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1994 Russian Acad. Sci. Izv. Math. 43 385

(http://iopscience.iop.org/1468-4810/43/2/A11)

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Сер. Матем. Том 57 (1993), № 5

Russian Acad. Sci. Izv. Math.
Vol. 43 (1994), No. 2

# ON COHOMOLOGY CLASSES DEFINED BY THE REAL POINTS OF A REAL ALGEBRAIC GM-SURFACE

UDC 513.6+517.6

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ABSTRACT. The cohomology classes $x_i = [X_i]^* \in H^2(X(\mathbb{C}), \mathbb{Z})$ are studied, where $X_1, \dots, X_m$ are the connected components of the set of real points $X(\mathbb{R})$ of a real algebraic GM-surface $X$ and $X(\mathbb{R}) = X_1 \cup \dots \cup X_m$ is assumed to be orientable. The results are applied to obtain congruences for the Euler characteristic of $X(\mathbb{R})$.

## INTRODUCTION

Let $X$ be a nonsingular projective real algebraic surface, and let $\tau\colon X(\mathbb{C}) \to X(\mathbb{C})$ be the involution of complex conjugation. Put $Y = X(\mathbb{C})/\tau$ and denote by $\pi\colon X(\mathbb{C}) \to Y$ the corresponding projection. Let $X_1, \dots, X_m$ be the connected components of $X(\mathbb{R})$, and let $x_i = [X_i]^* \in H^2(X(\mathbb{C}), \mathbb{F}_2)$ $(i \in \{1, \dots, m\})$ be their cohomology classes. We may assume that the components $X_1, \dots, X_m$ are embedded in $Y$; since $Y$ is a differentiable fourfold, the cohomology classes $y_i = [X_i]^*_Y \in H^2(Y, \mathbb{F}_2)$ are defined. In the present paper we prove, in particular, the following

**Theorem 1.** *If $X$ is a* GM-*surface, then the kernel of the natural homomorphism $\pi^*\colon H^2(Y, \mathbb{F}_2) \to H^2(X(\mathbb{C}), \mathbb{F}_2)$ is spanned by the cohomology classes $y_1, \dots, y_m$.*

Suppose that all the surfaces $X_1, \dots, X_m$ are orientable. We choose an orientation and denote by $X_1^+, \dots, X_m^+$ the oriented components. These oriented components define the cohomology classes $x_i^+ = [X_i^+]^* \in H^2(X(\mathbb{C}), \mathbb{Z})$ ($i \in \{1, \dots, m\}$). In addition to these cohomolgy classes we will consider the cohomology classes $\mathbf{x}_i^+ \in H^2(X(\mathbb{C}); G, \mathbb{Z})$, where $G = \{e, \tau\}$, also defined by the oriented surfaces $X_1^+, \dots, X_m^+$. A precise definition of these cohomology classes is given in §1.2, and now we only remark that under the canonical homomorphism $\alpha\colon H^2(X(\mathbb{C}); G, \mathbb{Z}) \to H^2(X(\mathbb{C}), \mathbb{Z})$ the cohomology classes $\mathbf{x}_1^+, \dots, \mathbf{x}_m^+$ are mapped to $x_1^+, \dots, x_m^+$.

In this paper we prove the following

**Theorem 2.**

$$H^2(X(\mathbb{C}); G, \mathbb{Z}) = \pi^* H^2(Y; G, \mathbb{Z}) + \mathscr{L}(\mathbf{x}_1^+, \dots, \mathbf{x}_m^+),$$

*where $\mathscr{L}(\mathbf{x}_1^+, \dots, \mathbf{x}_m^+)$ is the linear span of $\mathbf{x}_1^+, \dots, \mathbf{x}_m^+$ and the involution $\tau$ acts trivially on $Y$.*

The most important result for further applications is the following

**Corollary.** *If $X$ is a* GM-*surface and the group $H^*(X(\mathbb{C}), \mathbb{Z})$ is free, then the following equality holds:*

$$H^2(X(\mathbb{C}), \mathbb{Z})^{\tau^*} = \pi^* H^2(Y, \mathbb{Z}) + \mathscr{L}(x_1^+, \dots, x_m^+).$$

1991 *Mathematics Subject Classification.* Primary 14F25, 14F45, 14G30, 14J25, 32C05.

We denote by $\chi$ (resp. $\chi_i$) the Euler characteristic of $X(\mathbb{R})$ (resp. $X_i$) and by $\sigma$ the signature of $X(\mathbb{C})$. Using Theorem 1, in the present paper we prove the following

**Theorem 3.** *Let $X$ be a* GM*-surface such that the surfaces $X_1, \dots, X_{m-1}$ are orientable and the surface $X_m$ is not necessarily orientable, $\chi_i \equiv 0 \pmod 4$ for $i \in \{1, \dots, , m-1\}$, and $w_2(X(\mathbb{C})) = [X(\mathbb{R})]^*$. Then $\chi \equiv \sigma \pmod{16}$.*

Two special cases of this theorem for surfaces with $H_1(X(\mathbb{C}), \mathbb{Z}) = 0$ were proved in [1]. In the first case it is assumed that the quadratic form on $H_2(X(\mathbb{C}), \mathbb{Z})$ is even and the surface $X(\mathbb{R})$ is orientable. In the second case the assumption is that the quadratic form on $H_2(X(\mathbb{C}), \mathbb{Z})$ is odd and the surface $X(\mathbb{R})$ has one nonorientable component. The following result of A. L. Slepyan [2] is also a special case of Theorem 3: *If $C$ is a plane curve of degree $2k$ such that $C(\mathbb{R})$ disconnects $C(\mathbb{C})$ and the characteristic of each odd oval is even, then $p - n \equiv k^2 \pmod 8$.*

Using the corollary of Theorem 2, in the present paper we prove the following

**Theorem 4.** *Let $X$ be a surface such that $H^*(X(\mathbb{C}), \mathbb{Z})$ is a free group, $X(\mathbb{R}) = X_1 \cup \dots \cup X_m$ is an orientable surface, and $[X(\mathbb{R})] = 0$ in $H_2(X(\mathbb{C}), \mathbb{F}_2)$. Then the following assertions hold:*

(i) *if $X$ is an $M$-surface and $\chi_i \equiv 0 \pmod{2^\mu}$ for $i \in \{1, \dots, m\}$, then $\chi \equiv 0 \pmod{2^{\mu+3}}$;*

(ii) *if $X$ is an $(M-1)$-surface, $\chi_i \equiv 0 \pmod{2^\mu}$ for $i \in \{1, \dots, m\}$, and $\mu \geq 2$, then either $\chi \equiv 0 \pmod{2^{\mu+3}}$ or $\chi = 2^{2\nu+1} \cdot \chi'$, where $\chi'$ is an odd number.*

For surfaces with $H_1(X(\mathbb{C}), \mathbb{Z}) = 0$ this theorem was proved in [1]. We remark that in the proof of Theorem 4 we use constructions from [1], but, due to the corollary of Theorem 2, our proof does not rely on complicated theorems from number theory.

## §1. General theorems on cohomology classes defined by real points

1.1. **The kernel of the homomorphism** $\pi^* : H^2(Y, \mathbb{F}_2) \to H^2(X(\mathbb{C}), \mathbb{F}_2)$. Consider the spectral sequence

$$\mathrm{I}_2^{p,q} = H^p(Y, \mathscr{H}^q(G, \mathbb{F}_2)) \Rightarrow H^{p+q}(X(\mathbb{C}); G, \mathbb{F}_2).$$

By [3], this spectral sequence yields the following five-term exact sequence:

$$\begin{aligned} 0 \longrightarrow H^1(Y, \mathbb{F}_2) \longrightarrow H^1(X(\mathbb{C}); G, \mathbb{F}_2) \\ \longrightarrow H^0(X(\mathbb{R}), \mathbb{F}_2) \longrightarrow H^2(Y, \mathbb{F}_2) \longrightarrow H^2(X(\mathbb{C}); G, \mathbb{F}_2). \end{aligned} \tag{1}$$

We compute the homomorphism $H^0(X(\mathbb{R}), \mathbb{F}_2) \to H^2(Y, \mathbb{F}_2)$ in this sequence. To the components $X_1, \dots, X_m$ there correspond generators of the group $H^0(X(\mathbb{R}), \mathbb{F}_2)$ which will be denoted by $\tilde{X}_1, \dots, \tilde{X}_m$.

**Lemma 1.1.1.** *Under the homomorphism $H^0(X(\mathbb{R}), \mathbb{F}_2) \to H^2(Y, \mathbb{F}_2)$ the generators $\tilde{X}_1, \dots, \tilde{X}_m$ are mapped to $y_1, \dots, y_m$.*

*Proof.* Let $U_i$ be an open $\varepsilon$-neighborhood of $X_i$ in $X(\mathbb{C})$ with respect to a $\tau$-invariant Riemann metric and put $V_i = U_i/\tau$. Denote by $(\mathbb{F}_2)_{U_i}$ the kernel of the restriction homomorphism $\mathbb{F}_2 \to \mathbb{F}_2|_{X(\mathbb{C})-U_i}$; this is a sheaf on $X(\mathbb{C})$. In a similar way we define a sheaf $(\mathbb{F}_2)_{V_i}$ on $Y$. The inclusion of the sheaf $(\mathbb{F}_2)_{U_i}$ in $\mathbb{F}_2$ induces a homomorphism of the five-term exact sequence for the sheaf $(\mathbb{F}_2)_{U_i}$ to the exact sequence (1); in particular, we have a commutative diagram

$$\begin{array}{ccccc} H^0(X_i, \mathbb{F}_2) & \longrightarrow & H^2(Y, (\mathbb{F}_2)_{V_i}) & \longrightarrow & H^2(X(\mathbb{C}); G, (\mathbb{F}_2)_{U_i}) \\ \downarrow & & \downarrow & & \downarrow \\ H^2(X(\mathbb{R}), \mathbb{F}_2) & \longrightarrow & H^2(Y, \mathbb{F}_2) & \longrightarrow & H^2(X(\mathbb{C}); G, \mathbb{F}_2) \end{array} \tag{2}$$

The group $H^0(X_i, \mathbb{F}_2)$ has a single generator which is mapped to $\tilde{X}_i$ under the homomorphism $H^0(X_i, \mathbb{F}_2) \to H^0(X(\mathbb{R}), \mathbb{F}_2)$ in diagram (2). Similarly, the group $H^2(Y, (\mathbb{F}_2)_{V_i})$ has a single generator which is mapped to $y_i$ by the homomorphism $H^2(Y, (\mathbb{F}_2)_{V_i}) \to H^2(Y, \mathbb{F}_2)$. Hence it suffices to prove vanishing of the homomorphism $H^2(Y, (\mathbb{F}_2)_{V_i}) \to H^2(X(\mathbb{C}); G, (\mathbb{F}_2)_{U_i})$. To this end we consider the composition of homomorphisms

$$H^2(Y, (\mathbb{F}_2)_{V_i}) \to H^2(X(\mathbb{C}); G, (\mathbb{F}_2)_{U_i}) \xrightarrow{\alpha} H^2(X(\mathbb{C}), (\mathbb{F}_2)_{U_i}).$$

According to [3], this composition coincides with the homomorphism

$$\pi^*: H^2(Y, (\mathbb{F}_2)_{V_i}) \to H^2(X(\mathbb{C}), (\mathbb{F}_2)_{U_i}),$$

which is equal to zero. It remains to observe that the homomorphism

$$\alpha: H^2(X(\mathbb{C}); G, (\mathbb{F}_2)_{U_i}) \to H^2(X(\mathbb{C}), (\mathbb{F}_2)_{U_i})$$

is an isomorphism. This follows from the spectral sequence

$$\mathrm{II}_2^{p,q} = H^p(G, H^q(X(\mathbb{C}), (\mathbb{F}_2)_{U_i})) \Rightarrow H^{p+q}(X(\mathbb{C}); G, (\mathbb{F}_2)_{U_i}).$$

The lemma is proved.

**Theorem 1.1.2.** *Let $X$ be a* GM-*surface. Then the kernel of the homomorphism*

$$\pi^*: H^2(Y, \mathbb{F}_2) \to H^2(X(\mathbb{C}), \mathbb{F}_2)$$

*is generated by the cohomology classes $y_1, \dots, y_m$.*

*Proof.* From Lemma 1.1.1 and the exact sequence (1) it follows that the kernel of the homomorphism $H^2(Y, \mathbb{F}_2) \to H^2(X(\mathbb{C}); G, \mathbb{F}_2)$ is generated by the cohomology classes $y_1, \dots, y_m$. Consider the composition of maps

$$H^2(Y, \mathbb{F}_2) \to H^2(X(\mathbb{C}); G, \mathbb{F}_2) \xrightarrow{\alpha} H^2(X(\mathbb{C}), \mathbb{F}_2);$$

it coincides with the homomorphism $\pi: H^2(Y, \mathbb{F}_2) \to H^2(X(\mathbb{C}), \mathbb{F}_2)$.

Denote by $\{F^q H^2(X(\mathbb{C}); G, \mathbb{F}_2)\}$ the filtration in $H^2(X(\mathbb{C}); G, \mathbb{F}_2)$ induced by the spectral sequence

$$\mathrm{II}_2^{p,q} = H^p(G, H^q(X(\mathbb{C}), \mathbb{F}_2)) \Rightarrow H^{p+q}(X(\mathbb{C}); G, \mathbb{F}_2);$$

then $\ker\alpha = F^1 H^2(X(\mathbb{C}); G, \mathbb{F}_2)$. We need to show that the intersection of the image of $H^2(Y, \mathbb{F}_2) \to H^2(X(\mathbb{C}); G, \mathbb{F}_2)$ with $F^1 H^2(X(\mathbb{C}); G, \mathbb{F}_2)$ is trivial. To this end we consider the exact sequence of type (1) for the sheaf $\mathbb{F}_2$ on $X(\mathbb{R})$:

$$\begin{aligned} 0 \to H^1(X(\mathbb{R}), \mathbb{F}_2) &\to H^1(X(\mathbb{R}); G, \mathbb{F}_2) \\ &\to H^0(X(\mathbb{R}), \mathbb{F}_2) \to H^2(X(\mathbb{R}), \mathbb{F}_2) \to H^2(X(\mathbb{R}); G, \mathbb{F}_2). \end{aligned}$$

Since the composition of maps

$$H^2(X(\mathbb{R}), \mathbb{F}_2) \to H^2(X(\mathbb{R}); G, \mathbb{F}_2) \xrightarrow{\alpha} H^2(X(\mathbb{R}), \mathbb{F}_2)$$

coincides with the identity map, the image of the homomorphism $H^2(X(\mathbb{R}), \mathbb{F}_2) \to H^2(X(\mathbb{R}); G, \mathbb{F}_2)$ has trivial intersection with $F^1H^2(X(\mathbb{R}); G, \mathbb{F}_2)$. Consider now the commutative diagram

$$\begin{array}{ccc} H^2(Y, \mathbb{F}_2) & \longrightarrow & H^2(X(\mathbb{C}); G, \mathbb{F}_2) \\ \downarrow & & \downarrow \\ H^2(X(\mathbb{R}), \mathbb{F}_2) & \longrightarrow & H^2(X(\mathbb{R}); G, \mathbb{F}_2) \end{array}$$

where the vertical arrows are restriction homomorphisms. Since $X$ is a GM-surface, the restriction homomorphism $F^1H^2(X(\mathbb{C}); G, \mathbb{F}_2) \to F^1H^2(X(\mathbb{R}); G, \mathbb{F}_2)$ is an inclusion (cf. [4]). From the above observation on the image of the homomorphism $H^2(X(\mathbb{R}), \mathbb{F}_2) \to H^2(X(\mathbb{R}); G, \mathbb{F}_2)$ we infer a similar assertion about the image of the homomorphism $H^2(Y, \mathbb{F}_2) \to H^2(X(\mathbb{C}); G, \mathbb{F}_2)$. The theorem is proved.

1.2. **The structure of the group $H^2(X(\mathbb{C}), \mathbb{Z})^{\tau^*}$.** In this subsection we assume that $X(\mathbb{R})$ is an orientable surface. First of all we explain how to construct the cohomology classes $\mathbf{x}_1^+, \dots, \mathbf{x}_m^+ \in H^2(X(\mathbb{C}); G, \mathbb{Z})$. These classes are constructed as follows. The oriented components $X_1^+, \dots, X_m^+$ define via Poincaré-Lefschetz duality the cohomology classes $\tilde{x}_1^+, \dots, \tilde{x}_m^+ \in H^2(X(\mathbb{C}), X(\mathbb{C})\setminus X(\mathbb{R}); \mathbb{Z})$, which are mapped to $x_1^+, \dots, x_m^+$ under the homomorphism $H^2(X(\mathbb{C}), X(\mathbb{C})\setminus X(\mathbb{R}); \mathbb{Z}) \to H^2(X(\mathbb{C}), \mathbb{Z})$. The canonical homomorphism

$$\tilde{\alpha}\colon H^2(X(\mathbb{C}), X(\mathbb{C}) \setminus X(\mathbb{R}); G, \mathbb{Z}) \to H^2(X(\mathbb{C}), X(\mathbb{C}) \setminus X(\mathbb{R}); \mathbb{Z})$$

is an isomorphism; this follows from the second spectral sequence

$$\mathrm{II}_2^{p,q} = H^p\big(G, H^q(X(\mathbb{C}), X(\mathbb{C}) \setminus X(\mathbb{R}); \mathbb{Z})\big) \Rightarrow H^{p+q}\big(X(\mathbb{C}), X(\mathbb{C}) \setminus X(\mathbb{R}); G, \mathbb{Z}\big).$$

We denote by $\tilde{\mathbf{x}}_1^+, \dots, \tilde{\mathbf{x}}_m^+$ the preimages $\tilde{\alpha}^{-1}(\tilde{x}_1^+), \dots, \tilde{\alpha}^{-1}(\tilde{x}_m^+)$. By definition, their images under the homomorphism

$$H^2(X(\mathbb{C}), X(\mathbb{C}) \setminus X(\mathbb{R}); G, \mathbb{Z}) \to H^2(X(\mathbb{C}); G, \mathbb{Z})$$

are equal to $\mathbf{x}_1^+, \dots, \mathbf{x}_m^+$. We observe that the canonical homomorphism $\alpha\colon H^2(X(\mathbb{C}); G, \mathbb{Z}) \to H^2(X(\mathbb{C}), \mathbb{Z})$ maps the cohomology classes $\mathbf{x}_1^+, \dots, \mathbf{x}_m^+$ to $x_1^+, \dots, x_m^+$.

**Theorem 1.2.1.**

$$H^2(X(\mathbb{C}); G, \mathbb{Z}) = \pi^* H^2(Y; G, \mathbb{Z}) + \mathscr{L}(\mathbf{x}_1^+, \dots, \mathbf{x}_m^+), \tag{1}$$

*where the action of $\tau$ on $Y$ is trivial.*

A proof of this theorem will be given in the end of this section, and now we formulate a corollary, important for applications, and prove several lemmas used in the proof of the theorem.

**Corollary 1.2.2.** *Let $X$ be a* GM$\mathbb{Z}$*-surface. Then*

$$H^2(X(\mathbb{C}), \mathbb{Z})^{\tau^*} = \pi^* H^2(Y, \mathbb{Z}) + \mathscr{L}(x_1^+, \dots, x_m^+). \tag{2}$$

*Proof.* A definition of GM$\mathbb{Z}$-variety is given in [5], where it is shown that the spectral sequence $\mathrm{II}(X(\mathbb{C}); G, \mathbb{Z})$ for such a variety degenerates. From this it follows that the image of the homomorphism $\alpha\colon H^q(X(\mathbb{C}); G, \mathbb{Z}) \to H^q(X(\mathbb{C}), \mathbb{Z})$ is equal to $H^q(X(\mathbb{C}), \mathbb{Z})^{\tau^*}$. Applying $\alpha$ to equality (1), we get equality (2). In fact, as we already explained, $\alpha(H^2(X(\mathbb{C}); G, \mathbb{Z})) = H^2(X(\mathbb{C}), \mathbb{Z})^{\tau^*}$ and we have inclusions

$\alpha(\pi^* H^2(Y; G, \mathbb{Z})) \subset \pi^* H^2(Y, \mathbb{Z}) \subset H^2(X(\mathbb{C}), \mathbb{Z})^{\tau^*}$. Now it remains to observe that $\alpha(\mathscr{L}(\mathbf{x}_1^+, \dots, \mathbf{x}_m^+)) = \mathscr{L}(x_1^+, \dots, x_m^+)$. The corollary is proved.

Before formulating our lemmas we introduce additional notation. We choose a $\tau$-invariant Riemann metric on $X(\mathbb{C})$; let $U$ be the open $\varepsilon$-neighborhood of $X(\mathbb{R})$ with respect to this metric. Put $\mathscr{A} = \pi_* \mathbb{Z}_{X(\mathbb{C})}$; then $\mathbb{Z}_Y$ is a subsheaf of $\mathscr{A}$. Let $V = \pi(U) = U/\tau$ and let $\mathscr{A}_V$ be the kernel of the restriction homomorphism $\mathscr{A} \to \mathscr{A}|_{Y-V}$. We denote by $\tilde{\mathscr{A}}$ the subsheaf $\mathbb{Z}_Y + \mathscr{A}_V$ and by $\mathscr{B}$ the quotient sheaf $\mathscr{A}/\tilde{\mathscr{A}}$. We observe that $\tau$ acts on the sheaves $\mathscr{A}$, $\mathscr{A}_V$, $\tilde{\mathscr{A}}$, $\mathscr{B}$.

**Lemma 1.2.3.** *The homomorphism $H^2(Y; G, \tilde{\mathscr{A}}) \to H^2(Y; G, \mathscr{A})$ induced by the inclusion $\tilde{\mathscr{A}} \subset \mathscr{A}$ is an epimorphism.*

*Proof.* Consider the commutative diagram of exact sequences of sheaves

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \tilde{\mathscr{A}} & \longrightarrow & \mathscr{A} & \longrightarrow & \mathscr{B} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \tilde{\mathscr{A}}|_{Y-V} & \longrightarrow & \mathscr{A}|_{Y-V} & \longrightarrow & \mathscr{B}|_{Y-V} & \longrightarrow & 0 \end{array}$$

where the vertical arrows are restriction homomorphisms. This diagram yields the following commutative diagram of exact cohomology sequences:

(3)

$$\begin{array}{ccccccccc} \cdots \to & H^2(Y; G, \tilde{\mathscr{A}}) & \to & H^2(Y; G, \mathscr{A}) & \to & H^2(Y; G, \mathscr{B}) & \to \cdots \\ & \downarrow & & \downarrow & & {\scriptstyle s}\downarrow & \\ \cdots \to & H^2(Y; G, \tilde{\mathscr{A}}|_{Y-V}) & \to & H^2(Y; G, \mathscr{A}|_{Y-V}) & \to & H^2(Y; G, \mathscr{B}|_{Y-V}) & \to \cdots \end{array}$$

Since the sheaf $\mathscr{B}$ vanishes on $V$, the restriction homomorphism $H^2(Y; G, \mathscr{B}) \to H^2(Y; G, \mathscr{B}|_{Y-V})$ is an isomorphism. Now from diagram (3) it is evident that it suffices to show that the homomorphism $H^2(Y; G, \tilde{\mathscr{A}}|_{Y-V}) \to H^2(Y; G, \mathscr{A}|_{Y-V})$ is an epimorphism. Since $\tilde{\mathscr{A}}|_{Y-V} = \mathbb{Z}|_{Y-V}$, we have

$$H^2(Y; G, \tilde{\mathscr{A}}|_{Y-V}) = H^2(Y - V; G, \mathbb{Z}),$$

where

$$\mathrm{I}_{\infty}^{2,0}(Y; G, \tilde{\mathscr{A}}|_{Y-V}) = \mathrm{I}_{\infty}^{2,0}(Y - V; G, \mathbb{Z}) = H^2(Y - V, \mathbb{Z})$$

since $\mathrm{I}_2^{0,1}(Y - V; G, \mathbb{Z}) = H^0(Y - V, \mathscr{H}^1(G, \mathbb{Z})) = 0$.

On the other hand, $\mathscr{H}^q(G, \mathscr{A}|_{Y-V}) = 0$ for $q > 0$, hence

$$\begin{aligned} H^2(Y; G, \mathscr{A}|_{Y-V}) &= \mathrm{I}_{\infty}^{2,0}(Y; G, \mathscr{A}|_{Y-V}) \\ &= H^2(Y, \mathscr{H}^0(G, \mathscr{A}|_{Y-V})) = H^2(Y - V, \mathbb{Z}). \end{aligned}$$

Since the sheaf homomorphism $\mathscr{H}^0(G, \tilde{\mathscr{A}}|_{Y-V}) \to \mathscr{H}^0(G, \mathscr{A}|_{Y-V})$ is an isomorphism, the homomorphism

$$\mathrm{I}_{\infty}^{2,0}(Y; G, \tilde{\mathscr{A}}|_{Y-V}) \to \mathrm{I}_{\infty}^{2,0}(Y; G, \mathscr{A}|_{Y-V}) = H^2(Y; G, \mathscr{A}|_{Y-V})$$

is also an isomorphism. Hence the map $H^2(Y; G, \tilde{\mathscr{A}}|_{Y-V}) \to H^2(Y; G, \mathscr{A}|_{Y-V})$ is an epimorphism. The lemma is proved.

**Lemma 1.2.4.** *The homomorphism* $H^3(\bar{V}; G, \mathbb{Z}_V) \to H^3(\bar{V}; G, \mathscr{A}_V)$ *induced by the inclusion* $\mathbb{Z}_V \subset \mathscr{A}_V$, *where* $\bar{V}$ *is the closure of* $V$, *is a monomorphism.*

*Proof.* First we notice that the homomorphism

$$\pi^*: H^3(\bar{V}, \partial V; \mathbb{Z}) \to H^3(\bar{U}, \partial U; \mathbb{Z}) \tag{4}$$

is a monomorphism. In fact,

$$H^3(\bar{V}, \partial V; \mathbb{Z}) = H_1(X(\mathbb{R}), \mathbb{Z}), \qquad H^3(\bar{U}, \partial U; \mathbb{Z}) = H_1(X(\mathbb{R}), \mathbb{Z}),$$

and the homomorphism (4) is induced by multiplication by 2. Consider now the commutative diagram

$$\begin{tikzcd}
H^3(\bar{V}; G, \mathbb{Z}_V) \arrow[r] \arrow[d, "\alpha"'] & H^3(\bar{V}; G, \mathscr{A}_V) \arrow[r, "\pi^*", "\sim"'] \arrow[d, "\alpha"'] & H^3(\bar{U}; G, \mathbb{Z}_U) \arrow[d, "\alpha"'] \\
H^3(\bar{V}, \mathbb{Z}_V) \arrow[r] & H^3(\bar{V}, \mathscr{A}_V) \arrow[r, "\pi^*", "\sim"'] & H^3(\bar{U}, \mathbb{Z}_U)
\end{tikzcd}$$

where the composition of homomorphisms in the bottom row coincides with the homomorphism (4).

It remains to observe that the vertical homomorphisms in this diagram are monomorphisms. This follows from the second spectral sequence for the cohomology groups $H^n(\bar{V}; G, \mathbb{Z}_V)$, $H^n(\bar{U}; G, \mathbb{Z}_U)$. The lemma is proved.

**Lemma 1.2.5.** *The homomorphism*

$$H^2(Y; G, \mathbb{Z}) \oplus H^2(Y; G, \mathscr{A}_V) \to H^2(Y; G, \tilde{\mathscr{A}})$$

*induced by addition of sheaves* $\mathbb{Z} \oplus \mathscr{A}_V \to \tilde{\mathscr{A}}$ *is an epimorphism.*

*Proof.* Consider the commutative diagram of exact sequences of sheaves

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{Z}_V \arrow[r] \arrow[d] & \mathbb{Z}_Y \oplus \mathscr{A}_V \arrow[r] \arrow[d] & \tilde{\mathscr{A}} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathbb{Z}_V \arrow[r] & \mathbb{Z}_{\bar{V}} \oplus \mathscr{A}_V \arrow[r] & \tilde{\mathscr{A}}|_{\bar{V}} \arrow[r] & 0
\end{tikzcd}$$

where the vertical homomorphisms are induced by restricting the corresponding sheaves to $\bar{V}$. This diagram yields the following commutative diagram of exact cohomology sequences:

$$\begin{tikzcd}
\cdots \arrow[r] & H^2(Y; G, \mathbb{Z}) \oplus H^2(Y; G, \mathscr{A}_V) \arrow[r] \arrow[d] & H^2(Y; G, \tilde{\mathscr{A}}) \arrow[r] \arrow[d] & H^3(Y; G, \mathbb{Z}_V) \arrow[r] \arrow[d, "s"'] & \cdots \\
\cdots \arrow[r] & H^2(\bar{V}; G, \mathbb{Z}) \oplus H^2(\bar{V}; G, \mathscr{A}_V) \arrow[r] & H^2(\bar{V}; G, \tilde{\mathscr{A}}) \arrow[r] & H^3(\bar{V}; G, \mathbb{Z}_V) \arrow[r] & \cdots
\end{tikzcd}$$

Since the homomorphism $H^3(Y; G, \mathbb{Z}_V) \to H^3(\bar{V}; G, \mathbb{Z}_V)$ is an isomorphism, it suffices to verify that the homomorphism $H^2(\bar{V}; G, \tilde{\mathscr{A}}) \to H^3(\bar{V}; G, \mathbb{Z}_V)$ is equal to zero, which is equivalent to the homomorphism $H^3(\bar{V}; G, \mathbb{Z}_V) \to H^3(\bar{V}; G, \mathbb{Z}) \oplus H^3(\bar{V}; G, \mathscr{A}_V)$ being a monomorphism; this last assertion follows from Lemma 1.2.4. The lemma is proved.

*Proof of Theorem* 1.2.1. From Lemmas 1.2.3 and 1.2.5 it follows that the composition homomorphism

$$H^2(Y; G, \mathbb{Z}) \oplus H^2(Y; G, \mathscr{A}_V) \to H^2(Y; G, \mathscr{A})$$

is an epimorphism. It remains to observe that

$$H^2(Y; G, \mathscr{A}) = H^2(X(\mathbb{C}); G, \mathbb{Z}),$$
$$H^2(Y; G, \mathscr{A}_V) = H^2(X(\mathbb{C}), X(\mathbb{C}) \setminus U; G, \mathbb{Z}) = \mathscr{L}(\tilde{\mathbf{x}}_1^+, \dots, \tilde{\mathbf{x}}_m^+)$$

and the homomorphism (7) coincides with the homomorphism

$$H^2(Y; G, \mathbb{Z}) \oplus \mathscr{L}(\tilde{\mathbf{x}}_1^+, \dots, \tilde{\mathbf{x}}_m^+) \to H^2(X(\mathbb{C}); G, \mathbb{Z}).$$

The theorem is proved.

## §2. Application of general theorems to proving congruences

2.1. **Surfaces for which** $w_2(X(\mathbb{C})) = [X(\mathbb{R})]^*$. We begin with proving two lemmas.

**Lemma 2.1.1.** $y_1 + \dots + y_m = 0$.

*Proof.* Put $x = x_1 + \dots + x_m$, $y = y_1 + \dots + y_m$ and consider the linear form $l(z) = y \cdot z$ on $H^2(Y, \mathbb{F}_2)$ defined by the cup product; then

$$l(z) = y \cdot z = x \cdot \pi^*(z) = \tau^*(\pi^*(z)) \cdot \pi^*(z) = \pi^*(z) \cdot \pi^*(z) = 0.$$

The lemma is proved.

**Lemma 2.1.2.**

$$w_2(X(\mathbb{C})) = \pi^*(w_2(Y)) + x.$$

*Proof.* Consider a triangulation of $X(\mathbb{C})$ such that $\tau\colon X(\mathbb{C}) \to X(\mathbb{C})$ is a simplicial involution and $X(\mathbb{R})$ is a subcomplex. Then the quotient of this triangulation is a triangulation of $Y$. Let $C$ be the two-dimensional chain equal to the sum of two-dimensional simplices in the barycentric subdivision of the triangulation $X(\mathbb{C})$, and let $\pi(C)$ be the analogous chain for the corresponding triangulation of $Y$; then $C$ and $\pi(C)$ are cycles and $w_2(X(\mathbb{C})) = C^*$, $w_2(Y) = \pi(C)^*$ (cf. [6]). It remains to observe that $\pi^*(\pi(C)^*) = C^* + x$. The lemma is proved.

**Theorem 2.1.3.** *Let $X$ be a* GM-*surface such that the surfaces $X_1, \dots, X_{m-1}$ are orientable and the surface $X_m$ is not necessarily orientable. Suppose that $\chi_i \equiv 0 \pmod 4$ for $i \in \{1, \dots, m-1\}$ and that $w_2(X(\mathbb{C})) = [X(\mathbb{R})]^*$. Then $\chi \equiv \sigma \pmod{16}$.*

*Proof.* Since $\chi = \sigma - 2\sigma_+$ (cf. [7]) and $\sigma_+ = \sigma(Y)$, it suffices to show that $\sigma(Y) \equiv 0 \pmod 8$. From the equality $w_2(X(\mathbb{C})) = [X(\mathbb{R})]^*$, Lemmas 2.1.1 and 2.1.2, and Theorem 1.1.2 it follows that $w_2(Y) = y_{i_1} + \dots + y_{i_k}$, where $1 \le i_1 < \dots < i_k \le m-1$. We fix orientation on the surfaces $X_1, \dots, X_{m-1}$; then for $i \in \{1, \dots, m-1\}$ we get well-defined cohomology classes $x_i^+ \in H^2(X(\mathbb{C}), \mathbb{Z})$, $y_i^+ \in H^2(Y, \mathbb{Z})$. Since $(y_i^+)^2 = 2(x_i^+)^2 = -2\chi_i$, we have $(y_i^+)^2 \equiv 0 \pmod 8$. Therefore, $\sigma(y) \equiv (y_{i_1}^+)^2 + \dots + (y_{i_k}^+)^2 \equiv 0 \pmod 8$. The theorem is proved.

*Remark* 2.1.4. In [1] there are two congruences for the Euler characteristic $\chi$ of a surface $X$ with $H_1(X(\mathbb{C}), \mathbb{Z}) = 0$ such that $X(\mathbb{R})$ has only one nonorientable component. In the first congruence it is assumed that the quadratic form on $H_2(X(\mathbb{C}), \mathbb{Z})$ is even, in which case $X(\mathbb{R})$ is orientable (cf. [8, 9]); thus the set of surfaces for which this congruence is proved in [1] is empty. The second congruence is a special case of Theorem 2.1.3.

2.2. **The invariants** $\mu(X)$ **and** $\nu(X)$ **and the cohomology class** $h$. In what follows we will repeatedly apply Corollary 1.2.2. For this reason we assume that the surface $X(\mathbb{R})$ is orientable. The cohomology classes $x_1^+, \dots, x_m^+$ will be denoted by

$x_1, \dots, x_m$. Furthermore, suppose that the cohomology group $H^*(X(\mathbb{C}), \mathbb{Z})$ is free. Then $X$ is a GM$\mathbb{Z}$-surface if and only if $X$ is a GM-surface (cf. [5]). We also assume that the homology class $[X(\mathbb{R})]$ in $H_2(X(\mathbb{C}), \mathbb{F}_2)$ is trivial, i.e., $x_1 + \dots + x_m \equiv 0 \pmod 2$ in $H^2(X(\mathbb{C}), \mathbb{Z})$.

Denote by $L$ the group $H^2(X(\mathbb{C}), \mathbb{Z})$, denote by $\varphi$ the involution $\tau^*$ on $L$, and put $L^\pm = \{x \in L \mid \varphi(x) = \pm x\}$. On $L$ we consider two quadratic forms $Q, Q'$, where $Q(x, y) = x \cdot y$, $Q'(x, y) = x \cdot \varphi(y)$. The element $x_1 + \dots + x_m$ is characteristic for the form $Q'$ (cf. [10]). Since $x_1 + \dots + x_m \equiv 0 \pmod 2$, the form $Q'$ is even on $L$, and since $Q'|_{L^\pm} = \pm Q|_{L^\pm}$, the form $Q$ is even on $L^\pm$.

If $\chi_i = 0$ for all $i \in \{1, \dots, m\}$, then we put $\mu(X) = \infty$; otherwise $\mu(X)$ is defined to be the largest natural number $\mu$ such that $\chi_i \equiv 0 \pmod{2^\mu}$ for all $i \in \{1, \dots, m\}$. We denote by $\nu(X)$ the largest natural number $\nu$ for which there exist odd numbers $l_1, \dots, l_m$ such that

$$l_1 x_1 + \dots + l_m x_m \equiv 0 \pmod{2^\nu}.$$

If there exist numbers $\nu$ with this property bigger than any given number, then we put $\nu(X) = \infty$.

**Proposition 2.2.1.** *One always has* $\nu(X) \leq \mu(X)$.

*Proof.* We multiply the congruence $l_1 x_1 + \dots + l_m x_m \equiv 0 \pmod{2^\nu}$ by $x_i$; then we get the congruence $\chi_i \equiv 0 \pmod{2^\nu}$. The proposition is proved.

Suppose that $\nu(X) < \infty$. Then there exist odd numbers $l_1, \dots, l_m$ such that $l_1 x_1 + \dots + l_m x_m \equiv 0 \pmod{2^{\nu(X)}}$. In this case we put

$$h = \frac{1}{2^{\nu(X)}} \sum_{i=1}^m l_i x_i.$$

We notice that the cohomology class $h \in L^+$ is not uniquely defined.

**Proposition 2.2.2.** *If there exists a decomposition* $h = d + \varphi(d)$ *with* $d \in L$, *then* $\nu(X) < \mu(X)$.

*Proof.* From $h = d + \varphi(d)$ it follows that $h \cdot x_i = 2d \cdot x_i$. On the other hand,

$$h \cdot x_i = -\frac{1}{2^{\nu(X)}} l_i \chi_i,$$

hence $\chi_i \equiv 0 \pmod{2^{\nu(X)+1}}$. The proposition is proved.

**Proposition 2.2.3.** *Let* $X$ *be a* GM-*surface such that* $\nu(X) < \infty$. *Then there exist odd numbers* $l_1, \dots, l_m$ *such that* $l_1 x_1 + \dots + l_m x_m \equiv 0 \pmod{2^{\nu(X)}}$ *and*

$$h = \frac{1}{2^{\nu(X)}} \sum_{i=1}^m l_i x_i \in \pi^* H^2(Y, \mathbb{Z}).$$

*Proof.* Let $l'_1, \dots, l'_m$ be odd numbers such that $l'_1 x_1 + \dots + l'_m x_m \equiv 0 \pmod{2^{\nu(X)}}$. We put

$$h' = \frac{1}{2^{\nu(X)}} \sum_{i=1}^m l'_i x_i;$$

then $h' \in L^+$. By Corollary 1.2.2 there exist integers $n_1, \dots, n_m$ such that

$$h' + \sum_{i=1}^m n_i x_i \in \pi^* H^2(Y, \mathbb{Z}).$$

It remains to put $l_i = l'_i + n_i 2^{\nu(X)}$. The proposition is proved.

In what follows for a GM-surface $X$ with $\nu(X) < \infty$ we shall always assume that $h \in \pi^* H^2(Y, \mathbb{Z})$.

**Proposition 2.2.4.** *Let $X$ be a* GM-*surface such that $\nu(X) < \infty$ and $\nu(X) < \mu(X)$. Then there exists a decomposition $h = d + \varphi(d)$ with $d \in L$.*

*Proof.* First we observe that the existence of a decomposition $h = d + \varphi(d)$ is equivalent to the linear form $l(z) = h \cdot z$ on $L^+$ being even (cf. [4]). In view of Corollary 1.2.2, it suffices to show that this form is even on $\mathscr{L}(x_1, \dots, x_m)$ and on $\pi^* H^2(Y, \mathbb{Z})$. Since $\chi_i \equiv 0 \pmod{2^{\nu(X)+1}}$, it follows that $h \cdot x_i \equiv 0 \pmod 2$ and so the form $h \cdot z$ is even on $\mathscr{L}(x_1, \dots, x_m)$. Since $h \in \pi^* H^2(Y, \mathbb{Z})$, this form is even on $\pi^* H^2(Y, \mathbb{Z})$. The proposition is proved.

2.3. **Congruences for the Euler characteristic of an $M$-surface and $(M-1)$-surface.** In this section we keep the assumptions and notation of 2.2; furthermore, we denote by $a$ the dimension of the space $L/L^+ \oplus L^-$ over the field $\mathbb{F}_2$.

**Proposition 2.3.1.** *Let $X$ be a* GM-*surface such that $\mu(X) < \infty$. Then the following assertions hold*:

(i) *the congruence*

$$\chi \equiv 0 \pmod{2^{\min\{\mu(X)+3,\, 2\nu(X)+1\}}} \tag{1}$$

*always holds*;

(ii) *if the form $Q$ is even on $L$ and $\nu(X) < \mu(X)$, then*

$$\chi \equiv 0 \pmod{2^{\min\{\mu(X)+3,\, 2\nu(X)+2\}}}; \tag{2}$$

(iii) *if $a = 0$, then $\mu(X) = \nu(X)$ and*

$$\chi \equiv 0 \pmod{2^{\min\{\mu(X)+3,\, 2\mu(X)+1\}}}; \tag{3}$$

(iv) *if $a = 1$, $\mu(X) \geq 2$, and $\chi \not\equiv 0 \pmod{2^{\mu(X)+3}}$, then*

$$\chi \equiv -2^{2\nu(X)}\sigma \pmod{2^{\min\{\mu(X)+3,\, 2\nu(X)+3\}}}. \tag{4}$$

*Proof.* Since $Q$ is even on $L^+$, $h^2 \equiv 0 \pmod 2$, and therefore

$$l_1^2\chi_1 + \dots + l_m^2\chi_m \equiv 0 \pmod{2^{2\nu(X)+1}}. \tag{5}$$

On the other hand, $l_i^2 \equiv 1 \pmod{2^3}$, $\chi_i \equiv 0 \pmod{2^{\mu(X)}}$; hence

$$l_1^2\chi_1 + \dots + l_m^2\chi_m \equiv \chi_1 + \dots + \chi_m = \chi \pmod{2^{\mu(X)+3}}. \tag{6}$$

Congruence (1) immediately follows from congruences (5) and (6). We now turn to the proof of congruence (2). Since $\nu(X) < \mu(X)$, from Proposition 2.2.4 it follows that there exists a decomposition $h = d + \varphi(d)$, $d \in L$. Then $h^2 = 2d^2 + 2d \cdot \varphi(d)$, and since the forms $Q$ and $Q'$ are even on $L$, $h^2 \equiv 0 \pmod 4$; hence

$$l_1^2\chi_1 + \dots + l_m^2\chi_m \equiv 0 \pmod{2^{2\nu(X)+2}}.$$

Combining this congruence with congruence (6), we get congruence (2). To prove congruence (3) we observe that for $a = 0$ the element $h$ does not admit decomposition $h = d + \varphi(d)$. Hence from Proposition 2.2.4 it follows that $\nu(X) = \mu(X)$, and congruence (3) follows from congruence (1).

We turn to the proof of assertion (iv). Since $\chi \not\equiv 0 \pmod{2^{\mu(X)+3}}$ and $\mu(X) \geq 2$, congruence (1) yields the following inequalities: $2\nu(X) + 1 < \mu(X) + 3$, $\nu(X) < \mu(X)$. Hence from Proposition 2.2.4 it follows that there exists a decomposition $h = d + \varphi(d)$.

Now we observe that orientability of $X(\mathbb{R})$ implies the existence of a characteristic class $k$ of the form $Q$ with decomposition $k = c - \varphi(c)$. In fact, $k$ corresponds to a nonsingular real algebraic curve $K$ on $X$. From orientability of $X(\mathbb{R})$ it follows that $\mathrm{cl}_{\mathbb{R}}(K) = 0$, and in this case it is shown in [4] that $k = \mathrm{cl}_{\mathbb{C}}(K)$ admits a decomposition $k = c - \varphi(c)$. We claim that $2$ does not divide $k = c - \varphi(c)$, i.e., $Q$ is an odd form. In fact, by the Gudkov-Krakhnov-Kharlamov theorem $\chi \equiv \sigma \pm 2 \pmod{16}$ and by (1) $\chi \equiv 0 \pmod 8$. Hence $\sigma \equiv \pm 2 \pmod 8$ and $Q$ is an odd form.

Denote by $\tilde{c}$, $\tilde{d}$ the images of elements $c, d$ under the projection $L \to L/L^+ \oplus L^-$; then $\tilde{c} = \tilde{d}$. This follows from the equality $\dim_{\mathbb{F}_2}(L/L^+ \oplus L^-) = 1$ and the fact that $k = c - \varphi(c)$ and $h = d + \varphi(d)$ are odd elements. Since $\tilde{c} = \tilde{d}$, we have $h \equiv k \pmod 2$, i.e., $h$ is a characteristic element of the form $Q$. Hence $h^2 \equiv \sigma \pmod 8$ and therefore

$$l_1^2\chi_1 + \cdots + l_m^2\chi_m \equiv -2^{2\nu(X)}\sigma \pmod{2^{2\nu(X)+3}}.$$

Combining this with congruence (6) we obtain congruence (4). The proposition is proved.

*Remark* 2.3.2. From the proof of the above proposition it is clear that congruence (4) can be written in the form

$$\chi \equiv \pm 2^{2\nu(X)+1} \pmod{2^{\min\{\mu(X)+3,\, 2\nu(X)+3\}}},$$

from which it follows that $\chi = 2^{2\nu(X)+1} \cdot \chi'$, where $\chi'$ is an odd number.

**Theorem 2.3.3.** *Let $X$ be a surface such that $H^*(X(\mathbb{C}), \mathbb{Z})$ is a free group, $X(\mathbb{R}) = X_1 \cup \cdots \cup X_m$ is an orientable surface, and $[X(\mathbb{R})] = 0$ in $H_2(X(\mathbb{C}), \mathbb{F}_2)$. Then the following assertions hold*:

(i) *if $X$ is an $M$-surface and $\chi_i \equiv 0 \pmod{2^\mu}$ for $i \in \{1, \dots, m\}$, $\mu \geq 2$, then $\chi \equiv 0 \pmod{2^{\mu+3}}$*;

(ii) *if $X$ is an $(M-1)$-surface and $\chi_i \equiv 0 \pmod{2^\mu}$ for $i \in \{1, \dots, m\}$, $\mu \geq 2$, then either $\chi \equiv 0 \pmod{2^{\mu+3}}$ or $\chi \equiv -2^{2\nu}\sigma \pmod{2^{\min\{\mu+3,\, 2\nu+3\}}}$, where $1 \leq \nu < \mu/2 + 1$.*

*Proof*. Since all $M$-surfaces and $(M-1)$-surfaces are GM-surfaces (cf. [4]), the assertions of the theorem follow from Proposition 2.3.1, (iii) and Remark 2.3.2. The theorem is proved.

*Remark* 2.3.4. Theorem 2.3.3, (ii) implies the following claim:

(ii)′ *if $X$ is an $(M-1)$-surface and $\chi_i \equiv 0 \pmod 4$ for $i \in \{1, \dots, m\}$, but $\chi \not\equiv 0 \pmod{32}$, then $\chi \equiv -4\sigma \pmod{32}$.*

This claim yields the following Fidler congruence for $M$-curves of degree $4k$, where $k$ is an odd number: *If the characteristic of each even oval is even, then* $p - n \equiv -4 \pmod{16}$ (*cf.* [2, 11]).

However it is quite possible that assertion (ii)′ involves the empty set of surfaces, i.e., the following assertion holds:

(ii)″ *if $X$ is an $(M-1)$-surface and $\chi_i \equiv 0 \pmod 4$ for $i \in \{1, \dots, m\}$, then $\chi \equiv 0 \pmod{32}$.*

This last assertion certainly holds for surfaces with $H_1(X(\mathbb{C}), \mathbb{Z}) = 0$ since the condition $\chi_i \equiv 0 \pmod 4$ implies that $\chi \equiv 0 \pmod{16}$ (cf. [1]), and therefore the congruence $\chi \equiv -4\sigma = \pm 8 \pmod{32}$ is impossible. In particular, the assumptions of the Fidler congruence are not satisfied (this also follows from [2], viz. assertion (4.3) in that paper is incompatible with assertion (4.7)). We observe that assertion (ii)″ also holds if $\sigma \equiv \pm 2 \pmod{16}$ since in that case the Gudkov-Krakhnov-Kharlamov congruence shows that $\chi \equiv 0, \pm 4 \pmod{16}$, and therefore $\chi \not\equiv \pm 8 \pmod{32}$.

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Received 28/JUNE/91

Translated by F. L. ZAK