}); G, \mathbf{Z}(2))$, we have $\beta'(\Omega) \in F^0H^4(X(\mathbf{R}); G, \mathbf{F}_2)$. On the other hand,

$$\beta'(\mathrm{cl}(A^{k-2i}(X))) = \mathrm{cl}_{\mathbf{R}}(A^{k-2i}(X)) \subset H^{k-i}(X(\mathbf{R}), \mathbf{F}_2) \subset F^{k-i}H^{2k}(X(\mathbf{R}); G, \mathbf{F}_2),$$

and therefore

$$\beta'(F^{2k-1}H^{2k}(X(\mathbf{C}); G, \mathbf{Z}(k))) \subset F^{k-2}H^{2k}(X(\mathbf{R}); G, \mathbf{F}_2).$$

This proves the theorem.

Remark 4.3.8. Suppose X is a nonsingular projective real algebraic variety such that $H^{2q-1}(X(\mathbf{C}), \mathbf{Z}) = 0$ for all q, $H^{2q}(X(\mathbf{C}), \mathbf{Z})$ has no elements of order 2 for all

q, and the homomorphisms $\mathrm{cl}_{\mathbf{C}}\colon A^q(X) \to H^{2q}(X(\mathbf{C}), \mathbf{Z})$ are isomorphisms for all q. For example, X can be a variety with cellular decomposition. Then the proof of Theorem 4.3.4 results in a canonical decomposition

$$\begin{aligned} H^{2k}&(X(\mathbf{C}); G, \mathbf{Z}(k)) \\ &= A^k(X) \oplus A^{k-2}(X)/(2) \oplus A^{k-4}(X)/(2) \oplus \cdots \\ &= H^{2k}(X(\mathbf{C}), \mathbf{Z}) \oplus H^{2k-4}(X(\mathbf{C}), \mathbf{F}_2) \oplus H^{2k-8}(X(\mathbf{C}); \mathbf{F}_2) \oplus \cdots \\ &= H^{2k}(X(\mathbf{C}), \mathbf{Z}) \oplus H^{k-2}(X(\mathbf{R}), \mathbf{F}_2) \oplus H^{k-4}(X(\mathbf{R}), \mathbf{F}_2) \oplus \cdots . \end{aligned}$$

Before stating the next proposition, we note that the image of the homomorphism $\mathrm{cl}_{\mathbf{C}} : A^k(X) \to H^{2k}(X(\mathbf{C}), \mathbf{Z})$ is contained in $H^{2k}(X(\mathbf{C}), \mathbf{Z})^{(-1)^k\tau^*}$.

Proposition 4.3.9. *Let X be a nonsingular projective real algebraic variety such that the group $H^{2k}(X(\mathbf{C}), \mathbf{Z})$ has no elements of order 2 and the image of the homomorphism $\mathrm{cl}_{\mathbf{C}}\colon A^k(X) \to H^{2k}(X(\mathbf{C}), \mathbf{Z})$ is $H^{2k}(X(\mathbf{C}), \mathbf{Z})^{(-1)^k\tau^*}$; and suppose $y \in A^k(X)$. Then if $\mathrm{cl}_{\mathbf{C}}(y)$ is even in $H^{2k}(X(\mathbf{C}), \mathbf{Z})$, necessarily $\mathrm{cl}_{\mathbf{C}}(y)$ is even in $\mathrm{II}_\infty^{0,2k}(X(\mathbf{C}); G, \mathbf{Z}(k))$.*

Proof. Since $\mathrm{cl}_{\mathbf{C}} = \alpha \circ \mathrm{cl}$ and

$$\mathrm{Im}[\alpha\colon H^{2k}(X(\mathbf{C}); G, \mathbf{Z}(k)) \to H^{2k}(X(\mathbf{C}), \mathbf{Z})] = \mathrm{II}_\infty^{0,2k}(X(\mathbf{C}); G, \mathbf{Z}(k)),$$

the hypothesis implies that

$$\mathrm{II}_\infty^{0,2k}(X(\mathbf{C}); G, \mathbf{Z}(k)) = H^{2k}(X(\mathbf{C}), \mathbf{Z})^{(-1)^k\tau^*}.$$

It remains only to observe that evenness of $\mathrm{cl}_{\mathbf{C}}(y)$ in $H^{2k}(X(\mathbf{C}), \mathbf{Z})$ implies evenness of $\mathrm{cl}_{\mathbf{C}}(y)$ in $H^{2k}(X(\mathbf{C}), \mathbf{Z})^{(-1)^k\tau^*}$ if $H^{2k}(X(\mathbf{C}), \mathbf{Z})$ has no elements of order 2. This proves the proposition.

Proposition 4.3.10. *Let X be a nonsingular projective real algebraic GM-variety such that the group $H^q(X(\mathbf{C}), \mathbf{Z})$ has no elements of order 2 for $2 \le q \le 2k$. Then*

$$\mathrm{II}_\infty^{0,2k}(X(\mathbf{C}); G, \mathbf{Z}(k)) = H^{2k}(X(\mathbf{C}), \mathbf{Z})^{(-1)^k\tau^*}.$$

Proof. We must show that in the spectral sequence

$$\mathrm{II}_2^{p,q}(X(\mathbf{C}); G, \mathbf{Z}(k)) = H^p(G, H^q(X(\mathbf{C}), \mathbf{Z}(k))) \Rightarrow H^{p+q}(X(\mathbf{C}); G, \mathbf{Z}(k))$$

the differentials $d_r^{0,2k}(X(\mathbf{C}); G, \mathbf{Z}(k))$ are all zero. Consider the spectral sequence homomorphism

$$\mathrm{II}(X(\mathbf{C}); G, \mathbf{Z}(k)) \to \mathrm{II}(X(\mathbf{C}); G, \mathbf{F}_2)$$

given by the G-sheaf homomorphism $\mathbf{Z}(k) \to \mathbf{F}_2$. Since the groups $H^q(X(\mathbf{C}), \mathbf{Z})$ $(0 \le q \le 2k)$ have no elements of order 2, the homomorphism

$$H^p(G, H^q(X(\mathbf{C}), \mathbf{Z}(k))) \to H^p(G, H^q(X(\mathbf{C}), \mathbf{F}_2))$$

are mono for $p > 0$, $0 \le q < 2k$ (see [1]). Hence the equality $d_2^{0,2k}(X(\mathbf{C}); G, \mathbf{F}_2) = 0$ implies the equality $d_2^{0,2k}(X(\mathbf{C}); G, \mathbf{Z}(k)) = 0$. Continuing this argument for $r = 3, 4, \ldots$, we find that $d_r^{0,2k}(X(\mathbf{C}); G, \mathbf{Z}(k)) = 0$ for all r. This proves the proposition.

4.4. **A theorem on the Wu classes.** In this subsection we study the connection between the classes $v_k(X(\mathbf{R}))$ and $v_{2k}(X(\mathbf{C}))$. Let us first recall some definitions. If $w = 1 + w_1 + w_2 + \cdots$ is the total Stiefel-Whitney class of a variety, then

$$v = 1 + v_1 + v_2 + \cdots = \mathrm{Sq}^{-1}(w)$$

is the total Wu class of the variety. Here Sq^{-1} is the inverse operator to the operator

$$\mathrm{Sq} = 1 + \mathrm{Sq}^1 + \mathrm{Sq}^2 + \cdots : H^*(\cdot, \mathbf{F}_2) \to H^*(\cdot, \mathbf{F}_2).$$

We need also one further piece of notation. Let (X, τ) be a real topological space. Then there exists a canonical decomposition

$$H^{2q}(X^\tau; G, \mathbf{F}_2) = \bigoplus_{k=0}^{2q} H^k(X^\tau, \mathbf{F}_2).$$

This gives a canonical inclusion

$$H^q(X^\tau, \mathbf{F}_2) \hookrightarrow H^{2q}(X^\tau; G, \mathbf{F}_2).$$

This inclusion we denote by i^*, as also the inclusion of direct sums

$$H^*(X^\tau, \mathbf{F}_2) = \bigoplus_q H^q(X^\tau, \mathbf{F}_2) \hookrightarrow \bigoplus_q H^{2q}(X^\tau; G, \mathbf{F}_2) = H^{\mathrm{even}}(X^\tau; G, \mathbf{F}_2).$$

Next, we make the following auxiliary observation.

Lemma 4.4.1. *Let (X, τ) be a real topological space, where X is a finite* CW-*complex, and let (E, θ) be a Real vector bundle of rank m on X. Then there exists an element*

$$v(E, \theta) \in H^{\mathrm{even}}(X; G, \mathbf{F}_2)$$

such that

$$\tilde{\alpha}(v(E, \theta)) = \mathrm{Sq}^{-1}(\tilde{c}(E)), \qquad \tilde{\beta}'(v(E, \theta)) = i^*(\mathrm{Sq}^{-1}(w(E^\theta))).$$

Proof. Suppose first that E is a line bundle. Then $\widetilde{\mathrm{cw}}(E, \theta) = 1 + \widetilde{\mathrm{cw}}_1$, where $\widetilde{\mathrm{cw}}_1 = \widetilde{\mathrm{cw}}_1(E, \theta)$, and it suffices to put

$$v(E, \theta) = 1 + \widetilde{\mathrm{cw}}_1 + \widetilde{\mathrm{cw}}_1^2 + \widetilde{\mathrm{cw}}_1^4 + \widetilde{\mathrm{cw}}_1^8 + \cdots.$$

Indeed, we have then on the one hand that

$$\tilde{\alpha}(v(E, \theta)) = 1 + \tilde{c}_1 + \tilde{c}_1^2 + \tilde{c}_1^4 + \tilde{c}_1^8 + \cdots,$$

where $\tilde{c}_1 = \tilde{c}_1(E) = w_2(E)$, and

$$\tilde{\beta}'(v(E, \theta)) = 1 + w_1 + w_1^2 + w_1^4 + w_1^8 + \cdots,$$

where $w_1 = w_1(E^\theta)$, $w_1^{2^k} \in H^{2^k}(X^\tau, \mathbf{F}_2) \subset H^{2^{k+1}}(X^\tau; G, \mathbf{F}_2)$; and on the other hand

$$\mathrm{Sq}(1 + \tilde{c}_1 + \tilde{c}_1^2 + \tilde{c}_1^4 + \tilde{c}_1^8 + \cdots) = 1 + \tilde{c}_1 = \tilde{c}(E),$$
$$\mathrm{Sq}(1 + w_1 + w_1^2 + w_1^4 + w_1^8 + \cdots) = 1 + w_1 = w(E^\theta).$$

In the general case, consider a mapping $\pi: P \to X$ such that $\pi^*(E, \theta)$ splits into a sum of Real line bundles:

$$\pi^* E = E_1 \oplus \cdots \oplus E_m;$$

for example, take for $\pi: P \to X$ a composite of projectivizations. Then, putting

$$v(\pi^* E, \theta) = v(E_1, \theta) \cup \cdots \cup v(E_m, \theta),$$

we show that

$$v(\pi^* E, \theta) \in \pi^*(H^{\mathrm{even}}(X; G, \mathbf{F}_2))$$

this can be verified in the following fashion. From the definitions of $v(\pi^* E, \theta)$ and $v(E_1, \theta), \ldots, v(E_m, \theta)$ it follows that $v_{2q}(\pi^* E, \theta) \in H^{2q}(P; G, \mathbf{F}_2)$ is a symmetric polynomial in $\widetilde{\mathrm{cw}}_1(E_1, \theta), \ldots, \widetilde{\mathrm{cw}}_1(E_m, \theta)$. But the mixed characteristic classes $\widetilde{\mathrm{cw}}_k(\pi^*E, \theta) = \pi^*(\widetilde{\mathrm{cw}}_k(E, \theta))$ are elementary symmetric polynomials in $\widetilde{\mathrm{cw}}_1(E_1, \theta), \ldots \widetilde{\mathrm{cw}}_1(E_m, \theta)$. This means that $v_{2q}(\pi^*E, \theta)$ is a polynomial in $\widetilde{\mathrm{cw}}_1(\pi^*E, \theta), \ldots, \widetilde{\mathrm{cw}}_q(\pi^*E, \theta)$, and consequently $v_{2q}(\pi^*E, \theta)$ is contained in $\pi^*(H^{\mathrm{even}}(X; G, \mathbf{F}_2))$. Finally, observe that the homomorphism

$$\pi^*: H^{\mathrm{even}}(X; G, \mathbf{F}_2) \to H^{\mathrm{even}}(P; G, \mathbf{F}_2)$$

is an inclusion (see §3.3). This proves the lemma.

Remark 4.4.2. In proving the lemma we have found that the class $v_{2k}(E, \theta)$, as constructed, is a polynomial in $\widetilde{\mathrm{cw}}_1(E, \theta), \ldots, \widetilde{\mathrm{cw}}_k(E, \theta)$. So for a real algebraic bundle $E \to X$ the class $v_{2k}(E(\mathbf{C}), \theta)$ is algebraic; i.e., it is equal to $\mathrm{cl}(y) \bmod 2$, where $y \in A^k(X)$.

Remark 4.4.3. One can define a cohomology operation $\mathrm{Sq}^i: H^q(X; G, \mathbf{F}_2) \to H^{q+1}(X; G, \mathbf{F}_2)$, and for $v(E, \theta)$ take the cohomology class $\mathrm{Sq}^{-1}(\widetilde{\mathrm{cw}}(E, \theta))$. We have not pursued this path in proving the lemma, since it would involve a much more technical verification.

Theorem 4.4.4. *Let X be a nonsingular projective real algebraic* GM*-variety, with τ^* acting trivially on $H^{2k}(X(\mathbf{C}), \mathbf{F}_2)$. Then $v_k(X(\mathbf{R})) = 0$ implies $v_{2k}(X(\mathbf{C})) = 0$.*

The proof follows from Lemma 4.4.1, Remark 4.4.2, and Theorem 4.2.3.

4.5. **Corollaries of the aggregate theorems.** We apply the results first to complete intersections.

Corollary 4.5.1. *Let X be an n-dimensional nonsingular complete intersection in $\mathbb{P}^N$, and E a vector bundle on X. Then:*

1) *Suppose $2k < n$. If $c_k(E(\mathbf{C}))$ is even, then $w_k(E(\mathbf{R})) = 0$, and if $v_{2k}(X(\mathbf{C})) = 0$, then $v_k(X(\mathbf{R})) = 0$.*

2) *Suppose X is a* GM*-variety and $2k = n$. If $c_k(E(\mathbf{C}))$ is even, then $w_k(E(\mathbf{R})) = 0$ and if $v_{2k}(X(\mathbf{C})) = 0$, then $v_k(X(\mathbf{R})) = 0$.*

3) *Suppose X is a* GM*-variety and $2k < n$. If $w_k(E(\mathbf{R})) = 0$, then $c_k(E(\mathbf{C}))$ is even, and if $v_k(X(\mathbf{R})) = 0$, then $v_{2k}(X(\mathbf{C})) = 0$.*

4) *Suppose X is an M-variety. For every k, if $w_k(E(\mathbf{R})) = 0$, then $c_k(E(\mathbf{C}))$ is even, and if $v_k(X(\mathbf{R})) = 0$, then $v_{2k}(X(\mathbf{C})) = 0$.*

Proof. The first assertion follows from 4.3.3, 4.3.4, and 4.3.9. The second follows from 4.3.3, 4.3.4 and 4.3.10. The last two both follow from 4.2.2 and 4.4.4. This proves the corollary.

Before stating the second corollary, we make an observation concerning M-varieties.

Let X be an arbitrary n-dimensional nonsingular projective real algebraic variety. The set of real points $X(\mathbf{R})$ determines a homology class $[X(\mathbf{R})] \in H_n(X(\mathbf{C}), \mathbf{F}_2)$.

Lemma 4.5.2. *Let X be an M-variety. Then in $H^n(X(\mathbf{C}), \mathbf{F}_2)$ we have the equality*

$$[X(\mathbf{R})]^* = v_n(X(\mathbf{C})).$$

Proof. For brevity, denote the class $[X(\mathbf{R})]$ by a, and consider the quadratic form on $H_n(X(\mathbf{C}), \mathbf{F}_2)$ given by $A(x) = x \cdot \tau_*(x)$. Then a is the characteristic element of this quadratic form; i.e., $A(x) = a \cdot x$ (see [9]). Since X is an M-variety, we have $\tau_*(x) = x$, so that $A(x) = x^2$. But the characteristic element of this form is the homology class dual to the Wu class $v_n(X(\mathbf{C}))$. This proves the lemma.

Corollary 4.5.3. *Let X be a $2k$-dimensional nonsingular projective real algebraic M-variety. Then*:

1) *If $v_k(X(\mathbf{R})) = 0$, then $[X(\mathbf{R})] = 0$, and $\chi(X(\mathbf{R})) \equiv 0 \bmod 8$.*
2) *If X is a surface, and $X(\mathbf{R})$ an orientable surface, then $\chi(X(\mathbf{R})) \equiv 0 \bmod 16$.*
3) *If X is a complete intersection, then $[X(\mathbf{R})] = 0$ implies $v_k(X(\mathbf{R})) = 0$.*

Proof. If $v_k(X(\mathbf{R})) = 0$, it follows from Theorem 4.4.4 that $v_{2k}(X(\mathbf{C})) = 0$, and therefore from Lemma 4.5.2 that $[X(\mathbf{R})] = 0$. On the other hand, the equality $v_{2k}(X(\mathbf{C})) = 0$ implies that the quadratic form on $H_{2k}(X(\mathbf{C}), \mathbf{Z})/\mathrm{Tors}$ is even, so we have the congruence $\sigma(X(\mathbf{C})) \equiv 0 \bmod 8$. But we have in addition the congruence $\chi(X(\mathbf{R})) \equiv \sigma(X(\mathbf{C})) \bmod 16$, by Rokhlin's theorem in [10].

This proves the first part of the corollary. As for the second, we need only observe that for a surface the equality $v_2(X(\mathbf{C})) = 0$ implies the congruence $\sigma(X(\mathbf{C})) \equiv 0 \bmod 16$, by Rokhlin's theorem in [11]. Finally, the last part follows from 4.5.2 and 4.5.1. This completes the proof of the corollary.

BIBLIOGRAPHY

1. V. A. Krasnov, *Harnack-Thom inequalities for mappings of real algebraic varieties*, Izv. Akad. Nauk SSSR Ser. Mat. **47** (1983), 268–297; English transl. in Math. USSR Izv. **22** (1984).
2. V. A. Krasnov, *On homology classes determined by real points of a real algebraic variety*, Izv. Akad. Nauk. SSSR Ser. Mat. *55* (1991), 282–302; English transl. in Math. USSR Izv. **38** (1991).
3. Alexandre Grothendieck, *Sur quelques points d'algèbre homologique*, Tôhoku Math. J. (2) **9** (1957), 119–221.
4. M. F. Atiyah, *K-theory and reality*, Quart. J. Math. Oxford Ser. (2) **17** (1966), 367–386.
5. Andrew John Sommese, *Real algebraic spaces*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **4** (1977), 599–612.
6. V. A. Krasnov, *Orientability of real algebraic varieties*, Constructive Algebraic Geometry, Sb. Nauchn. Trudov Yaroslav. Gos. Ped. Inst. No. 194, (1981), 46–57. (Russian)
7. S. Łojasiewicz, *Triangulation of semi-analytic sets*, Ann. Scuola Norm. Sup. Pisa (3) **18** (1964), 449–474.
8. Dale Husemoller, *Fibre bundles*, McGraw-Hill, New York, 1966.
9. V. I. Arnol'd, *The situation of ovals of real plane algebraic curves, the involutions of four-dimensional smooth manifolds, and the arithmetic of integral quadratic forms*, Funktsional Anal. i Prilozhen. **5** (1971), no. 3, 1–9; English transl. in Functional Anal. Appl. **5** (1971).
10. V. A. Rokhlin, *Congruence modulo* 16 *in Hilbert's sixteenth problem*, I, II, Funktsional. Anal. i Prilozhen. **6** (1972), no. 4, 58–64, **7** (1973), no. 2, 91–92; English transl. in Functional Anal. Appl. **6** (1972), **7** (1973).
11. V. A. Rokhlin, *New results in the theory of four-dimensional manifolds*, Dokl. Akad. Nauk SSSR **84** (1952), 221–224. (Russian)
12. Armand Borel and André Haefliger, *La classe d'homologie fondamentale d'un espace analytique*, Bull. Soc. Math. France **89** (1961), 461–513.
13. William Fulton, *Intersection theory*, Springer-Verlag, Berlin, 1984.

Received 19/MAR/90

Translated by J. A. ZILBER