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COHOMOLOGICAL QUOTIENTS AND SMASHING LOCALIZATIONS

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Abstract. The quotient of a triangulated category modulo a subcategory was defined by Verdier. Motivated by the failure of the telescope conjecture, we introduce a new type of quotients for any triangulated category which generalizes Verdier's construction. Slightly simplifying this concept, the cohomological quotients are flat epimorphisms, whereas the Verdier quotients are Ore localizations. For any compactly generated triangulated category $\mathcal{S}$, a bijective correspondence between the smashing localizations of $\mathcal{S}$ and the cohomological quotients of the category of compact objects in $\mathcal{S}$ is established. We discuss some applications of this theory, for instance the problem of lifting chain complexes along a ring homomorphism. This is motivated by some consequences in algebraic K-theory and demonstrates the relevance of the telescope conjecture for derived categories. Another application leads to a derived analogue of an almost module category in the sense of Gabber-Ramero. It is shown that the derived category of an almost ring is of this form.

Introduction. The telescope conjecure from stable homotopy theory is a fascinating challenge for topologists and algebraists. It is a conjecture about smashing localizations, saying roughly that every smashing localization is a finite localization. The failure of this conjecture forces us to develop a general theory of smashing localizations which covers the ones which are not finite. This is precisely the subject of the first part of this paper. The second part discusses some applications of the general theory in the context of derived categories of associative rings. In fact, we demonstrate the relevance of the telescope conjecture for derived categories, by studying some applications in algebraic K-theory and in almost ring theory.

Let us describe the main concepts and results from this paper. We fix a compactly generated triangulated category $\mathcal{S}$, for example, the stable homotopy category of CW-spectra or the unbounded derived category of an associative ring. A *smashing localization functor* is by definition an exact functor $F\colon \mathcal{S} \to \mathcal{T}$ between triangulated categories having a right adjoint G which preserves all coproducts and satisfies $F \circ G \cong \mathrm{Id}_{\mathcal{T}}$. Such a functor induces an exact functor $F_c\colon \mathcal{S}_c \to \mathcal{T}_c$ between the full subcategories of compact objects, and the telescope conjecture [5, 31] claims that the induced functor $\mathcal{S}_c/\operatorname{Ker} F_c \to \mathcal{T}_c$ is an equivalence up to direct factors. Here, $\operatorname{Ker} F_c$ denotes the full triangulated subcategory of objects X in $\mathcal{S}_c$ such that $F_cX = 0$, and $\mathcal{S}_c/\operatorname{Ker} F_c$ is the quotient in the sense of Verdier [35]. The failure of the telescope conjecture [18, 23] motivates the following

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generalization of Verdier's definition of a quotient of a triangulated category. To be precise, there are examples of proper smashing localization functors F where $\operatorname{Ker} F_c = 0$. Nonetheless, the functor F_c is a cohomological quotient functor in the following sense.

Definition. Let $F\colon \mathcal{C} \to \mathcal{D}$ be an exact functor between triangulated categories. We call F a *cohomological quotient functor* if for every cohomological functor $H\colon \mathcal{C} \to \mathcal{A}$ satisfying $\operatorname{Ann} F \subseteq \operatorname{Ann} H$, there exists, up to a unique isomorphism, a unique cohomological functor $H'\colon \mathcal{D} \to \mathcal{A}$ such that $H = H' \circ F$.

Here, $\operatorname{Ann} F$ denotes the ideal of all maps ϕ in $\mathcal{C}$ such that $F\phi = 0$. The property of F to be a cohomological quotient functor can be expressed in many ways, for instance more elementary as follows: every object in $\mathcal{D}$ is a direct factor of some object in the image of F, and every map $\alpha\colon FX \to FY$ in $\mathcal{D}$ can be composed with a split epimorphism $F\pi\colon FX' \to FX$ such that $\alpha \circ F\pi$ belongs to the image of F.

Our main result shows a close relation between cohomological quotient functors and smashing localizations.

THEOREM 1. *Let $\mathcal{S}$ be a compactly generated triangulated category, and let $F\colon \mathcal{S}_c \to \mathcal{T}$ be a cohomological quotient functor. Denote by $\mathcal{R}$ the full subcategory of objects X in $\mathcal{S}$ such that every map $C \to X$ from a compact object C factors through some map in* $\operatorname{Ann} F$.

(1) *The category $\mathcal{R}$ is a triangulated subcategory of $\mathcal{S}$ and the quotient functor $\mathcal{S} \to \mathcal{S}/\mathcal{R}$ is a smashing localization functor which induces a fully faithful and exact functor $\mathcal{T} \to \mathcal{S}/\mathcal{R}$ making the following diagram commutative.*

$$\begin{array}{ccc} \mathcal{S}_c & \xrightarrow{F} & \mathcal{T} \\ \downarrow{\scriptstyle \mathrm{inc}} & & \downarrow \\ \mathcal{S} & \xrightarrow{\mathrm{can}} & \mathcal{S}/\mathcal{R} \end{array}$$

(2) *The triangulated category $\mathcal{S}/\mathcal{R}$ is compactly generated and the subcategory of compact objects is precisely the closure of the image of $\mathcal{T} \to \mathcal{S}/\mathcal{R}$ under forming direct factors.*

(3) *There exists a fully faithful and exact functor $G\colon \mathcal{T} \to \mathcal{S}$ such that*

$$\mathcal{S}(X, GY) \cong \mathcal{T}(FX, Y)$$

for all X in $\mathcal{S}_c$ and Y in $\mathcal{T}$.

One may think of this result as a generalization of the localization theorem of Neeman-Ravenel-Thomason-Trobaugh-Yao [27, 31, 34, 38]. To be precise, Nee-

man et al. considered cohomological quotient functors of the form $\mathcal{S}_c \to \mathcal{S}_c/\mathcal{R}_0$ for some triangulated subcategory $\mathcal{R}_0$ of $\mathcal{S}_c$ and analyzed the smashing localization functor $\mathcal{S} \to \mathcal{S}/\mathcal{R}$ where $\mathcal{R}$ denotes the localizing subcategory generated by $\mathcal{R}_0$.

Our theorem provides a bijective correspondence between smashing localizations of $\mathcal{S}$ and cohomological quotients of $\mathcal{S}_c$; it improves a similar correspondence [21] – the new ingredient in our proof being a recent variant [22] of Brown's Representability Theorem [6]. The essential invariant of a cohomological quotient functor $F: \mathcal{S}_c \to \mathcal{T}$ is the ideal $\operatorname{Ann} F$. The ideals of $\mathcal{S}_c$ which are of this form are called *exact* and are precisely those satisfying the following properties:

(1) $\mathfrak{I}^2 = \mathfrak{I}$.

(2) $\mathfrak{I}$ is *saturated*, that is, for every exact triangle $X' \xrightarrow{\alpha} X \xrightarrow{\beta} X'' \to \Sigma X'$ and every map $\phi: X \to Y$ in $\mathcal{S}_c$, we have that $\phi \circ \alpha, \beta \in \mathfrak{I}$ implies $\phi \in \mathfrak{I}$.

(3) $\Sigma\mathfrak{I} = \mathfrak{I}$.

Let us rephrase the telescope conjecture in terms of exact ideals and cohomological quotient functors. To this end, recall that a subcategory of $\mathcal{S}$ is *smashing* if it is of the form $\operatorname{Ker} F$ for some smashing localization functor $F: \mathcal{S} \to \mathcal{T}$.

COROLLARY. *The telescope conjecture for $\mathcal{S}$ is equivalent to each of the following statements.*

(1) *Every smashing subcategory of $\mathcal{S}$ is generated by compact objects.*

(2) *Every exact ideal is generated by idempotent elements.*

(3) *Every cohomological quotient functor $F: \mathcal{S}_c \to \mathcal{T}$ induces up to direct factors an equivalence $\mathcal{S}_c/\operatorname{Ker} F \to \mathcal{T}$.*

(4) *Every two-sided flat epimorphism $F: \mathcal{S}_c \to \mathcal{T}$ satisfying $\Sigma(\operatorname{Ann} F) = \operatorname{Ann} F$ is an Ore localization.*

This reformulation of the telescope conjecture is based on our approach to view a triangulated category as a ring with several object. In this setting, the cohomological quotient functors are the flat epimorphisms, whereas the Verdier quotient functors are the Ore localizations. The reformulation in terms of exact ideals refers to the classical problem from ring theory of finding idempotent generators for an idempotent ideal, studied for instance by Kaplansky [15] and Auslander [1]. We note that the telescope conjecture becomes a statement about the category of compact objects. Moreover, we see that the smashing subcategories of $\mathcal{S}$ form a complete lattice which is isomorphic to the lattice of exact ideals in $\mathcal{S}_c$.

The second part of this paper is devoted to studying noncommutative localizations of rings. We do this by using unbounded derived categories and demonstrate that the telescope conjecture is relevant in this context. This is inspired by recent work of Neeman and Ranicki [30]. They study the problem of lifting chain complexes up to homotopy along a ring homomorphism $R \to S$. To make this

precise, let us denote by $\mathbf{K}^b(R)$ the homotopy category of bounded complexes of finitely generated projective R-modules.

(1) We say that the *chain complex lifting problem* has a positive solution, if every complex Y in $\mathbf{K}^b(S)$ such that for each i we have $Y^i = P^i \otimes_R S$ for some finitely generated projective R-module P^i, is isomorphic to $X \otimes_R S$ for some complex X in $\mathbf{K}^b(R)$.

(2) We say that the *chain map lifting problem* has a positive solution, if for every pair X, Y of complexes in $\mathbf{K}^b(R)$ and every map $\alpha: X \otimes_R S \to Y \otimes_R S$ in $\mathbf{K}^b(S)$, there are maps $\phi: X' \to X$ and $\alpha': X' \to Y$ in $\mathbf{K}^b(R)$ such that $\phi \otimes_R S$ is invertible and $\alpha = \alpha' \otimes_R S \circ (\phi \otimes_R S)^{-1}$ in $\mathbf{K}^b(S)$.

Note that complexes can be lifted whenever maps can be lifted. For example, maps and complexes can be lifted if $R \to S$ is a commutative localization. However, there are obstructions in the noncommutative case, and this leads to the concept of a homological epimorphism. Recall from [13] that $R \to S$ is a *homological epimorphism* if $S \otimes_R S \cong S$ and $\operatorname{Tor}_i^R(S, S) = 0$ for all $i \geq 1$. For example, every commutative localization is a flat epimorphism and therefore a homological epimorphism. The following observation is crucial for both lifting problems.

PROPOSITION. *A ring homomorphism $R \to S$ is a homological epimorphism if and only if $- \otimes_R S: \mathbf{K}^b(R) \to \mathbf{K}^b(S)$ is a cohomological quotient functor.*

This shows that we can apply our theory of cohomological quotient functors, and we see that the telescope conjecture for the unbounded derived category $\mathbf{D}(R)$ of a ring R becomes relevant. In particular, we obtain a noncommutative analogue of Thomason-Trobaugh's localization theorem for algebraic K-theory [34].

THEOREM 2. *Let R be a ring such that the telescope conjecture holds true for $\mathbf{D}(R)$. Then the chain map lifting problem has a positive solution for a ring homomorphism $f: R \to S$ if and only if f is a homological epimorphism. Moreover, in this case f induces a sequence*

$$K(R, f) \longrightarrow K(R) \longrightarrow K(S)$$

of K-theory spectra which is a homotopy fibre sequence, up to failure of surjectivity of $K_0(R) \to K_0(S)$. In particular, there is induced a long exact sequence

$$\cdots \longrightarrow K_1(R) \longrightarrow K_1(S) \longrightarrow K_0(R, f) \longrightarrow K_0(R) \longrightarrow K_0(S)$$

of algebraic K-groups.

Unfortunately, not much seems to be known about the telescope conjecture for derived categories. Note that the telescope conjecture has been verified for $\mathbf{D}(R)$ provided R is commutative noetherian [26]. On the other hand, there are counter

examples which arise from homological epimorphisms where not all chain maps can be be lifted [18].

In the final part of this paper, we introduce the derived analogue of an almost module category in the sense of [10]. In fact, there is a striking parallel between almost rings and smashing localizations: both concepts depend on an idempotent ideal. Given a ring R and an idempotent ideal $\mathfrak{a}$, the category of *almost modules* is by definition the quotient

$$\operatorname{Mod}(R, \mathfrak{a}) = \operatorname{Mod} R/(\mathfrak{a}^{\perp}),$$

where $\operatorname{Mod} R$ denotes the category of right R-modules and $\mathfrak{a}^{\perp}$ denotes the Serre subcategory of R-modules annihilated by $\mathfrak{a}$. Given an idempotent ideal $\mathfrak{I}$ of $\mathbf{K}^b(R)$ which satifies $\Sigma\mathfrak{I} = \mathfrak{I}$, the objects in $\mathbf{D}(R)$ which are annihilated by $\mathfrak{I}$ form a triangulated subcategory, and we call the quotient category

$$\mathbf{D}(R, \mathfrak{I}) = \mathbf{D}(R)/(\mathfrak{I}^{\perp})$$

an *almost derived category*. It turns out that the almost derived categories are, up to equivalence, precisely the smashing subcategories of $\mathbf{D}(R)$. Moreover, as one should expect, the derived category of an almost ring is an almost derived category.

THEOREM 3. *Let R be a ring and $\mathfrak{a}$ be an idempotent ideal such that $\mathfrak{a} \otimes_R \mathfrak{a}$ is flat as left R-module. Then the maps in $\mathbf{K}^b(R)$ which annihilate all suspensions of the mapping cone of the natural map $\mathfrak{a} \otimes_R \mathfrak{a} \to R$ form an idempotent ideal $\mathfrak{A}$, and $\mathbf{D}(R, \mathfrak{A})$ is equivalent to the unbounded derived category of* $\operatorname{Mod}(R, \mathfrak{a})$.

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1. Modules. The homological properties of an additive category $\mathcal{C}$ are reflected by properties of functors from $\mathcal{C}$ to various abelian categories. In this context, the abelian category Ab of abelian groups plays a special role, and this leads to the concept of a $\mathcal{C}$-module. In this section we give definitions and fix some terminology.

Let $\mathcal{C}$ and $\mathcal{D}$ be additive categories. We denote by $\mathcal{H}om(\mathcal{C}, \mathcal{D})$ the category of functors from $\mathcal{C}$ to $\mathcal{D}$. The natural transformations between two functors form the morphisms in this category, but in general they do not form a set. A category will be called *large* to point out that the morphisms between fixed objects are not assumed to form a set.

A $\mathcal{C}$-*module* is by definition an additive functor $\mathcal{C}^{\mathrm{op}} \to \mathrm{Ab}$ into the category Ab of abelian groups, and we denote for $\mathcal{C}$-modules M and N by $\mathrm{Hom}_{\mathcal{C}}(M,N)$ the class of natural transformations $M \to N$. We write $\mathrm{Mod}\,\mathcal{C}$ for the category of $\mathcal{C}$-modules which is large, unless $\mathcal{C}$ is *small*, that is, the isomorphism classes of objects in $\mathcal{C}$ form a set. Note that $\mathrm{Mod}\,\mathcal{C}$ is an abelian category. A sequence $L \to M \to N$ of maps between $\mathcal{C}$-modules is *exact* if the sequence $LX \to MX \to NX$ is exact for all X in $\mathcal{C}$. We denote for every X in $\mathcal{C}$ by $H_X = \mathcal{C}(-,X)$ the corresponding representable functor and recall that $\mathrm{Hom}_{\mathcal{C}}(H_X,M) \cong MX$ for every module M by Yoneda's lemma. It follows that H_X is a projective object in $\mathrm{Mod}\,\mathcal{C}$.

A $\mathcal{C}$-module M is called *finitely presented* if it fits into an exact sequence

$$\mathcal{C}(-,X) \longrightarrow \mathcal{C}(-,Y) \longrightarrow M \to 0$$

with X and Y in $\mathcal{C}$. Note that $\mathrm{Hom}_{\mathcal{C}}(M,N)$ is a set for every finitely presented $\mathcal{C}$-module M by Yoneda's lemma. The finitely presented $\mathcal{C}$-modules form an additive category with cokernels which we denote by $\mathrm{mod}\,\mathcal{C}$.

Now let $F\colon \mathcal{C} \to \mathcal{D}$ be an additive functor. This induces the *restriction functor*

$$F_*\colon \mathrm{Mod}\,\mathcal{D} \longrightarrow \mathrm{Mod}\,\mathcal{C}, \quad M \longmapsto M \circ F,$$

and its left adjoint

$$F^*\colon \mathrm{Mod}\,\mathcal{C} \longrightarrow \mathrm{Mod}\,\mathcal{D}$$

which sends a $\mathcal{C}$-module M, written as a colimit $M = \mathrm{colim}_{\alpha \in MX}\, \mathcal{C}(-,X)$ of representable functors, to

$$F^*M = \mathrm{colim}_{\alpha \in MX}\, \mathcal{D}(-,FX).$$

Note that every $\mathcal{C}$-module can be written as a small colimit of representable functors provided $\mathcal{C}$ is small. The finitely presented $\mathcal{C}$-modules are precisely the finite colimits of representable functors. We denote the restriction of F^* by

$$F^\star\colon \mathrm{mod}\,\mathcal{C} \longrightarrow \mathrm{mod}\,\mathcal{D}$$

and observe that $F^\star$ is the unique right exact functor $\mathrm{mod}\,\mathcal{C} \to \mathrm{mod}\,\mathcal{D}$ sending $\mathcal{C}(-,X)$ to $\mathcal{D}(-,FX)$ for all X in $\mathcal{C}$.

Finally, we define

$$\begin{aligned} \mathrm{Ann}\, F &= \text{the ideal of all maps } \phi \in \mathcal{C} \text{ with } F\phi = 0, \text{ and} \\ \mathrm{Ker}\, F &= \text{the full subcategory of all objects } X \in \mathcal{C} \text{ with } FX = 0. \end{aligned}$$

Recall that an *ideal* $\mathcal{I}$ in $\mathcal{C}$ consists of subgroups $\mathcal{I}(X,Y)$ in $\mathcal{C}(X,Y)$ for every pair of objects X, Y in $\mathcal{C}$ such that for all ϕ in $\mathcal{I}(X,Y)$ and all maps $\alpha\colon X' \to X$

and $\beta: Y \to Y'$ in $\mathcal{C}$ the composition $\beta \circ \phi \circ \alpha$ belongs to $\mathfrak{I}(X', Y')$. Note that all ideals in $\mathcal{C}$ are of the form $\operatorname{Ann} F$ for some additive functor F.

Given any class Φ of maps in $\mathcal{C}$, we say that an object X in $\mathcal{C}$ is *annihilated* by Φ, if $\Phi \subseteq \operatorname{Ann} \mathcal{C}(-, X)$. We denote by $\Phi^{\perp}$ the full subcategory of objects in $\mathcal{C}$ which are annihilated by Φ.

2. Cohomological functors and ideals. Let $\mathcal{C}$ be an additive category and suppose $\operatorname{mod} \mathcal{C}$ is abelian. Note that $\operatorname{mod} \mathcal{C}$ is abelian if and only if every map $Y \to Z$ in $\mathcal{C}$ has a *weak kernel* $X \to Y$, that is, the sequence $\mathcal{C}(-, X) \to \mathcal{C}(-, Y) \to \mathcal{C}(-, Z)$ is exact. In particular, $\operatorname{mod} \mathcal{C}$ is abelian if $\mathcal{C}$ is triangulated. A functor $F: \mathcal{C} \to \mathcal{A}$ to an abelian category $\mathcal{A}$ is called *cohomological* if it sends every weak kernel sequence $X \to Y \to Z$ in $\mathcal{C}$ to an exact sequence $FX \to FY \to FZ$ in $\mathcal{A}$. If $\mathcal{C}$ is a triangulated category, then a functor $F: \mathcal{C} \to \mathcal{A}$ is cohomological if and only if F sends every exact triangle $X \to Y \to Z \to \Sigma X$ in $\mathcal{C}$ to an exact sequence $FX \to FY \to FZ \to F\Sigma X$ in $\mathcal{A}$. The Yoneda functor

$$H_{\mathcal{C}}: \mathcal{C} \longrightarrow \operatorname{mod} \mathcal{C}, \quad X \mapsto H_X = \mathcal{C}(-, X)$$

is the universal cohomological functor for $\mathcal{C}$. More precisely, for every abelian category $\mathcal{A}$, the functor

$$\mathcal{H}om\,(H_{\mathcal{C}}, \mathcal{A}): \mathcal{H}om\,(\operatorname{mod} \mathcal{C}, \mathcal{A}) \longrightarrow \mathcal{H}om\,(\mathcal{C}, \mathcal{A})$$

induces an equivalence

$$\mathcal{H}om_{\mathrm{ex}}\,(\operatorname{mod} \mathcal{C}, \mathcal{A}) \longrightarrow \mathcal{H}om_{\mathrm{coh}}\,(\mathcal{C}, \mathcal{A}),$$

where the subscripts ex = exact and coh = cohomological refer to the appropriate full subcategories; see [9, 35] and also [21, Lemma 2.1].

Following [21], we call an ideal $\mathfrak{I}$ in $\mathcal{C}$ *cohomological* if there exists a cohomological functor $F: \mathcal{C} \to \mathcal{A}$ such that $\mathfrak{I} = \operatorname{Ann} F$. For example, if $F: \mathcal{C} \to \mathcal{D}$ is an exact functor between triangulated categories, then $\operatorname{Ann} F$ is cohomological because $\operatorname{Ann} F = \operatorname{Ann}(H_{\mathcal{D}} \circ F)$. Note that the cohomological ideals of $\mathcal{C}$ form a complete lattice, provided $\mathcal{C}$ is small. For instance, given a family $(\mathfrak{I}_i)_{i \in \Lambda}$ of cohomological ideals, we have

$$\inf \mathfrak{I}_i = \bigcap_i \mathfrak{I}_i,$$

because $\bigcap_i \mathfrak{I}_i = \operatorname{Ann} F$ for

$$F: \mathcal{C} \longrightarrow \prod_i \mathcal{A}_i, \quad X \mapsto (F_i X)_{i \in \Lambda}$$

where each $F_i\colon \mathcal{C} \to \mathcal{A}_i$ is a cohomological functor satisfying $\mathfrak{I}_i = \operatorname{Ann} F_i$. We obtain $\sup \mathfrak{I}_i$ by taking the infimum of all cohomological ideals $\mathfrak{J}$ with $\mathfrak{I}_i \subseteq \mathfrak{J}$ for all $i \in \Lambda$.

3. Flat epimorphisms. The concept of a flat epimorphisms generalizes the classical notion of an Ore localization. We study flat epimorphisms of additive categories, following the idea that an additive category may be viewed as a ring with several objects. Given a flat epimorphism $\mathcal{C} \to \mathcal{D}$, it is shown that the maps in $\mathcal{D}$ are obtained from those in $\mathcal{C}$ by a generalized calculus of fractions. There is a close link between flat epimorphisms and quotients of abelian categories. It is the aim of this section to explain this connection which is summarized in Theorem 3.10. We start with a brief discussion of quotients of abelian categories.

Let $\mathcal{C}$ be an abelian category. A full subcategory $\mathcal{B}$ of $\mathcal{C}$ is called a *Serre subcategory* provided that for every exact sequence $0 \to X' \to X \to X'' \to 0$ in $\mathcal{C}$, the object X belongs to $\mathcal{B}$ if and only if X' and X'' belong to $\mathcal{B}$. The *quotient* $\mathcal{C}/\mathcal{B}$ with respect to a Serre subcategory $\mathcal{B}$ is by definition the localization $\mathcal{C}[\Phi^{-1}]$, where Φ denotes the class of maps ϕ in $\mathcal{C}$ such that $\operatorname{Ker}\phi$ and $\operatorname{Coker}\phi$ belong to $\mathcal{B}$; see [11, 12]. The localization functor $Q\colon \mathcal{C} \to \mathcal{C}/\mathcal{B}$ yields for every category $\mathcal{E}$ a functor

$$\mathcal{H}om\,(Q,\mathcal{E})\colon \mathcal{H}om\,(\mathcal{C}/\mathcal{B},\mathcal{E}) \longrightarrow \mathcal{H}om\,(\mathcal{C},\mathcal{E})$$

which induces an isomorphism onto the full subcategory of functors $F\colon \mathcal{C} \to \mathcal{E}$ such that $F\phi$ is invertible for all $\phi \in \Phi$. Note that $\mathcal{C}/\mathcal{B}$ is abelian and Q is exact with $\operatorname{Ker} Q = \mathcal{B}$. Up to an equivalence, a localization functor can be characterized as follows.

LEMMA 3.1. *Let $F\colon \mathcal{C} \to \mathcal{D}$ be an exact functor between abelian categories. Then the following are equivalent.*

(1) *F induces an equivalence $\mathcal{C}/\operatorname{Ker} F \to \mathcal{D}$.*

(2) *For every abelian category $\mathcal{A}$, the functor*

$$\mathcal{H}\mathrm{om}\,(F,\mathcal{A})\colon \mathcal{H}\mathrm{om}_{\mathrm{ex}}\,(\mathcal{D},\mathcal{A}) \longrightarrow \mathcal{H}\mathrm{om}_{\mathrm{ex}}\,(\mathcal{C},\mathcal{A})$$

induces an equivalence onto the full subcategory of exact functors $G\colon \mathcal{C} \to \mathcal{A}$ satisfying $\operatorname{Ker} F \subseteq \operatorname{Ker} G$.

Proof. See [11, III.1]. □

An exact functor between abelian categories satisfying the equivalent conditions of Lemma 3.1 is called an *exact quotient functor*. There is a further characterization in case the functor has a right adjoint.

LEMMA 3.2. *Let $F\colon \mathcal{C} \to \mathcal{D}$ be an exact functor between abelian categories and suppose there is a right adjoint $G\colon \mathcal{D} \to \mathcal{C}$. Then F is a quotient functor if and only*

if G is fully faithful. In this case, G identifies $\mathcal{D}$ with the full subcategory of objects X in $\mathcal{C}$ satisfying $\mathcal{C}(\operatorname{Ker} F, X) = 0$ and $\operatorname{Ext}^1_{\mathcal{C}}(\operatorname{Ker} F, X) = 0$.

Proof. See Proposition III.3 and Proposition III.5 in [11]. □

Next we analyze an additive functor $F\colon \mathcal{C} \to \mathcal{D}$ in terms of the induced functor $F^\star\colon \operatorname{mod}\mathcal{C} \to \operatorname{mod}\mathcal{D}$.

LEMMA 3.3. *Let $F\colon \mathcal{C} \to \mathcal{D}$ be an additive functor between additive categories and suppose $F^\star\colon \operatorname{mod}\mathcal{C} \to \operatorname{mod}\mathcal{D}$ is an exact quotient functor of abelian categories.*

(1) *Every object in $\mathcal{D}$ is a direct factor of some object in the image of F.*

(2) *For every map $\alpha\colon FX \to FY$ in $\mathcal{D}$, there are maps $\alpha'\colon X' \to Y$ and $\pi\colon X' \to X$ in $\mathcal{C}$ such that $F\alpha' = \alpha \circ F\pi$ and $F\pi$ is a split epimorphism.*

Proof. The functor $F^\star\colon \operatorname{mod}\mathcal{C} \to \operatorname{mod}\mathcal{D}$ is, up to an equivalence, a localization functor. Therefore the objects in $\operatorname{mod}\mathcal{D}$ coincide, up to isomorphism, with the objects in $\operatorname{mod}\mathcal{C}$. Moreover, the maps in $\operatorname{mod}\mathcal{D}$ are obtained via a calculus of fractions from the maps in $\operatorname{mod}\mathcal{C}$; see [12, I.2.5].

(1) Fix an object Y in $\mathcal{D}$. Then $\mathcal{D}(-, Y) \cong F^\star M$ for some M in $\operatorname{mod}\mathcal{C}$. If M is a quotient of $\mathcal{C}(-, X)$, then $F^\star M$ is a quotient of $\mathcal{D}(-, FX)$. Thus Y is a direct factor of FX.

(2) Fix a map $\alpha\colon FX \to FY$. The corresponding map $\mathcal{D}(-, \alpha)$ in $\operatorname{mod}\mathcal{D}$ is a fraction, that is, of the form

$$\mathcal{D}(-, FX) = F^\star\mathcal{C}(-, X) \xrightarrow{(F^\star\sigma)^{-1}} F^\star M \xrightarrow{F^\star\phi} F^\star\mathcal{C}(-, Y) = \mathcal{D}(-, FY)$$

for some M in $\operatorname{mod}\mathcal{C}$; see [12, I.2.5]. Choose an epimorphism $\rho\colon \mathcal{C}(-, X') \to M$ for some X' in $\mathcal{C}$. Now define $\alpha'\colon X' \to Y$ by $\mathcal{C}(-, \alpha') = \phi \circ \rho$, and define $\pi\colon X' \to X$ by $\mathcal{C}(-, \pi) = \sigma \circ \rho$. Clearly, $F\pi$ is a split epimorphism since $\mathcal{D}(-, FX)$ is a projective object in $\operatorname{mod}\mathcal{D}$. □

Remark 3.4. Conditions (1) and (2) in Lemma 3.3 imply that every map in $\mathcal{D}$ is a direct factor of some map in the image of F. To be precise, we say that a map $\alpha\colon X \to X'$ is a *direct factor* of a map $\beta\colon Y \to Y'$ if there is a commutative diagram

$$\begin{array}{ccccc} X & \xrightarrow{\varepsilon} & Y & \xrightarrow{\pi} & X \\ \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\alpha} \\ X' & \xrightarrow{\varepsilon'} & Y' & \xrightarrow{\pi'} & X' \end{array}$$

such that $\pi \circ \varepsilon = \mathrm{id}_X$ and $\pi' \circ \varepsilon' = \mathrm{id}_{X'}$.

Recall that an additive functor $F\colon \mathcal{C} \to \mathcal{D}$ is an *epimorphism of additive categories*, or simply an *epimorphism*, if $G \circ F = G' \circ F$ implies $G = G'$ for any pair $G, G'\colon \mathcal{D} \to \mathcal{E}$ of additive functors.

LEMMA 3.5. *Let $F\colon \mathcal{C} \to \mathcal{D}$ be an additive functor between additive categories having the following properties:*

(1) *Every object in $\mathcal{D}$ belongs to the image of F.*

(2) *For every map $\alpha\colon FX \to FY$ in $\mathcal{D}$, there are maps $\alpha'\colon X' \to Y$ and $\pi\colon X' \to X$ in $\mathcal{C}$ such that $F\alpha' = \alpha \circ F\pi$ and $F\pi$ is a split epimorphism.*

Then F is an epimorphism.

Proof. Let $G, G'\colon \mathcal{D} \to \mathcal{E}$ be a pair of additive functors satisfying $G \circ F = G' \circ F$. The first condition implies that G and G' coincide on objects, and the second condition implies that G and G' coincide on maps. Thus $G = G'$. □

Next we explain the notion of a flat functor. A $\mathcal{C}^{\mathrm{op}}$-module M is called *flat* if for every map $\beta\colon Y \to Z$ in $\mathcal{C}$ and every $y \in \operatorname{Ker}(M\beta)$, there exists a map $\alpha\colon X \to Y$ in $\mathcal{C}$ and some $x \in MX$ such that $(M\alpha)x = y$ and $\beta \circ \alpha = 0$. We call an additive functor $F\colon \mathcal{C} \to \mathcal{D}$ *flat* if the $\mathcal{C}^{\mathrm{op}}$-module $\mathcal{D}(X, F-)$ is flat for every X in $\mathcal{D}$. The functor F is *two-sided flat* if F and $F^{\mathrm{op}}\colon \mathcal{C}^{\mathrm{op}} \to \mathcal{D}^{\mathrm{op}}$ are flat.

We use the exact structure of a module category in order to characterize flat functors. Recall that $\operatorname{Mod}\mathcal{C}$ is an abelian category, and that a sequence $L \to M \to N$ of $\mathcal{C}$-modules is exact if the sequence $LX \to MX \to NX$ is exact for all X in $\mathcal{C}$. A sequence in $\operatorname{mod}\mathcal{C}$ is by definition exact if it is exact when viewed as a sequence in $\operatorname{Mod}\mathcal{C}$. We record without proof a number of equivalent conditions which justify our terminology.

LEMMA 3.6. *Let $F\colon \mathcal{C} \to \mathcal{D}$ be an additive functor between additive categories. Suppose* $\operatorname{mod}\mathcal{C}$ *is abelian. Then the following are equivalent.*

(1) *$\mathcal{D}(X, F-)$ is a flat $\mathcal{C}^{\mathrm{op}}$-module for every X in $\mathcal{D}$.*

(2) *F preserves weak kernels.*

(3) *$F^\star\colon \operatorname{mod}\mathcal{C} \to \operatorname{mod}\mathcal{D}$ sends exact sequences to exact sequences.*

LEMMA 3.7. *Let $F\colon \mathcal{C} \to \mathcal{D}$ be an additive functor between small additive categories. Then F is flat if and only if $F^*\colon \operatorname{Mod}\mathcal{C} \to \operatorname{Mod}\mathcal{D}$ is an exact functor.*

Given an additive functor $F\colon \mathcal{C} \to \mathcal{D}$, we continue with a criterion on F such that $F^\star\colon \operatorname{mod}\mathcal{C} \to \operatorname{mod}\mathcal{D}$ is an exact quotient functor.

LEMMA 3.8. *Let $F\colon \mathcal{C} \to \mathcal{D}$ be an additive functor between small additive categories. Suppose* $\operatorname{mod}\mathcal{C}$ *is abelian. If F is flat and $F_*\colon \operatorname{Mod}\mathcal{D} \to \operatorname{Mod}\mathcal{C}$ is fully faithful, then* $\operatorname{mod}\mathcal{D}$ *is abelian and $F^\star\colon \operatorname{mod}\mathcal{C} \to \operatorname{mod}\mathcal{D}$ is an exact quotient functor of abelian categories.*

Proof. The functor $F^*\colon \operatorname{Mod}\mathcal{C} \to \operatorname{Mod}\mathcal{D}$ is exact because F is flat, and it is a quotient functor because F_* is fully faithful. This follows from Lemma 3.2 since F_* is the right adjoint of F^*. We conclude that $\operatorname{mod}\mathcal{D}$ is abelian and that the restriction $F^\star = F^*|_{\operatorname{mod}\mathcal{C}}$ to the category of finitely presented modules is an exact quotient functor, for instance by [20, Theorem 2.6]. □

Definition 3.9. Let $F: \mathcal{C} \to \mathcal{D}$ be an additive functor between additive categories. We call F an *epimorphism up to direct factors*, if there exists a factorization $F = F_2 \circ F_1$ such that:

(1) F_1 is an epimorphism and bijective on objects, and

(2) F_2 is fully faithful and every object in $\mathcal{D}$ is a direct factor of some object in the image of F_2.

The following result summarizes our discussion and provides a characterization of flat epimorphisms.

THEOREM 3.10. *Let $F: \mathcal{C} \to \mathcal{D}$ be an additive functor between additive categories. Suppose* $\operatorname{mod}\mathcal{C}$ *is abelian and F is flat. Then the following are equivalent.*

(1) *The category* $\operatorname{mod}\mathcal{D}$ *is abelian and the exact functor $F^\star: \operatorname{mod}\mathcal{C} \to \operatorname{mod}\mathcal{D}$, sending $\mathcal{C}(-, X)$ to $\mathcal{D}(-, FX)$ for all X in $\mathcal{C}$, is a quotient functor of abelian categories.*

(2) *Every object in $\mathcal{D}$ is a direct factor of some object in the image of F. And for every map $\alpha: FX \to FY$ in $\mathcal{D}$, there are maps $\alpha': X' \to Y$ and $\pi: X' \to X$ in $\mathcal{C}$ such that $F\alpha' = \alpha \circ F\pi$ and $F\pi$ is a split epimorphism.*

(3) *F is an epimorphism up to direct factors.*

Proof. (1) $\Rightarrow$ (2): Apply Lemma 3.3.

(2) $\Rightarrow$ (3): We define a factorization

$$\mathcal{C} \xrightarrow{F_1} \mathcal{D}' \xrightarrow{F_2} \mathcal{D}$$

as follows. The objects of $\mathcal{D}'$ are those of $\mathcal{C}$ and F_1 is the identity on objects. Let

$$\mathcal{D}'(X, Y) = \mathcal{D}(FX, FY)$$

for all X, Y in $\mathcal{C}$, and let $F_1\alpha = F\alpha$ for each map α in $\mathcal{C}$. The functor F_2 equals F on objects and is the identity on maps. It follows from Lemma 3.5 that F_1 is an epimorphism. The functor F_2 is fully faithful by construction, and F_2 is surjective up to direct factors on objects by our assumption on F.

(3) $\Rightarrow$ (1): Assume that F is an epimorphism up to direct factors. We need to enlarge our universe so that $\mathcal{C}$ and $\mathcal{D}$ become small categories. Note that this does not affect our assumption on F, by Lemma A.6. It follows from Proposition A.5 that $F_*: \operatorname{Mod}\mathcal{D} \to \operatorname{Mod}\mathcal{C}$ is fully faithful, and Lemma 3.8 implies that $F^\star: \operatorname{mod}\mathcal{C} \to \operatorname{mod}\mathcal{D}$ is a quotient functor. □

4. Cohomological quotient functors. In this section we introduce the concept of a cohomological quotient functor between two triangulated categories. This concept generalizes the classical notion of a quotient functor $\mathcal{C} \to \mathcal{C}/\mathcal{B}$ which Verdier introduced for any triangulated subcategory $\mathcal{B}$ of $\mathcal{C}$; see [35].

Definition 4.1. Let $F\colon \mathcal{C} \to \mathcal{D}$ be an exact functor between triangulated categories. We call F a *cohomological quotient functor* if for every cohomological functor $H\colon \mathcal{C} \to \mathcal{A}$ satisfying $\operatorname{Ann} F \subseteq \operatorname{Ann} H$, there exists, up to a unique isomorphism, a unique cohomological functor $H'\colon \mathcal{D} \to \mathcal{A}$ such that $H = H' \circ F$.

Let us explain why a quotient funtor $\mathcal{C} \to \mathcal{C}/\mathcal{B}$ in the sense of Verdier is a cohomological quotient functor. To this end we need the following lemma.

LEMMA 4.2. *Let $F\colon \mathcal{C} \to \mathcal{D}$ be an exact functor between triangulated categories and suppose F induces an equivalence $\mathcal{C}/\mathcal{B} \to \mathcal{D}$ for some triangulated subcategory $\mathcal{B}$ of $\mathcal{C}$. Then* $\operatorname{Ann} F$ *is the ideal of all maps in $\mathcal{C}$ which factor through some object in $\mathcal{B}$.*

Proof. The quotient $\mathcal{C}/\mathcal{B}$ is by definition the localization $\mathcal{C}[\Phi^{-1}]$ where Φ is the class of maps $X \to Y$ in $\mathcal{C}$ which fit into an exact triangle $X \to Y \to Z \to \Sigma X$ with Z in $\mathcal{B}$. Now fix a map $\psi\colon Y \to Z$ in $\operatorname{Ann} F$. The maps in $\mathcal{C}/\mathcal{B}$ are described via a calculus of fractions. Thus $F\psi = 0$ implies the existence of a map $\phi\colon X \to Y$ in Φ such that $\psi \circ \phi = 0$. Complete ϕ to an exact triangle $X \to Y \to Z' \to \Sigma X$. Clearly, ψ factors through Z' and Z' belongs to $\mathcal{B}$. Thus $\operatorname{Ann} F$ is the ideal of maps which factor through some object in $\mathcal{B}$. □

Example 4.3. A quotient functor $F\colon \mathcal{C} \to \mathcal{C}/\mathcal{B}$ is a cohomological quotient functor. To see this, observe that a cohomological functor $H\colon \mathcal{C} \to \mathcal{A}$ with $\operatorname{Ker} H$ containing $\mathcal{B}$ factors uniquely through F via some cohomological functor $H'\colon \mathcal{C}/\mathcal{B} \to \mathcal{A}$; see [35, Corollaire II.2.2.11]. Now use that

$$\mathcal{B} \subseteq \operatorname{Ker} H \iff \operatorname{Ann} F \subseteq \operatorname{Ann} H,$$

which follows from Lemma 4.2.

It turns out that cohomological quotients are closely related to quotients of additive and abelian categories. The following result makes this relation precise and provides a number of characterizations for a functor to be a cohomological quotient functor.

THEOREM 4.4. *Let $F\colon \mathcal{C} \to \mathcal{D}$ be an exact functor between triangulated categories. Then the following are equivalent.*

(1) *F is a cohomological quotient functor.*

(2) *The exact functor $F^{\star}\colon \operatorname{mod} \mathcal{C} \to \operatorname{mod} \mathcal{D}$, sending $\mathcal{C}(-, X)$ to $\mathcal{D}(-, FX)$ for all X in $\mathcal{C}$, is a quotient functor of abelian categories.*

(3) *Every object in $\mathcal{D}$ is a direct factor of some object in the image of F. And for every map $\alpha\colon FX \to FY$ in $\mathcal{D}$, there are maps $\alpha'\colon X' \to Y$ and $\pi\colon X' \to X$ in $\mathcal{C}$ such that $F\alpha' = \alpha \circ F\pi$ and $F\pi$ is a split epimorphism.*

(4) *F is up to direct factors an epimorphism of additive categories.*

Proof. All we need to show is the equivalence of (1) and (2). The rest then follows from Theorem 3.10. Fix an abelian category $\mathcal{A}$ and consider the following commutative diagram

$$\begin{array}{ccc} \mathcal{H}om_{\mathrm{ex}}(\operatorname{mod}\mathcal{D},\mathcal{A}) & \xrightarrow{\mathcal{H}om(F^\star,\mathcal{A})} & \mathcal{H}om_{\mathrm{ex}}(\operatorname{mod}\mathcal{C},\mathcal{A}) \\ \downarrow{\scriptstyle \mathcal{H}om(H_{\mathcal{D}},\mathcal{A})} & & \downarrow{\scriptstyle \mathcal{H}om(H_{\mathcal{C}},\mathcal{A})} \\ \mathcal{H}om_{\mathrm{coh}}(\mathcal{D},\mathcal{A}) & \xrightarrow{\mathcal{H}om(F,\mathcal{A})} & \mathcal{H}om_{\mathrm{coh}}(\mathcal{C},\mathcal{A}), \end{array}$$

where the vertical functors are equivalences. Observe that $\mathcal{H}om(H_{\mathcal{C}},\mathcal{A})$ identifies the exact functors $G\colon \operatorname{mod}\mathcal{C} \to \mathcal{A}$ satisfying $\operatorname{Ker}F^\star \subseteq \operatorname{Ker}G$ with the cohomological functors $H\colon \mathcal{C} \to \mathcal{A}$ satisfying $\operatorname{Ann}F \subseteq \operatorname{Ann}H$. This follows from the fact that each M in $\operatorname{mod}\mathcal{C}$ is of the form $M = \operatorname{Im}\mathcal{C}(-,\phi)$ for some map ϕ in $\mathcal{C}$. We conclude that the property of $F^\star$ to be an exact quotient functor, is equivalent to the property of F to be a cohomological quotient functor. □

We complement the description of cohomological quotient functors by a characterization of quotient functors in the sense of Verdier.

PROPOSITION 4.5. *Let $F\colon \mathcal{C} \to \mathcal{D}$ be an exact functor between triangulated categories. Then the following are equivalent.*

(1) *F induces an equivalence $\mathcal{C}/\operatorname{Ker}F \to \mathcal{D}$.*

(2) *Every object in $\mathcal{D}$ is isomorphic to some object in the image of F. And for every map $\alpha\colon FX \to FY$ in $\mathcal{D}$, there are maps $\alpha'\colon X' \to Y$ and $\pi\colon X' \to X$ in $\mathcal{C}$ such that $F\alpha' = \alpha \circ F\pi$ and $F\pi$ is an isomorphism.*

Proof. Let $\mathcal{B} = \operatorname{Ker}F$ and denote by $Q\colon \mathcal{C} \to \mathcal{C}/\mathcal{B}$ the quotient functor, which is the identity on objects. Given objects X and Y in $\mathcal{C}$, the maps $X \to Y$ in $\mathcal{C}/\mathcal{B}$ are fractions of the form

$$X \xrightarrow{(Q\pi)^{-1}} X' \xrightarrow{Q\alpha'} Y$$

such that $F\pi$ is an isomorphism. This shows that (1) implies (2). To prove the converse, denote by $G\colon \mathcal{C}/\mathcal{B} \to \mathcal{D}$ the functor which is induced by F. The description of the maps in $\mathcal{C}/\mathcal{B}$ implies that G is full. It remains to show that G is faithful. To this end choose a map $\psi\colon Y \to Z$ such that $F\psi = 0$. We complete ψ to an exact triangle

$$X \xrightarrow{\phi} Y \xrightarrow{\psi} Z \longrightarrow \Sigma X$$

and observe that $F\phi$ is a split epimorphism. Choose an inverse $\alpha\colon FY \to FX$ and write it as $F\alpha' \circ (F\pi)^{-1}$, using (2). Thus $Q(\phi \circ \alpha')$ is invertible, and $\psi \circ \phi \circ \alpha' = 0$ implies $Q\psi = 0$ in $\mathcal{C}/\mathcal{B}$. We conclude that G is faithful, and this completes the proof. □

5. Flat epimorphic quotients. In this section we establish a triangulated structure for every additive category which is a flat epimorphic quotient of some triangulated category.

THEOREM 5.1. *Let $\mathcal{C}$ be a triangulated category, and let $\mathcal{D}$ be an additive category with split idempotents. Suppose $F\colon \mathcal{C} \to \mathcal{D}$ is a two-sided flat epimorphism up to direct factors satisfying $\Sigma(\operatorname{Ann} F) = \operatorname{Ann} F$. Then there exists a unique triangulated structure on $\mathcal{D}$ such that F is exact. Moreover, a triangle Δ in $\mathcal{D}$ is exact if and only if there is an exact triangle Γ in $\mathcal{C}$ such that Δ is a direct factor of $F\Gamma$.*

Note that an interesting application arises if one takes for $\mathcal{D}$ the idempotent completion of $\mathcal{C}$. In this case, one obtains the main result of [2].

The proof of Theorem 5.1 is given in several steps and requires some preparation. Assuming the suspension $\Sigma\colon \mathcal{D} \to \mathcal{D}$ is already defined, let us define the exact triangles in $\mathcal{D}$. We call a triangle Δ in $\mathcal{D}$ *exact*, if there exists an exact triangle Γ in $\mathcal{C}$ such that Δ is a direct factor of $F\Gamma$, that is, there are triangle maps $\phi\colon \Delta \to F\Gamma$ and $\psi\colon F\Gamma \to \Delta$ such that $\psi \circ \phi = \mathrm{id}_\Delta$.

From now on assume that $F\colon \mathcal{C} \to \mathcal{D}$ is a two-sided flat epimorphism up to direct factors, satisfying $\Sigma(\operatorname{Ann} F) = \operatorname{Ann} F$. We simplify our notation and identify $\mathcal{C}$ with the image of the Yoneda functor $\mathcal{C} \to \operatorname{mod}\mathcal{C}$. The same applies to the Yoneda functor $\mathcal{D} \to \operatorname{mod}\mathcal{D}$. Moreover, we identify $F^\star = F$ and $\Sigma^\star = \Sigma$. Note that $\operatorname{mod}\mathcal{D}$ is abelian and that $F\colon \operatorname{mod}\mathcal{C} \to \operatorname{mod}\mathcal{D}$ is an exact quotient functor by Theorem 3.10. In particular, the maps in $\operatorname{mod}\mathcal{D}$ are obtained from maps in $\operatorname{mod}\mathcal{C}$ via a calculus of fractions.

LEMMA 5.2. *The category* $\operatorname{mod}\mathcal{D}$ *is an abelian* Frobenius category*, that is, there are enough projectives and enough injectives, and both coincide.*

Proof. We know from Lemma B.1 that $\operatorname{mod}\mathcal{C}$ is a Frobenius category because we have an equivalence $I\colon (\operatorname{mod}\mathcal{C})^{\mathrm{op}} \to \operatorname{mod}(\mathcal{C}^{\mathrm{op}})$ which extends the identity functor $\mathcal{C}^{\mathrm{op}} \to \mathcal{C}^{\mathrm{op}}$. The functor I identifies $\operatorname{Ker} F$ with $\operatorname{Ker}(F^{\mathrm{op}})$. In fact, a module $M = \operatorname{Im}\phi$ in $\operatorname{mod}\mathcal{C}$ with ϕ in $\mathcal{C}$ belongs to $\operatorname{Ker} F$ if and only if $F\phi = 0$. Thus I induces an equivalence $(\operatorname{mod}\mathcal{D})^{\mathrm{op}} \to \operatorname{mod}(\mathcal{D}^{\mathrm{op}})$. It follows that $\operatorname{mod}\mathcal{D}$ is a Frobenius category. □

Let us construct the suspension for $\mathcal{D}$.

LEMMA 5.3. *There is an equivalence $\Sigma'\colon \operatorname{mod}\mathcal{D} \to \operatorname{mod}\mathcal{D}$ making the following diagram commutative.*

$$\begin{array}{ccc} \operatorname{mod}\mathcal{C} & \xrightarrow{F} & \operatorname{mod}\mathcal{D} \\ \downarrow{\scriptstyle \Sigma} & & \downarrow{\scriptstyle \Sigma'} \\ \operatorname{mod}\mathcal{C} & \xrightarrow{F} & \operatorname{mod}\mathcal{D} \end{array}$$

The equivalence Σ' is unique up to a unique isomorphism.

Proof. Every object in $\operatorname{mod}\mathcal{D}$ is isomorphic to FM for some M in $\operatorname{mod}\mathcal{C}$. And every map $\alpha: FM \to FN$ is a fraction, that is, of the form

$$FM \xrightarrow{F\phi} FN' \xleftarrow{(F\sigma)^{-1}} FN.$$

Now define $\Sigma'(FM) = F(\Sigma M)$ and $\Sigma'\alpha = F(\Sigma\sigma)^{-1} \circ F(\Sigma\phi)$. □

We shall abuse notation and identify $\Sigma' = \Sigma$. Now fix M, N in $\operatorname{mod}\mathcal{D}$. We may assume that $M = FM'$ and $N = FN'$. We have a natural map

$$\kappa_{M',N'}: \operatorname{Hom}_{\mathcal{C}}(M', N') \longrightarrow \operatorname{Ext}^3_{\mathcal{C}}(\Sigma M', N')$$

which is induced from the triangulated structure on $\mathcal{C}$; see Appendix 16. This map induces a natural map

$$\kappa_{M,N}: \operatorname{Hom}_{\mathcal{D}}(M, N) \longrightarrow \operatorname{Ext}^3_{\mathcal{D}}(\Sigma M, N)$$

since every map $FM' \to FN'$ is a fraction of maps in the image of F. Recall that $\kappa_M = \kappa_{M,M}(\operatorname{id}_M)$. Let

$$\Delta: X \xrightarrow{\alpha} Y \xrightarrow{\beta} Z \xrightarrow{\gamma} \Sigma X$$

be a triangle in $\mathcal{D}$ and put $M = \operatorname{Ker}\alpha$. We call Δ *pre-exact*, if γ induces a map $Z \to \Sigma M$ such that the sequence

$$0 \longrightarrow M \longrightarrow X \xrightarrow{\alpha} Y \xrightarrow{\beta} Z \longrightarrow \Sigma M \longrightarrow 0$$

is exact in $\operatorname{mod}\mathcal{D}$ and represents $\kappa_M \in \operatorname{Ext}^3_{\mathcal{D}}(\Sigma M, M)$.

We know from Proposition B.2 that a triangle in $\mathcal{C}$ is exact if and only if it is pre-exact. The exact triangles in $\mathcal{D}$ arise by definition from exact triangles in $\mathcal{C}$. Also, pre-exact triangles are preserved by $F: \mathcal{C} \to \mathcal{D}$, and they are preserved under taking direct factors. It follows that every exact triangle in $\mathcal{D}$ is pre-exact.

LEMMA 5.4. *Given a commutative diagram*

(5.1)
$$\begin{array}{ccccccc} X & \xrightarrow{\alpha} & Y & \xrightarrow{\beta} & Z & \xrightarrow{\gamma} & \Sigma X \\ \downarrow{\scriptstyle\phi} & & \downarrow{\scriptstyle\psi} & & & & \downarrow{\scriptstyle\Sigma\phi} \\ X' & \xrightarrow{\alpha'} & Y' & \xrightarrow{\beta'} & Z' & \xrightarrow{\gamma'} & \Sigma X' \end{array}$$

in $\mathcal{D}$ such that both rows are pre-exact triangles, there exists a map $\rho: Z \to Z'$ such

that the completed diagram commutes. Moreover, if $\phi^2 = \phi$ and $\psi^2 = \psi$, then there exists a choice for ρ such that $\rho^2 = \rho$.

Proof. Let $M = \operatorname{Ker}\alpha$ and $M' = \operatorname{Ker}\alpha'$. The pair ϕ, ψ induces a map $\mu\colon M \to M'$ and we obtain the following diagram in $\operatorname{mod}\mathcal{D}$.

$$\begin{array}{lccccccc} \kappa_M\colon & 0 & \longrightarrow & \Omega^{-2}M & \longrightarrow & Z & \longrightarrow & \Sigma M \longrightarrow 0 \\ & & & \downarrow{\scriptstyle \Omega^{-2}\mu} & & & & \downarrow{\scriptstyle \Sigma\mu} \\ \kappa_{M'}\colon & 0 & \longrightarrow & \Omega^{-2}M' & \longrightarrow & Z' & \longrightarrow & \Sigma M' \longrightarrow 0. \end{array} \tag{5.2}$$

Here we use a dimension shift to represent κ_M and $\kappa_{M'}$ by short exact sequences. This is possible since $\operatorname{mod}\mathcal{D}$ is a Frobenius category. The map $\kappa_{M,N}$ is natural in M and N, and therefore $\kappa_{M,\mu}(\kappa_M) = \kappa_{\mu,M'}(\kappa_{M'})$. This implies the existence of a map $\rho\colon Z \to Z'$ making the diagram (5.2) commutative. Note that we can choose ρ to be idempotent if μ is idempotent. It follows that the map ρ completes the diagram (5.1) to a map of triangles. □

LEMMA 5.5. *Every map $X \to Y$ in $\mathcal{D}$ can be completed to an exact triangle $X \to Y \to Z \to \Sigma X$.*

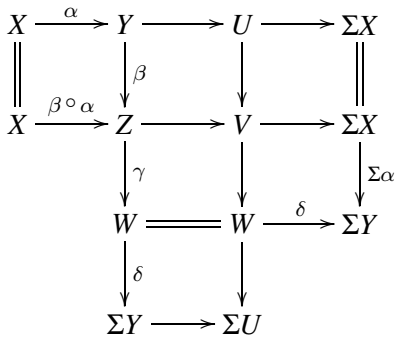
Proof. A map in $\mathcal{D}$ is a direct factor of some map in the image of F by Theorem 3.10; see also Remark 3.4. Thus we have a commutative square

$$\begin{array}{ccc} FX' & \xrightarrow{F\alpha} & FY' \\ \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle \psi} \\ FX' & \xrightarrow{F\alpha} & FY' \end{array}$$

such that ϕ and ψ are idempotent and the map $X \to Y$ equals the map $\operatorname{Im}\phi \to \operatorname{Im}\psi$ induced by $F\alpha$. We complete α to an exact triangle Δ in $\mathcal{C}$ and extend the pair ϕ, ψ to an idempotent triangle map $\varepsilon\colon F\Delta \to F\Delta$, which is possible by Lemma 5.4. The image $\operatorname{Im}\varepsilon$ is an exact triangle in $\mathcal{D}$, which completes the map $X \to Y$. □

We are now in the position to prove the octahedral axiom for $\mathcal{D}$. Note that we have already established that $\mathcal{D}$ is a pre-triangulated category. We say that a pair of composable maps $\alpha\colon X \to Y$ and $\beta\colon Y \to Z$ can be *completed to an octahedron*

if there exists a commutative diagram of the form



such that all triangles which occur are exact.

We shall use the following result due to Balmer and Schlichting.

LEMMA 5.6. *Let $\alpha: X \to Y$ and $\beta: Y \to Z$ be maps in a pre-triangulated category. Suppose there are objects X', Y', Z' such that*

$$X \amalg X' \xrightarrow{\begin{bmatrix} \alpha & 0 \\ 0 & 0 \end{bmatrix}} Y \amalg Y' \quad \text{and} \quad Y \amalg Y' \xrightarrow{\begin{bmatrix} \beta & 0 \\ 0 & 0 \end{bmatrix}} Z \amalg Z'$$

can be completed to an octahedron. Then α and β can be completed to an octahedron.

Proof. See the proof of Theorem 1.12 in [2]. □

LEMMA 5.7. *Every pair of composable maps in $\mathcal{D}$ can be completed to an octahedron.*

Proof. Fix two maps $\alpha: X \to Y$ and $\beta: Y \to Z$ in $\mathcal{D}$. We proceed in two steps. First assume that $X = FA$, $Y = FB$, and $Z = FC$. We use the description of the maps in $\mathcal{D}$ which is given in Theorem 3.10. We consider the map $\beta: Y \to Z$ and obtain new maps $\psi: B' \to C$ and $\pi: B' \to B$ in $\mathcal{C}$ such that $F\psi = \beta \circ F\pi$ and $F\pi$ is a split epimorphism. We get a decomposition $FB' = Y \amalg Y'$ and an automorphism $\varepsilon: Y \amalg Y' \to Y \amalg Y'$ such that $F\psi \circ \varepsilon = \begin{bmatrix} \beta & 0 \end{bmatrix}$. The same argument, applied to the composite

$$X \xrightarrow{\alpha} Y \xrightarrow{\begin{bmatrix} \mathrm{id}_Y \\ 0 \end{bmatrix}} Y \amalg Y',$$

gives a map $\phi: A' \to B'$ in $\mathcal{C}$, a decomposition $FA' = X \amalg X'$, and an automorphism $\delta: X \amalg X' \to X \amalg X'$ such that $F\phi \circ \delta = \begin{bmatrix} \alpha & 0 \\ 0 & 0 \end{bmatrix}$. We know that the pair ϕ, ψ in $\mathcal{C}$ can be completed to an octahedron. Thus $F\phi$ and $F\psi$ can be completed to an octahedron

in $\mathcal{D}$. It follows that $\begin{bmatrix} \alpha & 0 \\ 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} \beta & 0 \end{bmatrix}$ can be completed to an octahedron. Using Lemma 5.6, we conclude that the pair α, β can be completed to an octahedron.

In the second step of the proof, we assume that the objects X, Y, and Z are arbitrary. Applying again the description of the maps in $\mathcal{D}$, we find objects X', Y', and Z' in $\mathcal{D}$ such that $X \amalg X'$, $Y \amalg Y'$, and $Z \amalg Z'$ belong to the image of F. We know from the first part of the proof that the maps

$$X \amalg X' \xrightarrow{\begin{bmatrix} \alpha & 0 \\ 0 & 0 \end{bmatrix}} Y \amalg Y' \quad \text{and} \quad Y \amalg Y' \xrightarrow{\begin{bmatrix} \beta & 0 \\ 0 & 0 \end{bmatrix}} Z \amalg Z'$$

can be completed to an octahedron. From this it follows that α and β can be completed to an octahedron, using again Lemma 5.6. This finishes the proof of the octahedral axiom for $\mathcal{D}$. □

Let us complete the proof of Theorem 5.1.

Proof of Theorem 5.1. We have constructed an equivalence $\Sigma\colon \mathcal{D} \to \mathcal{D}$, and the exact triangles in $\mathcal{D}$ are defined as well. We need to verify the axioms (TR1) – (TR4) from [35]. Let us concentrate on the properties of $\mathcal{D}$, which are not immediately clear from our set-up. In Lemma 5.5, it is shown that every map in $\mathcal{D}$ can be completed to an exact triangle. In Lemma 5.4, it is shown that every partial map between exact triangles can be completed to a full map. Finally, the octahedral axiom (TR4) is established in Lemma 5.7. □

Remark 5.8. The crucial step in the proof of Theorem 5.1 is the verification of the octahedral axiom. An elegant alternative proof has been pointed out by Amnon Neeman. We sketch this proof which uses the following equivalent formulation of the octahedral axiom [29, Definition 1.3.13]: *Every partial map $\phi\colon \Gamma \to \Delta$ between exact triangles can be completed to a map of triangles such that its mapping cone is an exact triangle.*

Suppose now that $\phi\colon \Gamma \to \Delta$ is a partial map between exact triangles in $\mathcal{D}$. We use the description of the maps in $\mathcal{D}$ which is given in Theorem 3.10. Thus we find two contractible exact triangles Γ', Δ' in $\mathcal{D}$ and a partial map $\bar{\phi}\colon \bar{\Gamma} \to \bar{\Delta}$ between exact triangles in $\mathcal{C}$ such that $F\bar{\phi}$ is isomorphic to

$$\Gamma \amalg \Gamma' \xrightarrow{\begin{bmatrix} \phi & 0 \\ 0 & 0 \end{bmatrix}} \Delta \amalg \Delta'.$$

The partial map $\bar{\phi}$ can be completed in $\mathcal{C}$ to a map such that its mapping cone is an exact triangle $\bar{E}$. Thus $\begin{bmatrix} \phi & 0 \\ 0 & 0 \end{bmatrix}$ can be completed to a map ψ in $\mathcal{D}$ such that

its mapping cone is isomorphic to $F\bar{E}$ and therefore exact. The composite

$$\Gamma \xrightarrow{\text{can}} \Gamma \amalg \Gamma' \xrightarrow{\psi} \Delta \amalg \Delta' \xrightarrow{\text{can}} \Delta$$

extends ϕ and we claim that its mapping cone E is exact. In fact, Γ' and Δ' are contractible and therefore $F\bar{E} \cong E \amalg E'$ where E' is contractible.

6. A criterion for exactness. Given an additive functor $\mathcal{C} \to \mathcal{D}$ between triangulated categories, it is a natural question to ask when this functor is exact. We provide a criterion in terms of the induced functor $\operatorname{mod}\mathcal{C} \to \operatorname{mod}\mathcal{D}$ and the extension κ_M in $\operatorname{Ext}^3(\Sigma^\star M, M)$ defined for each M in $\operatorname{mod}\mathcal{C}$; see Appendix 16. There is an interesting consequence. Given any factorization $F = F_2 \circ F_1$ of an exact functor, the functor F_2 is exact provided that F_1 is a cohomological quotient functor.

PROPOSITION 6.1. *Let $F\colon \mathcal{C} \to \mathcal{D}$ be an additive functor between triangulated categories. Then F is exact if and only if the following holds:*

(1) *The right exact functor $F^\star\colon \operatorname{mod}\mathcal{C} \to \operatorname{mod}\mathcal{D}$, sending $\mathcal{C}(-,X)$ to $\mathcal{D}(-,FX)$ for all X in $\mathcal{C}$, is exact.*

(2) *There is a natural isomorphism $\eta\colon F \circ \Sigma_{\mathcal{C}} \to \Sigma_{\mathcal{D}} \circ F$.*

(3) *$F^\star\kappa_M = \operatorname{Ext}^3_{\mathcal{D}}(\eta^\star_M, F^\star M)(\kappa_{F^\star M})$ for all M in $\operatorname{mod}\mathcal{C}$.*

Here, we denote by $\eta^\star$ the natural isomorphism $F^\star \circ \Sigma^\star_{\mathcal{C}} \to \Sigma^\star_{\mathcal{D}} \circ F^\star$ which extends η, that is, $\eta^\star_{\mathcal{C}(-,X)} = \mathcal{D}(-,\eta_X)$ for all X in $\mathcal{C}$.

Proof. Suppose first that (1) – (3) hold. Let

$$\Delta\colon X \xrightarrow{\alpha} Y \longrightarrow Z \longrightarrow \Sigma_{\mathcal{C}} X$$

be an exact triangle in $\mathcal{C}$. We need to show that F sends this triangle to an exact triangle in $\mathcal{D}$. To this end complete the map $F\alpha$ to an exact triangle

$$FX \xrightarrow{F\alpha} FY \longrightarrow Z' \longrightarrow \Sigma_{\mathcal{D}}(FX).$$

Now let $M = \operatorname{Ker}\mathcal{C}(-,\alpha)$. We use a dimension shift to represent the class κ_M by a short exact sequence corresponding to an element in $\operatorname{Ext}^1_{\mathcal{C}}(\Sigma^\star_{\mathcal{C}} M, \Omega^{-2} M)$. Analogously, we represent $\kappa_{F^\star M}$ by a short exact sequence. Next we use the exactness of $F^\star$ to obtain the following diagram in $\operatorname{mod}\mathcal{D}$:

$$\begin{array}{lccccccc}
F^\star\kappa_M\colon & 0 \longrightarrow & F^\star(\Omega^{-2}M) & \longrightarrow & \mathcal{D}(-,FZ) & \longrightarrow & F^\star(\Sigma^\star_{\mathcal{C}} M) & \longrightarrow 0 \\
 & & \| & & & & \downarrow{\scriptstyle \eta^\star_M} & \\
\kappa_{F^\star M}\colon & 0 \longrightarrow & \Omega^{-2}(F^\star M) & \longrightarrow & \mathcal{D}(-,Z') & \longrightarrow & \Sigma^\star_{\mathcal{D}}(F^\star M) & \longrightarrow 0.
\end{array}$$

The diagram can be completed by a map $\mathcal{D}(-,FZ) \to \mathcal{D}(-,Z')$ because $F^\star\kappa_M = \operatorname{Ext}^3_{\mathcal{D}}(\eta^\star_M, F^\star M)(\kappa_{F^\star M})$. Let $\phi\colon FZ \to Z'$ be the new map which is an isomorphism, since $\eta^\star_M$ is an isomorphism. We obtain the following commutative diagram

$$\begin{array}{ccccccc} FX & \xrightarrow{F\alpha} & FY & \longrightarrow & FZ & \longrightarrow & F(\Sigma_{\mathcal{C}}X) \\ \| & & \| & & \downarrow{\scriptstyle\phi} & & \downarrow{\scriptstyle\eta_X} \\ FX & \xrightarrow{F\alpha} & FY & \longrightarrow & Z' & \longrightarrow & \Sigma_{\mathcal{D}}(FX) \end{array}$$

and therefore $F\Delta$ is an exact triangle. Thus F is an exact functor. It is not difficult to show that an exact functor F satisfies (1) – (3), and therefore the proof is complete. □

COROLLARY 6.2. *Let $F\colon \mathcal{C} \to \mathcal{D}$ and $G\colon \mathcal{D} \to \mathcal{E}$ be additive functors between triangulated categories. Suppose F and $G \circ F$ are exact. Suppose in addition that F is a cohomological quotient functor. Then G is exact.*

Proof. We apply Proposition 6.1. First observe that $G^\star\colon \operatorname{mod}\mathcal{D} \to \operatorname{mod}\mathcal{E}$ is exact because the composite $G^\star F^\star = (GF)^\star$ is exact and $F^\star$ is an exact quotient functor, by Theorem 4.4. Denote by $\eta^F\colon F^\star\Sigma^\star_{\mathcal{C}} \to \Sigma^\star_{\mathcal{D}}F^\star$ and $\eta^{GF}\colon (G^\star F^\star)\Sigma^\star_{\mathcal{C}} \to \Sigma^\star_{\mathcal{E}}(G^\star F^\star)$ the natural isomorphisms which exists because F and GF are exact. In order to define $\eta^G\colon G^\star\Sigma^\star_{\mathcal{D}} \to \Sigma^\star_{\mathcal{E}}G^\star$, we use again the fact that $F^\star\colon \operatorname{mod}\mathcal{C} \to \operatorname{mod}\mathcal{D}$ is an exact quotient functor. Thus every object in $\operatorname{mod}\mathcal{D}$ is isomorphic to $F^\star M$ for some M in $\operatorname{mod}\mathcal{C}$. Moreover, any morphism $F^\star M \to F^\star N$ is a fraction, that is, of the form

$$F^\star M \xrightarrow{F^\star\phi} F^\star N' \xrightarrow{(F^\star\sigma)^{-1}} F^\star N.$$

Now define $\eta^G_{F^\star M}$ as the composite

$$\eta^G_{F^\star M}\colon (G^\star\Sigma^\star_{\mathcal{D}})F^\star M \xrightarrow{(G^\star\eta^F_M)^{-1}} (G^\star F^\star\Sigma^\star_{\mathcal{C}})M \xrightarrow{\eta^{GF}_M} (\Sigma^\star_{\mathcal{E}}G^\star)F^\star M.$$

The map is natural, because η^F and η^{GF} are natural transformations, and maps $F^\star M \to F^\star N$ come from maps in $\operatorname{mod}\mathcal{C}$. A straightforward calculation shows that $G^\star\kappa_N = \operatorname{Ext}^3_{\mathcal{D}}(\eta^G_N, G^\star N)(\kappa_{G^\star N})$ for all $N = F^\star M$ in $\operatorname{mod}\mathcal{D}$. Thus G is exact by Proposition 6.1. □

7. Exact quotient functors. The definition of a cohomological quotient functor between two triangulated categories involves cohomological functors to an abelian category. It is natural to study the analogue where the cohomological functors are replaced by exact functors to a triangulated category.

Definition 7.1. Let $F\colon \mathcal{C} \to \mathcal{D}$ be an exact functor between triangulated categories. We call F an *exact quotient functor* if for every triangulated category $\mathcal{E}$

and every exact functor $G\colon \mathcal{C} \to \mathcal{E}$ satisfying $\operatorname{Ann} F \subseteq \operatorname{Ann} G$, there exists, up to a unique isomorphism, a unique exact functor $G'\colon \mathcal{D} \to \mathcal{E}$ such that $G = G' \circ F$.

The motivating examples for this definition are the quotient functors in the sense of Verdier.

Example 7.2. A quotient functor $F\colon \mathcal{C} \to \mathcal{C}/\mathcal{B}$ is an exact quotient functor. To see this, observe that an exact functor $G\colon \mathcal{C} \to \mathcal{D}$ with $\operatorname{Ker} G$ containing $\mathcal{B}$ factors uniquely through F via some exact functor $G'\colon \mathcal{C}/\mathcal{B} \to \mathcal{D}$; see [35, Corollaire II.2.2.11]. Now use that

$$\mathcal{B} \subseteq \operatorname{Ker} G \iff \operatorname{Ann} F \subseteq \operatorname{Ann} G,$$

which follows from Lemma 4.2.

We want to relate cohomological and exact quotient functors.

LEMMA 7.3. *Let $F\colon \mathcal{C} \to \mathcal{D}$ be a cohomological quotient functor, and denote by $\mathcal{D}'$ the smallest full triangulated subcategory containing the image of F. Then the restriction $F'\colon \mathcal{C} \to \mathcal{D}'$ of F has the following properties.*

(1) *F' is a cohomological quotient functor.*

(2) *F' is an exact quotient functor.*

Proof. (1) Use the characterization of cohomological quotient functors in Theorem 4.4.

(2) For simplicity we assume $\mathcal{D}' = \mathcal{D}$. Let $G\colon \mathcal{C} \to \mathcal{E}$ be an exact functor satisfying $\operatorname{Ann} F \subseteq \operatorname{Ann} G$. Then the composite $H_{\mathcal{E}} \circ G$ with the Yoneda functor factors through F because F is a cohomological quotient functor. We have the following sequence of inclusions

$$\mathcal{E} \subseteq \bar{\mathcal{E}} \subseteq \operatorname{mod} \mathcal{E}$$

where $\bar{\mathcal{E}}$ denotes the idempotent completion of $\mathcal{E}$. We obtain a functor $\mathcal{D} \to \operatorname{mod} \mathcal{E}$ and its image lies in $\bar{\mathcal{E}}$, since every object in $\mathcal{D}$ is a direct factor of some object in the image of F. Thus we have a functor $G'\colon \mathcal{D} \to \bar{\mathcal{E}}$ which is exact by Corollary 6.2. Our additional assumption on F implies that $\operatorname{Im} G' \subseteq \mathcal{E}$. We conclude that G factors through F via an exact functor $\mathcal{D} \to \mathcal{E}$. □

The following example has been suggested by B. Keller. It shows that there are exact quotient functors which are not cohomological quotient functors.

Example 7.4. Let A be the algebra of upper 2×2 matrices over a field k, and let $B = k \times k$. We consider the bounded derived categories $\mathcal{C} = \mathbf{D}^b(\operatorname{mod} A)$ and $\mathcal{D} = \mathbf{D}^b(\operatorname{mod} B)$. Restriction along the algebra homomorphism $f\colon B \to A$, $(x, y) \mapsto \begin{bmatrix} x & 0 \\ 0 & y \end{bmatrix}$, induces an exact functor $F\colon \mathcal{C} \to \mathcal{D}$ which is an exact quotient

functor but not a cohomological quotient functor. In fact, f has a left inverse $A \to B$, $\begin{bmatrix} x & z \\ 0 & y \end{bmatrix} \mapsto (x, y)$, which induces a right inverse $G\colon \mathcal{D} \to \mathcal{C}$ for F. Thus every exact functor $F'\colon \mathcal{C} \to \mathcal{E}$ satisfying $\operatorname{Ann} F \subseteq \operatorname{Ann} F'$ factors uniquely through F, by Lemma 7.5 below. However, the exact functor $F^\star\colon \operatorname{mod}\mathcal{C} \to \operatorname{mod}\mathcal{D}$ extending F does not induce an equivalence $\operatorname{mod}\mathcal{C}/\operatorname{Ker} F^\star \to \operatorname{mod}\mathcal{D}$.

Let us describe $\operatorname{mod}\mathcal{C}$. To this end denote by $0 \to X_1 \to X_3 \to X_2 \to 0$ the unique non-split exact sequence in $\operatorname{mod} A$ involving the simple A-modules X_1 and X_2. This sequence induces an exact triangle $X_1 \to X_3 \to X_2 \to \Sigma X_1$ in $\mathcal{C}$. Note that each indecomposable object in $\mathcal{C}$ is determined by its cohomology and is therefore of the form $\Sigma^n X_i$ for some $n \in \mathbb{Z}$ and some $i \in \{1, 2, 3\}$. Thus the indecomposable objects in $\operatorname{mod}\mathcal{C}$ are precisely the objects of the form

$$\mathcal{C}(-, \Sigma^n X_i)/\operatorname{rad}^j \mathcal{C}(-, \Sigma^n X_i) \quad \text{with } n \in \mathbb{Z},\ i \in \{1, 2, 3\},\ j \in \{0, 1\},$$

where $\operatorname{rad}^0 M = M$ and $\operatorname{rad}^1 M$ is the intersection of all maximal subobjects of M. The restriction functor $\operatorname{mod} A \to \operatorname{mod} B$ sends $0 \to X_1 \to X_3 \to X_2 \to 0$ to a split exact sequence. Thus F kills the map $X_2 \to \Sigma X_1$ in $\mathcal{C}$, and we have that $F^\star M = 0$ for some indecomposable M in $\operatorname{mod}\mathcal{C}$ if and only if $M \cong \mathcal{C}(-, \Sigma^n X_2)/\operatorname{rad}\mathcal{C}(-, \Sigma^n X_2)$ for some $n \in \mathbb{Z}$. It follows that the canonical functor

$$\coprod_{n\in\mathbb{Z}} \operatorname{mod} A \longrightarrow \operatorname{mod}\mathcal{C}/\operatorname{Ker} F^\star, \quad (M_n)_{n\in\mathbb{Z}} \longmapsto \coprod_{n\in\mathbb{Z}} \mathcal{C}(-, \Sigma^n M_n)$$

is an equivalence.

We have seen that $\operatorname{mod}\mathcal{C}/\operatorname{Ker} F^\star$ is not a semi-simple category, whereas in $\operatorname{mod}\mathcal{D}$ every object is semi-simple. More specifically, the cohomological functor

$$H\colon \mathcal{C} \longrightarrow \operatorname{mod}\mathcal{C} \longrightarrow \operatorname{mod}\mathcal{C}/\operatorname{Ker} F^\star$$

does not factor through F via some cohomological functor $\mathcal{D} \to \operatorname{mod}\mathcal{C}/\operatorname{Ker} F^\star$, even though $\operatorname{Ann} H = \operatorname{Ann} F$. We conclude that F is not a cohomological quotient functor.

LEMMA 7.5. *Let $F\colon \mathcal{C} \to \mathcal{D}$ be an additive functor between additive categories which admits a right inverse $G\colon \mathcal{D} \to \mathcal{C}$, that is, $F \circ G = \mathrm{Id}_{\mathcal{D}}$. Suppose $F'\colon \mathcal{C} \to \mathcal{E}$ is an additive functor satisfying*

(1) $\operatorname{Ann} F \subseteq \operatorname{Ann} F'$, *and*

(2) *for all X, Y in $\mathcal{C}$, $FX = FY$ implies $F'X = F'Y$.*

Then F' factors uniquely through F via the functor $F' \circ G\colon \mathcal{D} \to \mathcal{E}$.

Proof. We have for an object X in $\mathcal{C}$ that $FGFX = FX$. Thus $F'GFX = F'X$. Given a map ϕ in $\mathcal{C}$, we have $F((GF\phi) - \phi) = 0$ and therefore $F'((GF\phi) - \phi) = 0$. Thus $F'GF\phi = F'\phi$. It follows that $(F' \circ G) \circ F = F'$. The uniqueness

of the factorization follows from the fact that F is surjective on objects and morphisms. □

8. Exact ideals. Given a cohomological quotient functor $F: \mathcal{C} \to \mathcal{D}$, the ideal $\operatorname{Ann} F$ is an important invariant. In this section we investigate the collection of all ideals which are of this form.

Definition 8.1. Let $\mathcal{C}$ be a triangulated category. An ideal $\mathcal{I}$ of $\mathcal{C}$ is called *exact* if there exists a cohomological quotient functor $F: \mathcal{C} \to \mathcal{D}$ such that $\mathcal{I} = \operatorname{Ann} F$.

The exact ideals are partially ordered by inclusion and we shall investigate the structure of this poset. Recall that an ideal $\mathcal{I}$ in a triangulated category $\mathcal{C}$ is *cohomological*, if there exists a cohomological functor $F: \mathcal{C} \to \mathcal{A}$ such that $\mathcal{I} = \operatorname{Ann} F$.

THEOREM 8.2. *Let $\mathcal{C}$ be a small triangulated category. Then the exact ideals in $\mathcal{C}$ form a complete lattice, that is, given a family $(\mathcal{I}_i)_{i\in\Lambda}$ of exact ideals, the supremum* $\sup \mathcal{I}_i$ *and the infimum* $\inf \mathcal{I}_i$ *exist. Moreover, the supremum coincides with the supremum in the lattice of cohomological ideals.*

Our strategy for the proof is to use a bijection between the cohomological ideals of $\mathcal{C}$ and the Serre subcategories of $\operatorname{mod}\mathcal{C}$. We proceed in several steps and start with a few definitions. Given an ideal $\mathcal{I}$ of $\mathcal{C}$, we define

$$\operatorname{Im}\mathcal{I} = \{M \in \operatorname{mod}\mathcal{C} \mid M \cong \operatorname{Im}\mathcal{C}(-,\phi) \text{ for some } \phi \in \mathcal{I}\}.$$

The next definition is taken from [3].

Definition 8.3. Let $\mathcal{C}$ be a triangulated category. An ideal $\mathcal{I}$ of $\mathcal{C}$ is called *saturated* if for every exact triangle $X' \xrightarrow{\alpha} X \xrightarrow{\beta} X'' \to \Sigma X'$ and every map $\phi: X \to Y$ in $\mathcal{S}_c$, we have that $\phi\circ\alpha, \beta \in \mathcal{I}$ implies $\phi \in \mathcal{I}$.

The following characterization combines [21, Lemma 3.2] and [3, Theorem 3.1].

LEMMA 8.4. *Let $\mathcal{C}$ be a triangulated category. Then the following are equivalent for an ideal $\mathcal{I}$ of $\mathcal{C}$.*

(1) $\mathcal{I}$ *is cohomological.*

(2) $\mathcal{I}$ *is saturated.*

(3) $\operatorname{Im}\mathcal{I}$ *is a Serre subcategory of* $\operatorname{mod}\mathcal{C}$.

Moreover, the map $\mathcal{I} \mapsto \operatorname{Im}\mathcal{I}$ induces a bijection between the cohomological ideals of $\mathcal{C}$ and the Serre subcategories of $\operatorname{mod}\mathcal{C}$.

Proof. (1) $\Rightarrow$ (2): Let $\mathcal{I} = \operatorname{Ann} F$ for some cohomological functor $F: \mathcal{C} \to \mathcal{A}$. Fix an exact triangle $X' \xrightarrow{\alpha} X \xrightarrow{\beta} X'' \to \Sigma X'$ and a map $\phi: X \to Y$ in $\mathcal{C}$. Suppose $\phi\circ\alpha, \beta \in \mathcal{I}$. Then $F\alpha$ is an epimorphism, and therefore $F\phi\circ F\alpha = 0$ implies $F\phi = 0$. Thus $\phi \in \mathcal{I}$.

(2) $\Rightarrow$ (3): Let $0 \to F' \to F \to F'' \to 0$ be an exact sequence in $\operatorname{mod}\mathcal{C}$. Using that $\mathfrak{I}$ is an ideal, it is clear that $F \in \operatorname{Im}\mathfrak{I}$ implies $F', F'' \in \operatorname{Im}\mathfrak{I}$. Now suppose that $F', F'' \in \operatorname{Im}\mathfrak{I}$. Using that $\operatorname{mod}\mathcal{C}$ is a Frobenius category, we find maps $\phi\colon X \to Y$ and $\alpha\colon X' \to X$ such that $F = \operatorname{Im}\mathcal{C}(-,\phi)$ and $F' = \operatorname{Im}\mathcal{C}(-,\phi\circ\alpha)$. Now form exact triangles

$$W \xrightarrow{\chi} X \xrightarrow{\phi} Y \longrightarrow \Sigma W \quad \text{and} \quad X' \amalg W \xrightarrow{\begin{bmatrix}\alpha & \chi\end{bmatrix}} X \xrightarrow{\beta} X'' \longrightarrow \Sigma(X' \amalg W)$$

in $\mathcal{C}$, and observe that $F'' = \operatorname{Im}\mathcal{C}(-,\beta)$. We have $\phi \circ \begin{bmatrix}\alpha & \chi\end{bmatrix}$ and β in $\mathfrak{I}$. Thus $\phi \in \mathfrak{I}$ since $\mathfrak{I}$ is saturated. It follows that $F = \operatorname{Im}\mathcal{C}(-,\phi)$ belongs to $\operatorname{Im}\mathfrak{I}$.

(3) $\Rightarrow$ (1): Let F be the composite of the Yoneda functor $\mathcal{C} \to \operatorname{mod}\mathcal{C}$ with the quotient functor $\operatorname{mod}\mathcal{C} \to \operatorname{mod}\mathcal{C}/\operatorname{Im}\mathfrak{I}$. This functor is cohomological and we have $\mathfrak{I} = \operatorname{Ann} F$. □

We need some more terminology. Fix an abelian category $\mathcal{A}$. A Serre subcategory $\mathcal{B}$ of $\mathcal{A}$ is called *localizing* if the quotient functor $\mathcal{A} \to \mathcal{A}/\mathcal{B}$ has a right adjoint. If $\mathcal{A}$ is a Grothendieck category, then $\mathcal{B}$ is localizing if and only if $\mathcal{B}$ is closed under taking coproducts [11, Proposition III.8]. We denote for any subcategory $\mathcal{B}$ by $\varinjlim\mathcal{B}$ the full subcategory of filtered colimits $\varinjlim X_i$ in $\mathcal{A}$ such that X_i belongs to $\mathcal{B}$ for all i.

Now let $\mathcal{C}$ be a small additive category and suppose $\operatorname{mod}\mathcal{C}$ is abelian. Given a Serre subcategory $\mathcal{S}$ of $\operatorname{mod}\mathcal{C}$, then $\varinjlim\mathcal{S}$ is a localizing subcategory of $\operatorname{Mod}\mathcal{C}$; see [20, Theorem 2.8]. This has the following consequence which we record for later reference.

LEMMA 8.5. *Let $\mathcal{C}$ be a small triangulated category and $\mathfrak{I}$ be a cohomological ideal of $\mathcal{C}$. Then* $\varinjlim\operatorname{Im}\mathfrak{I}$ *is a localizing subcategory of* $\operatorname{Mod}\mathcal{C}$.

Proof. Use Lemma 8.4. □

We call a Serre subcategory $\mathcal{S}$ of $\operatorname{mod}\mathcal{C}$ *perfect* if the right adjoint of the quotient functor $\operatorname{Mod}\mathcal{C} \to \operatorname{Mod}\mathcal{C}/\varinjlim\mathcal{S}$ is an exact functor. We have a correspondence between perfect Serre subcategories of $\operatorname{mod}\mathcal{C}$ and flat epimorphisms starting in $\mathcal{C}$. To make this precise, we call a pair $F_1\colon\mathcal{C} \to \mathcal{D}_1$ and $F_2\colon\mathcal{C} \to \mathcal{D}_2$ of flat epimorphisms *equivalent* if $\operatorname{Ker} F_1^\star = \operatorname{Ker} F_2^\star$.

LEMMA 8.6. *Let $\mathcal{C}$ be a small additive category and suppose* $\operatorname{mod}\mathcal{C}$ *is abelian. Then the map*

$$(F\colon\mathcal{C} \to \mathcal{D}) \quad\longmapsto\quad \operatorname{Ker} F^\star$$

induces a bijection between the equivalence classes of flat epimorphisms starting in $\mathcal{C}$, and the perfect Serre subcategories of $\operatorname{mod}\mathcal{C}$.

Proof. We construct the inverse map as follows. Let $\mathcal{S}$ be a perfect Serre subcategory of $\operatorname{mod}\mathcal{C}$ and consider the quotient functor $Q\colon \operatorname{Mod}\mathcal{C} \to \operatorname{Mod}\mathcal{C}/\varinjlim\mathcal{S}$. Observe that Q preserves projectivity since the right adjoint of Q is exact. Now define $\mathcal{D}$ to be the full subcategory formed by the objects $Q\mathcal{C}(-,X)$ with X in $\mathcal{C}$, and let $F\colon\mathcal{C}\to\mathcal{D}$ be the functor which sends X to $Q\mathcal{C}(-,X)$. It follows that $F^*\colon\operatorname{Mod}\mathcal{C}\to\operatorname{Mod}\mathcal{D}$ induces an equivalence $\operatorname{Mod}\mathcal{C}/\varinjlim\mathcal{S}\to\operatorname{Mod}\mathcal{D}$. Thus F is a flat epimorphism satisfying $\mathcal{S}=\operatorname{Ker}F^\star$. □

LEMMA 8.7. *Let $\mathcal{C}$ be a small additive category and suppose* $\operatorname{mod}\mathcal{C}$ *is abelian. If $(\mathcal{S}_i)_{i\in\Lambda}$ is a family of perfect Serre subcategories of* $\operatorname{mod}\mathcal{C}$*, then the smallest Serre subcategory of* $\operatorname{mod}\mathcal{C}$ *containing all $\mathcal{S}_i$ is perfect.*

Proof. For each $i\in\Lambda$, let $\mathcal{M}_i$ be the full subcategory of $\mathcal{C}$-modules M satisfying $\operatorname{Hom}_{\mathcal{C}}(\varinjlim\mathcal{S}_i,M)=0=\operatorname{Ext}^1_{\mathcal{C}}(\varinjlim\mathcal{S}_i,M)$. Note that the right adjoint adjoint of the quotient functor $Q\colon\operatorname{Mod}\mathcal{C}\to\operatorname{Mod}\mathcal{C}/\varinjlim\mathcal{S}_i$ identifies $\operatorname{Mod}\mathcal{C}/\varinjlim\mathcal{S}_i$ with $\mathcal{M}_i$; see Lemma 3.2. Let $\mathcal{S}=\sup\mathcal{S}_i$. Then the full subcategory $\varinjlim\mathcal{S}$ is the smallest localizing subcategory of $\operatorname{Mod}\mathcal{C}$ containing all $\mathcal{S}_i$. Let $\mathcal{M}$ be the full subcategory of $\mathcal{C}$-modules M satisfying $\operatorname{Hom}_{\mathcal{C}}(\varinjlim\mathcal{S},M)=0=\operatorname{Ext}^1_{\mathcal{C}}(\varinjlim\mathcal{S},M)$. We claim that $\mathcal{M}=\bigcap_i\mathcal{M}_i$. To see this, let $\mathcal{I}_i$ be the full subcategory of injective objects in $\mathcal{M}_i$. Note that a $\mathcal{C}$-module M belongs to $\varinjlim\mathcal{S}_i$ if and only if $\operatorname{Hom}_{\mathcal{C}}(M,\mathcal{I}_i)=0$, and M belongs to $\mathcal{M}_i$ if and only if the modules I_0,I_1 in a minimal injective resolution $0\to M\to I_0\to I_1$ belong to $\mathcal{I}_i$. Let $\mathcal{I}=\bigcap_i\mathcal{I}_i$. Then we have that a $\mathcal{C}$-module M belongs to $\varinjlim\mathcal{S}$ if and only if $\operatorname{Hom}_{\mathcal{C}}(M,\mathcal{I})=0$, and M belongs to $\bigcap_i\mathcal{M}_i$ if and only if the modules I_0,I_1 in a minimal injective resolution $0\to M\to I_0\to I_1$ belong to $\mathcal{I}$. This proves $\mathcal{M}=\bigcap_i\mathcal{M}_i$. It follows that the inclusion $\mathcal{M}\to\operatorname{Mod}\mathcal{C}$ is exact because each inclusion $\mathcal{M}_i\to\operatorname{Mod}\mathcal{C}$ is exact. Thus $\mathcal{S}$ is a perfect Serre subcateory. □

PROPOSITION 8.8. *Let $\mathcal{C}$ be a small triangulated category, and let $\mathfrak{I}$ be an ideal in $\mathcal{C}$ satisfying $\Sigma\mathfrak{I}=\mathfrak{I}$. Then the following are equivalent.*

(1) *$\mathfrak{I}$ is an exact ideal.*

(2) $\operatorname{Im}\mathfrak{I}$ *is a perfect Serre subcategory of* $\operatorname{mod}\mathcal{C}$.

(3) *There exists a flat epimorphism $F\colon\mathcal{C}\to\mathcal{D}$ such that* $\operatorname{Ann}F=\mathfrak{I}$.

Proof. (1) ⇒ (2): Suppose $\mathfrak{I}$ is an exact ideal, that is, there is a cohomological quotient functor $F\colon\mathcal{C}\to\mathcal{D}$ such that $\mathfrak{I}=\operatorname{Ann}F$. Then F is a flat epimorphism by Theorem 4.4. Now observe that $\operatorname{Im}\mathfrak{I}=\operatorname{Ker}F^\star$. Thus $\operatorname{Im}\mathfrak{I}$ is perfect by Lemma 8.6.

(2) ⇒ (3): Apply again Lemma 8.6 to obtain a flat epimorphisms $F\colon\mathcal{C}\to\mathcal{D}$ with $\operatorname{Ann}F=\mathfrak{I}$.

(3) ⇒ (1): We may assume that idempotents in $\mathcal{D}$ split. It follows from Theorem 5.1 that $\mathcal{D}$ is a triangulated category and that F is an exact functor. Moreover, Theorem 4.4 implies that F is a cohomological quotient functor. Thus $\mathfrak{I}$ is an exact ideal. □

We collect our findings to obtain the proof of the theorem from the beginning of this section.

Proof of Theorem 8.2. Let $(\mathfrak{I}_i)_{i\in\Lambda}$ be a family of exact ideals in $\mathcal{C}$ and consider the corresponding Serre subcategories $\mathcal{S}_i = \operatorname{Im}\mathfrak{I}_i$ of $\operatorname{mod}\mathcal{C}$ which are perfect by Proposition 8.8. It follows from Lemma 8.7 that $\mathcal{S} = \sup\mathcal{S}_i$ is perfect. There is a cohomological ideal $\mathfrak{I}$ in $\mathcal{C}$ satisfying $\mathcal{S} = \operatorname{Im}\mathfrak{I}$ and we have $\mathfrak{I} = \sup\mathfrak{I}_i$ in the lattice of cohomological ideals by Lemma 8.4. Applying again Proposition 8.8, we see that the ideal $\mathfrak{I}$ is exact. This completes the proof. □

9. Factorizations. Let $\mathcal{C}$ be a triangulated category. If $\mathcal{B}$ is a full triangulated subcategory of $\mathcal{C}$, then the quotient functor $Q\colon\mathcal{C} \to \mathcal{C}/\mathcal{B}$ in the sense of Verdier is a cohomological and exact quotient functor (in the sense of Definition 4.1 and Definition 7.1). This fact motivates the following definition.

Definition 9.1. Let $F\colon\mathcal{C} \to \mathcal{D}$ be an exact functor between triangulated categories. We call F a *CE-quotient functor* if F is a cohomological quotient functor and an exact quotient functor.

The terminology refers to the properties "cohomological" and "exact." In addition, we wish to honor Cartan and Eilenberg. In this section we study the collection of all CE-quotient functors starting in a fixed triangulated category. Given a pair $F_1\colon\mathcal{C} \to \mathcal{D}_1$ and $F_2\colon\mathcal{C} \to \mathcal{D}_2$ of CE-quotient functors, we define

$$F_1 \sim F_2 \iff \text{there exists an equivalence } G\colon\mathcal{D}_1 \to \mathcal{D}_2 \text{ such that } F_2 = G\circ F_1,$$
$$F_1 \geq F_2 \iff \text{there exists an exact functor } G\colon\mathcal{D}_1 \to \mathcal{D}_2 \text{ such that } F_2 = G\circ F_1.$$

We obtain a partial ordering on the equivalence classes of CE-quotient functors, which may be rephrased as follows.

$$F_1 \sim F_2 \iff \operatorname{Ann} F_1 = \operatorname{Ann} F_2,$$
$$F_1 \geq F_2 \iff \operatorname{Ann} F_1 \subseteq \operatorname{Ann} F_2.$$

The ideals of the form $\operatorname{Ann} F$ arising from CE-quotient functors $F\colon\mathcal{C} \to \mathcal{D}$ form a complete lattice. This has been established in Theorem 8.2, and we obtain the following immediate consequence.

THEOREM 9.2. *Let $\mathcal{C}$ be a small triangulated category.*

(1) *The equivalence classes of CE-quotient functors starting in $\mathcal{C}$ form a complete lattice.*

(2) *The assignment $F \mapsto \operatorname{Ann} F$ induces an anti-isomorphism between the lattice of CE-quotient functors starting in $\mathcal{C}$ and the lattice of exact ideals of $\mathcal{C}$.*

(3) *Given a family* $(F_i)_{i\in\Lambda}$ *of CE-quotient functors, we have*

$$\operatorname{Ann}\Big(\inf_{i\in\Lambda} F_i\Big) = \sup_{i\in\Lambda}\big(\operatorname{Ann} F_i\big)$$

where the supremum is taken in the lattice of cohomological ideals of $\mathcal{C}$.

Proof. The ideals of the form $\operatorname{Ann} F$ for some CE-quotient functor $F\colon \mathcal{C} \to \mathcal{D}$ are precisely the exact ideals of $\mathcal{C}$. This follows from Lemma 7.3. Now apply Theorem 8.2. □

The completeness of the CE-quotient functor lattice yields a canonical factorization for every exact functor between two triangulated categories.

COROLLARY 9.3. *Let* $\mathcal{C}$ *be a small triangulated category. Then every exact functor* $F\colon \mathcal{C} \to \mathcal{D}$ *to a triangulated category* $\mathcal{D}$ *has a factorization*

$$\mathcal{C} \xrightarrow{Q} \mathcal{C}' \xrightarrow{F'} \mathcal{D}$$

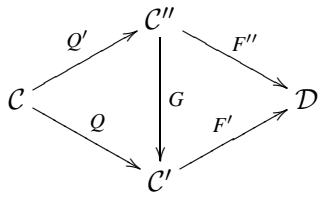
having the following properties:

(1) Q *is a CE-quotient functor and* F' *is exact.*

(2) *Given a factorization*

$$\mathcal{C} \xrightarrow{Q'} \mathcal{C}'' \xrightarrow{F''} \mathcal{D}$$

of F *such that* Q' *is a CE-quotient functor and* F'' *is exact, there exists, up to a unique isomorphism, a unique exact functor* $G\colon \mathcal{C}'' \to \mathcal{C}'$



such that $Q = G \circ Q'$ *and* $F'' \cong F' \circ G$.

Proof. We obtain the CE-quotient functor $Q\colon \mathcal{C} \to \mathcal{C}'$ by taking the infimum over all CE-quotient functors $Q'\colon \mathcal{C} \to \mathcal{C}''$ admitting a factorization

$$\mathcal{C} \xrightarrow{Q'} \mathcal{C}'' \xrightarrow{F''} \mathcal{D}.$$

Note that F factors through Q because $\operatorname{Ann} Q \subseteq \operatorname{Ann} F$. This follows from the fact that $\operatorname{Ann} F$ is cohomological and $\operatorname{Ann} Q' \subseteq \operatorname{Ann} F$ for all Q'. □

10. Compactly generated triangulated categories and Brown representability. We recall the definition of a compactly generated triangulated category, and we review a variant of Brown's Representability Theorem which will be needed later on.

Let $\mathcal{S}$ be a triangulated category and suppose that arbitrary coproducts exist in $\mathcal{S}$. An object X in $\mathcal{S}$ is called *compact* if for every family $(Y_i)_{i\in I}$ in $\mathcal{S}$, the canonical map $\coprod_i \mathcal{S}(X, Y_i) \to \mathcal{S}(X, \coprod_i Y_i)$ is an isomorphism. We denote by $\mathcal{S}_c$ the full subcategory of compact objects in $\mathcal{S}$ and observe that $\mathcal{S}_c$ is a triangulated subcategory of $\mathcal{S}$. Following [28], the category $\mathcal{S}$ is called *compactly generated* provided that the isomorphism classes of objects in $\mathcal{S}_c$ form a set, and for every object X in $\mathcal{S}$ we have $\mathcal{S}(C, X) \neq 0$ for some C in $\mathcal{S}_c$.

A basic tool for studying a compactly generated triangulated category $\mathcal{S}$ is the cohomological functor

$$H_{\mathcal{S}}\colon \mathcal{S} \longrightarrow \operatorname{Mod}\mathcal{S}_c, \quad X \mapsto H_X = \mathcal{S}(-, X)|_{\mathcal{S}_c}$$

which we call *restricted Yoneda functor*. Our notation does not distinguish between the Yoneda functor $H_{\mathcal{S}}\colon \mathcal{S} \to \operatorname{mod}\mathcal{S}$ and the restricted Yoneda functor. However, the meaning of $H_{\mathcal{S}}$ and H_X for some X in $\mathcal{S}$ will be clear from the context.

Next we recall from [22] a variant of Brown's Representability Theorem [6]; see also [17, 28, 8]. Let $\mathcal{S}$ be a triangulated category with arbitrary products. An object U in $\mathcal{S}$ is called a *perfect cogenerator* if $\mathcal{S}(X, U) = 0$ implies $X = 0$ for every object X in $\mathcal{S}$, and for every countable family of maps $X_i \to Y_i$ in $\mathcal{S}$, the induced map

$$\mathcal{S}\left(\prod_i Y_i, U\right) \longrightarrow \mathcal{S}\left(\prod_i X_i, U\right)$$

is surjective provided that the map $\mathcal{S}(Y_i, U) \to \mathcal{S}(X_i, U)$ is surjective for all i.

PROPOSITION 10.1 (Brown representability). *Let $\mathcal{S}$ be a triangulated category with arbitrary products and a perfect cogenerator U.*

(1) *A functor $H\colon \mathcal{S} \to$ Ab is cohomological and preserves all products if and only if $H \cong \mathcal{S}(X, -)$ for some object X in $\mathcal{S}$.*

(2) *$\mathcal{S}$ coincides with its smallest full triangulated subcategory which contains U and is closed under taking all products.*

Proof. See Theorem A in [22]. □

There is an immediate consequence which we shall use.

COROLLARY 10.2. *Let $\mathcal{S}$ be a triangulated category with arbitrary products and a perfect cogenerator U. An exact functor $\mathcal{S} \to \mathcal{T}$ between triangulated categories preserves all products if and only if it has a left adjoint.*

Proof. The left adjoint of a functor $F\colon \mathcal{S} \to \mathcal{T}$ sends an object X in $\mathcal{T}$ to the object in $\mathcal{S}$ representing $\mathcal{T}(X, F-)$. □

Take as an example for $\mathcal{S}$ a compactly generated triangulated category. Then

$$U = \coprod_{C \in \mathcal{S}_c} C$$

is a perfect cogenerator for $\mathcal{S}^{\mathrm{op}}$. If I is an injective cogenerator for $\operatorname{Mod} \mathcal{S}_c$, then the object V satisfying

$$\operatorname{Hom}_{\mathcal{S}_c}(H_{\mathcal{S}}-, I) \cong \mathcal{S}(-, V)$$

is a perfect cogenerator for $\mathcal{S}$. Note that V exists because $\mathcal{S}^{\mathrm{op}}$ is perfectly cogenerated.

11. Smashing localizations. We establish for any compactly generated triangulated category $\mathcal{S}$ a bijective correspondence between the smashing localizations of $\mathcal{S}$ and the cohomological quotients of $\mathcal{S}_c$.

This result is divided into two parts. In this section we show that any smashing localization induces a cohomological quotient. Let us recall the relevant definitions.

An exact functor $F\colon \mathcal{S} \to \mathcal{T}$ between triangulated categories is a *localization functor* if it has a right adjoint G such that $F \circ G \cong \mathrm{Id}_{\mathcal{T}}$. Note that the condition $F \circ G \cong \mathrm{Id}_{\mathcal{T}}$ is equivalent to the fact that F induces an equivalence $\mathcal{S}/\operatorname{Ker} F \to \mathcal{T}$, where $\mathcal{S}/\operatorname{Ker} F$ denotes quotient in the sense of Verdier [35]. It is often useful to identify a localization functor $F\colon \mathcal{S} \to \mathcal{T}$ with the idempotent functor $L\colon \mathcal{S} \to \mathcal{S}$ defined by $L = G \circ F$. The *L-acyclic* objects are those in $\operatorname{Ker} F$ and the *L-local* object are those which are isomorphic to some object in the image of G. The localization F is called *smashing* if G preserves all coproducts which exist in $\mathcal{T}$. (If $\mathcal{S}$ carries a smash product $\wedge\colon \mathcal{S} \times \mathcal{S} \to \mathcal{S}$ with unit S, then $LX = X \wedge LS$ provided F is smashing.)

THEOREM 11.1. *Let $\mathcal{S}$ be a compactly generated triangulated category and $F\colon \mathcal{S} \to \mathcal{T}$ be an exact functor between triangulated categories. Then F is a smashing localization if and only if the following holds:*

(1) *$\mathcal{T}$ is a compactly generated triangulated category.*

(2) *F preserves coproducts.*

(3) *F induces a functor $F_c\colon \mathcal{S}_c \to \mathcal{T}_c$ which is a cohomological quotient functor.*

We need some preparation before we can give the proof of this result.

LEMMA 11.2. *Let $\mathcal{S}$ be a compactly generated triangulated category and $F\colon \mathcal{S} \to \mathcal{T}$ be an exact functor between triangulated categories. Suppose F preserves coproducts. Then the right adjoint of F preserves coproducts if and only if F preserves compactness.*

Proof. Combine the definition of compactness and the adjointness isomorphism; see [28, Theorem 5.1]. □

LEMMA 11.3. *Let $F\colon \mathcal{S} \to \mathcal{T}$ be an exact functor between compactly generated triangulated categories. Suppose F has a right adjoint $G\colon \mathcal{T} \to \mathcal{S}$ which preserves coproducts. Then the following diagram commutes:*

$$\begin{array}{ccccc} \mathcal{S} & \xrightarrow{F} & \mathcal{T} & \xrightarrow{G} & \mathcal{S} \\ \downarrow{\scriptstyle H_{\mathcal{S}}} & & \downarrow{\scriptstyle H_{\mathcal{T}}} & & \downarrow{\scriptstyle H_{\mathcal{S}}} \\ \operatorname{Mod}\mathcal{S}_c & \xrightarrow{(F_c)^*} & \operatorname{Mod}\mathcal{T}_c & \xrightarrow{(F_c)_*} & \operatorname{Mod}\mathcal{S}_c. \end{array}$$

Proof. See Proposition 2.6 in [21]. □

We are now in the position that we can prove the main result of this section.

Proof of Theorem 11.1. Suppose first that F is a smashing localization. Thus F has a right adjoint G which preserves coproducts. This implies that F induces a functor $F_c\colon \mathcal{S}_c \to \mathcal{T}_c$, by Lemma 11.2. Using the adjointness formula $\mathcal{T}(FX, Y) \cong \mathcal{S}(X, GY)$, one sees that $\mathcal{T}$ is generated by the image of F_c. Thus $\mathcal{T}$ is compactly generated. It remains to show that F_c is a cohomological quotient functor. To this end denote by $\mathcal{M}$ the class of $\mathcal{T}_c$-modules M such that the natural map $((F_c)^* \circ (F_c)_*)M \to M$ is an isomorphism. Observe that $(F_c)^* \circ (F_c)_*$ composed with the Yoneda embedding $\mathcal{T}_c \to \operatorname{Mod}\mathcal{T}_c$ equals the composite $H_{\mathcal{T}} \circ F \circ G|_{\mathcal{T}_c}$, by Lemma 11.3. Our assumption implies $F \circ G \cong \mathrm{Id}_{\mathcal{T}}$, and therefore $\mathcal{M}$ contains all representable functors. Note that $(F_c)_*$ preserves colimits because they are defined pointwise, and $(F_c)^*$ preserves colimits because it is a left adjoint. Thus the composite $(F_c)^* \circ (F_c)_*$ preserves all colimits and therefore $\mathcal{M}$ is closed under taking colimits. We conclude that

$$(F_c)^* \circ (F_c)_* \cong \mathrm{Id}_{\operatorname{Mod}\mathcal{T}_c}$$

since every module is a colimit of representable functors. Thus F_c is up to direct factors an epimorphism by Proposition A.5, and therefore a cohomological quotient functor by Theorem 4.4.

Now suppose that F satisfies (1) – (3). An application of Brown's Representability Theorem shows that F has a right adjoint G, since F preserves coproducts. Moreover, G preserves coproducts by Lemma 11.2, since F preserves compactness. It remains to show that $F \circ G \cong \mathrm{Id}_{\mathcal{T}}$. To this end denote by $\mathcal{T}'$ the class of objects X in $\mathcal{T}$ such that the natural map $(F \circ G)X \to X$ is an isomorphism. Our assumption on F_c implies

$$(F_c)^* \circ (F_c)_* \cong \mathrm{Id}_{\operatorname{Mod}\mathcal{T}_c}.$$

Using again Lemma 11.3, we see that $\mathcal{T}_c \subseteq \mathcal{T}'$. The objects in $\mathcal{T}'$ form a triangulated subcategory which is closed under taking coproducts. It follows that $\mathcal{T}' = \mathcal{T}$ since $\mathcal{T}$ is compactly generated. This finishes the proof. □

12. Smashing subcategories. Let $\mathcal{S}$ be a compactly generated triangulated category. In this section we complete the correspondence between smashing localizations of $\mathcal{S}$ and cohomological quotients of $\mathcal{S}_c$. In order to formulate this, let us define the following full subcategories of $\mathcal{S}$ for any ideal $\mathcal{I}$ in $\mathcal{S}_c$:

$$\operatorname{Filt}\mathcal{I} = \{X \in \mathcal{S} \mid \text{every map } C \to X,\ C \in \mathcal{S}_c, \text{ factors through some map in } \mathcal{I}\},$$
$$\mathcal{I}^{\perp} = \{X \in \mathcal{S} \mid \mathcal{S}(\phi, X) = 0 \text{ for all } \phi \in \mathcal{I}\}.$$

THEOREM 12.1. *Let $\mathcal{S}$ be a compactly generated triangulated category, and let $\mathcal{I}$ be an exact ideal in $\mathcal{S}_c$. Then there exists a smashing localization $F\colon \mathcal{S} \to \mathcal{T}$ having the following properties:*

(1) *The right adjoint of F identifies $\mathcal{T}$ with $\mathcal{I}^{\perp}$.*
(2) $\operatorname{Ker} F = \operatorname{Filt}\mathcal{I}$.
(3) $\mathcal{S}_c \cap \operatorname{Ann} F = \mathcal{I}$.

The proof of this result requires some preparation. We start with descriptions of $\operatorname{Filt}\mathcal{I}$ and $\mathcal{I}^{\perp}$ which we take from [21].

LEMMA 12.2. *Let $\mathcal{I}$ be an ideal in $\mathcal{S}_c$ and X be an object in $\mathcal{S}$.*
(1) *$X \in \operatorname{Filt}\mathcal{I}$ if and only if $H_X \in \varinjlim \operatorname{Im}\mathcal{I}$.*
(2) *$X \in \mathcal{I}^{\perp}$ if and only if $\operatorname{Hom}_{\mathcal{S}_c}(\operatorname{Im}\mathcal{I}, H_X) = 0$.*

Proof. For (1), see Lemma 3.9 in [21]. (2) follows from the fact that $\operatorname{Hom}_{\mathcal{S}_c}(-, H_X)$ is exact when restricted to $\operatorname{mod}\mathcal{S}_c$. □

Now suppose that $\mathcal{I}$ is a cohomological ideal in $\mathcal{S}_c$ and observe that $\mathcal{L} = \varinjlim \operatorname{Im}\mathcal{I}$ is a localizing subcategory of $\operatorname{Mod}\mathcal{S}_c$, by Lemma 8.5. Thus we obtain a quotient functor $Q\colon \operatorname{Mod}\mathcal{S}_c \to \operatorname{Mod}\mathcal{S}_c/\mathcal{L}$ which has a right adjoint R; see [11, Proposition III.8]. Note that R identifies $\operatorname{Mod}\mathcal{S}_c/\mathcal{L}$ with the full subcategory $\mathcal{M}$ of $\mathcal{S}_c$-modules M satisfying $\operatorname{Hom}_{\mathcal{S}_c}(\mathcal{L}, M) = 0 = \operatorname{Ext}^1_{\mathcal{S}_c}(\mathcal{L}, M)$; see Lemma 3.2. Moreover, every $\mathcal{S}_c$-module M fits into an exact sequence

$$0 \longrightarrow M' \longrightarrow M \longrightarrow (R \circ Q)M \longrightarrow M'' \longrightarrow 0$$

with M', M'' in $\mathcal{L}$.

LEMMA 12.3. *An object X in $\mathcal{S}$ belongs to $\mathcal{I}^{\perp}$ if and only if $\operatorname{Hom}_{\mathcal{S}_c}(\mathcal{L}, H_X) = 0 = \operatorname{Ext}^1_{\mathcal{S}_c}(\mathcal{L}, H_X)$.*

Proof. Suppose $X \in \mathfrak{I}^{\perp}$. Then we have $\operatorname{Hom}_{\mathcal{S}_c}(\mathcal{L}, H_X) = 0$ because $\operatorname{Hom}_{\mathcal{S}_c}(\operatorname{Im}\mathfrak{I}, H_X) = 0$ by Lemma 12.2. Thus we have an exact sequence

$$0 \longrightarrow H_X \longrightarrow (R \circ Q)H_X \longrightarrow M \longrightarrow 0.$$

We claim that $M = 0$. For this it is sufficient to show that every map $\phi\colon M' \to M$ from a finitely presented module M' is zero. The map ϕ factors through some M'' in $\operatorname{Im}\mathfrak{I}$ because $M \in \mathcal{L}$. Now we use that $\operatorname{Ext}^1_{\mathcal{S}_c}(-, H_X)$ vanishes on finitely presented modules; see [21, Lemma 1.6]. Thus $M'' \to M$ factors through $(R \circ Q)H_X$. However, $\operatorname{Hom}_{\mathcal{S}_c}(\mathcal{L}, \mathcal{M}) = 0$, and this implies $\phi = 0$. Therefore $M = 0$, and H_X belongs to $\mathcal{M}$. Thus the proof is complete because the other implication is trivial. □

LEMMA 12.4. *Let $\mathfrak{I}$ be an ideal in $\mathcal{S}_c$ satisfying $\Sigma\mathfrak{I} = \mathfrak{I}$. If $\mathfrak{I}$ is exact or $\mathfrak{I}^2 = \mathfrak{I}$, then $\mathfrak{I}^{\perp}$ is a triangulated subcategory of $\mathcal{S}$ which is closed under taking products and coproducts.*

Proof. Clearly, $\mathfrak{I}^{\perp}$ is closed under taking products and coproducts. Also, $\Sigma(\mathfrak{I}^{\perp}) = \mathfrak{I}^{\perp}$ is clear. It remains to show that $\mathfrak{I}^{\perp}$ is closed under forming extensions. Let

$$X \xrightarrow{\alpha} Y \xrightarrow{\beta} Z \xrightarrow{\gamma} \Sigma X$$

be a triangle in $\mathcal{S}$ with X and Y in $\mathfrak{I}^{\perp}$, which induces an exact sequence

$$0 \longrightarrow \operatorname{Coker} H_\alpha \longrightarrow H_Z \longrightarrow \operatorname{Ker} H_{\Sigma\alpha} \longrightarrow 0$$

in $\operatorname{Mod}\mathcal{S}_c$. The modules annihilated by $\mathfrak{I}$ form a subcategory which is automatically closed under subobjects and quotients. The subcategory is closed under extensions if $\mathfrak{I}^2 = \mathfrak{I}$. Thus H_Z is annihilated by $\mathfrak{I}$ because $\operatorname{Coker} H_\alpha$ and $\operatorname{Ker} H_{\Sigma\alpha}$ are annihilated by $\mathfrak{I}$. It follows that Z belongs to $\mathfrak{I}^{\perp}$. Now suppose that $\mathfrak{I}$ is exact. We apply Proposition 8.8 to see that the category $\mathcal{M}$ of $\mathcal{S}_c$-modules M satisfying $\operatorname{Hom}_{\mathcal{S}_c}(\mathcal{L}, M) = 0 = \operatorname{Ext}^1_{\mathcal{S}_c}(\mathcal{L}, M)$ is closed under taking kernels, cokernels, and extensions. Thus $\operatorname{Coker} H_\alpha$ and $\operatorname{Ker} H_{\Sigma\alpha}$ belong to $\mathcal{M}$, and therefore H_Z as well. We conclude again that Z belongs to $\mathfrak{I}^{\perp}$. This finishes the proof. □

We are now in the position that we can prove Theorem 12.1.

Proof of Theorem 12.1. We know from Lemma 8.5 that $\mathcal{L} = \varinjlim \operatorname{Im}\mathfrak{I}$ is a localizing subcategory of $\operatorname{Mod}\mathcal{S}_c$. Denote by R the right adjoint of the quotient functor $\operatorname{Mod}\mathcal{S}_c \to \operatorname{Mod}\mathcal{S}_c/\mathcal{L}$. Recall that R identifies $\operatorname{Mod}\mathcal{S}_c/\mathcal{L}$ with the full subcategory $\mathcal{M}$ of $\mathcal{S}_c$-modules M satisfying $\operatorname{Hom}_{\mathcal{S}_c}(\mathcal{L}, M) = 0 = \operatorname{Ext}^1_{\mathcal{S}_c}(\mathcal{L}, M)$. The quotient $\operatorname{Mod}\mathcal{S}_c/\mathcal{L}$ is an abelian Grothendieck category. Thus there is an

injective cogenerator, say I, and we denote by U the object in $\mathcal{S}$ satisfying

$$\mathrm{Hom}_{\mathcal{S}_c}(H_{\mathcal{S}}-, RI) \cong \mathcal{S}(-, U).$$

Let $\mathcal{T} = \mathfrak{I}^{\perp}$, which is a triangulated subcategory of $\mathcal{S}$ and closed under taking products and coproducts by Lemma 12.4. We claim that U is a perfect cogenerator for $\mathcal{T}$. To see this, let X be an object in $\mathcal{T}$ satisfying $\mathcal{S}(X, U) = 0$. We have $H_X \in \mathcal{M}$ by Lemma 12.3, and therefore $H_X = 0$ since RI is a cogenerator for $\mathcal{M}$. Thus $X = 0$. Now let $X_i \to Y_i$ be a set of maps in $\mathcal{T}$ such that the map $\mathcal{S}(Y_i, U) \to \mathcal{S}(X_i, U)$ is surjective for all i. Using the fact that RI is an injective cogenerator for $\mathcal{M}$, we see that each map $H_{X_i} \to H_{Y_i}$ is a monomorphism. Thus their product is a monomorphism, and therefore the map $H_{\prod_i X_i} \to H_{\prod_i Y_i}$ is a monomorphism. We conclude that the map $\mathcal{S}(\prod_i Y_i, U) \to \mathcal{S}(\prod_i X_i, U)$ is surjective, since RI is an injective object. Now we can apply Corollary 10.2. It follows that the inclusion $\mathcal{T} \to \mathcal{S}$ has a left adjoint F which is a smashing localization since $\mathcal{T}$ is closed under taking coproducts.

It remains to describe $\operatorname{Ker} F$ and $\operatorname{Ann} F$. To this end let X be an object in $\mathcal{S}$. We have $FX = 0$ iff $\mathcal{S}(X, U) = 0$ iff $\mathrm{Hom}_{\mathcal{S}_c}(H_X, RI) = 0$ iff $H_X \in \mathcal{L}$ iff $X \in \operatorname{Filt} \mathfrak{I}$, by Lemma 12.2. Now let $\phi: X \to Y$ be a map in $\mathcal{S}_c$. Suppose first that $F\phi = 0$. Then $\mathcal{S}(\phi, U) = 0$. Now we have that $\mathcal{S}(\phi, U) = 0$ iff $\mathrm{Hom}_{\mathcal{S}_c}(H_\phi, RI) = 0$ iff $\operatorname{Im} H_\phi \in \mathcal{L}$ iff $\operatorname{Im} H_\phi \in \operatorname{Im} \mathfrak{I}$ iff $\phi \in \mathfrak{I}$. Conversely, suppose $\phi \in \mathfrak{I}$. Then $FY \in \mathfrak{I}^{\perp}$ implies $\mathcal{S}(\phi, FY) \cong \mathcal{T}(F\phi, FY) = 0$. Clearly, this implies $F\phi = 0$. Thus $\mathcal{S}_c \cap \operatorname{Ann} F = \mathfrak{I}$, and the proof is complete. □

Combining Theorem 11.1 and Theorem 12.1, one obtains a bijection between smashing localizations of $\mathcal{S}$ and exact ideals of $\mathcal{S}_c$. It is convenient to formulate this in terms of smashing subcategories. Recall that a subcategory of $\mathcal{S}$ is *smashing* if it of the form $\operatorname{Ker} F$ for some smashing localization functor $F: \mathcal{S} \to \mathcal{T}$. Note that the kernel $\operatorname{Ker} F$ of any localization functor F is a *localizing* subcategory, that is, $\operatorname{Ker} F$ is a full triangulated subcategory which is closed under taking coproducts. Thus a subcategory $\mathcal{R}$ of $\mathcal{S}$ is smashing if and only if $\mathcal{R}$ is a localizing subcategory admitting a right adjoint for the inclusion $\mathcal{R} \to \mathcal{S}$ which preserves coproducts.

COROLLARY 12.5. *Let $\mathcal{S}$ be a compactly generated triangulated category. Then the maps*

$$\mathfrak{I} \mapsto \operatorname{Filt} \mathfrak{I} \quad \textit{and} \quad \mathcal{R} \mapsto \{\phi \in \mathcal{S}_c \mid \phi \textit{ factors through some object in } \mathcal{R}\}$$

induce mutually inverse bijections between the set of exact ideals of $\mathcal{S}_c$ and the set of smashing subcategories of $\mathcal{S}$.

A similar result appears as Theorem 4.9 in [21]. However, the proof given there is not correct for two reasons: it uses an unnecessary assumption and relies

on an erroneous definition of an exact ideal. (The error occurs in Lemma 4.10. The claim that $f^*(\mu)$ is an isomorphism is only correct if f is a cohomological quotient functor.)

Let us formulate further consequences of Theorem 12.1 about cohomological quotient functors. I am grateful to B. Keller for pointing out to me the following simple description of the exact ideals of $\mathcal{S}_c$.

COROLLARY 12.6. *Let $\mathcal{S}$ be a compactly generated triangulated category. Then an ideal $\mathcal{I}$ of $\mathcal{S}_c$ is exact if and only if the following conditions hold.*

(1) $\Sigma\mathcal{I} = \mathcal{I}$.

(2) *$\mathcal{I}$ is saturated.*

(3) *$\mathcal{I}$ is idempotent, that is, $\mathcal{I}^2 = \mathcal{I}$.*

Proof. Suppose first that $\mathcal{I}$ is exact. Applying Corollary 12.5, the ideal $\mathcal{I}$ is the collection of maps in $\mathcal{S}_c$ which factor through an object in $\operatorname{Filt}\mathcal{I}$. Now fix a map $\phi\colon X \to Y$ in $\mathcal{I}$. Then ϕ factors through an object Y' in $\operatorname{Filt}\mathcal{I}$ via a map $\phi'\colon X \to Y'$, and ϕ' factors through a map $\phi_1\colon X \to Y''$ in $\mathcal{I}$ since Y' belongs to $\operatorname{Filt}\mathcal{I}$. Thus $\phi = \phi_2 \circ \phi_1$, and $\phi_2\colon Y'' \to Y$ belongs to $\mathcal{I}$ because it factors through an object in $\operatorname{Filt}\mathcal{I}$.

Now suppose that $\mathcal{I}$ satisfies (1) – (3). The proof of Theorem 12.1 works with these assumptions, thanks to Lemma 12.4. The conclusion of Theorem 12.1 shows that $\mathcal{I} = \operatorname{Ann} F_c$ for some smashing localization $F\colon \mathcal{S} \to \mathcal{T}$. Thus $\mathcal{I}$ is exact because F_c is a cohomological quotient functor by Theorem 11.1. □

One may think of the following result as a generalization of the localization theorem of Neeman-Ravenel-Thomason-Trobaugh-Yao [27, 31, 34, 38]. To be precise, Neeman et al. considered cohomological quotient functors of the form $\mathcal{S}_c \to \mathcal{S}_c/\mathcal{R}_0$ for some triangulated subcategory $\mathcal{R}_0$ of $\mathcal{S}_c$ and analyzed the smashing localization functor $\mathcal{S} \to \mathcal{S}/\mathcal{R}$ where $\mathcal{R}$ denotes the localizing subcategory generated by $\mathcal{R}_0$.

COROLLARY 12.7. *Let $\mathcal{S}$ be a compactly generated triangulated category, and let $F\colon \mathcal{S}_c \to \mathcal{T}$ be a cohomological quotient functor.*

(1) *The category $\mathcal{R} = \operatorname{Filt}(\operatorname{Ann} F)$ is a smashing localizing subcategory of $\mathcal{S}$ and the quotient functor $\mathcal{S} \to \mathcal{S}/\mathcal{R}$ induces a fully faithful and exact functor $\mathcal{T} \to \mathcal{S}/\mathcal{R}$ making the following diagram commutative.*

$$\begin{array}{ccc} \mathcal{S}_c & \xrightarrow{F} & \mathcal{T} \\ \downarrow{\scriptstyle \mathrm{inc}} & & \downarrow \\ \mathcal{S} & \xrightarrow{\mathrm{can}} & \mathcal{S}/\mathcal{R} \end{array}$$

(2) *The triangulated category $\mathcal{S}/\mathcal{R}$ is compactly generated and $(\mathcal{S}/\mathcal{R})_c$ is the closure of the image of $\mathcal{T} \to \mathcal{S}/\mathcal{R}$ under forming direct factors.*

(3) *There exists a fully faithful and exact functor* $G\colon \mathcal{T} \to \mathcal{S}$ *such that*

$$\mathcal{S}(X, GY) \cong \mathcal{T}(FX, Y)$$

for all X *in* $\mathcal{S}_c$ *and* Y *in* $\mathcal{T}$.

Proof. The ideal $\operatorname{Ann} F$ is exact and we obtain from Theorem 12.1 a smashing localization functor $Q\colon \mathcal{S} \to \mathcal{S}/\mathcal{R}$. The induced functor $Q_c\colon \mathcal{S}_c \to (\mathcal{S}/\mathcal{R})_c$ is a cohomological quotient functor with $\operatorname{Ann} Q_c = \operatorname{Ann} F$, by Theorem 11.1. The proof of Lemma 7.3 shows that Q_c factors through F, since idempotents in $(\mathcal{S}/\mathcal{R})_c$ split. Moreover, the functor $\mathcal{T} \to \mathcal{S}/\mathcal{R}$ is fully faithful since it induces an equivalence $\operatorname{mod} \mathcal{T} \to \operatorname{mod} (\mathcal{S}/\mathcal{R})_c$. Note that every compact object in $\mathcal{S}/\mathcal{R}$ is a direct factor of some object in the image of Q_c by Theorem 4.4. To obtain the functor $G\colon \mathcal{T} \to \mathcal{S}$, take the fully faithful right adjoint $\mathcal{S}/\mathcal{R} \to \mathcal{S}$ of Q, and compose this with the functor $\mathcal{T} \to \mathcal{S}/\mathcal{R}$. □

13. The telescope conjecture. The telescope conjecture due to Bousfield and Ravenel is originally formulated for the stable homotopy category of CW-spectra; see [5, 3.4], [31, 1.33] (and [23] for an unsuccessful attempt to disprove the conjecture). The stable homotopy category is a compactly generated triangulated category. This fact suggests the following formulation of the telescope conjecture for a specific triangulated category $\mathcal{S}$ which is compactly generated.

TELESCOPE CONJECTURE. *Every smashing subcategory of* $\mathcal{S}$ *is generated as a localizing subcategory by objects which are compact in* $\mathcal{S}$.

Recall that a subcategory of $\mathcal{S}$ is *smashing* if it is of the form $\operatorname{Ker} F$ for some smashing localization functor $F\colon \mathcal{S} \to \mathcal{T}$. Note that $\operatorname{Ker} F$ is a *localizing* subcategory of $\mathcal{S}$, that is, $\operatorname{Ker} F$ is a full triangulated subcategory which is closed under taking coproducts. A localizing subcategory of $\mathcal{S}$ is *generated* by a class $\mathcal{X}$ of objects if it is the smallest localizing subcategory of $\mathcal{S}$ which contains $\mathcal{X}$.

The telescope conjecture in this general form is known to be false. In fact, Keller gives an example of a smashing subcategory which contains no non-zero compact object [18]; see also Section 15. However, there are classes of compactly generated triangulated categories where the conjecture has been verified.

We have seen that smashing subcategories of $\mathcal{S}$ are closely related to cohomological quotients of $\mathcal{S}_c$. It is therefore natural to translate the telescope conjecture into a statement about cohomological quotients. Roughly speaking, the telescope conjecture for $\mathcal{S}$ is equivalent to the assertion that every flat epimorphism $\mathcal{S}_c \to \mathcal{T}$ is an Ore localization. We need some preparation in order to make this precise.

LEMMA 13.1. *Let* $\mathcal{S}$ *be a compactly generated triangulated category and* $F\colon \mathcal{S} \to \mathcal{T}$ *be a smashing localization functor. Then the following are equivalent.*

(1) *The localizing subcategory* $\operatorname{Ker} F$ *is generated by objects which are compact in* $\mathcal{S}$.

(2) *The ideal $\mathcal{S}_c \cap \operatorname{Ann} F$ of $\mathcal{S}_c$ is generated by identity maps.*

Proof. Let $\mathcal{R} = \operatorname{Ker} F$ and $\mathfrak{I} = \mathcal{S}_c \cap \operatorname{Ann} F$.

(1) $\Rightarrow$ (2): Suppose $\mathcal{R}$ is generated by compact objects. Then every object in $\mathcal{R}$ is a homotopy colimit of objects in $\mathcal{R} \cap \mathcal{S}_c$; see [28]. Let $\phi\colon X \to Y$ be a map in $\mathfrak{I}$. It follows from Lemma 4.2 that ϕ factors through a homotopy colimit of objects in $\mathcal{R} \cap \mathcal{S}_c$. Thus ϕ factors through some object in $\mathcal{R} \cap \mathcal{S}_c$ since X is compact. We conclude that $\mathfrak{I}$ is generated by the identity maps of all objects in $\mathcal{R} \cap \mathcal{S}_c$.

(2) $\Rightarrow$ (1): Let $\mathcal{R}_0$ be a class of compact objects and suppose $\mathfrak{I}$ is generated by the identity maps of all objects in $\mathcal{R}_0$. Let $\mathcal{R}'$ be the localizing subcategory which is generated by $\mathcal{R}_0$. This category is smashing and we have a localization functor $F'\colon \mathcal{S} \to \mathcal{T}'$ with $\mathcal{R}' = \operatorname{Ker} F'$. Clearly, $\operatorname{Ann} F_c \subseteq \operatorname{Ann} F'_c$ since id_X belongs to $\operatorname{Ann} F'_c$ for all X in $\mathcal{R}_0$. On the other hand, $\mathcal{R}' \subseteq \mathcal{R}$ since $\mathcal{R}_0 \subseteq \mathcal{R}$. Thus $\operatorname{Ann} F'_c \subseteq \operatorname{Ann} F_c$ by Lemma 4.2, and $\operatorname{Ann} F'_c = \operatorname{Ann} F_c$ follows. We conclude that $\mathcal{R}' = \mathcal{R}$ because Theorem 12.1 states that a smashing subcategory is determined by the corresponding exact ideal in $\mathcal{S}_c$. Thus $\mathcal{R}$ is generated by compact objects. □

PROPOSITION 13.2. *Let $F\colon \mathcal{C} \to \mathcal{D}$ be an exact functor between triangulated categories. Then the following are equivalent.*

(1) *F induces an equivalence $\mathcal{C}/\operatorname{Ker} F \to \mathcal{D}$.*

(2) *F induces an equivalence $\mathcal{C}[\Phi^{-1}] \to \mathcal{D}$ where $\Phi = \{\phi \in \mathcal{C} \mid F\phi \text{ is an iso}\}$.*

(3) *F is a CE-quotient functor and the ideal $\operatorname{Ann} F$ is generated by identity maps.*

Proof. We put $\mathcal{B} = \operatorname{Ker} F$ and denote by $Q\colon \mathcal{C} \to \mathcal{C}/\mathcal{B}$ the quotient functor.

(1) $\Leftrightarrow$ (2): The quotient $\mathcal{C}/\mathcal{B}$ is by definition $\mathcal{C}[\Psi^{-1}]$ where Ψ is the class of maps $X \to Y$ in $\mathcal{C}$ which fit into an exact triangle $X \to Y \to Z \to \Sigma X$ with Z in $\mathcal{B}$. The exactness of F implies that Ψ is precisely the class of maps ϕ in $\mathcal{C}$ such that $F\phi$ is invertible.

(1) $\Rightarrow$ (3): We have seen in Example 4.3 and Example 7.2 that Q is a CE-quotient functor. Lemma 4.2 implies that the ideal $\operatorname{Ann} Q$ is generated by the identity maps of all objects in $\mathcal{B}$.

(3) $\Rightarrow$ (1): The functor F induces an exact functor $\mathcal{C}/\mathcal{B} \to \mathcal{D}$. Now suppose that $\operatorname{Ann} F$ is generated by identity maps. Then $\operatorname{Ann} F$ is generated by the identity maps of all objects in $\mathcal{B}$, and therefore $\operatorname{Ann} Q = \operatorname{Ann} F$ by Lemma 4.2. Thus Q factors through F by an exact functor $\mathcal{D} \to \mathcal{C}/\mathcal{B}$ since F is an exact quotient functor. The uniqueness of $\mathcal{D} \to \mathcal{C}/\mathcal{B}$ and $\mathcal{C}/\mathcal{B} \to \mathcal{D}$ implies that both functors are mutually inverse equivalences. □

LEMMA 13.3. *Let $F\colon \mathcal{C} \to \mathcal{D}$ be a cohomological quotient functor. Then the following are equivalent.*

(1) *F induces a fully faithful functor $\mathcal{C}/\operatorname{Ker} F \to \mathcal{D}$.*

(2) *The ideal $\operatorname{Ann} F$ is generated by identity maps.*

Proof. Let $\mathcal{D}'$ be the smallest full triangulated subcategory of $\mathcal{D}$ containing the image of F. It follows from Lemma 7.3 that the induced functor $F'\colon \mathcal{C} \to \mathcal{D}'$ is a CE-quotient functor. Now the assertion follows from Proposition 13.2 since $\operatorname{Ann} F = \operatorname{Ann} F'$. □

We obtain the following reformulation of the telescope conjecture. Note in particular, that the telescope conjecture becomes a statement about the category of compact objects.

THEOREM 13.4. *Let $\mathcal{S}$ be a compactly generated triangulated category. Then the following are equivalent.*

(1) *Every smashing subcategory of $\mathcal{S}$ is generated by objects which are compact in $\mathcal{S}$.*

(2) *Every smashing subcategory of $\mathcal{S}$ is a compactly generated triangulated category.*

(3) *Every exact ideal in $\mathcal{S}_c$ is generated by idempotent elements.*

(4) *Every CE-quotient functor $F\colon \mathcal{S}_c \to \mathcal{T}$ induces an equivalence $\mathcal{S}_c/\operatorname{Ker} F \to \mathcal{T}$.*

(5) *Every cohomological quotient functor $F\colon \mathcal{S}_c \to \mathcal{T}$ induces a fully faithful functor $\mathcal{S}_c/\operatorname{Ker} F \to \mathcal{T}$.*

(6) *Every two-sided flat epimorphism $F\colon \mathcal{S}_c \to \mathcal{T}$ satisfying $\Sigma(\operatorname{Ann} F) = \operatorname{Ann} F$ induces an equivalence $\mathcal{S}_c[\Phi^{-1}] \to \mathcal{T}$ where $\Phi = \{\phi \in \mathcal{S}_c \mid F\phi \text{ is an iso}\}$.*

Proof. We use the bijection between smashing subcategories of $\mathcal{S}$ and exact ideals of $\mathcal{S}_c$; see Corollary 12.5. Recall that an ideal is by definition exact if it is of the form $\operatorname{Ann} F$ for some cohomological functor $F\colon \mathcal{S}_c \to \mathcal{T}$.

(1) $\Leftrightarrow$ (2): The inclusion $\mathcal{R} \to \mathcal{S}$ of a smashing subcategory preserves compactness.

(1) $\Leftrightarrow$ (3): Apply Lemma 13.1. Note that any ideal in $\mathcal{S}_c$ which is generated by idempotent maps is also generated by identity maps. This follows from the fact that idempotents in $\mathcal{S}_c$ split.

(3) $\Leftrightarrow$ (4): Apply Proposition 13.2.

(3) $\Leftrightarrow$ (5): Apply Lemma 13.3.

(5) $\Rightarrow$ (6): Let $F\colon \mathcal{S}_c \to \mathcal{T}$ be a two-sided flat epimorphism satisfying $\Sigma(\operatorname{Ann} F) = \operatorname{Ann} F$. Composing it with the idempotent completion $\mathcal{T} \to \bar{\mathcal{T}}$ gives a cohomological quotient functor $\mathcal{S}_c \to \bar{\mathcal{T}}$, by Theorem 4.4 and Theorem 5.1. Now use that $\mathcal{S}_c[\Phi^{-1}] = \mathcal{S}_c/\operatorname{Ker} F$. Thus $\mathcal{S}_c[\Phi^{-1}] \to \mathcal{T}$ is fully faithful, and it is an equivalence since F is surjective on objects by Lemma A.3.

(6) $\Rightarrow$ (5): Let $F\colon \mathcal{S}_c \to \mathcal{T}$ be a cohomological quotient functor, and denote by $\mathcal{T}'$ the full subcategory of $\mathcal{T}$ whose objects are those in the image of F. The induced functor $\mathcal{S}_c \to \mathcal{T}'$ is a flat epimorphism. Now use again that $\mathcal{S}_c[\Phi^{-1}] = \mathcal{S}_c/\operatorname{Ker} F$. Thus the induced functor $\mathcal{S}_c/\operatorname{Ker} F \to \mathcal{T}$ is fully faithful. □

Remark 13.5. Let $\mathcal{C}$ be a triangulated category and $F\colon \mathcal{C} \to \mathcal{D}$ be a flat functor satisfying $\Sigma(\operatorname{Ann} F) = \operatorname{Ann} F$. Then $\Phi = \{\phi \in \mathcal{C} \mid F\phi \text{ is an iso}\}$ is a *multiplica-*

tive system, that is, Φ admits a calculus of left and right fractions in the sense of [12].

We say that an additive functor $\mathcal{C} \to \mathcal{D}$ is an *Ore localization* if it induces an equivalence $\mathcal{C}[\Phi^{-1}] \to \mathcal{D}$ for some multiplicative system Φ in $\mathcal{C}$. Using this terminology, Theorem 13.4 suggests the following reformulation of the telescope conjecture.

COROLLARY 13.6. *The telescope conjecture holds true for a compactly generated triangulated category $\mathcal{S}$ if and only if every two-sided flat epimorphism $F\colon \mathcal{S}_c \to \mathcal{T}$ satisfying $\Sigma(\operatorname{Ann} F) = \operatorname{Ann} F$ is an Ore localization.*

The reformulation of the telescope conjecture in terms of exact ideals raises the question when an idempotent ideal is generated by idempotent elements. This follows from Corollary 12.6 where it is shown that the exact ideals are precisely the idempotent ideals which satisfy some natural extra conditions.

COROLLARY 13.7. *The telescope conjecture holds true for a compactly generated triangulated category $\mathcal{S}$ if and only if every idempotent and saturated ideal $\mathcal{I}$ of $\mathcal{S}_c$ satisfying $\Sigma\mathcal{I} = \mathcal{I}$ is generated by idempotent elements.*

The problem of finding idempotent generators for an idempotent ideal is a very classical one from ring theory. For instance, Kaplansky introduced the class of SBI rings, where SBI stands for "suitable for building idempotent elements" [15, III.8]. Also, Auslander asked the question for which rings every idempotent ideal is generated by an idempotent element [1, p. 241]. One can show for an additive category $\mathcal{C}$, that every idempotent ideal is generated by idempotent elements provided that $\mathcal{C}$ is perfect in the sense of Bass [24]. Recall that $\mathcal{C}$ is *perfect* if every object in $\mathcal{C}$ decomposes into a finite coproduct of indecomposable objects with local endomorphism rings, and, for every sequence

$$X_1 \xrightarrow{\phi_1} X_2 \xrightarrow{\phi_2} X_3 \xrightarrow{\phi_3} \cdots$$

of non-isomorphisms between indecomposable objects, the composition $\phi_n \circ \ldots \circ \phi_2 \circ \phi_1$ is zero for n sufficiently large. Note that the category $\mathcal{S}_c$ of compact objects is perfect if and only if every object in $\mathcal{S}$ is a coproduct of indecomposable objects with local endomorphism rings [21, Theorem 2.10].

14. Homological epimorphisms of rings. A commutative localization $R \to S$ of an associative ring R is always a flat epimorphism. For noncommutative localizations, there is a weaker condition which is often satisfied. Recall from [13] that a ring homomorphism $R \to S$ is a *homological epimorphism* if $S \otimes_R S \cong S$ and $\operatorname{Tor}_i^R(S, S) = 0$ for all $i \geq 1$. Homological epimorphisms frequently arise in representation theory of finite dimensional algebras, in particular via universal localizations [7, 32].

In recent work of Neeman and Ranicki [30], homological epimorphisms appear when they study the following chain complex lifting problem for a ring homomorphism $R \to S$. We denote by $\mathbf{K}^b(R)$ the homotopy category of bounded complexes of finitely generated projective R-modules.

Definition 14.1. Fix a ring homomorphism $R \to S$.

(1) We say that the *chain complex lifting problem* has a positive solution, if every complex Y in $\mathbf{K}^b(S)$ such that for each i we have $Y^i = P^i \otimes_R S$ for some finitely generated projective R-module P^i, is isomorphic to $X \otimes_R S$ for some complex X in $\mathbf{K}^b(R)$.

(2) We say that the *chain map lifting problem* has a positive solution, if for every pair X, Y of complexes in $\mathbf{K}^b(R)$ and every map $\alpha: X \otimes_R S \to Y \otimes_R S$ in $\mathbf{K}^b(S)$, there are maps $\phi: X' \to X$ and $\alpha': X' \to Y$ in $\mathbf{K}^b(R)$ such that $\phi \otimes_R S$ is invertible and $\alpha = \alpha' \otimes_R S \circ (\phi \otimes_R S)^{-1}$ in $\mathbf{K}^b(S)$.

The following observation shows that both lifting problems are closely related. In fact, it seems that the more general problem of lifting maps is the more natural one.

LEMMA 14.2. *Given a ring homomorphism $R \to S$, the chain complex lifting problem has a positive solution whenever the chain map lifting problem has a positive solution.*

Proof. Fix a complex Y in $\mathbf{K}^b(S)$. We proceed by induction on its length $\ell(Y) = n$. If $n = 0$, then Y is concentrated in one degree, say i, and therefore $Y = X \otimes_R S$ for $X = P^i$. If $n > 0$, choose an exact triangle $Y_1 \to Y_2 \to Y \to \Sigma Y_1$ with $\ell(Y_i) < n$ for $i = 1, 2$. By our assumption, we have $Y_i \cong X_i \otimes_R S$ for some complexes X_1 and X_2 in $\mathbf{K}^b(R)$. Moreover, using the positive solution of the chain map lifting problem, the map $X_1 \otimes_R S \to X_2 \otimes_R S$ is of the form $\alpha \otimes_R S \circ (\phi \otimes_R S)^{-1}$ for some maps $\phi: X_1' \to X_1$ and $\alpha: X_1' \to X_2$ in $\mathbf{K}^b(R)$. We complete α to an exact triangle $X_1' \to X_2 \to X \to \Sigma X_1'$ and conclude that $Y \cong X \otimes_R S$. □

The example of a proper field extension $k \to K$ shows that both lifting problems are not equivalent. In fact, the chain complex lifting problem for $k \to K$ has a positive solution, but the chain map lifting problem does not.

The proof of Lemma 14.2 suggests the following reformulation of the chain complex lifting problem.

LEMMA 14.3. *Given a ring homomorphism $R \to S$, the chain complex lifting problem has a positive solution if and only if the full subcategory of $\mathbf{K}^b(S)$ formed by the objects in the image of $- \otimes_R S$ is a triangulated subcategory.*

Proof. Clear. □

We continue with a reformulation of the chain map lifting problem.

PROPOSITION 14.4. *Let $R \to S$ be a ring homomorphism and denote by $\mathcal{K}$ the full subcategory of $\mathbf{K}^b(R)$ formed by the complexes X such that $X \otimes_R S = 0$. Then the following are equivalent.*

(1) *The chain map lifting problem has a positive solution.*

(2) *The functor $- \otimes_R S$ induces a fully faithful functor $\mathbf{K}^b(R)/\mathcal{K} \to \mathbf{K}^b(S)$.*

Proof. (1) $\Rightarrow$ (2): Denote by $\mathcal{D}$ the full subcategory of $\mathbf{K}^b(S)$ which is formed by all objects in the image of $- \otimes_R S$. Using the description of the maps in $\mathcal{D}$, we observe that $\mathcal{D}$ is a triangulated subcategory of $\mathbf{K}^b(S)$. It follows from Proposition 4.5 that F induces an equivalence $\mathbf{K}^b(R)/\mathcal{K} \to \mathcal{D}$.

(2) $\Rightarrow$ (1): The maps in $\mathbf{K}^b(R)/\mathcal{K}$ can be described as fractions; see for instance Proposition 4.5. The functor $\mathbf{K}^b(R)/\mathcal{K} \to \mathbf{K}^b(S)$ being full implies the positive solution of the chain map lifting problem. □

Given a ring homomorphism $R \to S$, we shall see that the problem of lifting complexes and their maps is closely related to the question, when the derived functor $- \otimes^{\mathbf{L}}_R S$ is a smashing localization.

Let us denote by $\mathbf{D}(R)$ the unbounded derived category of R. Note that the inclusion $\mathbf{K}^b(R) \to \mathbf{D}(R)$ induces an equivalence $\mathbf{K}^b(R) \to \mathbf{D}(R)_c$.

THEOREM 14.5. *For a ring homomorphism $R \to S$ the following are equivalent.*

(1) *The derived functor $- \otimes^{\mathbf{L}}_R S \colon \mathbf{D}(R) \to \mathbf{D}(S)$ is a smashing localization.*

(2) *The functor $- \otimes_R S \colon \mathbf{K}^b(R) \to \mathbf{K}^b(S)$ is a cohomological quotient functor.*

(3) *The map $R \to S$ is a homological epimorphism.*

Proof. The functor $F = - \otimes^{\mathbf{L}}_R S \colon \mathbf{D}(R) \to \mathbf{D}(S)$ has a right adjoint $G \colon \mathbf{D}(S) \to \mathbf{D}(R)$ which is simply restriction of scalars, that is, $G = \mathbf{R}\operatorname{Hom}_S(S, -)$. Clearly, G preserves coproducts. Thus F is a smashing localization if and only if F is a localization functor. Note that F is a localization functor if and only if $F \circ G \cong \mathrm{Id}_{\mathbf{D}(S)}$. Moreover, $F \circ G$ is exact and preserves coproducts. Using infinite devissage, one sees that $F \circ G \cong \mathrm{Id}_{\mathbf{D}(S)}$ if and only if the canonical map $X \otimes^{\mathbf{L}}_R S \to X$ is an isomorphism for the complex $X = S$ which is concentrated in degree 0. Clearly, this condition is equivalent to $S \otimes_R S \cong S$ and $\operatorname{Tor}^R_i(S, S) = 0$ for all $i \geq 1$. This proves the equivalence of (1) and (3). The equivalence of (1) and (2) follows from Theorem 11.1 since $F|_{\mathbf{K}^b(R)} = - \otimes_R S$. □

We obtain the following conditions for solving the chain map lifting problem.

THEOREM 14.6. *Given a ring homomorphism $R \to S$, the chain map lifting problem has a positive solution if and only if*

(1) *$R \to S$ is a homological epimorphism, and*

(2) *every map ϕ in $\mathbf{K}^b(R)$ satisfying $\phi \otimes_R S = 0$ factors through some X in $\mathbf{K}^b(R)$ such that $X \otimes_R S = 0$.*

Proof. Suppose first that (1) and (2) hold. Condition (1) says that $F = - \otimes_R S \colon \mathbf{K}^b(R) \to \mathbf{K}^b(S)$ is a cohomological quotient functor. This follows from

Theorem 14.5. Applying Lemma 13.3, we conclude from (2) that F induces a fully faithful functor $\mathbf{K}^b(R)/\operatorname{Ker} F \to \mathbf{K}^b(S)$. The positive solution of the chain map lifting problem follows from Proposition 14.4.

Now suppose that the chain map lifting problem has a positive solution. The description of the maps in the image $\operatorname{Im} F$ of F implies that the full subcategory formed by the objects in $\operatorname{Im} F$ is a triangulated subcategory of $\mathbf{K}^b(S)$. It contains a generator of $\mathbf{K}^b(S)$ and therefore every object in $\mathbf{K}^b(S)$ is a direct factor of some object in $\operatorname{Im} F$. Now we apply Theorem 4.4 and see that F is a cohomological quotient functor. Thus (1) holds by Theorem 14.5. The induced functor $\mathbf{K}^b(R)/\operatorname{Ker} F \to \mathbf{K}^b(S)$ is fully faithful by Proposition 14.4. It follows from Lemma 13.3 that (2) holds. This finishes the proof. □

COROLLARY 14.7. *Let R be a ring such that the telescope conjecture holds true for $\mathbf{D}(R)$. Then the chain map lifting problem has a positive solution for a ring homomorphism $f\colon R \to S$ if and only if f is a homological epimorphism.*

Note that the telescope conjecture has been verified for $\mathbf{D}(R)$ provided R is commutative noetherian [26]. On the other hand, Keller has given an example of a ring R such that the telescope conjecture for $\mathbf{D}(R)$ does not hold [18]. Let us mention that the validity of the telescope conjecture is preserved under homological epimorphims.

PROPOSITION 14.8. *Let $R \to S$ be a homological epimorphism. If the telescope conjecture holds for $\mathbf{D}(R)$, then the telescope conjecture holds for $\mathbf{D}(S)$.*

Proof. The derived functor $F = - \otimes^{\mathbf{L}}_R S\colon \mathbf{D}(R) \to \mathbf{D}(S)$ is a smashing localization by Theorem 14.5. Now suppose that $G\colon \mathbf{D}(S) \to \mathcal{T}$ is a smashing localization. A composite of smashing localizations is a smashing localizations. Thus $\operatorname{Ker} F$ is generated by a class $\mathcal{X}$ of compact objects since the telescope conjecture holds for $\mathbf{D}(R)$. It follows that $\operatorname{Ker} G$ is generated by $F\mathcal{X}$. □

The work of Neeman and Ranicki [30] on the problem of lifting chain complexes is motivated by some applications in algebraic K-theory. In fact, they generalize the classical long exact sequence which is induced by an injective Ore localization. More precisely, they show that every universal localization $f\colon R \to S$ which is a homological epimorphism induces a long exact sequence

$$\cdots \longrightarrow K_1(R) \longrightarrow K_1(S) \longrightarrow K_0(R,f) \longrightarrow K_0(R) \longrightarrow K_0(S)$$

in algebraic K-theory [30, Theorem 10.11]. Our analysis of the chain map lifting problem suggests that being a homological epimorphism and satisfying the additional hypothesis (2) in Theorem 14.6 is the crucial property for such a sequence. We sketch the construction of this sequence which uses the machinery developed by Waldhausen in [36]. Our exposition follows closely the ideas of Thomason-Trobaugh [34] and Neeman-Ranicki [30].

We fix a ring homomorphism $f\colon R \to S$. Denote by $\mathbf{W}(R)$ the complicial bi-Waldhausen category of bounded chain complexes of finitely generated projective R-modules [34, 1.2.11]. We denote by $K(R)$ the corresponding Waldhausen K-theory spectrum $K(\mathbf{W}(R))$; see [34, 1.5.2]. Note that $K(R)$ is homotopy equivalent to the Quillen K-theory spectrum of the exact category $\operatorname{proj} R$ of finitely generated projective R-modules [34, 1.11.2]. The algebraic K-groups $K_n(R) = \pi_n K(R)$ are by definition the homotopy groups of the spectrum $K(R)$. Now let $\mathbf{W}(R,f)$ be the complicial biWaldhausen subcategory of $\mathbf{W}(R)$ consisting of those complexes X in $\mathbf{W}(R)$ such that $X \otimes_R S$ is acyclic, and put $K(R,f) = K(\mathbf{W}(R,f))$.

THEOREM 14.9. *Let $f\colon R \to S$ be a homological epimorphism and suppose f satisfies condition (2) in Theorem 14.6. Then f induces a sequence*

$$\mathbf{W}(R,f) \longrightarrow \mathbf{W}(R) \longrightarrow \mathbf{W}(S)$$

of exact functors such that

$$K(R,f) \longrightarrow K(R) \longrightarrow K(S)$$

is a homotopy fibre sequence, up to failure of surjectivity of $K_0(R) \to K_0(S)$. In particular, there is induced a long exact sequence

$$\cdots \longrightarrow K_1(R) \longrightarrow K_1(S) \longrightarrow K_0(R,f) \longrightarrow K_0(R) \longrightarrow K_0(S)$$

of algebraic K-groups.

Proof. The proof is modeled after that of Thomason-Trobaugh's localization theorem [34, Theorem 5.1]. We recall that a complicial biWaldhausen category comes equipped with cofibrations and weak equivalences [34, 1.2.11]. The cofibrations of $\mathbf{W}(R)$ are by definition the chain maps which are split monomorphism in each degree, and the weak equivalences are the quasi-isomorphisms. We define a new complicial biWaldhausen category $\mathbf{W}(R/f)$ as follows. The underlying category is that of $\mathbf{W}(R)$, the cofibrations are those of $\mathbf{W}(R)$, and the weak equivalences are the chain maps whose mapping cone lies in $\mathbf{W}(R,f)$. We denote by $K(R/f)$ the K-theory spectrum of $\mathbf{W}(R/f)$ and obtain an induced sequence

$$\mathbf{W}(R,f) \longrightarrow \mathbf{W}(R) \longrightarrow \mathbf{W}(R/f)$$

of exact functors such that

$$K(R,f) \longrightarrow K(R) \longrightarrow K(R/f)$$

is a homotopy fibre sequence by Waldhausen's localization theorem [34, 1.8.2]. The functor $\mathbf{W}(R) \to \mathbf{W}(S)$ factors through $\mathbf{W}(R) \to \mathbf{W}(R/f)$ and induces an

exact functor $\mathbf{W}(R/f) \to \mathbf{W}(S)$. Note that any exact functor $\mathbf{A} \to \mathbf{B}$ between complicial biWaldhausen categories induces a homotopy equivalence of K-theory spectra $K(\mathbf{A}) \to K(\mathbf{B})$ provided the functor induces an equivalence $\mathrm{Ho}\,(\mathbf{A}) \to \mathrm{Ho}\,(\mathbf{B})$ of the derived homotopy categories [34, 1.9.8]. Observe that $\mathrm{Ho}\,(\mathbf{W}(R)) = \mathbf{K}^b(R)$. Moreover, $\mathbf{W}(R) \to \mathbf{W}(R/f)$ induces an equivalence

$$\mathrm{Ho}\,(\mathbf{W}(R))/\mathrm{Ho}\,(\mathbf{W}(R,f)) \xrightarrow{\sim} \mathrm{Ho}\,(\mathbf{W}(R/f)).$$

Thus we have the following commutative diagram

$$\begin{array}{ccccccc}
\mathrm{Ho}\,(\mathbf{W}(R,f)) & \longrightarrow & \mathrm{Ho}\,(\mathbf{W}(R)) & \longrightarrow & \mathrm{Ho}\,(\mathbf{W}(R/f)) & \longrightarrow & \mathrm{Ho}\,(\mathbf{W}(S)) \\
\downarrow\wr & & \downarrow\wr & & \downarrow\wr & & \downarrow\wr \\
\mathcal{K} & \longrightarrow & \mathbf{K}^b(R) & \longrightarrow & \mathbf{K}^b(R)/\mathcal{K} & \longrightarrow & \mathbf{K}^b(S),
\end{array}$$

where $\mathcal{K}$ denotes the full subcategory of $\mathbf{K}^b(R)$ consisting of all complexes X such that $X \otimes_R S = 0$. Next we use our assumption about the ring homomorphism f and apply Proposition 14.4 and Theorem 14.6. It follows that $\mathbf{W}(R/f) \to \mathbf{W}(S)$ induces a functor

$$\mathrm{Ho}\,(\mathbf{W}(R/f)) \longrightarrow \mathrm{Ho}\,(\mathbf{W}(S))$$

which is an equivalence up to direct factors. We conclude from the cofinality theorem [34, 1.10.1] that

$$K(R,f) \longrightarrow K(R) \longrightarrow K(S)$$

is a homotopy fibre sequence, up to failure of surjectivity of $K_0(R) \to K_0(S)$. □

15. Homological localizations of rings. Let R be an associative ring and let Φ be a class of maps between finitely generated projective R-modules. The *universal localization* of R with respect to Φ is the universal ring homomorphism $R \to S$ such that $\phi \otimes_R S$ is an isomorphism of S-modules for all ϕ in Φ; see [7, 32]. To construct S, one formally inverts all maps from Φ in the category $\mathcal{C} = \mathrm{proj}\,R$ of finitely generated projective R-modules and puts $S = \mathcal{C}[\Phi^{-1}](R,R)$. The following concept generalizes universal localizations.

Definition 15.1. We call a ring homomorphism $f\colon R \to S$ a *homological localization* with respect to a class Φ of complexes in $\mathbf{K}^b(R)$ if

(1) $X \otimes_R S = 0$ in $\mathbf{K}^b(S)$ for all X in Φ, and

(2) given any ring homomorphism $f'\colon R \to S'$ such that $X \otimes_R S' = 0$ in $\mathbf{K}^b(S')$ for all X in Φ, there exists a unique homomorphism $g\colon R \to R'$ such that $f' = g \circ f$.

Any universal localization is a homological localization. In fact, any map $\phi\colon P \to Q$ between finitely generated projective R-modules may be viewed as a

complex of length one by taking its mapping cone $\operatorname{Cone}\phi$. If $R \to S$ is a ring homomorphism, then $\phi \otimes_R S$ is an isomorphism if and only if $(\operatorname{Cone}\phi) \otimes_R S = 0$ in $\mathbf{K}^b(S)$.

The following example, which I learned from A. Neeman, shows that a homological localization need not exist.

Example 15.2. Let k be a field and $R = k[x, y]$. Let P be the complex

$$\cdots \longrightarrow 0 \longrightarrow R \xrightarrow{\begin{bmatrix} x \\ y \end{bmatrix}} R \amalg R \xrightarrow{\begin{bmatrix} y & -x \end{bmatrix}} R \longrightarrow 0 \longrightarrow \cdots$$

which is a projective resolution of $R/(x, y)$. Then we have $P \otimes_R R[x^{-1}] = 0$ and $P \otimes_R R[y^{-1}] = 0$. Now suppose there is a homological localization $R \to S$ with respect to P. Viewing $R[x^{-1}]$ and $R[y^{-1}]$ as subrings of $k(x, y)$, we have $R[x^{-1}] \cap R[y^{-1}] = R$. Therefore the identity $R \to R$ factors through $R \to S$. This is a contradiction and shows that the homological localization with respect to P cannot exist.

Next we consider an example of a homological epimorphism which is not a homological localization. Keller used this example in order to disprove the telescope conjecture for the derived category of a ring [18]. Let us explain the idea of Keller's example. He uses the following observation.

LEMMA 15.3. *Let R be a ring and $\mathfrak{a}$ be a two-sided ideal which is contained in the Jacobson radical of R. Then $X \otimes_R R/\mathfrak{a} = 0$ implies $X = 0$ for every X in $\mathbf{K}^b(R)$.*

Proof. Using induction on the length of the complex X, the assertion follows from Nakayama's lemma. □

In [37], Wodzicki has constructed an example of a ring R such that the Jacobson radical $\mathfrak{r}$ is non-zero and satisfies

$$\operatorname{Tor}_i^R(R/\mathfrak{r}, R/\mathfrak{r}) = 0 \quad \text{for all} \quad i \geq 1.$$

Thus $R \to R/\mathfrak{r}$ is a homological epimorphism which induces a cohomological quotient functor

$$F = - \otimes_R R/\mathfrak{r} \colon \mathbf{K}^b(R) \longrightarrow \mathbf{K}^b(R/\mathfrak{r})$$

satisfying $\operatorname{Ker} F = 0$ and $\operatorname{Ann} F \neq 0$. It follows that $R \to R/\mathfrak{r}$ is not a homological localization since $\operatorname{Ker} F = 0$. Moreover, Theorem 13.4 shows that the telescope conjecture does not hold for $\mathbf{D}(R)$.

One can find more examples along these lines, as R. Buchweitz kindly pointed out to me. Take any *Bézout domain* R, that is, an integral domain such that every

finitely generated ideal is principal. We have for every ideal $\mathfrak{a}$

$$\operatorname{Tor}_1^R(R/\mathfrak{a}, R/\mathfrak{a}) \cong \mathfrak{a}/\mathfrak{a}^2 \quad \text{and} \quad \operatorname{Tor}_i^R(R/\mathfrak{a}, -) = 0 \quad \text{for all} \quad i > 1.$$

Thus for any idempotent ideal $\mathfrak{a}$, the natural map $R \to R/\mathfrak{a}$ is a homological epimorphism. Specific examples arise from valuation domains, which are precisely the local Bézout domains.

Our interest in homological localizations is motivated by the following observation, which shows that the positive solution of the chain map lifting problem forces a ring homomorphism to be a homological localization.

PROPOSITION 15.4. *Let $f: R \to S$ be a ring homomorphism and suppose the chain map lifting problem has a positive solution. Then f is a homological localization.*

Proof. Denote by Φ the set of complexes X in $\mathbf{K}^b(R)$ such that $X \otimes_R S = 0$, and denote by $\mathcal{K}$ the corresponding full subcategory. We have seen in Proposition 14.4 that $- \otimes_R S: \mathbf{K}^b(R) \to \mathbf{K}^b(S)$ induces a fully faithful functor $\mathbf{K}^b(R)/\mathcal{K} \to \mathbf{K}^b(S)$. Now suppose that $f': R \to S'$ is a ring homomorphism satisfying $X \otimes_R S' = 0$ for all X in Φ. Then $- \otimes_R S'$ factors through the quotient functor $\mathbf{K}^b(R) \to \mathbf{K}^b(R)/\mathcal{K}$ via some functor $G: \mathbf{K}^b(R)/\mathcal{K} \to \mathbf{K}^b(S')$. Clearly, G induces a homomorphism $g: S \to S'$ such that $f' = g \circ f$. The uniqueness of g follows from the uniqueness of G. □

In [30], Neeman and Ranicki show that the chain complex lifting problem has a positive solution for every universal localization which is a homological epimorphism. We give an alternative proof of this result which is based on the criterion for lifting chain maps in Theorem 14.6.

THEOREM 15.5. *The chain map lifting problem has a positive solution for every homological epimorphism $R \to S$ which is a universal localization.*

We need some preparation for the proof of this result. Fix a homological epimorphism $R \to S$, and suppose it is the universal localization with respect to a class Φ of maps in the category $\mathcal{C} = \operatorname{proj} R$ of finitely generated projective R-modules. Thus we have $\operatorname{proj} S = \mathcal{C}[\Phi^{-1}]$. We denote by $\operatorname{Cone}\Phi = \{\operatorname{Cone}\phi \mid \phi \in \Phi\}$ the corresponding class of complexes of length one in $\mathbf{K}^b(R) = \mathbf{K}^b(\mathcal{C})$, and we write $\langle \operatorname{Cone}\Phi \rangle$ for the thick subcategory generated by $\operatorname{Cone}\Phi$. Finally, denote by $\mathcal{T}$ the idempotent completion of the quotient $\mathbf{K}^b(\mathcal{C})/\langle \operatorname{Cone}\Phi \rangle$, which one obtains for instance from Corollary 12.7 by embedding $\mathbf{K}^b(\mathcal{C})/\langle \operatorname{Cone}\Phi \rangle$ into the derived category $\mathbf{D}(R)$.

LEMMA 15.6. *The composite $Q: \mathbf{K}^b(\mathcal{C}) \to \mathbf{K}^b(\mathcal{C})/\langle \operatorname{Cone}\Phi \rangle \to \mathcal{T}$ has the following properties.*

(1) $\mathcal{T}(\Sigma^n(QX), QY) = 0$ *for all X, Y in $\mathcal{C}$ and $n > 0$.*

(2) *The functor* $Q|_{\mathcal{C}}\colon \mathcal{C} \to \mathcal{T}$ *factors through the localization* $\mathcal{C} \to \mathcal{C}[\Phi^{-1}]$.
(3) *The functor* $\mathcal{C}[\Phi^{-1}] \to \mathcal{T}$ *extends to an exact functor* $\mathbf{K}^b(\mathcal{C}[\Phi^{-1}]) \to \mathcal{T}$.
(4) *The functor* $- \otimes_R S\colon \mathbf{K}^b(\mathcal{C}) \to \mathbf{K}^b(\mathcal{C}[\Phi^{-1}])$ *factors through* $Q\colon \mathbf{K}^b(\mathcal{C}) \to \mathcal{T}$.

Proof. (1) The functor $Q\colon \mathbf{K}^b(\mathcal{C}) \to \mathcal{T}$ is a cohomological quotient functor. Thus we can apply Corollary 12.7 and obtain a fully faithful and exact "right adjoint" $Q'\colon \mathcal{T} \to \mathbf{D}(R)$ such that

$$\mathcal{T}(QA, B) \cong \mathbf{D}(R)(A, Q'B) \quad \text{for all } A \in \mathbf{K}^b(R) \text{ and } B \in \mathcal{T}.$$

To compute $\mathcal{T}(\Sigma^n(QX), QY)$ for X, Y in $\mathcal{C}$, it is sufficient to consider the case $X = R = Y$. We have

$$\mathcal{T}(\Sigma^n(QR), QR) \cong \mathbf{D}(R)(\Sigma^n R, (Q' \circ Q)R) \cong H^{-n}((Q' \circ Q)R).$$

Now we apply Corollary 3.31 from [30] which says that $\operatorname{Tor}_n^R(S, S) = 0$ for all $n > 0$ implies $H^{-n}((Q' \circ Q)R) = 0$ for all $n > 0$.

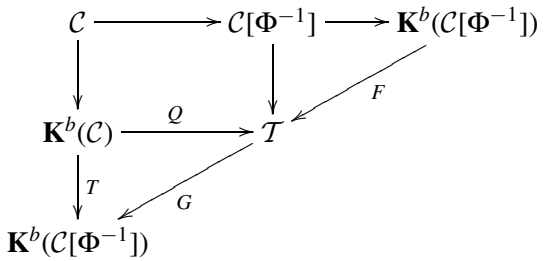
(2) The functor $\mathbf{K}^b(\mathcal{C}) \to \mathbf{K}^b(\mathcal{C})/\langle \operatorname{Cone}\Phi \rangle$ makes the maps in Φ invertible by sending the objects in $\operatorname{Cone}\Phi$ to zero. Therefore $\mathcal{C} \to \mathcal{T}$ factors through the localization $\mathcal{C} \to \mathcal{C}[\Phi^{-1}]$.

(3) This follows from the "universal property" of the homotopy category $\mathbf{K}^b(\mathcal{C}[\Phi^{-1}])$ which is the main result in [19]. More precisely, any additive functor $F\colon \mathcal{D} \to \bar{\mathcal{A}}$ from an additive category $\mathcal{D}$ to the stable category of a Frobenius category $\mathcal{A}$ extends to an exact functor $\mathbf{K}^b(\mathcal{D}) \to \bar{\mathcal{A}}$ provided that $\bar{\mathcal{A}}(\Sigma^n(FX), FY) = 0$ for all X, Y in $\mathcal{D}$ and $n > 0$. Note that we are using (1) and the fact that $\mathcal{T}$ can be embedded into the stable category of a Frobenius category.

(4) We have $\mathbf{K}^b(S) = \mathbf{K}^b(\mathcal{C}[\Phi^{-1}])$ and $X \otimes_R S = 0$ for all $X \in \operatorname{Cone}\Phi$ since $R \to S$ is the universal localization with respect to Φ. Thus $- \otimes_R S$ factors through the quotient functor $\mathbf{K}^b(\mathcal{C}) \to \mathbf{K}^b(\mathcal{C})/\langle \operatorname{Cone}\Phi \rangle$. Moreover, $- \otimes_R S$ factors through $\mathcal{T}$ since idempotents in $\mathbf{K}^b(\mathcal{C}[\Phi^{-1}])$ split. □

The following commutative diagram summarizes our findings from Lemma 15.6.

(15.1)



Proof of Theorem 15.5. We want to apply Theorem 14.6 and use the diagram (15.1). More precisely, we need to show that $\operatorname{Ann} T = \operatorname{Ann} Q$, because this implies condition (2) in Theorem 14.6 since $T = - \otimes_R S$ and $\operatorname{Ann} Q$ is generated by identity maps. We claim that $Q = F \circ T$. This follows from the "universal property" of the homotopy category $\mathbf{K}^b(\mathcal{C})$ since $Q|_{\mathcal{C}} = F \circ T|_{\mathcal{C}}$; see [16]. We obtain that $Q = F \circ G \circ Q$, and this implies $F \circ G \cong \mathrm{Id}_{\mathcal{T}}$ since both functors agree on $\mathbf{K}^b(\mathcal{C})/\langle \operatorname{Cone} \Phi \rangle$. Thus G is faithful and we conclude that $\operatorname{Ann} T = \operatorname{Ann} Q$. This completes the proof. □

Remark 15.7. I conjecture that Theorem 15.5 remains true if one replaces "universal localization" by "homological localization."

16. Almost derived categories. Almost rings and modules have been introduced by Gabber and Ramero [10]. Here, we analyze their formal properties and introduce their analogue for derived categories. Let us start with a piece of notation. Given a class Φ of maps in some additive category $\mathcal{C}$, we denote by

$$\Phi^{\perp} = \{X \in \mathcal{C} \mid \mathcal{C}(\phi, X) = 0 \text{ for all } \phi \in \Phi\}$$

the full subcategory of objects which are *annihilated* by Φ.

Throughout this section we fix an associative ring R. We view elements of R as maps $R \to R$. Thus $\mathfrak{a}^{\perp}$ for any ideal $\mathfrak{a}$ of R denotes the category of R modules which are annihilated by R.

The formal essence of an almost module category can be formulated as follows.

PROPOSITION 16.1. *Let $\mathcal{A}$ be a full subcategory of a module category* $\operatorname{Mod} R$. *Then the following are equivalent.*

(1) *$\mathcal{A}$ is a Serre subcategory, and the inclusion has a left and a right adjoint.*

(2) *There exists an idempotent ideal $\mathfrak{a}$ of R such that $\mathcal{A} = \mathfrak{a}^{\perp}$.*

In this case, the quotient category $\operatorname{Mod} R/\mathcal{A}$ *is the category of almost modules with respect to $\mathfrak{a}$, which is denoted by* $\operatorname{Mod}(R, \mathfrak{a})$.

Proof. The proof of the first part is straightforward; see for instance [1, Proposition 7.1]. The second part is just the definition of an almost module category from [10]. □

The following result is the analogue of Proposition 16.1 for triangulated categories.

THEOREM 16.2. *Let $\mathcal{R}$ be a full subcategory of a compactly generated triangulated category $\mathcal{S}$. Then the following are equivalent.*

(1) *$\mathcal{R}$ is a triangulated subcategory, and the inclusion has a left and a right adjoint.*

(2) *There exists an idempotent ideal $\mathfrak{I}$ of $\mathcal{S}_c$ satisfying $\Sigma\mathfrak{I} = \mathfrak{I}$, such that $\mathcal{R} = \mathfrak{I}^{\perp}$.*

In this case, the left adjoint of the quotient functor $\mathcal{S} \to \mathcal{S}/\mathcal{R}$ identifies $\mathcal{S}/\mathcal{R}$ with a smashing subcategory of $\mathcal{S}$. Moreover, every smashing subcategory of $\mathcal{S}$ arises in this way.

Proof. Let us denote by $F\colon \mathcal{R} \to \mathcal{S}$ the inclusion functor.

(1) $\Rightarrow$ (2): The left adjoint $E\colon \mathcal{S} \to \mathcal{R}$ of the inclusion $\mathcal{R} \to \mathcal{S}$ is a smashing localization functor since $E \circ F \cong \mathrm{Id}_{\mathcal{R}}$. It follows from Theorem 12.1 and its Corollary 12.6 that $\mathcal{I} = \mathcal{S}_c \cap \operatorname{Ann} E$ is an idempotent ideal satisfying $\Sigma\mathcal{I} = \mathcal{I}$ and $\mathcal{I}^{\perp} = \mathcal{R}$.

(2) $\Rightarrow$ (1): Lemma 12.4 implies that $\mathcal{I}^{\perp} = \mathcal{R}$ is a triangulated subcategory. Let us replace $\mathcal{I}$ by the ideal $\mathcal{J}$ of all maps in $\mathcal{S}_c$ annihilating $\mathcal{R}$. Thus $\mathcal{J}$ is a cohomological ideal satisfying $\Sigma\mathcal{J} = \mathcal{J}$ and $\mathcal{J}^{\perp} = \mathcal{R}$. The proof of Theorem 12.1 shows that $\mathcal{R}$ is perfectly cogenerated. Thus the inclusion $F\colon \mathcal{R} \to \mathcal{S}$ has a left adjoint by Corollary 10.2, since F preserves all products. Theorem 11.1 implies that $\mathcal{R}$ is compactly generated. Thus F has a right adjoint by the dual of Corollary 10.2, since F preserves all coproducts.

Now let us prove the second part. Suppose first that (1) – (2) hold. Then the left adjoint $E\colon \mathcal{S} \to \mathcal{R}$ of the inclusion $\mathcal{R} \to \mathcal{S}$ is a smashing localization functor. It follows that the left adjoint of the quotient functor $\mathcal{S} \to \mathcal{S}/\mathcal{R}$ identifies $\mathcal{S}/\mathcal{R}$ with $\operatorname{Ker} E$, which is by definition a smashing subcategory.

Finally, suppose that $\mathcal{T}$ is a smashing subcategory of $\mathcal{S}$. Let $\mathcal{I}$ be the idempotent ideal of all maps in $\mathcal{S}_c$ which factor through some object in $\mathcal{T}$. Then $\mathcal{R} = \mathcal{I}^{\perp}$ is a triangulated subcategory of $\mathcal{S}$, and the inclusion $\mathcal{R} \to \mathcal{S}$ has a left and a right adjoint. It follows that the left adjoint of the quotient functor $\mathcal{S} \to \mathcal{S}/\mathcal{R}$ identifies $\mathcal{S}/\mathcal{R}$ with $\mathcal{T}$. This finishes the proof. □

Let us complete the parallel between module categories and derived categories. Thus we consider the unbounded derived category $\mathbf{D}(R)$ of the module category $\operatorname{Mod} R$. Comparing the statements of Proposition 16.1 and Theorem 16.2, we see that the formal analogue of an almost module category is a triangulated category of the form

$$\mathbf{D}(R, \mathcal{I}) = \mathbf{D}(R)/(\mathcal{I}^{\perp})$$

for some idempotent ideal $\mathcal{I}$ of $\mathbf{K}^b(R)$ satisfying $\Sigma\mathcal{I} = \mathcal{I}$. We call such a category an *almost derived category*.

Next we show that the derived category of an almost module category is an almost derived category.

COROLLARY 16.3. *Let R be a ring and $\mathfrak{a}$ be an idempotent ideal such that $\mathfrak{a} \otimes_R \mathfrak{a}$ is flat as left R-module. Denote by $\mathfrak{A}$ the maps in $\mathbf{K}^b(R)$ which annihilate all suspensions of the mapping cone of the natural map $\mathfrak{a} \otimes_R \mathfrak{a} \to R$. Then we have*

$$\mathfrak{A}^2 = \mathfrak{A} \quad \textit{and} \quad \mathbf{D}(\operatorname{Mod}(R, \mathfrak{a})) = \mathbf{D}(R, \mathfrak{A}).$$

Proof. The quotient functor $F\colon \operatorname{Mod} R \to \operatorname{Mod}(R, \mathfrak{a})$ has a left adjoint E, and we have

$$(E \circ F)M = M \otimes_R (\mathfrak{a} \otimes_R \mathfrak{a});$$

see for instance [33, p. 200]. The extra assumption on $\mathfrak{a}$ implies that E is exact. Taking derived functors, we obtain an adjoint pair of exact functors

$$\mathbf{R}F\colon \mathbf{D}(\operatorname{Mod} R) \longrightarrow \mathbf{D}(\operatorname{Mod}(R, \mathfrak{a})) \quad \text{and} \quad \mathbf{L}E\colon \mathbf{D}(\operatorname{Mod}(R, \mathfrak{a})) \longrightarrow \mathbf{D}(\operatorname{Mod} R)$$

such that $\mathbf{R}F \circ \mathbf{L}E \cong \operatorname{Id}_{\mathbf{D}(\operatorname{Mod}(R,\mathfrak{a}))}$. It follows that $\mathbf{L}E$ identifies $\mathbf{D}(\operatorname{Mod}(R, \mathfrak{a}))$ with a smashing subcategory $\mathcal{R}$ of $\mathbf{D}(R)$. Now observe that the mapping cone $\operatorname{Cone}(\mathfrak{a} \otimes_R \mathfrak{a} \to R)$ generates $\mathbf{D}(R)/\mathcal{R}$. In fact, the canonical map

$$R \longrightarrow \operatorname{Cone}(\mathfrak{a} \otimes_R \mathfrak{a} \to R)$$

is an isomorphism in $\mathbf{D}(R)/\mathcal{R}$ since $(\mathbf{L}E \circ \mathbf{R}F)R = \mathfrak{a} \otimes_R \mathfrak{a}$. Therefore $\mathfrak{A}$ is the exact ideal corresponding to $\mathcal{R}$ which is idempotent by Corollary 12.6. Moreover, the localization functor $\mathbf{RF}$ identifies $\mathbf{D}(R)/(\mathfrak{A}^{\perp})$ with $\mathbf{D}(\operatorname{Mod}(R, \mathfrak{a}))$. □

We know that the derived category of a ring is compactly generated. This is no longer true for almost derived categories [18]. In deed, the telescope conjecture expresses the fact that all almost derived categories are compactly generated.

COROLLARY 16.4. *The telescope conjecture holds for the derived category* $\mathbf{D}(R)$ *of a ring* R *if and only if every almost derived category* $\mathbf{D}(R, \mathfrak{I})$ *is a compactly generated triangulated category.*

Appendix A. Epimorphisms of additive categories. An additive functor $F\colon \mathcal{C} \to \mathcal{D}$ between additive categories is called an *epimorphism of additive categories*, or simply an *epimorphism*, if $G \circ F = G' \circ F$ implies $G = G'$ for any pair $G, G'\colon \mathcal{D} \to \mathcal{E}$ of additive functors. In this section we characterize epimorphisms of additive categories in terms of functors between their module categories. This material is classical [25], but we need it in a form which slightly generalizes the usual approach.

LEMMA A.1. *Let* $F\colon \mathcal{C} \to \mathcal{D}$ *be an additive functor between additive categories. Suppose the restriction* $F_*\colon \operatorname{Mod} \mathcal{D} \to \operatorname{Mod} \mathcal{C}$ *is full and* F *is surjective on objects. Then* F *is an epimorphism.*

Proof. Let $G, G'\colon \mathcal{D} \to \mathcal{E}$ be a pair of additive functors satisfying $G \circ F = G' \circ F$. Clearly, G and G' coincide on objects since F is surjective on objects. Now choose a map $\alpha\colon X \to Y$ in $\mathcal{D}$. We need to show that $G\alpha = G'\alpha$. The functor G' induces a $\mathcal{C}$-linear map

$$\gamma\colon F_*\mathcal{D}(-, Y) \longrightarrow (F_* \circ G_*)\mathcal{E}(-, GY),$$

which is defined by

$$\gamma_C\colon \mathcal{D}(FC, Y) \longrightarrow \mathcal{E}(G(FC), GY), \quad \phi \mapsto G'\phi$$

for each C in $\mathcal{C}$. The fact that F_* is full implies that $\gamma = F_*\delta$ for some $\mathcal{D}$-linear map $\delta\colon \mathcal{D}(-, Y) \to G_*\mathcal{E}(-, GY)$. In particular, $\delta_X = \gamma_C$ for some C in $\mathcal{C}$ satisfying $FC = X$. Thus we obtain the following commutative diagram

$$\begin{array}{ccc} \mathcal{D}(Y,Y) & \xrightarrow{\delta_Y} & \mathcal{E}(GY,GY) \\ \downarrow{\scriptstyle \mathcal{D}(\alpha,Y)} & & \downarrow{\scriptstyle \mathcal{E}(G\alpha,GY)} \\ \mathcal{D}(X,Y) & \xrightarrow{\delta_X} & \mathcal{E}(GX,GY) \end{array}$$

which shows $G\alpha = G'\alpha$ if we apply it to id_Y. We conclude that $G = G'$. □

Lemma A.2. *Let $F\colon \mathcal{C} \to \mathcal{D}$ be an additive functor between additive categories. Suppose F is an epimorphism and bijective on objects. Then the restriction $F_*\colon \operatorname{Mod}\mathcal{D} \to \operatorname{Mod}\mathcal{C}$ is fully faithful.*

Proof. Let M, N be a pair of $\mathcal{D}$-modules. We need to show that the canonical map

$$(F_*)_{M,N}\colon \operatorname{Hom}_{\mathcal{D}}(M, N) \longrightarrow \operatorname{Hom}_{\mathcal{C}}(F_*M, F_*N)$$

is bijective. Given a family $\phi = (\phi_X)_{X\in\mathcal{D}}$ of maps $\phi_X\colon MX \to NX$, we define a $\mathcal{D}$-module H_ϕ by

$$H_\phi X = MX \amalg NX \quad \text{and} \quad H_\phi\alpha = \begin{bmatrix} M\alpha & 0 \\ N\alpha\circ\phi_Y - \phi_X\circ M\alpha & N\alpha \end{bmatrix}$$

for each object X and each map $\alpha\colon X \to Y$ in $\mathcal{D}$. Note that

$$(\phi_X)_{X\in\mathcal{D}}\colon M \longrightarrow N$$

is $\mathcal{D}$-linear if and only if $H_\phi = M \amalg N$.

To prove that $(F_*)_{M,N}$ is surjective, fix a $\mathcal{C}$-linear map

$$\psi = (\psi_X)_{X\in\mathcal{C}}\colon F_*M \longrightarrow F_*N.$$

For each X in $\mathcal{D}$ put $\phi_X = \psi_{F^{-1}X}$. We have $H_\phi \circ F = (M \amalg N)\circ F$ since ψ is $\mathcal{C}$-linear. Thus ϕ is $\mathcal{D}$-linear because $H_\phi \circ F = (M \amalg N)\circ F$ implies $H_\phi = M \amalg N$. We have $F_*\phi = \psi$ and conclude that the map $(F_*)_{M,N}$ is surjective.

To prove that $(F_*)_{M,N}$ is injective, choose a non-zero map $\phi\colon M \to N$. Thus $\operatorname{Im}\phi \neq 0$. We have $\operatorname{Im}(F_*\phi) = F_*(\operatorname{Im}\phi) \neq 0$ and therefore $F_*\phi \neq 0$. It follows that $(F_*)_{M,N}$ is injective. □

LEMMA A.3. *Let $F\colon \mathcal{C} \to \mathcal{D}$ be an additive functor between additive categories. If F is an epimorphism, then F is surjective on objects.*

Proof. Suppose there is an object D in $\mathcal{D}$ which does not belong to the image of F. We construct a new additive category $\mathcal{E}$ which contains $\mathcal{D}$ as a full subcategory and has one additional object, denoted by D'. Let $\mathcal{E}(X, D') = \mathcal{D}(X, D)$ and $\mathcal{E}(D', X) = \mathcal{D}(D, X)$ for all X in $\mathcal{D}$, and let $\mathcal{E}(D', D') = \mathcal{D}(D, D)$. Now define $G\colon \mathcal{D} \to \mathcal{E}$ to be the inclusion, and define $G'\colon \mathcal{D} \to \mathcal{E}$ by $G'X = GX$ for all X in $\mathcal{D}$, except for $X = D$, where we put $G'D = D'$. Clearly, $G \circ F = G' \circ F$ but $G \neq G'$. Thus an epimorphism is surjective on objects. □

LEMMA A.4. *Let $F\colon \mathcal{C} \to \mathcal{D}$ be an additive functor between small additive categories. If the restriction $F_*\colon \operatorname{Mod}\mathcal{D} \to \operatorname{Mod}\mathcal{C}$ is faithful, then every object in $\mathcal{D}$ is a direct factor of some object in the image of F.*

Proof. The restriction F_* has a left adjoint $F^*\colon \operatorname{Mod}\mathcal{C} \to \operatorname{Mod}\mathcal{D}$. The assumption on F_* implies that for each $\mathcal{D}$-module M the natural map $(F^* \circ F_*)M \to M$ is an epimorphism. Now fix an object Y in $\mathcal{D}$. Every module is a quotient of a coproduct of representable functors. Thus we have an epimorphism

$$\coprod_{i\in\Lambda} \mathcal{C}(-, X_i) \longrightarrow F_*\mathcal{D}(-, Y),$$

and applying F^* induces an epimorphism

$$\coprod_{i\in\Lambda} \mathcal{D}(-, FX_i) \longrightarrow (F^* \circ F_*)\mathcal{D}(-, Y) \longrightarrow \mathcal{D}(-, Y).$$

Using Yoneda's lemma, we see that Y is a direct factor of $F(\coprod_{i\in\Gamma} X_i)$ for some finite subset $\Gamma \subseteq \Lambda$. □

PROPOSITION A.5. *Let $F\colon \mathcal{C} \to \mathcal{D}$ be an additive functor between small additive categories. Then $F_*\colon \operatorname{Mod}\mathcal{D} \to \operatorname{Mod}\mathcal{C}$ is fully faithful if and only if there is a factorization $F = F_2 \circ F_1$ such that*

(1) *F_1 is an epimorphism and bijective on objects, and*

(2) *F_2 is fully faithful and every object in $\mathcal{D}$ is a direct factor of some object in the image of F_2.*

Proof. Suppose first that F_* is fully faithful. We define a factorization

$$\mathcal{C} \xrightarrow{F_1} \mathcal{D}' \xrightarrow{F_2} \mathcal{D}$$

as follows. The objects of $\mathcal{D}'$ are those of $\mathcal{C}$ and F_1 is the identity on objects. Let

$$\mathcal{D}'(X, Y) = \mathcal{D}(FX, FY)$$

for all X, Y in $\mathcal{C}$, and let $F_1\alpha = F\alpha$ for each map α in $\mathcal{C}$. The functor F_2 equals F on objects and is the identity on maps. It follows that F_2 is fully faithful and surjective up to direct factors on objects, by Lemma A.4. Thus $(F_2)_*$ is fully faithful, and this implies that $(F_1)_*$ is fully faithful, since $F_* = (F_2)_* \circ (F_1)_*$. We conclude from Lemma A.1 that F_1 is an epimorphism.

Now suppose F admits a factorization $F = F_2 \circ F_1$ satisfying (1) and (2). Then $(F_1)_*$ is fully faithful by Lemma A.2, and $(F_2)_*$ is automatically fully faithful. Thus F_* is fully faithful. □

The property of being an epimorphism is invariant under enlarging the universe.

LEMMA A.6. *Let $\mathfrak{U}$ and $\mathfrak{V}$ be universes in the sense of Grothendieck [11, I.1], and suppose $\mathfrak{U} \subseteq \mathfrak{V}$. If $F\colon \mathcal{C} \to \mathcal{D}$ is an epimorphism of additive $\mathfrak{U}$-categories, then F is an epimorphism of additive $\mathfrak{V}$-categories.*

Proof. Let $G, G'\colon \mathcal{D} \to \mathcal{E}$ be a pair of additive functors into a $\mathfrak{V}$-category $\mathcal{E}$ satisfying $G \circ F = G' \circ F$. We denote by $\mathcal{F}$ the smallest additive subcategory of $\mathcal{E}$ containing the image of G and G'. Observe that $\mathcal{F}$ is a $\mathfrak{U}$-category since $\mathcal{D}$ is a $\mathfrak{U}$-category. Thus the restrictions $\mathcal{D} \to \mathcal{F}$ of G and G' agree by our assumption on F. It follows that $G = G'$. □

Appendix B. The abelianization of a triangulated category. Let $\mathcal{C}$ be a triangulated category. In this section we discuss some properties of the *abelianization* $\operatorname{mod}\mathcal{C}$ of $\mathcal{C}$. Most of this material can be found in work of Freyd [9] and Heller [14] about the formal properties of the stable homotopy category.

LEMMA B.1. *Let $\mathcal{C}$ be a triangulated category. Then the category* $\operatorname{mod}\mathcal{C}$ *is an abelian* Frobenius category*, that is, there are enough projectives and enough injectives, and both coincide.*

Proof. The representable functors are projective objects in $\operatorname{mod}\mathcal{C}$ by Yoneda's lemma. Thus $\operatorname{mod}\mathcal{C}$ has enough projectives. Using the fact that the Yoneda functors $\mathcal{C} \to \operatorname{mod}\mathcal{C}$ and $\mathcal{C}^{\mathrm{op}} \to \operatorname{mod}(\mathcal{C}^{\mathrm{op}})$ are universal cohomological functors, we obtain an equivalence $(\operatorname{mod}\mathcal{C})^{\mathrm{op}} \to \operatorname{mod}(\mathcal{C}^{\mathrm{op}})$ which sends $\mathcal{C}(-, X)$ to $\mathcal{C}(X, -)$ for all X in $\mathcal{C}$. Thus the representable functors are injective objects, and $\operatorname{mod}\mathcal{C}$ has enough injectives. □

The triangulated structure of $\mathcal{C}$ induces some additional structure on $\operatorname{mod}\mathcal{C}$. This involves the equivalence $\Sigma^\star\colon \operatorname{mod}\mathcal{C} \to \operatorname{mod}\mathcal{C}$ which extends $\Sigma\colon \mathcal{C} \to \mathcal{C}$. By abuse of notation, we identify $\Sigma^\star = \Sigma$. Using this internal grading, the category $\operatorname{mod}\mathcal{C}$ is $(3, -1)$-periodic [9]. Thus we obtain a canonical extension κ_M in $\operatorname{Ext}^3_{\mathcal{C}}(\Sigma M, M)$ for every module M in $\operatorname{mod}\mathcal{C}$. Under some additional assumptions, this extension is induced by a Hochschild cocycle of degree $(3, -1)$; it plays a crucial role in [4].

PROPOSITION B.2. *Let $\mathcal{C}$ be a triangulated category.*
(1) *Given a pair M, N of objects in* $\operatorname{mod}\mathcal{C}$*, there is a natural map*

$$\kappa_{M,N}\colon \operatorname{Hom}_{\mathcal{C}}(M,N) \longrightarrow \operatorname{Ext}^3_{\mathcal{C}}(\Sigma M, N)$$

and we write $\kappa_N = \kappa_{N,N}(\operatorname{id}_N)$.
(2) *Let*

$$\Delta\colon X \xrightarrow{\alpha} Y \xrightarrow{\beta} Z \xrightarrow{\gamma} \Sigma X$$

be a sequence of maps in $\mathcal{C}$ and let $N = \operatorname{Ker}\mathcal{C}(-,\alpha)$. Then Δ is an exact triangle if and only if the map γ induces a map $\mathcal{C}(-,Z) \longrightarrow \Sigma N$ such that the sequence

$$0 \longrightarrow N \longrightarrow \mathcal{C}(-,X) \xrightarrow{\mathcal{C}(-,\alpha)} \mathcal{C}(-,Y) \xrightarrow{\mathcal{C}(-,\beta)} \mathcal{C}(-,Z) \longrightarrow \Sigma N \longrightarrow 0$$

is exact in $\operatorname{mod}\mathcal{C}$ *and represents κ_N.*

Proof. (1) Let $M = \operatorname{Coker}\mathcal{C}(-,\beta)$ be a an object in $\operatorname{mod}\mathcal{C}$ and complete $\beta\colon Y \to Z$ to an exact triangle $X \to Y \to Z \to \Sigma X$ to obtain a projective resolution

$$\cdots \to \mathcal{C}(-,Y) \to \mathcal{C}(-,Z) \to \mathcal{C}(-,\Sigma X) \to \mathcal{C}(-,\Sigma Y) \to \mathcal{C}(-,\Sigma Z) \to \Sigma M \to 0$$

of ΣM. The map $\kappa_{M,N}$ takes by definition a map $\phi\colon M \to N$ to the element in $\operatorname{Ext}^3_{\mathcal{C}}(\Sigma M, N)$ which is represented the composition of ϕ with the projection $\mathcal{C}(-,Z) \to M$.
(2) Fix a sequence

$$\Delta\colon X \xrightarrow{\alpha} Y \xrightarrow{\beta} Z \xrightarrow{\gamma} \Sigma X$$

in $\mathcal{C}$ and let $N = \operatorname{Ker}\mathcal{C}(-,\alpha)$. Suppose first that Δ is an exact triangle. The definition of κ_N implies that the induced sequence

$$\varepsilon_\Delta\colon 0 \longrightarrow N \longrightarrow \mathcal{C}(-,X) \xrightarrow{\mathcal{C}(-,\alpha)} \mathcal{C}(-,Y) \xrightarrow{\mathcal{C}(-,\beta)} \mathcal{C}(-,Z) \longrightarrow \Sigma N \longrightarrow 0$$

is exact in $\operatorname{mod}\mathcal{C}$ and represents κ_N. Conversely, suppose that ε_Δ is exact and represents κ_N. Complete α to an exact triangle

$$\Delta'\colon X \xrightarrow{\alpha} Y \xrightarrow{\beta'} Z' \xrightarrow{\gamma'} \Sigma X$$

in $\mathcal{C}$. We use dimension shift and replace both sequences ε_Δ and $\varepsilon_{\Delta'}$ by short exact sequences

$$0 \to \Omega^{-2}N \to \mathcal{C}(-,Z) \to \Sigma N \to 0 \quad \text{and} \quad 0 \to \Omega^{-2}N \to \mathcal{C}(-,Z') \to \Sigma N \to 0.$$

These represent the same element in $\operatorname{Ext}^1_{\mathcal{C}}(\Sigma N, \Omega^{-2}N)$, and we obtain therefore an isomorphism $\phi\colon Z \to Z'$ which induces an isomorphism of triangles $\Delta \to \Delta'$. Thus Δ is an exact triangle. □

Let us explain a more conceptual way to understand the natural map $\kappa_{M,N}$. To this end denote by $\underline{\operatorname{mod}}\,\mathcal{C}$ the *stable category* of $\operatorname{mod}\mathcal{C}$, that is, the objects are those of $\operatorname{mod}\mathcal{C}$ and

$$\underline{\operatorname{Hom}}_{\mathcal{C}}(M,N) = \operatorname{Hom}_{\mathcal{C}}(M,N)/\mathfrak{P}(M,N)$$

where $\mathfrak{P}$ denotes the ideal of all maps in $\operatorname{mod}\mathcal{C}$ which factor through some projective object. Taking syzygies in $\operatorname{mod}\mathcal{C}$ induces an equivalence

$$\Omega\colon \underline{\operatorname{mod}}\,\mathcal{C} \longrightarrow \underline{\operatorname{mod}}\,\mathcal{C}$$

since $\operatorname{mod}\mathcal{C}$ is a Frobenius category. Moreover,

$$\underline{\operatorname{Hom}}_{\mathcal{C}}(\Omega^n M, N) \cong \operatorname{Ext}^n_{\mathcal{C}}(M,N) \cong \underline{\operatorname{Hom}}_{\mathcal{C}}(M, \Omega^{-n}N) \quad \text{for all } M, N \text{ and } n > 0.$$

The map $\kappa_{M,N}$ induces a natural isomorphism

$$\underline{\operatorname{Hom}}_{\mathcal{C}}(M,N) \longrightarrow \operatorname{Ext}^3_{\mathcal{C}}(\Sigma M, N),$$

and composing this with the natural isomorphism

$$\operatorname{Ext}^3_{\mathcal{C}}(\Sigma M, N) \longrightarrow \underline{\operatorname{Hom}}_{\mathcal{C}}(\Sigma M, \Omega^{-3}N)$$

induces a natural isomorphism between

$$\bar{\Sigma}\colon \underline{\operatorname{mod}}\,\mathcal{C} \longrightarrow \underline{\operatorname{mod}}\,\mathcal{C} \quad \text{and} \quad \Omega^{-3}\colon \underline{\operatorname{mod}}\,\mathcal{C} \longrightarrow \underline{\operatorname{mod}}\,\mathcal{C}.$$

Note that the natural map $\kappa_{M,N}$ can be reconstructed from the natural isomorphism $\bar{\Sigma} \cong \Omega^{-3}$.

Added in Proof. The paper [30] of Neeman and Ranicki which inspired the present work has been substantially revised and is now published with a slightly different title: Noncommutative localisation in algebraic K-theory. I, *Geom. Topol.* **8** (2004), 1385–1425.

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