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1992 Math. USSR Izv. 38 277

(http://iopscience.iop.org/0025-5726/38/2/A03)

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Сер. Матем. Том 55 (1991), № 2

Math. USSR Izvestiya
Vol. 38 (1992), No. 2

ON HOMOLOGY CLASSES DETERMINED BY REAL POINTS OF A REAL ALGEBRAIC VARIETY

UDC 513.6+517.6

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ABSTRACT. For a nonsingular n-dimensional real projective algebraic variety X the set $X(\mathbf{R})$ of its real points is the union of connected components $X_1 \cup \cdots \cup X_m$. Those components give rise to homology classes $[X_1], \dots, [X_m] \in H_n(X(\mathbf{C}, \mathbf{F}_2)$. In this paper a bound on the number of relations between those homology classes is obtained.

INTRODUCTION

In this paper we obtain a bound on the number of relations between the homology classes which arise from the connected components of the set of real points of a real algebraic variety, where the set of real points is considered as a subset of the set of complex points. To give a more precise statement of the problem we introduce the following notation. Let X be an n-dimensional real projective algebraic variety which we assume to be nonsingular and complete. Then $X(\mathbf{R})$ will denote the set of real points and $X(\mathbf{C})$ the set of complex points of X. The set $X(\mathbf{R})$ may consist of several connected components; we denote them $X_1, \dots, X_m$. They give rise to homology classes $[X_1], \dots, [X_m] \in H_n(X(\mathbf{C}), \mathbf{F}_2)$. Let $\kappa(X)$ denote the number of relations between those classes. The main goal of this paper is to obtain bounds on $\kappa(X)$.

First we should state the results already known. For curves the result is classical: $\kappa(X) \leq 1$. The same holds for surfaces which satisfy the condition $H_1(X(\mathbf{C}), \mathbf{F}_2) = 0$. This is a more recent result, the first published proof of which is due to V. M. Kharlamov. We should remark that this proof already requires a theory, whereas in the case of a curve the bound is obtained by elementary arguments. This theory rests on the Smith exact sequence. However, to use this sequence for this problem in the general case would be, in my opinion, rather difficult. With the aid of Galois-Grothendieck cohomology I obtained in [1] the following general bound:

$$(*) \qquad \kappa(X) \leq \sum_{q>0} \dim_{\mathbf{F}_2} H^1(G, H^{n+q}(X(\mathbf{C}), \mathbf{F}_2)),$$

where $G = \{e, \tau\}$ is the group of order two generated by the complex conjugation $\tau \colon X(\mathbf{C}) \to X(\mathbf{C})$. Notice that the known bounds for curves and surfaces can be deduced from $(*)$; but, as we will see later, for higher dimensions this formula is clearly imprecise.

Before stating the new results we introduce some more notation. Notice that some components of $X(\mathbf{R})$ may be nonorientable. Let $X_1, \dots, X_s$ be the orientable components of $X(\mathbf{R})$. By $\kappa_+(X)$ we shall denote the number of relations between the

1991 *Mathematics Subject Classification*. Primary 14J99, 55N99; Secondary 14G27.

homology classes $[X_1], \ldots, [X_s] \in H_n(X(\mathbf{C}), \mathbf{F}_2)$, and by $\kappa_-(X)$ the corresponding number for the nonorientable components.

We now state the new results. In the first theorem (which is only partially new) we deal with surfaces. More precisely, we have

Theorem. *Let X be a surface. Then the following assertions are true:*

(i) *If the homology group $H_1(X(\mathbf{C}), \mathbf{Z})$ is free, then*

$$\kappa(X) \leq 1 + q(X),$$

where $q(X)$ is the irregularity of the surface.

(ii) *If* $\deg X$ *is odd and $H_1(X(\mathbf{C}), \mathbf{F}_2) = 0$, then $\kappa(X) = 0$.*

(iii) *If* $\deg X$ *is odd and $H_1(X(\mathbf{C}), \mathbf{Z})$ is free, then $\kappa(X) \leq q(X)$.*

(iv) *If $H_1(X(\mathbf{C}), \mathbf{Z}) = 0$, then $\kappa_-(X) = 0$.*

In the next theorem we consider complete intersections of arbitrary dimension.

Theorem. *Let X be a complete intersection. Then the following assertions are true:*

(i) $\kappa(X) \leq 1$.

(ii) *If* $\deg X$ *is odd, then $\kappa_-(X) = 0$.*

(iii) *If* $\deg X$ *is odd and* $\dim X$ *is even, then $\kappa(X) = 0$.*

Let

$$L\colon H^q(X(\mathbf{C}), \mathbf{F}_2) \to H^{q+2}(X(\mathbf{C}), \mathbf{F}_2)$$

denote the Lefschetz operator. We then have

Theorem. *Let X be a variety such that the homomorphisms*

$$L^q\colon H^{n-q}(X(\mathbf{C}), \mathbf{F}_2) \to H^{n+q}(X(\mathbf{C}), \mathbf{F}_2)$$

are isomorphisms for $0 \leq q \leq n$. Then

$$\kappa(X) \leq \dim H^{n+1}(X(\mathbf{C}), \mathbf{F}_2) + \dim H^{n+2}(X(\mathbf{C}), \mathbf{F}_2).$$

The Lefschetz operator is given by multiplication by the cohomology class corresponding to a hyperplane section. We now consider multiplication by the cohomology class corresponding to the canonical divisor. Let

$$K\colon H^q(X(\mathbf{C}), \mathbf{F}_2) \to H^{q+2}(X(\mathbf{C}), \mathbf{F}_2)$$

by the resulting homomorphism. We now have

Theorem. *Let X be a* GM*-variety such that the homomorphisms*

$$K^q\colon H^{n-q}(X(\mathbf{C}), \mathbf{F}_2) \to H^{n+q}(X(\mathbf{C}), \mathbf{F}_2)$$

are isomorphisms for $0 \leq q \leq n$. Then $\kappa_+(X) = 0$.

In the next two theorems we use the Hodge structure on $X(\mathbf{C})$.

Theorem. *Suppose that the homology groups $H_q(X(\mathbf{C}), \mathbf{Z})$ $(0 < q < n)$ are free and $k = [(n-1)/2]$. Then*

$$\kappa(X) \leq \sum_{\substack{0<r+s<n \\ r>s}} h^{r,s}(X(\mathbf{C})) + \sum_{i \geq 0} h^{k-i,k-i}(X(\mathbf{C})).$$

A more precise bound, although under some additional assumptions, is given by the following

Theorem. *Let Y be a hyperplane section of X and suppose that the homology groups $H_q(X(\mathbf{C}) - Y(\mathbf{C}), \mathbf{Z})$ $(0 < q < n)$ are free. Then*

$$\kappa(X) \leq \sum_{\substack{n-2\leq r+s\leq n-1 \\ r>s}} h^{r,s}(X(\mathbf{C})) + h^{k,k}(X(\mathbf{C})).$$

With this we conclude the list of the main results, but we should remark that this paper contains some other new results. Finally, we remark that we use a continuous numbering of theorems, propositions, corollaries, and remarks.

§1. General bounds

In this section X will denote an n-dimensional real algebraic variety which may be singular and noncomplete. The set $X(\mathbf{C})$ of complex points is an n-dimensional complex-analytic space, and the set $X(\mathbf{R})$ of real points is a real-analytic space which is not necessarily connected. The dimensions of its connected components do not exceed n. Let $\tau\colon X(\mathbf{C}) \to X(\mathbf{C})$ denote the complex conjugation and $G = \{e, \tau\}$ the group of order two generated by the involution τ. Let $\kappa(X)$ denote the kernel dimension of the homomorphism

$$H_n(X(\mathbf{R}), \mathbf{F}_2) \to H_n(X(\mathbf{C}), \mathbf{F}_2),$$

induced by the embedding $X(\mathbf{R}) \hookrightarrow X(\mathbf{C})$.

1.1. **Old bounds.** The following bounds were obtained, among other results, in [1]:

$$\kappa(X) \leq \sum_{q>0} \dim_{\mathbf{F}_2} H^1(G, H^{n+q}(X(\mathbf{C}), \mathbf{F}_2)), \tag{1.1.1}$$

$$\kappa(X) \leq \sum_{q>0} \dim_{\mathbf{F}_2} H^{\varepsilon(q)}(G, H^{n+q}(X(\mathbf{C}), \mathbf{Z})), \tag{1.1.2}$$

where $\varepsilon(q)$ equals 1 when q is odd and 2 when q is even (see [1], Proposition 2.6, 3.9).

In this section we shall generalize (1.1.1) and (1.1.2). Before doing that we should remark, however, that those bounds have relative versions. Let Y be a subvariety of X and let $\kappa(X, Y)$ denote the kernel dimension of the homomorphism

$$H_n(X(\mathbf{R}), Y(\mathbf{R}); \mathbf{F}_2) \to H_n(X(\mathbf{C}), Y(\mathbf{C}); \mathbf{F}_2).$$

We now have the following bounds:

$$\kappa(X, Y) \leq \sum_{q>0} \dim_{\mathbf{F}_2} H^1(G, H^{n+q}(X(\mathbf{C}), Y(\mathbf{C}); \mathbf{F}_2)), \tag{1.1.3}$$

$$\kappa(X, Y) \leq \sum_{q>0} \dim_{\mathbf{F}_2} H^{\varepsilon(q)}(G, H^{n+q}(X(\mathbf{C}), Y(\mathbf{C}); \mathbf{Z})). \tag{1.1.4}$$

Although these results cannot be found in [1], their proof is identical to that of (1.1.1) and (1.1.2). Notice also that, examining the homomorphism

$$\begin{array}{ccccccccc}
0 & \longrightarrow & H_n(X(\mathbf{R})) & \longrightarrow & H_n(X(\mathbf{R}), Y(\mathbf{R})) & \longrightarrow & H_{n-1}(Y(\mathbf{R})) & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & H_n(X(\mathbf{C})) & \longrightarrow & H_n(X(\mathbf{C}), Y(\mathbf{C})) & \longrightarrow & H_{n-1}(Y(\mathbf{C})) & \longrightarrow & \cdots
\end{array}$$

between the long homology exact sequences of $(X(\mathbf{R}), Y(\mathbf{R}))$ and $(X(\mathbf{C}), Y(\mathbf{C}))$, one has that

$$\kappa(X) \leq \kappa(X, Y). \tag{1.1.5}$$

1.2. **New bounds.** Let $f\colon X \to Z$ be a map of real algebraic varieties. Consider the homomorphism

$$H_n(X(\mathbf{R}), \mathbf{F}_2) \to H_n(X(\mathbf{C}), \mathbf{F}_2) \oplus H_n(Z(\mathbf{R}), \mathbf{F}_2), \tag{1.2.1}$$

which is the direct sum of the homomorphisms

$$H_n(X(\mathbf{R}), \mathbf{F}_2) \to H_n(X(\mathbf{C}), \mathbf{F}_2), \tag{1.2.2}$$

and

$$H_n(X(\mathbf{R}), \mathbf{F}_2) \to H_n(Z(\mathbf{R}), \mathbf{F}_2), \tag{1.2.3}$$

induced by the maps $X(\mathbf{R}) \hookrightarrow X(\mathbf{C})$ and $X(\mathbf{R}) \to Z(\mathbf{R})$. Let $\kappa(f)$ denote the kernel dimension of the homomorphism (1.2.1) and $\delta(f)$ the image dimension of the homomorphism (1.2.3). We then have that

$$\kappa(X) - \delta(f) \le \kappa(f) \le \kappa(X). \tag{1.2.4}$$

In some cases one can find a bound on $\delta(f)$. For example, if Z is a projective space, then $\delta(f) \le 1$. Thus if we find a bound on $\kappa(f)$, then (1.2.4) would yield a bound on $\kappa(X)$. Before we formulate the corresponding results we recall some facts about Galois-Grothendieck cohomology groups $H^N(X(\mathbf{C}), G; \mathbf{F}_2)$ and the filtration $\{F^k(X(\mathbf{C}), G; \mathbf{F}_2)\}$ of them given by the spectral sequence

$$I_2^{p,q}(X(\mathbf{C}), G; \mathbf{F}_2) = H^p(G, H^q(X(\mathbf{C}); \mathbf{F}_2)) \Rightarrow H^{p+q}(X(\mathbf{C}), G; \mathbf{F}_2) \tag{1.2.5}$$

(see [1]). All those facts are contained in the following theorem.

Theorem 1. *The following assertions are true*:

(i) *If* $N > 2n$, *then the homomorphism*

$$H^N(X(\mathbf{C}), G; \mathbf{F}_2) \to H^N(X(\mathbf{R}), G; \mathbf{F}_2),$$

induced by the embedding $X(\mathbf{R}) \hookrightarrow X(\mathbf{C})$ *is an isomorphism.*

(ii) *There is a canonical isomorphism*

$$H^N(X(\mathbf{R}), G; \mathbf{F}_2) = \bigoplus_{q=0}^{N} H^q(X(\mathbf{R}), \mathbf{F}_2).$$

(iii) *If* $k \le N$, *then there are canonical isomorphisms*

$$F^k H^N(X(\mathbf{R}), G; \mathbf{F}_2) = \bigoplus_{q=0}^{k} H^q(X(\mathbf{R}), \mathbf{F}_2).$$

We recall also that the variety X is called a GM-*variety* if the spectral sequence $I(X(\mathbf{C}), G; \mathbf{F}_2)$ degenerates.

We now formulate a new result.

Theorem 2. *Suppose that* Z *is a* GM-*variety. Then*

$$\kappa(f) \le \sum_{q>0} \dim_{\mathbf{F}_2} \operatorname{Coker}[H^1(G, H^{n+q}(Z(\mathbf{C}), \mathbf{F}_2)) \to H^1(G, H^{n+q}(X(\mathbf{C}), \mathbf{F}_2))].$$

Proof. Let $m = \max\{\dim X, \dim Z\}$. We will make use of the Galois-Grothendieck cohomology group $H^{2m+1}(\cdot, G; \mathbf{F}_2)$ and the filtration $\{F^k(\cdot, G; \mathbf{F}_2)\}$ given by spectral sequence (1.2.5). First we remark that

$$\begin{aligned} &\operatorname{Im}[I_\infty^{2m-n+1,n}(X(\mathbf{C})) \oplus I_\infty^{2m-n+1,n}(Z(\mathbf{R})) \to I_\infty^{2m-n+1,n}(X(\mathbf{R}))] \\ &\quad = \operatorname{Im}[I_\infty^{2m-n+1,n}(X(\mathbf{C})) \oplus H^n(Z(\mathbf{R})) \to H^n(X(\mathbf{R}))], \end{aligned} \tag{1.2.6}$$

since, by virtue of Theorem 1,

$$I_\infty^{2m-n+1,n}(Z(\mathbf{R})) = H^n(Z(\mathbf{R})), \qquad I_\infty^{2m-n+1,n}(X(\mathbf{R})) = H^n(X(\mathbf{R})).$$

We also have

$$\begin{aligned} &\operatorname{Im}[I_\infty^{2m-n+1,n}(X(\mathbf{C})) \oplus I_\infty^{2m-n+1,n}(Z(\mathbf{R})) \to I_\infty^{2m-n+1,n}(X(\mathbf{R}))] \\ &\quad = \operatorname{Im}[F^n(X(\mathbf{C}))/F^{n-1}(X(\mathbf{C})) \oplus F^n(Z(\mathbf{R}))/F^{n-1}(Z(\mathbf{R})) \\ &\qquad\qquad \to F^n(X(\mathbf{R}))/F^{n-1}(X(\mathbf{R}))] \\ &\quad = \operatorname{Im}[F^n(X(\mathbf{C})) \oplus F^n(Z(\mathbf{R})) \to F^n(X(\mathbf{R}))/F^{n-1}(X(\mathbf{R}))]. \end{aligned} \tag{1.2.7}$$

Since

$$\begin{aligned} \kappa(f) &= \dim \operatorname{Ker}[H_n(X(\mathbf{R})) \to H_n(X(\mathbf{C})) \oplus H_n(Z(\mathbf{R}))] \\ &= \dim \operatorname{Coker}[H^n(X(\mathbf{C})) \oplus H^n(Z(\mathbf{R})) \to H^n(X(\mathbf{R}))], \end{aligned}$$

it follows from (1.2.6) and (1.2.7) that

$$\kappa(f) \le \dim \operatorname{Coker}[F^n(X(\mathbf{C})) \oplus F^n(Z(\mathbf{R})) \to F^n(X(\mathbf{R}))/F^{n-1}(X(\mathbf{R}))]. \tag{1.2.8}$$

Consider the commutative diagram

$$\begin{array}{ccc} F^n(X(\mathbf{C})) \oplus F^{2m+1}(Z(\mathbf{C})) & \longrightarrow & F^n(X(\mathbf{C})) \oplus F^{2m+1}(Z(\mathbf{R})) \\ \downarrow & & \downarrow \\ F^{2m+1}(X(\mathbf{C})) & \longrightarrow & F^{2m+1}(X(\mathbf{R}))/F^{n-1}(X(\mathbf{R})), \end{array} \tag{1.2.9}$$

where the upper homomorphism equals the sum of the homomorphisms

$$\mathrm{id}\colon F^n(X(\mathbf{C})) \to F^n(X(\mathbf{C})), \qquad F^{2m+1}(Z(\mathbf{C})) \to F^{2m+1}(Z(\mathbf{R})),$$

and therefore is an isomorphism by Theorem 1(i) since

$$F^{2m+1}(Z(\mathbf{C})) = H^{2m+1}(Z(\mathbf{C}), G; \mathbf{F}_2), \qquad F^{2m+1}(Z(\mathbf{R})) = H^{2m+1}(Z(\mathbf{R}), G; \mathbf{F}_2).$$

The lower homomorphism in (1.2.9) is, by a similar argument, an epimorphism. Therefore it follows from (1.2.9) that

$$\begin{aligned} &\dim \operatorname{Coker}[F^n(X(\mathbf{C})) \oplus F^{2m+1}(Z(\mathbf{R})) \to F^{2m+1}(X(\mathbf{R}))/F^{n-1}(X(\mathbf{R}))] \\ &\quad \le \dim \operatorname{Coker}[F^n(X(\mathbf{C})) \oplus F^{2m+1}(Z(\mathbf{C})) \to F^{2m+1}(X(\mathbf{C}))] \\ &\quad = \dim \operatorname{Coker}[F^{2m+1}(Z(\mathbf{C})) \to F^{2m+1}(X(\mathbf{C}))/F^n(X(\mathbf{C}))]. \end{aligned} \tag{1.2.10}$$

On the other hand,

$$\begin{aligned} &\dim \operatorname{Coker}[F^n(X(\mathbf{C})) \oplus F^{2m+1}(Z(\mathbf{R})) \to F^{2m+1}(X(\mathbf{R}))/F^{n-1}(X(\mathbf{R}))] \\ &\quad = \dim \operatorname{Coker}[F^n(X(\mathbf{C})) \oplus F^n(Z(\mathbf{R})) \to F^n(X(\mathbf{R}))/F^{n-1}(X(\mathbf{R}))], \end{aligned} \tag{1.2.11}$$

since

$$F^{2m+1}(X(\mathbf{R})) = \bigoplus_{q=0}^{n} H^q(X(\mathbf{R})), \qquad F^{2m+1}(X(\mathbf{R}))/F^{n-1}(X(\mathbf{R})) = H^n(X(\mathbf{R})),$$

$$F^{2m+1}(Z(\mathbf{R})) = \bigoplus_{q=0}^{m} H^q(Z(\mathbf{R})),$$

and the homomorphism $F^{2m+1}(Z(\mathbf{R})) \to F^{2m+1}(X(\mathbf{R}))$ is the direct sum of the homomorphisms $H^q(Z(\mathbf{R})) \to H^q(X(\mathbf{R}))$. It now follows from (1.2.8), (1.2.10), and (1.2.11) that

$$\kappa(f) \le \dim \operatorname{Coker}[F^{2m+1}(Z(\mathbf{C})) \to F^{2m+1}(X(\mathbf{C}))/F^n(X(\mathbf{C}))]. \tag{1.2.12}$$

We now notice that

$$\begin{aligned}(1.2.13)\qquad &\dim \operatorname{Coker}[F^{2m+1}(Z(\mathbf{C})) \to F^{2m+1}(X(\mathbf{C}))/F^n(X(\mathbf{C}))] \\ &\quad \le \sum_{q>n} \dim \operatorname{Coker}[I_\infty^{2m-q+1,q}(Z(\mathbf{C})) \to I_\infty^{2m-q+1,q}(X(\mathbf{C}))].\end{aligned}$$

Furthermore, since Z is a GM-variety,

$$\begin{aligned}(1.2.14)\qquad &\dim \operatorname{Coker}[I_\infty^{2m-q+1,q}(Z(\mathbf{C})) \to I_\infty^{2m-q+1,q}(X(\mathbf{C}))] \\ &\quad \le \dim \operatorname{Coker}[I_2^{2m-q+1,q}(Z(\mathbf{C})) \to I_2^{2m-q+1,q}(X(\mathbf{C}))] \\ &\quad = \dim \operatorname{Coker}[H^{2m-q+1}(G, H^q(Z(\mathbf{C}))) \to H^{2m-q+1}(G, H^q(X(\mathbf{C})))] \\ &\quad = \dim \operatorname{Coker}[H^1(G, H^q(Z(\mathbf{C}))) \to H^1(G, H^q(X(\mathbf{C})))],\end{aligned}$$

where the last equality follows from the equality

$$H^p(G, A) = A^\tau/\{a+\tau(a)\}, \qquad p > 0$$

(here the group A consists of elements of order two; see [1]). It now remains to apply (1.2.12)–(1.2.14), and the theorem is proved.

Before we state another result about $\kappa(f)$ we recall the necessary facts about the cohomology groups $H^N(X(\mathbf{C}), G; \mathbf{Z})$ and the filtration $\{F^k(X(\mathbf{C}), G; \mathbf{Z})\}$ of them given by the spectral sequence

$$I_2^{p,q}(X(\mathbf{C}), G; \mathbf{Z}) = H^p(G, H^q(X(\mathbf{C}); \mathbf{Z})) \Rightarrow H^{p+q}(X(\mathbf{C}), G; \mathbf{Z})$$

(see [1]). All those facts are contained in the following theorem.

Theorem 3. *The following assertions are true*:

(i) *If $N > 2n$, then the homomorphism*

$$H^N(X(\mathbf{C}), G; \mathbf{Z}) \to H^N(X(\mathbf{R}), G; \mathbf{Z})$$

is an isomorphism.

(ii) *If $2M > n$, then there is a canonical isomorphism*

$$H^{2M}(X(\mathbf{R}), G; \mathbf{Z}) = \bigoplus_{q=0}^{M} H^{2q}(X(\mathbf{R}), \mathbf{F}_2),$$

and if $2m \ge n$, then there is a canonical isomorphism

$$H^{2M+1}(X(\mathbf{R}), G; \mathbf{Z}) = \bigoplus_{q=0}^{M} H^{2q+1}(X(\mathbf{R}), \mathbf{F}_2).$$

(iii) *There are canonical isomorphisms*

$$F^{2k}H^{2M}(X(\mathbf{R}), G; \mathbf{Z}) = \left(\bigoplus_{q=0}^{k-1} H^{2q}(X(\mathbf{R}), \mathbf{F}_2)\right) \oplus H^{2k}(X(\mathbf{R}), \mathbf{Z})/(2), \qquad 2M > n,$$

$$F^{2k+1}H^{2M}(X(\mathbf{R}), G; \mathbf{Z}) = \bigoplus_{q=0}^{k} H^{2q}(X(\mathbf{R}), \mathbf{F}_2), \qquad 2M > n,$$

$$F^{2k}H^{2M+1}(X(\mathbf{R}), G; \mathbf{Z}) = \bigoplus_{q=0}^{k-1} H^{2q+1}(X(\mathbf{R}), \mathbf{F}_2), \qquad 2M \ge n,$$

$$F^{2k+1}H^{2M+1}(X(\mathbf{R}), G; \mathbf{Z}) = \left(\bigoplus_{q=0}^{k-1} H^{2q+1}(X(\mathbf{R}), \mathbf{F}_2)\right) \oplus H^{2k+1}(X(\mathbf{R}), \mathbf{Z})/(2),$$

$$2M \ge n.$$

We recall also that the variety X is called a GM **Z**-*variety* if the spectral sequence $I(X(\mathbf{C}), G; \mathbf{Z})$ degenerates.

We now state another new result.

Theorem 4. *Suppose that* Z *is a* GM **Z**-*variety. Then*

(1.2.15)
$$\kappa(f) \leq \sum_{q>0} \dim \operatorname{Coker}[H^{\varepsilon(q)}(G, H^{n+q}(Z(\mathbf{C}), \mathbf{Z})) \to H^{\varepsilon(q)}(G, H^{n+q}(X(\mathbf{C}), \mathbf{Z}))].$$

Proof. We shall use the Galois-Grothendieck cohomology groups $H^{2m+1}(\cdot, G; \mathbf{Z})$ if n is odd, and $H^{2m+2}(\cdot, G; \mathbf{Z})$ if n is even. When n is odd the argument is almost identical to that of Theorem 2. When n is even some modifications are required. Since

$$I_\infty^{2m-n+1,n}(X(\mathbf{R}), G; \mathbf{Z}) = H^{2m-n+1}(G, H^n(X(\mathbf{R}), \mathbf{Z})) = H^n(X(\mathbf{R}), \mathbf{Z})_2,$$

where $H^n(X(\mathbf{R}), \mathbf{Z})_2$ is the subset of elements of $H^n(X(\mathbf{R}), \mathbf{Z})$ of order two, we should replace $H^{2m+1}(\cdot, G; \mathbf{Z})$ with $H^{2m+2}(\cdot, G; \mathbf{Z})$. We then have that

$$\begin{aligned} I_\infty^{2m-n+2,n}(X(\mathbf{R}), G; \mathbf{Z}) &= H^{2m-n+2}(G, H^n(X(\mathbf{R}), \mathbf{Z})) \\ &= H^n(X(\mathbf{R}), \mathbf{Z})/(2) = H^n(X(\mathbf{R}), \mathbf{F}_2). \end{aligned}$$

Now we adapt the proof of Theorem 2 to our situation. We start at the end of it and proceed backwards.

Since Z is a GM **Z**-variety, we have that

$$\begin{aligned} &\dim \operatorname{Coker}[I_\infty^{2m-q+2,q}(Z(\mathbf{C})) \to I_\infty^{2m-q+2,q}(X(\mathbf{C}))] \\ &\quad \leq \dim \operatorname{Coker}[H^{\varepsilon(q)}(G, H^q(Z(\mathbf{C}))) \to H^{\varepsilon(q)}(G, H^q(X(\mathbf{C})))], \end{aligned} \tag{1.2.16}$$

where the cohomology of $X(\mathbf{C})$ and $Z(\mathbf{C})$ is considered with integer coefficients. Notice that we also have

$$\begin{aligned} &\dim \operatorname{Coker}[F^{2m+2}(Z(\mathbf{C})) \to F^{2m+2}(X(\mathbf{C}))/F^n(X(\mathbf{C}))] \\ &\quad \leq \sum_{q>n} \dim \operatorname{Coker}[I_\infty^{2m-q+2,q}(Z(\mathbf{C})) \to I_\infty^{2m-q+2,q}(X(\mathbf{C}))]. \end{aligned} \tag{1.2.17}$$

Similarly to (1.2.10), one can prove that

$$\begin{aligned} &\dim \operatorname{Coker}[F^n(X(\mathbf{C})) \oplus F^{2m+2}(Z(\mathbf{R})) \to F^{2m+2}(X(\mathbf{R}))/F^{n-1}(X(\mathbf{R}))] \\ &\quad \leq \dim \operatorname{Coker}[F^{2m+2}(Z(\mathbf{C})) \to F^{2m+2}(X(\mathbf{C}))/F^n(X(\mathbf{C}))]. \end{aligned} \tag{1.2.18}$$

Instead of (1.2.11), one has

$$\begin{aligned} &\dim \operatorname{Coker}[F^n(X(\mathbf{C})) \oplus F^{2m+2}(Z(\mathbf{R})) \to F^{2m+2}(X(\mathbf{R}))/F^{n-1}(X(\mathbf{R}))] \\ &\quad = \dim \operatorname{Coker}[F^n(X(\mathbf{C})) \oplus F^{n+1}(Z(\mathbf{R})) \to F^n(X(\mathbf{R}))/F^{n-1}(X(\mathbf{R}))], \end{aligned} \tag{1.2.19}$$

i.e., $F^n(Z(\mathbf{R}))$ must be replaced by $F^{n+1}(Z(\mathbf{R}))$, and (1.2.8) must be replaced by

$$\kappa(f) \leq \dim \operatorname{Coker}[F^n(X(\mathbf{C})) \oplus F^{n+1}(Z(\mathbf{R})) \to F^n(X(\mathbf{R}))/F^{n-1}(X(\mathbf{R}))]. \tag{1.2.20}$$

Now (1.2.15) follows from (1.2.20), (1.2.19), (1.2.17), and (1.2.16). The theorem is proved.

§2. Applications of the general results

In this section X will be an n-dimensional real algebraic variety, which is assumed to be nonsingular and complete. Under these assumptions, the set $X(\mathbf{C})$ of complex points is an n-dimensional compact complex variety, and the set $X(\mathbf{R})$ of real points if it is nonempty, is an n-dimensional real-analytic variety which need not be connected. Let $X_1, \dots, X_m$ denote the connected components $X(\mathbf{R})$. They give rise to homology classes $[X_1], \dots, [X_m] \in H_n(X(\mathbf{C}), \mathbf{F}_2)$. The number of linearly independent relations between them is equal to the number $\kappa(X)$ introduced in §1.

2.1. **Curves and surfaces.** First we shall re-prove two known results.

Proposition 1. *Let X be a curve. Then $\kappa(X) \leq 1$ and if $\kappa(X) = 1$ then the only relation is of the form*

$$[X_1] + \cdots + [X_m] = 0. \tag{2.1.1}$$

Proof. In the case of a curve inequality (1.1.1) yields

$$\kappa(X) \leq \dim H^1(G, H^2(X(\mathbf{C}), \mathbf{F}_2)) = 1.$$

Hence only one relation is possible. To show that this relation is indeed of the form (2.1.1) we consider the noncomplete curve $\widetilde{X}$ which is obtained from X by deleting a real point. We then have that

$$\kappa(\widetilde{X}) \leq \dim H^1(G, H^2(\widetilde{X}(\mathbf{C}), \mathbf{F}_2)) = 0,$$

i.e., $\kappa(\widetilde{X}) = 0$ and the homomorphism $H_1(\widetilde{X}(\mathbf{C}), \mathbf{F}_2) \to H_1(X(\mathbf{C}), \mathbf{F}_2)$ is an isomorphism. This means that if we delete a point from X_1 then the homology classes $[X_2], \dots, [X_m]$ are linearly independent in $H_1(X(\mathbf{C}), \mathbf{F}_2)$. The proposition is proved.

Proposition 2. *Let X be a surface such that $H_1(X(\mathbf{C}), \mathbf{F}_2) = 0$. Then $\kappa(X) \leq 1$, and if $\kappa(X) = 1$ then the only relation is of the form* (2.1.1).

The proof is identical to that of Proposition 1.

Remark 1. Proposition 1 is a classical result. An elementary proof of it can be found, for example, in [2]. Another proof of Proposition 2 can be found in [3].

Proposition 3. *Let X be a surface such that the homology group $H_1(X(\mathbf{C}), \mathbf{Z})$ is free. Then*

$$\kappa(X) \leq 1 + q(X), \tag{2.1.2}$$

where $q(X)$ is the irregularity of the surface.

Proof. In the case of a surface inequality (1.1.2) becomes

$$\kappa(X) \leq 1 + \dim_{\mathbf{F}_2} H^1(G, H^3(X(\mathbf{C}), \mathbf{Z})). \tag{2.1.3}$$

Since the group $H^3(X(\mathbf{C}), \mathbf{Z})$ is free,

$$\dim_{\mathbf{F}_2} H^1(G, H^3(X(\mathbf{C}), \mathbf{Z})) = \dim_{\mathbf{F}_2} H^2(G, H^3(X(\mathbf{C}), \mathbf{Z}))$$

(see [4]). Therefore

$$\dim_{\mathbf{F}_2} H^1(G, H^3(X(\mathbf{C}), \mathbf{Z})) \leq \tfrac{1}{2} \operatorname{rk} H^3(X(\mathbf{C}), Z) = q(X),$$

and the proposition is proved.

Remark 2. The bound (2.1.2) is sharp. To see that, consider the surface X which is the product of a curve M of genus g by the projective line. Then $\kappa(X) = g + 1 = 1 + q(X)$.

Remark 3. Let $A(X)$ be the Albanese variety of a surface X and let $|A(X)|$ denote the number of connected components of the set of real points of $A(X)$. Then

$$\dim_{\mathbf{F}_2} H^1(G, H^3(X(\mathbf{C}), \mathbf{Z})) = \log_2 |A(X)|$$

(see [4]), and therefore (2.1.3) can be rewritten as

$$\kappa(X) \leq 1 + \log_2 |A(X)|.$$

Remark 4. If the equality $\kappa(X) = 1 + q(X)$ holds, then $\log_2 |A(X)| = q(X)$, which means that $A(X)$ is an M-variety.

Proposition 4. *Let X be a projective surface of odd degree such that $H_1(X(\mathbf{C}), \mathbf{F}_2) = 0$. Then $\kappa(X) = 0$.*

Proof. Consider an embedding map $f \colon X \to \mathbf{P}^N$ and the commutative diagram

$$\begin{array}{ccc} H_2(X(\mathbf{R})) & \longrightarrow & H_2(\mathbf{P}^N(\mathbf{R})) \\ \downarrow & & \downarrow \\ H_2(X(\mathbf{C})) & \xrightarrow{f^*} & H_2(\mathbf{P}^N(\mathbf{C})) \end{array} \tag{2.1.4}$$

Since X is of odd degree, this diagram shows that f_* sends the homology class $[X_1]+\cdots+[X_m]$ to a generator of the group $H_2(\mathbf{P}^N(\mathbf{C}))$. Therefore $[X_1]+\cdots+[X_m] \neq 0$, and the proposition is proved.

Proposition 5. *Let X be a projective surface of odd degree such that the group $H_1(X(\mathbf{C})\mathbf{Z})$ is free. Then*

$$\kappa(X) \leq \log_2 |A(X)|, \qquad \kappa(X) \leq q(X).$$

Proof. Consider an embedding map $f \colon X \to \mathbf{P}^N$. It follows from (1.2.15) that

$$\kappa(f) \leq \dim_{\mathbf{F}_2} H^1(G, H^3(X(\mathbf{C}), \mathbf{Z})) = \log_2 |A(X)|.$$

We shall now show that $\kappa(X) = \kappa(f)$. For this we have to show that if the image of a homology class $a \in H_2(X(\mathbf{R}))$ under the map $H_2(X(\mathbf{R})) \to H_2(\mathbf{P}^N(\mathbf{R}))$ is different from zero, then it is also nonzero under the map $H_2(X(\mathbf{R})) \to H_2(X(\mathbf{C}))$. But the latter can be seen from the diagram (2.1.4). The proposition is proved.

Before we formulate the next proposition we make some remarks. Let X be a surface, Y a curve on X, and suppose that $\kappa(Y) = 1$ and the cycle $Y(\mathbf{R})$ is homologous to zero in $X(\mathbf{R})$. We can now form a new topological surface W as follows. The set $Y(\mathbf{R})$ splits $Y(\mathbf{C})$ into two parts. Choose one of them. On the other hand, take a part of $X(\mathbf{R})$ whose boundary equals $Y(\mathbf{R})$. Then W is the union of the chosen parts of $Y(\mathbf{C})$ and $X(\mathbf{R})$ with common boundary $Y(\mathbf{R})$. Notice that W is not defined uniquely.

Proposition 6. *Let X be a surface such that $H_1(X(\mathbf{C}), \mathbf{F}_2) = 0$, $Y \subset X$, and let $Y \subset X$ be a nonsingular curve. Then $\kappa(X, Y) \leq 2$ and $\kappa(X, Y) = 2$ only when the following conditions hold: $\kappa(X) = \kappa(Y) = 1$, the cycle $Y(\mathbf{R})$ is homologous to zero in $X(\mathbf{R})$, and, for some choice of the part of $X(\mathbf{R})$, the topological surface W is homologous to zero in $X(\mathbf{C})$, where the homology is considered with coefficients in $\mathbf{F}_2$.*

Proof. If the homomorphism

$$H_2(Y(\mathbf{C})) \to H_2(X(\mathbf{C})) \tag{2.1.5}$$

is a monomorphism, then it follows from the long homology exact sequence of the pair $(X(\mathbf{C}), Y(\mathbf{C}))$ that $H_3(X(\mathbf{C}), Y(\mathbf{C})) = 0$. Therefore (1.1.3) implies that $\kappa(X, Y) \leq 1$. If the homomorphism (2.1.5) is zero, then $H_3(X(\mathbf{C}), Y(\mathbf{C})) = \mathbf{F}_2$ and similarly to the above we have that $\kappa(X, Y) \leq 2$. Hence we always have that

$\kappa(X, Y) \leq 2$. Consider now the homomorphism between the long homology exact sequences of the pairs $(X(\mathbf{R}), Y(\mathbf{R}))$ and $(X(\mathbf{C}), Y(\mathbf{C}))$. It gives rise to the commutative diagram

(2.1.6)
$$\begin{array}{ccccccccc} & & 0 & \longrightarrow & H_2(X(\mathbf{R})) & \longrightarrow & H_2(X(\mathbf{R}), Y(\mathbf{R})) & \longrightarrow & H_1(Y(\mathbf{R})) & \longrightarrow & H_1(X(\mathbf{R})) \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \cdots & \longrightarrow & \mathbf{F}_2 & \longrightarrow & H_2(X(\mathbf{C})) & \longrightarrow & H_2(X(\mathbf{C}), Y(\mathbf{C})) & \longrightarrow & H_1(Y(\mathbf{C})) & \longrightarrow & 0 \end{array}$$

Let $\kappa(X, Y) = 2$. Then it follows immediately from this diagram that $\kappa(X) = \kappa(Y) = 1$. Furthermore, since the only relation between the homology classes $[Y_1], \dots, [Y_m]$ is of the form $[Y_1] + \cdots + [Y_m] = [Y(\mathbf{R})] = 0$, we have that if $\kappa(X, Y) = 2$ then the curve $Y(\mathbf{R})$ splits $X(\mathbf{R})$ in such a way that for some choice of the part of $X(\mathbf{R})$ this part is homologous to zero in $H_2(X(\mathbf{C}), Y(\mathbf{C}))$; all this follows from (2.1.6). If we now take the corresponding surface W, then it will be homologous to zero in $H_2(X(\mathbf{C}))$. Thus the condition $\kappa(X, Y) = 2$ implies the conditions $\kappa(X) = \kappa(Y) = 1$ and $W \sim 0$. Diagram (2.1.6) shows that the converse is also true. The proposition is proved.

2.2. **Complete intersections and double coverings.**

Theorem 5. *Let X be a nonsingular complete intersection $\mathbf{P}^N$. Then the following assertions are true:*

(i) *$\kappa(X) \leq 1$, and the only relation is of the form $[X(\mathbf{R})] = 0$.*

(ii) *If $\deg X$ is odd and $\dim X = n$ is even, then $\kappa(X) = 0$.*

Proof. If Y is a hyperplane section of X, then $H_q(X(\mathbf{C}), Y(\mathbf{C})) = 0$ for $n < q < 2n$. Therefore (1.1.3) implies that

$$\kappa(X, Y) \leq \dim H^1(G, H^{2n}(X(\mathbf{C}), Y(\mathbf{C}))) = 1.$$

Now repeating the argument with a deleted pointed $x_0 \in X(\mathbf{R})\backslash Y(\mathbf{R})$ from the proof of Proposition 1, we see that the only relation between the relative homology classes is of the form

$$[X(\mathbf{R})] \mod Y(\mathbf{R}) = 0. \tag{2.2.1}$$

After that we consider the homomorphism between the long homology exact sequences of the pairs $(X(\mathbf{R}), Y(\mathbf{R}))$ and $(X(\mathbf{C}), Y(\mathbf{C}))$. We then have that $\kappa(X) \leq \kappa(X, Y) \leq 1$, and the only relation will be, by virtue of (2.2.1), of the form $[X(\mathbf{R})] = 0$. Thus the first assertion of the theorem is proved. The second assertion can be proved by an argument similar to the one in the proof of Proposition 4. The theorem is proved.

Theorem 6. *Let X be a two-fold covering of a nonsingular complete intersection $Z \subset \mathbf{P}^N$ with branching along a regular intersection by a nonsingular hypersurface in $\mathbf{P}^N$. Then $\kappa(X) \leq 1$, and the only relation is of the form $[X(\mathbf{R})] = 0$.*

Proof. Let Y be the preimage of a general hyperplane section of Z under the projection $X \to Z$. Then the desired result follows from the argument in the proof of the first assertion of Theorem 5. The theorem is proved.

2.3. **Hyperplane sections.**

Proposition 7. *Let Z be a nonsingular projective* GM*-variety, X a hyperplane section of Z such that the operators*

$$L^q: H^{n-q}(X(\mathbf{C}), \mathbf{F}_2) \to H^{n+q}(X(\mathbf{C}), \mathbf{F}_2)$$

are isomorphisms for $0 \le q \le n$, *where* L *is the Lefschetz operator, and* $f\colon X \hookrightarrow Z$ *the embedding map. Then* $\kappa(f) = 0$.

Proof. We shall use the bound from Theorem 2:

$$\kappa(f) \le \sum_{q>0} \dim \operatorname{Coker}[H^1(G, H^{n+q}(Z(\mathbf{C}))) \to H^1(G, H^{n+q}(X(\mathbf{C})))].$$

It suffices to show that the homomorphisms in the right-hand side are epimorphisms. To this end consider the commutative diagram

$$\begin{array}{ccc} L^q\colon H^{n-q}(Z(\mathbf{C})) & \longrightarrow & H^{n+q}(Z(\mathbf{C})) \\ \downarrow \wr & & \downarrow \\ L^q\colon H^{n-q}(X(\mathbf{C})) & \xrightarrow{\sim} & H^{n+q}(X(\mathbf{C})). \end{array}$$

It gives rise to the commutative diagram

$$\begin{array}{ccc} L^q\colon H^1(G, H^{n-q}(Z(\mathbf{C}))) & \longrightarrow & H^1(G, H^{n+q}(Z(\mathbf{C}))) \\ \downarrow \wr & & \downarrow \\ L^q\colon H^1(G, H^{n-q}(X(\mathbf{C})) & \xrightarrow{\sim} & H^1(G, H^{n+q}(X(\mathbf{C}))), \end{array}$$

whence the desired assertion. The theorem is proved.

Remark 5. In the above proposition X can be replaced by a complete intersection of Z.

Theorem 7. *Let* X *be a projective variety such that the operators*

$$L^q\colon H^{n-q}(X(\mathbf{C}), \mathbf{F}_2) \to H^{n+q}(X(\mathbf{C}), \mathbf{F}_2)$$

are isomorphisms for $0 \le q \le n$, *and* Y *a hyperplane section. Then*

$$\kappa(X, Y) \le \dim H^1(G, H^{n-2}(X(\mathbf{C}), \mathbf{F}_2)) + \dim H^1(G, H^{n-1}(X(\mathbf{C}), \mathbf{F}_2)).$$

Proof. By (1.1.3),

$$\begin{aligned} \kappa(X, Y) &\le \sum_{q>0} \dim H^1(G, H^{n+q}(X(\mathbf{C}), Y(\mathbf{C}); \mathbf{F}_2)) \\ &= \sum_{q>0} \dim H^1(G, H^{n+q}(X(\mathbf{C}), Y(\mathbf{C}); \mathbf{F}_2)) \\ &= \sum_{q=0}^{n=1} \dim H^1(G, H^q(X(\mathbf{C}) - Y(\mathbf{C}), \mathbf{F}_2)). \end{aligned}$$

Now we want to compute the cohomology group $H^q(X(\mathbf{C}) - Y(\mathbf{C}), \mathbf{F}_2)$ for $q \le n - 1$. Applying Poincaré duality to the long homology exact sequence of the pair $(X(\mathbf{C}), Y(\mathbf{C}))$, we have the exact sequence

$$\begin{aligned} \cdots \to H^{q-2}(Y(\mathbf{C})) \to H^q(X(\mathbf{C})) \to H^q(X(\mathbf{C}) - Y(\mathbf{C})) \\ \to H^{q-1}(Y(\mathbf{C})) \to H^{q+1}(X(\mathbf{C})) \to \cdots. \end{aligned}$$

Since $H^k(Y(\mathbf{C})) = H^k(X(\mathbf{C}))$ for $k < n - 1$, it follows from the above sequence that the sequence

(2.3.1)
$$\begin{aligned} H^{q-2}(X(\mathbf{C})) \xrightarrow{L} H^q(X(\mathbf{C})) \to H^q(X(\mathbf{C}) - Y(\mathbf{C})) \\ \to H^{q-1}(X(\mathbf{C})) \xrightarrow{L} H^{q+1}(X(\mathbf{C})) \end{aligned}$$

is exact for $q \leq n-1$. Let $P^q(X(\mathbf{C}), \mathbf{F}_2)$ denote the primitive part of $H^q(X(\mathbf{C}), \mathbf{F}_2)$ i.e.,

$$P^q(X(\mathbf{C}), \mathbf{F}_2) = \operatorname{Ker}[L^{n-q+1}\colon H^q(X(\mathbf{C}), \mathbf{F}_2) \to H^{2n-q+1}(X(\mathbf{C}), \mathbf{F}_2)].$$

The condition that $L^q\colon H^{n-q}(X(\mathbf{C})) \to H^{n+q}(X(\mathbf{C}))$ are isomorphisms for $0 \leq q \leq n$ gives rise to the Lefschetz decomposition

$$H^q(X(\mathbf{C})) = \bigoplus_{k=0}^{[q/2]} L^k P^{q-2k}(X(\mathbf{C})). \tag{2.3.2}$$

Therefore it follows from (2.3.1) that

$$H^q(X(\mathbf{C}) - Y(\mathbf{C})) = P^q(X(\mathbf{C}))$$

for $q \leq n-1$. Hence

$$\sum_{q=0}^{n-1} \dim H^1(G, H^q(X(\mathbf{C}) - Y(\mathbf{C}))) = \sum_{q=0}^{n-1} \dim H^1(G, P^q(X(\mathbf{C}))).$$

But, according to (2.3.2), this sum is equal to

$$\dim H^1(G, H^{n-2}(X(\mathbf{C}))) + \dim H^1(G, H^{n-1}(X(\mathbf{C}))).$$

The theorem is proved.

Corollary 1. *Let X be a projective variety such that the operators*

$$L^q\colon H^{n-q}(X(\mathbf{C}), \mathbf{F}_2) \to H^{n+q}(X(\mathbf{C}), \mathbf{F}_2)$$

are isomorphisms for $0 \leq q \leq n$. Then

$$\kappa(X) \leq \dim H^1(G, H^{n-2}(X(\mathbf{C}))) + \dim H^1(G, H^{n-1}(X(\mathbf{C}))).$$

To prove this, just apply (1.1.5).

2.4. **Applications of the Hodge structure.**

Theorem 8. *Let X be a projective variety such that the groups $H_q(X(\mathbf{C}), \mathbf{Z})$ $(0 < q < n)$ are free and $k = [(n-1)/2]$. Then*

$$\kappa(X) \leq \sum_{\substack{0<r+s<n \\ r>s}} h^{r,s}(X(\mathbf{C})) + \sum_{i\geq 0} h^{k-2i,k-2i}(X(\mathbf{C})), \tag{2.4.1}$$

where the $h^{r,s}(X(\mathbf{C}))$ are the Hodge numbers.

Proof. We shall show that the right-hand side of the inequality

$$\kappa(X) \leq \sum_{q>0} \dim H^{\varepsilon(q)}(G, H^{n+q}(G, H^{n+q}(X(\mathbf{C}), \mathbf{Z}))$$

(see (1.1.2)) does not exceed the right-hand side of (2.4.1). To this end, note that

$$\begin{aligned} \dim H^{\varepsilon(q)}(G, H^{n+q}(X(\mathbf{C}), \mathbf{Z})) &= \dim H^1(G, H^{n+q}(X(\mathbf{C}), \mathbf{Z})) \\ &\leq \operatorname{rk} H^{n+q}(X(\mathbf{C}), \mathbf{Z})^{-\tau^*} = \dim_{\mathbf{C}} H^{n+q}(X(\mathbf{C}), \mathbf{C})^{-\tau^*} \end{aligned} \tag{2.4.2}$$

when q is odd, and

$$\begin{aligned} \dim H^{\varepsilon(q)}(G, H^{n+q}(X(\mathbf{C}), \mathbf{Z})) &= \dim H^2(G, H^{n+q}(X(\mathbf{C}), \mathbf{Z})) \\ &\leq \operatorname{rk} H^{n+q}(X(\mathbf{C}), \mathbf{Z})^{\tau^*} = \dim H^{n+q}(X(\mathbf{C}), \mathbf{C})^{\tau^*} \end{aligned} \tag{2.4.3}$$

when q is even. Note also that when q is odd the Lefschetz isomorphism

$$L^q\colon H^{n-q}(X(\mathbf{C}), \mathbf{C}) \xrightarrow{\sim} H^{n+q}(X(\mathbf{C}), \mathbf{C})$$

induces the isomorphism

$$L^q : H^{n-q}(X(\mathbf{C}), \mathbf{C})^{\tau^*} \xrightarrow{\sim} H^{n+q}(X(\mathbf{C}), \mathbf{C})^{-\tau^*}, \tag{2.4.4}$$

because

$$L \circ \tau^* = -\tau^* \circ L, \tag{2.4.5}$$

and when q is even we have the isomorphism

$$L^q : H^{n-q}(X(\mathbf{C}), \mathbf{C})^{\tau^*} \xrightarrow{\sim} H^{n+q}(X(\mathbf{C}), \mathbf{C})^{\tau^*}. \tag{2.4.6}$$

It now follows from (2.4.2)–(2.4.4) and (2.4.6) that

$$\sum_{q>0} \dim H^{\varepsilon(q)}(C, H^{n+q}(X(\mathbf{C}), \mathbf{Z})) \leq \sum_{q=1}^{n} \dim H^{n-q}(X(\mathbf{C}), \mathbf{Z})^{\tau^*}. \tag{2.4.7}$$

Consider now the Hodge decomposition

$$H^{n-q}(X(\mathbf{C}), \mathbf{C}) = \bigoplus_{r+s=n-q} H^{r,s}(X(\mathbf{C})).$$

Since $\tau^*(H^{r,s}(X(\mathbf{C}))) = H^{s,r}(X(\mathbf{C}))$, it follows from this decomposition that

$$\dim H^{n-q}(X(\mathbf{C}), \mathbf{C})^{\tau^*} = \sum_{\substack{r+s=n-q \\ r>s}} h^{r,s}(X(\mathbf{C})) \tag{2.4.8}$$

when $n-q$ is odd, and

$$\dim H^{n-q}(X(\mathbf{C}), \mathbf{C})^{\tau^*} = \sum_{\substack{r+s=n-q \\ r>s}} h^{r,s}(X(\mathbf{C})) + \dim H^{t,t}(X(\mathbf{C}))^{\tau^*} \tag{2.4.9}$$

when $n-q = 2t$.

It follows from (1.1.2) and (2.4.7)—(2.4.9) that to finish the proof of (2.4.1) it remains to show that

$$\sum_{0\leq t\leq k} \dim H^{t,t}(X(\mathbf{C}))^{\tau^*} \leq \sum_{i\geq 0} h^{k-2i,k-2i}(X(\mathbf{C})). \tag{2.4.10}$$

To abbreviate our notation let $H^{t,t} = H^{t,t}(X(\mathbf{C}))$, and let $P^{t,t}$ denote the primitive part of $H^{t,t}$, i.e., the kernel of the operator

$$L^{n-2t+1} : H^{t,t} \to H^{n-t+1,n-t+1},$$

$$H_+^{t,t} = (H^{t,t})^{\tau^*}, \; P_\pm^{t,t} = (P^{t,t})^{\pm\tau^*}, \quad h_+^{t,t} = \dim H_+^{t,t} \quad P_\pm^{t,t} = \dim P_\pm^{t,t}.$$

By virtue of (2.4.5), it follows from the Lefschetz decomposition

$$H^{t,t} = P^{t,t} \oplus LP^{t-1,t-1} \oplus L^2P^{t-2,t-2} \oplus L^3P^{t-3,t-3} \oplus \cdots$$

that

$$H_+^{t,t} = P_+^{t,t} \oplus LP_-^{t-1,t-1} \oplus L^2P_+^{t-2,t-2} \oplus L^3P_-^{t-3,t-3} \oplus \cdots,$$

which in turn implies that

$$h_+^{t,t} = p_+^{t,t} + p_-^{t-1,t-1} + p_+^{t-2,t-2} + p_-^{t-3,t-3} + \cdots.$$

Replacing t by $t-1$ in this equality, we get

$$h_+^{t-1,t-1} = p_+^{t-1,t-1} + p_-^{t-2,t-2} + p_+^{t-3,t-3} + p_-^{t-4,t-4} + \cdots.$$

Adding the two equalities, we have

$$h_+^{t,t} + h_+^{t-1,t-1} = p_+^{t,t} + p^{t-1,t-1}, + p^{t-2,t-2} + p^{t-3,t-3} + \cdots,$$

which implies, since $p_+^{t,t} \leq p^{t,t}$, that

$$h_+^{t,t} + h_+^{t-1,t-1} \leq \sum_{i \leq t} p^{i,i} = h^{t,t}.$$

Therefore

$$\sum_{0 \leq t \leq k} h_+^{t,t} \leq \sum_{i \geq 0} h^{k-2i,k-2i},$$

which is (2.4.10). The theorem is proved.

Remark 6. In the surface case Theorem 8 yields (2.1.2).

In the next theorem we will need the following notation. Consider the set $0, 1, \ldots, n-1$, and count the number of elements of it of the form $4k$. If n is odd, subtract one. The number thus obtained is denoted by $\Delta(n)$. Note that

$$\Delta(n) = \begin{cases} n/4, & n \equiv 0 \bmod 4, \\ (n-1)/4, & n \equiv 1 \bmod 4, \\ (n+2)/4, & n \equiv 2 \bmod 4, \\ (n-3)/4, & n \equiv 3 \bmod 4. \end{cases}$$

Theorem 9. *Let X be a projective variety of odd degree such that the homology groups $H_q(X(\mathbf{C}), \mathbf{Z})$ $(0 < q < n)$ are free, and let $k = [(n-1)/2]$. Then*

$$\kappa(X) \leq \sum_{\substack{0<r+s<n \\ r>s}} h^{r,s}(X(\mathbf{C})) + \sum_{i \geq 0} h^{k-2i,k-2i}(X(\mathbf{C})) - \Delta(n).$$

Proof. Let $f\colon X \to \mathbf{P}^N$ be the embedding map. Applying (1.2.5), we have

(2.4.11)
$$\kappa(f\,f) \leq \sum_{q=1}^{n} \dim H^{\varepsilon(q)}(G, H^{n+q}(X(\mathbf{C}), \mathbf{Z})) - \sum_{q=1}^{n} \dim \operatorname{Im}[H^{\varepsilon(q)}(G, H^{n+q}(\mathbf{P}^N(\mathbf{C}), \mathbf{Z})) \to H^{\varepsilon(q)}(G, H^{n+q}(X(\mathbf{C}), \mathbf{Z}))],$$

where the right-hand side of (1.2.5) was rewritten in a different way. Note now that

$$H^{\varepsilon(q)}(G, H^{n+q}(\mathbf{P}^N(\mathbf{C}), \mathbf{Z})) = \mathbf{F}_2$$

only when $n - q \equiv 0 \bmod 4$; otherwise this group is zero. We shall now show that

(2.4.12) $$\dim \operatorname{Im}[H^{\varepsilon(q)}(G, H^{n+q}(\mathbf{P}^N(\mathbf{C}), \mathbf{Z})) \to H^{\varepsilon(q)}(G, H^{n+q}(X(\mathbf{C}), \mathbf{Z}))] = 1$$

whenever $n - q \equiv 0 \bmod 4$. Let α denote a generator of $H^1(G, H^2(\mathbf{P}^N(\mathbf{C}), \mathbf{Z}))$. Then the image of α^n under the homomorphism

$$H^n(G, H^{2n}(\mathbf{P}^N(\mathbf{C}), \mathbf{Z})) \to H^n(G, H^{2n}(X(\mathbf{C}), \mathbf{Z}))$$

is different from zero, since the degree of X is odd. Hence the image of α^m under the homomorphism

$$H^m(G, H^{2m}(\mathbf{P}^N(\mathbf{C}), \mathbf{Z})) \to H^m(G, H^{2m}(X(\mathbf{C}), \mathbf{Z}))$$

is also different from zero for $m = 1, \ldots, n$. This gives us (2.4.12). Finally, we remark that $\kappa(f) = \kappa(X)$ for n even—this can be checked as in the proof of

Proposition 5—and $\kappa(X) \leq \kappa(f)+1$ for n odd. It remains to apply (2.4.11), (2.4.12), and the inequality

$$\sum_{q=1}^{n} \dim H^{\varepsilon(q)}(G, H^{n+q}(X(\mathbf{C}), \mathbf{Z})) \leq \sum_{\substack{0<r+s<n \\ r>s}} h^{r,s}(X(\mathbf{C})) + \sum_{i\geq 0} h^{k-2i,k-2i}(X(\mathbf{C})),$$

obtained in the proof of Theorem 8. The theorem is proved.

Now we state the relative version of Theorem 8.

Theorem 10. *Let X be a projective variety and Y a hyperplane section of X. Suppose that the homology groups $H_q(X(\mathbf{C}) - Y(\mathbf{C}), \mathbf{Z})$ are free for $0 < q < n$, and let $k = [(n-1)/2]$. Then*

$$\kappa(X, Y) \leq \sum_{\substack{n-2\leq r+s\leq n-1 \\ r>s}} h^{r,s}(X(\mathbf{C})) + h^{k,k}(X(\mathbf{C})).$$

Proof. By the Poincaré isomorphism, we can identify homology with cohomology. Since the complex conjugation changes the orientation of $X(\mathbf{C})$ only when n is odd, we have that

$$\tau_* = (-1)^n \tau^*.$$

By virtue of (1.1.4), this yields

$$\kappa(X, Y) \leq \sum_{q>0} \dim H^{\varepsilon(q)}(G, H^{n+q}(X(\mathbf{C}), Y(\mathbf{C}); \mathbf{Z}))$$

(2.4.13)
$$= \sum_{q=0}^{n-1} \dim H^{\varepsilon(q)}(G, H_q(X(\mathbf{C}) - Y(\mathbf{C}), \mathbf{Z}))$$

$$= \sum_{q=0}^{n-1} \dim H^{\varepsilon(q)}(G, H^q(X(\mathbf{C}) - Y(\mathbf{C}), \mathbf{Z})),$$

where in the last equality we used the freeness of the group $H_q(X(\mathbf{C}) - Y(\mathbf{C}), \mathbf{Z})$, $0 \leq q \leq n-1$. Now we make several observations. First,

(2.4.14)
$$\dim_{\mathbf{F}_2} H^{\varepsilon(q)}(G, H^q(X(\mathbf{C}) - Y(\mathbf{C}), \mathbf{Z})) \leq \dim_{\mathbf{C}} H^q(X(\mathbf{C}) - Y(\mathbf{C}), \mathbf{C})^{(-1)^q \tau^*}$$

Second, for $0 \leq q \leq n-1$ the equality

(2.4.15)
$$H^q(X(\mathbf{C}) - Y(\mathbf{C}), \mathbf{C}) = P^q(X(\mathbf{C}), \mathbf{C})$$

holds. And finally

(2.4.16)
$$\dim P^q(X(\mathbf{C}), \mathbf{C})^{-\tau^*} = \sum_{\substack{r+s=q \\ r>s}} p^{r,s} + p^{q/2,q/2}$$

when q is odd, and

(2.4.17)
$$\dim P^q(X(\mathbf{C}), \mathbf{C})^{\tau^*} \leq \sum_{\substack{r+s=q \\ r>s}} p^{r,s} + p^{q/2,q/2}$$

when q is even.

Similar equalities and inequalities have been obtained before, and we omit the details. Thus we deduce from (2.4.13)—(2.4.17) that

$$\kappa(X, Y) \leq \sum_{\substack{0<r+s<n \\ r>s}} p^{r,s} + \sum_{0\leq i\leq k} p^{i,i} = \sum_{\substack{n-2\leq r+s\leq n-1 \\ r>s}} h^{r,s} + h^{k,k}.$$

The theorem is proved.

Corollary 2. *Under the assumptions of Theorem* 10,

$$\kappa(X) \leq \sum_{\substack{n-2\leq r+s\leq n-1 \\ r>s}} h^{r,s}(X(\mathbf{C})) + h^{k,k}(X(\mathbf{C})).$$

This follows from (1.15).

§3. Orientable and nonorientable components

Among the components $X_1, \dots, X_m$ there can be both orientable and nonorientable ones. Let $X_1, \dots, X_s$ be the orientable components, and the rest nonorientable. The number of relations between the homology classes $[X_1], \dots, [X_s] \in H_n(X(\mathbf{C}), \mathbf{F}_2)$ will be denoted by $\kappa_+(X)$, and the number of relations between the remaining classes by $\kappa_-(X)$. It is clear that

$$\kappa_+(X) + \kappa_-(X) \leq \kappa(X),$$

and therefore any bound on $\kappa(X)$ yields bounds on $\kappa_+(X)$ and $\kappa_-(X)$. In this section we shall obtain more precise bounds than those derived from the above inequality. In order to get information about $\kappa_+(X)$ we need to recall some facts about complex line bundles with real structure.

3.1. **Characteristic classes.** Let $L \to X$ be a line bundle, where L and X are real algebraic varieties. Then the map $L(\mathbf{C}) \to X(\mathbf{C})$ between the sets of complex points in a complex-analytic line bundle and the map $L(\mathbf{R}) \to X(\mathbf{R})$ is a real-analytic bundle. Associated with $L(\mathbf{C})$ there is the Chern class $c(L(\mathbf{C})) \in H^2(X(\mathbf{C}), \mathbf{Z})$ and with $L(\mathbf{R})$-the Stiefel-Whitney characteristic class $w(L(\mathbf{R})) \in H^1(X(\mathbf{R}), \mathbf{F}_2)$. In this subsection we shall define a new characteristic class $cw(L) \in H^2(X(\mathbf{C}), G; \mathbf{Z}(1))$ which is a "mixture" of the characteristic classes $c(L(\mathbf{C}))$ and $w(L(\mathbf{R}))$. Here $\mathbf{Z}(1)$ is the constant sheaf on $X(\mathbf{C})$ with fiber $\mathbf{Z}$ on which involution τ acts by multiplication by -1. We shall define the characteristic class $cw(L)$ with the aid of the exponential exact sequence

(3.1.1) $$0 \to \mathbf{Z}(1) \to \mathscr{O}_{X(\mathbf{C})} \to \mathscr{O}^*_{X(C)} \to 1$$

of G-sheaves, where the homomorphism $\mathbf{Z}(1) \to \mathscr{O}_{X(\mathbf{C})}$ sends k to $2\pi i k$, and the other homomorphism sends f to $\exp(f)$. The group $H^1(X(\mathbf{C}), G; \mathscr{O}^*_{X(\mathbf{C})})$ is the group of complex-analytic bundles on $X(\mathbf{C})$ with real structure. It coincides with $\operatorname{Pic} X$. From the exact sequence (3.1.1) we obtain the coboundary operator

$$\delta\colon H^1(X(\mathbf{C}), G; \mathscr{O}^*_{X(\mathbf{C})}) \to H^2(X(\mathbf{C}), G; \mathbf{Z}(1)).$$

We now set

(3.1.2) $$cw(L) = \delta(L(\mathbf{C})).$$

We can also define homomorphisms

$$\alpha\colon H^2(X(\mathbf{C}), G; \mathbf{Z}(1)) \to H^2(X(\mathbf{C}), \mathbf{Z}),$$
$$\beta\colon H^2(X(\mathbf{C}), G; \mathbf{Z}(1)) \to H^1(X(\mathbf{R}), \mathbf{F}_2).$$

The map α is the composition of the projection onto $I_\infty^{0,2}(X(\mathbf{C}), G; \mathbf{Z}(1))$ and the embedding

$$I_\infty^{0,2}(X(\mathbf{C}), G; \mathbf{Z}(1)) \subset H^2(X(\mathbf{C}), \mathbf{Z}(1)) = H^2(X(\mathbf{C}), \mathbf{Z}).$$

Before defining the homomorphism β we examine the spectral sequence

$$I_2^{p,q}(X(\mathbf{R}), G; \mathbf{Z}(1)) = H^p(X(\mathbf{R}), \mathscr{H}^q(\mathbf{Z}(1))) \Rightarrow H^{p+q}(X(\mathbf{R}), G; \mathbf{Z}(1)).$$

Since

$$\mathscr{H}^q(\mathbf{Z}(1)) = \begin{cases} 0, & q \text{ even}, \\ \mathbf{F}_2, & q \text{ is odd}, \end{cases}$$

we have the canonical homomorphism

$$H^2(X(\mathbf{R}), G; \mathbf{Z}(1)) \xrightarrow{\sim} H^1(X(\mathbf{R}), \mathbf{F}_2).$$

The composition

$$H^2(X(\mathbf{C}), G; \mathbf{Z}(1)) \to H^2(X(\mathbf{R}), G; \mathbf{Z}(1)) \xrightarrow{\sim} H^1(X(\mathbf{R}), \mathbf{F}_2)$$

will be denoted by β.

Proposition 8.

$$\alpha(cw(L)) = c(L(\mathbf{C})), \qquad \beta(cw(L)) = w(L(\mathbf{R})).$$

Proof. First we remark that the exact sequence of sheaves (3.11) gives rise to the exact sequence of Galois-Grothendieck cohomology groups and the exact sequence of usual cohomology of sheaves. Moreover, there is a homomorphism from the first sequence into the second one. In particular, we have the commutative diagram

$$\begin{array}{ccc} H^1(X(\mathbf{C}), G; \mathscr{O}^*) & \xrightarrow{\delta} & H^2(X(\mathbf{C}), G; \mathbf{Z}(1)) \\ \downarrow & & \downarrow \\ H^1(X(\mathbf{C}), \mathscr{O}^*) & \xrightarrow{\delta} & H^2(X(\mathbf{C}), \mathbf{Z}(1)), \end{array}$$

which implies the first equality in the proposition. To prove the other equality, consider the exact sequence of G-sheaves

(3.1.3) $$0 \to \mathscr{H}^0(G, \mathscr{O}|_{X(\mathbf{R})} \to \mathscr{H}^0(G, \mathscr{O}^*)|_{X(\mathbf{R})} \xrightarrow{\delta} \mathscr{H}^1(G, \mathbf{Z}(1)) \to 0,$$

which is obtained from (3.1.1). It coincides with another exact exponential sequence

(3.1.4) $$0 \to \mathscr{A} \xrightarrow{\exp} \mathscr{A}^* \xrightarrow{\text{sgn}} \mathbf{F}_2 \to 0$$

on $X(\mathbf{R})$, where $\mathscr{A}$ is the sheaf of germs of real-valued complex-analytic functions on $X(\mathbf{R})$ and $\mathscr{A}^*$ is the sheaf of germs of invertible functions. Thus exact sequences (3.1.3) and (3.1.4) give rise to the commutative diagram

(3.1.5) $$\begin{array}{ccc} H^1(X(\mathbf{R}), \mathscr{H}^0(G, \mathscr{O}^*)) & \longrightarrow & H^1(X(\mathbf{R}), \mathscr{H}^1(G, \mathbf{Z}(1)) \\ \| & & \| \\ H^1(X(\mathbf{R}), \mathscr{A}^*) & \xrightarrow{\text{sgn}} & H^1(X(\mathbf{R}), \mathbf{F}_2) \end{array}$$

Note now that the homomorphism

$$H^1(X(\mathbf{R}), \mathscr{H}^0(G, \mathscr{O}^*)) \to H^1(X(\mathbf{R}), \mathbf{F}_2),$$

which is derived from (3.1.5), coincides with the composition

$$H^1(X(\mathbf{R}), \mathscr{H}^0(G, \mathscr{O}^*)) \hookrightarrow H^1(X(\mathbf{R}), G; \mathscr{O}^*)$$
$$\xrightarrow{\delta} H^2(X(\mathbf{R}), G; \mathbf{Z}(1)) \xrightarrow{\beta} H^1(X(\mathbf{R})\mathbf{F}_2).$$

In particular,

$$\beta(\delta(L(\mathbf{C})|_{X(\mathbf{R})})) = \operatorname{sgn}(L(\mathbf{R})). \tag{3.1.6}$$

Since

$$\operatorname{sgn}(L(\mathbf{R})) = w(L(\mathbf{R})), \qquad \beta(\delta(L(\mathbf{C})|_{X(\mathbf{R})})) = \beta(cw(L)),$$

the second equality follows from (3.1.6). The proposition is proved.

3.2. **Orientable components.** First we introduce some notation. Let $X_+(\mathbf{R})$ denote the union of the orientable components of $X(\mathbf{R})$. For brevity we set

$$F^q H^{2n+1}(\cdot, G; \mathbf{F}_2) = F^q(\cdot),$$

where $F^q H^{2n+1}(\cdot, G; \mathbf{F}_2)$ denotes the filtration obtained from the spectral sequence

$$I_2^{p,q}(\cdot, G; \mathbf{F}_2) = H^p(G, H^q(\cdot, \mathbf{F}_2)) \Rightarrow H^{2n+1}(\cdot, G; \mathbf{F}_2).$$

This abbreviation will be used for the spaces $X(\mathbf{C})$, $X(\mathbf{R})$, and $X_+(\mathbf{R})$. Moreover, let

$$\mathscr{K} \in H^2(X(\mathbf{C}), G; \mathbf{Z}(1)), \quad K \in H^2(X(\mathbf{C}), \mathbf{Z}), \quad k \in H^1(X(\mathbf{R}), \mathbf{F}_2)$$

be the characteristic classes of the canonical bundles on X, $X(\mathbf{C})$, and $X(\mathbf{R})$, respectively. Note that $K = \alpha(\mathscr{K})$, $k = \beta(\mathscr{K})$, and $k|_{X_+(\mathbf{R})} = 0$. Also let

$$K^q\colon H^{n-q}(X(\mathbf{C}), \mathbf{F}_2) \to H^{n+q}(X(\mathbf{C}), \mathbf{F}_2) \tag{3.2.1}$$

denote the homomorphism given by multiplication by K^q.

Theorem 11. *Let X be a* GM-*variety such that the homomorphisms* (3.2.1) *are isomorphisms for $0 \le q \le n$. Then $\kappa_+(X) = 0$.*

Proof. Consider the homomorphism

$$F^n(X(\mathbf{C}))/F^{n-1}(X(\mathbf{C})) \to F^n(X_+(\mathbf{R}))/F^{n-1}(X_+(\mathbf{R})). \tag{3.2.2}$$

It coincides with

$$H^{n+1}(G, H^n(X(\mathbf{C}), \mathbf{F}_2)) \to H^n(X_+(\mathbf{R}), \mathbf{F}_2), \tag{3.2.3}$$

since X is a GM-variety. In turn, (3.2.3) is determined by the homomorphism

$$H^n(X(\mathbf{C}), \mathbf{F}_2) \to H^n(X_+(\mathbf{R}), \mathbf{F}_2), \tag{3.2.4}$$

induced by the embedding $X_+(\mathbf{R}) \subset X(\mathbf{C})$. We shall prove that (3.2.2) is an epimorphism. This would imply that (3.2.4) is also an epimorphism, and the latter means that $\kappa_+(X) = 0$. To prove that (3.2.2) is an epimorphism we first remark that the homomorphism

$$F^{2n+1}(X(\mathbf{C}))/F^{n-1}(X(\mathbf{C})) \to F^{2n+1}(X_+(\mathbf{R}))/F^{n-1}(X_+(\mathbf{R})) \tag{3.2.5}$$

is an epimorphism because the map

$$X^{2n+1}(X(\mathbf{C}), G; \mathbf{F}_2) \to H^{2n+1}(X(\mathbf{R}), G; \mathbf{F}_2)$$

is an isomorphism and the map

$$H^{2n+1}(X(\mathbf{R}), G; \mathbf{F}_2) \to H^{2n+1}(X_+(\mathbf{R}), G; \mathbf{F}_2)$$

is an epimorphism. Now we prove by induction on q that

$$\begin{aligned}(3.2.6)\qquad &\operatorname{Im}[F^{n+q}(X(\mathbf{C}))/F^{n-1}(X(\mathbf{C})) \to F^{n+q}(X_+(\mathbf{R}))/F^{n-1}(X_+(\mathbf{R}))] \\ &= \operatorname{Im}[F^{n}(X(\mathbf{C}))/F^{n-1}(X(\mathbf{C})) \to F^{n}(X_+(\mathbf{R}))/F^{n-1}(X_+(\mathbf{R}))],\end{aligned}$$

where $q \geq 0$. Because

$$F^{n+q}(X_+(\mathbf{R}))/F^{n-1}(X_+(\mathbf{R})) = F^{n}(X_+(\mathbf{R}))/F^{n-1}(X_+(\mathbf{R})),$$

the surjectivity of (3.2.2) follows from (3.2.6) and the surjectivity of (3.2.5). When $q = 0$ both sides of (3.2.6) are the same. Assume that this is also true for $q = m - 1 \geq 0$. We now show that this is true for $q = m$. To this end, consider the homomorphism

$$\begin{aligned}(3.2.7)\qquad &F^{n-m}(H^{2n-2m+1}(X(\mathbf{C}), G; \mathbf{F}_2))/F^{n-m-1} \\ &\xrightarrow{\cup \mathscr{K}^m} F^{n+m}(H^{2n+1}(X(\mathbf{C}), G, \mathbf{F}_2))/F^{n+m-1}.\end{aligned}$$

Since X is a GM-variety, this homomorphism coincides with

$$H^{n-m+1}(G, H^{n-m}(X(\mathbf{C}), \mathbf{F}_2)) \xrightarrow{\cup K^m} H^{n-m+1}(G, H^{n+m}(X(\mathbf{C}), \mathbf{F}_2)).$$

It now follows from the hypothesis of the theorem that (3.2.7) is an isomorphism. Therefore

$$F^{n+m}(X(\mathbf{C})) = F^{n+m-1}(X(\mathbf{C})) + \mathscr{K}^m \cup F^{n-m}(H^{2n-2m+1}(X(\mathbf{C}), G; \mathbf{F}_2)).$$

The image of $\mathscr{K}^m \cup F^{n-m}(H^{2n-2m+1}(X(\mathbf{C}), G; \mathbf{F}_2))$ under the homomorphism $F^{n+m}(X(\mathbf{C})) \to F^{n+m}(X_+(\mathbf{R}))$ is zero because $\mathscr{K}$ goes to k. Therefore the images of $F^{n+m}(X(\mathbf{C}))$ and $F^{n+m-1}(X(\mathbf{C}))$ coincide. The theorem is proved.

Remark 7. Under the assumptions of Theorem 11 the number K^n is odd, and therefore $X(\mathbf{R})$ contains at least one nonorientable component because in this case $k^n \neq 0$.

3.3. **Nonorientable components.** First we obtain two general bounds, and then we apply them to certain types of varieties.

Proposition 9.

$$\kappa_-(X) \leq \sum_{q>0} \dim_{\mathbf{F}_2} H^{\delta(q)}(G, H^{n+q}(X(\mathbf{C}), \mathbf{Z})),$$

where

$$\delta(q) = \begin{cases} 1 & q \text{ even}, \\ 2 & q \text{ odd} \end{cases}$$

Proof. We use the Galois-Grothendieck cohomology groups $H^{2n+2}(\cdot, G; \mathbf{Z})$ when n is odd, and $H^{2n+1}(\cdot, G; \mathbf{Z})$ when n is even. Consider the case of odd n. Note first that

$$I_\infty^{n+2,n}(X(\mathbf{R}), G; \mathbf{Z}) = H^{n+2}(G, H^n(X(\mathbf{R}), \mathbf{Z})) = H^n(X(\mathbf{R}), \mathbf{Z})_2,$$

where $H^n(X(\mathbf{R}), \mathbf{Z})_2$ is the subgroup of $H^n(X(\mathbf{R}), \mathbf{Z})$, consisting of elements of order two. On the other hand,

$$\begin{aligned}\kappa_-(X) &\leq \dim \operatorname{Coker}[H^n(X(\mathbf{C}), \mathbf{Z})^{-\tau^*} \to H^n(X(\mathbf{R}), \mathbf{Z})_2] \\ &\leq \dim \operatorname{Coker}[H^{n+2}(G, H^n(X(\mathbf{C}), \mathbf{Z})) \to H^{n+2}(G, H^n(X(\mathbf{R}), \mathbf{Z}))] \\ &\leq \dim \operatorname{Coker}[I_\infty^{n+2,n}(X(\mathbf{C}), G; \mathbf{Z}) \to I_\infty^{n+2,n}(X(\mathbf{R}), G; \mathbf{Z})].\end{aligned}$$

Since the homomorphism $H^{2n+2}(X(\mathbf{C}), G; \mathbf{Z}) \to H^{2n+2}(X(\mathbf{R}), G; \mathbf{Z})$ is an isomorphism, the homomorphism

$$\begin{aligned} &H^{2n+2}(X(\mathbf{C}), G; \mathbf{Z})/F^{n-1}(X(\mathbf{C}), G; \mathbf{Z}) \\ &\quad \to H^{2n+2}(X(\mathbf{R}), G; \mathbf{Z})/F^{n-1}(X(\mathbf{R}), G; \mathbf{Z}) \end{aligned}$$

is an epimorphism. Therefore

$$\begin{aligned} \dim \operatorname{Coker}[&I_\infty^{n+2,n}(X(\mathbf{C}), G; \mathbf{Z}) \to I_\infty^{n+2,n}(X(\mathbf{R}), G; \mathbf{Z})] \\ &\le \dim H^{2n+2}(X(\mathbf{C}), G; \mathbf{Z})/F^n(X(\mathbf{C}), G; \mathbf{Z}) \\ &\le \sum_{q>0} \dim H^{n-q+2}(G, H^{n+q}(X(\mathbf{C}), \mathbf{Z})) \\ &= \sum_{q>0} \dim H^{\delta(q)}(G, H^{n+q}(X(\mathbf{C}), \mathbf{Z})). \end{aligned}$$

The proposition is proved.

Remark 8. Similar arguments yield the relative version of Proposition 9:

$$\kappa_-(X, Y) \le \sum_{q>0} \dim_{\mathbf{F}_2} H^{\delta(q)}(G, H^{n+q}(X(\mathbf{C}), Y(\mathbf{C}); \mathbf{Z})).$$

Notice that under the map

$$H^n(X(\mathbf{C}), \mathbf{Z}) \to H^n(X(\mathbf{R}), \mathbf{Z})$$

the subgroup $H^n(X(\mathbf{C}), \mathbf{Z})^{-\tau^*}$ goes to the subgroup $H^n(X(\mathbf{R}), \mathbf{Z})_2$. Therefore the number

$$\kappa_-(f) = \dim \operatorname{Coker}[H^n(X(\mathbf{C}), \mathbf{Z})^{-\tau^*} \oplus H^n(Z(\mathbf{R}), \mathbf{Z})_2 \to H^n(X(\mathbf{R}), \mathbf{Z})_2]$$

can be defined for a map $f\colon X \to Z$ of real algebraic varieties.

Theorem 12. *Suppose that Z is a* GM**Z**-*variety. Then*

$$\kappa_-(f) \le \sum_{q>0} \dim_{\mathbf{F}_2} \operatorname{Coker}[H^{\delta(q)}(G, H^{n+q}(Z(\mathbf{C}), \mathbf{Z})) \to H^{\delta(q)}(G, H^{n+q}(X(\mathbf{C}), \mathbf{Z}))].$$

The proof is similar to that of Theorem 4. One should only use the Galois-Grothendieck cohomology groups $H^{2m+2}(\cdot, G; Z)$, n is odd, and $H^{2m+1}(\cdot, G; \mathbf{Z})$, n is even.

We now consider some consequences of our results.

Corollary 3. *Let X be a surface such that $H_1(X(\mathbf{C}), \mathbf{Z}) = 0$. Then $\kappa_-(X) = 0$.*

This follows from Proposition 9.

Corollary 4. *Let X be a complete intersection of odd degree. Then $\kappa_-(X) = 0$.*

Proof. If n is even, then $\kappa(X) = 0$ by Theorem 5, and therefore $\kappa_-(X) = 0$. If n is odd, then on applying Theorem 12 to the embedding $f\colon X \hookrightarrow \mathbf{P}^N$ we have the equality $\kappa_-(f) = 0$. On the other hand, $H^n(\mathbf{P}^N(\mathbf{C}), \mathbf{Z}) = 0$ when n is odd, and therefore

$$0 = \kappa_-(f) = \dim \operatorname{Coker}[H^n(X(\mathbf{C}), \mathbf{Z})^{-\tau^*} \to H^n(X(\mathbf{R}), \mathbf{Z})_2].$$

It now follows that $\kappa_-(X) = 0$. The corollary is proved.

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Received 5/MAY/88

Translated by A. MARTSINKOVSKY