LOCALIZATION THEORY FOR TRIANGULATED CATEGORIES

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CONTENTS

1. INTRODUCTION

These notes provide an introduction to the theory of localization for triangulated categories. Localization is a machinery to formally invert morphisms in a category. We explain this formalism in some detail and we show how it is applied to triangulated categories.

There are basically two ways to approach the localization theory for triangulated categories and both are closely related to each other. To explain this, let us fix a triangulated category $\mathcal{T}$. The first approach is *Verdier localization*. For this one chooses a full triangulated subcategory $\mathcal{S}$ of $\mathcal{T}$ and constructs a universal exact functor $\mathcal{T} \to \mathcal{T}/\mathcal{S}$ which annihilates the objects belonging to $\mathcal{S}$. In fact, the quotient category $\mathcal{T}/\mathcal{S}$ is obtained by formally inverting all morphisms σ in $\mathcal{T}$ such that the cone of σ belongs to $\mathcal{S}$.

On the other hand, there is *Bousfield localization*. In this case one considers an exact functor $L \colon \mathcal{T} \to \mathcal{T}$ together with a natural morphism $\eta X \colon X \to LX$ for all X in $\mathcal{T}$ such that $L(\eta X) = \eta(LX)$ is invertible. There are two full triangulated subcategories arising from such a localization functor L. We have the subcategory $\operatorname{Ker} L$ formed by all L-acyclic objects, and we have the essential image $\operatorname{Im} L$ which coincides with the subcategory formed by all L-local objects. Note that L, $\operatorname{Ker} L$, and $\operatorname{Im} L$ determine each other. Moreover, L induces an equivalence $\mathcal{T}/\operatorname{Ker} L \xrightarrow{\sim} \operatorname{Im} L$. Thus a Bousfield localization functor $\mathcal{T} \to \mathcal{T}$ is nothing but the composite of a Verdier quotient functor $\mathcal{T} \to \mathcal{T}/\mathcal{S}$ with a fully faithful right adjoint $\mathcal{T}/\mathcal{S} \to \mathcal{T}$.

Having introduced these basic objects, there are a number of immediate questions. For example, given a triangulated subcategory $\mathcal{S}$ of $\mathcal{T}$, can we find a localization functor

$L\colon \mathcal{T} \to \mathcal{T}$ satisfying $\operatorname{Ker} L = \mathcal{S}$ or $\operatorname{Im} L = \mathcal{S}$? On the other hand, if we start with L, which properties of $\operatorname{Ker} L$ and $\operatorname{Im} L$ are inherited from $\mathcal{T}$? It turns out that well generated triangulated categories in the sense of Neeman [33] provide an excellent setting for studying these questions.

Let us discuss briefly the relevance of well generated categories. The concept generalizes that of a compactly generated triangulated category. For example, the derived category of unbounded chain complexes of modules over some fixed ring is compactly generated. Also, the stable homotopy category of CW-spectra is compactly generated. Given any localization functor L on a compactly generated triangulated category, it is rare that $\operatorname{Ker} L$ or $\operatorname{Im} L$ are compactly generated. However, in all known examples $\operatorname{Ker} L$ and $\operatorname{Im} L$ are well generated. The following theorem provides a conceptual explanation; it combines several results from Section 7.

Theorem. *Let $\mathcal{T}$ be a well generated triangulated category and $\mathcal{S}$ a full triangulated subcategory which is closed under small coproducts. Then the following are equivalent.*

(1) *The triangulated category $\mathcal{S}$ is well generated.*
(2) *The triangulated category $\mathcal{T}/\mathcal{S}$ is well generated.*
(3) *There exists a cohomological functor $H\colon \mathcal{T} \to \mathcal{A}$ into a locally presentable abelian category such that H preserves small coproducts and $\mathcal{S} = \operatorname{Ker} H$.*
(4) *There exists a small set $\mathcal{S}_0$ of objects in $\mathcal{S}$ such that $\mathcal{S}$ admits no proper full triangulated subcategory closed under small coproducts and containing $\mathcal{S}_0$.*

Moreover, in this case there exists a localization functor $L\colon \mathcal{T} \to \mathcal{T}$ such that $\operatorname{Ker} L = \mathcal{S}$.

Note that every abelian Grothendieck category is locally presentable; in particular every module category is locally presentable.

Our approach for studying localization functors on well generated triangulated categories is based on the interplay between triangulated and abelian structure. A well known construction due to Freyd provides for any triangulated category $\mathcal{T}$ an abelian category $A(\mathcal{T})$ together with a universal cohomological functor $\mathcal{T} \to A(\mathcal{T})$. However, the category $A(\mathcal{T})$ is usually far too big and therefore not manageable. If $\mathcal{T}$ is well generated, then we have a canonical filtration

$$A(\mathcal{T}) = \bigcup_{\alpha} A_\alpha(\mathcal{T})$$

indexed by all regular cardinals, such that for each α the category $A_\alpha(\mathcal{T})$ is abelian and locally α-presentable in the sense of Gabriel and Ulmer [17]. Moreover, each inclusion $A_\alpha(\mathcal{T}) \to A(\mathcal{T})$ admits an exact right adjoint and the composite

$$H_\alpha\colon \mathcal{T} \longrightarrow A(\mathcal{T}) \longrightarrow A_\alpha(\mathcal{T})$$

is the universal cohomological functor into a locally α-presentable abelian category. Thus we may think of the functors $\mathcal{T} \to A_\alpha(\mathcal{T})$ as successive approximations of $\mathcal{T}$ by locally presentable abelian categories. For instance, there exists for each object X in $\mathcal{T}$ some cardinal $\alpha(X)$ such that the induced map $\mathcal{T}(X, Y) \to A_\beta(\mathcal{T})(H_\beta X, H_\beta Y)$ is bijective for all Y in $\mathcal{T}$ and all $\beta \geq \alpha(X)$.

These notes are organized as follows. We start off with an introduction to categories of fractions and localization functors for arbitrary categories. Then we apply this to triangulated categories. First we treat arbitrary triangulated categories and explain the localization in the sense of Verdier and Bousfield. Then we pass to compactly

and well generated triangulated categories where Brown representability provides an indispensable tool for constructing localization functors. Module categories and their derived categories are used to illustrate most of the concepts; see [12] for complementary material from topology. The results on well generated categories are based on facts from the theory of locally presentable categories; we have collected these in a separate appendix.

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2. Categories of fractions and localization functors

2.1. **Categories.** Throughout we fix a universe of sets in the sense of Grothendieck [19]. The members of this universe will be called *small sets*.

Let $\mathcal{C}$ be a category. We denote by $\mathrm{Ob}\,\mathcal{C}$ the set of objects and by $\mathrm{Mor}\,\mathcal{C}$ the set of morphisms in $\mathcal{C}$. Given objects X, Y in $\mathcal{C}$, the set of morphisms $X \to Y$ will be denoted by $\mathcal{C}(X, Y)$. The identity morphism of an object X is denoted by $\mathrm{id}_{\mathcal{C}} X$ or just $\mathrm{id}\, X$. If not stated otherwise, we always assume that the morphisms between two fixed objects of a category form a small set.

A category $\mathcal{C}$ is called *small* if the isomorphism classes of objects in $\mathcal{C}$ form a small set. In that case we define the *cardinality* of $\mathcal{C}$ as $\mathrm{card}\,\mathcal{C} = \sum_{X,Y \in \mathcal{C}_0} \mathrm{card}\,\mathcal{C}(X, Y)$ where $\mathcal{C}_0$ denotes a representative set of objects of $\mathcal{C}$, meeting each isomorphism class exactly once.

Let $F \colon \mathcal{I} \to \mathcal{C}$ be a functor from a small (indexing) category $\mathcal{I}$ to a category $\mathcal{C}$. Then we write $\varinjlim_{i \in \mathcal{I}} Fi$ for the colimit of F, provided it exists. Given a cardinal α, the colimit of F is called α-*colimit* if $\mathrm{card}\,\mathcal{I} < \alpha$. An example of a colimit is the coproduct $\coprod_{i \in I} X_i$ of a family $(X_i)_{i \in I}$ of objects in $\mathcal{C}$ where the indexing set I is always assumed to be small. We say that a category $\mathcal{C}$ *admits small coproducts* if for every family $(X_i)_{i \in I}$ of objects in $\mathcal{C}$ which is indexed by a small set I the coproduct $\coprod_{i \in I} X_i$ exists in $\mathcal{C}$. Analogous terminology is used for limits and products.

2.2. **Categories of fractions.** Let $F \colon \mathcal{C} \to \mathcal{D}$ be a functor. We say that F makes a morphism σ of $\mathcal{C}$ invertible if $F\sigma$ is invertible. The set of all those morphisms which F inverts is denoted by $\Sigma(F)$.

Given a category $\mathcal{C}$ and any set Σ of morphisms of $\mathcal{C}$, we consider the *category of fractions* $\mathcal{C}[\Sigma^{-1}]$ together with a canonical *quotient functor*

$$Q_\Sigma \colon \mathcal{C} \longrightarrow \mathcal{C}[\Sigma^{-1}]$$

having the following properties.

(Q1) Q_Σ makes the morphisms in Σ invertible.

(Q2) If a functor $F\colon \mathcal{C} \to \mathcal{D}$ makes the morphisms in Σ invertible, then there is a unique functor $\bar{F}\colon \mathcal{C}[\Sigma^{-1}] \to \mathcal{D}$ such that $F = \bar{F} \circ Q_\Sigma$.

Note that $\mathcal{C}[\Sigma^{-1}]$ and Q_Σ are essentially unique if they exists. Now let us sketch the construction of $\mathcal{C}[\Sigma^{-1}]$ and Q_Σ. At this stage, we ignore set-theoretic issues, that is, the morphisms between two objects of $\mathcal{C}[\Sigma^{-1}]$ need not to form a small set. We put $\mathrm{Ob}\,\mathcal{C}[\Sigma^{-1}] = \mathrm{Ob}\,\mathcal{C}$. To define the morphisms of $\mathcal{C}[\Sigma^{-1}]$, consider the quiver (i.e. oriented graph) with set of vertices $\mathrm{Ob}\,\mathcal{C}$ and with set of arrows the disjoint union $(\mathrm{Mor}\,\mathcal{C}) \amalg \Sigma^{-1}$, where $\Sigma^{-1} = \{\sigma^{-1}\colon Y \to X \mid \Sigma \ni \sigma\colon X \to Y\}$. Let $\mathcal{P}$ be the set of paths in this quiver (i.e. finite sequences of composable arrows), together with the obvious composition which is the concatenation operation and denoted by $\circ_\mathcal{P}$. We define $\mathrm{Mor}\,\mathcal{C}[\Sigma^{-1}]$ as the quotient of $\mathcal{P}$ modulo the following relations:

(1) $\beta \circ_\mathcal{P} \alpha = \beta \circ \alpha$ for all composable morphisms $\alpha, \beta \in \mathrm{Mor}\,\mathcal{C}$.
(2) $\mathrm{id}_\mathcal{P} X = \mathrm{id}_\mathcal{C} X$ for all $X \in \mathrm{Ob}\,\mathcal{C}$.
(3) $\sigma^{-1} \circ_\mathcal{P} \sigma = \mathrm{id}_\mathcal{P} X$ and $\sigma \circ_\mathcal{P} \sigma^{-1} = \mathrm{id}_\mathcal{P} Y$ for all $\sigma\colon X \to Y$ in Σ.

The composition in $\mathcal{P}$ induces the composition of morphisms in $\mathcal{C}[\Sigma^{-1}]$. The functor Q_Σ is the identity on objects and on $\mathrm{Mor}\,\mathcal{C}$ the composite

$$\mathrm{Mor}\,\mathcal{C} \xrightarrow{\mathrm{inc}} (\mathrm{Mor}\,\mathcal{C}) \amalg \Sigma^{-1} \xrightarrow{\mathrm{inc}} \mathcal{P} \xrightarrow{\mathrm{can}} \mathrm{Mor}\,\mathcal{C}[\Sigma^{-1}].$$

Having completed the construction of the category of fractions $\mathcal{C}[\Sigma^{-1}]$, let us mention that it is also called *quotient category* or *localization* of $\mathcal{C}$ with respect to Σ.

2.3. **Adjoint functors.** Let $F\colon \mathcal{C} \to \mathcal{D}$ and $G\colon \mathcal{D} \to \mathcal{C}$ be a pair of functors and assume that F is left adjoint to G. We denote by

$$\theta\colon F \circ G \to \mathrm{Id}\,\mathcal{D} \quad \text{and} \quad \eta\colon \mathrm{Id}\,\mathcal{C} \to G \circ F$$

the corresponding adjunction morphisms. Let $\Sigma = \Sigma(F)$ denote the set of morphisms σ of $\mathcal{C}$ such that $F\sigma$ is invertible. Recall that a morphism $\mu\colon F \to F'$ between two functors is invertible if for each object X the morphism $\mu X\colon FX \to F'X$ is invertible.

Proposition 2.3.1. *The following statements are equivalent.*

(1) *The functor G is fully faithful.*
(2) *The morphism $\theta\colon F \circ G \to \mathrm{Id}\,\mathcal{D}$ is invertible.*
(3) *The functor $\bar{F}\colon \mathcal{C}[\Sigma^{-1}] \to \mathcal{D}$ satisfying $F = \bar{F} \circ Q_\Sigma$ is an equivalence.*

Proof. See [18, I.1.3]. $\square$

2.4. **Localization functors.** A functor $L\colon \mathcal{C} \to \mathcal{C}$ is called a *localization functor* if there exists a morphism $\eta\colon \mathrm{Id}\,\mathcal{C} \to L$ such that $L\eta\colon L \to L^2$ is invertible and $L\eta = \eta L$. Note that we only require the existence of η; the actual morphism is not part of the definition of L. However, we will see that η is determined by L, up to a unique isomorphism $L \to L$.

Proposition 2.4.1. *Let $L\colon \mathcal{C} \to \mathcal{C}$ be a functor and $\eta\colon \mathrm{Id}\,\mathcal{C} \to L$ be a morphism. Then the following are equivalent.*

(1) $L\eta\colon L \to L^2$ *is invertible and $L\eta = \eta L$.*
(2) *There exists a functor $F\colon \mathcal{C} \to \mathcal{D}$ and a fully faithful right adjoint $G\colon \mathcal{D} \to \mathcal{C}$ such that $L = G \circ F$ and $\eta\colon \mathrm{Id}\,\mathcal{C} \to G \circ F$ is the adjunction morphism.*

Proof. $(1) \Rightarrow (2)$: Let $\mathcal{D}$ denote the full subcategory of $\mathcal{C}$ formed by all objects X such that ηX is invertible. For each $X \in \mathcal{D}$, let $\theta X \colon LX \to X$ be the inverse of ηX. Define $F \colon \mathcal{C} \to \mathcal{D}$ by $FX = LX$ and let $G \colon \mathcal{D} \to \mathcal{C}$ be the inclusion. We claim that F and G form an adjoint pair. In fact, it is straightforward to check that the maps

$$\mathcal{D}(FX, Y) \longrightarrow \mathcal{C}(X, GY), \quad \alpha \mapsto G\alpha \circ \eta X,$$

and

$$\mathcal{C}(X, GY) \longrightarrow \mathcal{D}(FX, Y), \quad \beta \mapsto \theta Y \circ F\beta,$$
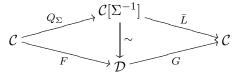
are mutually inverse bijections.

$(2) \Rightarrow (1)$: Let $\theta \colon FG \to \operatorname{Id}\mathcal{D}$ denote the second adjunction morphism. Then the composites

$$F \xrightarrow{F\eta} FGF \xrightarrow{\theta F} F \quad \text{and} \quad G \xrightarrow{\eta G} GFG \xrightarrow{G\theta} G$$

are identity morphisms; see [27, IV.1]. We know from Proposition 2.3.1 that θ is invertible because G is fully faithful. Therefore $L\eta = GF\eta$ is invertible. Moreover, we have

$$L\eta = GF\eta = (G\theta F)^{-1} = \eta GF = \eta L. \qquad \square$$

Corollary 2.4.2. *A functor $L \colon \mathcal{C} \to \mathcal{C}$ is a localization functor if and only if there exists a functor $F \colon \mathcal{C} \to \mathcal{D}$ and a fully faithful right adjoint $G \colon \mathcal{D} \to \mathcal{C}$ such that $L = G \circ F$. In that case there exist a unique equivalence $\mathcal{C}[\Sigma^{-1}] \to \mathcal{D}$ making the following diagram commutative*

where Σ denotes the set of morphisms σ in $\mathcal{C}$ such that $L\sigma$ is invertible.

Proof. The characterization of a localization functor follows from Proposition 2.4.1. Now observe that Σ equals the set of morphisms σ in $\mathcal{C}$ such that $F\sigma$ is invertible since G is fully faithful. Thus we can apply Proposition 2.3.1 to obtain the equivalence $\mathcal{C}[\Sigma^{-1}] \to \mathcal{D}$ making the diagram commutative. $\qquad \square$

2.5. **Local objects.** Given a localization functor $L \colon \mathcal{C} \to \mathcal{C}$, we wish to describe those objects X in $\mathcal{C}$ such that $X \xrightarrow{\sim} LX$. To this end, it is convenient to make the following definition. An object X in a category $\mathcal{C}$ is called *local* with respect to a set Σ of morphisms if for every morphism $W \to W'$ in Σ the induced map $\mathcal{C}(W', X) \to \mathcal{C}(W, X)$ is bijective. Now let $F \colon \mathcal{C} \to \mathcal{D}$ be a functor and let $\Sigma(F)$ denote the set of morphisms σ of $\mathcal{C}$ such that $F\sigma$ is invertible. An object X in $\mathcal{C}$ is called *F-local* if it is local with respect to $\Sigma(F)$.

Lemma 2.5.1. *Let $F \colon \mathcal{C} \to \mathcal{D}$ be a functor and X an object of $\mathcal{C}$. Suppose there are two morphisms $\eta_1 \colon X \to Y_1$ and $\eta_2 \colon X \to Y_2$ such that $F\eta_i$ is invertible and Y_i is F-local for $i = 1, 2$. Then there exists a unique isomorphism $\phi \colon Y_1 \to Y_2$ such that $\eta_2 = \phi \circ \eta_1$.*

Proof. The morphism η_1 induces a bijection $\mathcal{C}(Y_1, Y_2) \to \mathcal{C}(X, Y_2)$ and we take for ϕ the unique morphism which is sent to η_2. Exchanging the roles of η_1 and η_2, we obtain the inverse for ϕ. $\qquad \square$

Proposition 2.5.2. *Let $L\colon \mathcal{C} \to \mathcal{C}$ be a localization functor and $\eta\colon \operatorname{Id}\mathcal{C} \to L$ a morphism such that $L\eta$ is invertible. Then the following are equivalent for an object X in $\mathcal{C}$.*

1. *The object X is L-local.*
2. *The map $\mathcal{C}(LW, X) \to \mathcal{C}(W, X)$ induced by ηW is bijective for all W in $\mathcal{C}$.*
3. *The morphism $\eta X\colon X \to LX$ is invertible.*
4. *The map $\mathcal{C}(W, X) \to \mathcal{C}(LW, LX)$ induced by L is bijective for all W in $\mathcal{C}$.*
5. *The object X is isomorphic to LX' for some object X' in $\mathcal{C}$.*

Proof. $(1) \Rightarrow (2)$: The morphism ηW belongs to $\Sigma(L)$ and therefore $\mathcal{C}(\eta W, X)$ is bijective if X is L-local.

$(2) \Rightarrow (3)$: Put $W = X$. We obtain a morphism $\phi\colon LX \to X$ which is an inverse for ηX. More precisely, we have $\phi \circ \eta X = \operatorname{id} X$. On the other hand,

$$\eta X \circ \phi = L\phi \circ \eta LX = L\phi \circ L\eta X = L(\phi \circ \eta X) = \operatorname{id} LX.$$

Thus ηX is invertible.

$(3) \Leftrightarrow (4)$: We use the factorization $\mathcal{C} \xrightarrow{F} \mathcal{D} \xrightarrow{G} \mathcal{C}$ of L from Proposition 2.4.1. Then we obtain for each W in $\mathcal{C}$ a factorization

$$\mathcal{C}(W, X) \longrightarrow \mathcal{C}(W, LX) \xrightarrow{\sim} \mathcal{C}(FW, FX) \xrightarrow{\sim} \mathcal{C}(LW, LX)$$

of the map $f_W\colon \mathcal{C}(W, X) \to \mathcal{C}(LW, LX)$ induced by L. Here, the first map is induced by ηX, the second follows from the adjunction, and the third is induced by G. Thus f_W is bijective for all W iff the first map is bijective for all W iff ηX is invertible.

$(3) \Rightarrow (5)$: Take $X' = X$.

$(5) \Rightarrow (1)$: We use again the factorization $\mathcal{C} \xrightarrow{F} \mathcal{D} \xrightarrow{G} \mathcal{C}$ of L from Proposition 2.4.1. Fix σ in $\Sigma(L)$ and observe that $F\sigma$ is invertible. Then we have $\mathcal{C}(\sigma, X) \cong \mathcal{C}(\sigma, G(FX')) \cong \mathcal{D}(F\sigma, FX')$ and this implies that $\mathcal{C}(\sigma, X)$ is bijective since $F\sigma$ is invertible. $\square$

Given a functor $F\colon \mathcal{C} \to \mathcal{D}$, we denote by $\operatorname{Im} F$ the *essential image* of F, that is, the full subcategory of $\mathcal{D}$ which is formed by all objects isomorphic to FX for some object X in $\mathcal{C}$.

Corollary 2.5.3. *Let $L\colon \mathcal{C} \to \mathcal{C}$ be a localization functor. Then L induces an equivalence $\mathcal{C}[\Sigma(L)^{-1}] \xrightarrow{\sim} \operatorname{Im} L$ and $\operatorname{Im} L$ is the full subcategory of $\mathcal{C}$ consisting of all L-local subobjects.*

Proof. Write L as composite $\mathcal{C} \xrightarrow{F} \operatorname{Im} L \xrightarrow{G} \mathcal{C}$ of two functors, where $FX = LX$ for all X in $\mathcal{C}$ and G is the inclusion functor. Then it follows from Corollary 2.4.2 that F induces an equivalence $\mathcal{C}[\Sigma(L)^{-1}] \xrightarrow{\sim} \operatorname{Im} L$. The second assertion is an immediate consequence of Proposition 2.5.2. $\square$

Given a localization functor $L\colon \mathcal{C} \to \mathcal{C}$ and an object X in $\mathcal{C}$, the morphism $X \to LX$ is initial among all morphisms to an object in $\operatorname{Im}\mathcal{D}$ and terminal among all morphisms in $\Sigma(L)$. The following statement makes this precise.

Corollary 2.5.4. *Let $L\colon \mathcal{C} \to \mathcal{C}$ be a localization functor and $\eta\colon \operatorname{Id}\mathcal{C} \to L$ a morphism such that $L\eta$ is invertible. Then for each morphism $\eta X\colon X \to LX$ the following holds.*

1. *The object LX belongs to $\operatorname{Im} L$ and every morphism $X \to Y$ with Y in $\operatorname{Im} L$ factors uniquely through ηX.*
2. *The morphism ηX belongs to $\Sigma(L)$ and factors uniquely through every morphism $X \to Y$ in $\Sigma(L)$.*

Proof. Apply Proposition 2.5.2. □

Remark 2.5.5. (1) Let $L\colon \mathcal{C} \to \mathcal{C}$ be a localization functor and suppose there are two morphisms $\eta_i\colon \operatorname{Id}\mathcal{C} \to L$ such that $L\eta_i$ is invertible for $i = 1, 2$. Then there exists a unique isomorphism $\phi\colon L \xrightarrow{\sim} L$ such that $\eta_2 = \phi \circ \eta_1$. This follows from Lemma 2.5.1.

(2) Given any functor $F\colon \mathcal{C} \to \mathcal{D}$, the full subcategory of F-local objects is closed under taking all limits which exist in $\mathcal{C}$.

2.6. **Existence of localization functors.** We provide a criterion for the existence of a localization functor L; it explains how L is determined by the category of L-local objects.

Proposition 2.6.1. *Let $\mathcal{C}$ be a category and $\mathcal{D}$ a full subcategory. Suppose that every object in $\mathcal{C}$ isomorphic to one in $\mathcal{D}$ belongs to $\mathcal{D}$. Then the following are equivalent.*

(1) *There exists a localization functor $L\colon \mathcal{C} \to \mathcal{C}$ with $\operatorname{Im} L = \mathcal{D}$.*
(2) *For every object X in $\mathcal{C}$ there exists a morphism $\eta X\colon X \to X'$ with X' in $\mathcal{D}$ such that every morphism $X \to Y$ with Y in $\mathcal{D}$ factors uniquely through ηX.*
(3) *The inclusion functor $\mathcal{D} \to \mathcal{C}$ admits a left adjoint.*

Proof. (1) $\Rightarrow$ (2): Suppose there exists a localization functor $L\colon \mathcal{C} \to \mathcal{C}$ with $\operatorname{Im} L = \mathcal{D}$ and let $\eta\colon \operatorname{Id}\mathcal{C} \to L$ be a morphism such that $L\eta$ is invertible. Then Proposition 2.5.2 shows that $\mathcal{C}(\eta X, Y)$ is bijective for all Y in $\mathcal{D}$.

(2) $\Rightarrow$ (3): The morphisms ηX provide a functor $F\colon \mathcal{C} \to \mathcal{D}$ by sending each X in $\mathcal{C}$ to X'. It is straightforward to check that F is a left adjoint for the inclusion $\mathcal{D} \to \mathcal{C}$.

(3) $\Rightarrow$ (1): Let $G\colon \mathcal{D} \to \mathcal{C}$ denote the inclusion and F its right adjoint. Then $L = G \circ F$ is a localization functor with $\operatorname{Im} L = \mathcal{D}$ by Proposition 2.4.1. □

2.7. **Localization functors preserving coproducts.** We characterize the fact that a localization functor preserves small coproducts.

Proposition 2.7.1. *Let $L\colon \mathcal{C} \to \mathcal{C}$ be a localization functor and suppose the category $\mathcal{C}$ admits small coproducts. Then the following are equivalent.*

(1) *The functor L preserves small coproducts.*
(2) *The L-local objects are closed under taking small coproducts in $\mathcal{C}$.*
(3) *The right adjoint of the quotient functor $\mathcal{C} \to \mathcal{C}[\Sigma(L)^{-1}]$ preserves small coproducts.*

Proof. (1) $\Rightarrow$ (2): Let $(X_i)_{i \in I}$ be a family of L-local objects. Thus the natural morphisms $X_i \to LX_i$ are invertible by Proposition 2.5.2 and they induce an isomorphism

$$\coprod_i X_i \xrightarrow{\sim} \coprod_i LX_i \xrightarrow{\sim} L(\coprod_i X_i).$$

It follows that $\coprod_i X_i$ is L-local.

(2) $\Leftrightarrow$ (3): We can identify $\mathcal{C}[\Sigma(L)^{-1}] = \operatorname{Im} L$ by Corollary 2.5.3 and then the right adjoint of the quotient functor identifies with the inclusion $\operatorname{Im} L \to \mathcal{C}$. Thus the right adjoint preserves small coproducts if and only if the inclusion $\operatorname{Im} L \to \mathcal{C}$ preserves small coproducts.

(3) $\Rightarrow$ (1): Write L as composite $\mathcal{C} \to \mathcal{C}[\Sigma(L)^{-1}] \to \mathcal{C}$ of the quotient functor Q with its right adjoint $\bar{L}$. Then Q preserves small coproducts since it is a left adjoint. It follows that L preserves small coproducts if $\bar{L}$ preserves small coproducts. □

2.8. **Colocalization functors.** A functor $\Gamma\colon \mathcal{C} \to \mathcal{C}$ is called *colocalization functor* if its opposite functor $\Gamma^{\mathrm{op}}\colon \mathcal{C}^{\mathrm{op}} \to \mathcal{C}^{\mathrm{op}}$ is a localization functor. We call an object X in $\mathcal{C}$ Γ-*colocal* if it is Γ^{op}-local when viewed as an object of $\mathcal{C}^{\mathrm{op}}$. Note that a colocalization functor $\Gamma\colon \mathcal{C} \to \mathcal{C}$ induces an equivalence

$$\mathcal{C}[\Sigma(\Gamma)^{-1}] \xrightarrow{\sim} \mathrm{Im}\,\Gamma$$

and the essential image $\mathrm{Im}\,\Gamma$ equals the full subcategory of $\mathcal{C}$ consisting of all Γ-colocal objects.

Remark 2.8.1. We think of Γ as L turned upside down; this explains our notation. Another reason for the use of Γ is the interpretation of local cohomology as colocalization.

2.9. **Example: Localization of modules.** Let A be an associative ring and denote by $\mathrm{Mod}\,A$ the category of (right) A-modules. Suppose that A is commutative and let $S \subseteq A$ be a multiplicatively closed subset, that is, $1 \in S$ and $st \in S$ for all $s, t \in S$. We denote by

$$S^{-1}A = \{x/s \mid x \in A \text{ and } s \in S\}$$

the ring of fractions. For each A-module M, let

$$S^{-1}M = \{x/s \mid x \in M \text{ and } s \in S\}$$

be the localized module. An $S^{-1}A$-module N becomes an A-module via restriction of scalars along the canonical ring homomorpism $A \to S^{-1}A$. We obtain a pair of functors

$$F\colon \mathrm{Mod}\,A \longrightarrow \mathrm{Mod}\,S^{-1}A, \quad M \mapsto S^{-1}M \cong M \otimes_A S^{-1}A,$$

$$G\colon \mathrm{Mod}\,S^{-1}A \longrightarrow \mathrm{Mod}\,A, \quad N \mapsto N \cong \mathrm{Hom}_{S^{-1}A}(S^{-1}A, N).$$

Moreover, for each pair of modules M over A and N over $S^{-1}A$, we have natural morphisms

$$\eta M\colon M \longrightarrow (G \circ F)M = S^{-1}M, \quad x \mapsto x/1,$$

$$\theta N\colon S^{-1}N = (F \circ G)N \longrightarrow N, \quad x/s \mapsto xs^{-1}.$$

These natural morphisms induce mutually inverse bijections as follows:

$$\mathrm{Hom}_A(M, GN) \xrightarrow{\sim} \mathrm{Hom}_{S^{-1}A}(FM, N), \quad \alpha \mapsto \theta N \circ F\alpha,$$

$$\mathrm{Hom}_{S^{-1}A}(FM, N) \xrightarrow{\sim} \mathrm{Hom}_A(M, GN), \quad \beta \mapsto G\beta \circ \eta M.$$

It is clear that the functors F and G form an adjoint pair, that is, F is a left adjoint of G and G is a right adjoint of F. Moreover, the adjunction morphism $\theta\colon F \circ G \to \mathrm{Id}$ is invertible. Therefore the composite $L = G \circ F$ is a localization functor.

Let us formulate this slightly more generally. Fix a ring homomorphism $f\colon A \to B$. Then it is well known that the restriction functor $\mathrm{Mod}\,B \to \mathrm{Mod}\,A$ is fully faithful if and only if f is an epimorphism; see [45, Proposition XI.1.2]. Thus the functor $\mathrm{Mod}\,A \to \mathrm{Mod}\,A$ taking a module M to $M \otimes_A B$ is a localization functor provided that f is an epimorphism.

2.10. **Example: Localization of spectra.** A *spectrum E* is a sequence of based topo-logical spaces E_n and based homeomorphisms $E_n \to \Omega E_{n+1}$. A morphism of spectra $E \to F$ is a sequence of based continuous maps $E_n \to F_n$ strictly compatible with the given structural homeomorphisms. The homotopy groups of a spectrum E are the groups $\pi_n E = \pi_{n+i}(E_i)$ for $i \geq 0$ and $n + i \geq 0$. A morphism between spectra is a *weak equivalence* if it induces an isomorphism on homotopy groups. The *stable homotopy category* $\mathrm{Ho}\,\mathcal{S}$ is obtained from the category $\mathcal{S}$ of spectra by formally inverting the weak equivalences. Thus $\mathrm{Ho}\,\mathcal{S} = \mathcal{S}[\Sigma^{-1}]$ where Σ denotes the set of weak equivalences. We refer to [2, 39] for details.

2.11. **Notes.** The category of fractions is introduced by Gabriel and Zisman in [18], but the idea of formally inverting elements can be traced back much further; see for instance [36]. The appropriate context for localization functors is the theory of monads; see [27].

3. Calculus of fractions

3.1. **Calculus of fractions.** Let $\mathcal{C}$ be a category and Σ a set of morphisms in $\mathcal{C}$. The category of fractions $\mathcal{C}[\Sigma^{-1}]$ admits an elementary description if some extra assumptions on Σ are satisfied. We say that Σ *admits a calculus of left fractions* if the following holds.

(LF1) If σ, τ are composable morphisms in Σ, then $\tau \circ \sigma$ is in Σ. The identity morphism $\mathrm{id}\, X$ is in Σ for all X in $\mathcal{C}$.

(LF2) Each pair of morphisms $X' \xleftarrow{\sigma} X \xrightarrow{\alpha} Y$ with σ in Σ can be completed to a commutative square

$$\begin{array}{ccc} X & \xrightarrow{\alpha} & Y \\ \downarrow{\sigma} & & \downarrow{\sigma'} \\ X' & \xrightarrow{\alpha'} & Y' \end{array}$$
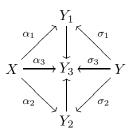
such that σ' is in Σ.

(LF3) Let $\alpha, \beta \colon X \to Y$ be morphisms in $\mathcal{C}$. If there is a morphism $\sigma \colon X' \to X$ in Σ with $\alpha \circ \sigma = \beta \circ \sigma$, then there exists a morphism $\tau \colon Y \to Y'$ in Σ with $\tau \circ \alpha = \tau \circ \beta$.

Now assume that Σ admits a calculus of left fractions. Then one obtains a new category $\Sigma^{-1}\mathcal{C}$ as follows. The objects are those of $\mathcal{C}$. Given objects X and Y, we call a pair (α, σ) of morphisms

$$X \xrightarrow{\alpha} Y' \xleftarrow{\sigma} Y$$

in $\mathcal{C}$ with σ in Σ a *left fraction*. The morphisms $X \to Y$ in $\Sigma^{-1}\mathcal{C}$ are equivalence classes $[\alpha, \sigma]$ of such left fractions, where two diagrams (α_1, σ_1) and (α_2, σ_2) are equivalent if there exists a commutative diagram

$$\begin{array}{ccccc} & & Y_1 & & \\ & \nearrow^{\alpha_1} & \downarrow & \nwarrow^{\sigma_1} & \\ X & \xrightarrow{\alpha_3} & Y_3 & \xleftarrow{\sigma_3} & Y \\ & \searrow_{\alpha_2} & \uparrow & \swarrow_{\sigma_2} & \\ & & Y_2 & & \end{array}$$

with σ_3 in Σ. The composition of two equivalence classes $[\alpha, \sigma]$ and $[\beta, \tau]$ is by definition the equivalene class $[\beta' \circ \alpha, \sigma' \circ \tau]$ where σ' and β' are obtained from condition (LF2) as in the following commutative diagram.

$$
\begin{array}{ccccc}
 & & Z'' & & \\
 & \overset{\beta'}{\nearrow} & & \overset{\sigma'}{\nwarrow} & \\
 & Y' & & Z' & \\
\overset{\alpha}{\nearrow} & & \overset{\sigma}{\nwarrow} \quad \overset{\beta}{\nearrow} & & \overset{\tau}{\nwarrow} \\
X & & Y & & Z
\end{array}
$$

We obtain a canonical functor

$$P_\Sigma \colon \mathcal{C} \longrightarrow \Sigma^{-1}\mathcal{C}$$

by taking the identity map on objects and by sending a morphism $\alpha \colon X \to Y$ to the equivalence class $[\alpha, \operatorname{id} Y]$. Let us compare P_Σ with the quotient functor $Q_\Sigma \colon \mathcal{C} \to \mathcal{C}[\Sigma^{-1}]$.

Proposition 3.1.1. *The functor $F \colon \Sigma^{-1}\mathcal{C} \to \mathcal{C}[\Sigma^{-1}]$ which is the identity map on objects and which takes a morphism $[\alpha, \sigma]$ to $(Q_\Sigma \sigma)^{-1} \circ Q_\Sigma \alpha$ is an isomorphism.*

Proof. The functor P_Σ inverts all morphisms in Σ and factors therefore through Q_Σ via a functor $G \colon \mathcal{C}[\Sigma^{-1}] \to \Sigma^{-1}\mathcal{C}$. It is straightforward to check that $F \circ G = \operatorname{Id}$ and $G \circ F = \operatorname{Id}$. $\qquad\square$

From now on, we will identify $\Sigma^{-1}\mathcal{C}$ with $\mathcal{C}[\Sigma^{-1}]$ whenever Σ admits a calculus of left fractions. A set of morphisms Σ in $\mathcal{C}$ *admits a calculus of right fractions* if the dual conditions of (LF1) – (LF3) are satisfied. Moreover, Σ is called a *multiplicative system* if it admits both, a calculus of left fractions and a calculus of right fractions. Note that all results about sets of morphisms admitting a calculus of left fractions have a dual version for sets of morphisms admitting a calculus of right fractions.

3.2. **Calculus of fractions and adjoint functors.** Given a category $\mathcal{C}$ and a set of morphisms Σ, it is an interesting question to ask when the quotient functor $\mathcal{C} \to \mathcal{C}[\Sigma^{-1}]$ admits a right adjoint. It turns out that this problem is closely related to the property of Σ to admit a calculus of left fractions.

Lemma 3.2.1. *Let $F \colon \mathcal{C} \to \mathcal{D}$ and $G \colon \mathcal{D} \to \mathcal{C}$ be a pair of adjoint functors. Assume that the right adjoint G is fully faithful and let Σ be the set of morphisms σ in $\mathcal{C}$ such that $F\sigma$ is invertible. Then Σ admits a calculus of left fractions.*

Proof. We need to check the conditions (LF1) – (LF3). Observe first that $L = G \circ F$ is a localization functor so that we can apply Proposition 2.5.2.

(LF1): This condition is clear because F is a functor.

(LF2): Let $X' \overset{\sigma}{\leftarrow} X \overset{\alpha}{\to} Y$ be a pair of morphisms with σ in Σ. This can be completed to a commutative square

$$
\begin{array}{ccc}
X & \overset{\alpha}{\longrightarrow} & Y \\
\downarrow{\scriptstyle\sigma} & & \downarrow{\scriptstyle\sigma'} \\
X' & \underset{\alpha'}{\longrightarrow} & Y'
\end{array}
$$

if we take for σ' the morphism $\eta Y \colon Y \to LY$ in Σ, because the map $\mathcal{C}(\sigma, LY)$ is surjective by Proposition 2.5.2.

(LF3): Let $\alpha, \beta \colon X \to Y$ be morphisms in $\mathcal{C}$ and suppose there is a morphism $\sigma \colon X' \to X$ in Σ with $\alpha \circ \sigma = \beta \circ \sigma$. Then we take $\tau = \eta Y$ in Σ and have $\tau \circ \alpha = \tau \circ \beta$, because the map $\mathcal{C}(\sigma, LY)$ is injective by Proposition 2.5.2. $\qquad\square$

Lemma 3.2.2. *Let $\mathcal{C}$ be a category and Σ a set of morphisms admitting a calculus of left fractions. Then the following are equivalent for an object X in $\mathcal{C}$.*

(1) *X is local with respect to Σ.*
(2) *The quotient functor induces a bijection $\mathcal{C}(W, X) \to \mathcal{C}[\Sigma^{-1}](W, X)$ for all W.*

Proof. (1) $\Rightarrow$ (2): To show that $f_W \colon \mathcal{C}(W, X) \to \mathcal{C}[\Sigma^{-1}](W, X)$ is surjective, choose a left fraction $W \xrightarrow{\alpha} X' \xleftarrow{\sigma} X$ with σ in Σ. Then there exists $\tau \colon X' \to X$ with $\tau \circ \sigma = \mathrm{id}\, X$ since X is local. Thus $f_W(\tau \circ \alpha) = [\alpha, \sigma]$. To show that f_W is injective, suppose that $f_W(\alpha) = f_W(\beta)$. Then we have $\sigma \circ \alpha = \sigma \circ \beta$ for some $\sigma \colon X \to X'$ in Σ. The morphism σ is a section because X is local, and therefore $\alpha = \beta$.

(2) $\Rightarrow$ (1): Let $\sigma \colon W \to W'$ be a morphism in Σ. Then we have $\mathcal{C}(\sigma, X) \cong \mathcal{C}[\Sigma^{-1}]([\sigma, \mathrm{id}\, W'], X)$. Thus $\mathcal{C}(\sigma, X)$ is bijective since $[\sigma, \mathrm{id}\, W']$ is invertible. $\qquad\square$

Proposition 3.2.3. *Let $\mathcal{C}$ be a category, Σ a set of morphisms admitting a calculus of left fractions, and $Q \colon \mathcal{C} \to \mathcal{C}[\Sigma^{-1}]$ the quotient functor. Then the following are equivalent.*

(1) *The functor Q has a right adjoint (which is then fully faithful).*
(2) *For each object X in $\mathcal{C}$, there exist a morphism $\eta X \colon X \to X'$ such that X' is local with respect to Σ and $Q(\eta X)$ is invertible.*

Proof. (1) $\Rightarrow$ (2): Denote by Q_ρ the right adjoint of Q and by $\eta \colon \mathrm{Id}\, \mathcal{C} \to Q_\rho Q$ the adjunction morphism. We take for each object X in $\mathcal{C}$ the morphism $\eta X \colon X \to Q_\rho Q X$. Note that $Q_\rho Q X$ is local by Proposition 2.5.2.

(2) $\Rightarrow$ (1): We fix objects X and Y. Then we have two natural bijections

$$\mathcal{C}[\Sigma^{-1}](X, Y) \xrightarrow{\sim} \mathcal{C}[\Sigma^{-1}](X, Y') \xleftarrow{\sim} \mathcal{C}(X, Y').$$

The first is induced by $\eta Y \colon Y \to Y'$ and is bijective since $Q(\eta Y)$ is invertible. The second map is bijective by Lemma 3.2.2, since Y' is local with respect to Σ. Thus we obtain a right adjoint for Q by sending each object Y of $\mathcal{C}[\Sigma^{-1}]$ to Y'. $\qquad\square$

3.3. **A criterion for the fractions to form a small set.** Let $\mathcal{C}$ be a category and Σ a set of morphisms in $\mathcal{C}$. Suppose that Σ admits a calculus of left fractions. From the construction of $\mathcal{C}[\Sigma^{-1}]$ we cannot expect that for any given pair of objects X and Y the equivalence classes of fractions in $\mathcal{C}[\Sigma^{-1}](X, Y)$ form a small set. The situation is different if the category $\mathcal{C}$ is small. Then it is clear that $\mathcal{C}[\Sigma^{-1}](X, Y)$ is a small set for all objects X, Y. The following criterion generalizes this simple observation.

Lemma 3.3.1. *Let $\mathcal{C}$ be a category and Σ a set of morphisms in $\mathcal{C}$ which admits a calculus of left fractions. Let Y be an object in $\mathcal{C}$ and suppose that there exists a small set $S = S(Y, \Sigma)$ of objects in $\mathcal{C}$ such that for every morphism $\sigma \colon Y \to Y'$ in Σ there is a morphism $\tau \colon Y' \to Y''$ with $\tau \circ \sigma$ in Σ and Y'' in S. Then $\mathcal{C}[\Sigma^{-1}](X, Y)$ is a small set for every object X in $\mathcal{C}$.*

Proof. The condition on Y implies that every fraction $X \xrightarrow{\alpha} Y' \xleftarrow{\sigma} Y$ is equivalent to one of the form $X \xrightarrow{\alpha'} Y'' \xleftarrow{\sigma'} Y$ with Y'' in S. Clearly, the fractions of the form (α', σ') with $\sigma' \in \mathcal{C}(Y, Y'')$ and $Y'' \in S$ form a small set. $\qquad \square$

3.4. **Calculus of fractions for subcategories.** We provide a criterion such that the calculus of fractions for a set of morphisms in a category $\mathcal{C}$ is compatible with the passage to a subcategory of $\mathcal{C}$.

Lemma 3.4.1. *Let $\mathcal{C}$ be a category and Σ a set of morphisms admitting a calculus of left fractions. Suppose $\mathcal{D}$ is a full subcategory of $\mathcal{C}$ such that for every morphism $\sigma \colon Y \to Y'$ in Σ with Y in $\mathcal{D}$ there is a morphism $\tau \colon Y' \to Y''$ with $\tau \circ \sigma$ in $\Sigma \cap \mathcal{D}$. Then $\Sigma \cap \mathcal{D}$ admits a calculus of left fractions and the induced functor $\mathcal{D}[(\Sigma \cap \mathcal{D})^{-1}] \to \mathcal{C}[\Sigma^{-1}]$ is fully faithful.*

Proof. It is straightforward to check (LF1) – (LF3) for $\Sigma \cap \mathcal{D}$. Now let X, Y be objects in $\mathcal{D}$. Then we need to show that the induced map

$$f \colon \mathcal{D}[(\Sigma \cap \mathcal{D})^{-1}](X, Y) \longrightarrow \mathcal{C}[\Sigma^{-1}](X, Y)$$

is bijective. The map sends the equivalence class of a fraction to the equivalence class of the same fraction. If $[\alpha, \sigma]$ belongs to $\mathcal{C}[\Sigma^{-1}](X, Y)$ and τ is a morphism with $\tau \circ \sigma$ in $\Sigma \cap \mathcal{D}$, then $[\tau \circ \alpha, \tau \circ \sigma]$ belongs to $\mathcal{D}[(\Sigma \cap \mathcal{D})^{-1}](X, Y)$ and f sends it to $[\alpha, \sigma]$. Thus f is surjective. A similar argument shows that f is injective. $\qquad \square$

Example 3.4.2. Let A be a commutative noetherian ring and $S \subseteq A$ a multiplicatively closed subset. Denote by Σ the set of morphisms σ in $\operatorname{Mod} A$ such that $S^{-1}\sigma$ is invertible. Then Σ is a multiplicative system and one can show directly that for the subcategory $\operatorname{mod} A$ of finitely generated A-modules and $T = \Sigma \cap \operatorname{mod} A$ the dual of the condition in Lemma 3.4.1 holds. Thus the induced functor

$$(\operatorname{mod} A)[T^{-1}] \longrightarrow (\operatorname{Mod} A)[\Sigma^{-1}]$$

is fully faithful.

3.5. **Calculus of fractions and coproducts.** We provide a criterion for the quotient functor $\mathcal{C} \to \mathcal{C}[\Sigma^{-1}]$ to preserve small coproducts.
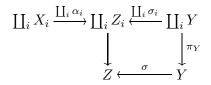
Proposition 3.5.1. *Let $\mathcal{C}$ be a category which admits small coproducts. Suppose that Σ is a set of morphisms in $\mathcal{C}$ which admits a calculus of left fractions. If $\coprod_i \sigma_i$ belongs to Σ for every family $(\sigma_i)_{i \in I}$ in Σ, then the category $\mathcal{C}[\Sigma^{-1}]$ admits small coproducts and the quotient functor $\mathcal{C} \to \mathcal{C}[\Sigma^{-1}]$ preserves small coproducts.*

Proof. Let $(X_i)_{i \in I}$ be a family of objects in $\mathcal{C}[\Sigma^{-1}]$ which is indexed by a small set I. We claim that the coproduct $\coprod_i X_i$ in $\mathcal{C}$ is also a coproduct in $\mathcal{C}[\Sigma^{-1}]$. Thus we need to show that for every object Y, the canonical map

$$(3.5.1) \qquad \mathcal{C}[\Sigma^{-1}](\coprod_i X_i, Y) \longrightarrow \prod_i \mathcal{C}[\Sigma^{-1}](X_i, Y)$$

is bijective.

To check surjectivity of (3.5.1), let $(X_i \xrightarrow{\alpha_i} Z_i \xleftarrow{\sigma_i} Y)_{i \in I}$ be a family of left fractions. Using (LF2), we obtain a commutative diagram

$$
\begin{array}{ccc}
\coprod_i X_i \xrightarrow{\coprod_i \alpha_i} \coprod_i Z_i \xleftarrow{\coprod_i \sigma_i} \coprod_i Y \\
\downarrow \qquad\qquad \downarrow {\scriptstyle \pi_Y} \\
Z \xleftarrow{\quad \sigma \quad} Y
\end{array}
$$

where $\pi_Y \colon \coprod_i Y \to Y$ is the summation morphism and $\sigma \in \Sigma$. It is easily checked that

$$(X_i \to Z \xleftarrow{\sigma} Y) \sim (X_i \xrightarrow{\alpha_i} Z_i \xleftarrow{\sigma_i} Y)$$

for all $i \in I$, and therefore (3.5.1) sends $\coprod_i X_i \to Z \xleftarrow{\sigma} Y$ to the family $(X_i \xrightarrow{\alpha_i} Z_i \xleftarrow{\sigma_i} Y_i)_{i \in I}$.

To check injectivity of (3.5.1), let $\coprod_i X_i \xrightarrow{\alpha'} Z' \xleftarrow{\sigma'} Y$ and $\coprod_i X_i \xrightarrow{\alpha''} Z'' \xleftarrow{\sigma''} Y$ be left fraction such that

$$(X_i \xrightarrow{\alpha_i'} Z' \xleftarrow{\sigma'} Y) \sim (X_i \xrightarrow{\alpha_i''} Z'' \xleftarrow{\sigma''} Y)$$

for all i. We may assume that $Z' = Z = Z''$ and $\sigma' = \sigma = \sigma''$ since we can choose morphisms $\tau' \colon Z' \to Z$ and $\tau'' \colon Z'' \to Z$ with $\tau' \circ \sigma' = \tau'' \circ \sigma'' \in \Sigma$. Thus there are morphisms $\beta_i \colon Z \to Z_i$ with $\beta_i \circ \alpha_i' = \beta_i \circ \alpha_i''$ and $\beta_i \circ \sigma \in \Sigma$ for all i. Each β_i belongs to the *saturation* $\bar{\Sigma}$ of Σ which is the set of all morphisms in $\mathcal{C}$ which become invertible in $\mathcal{C}[\Sigma^{-1}]$. Note that a morphism ϕ in $\mathcal{C}$ belongs to $\bar{\Sigma}$ if and only if there are morphisms ϕ' and ϕ'' such that $\phi \circ \phi'$ and $\phi'' \circ \phi$ belong to Σ. Therefore $\bar{\Sigma}$ is also closed under taking coproducts. Moreover, $\bar{\Sigma}$ admits a calculus of left fractions, and we obtain therefore a commutative diagram

$$
\begin{array}{ccc}
\coprod_i X_i \longrightarrow \coprod_i Z \xrightarrow{\pi_Z} Z \\
\qquad\quad \downarrow {\scriptstyle \coprod_i \beta_i} \qquad \downarrow {\scriptstyle \tau} \\
\coprod_i Z_i \longrightarrow Z^*
\end{array}
$$

with $\tau \in \bar{\Sigma}$. Thus $\tau \circ \sigma \in \bar{\Sigma}$, and we have

$$\left(\coprod_i X_i \xrightarrow{\alpha'} Z \xleftarrow{\sigma} Y\right) \sim \left(\coprod_i X_i \xrightarrow{\alpha''} Z \xleftarrow{\sigma} Y\right)$$

since $\pi_Z \circ \coprod_i \alpha_i' = \alpha'$ and $\pi_Z \circ \coprod_i \alpha_i'' = \alpha'$. Therefore the map (3.5.1) is also injective, and this completes the proof. $\square$

Example 3.5.2. Let $\mathcal{C}$ be a category which admits small coproducts and $L \colon \mathcal{C} \to \mathcal{C}$ be a localization functor. Then a morphism σ in $\mathcal{C}$ belongs to $\Sigma = \Sigma(L)$ if and only if the induced map $\mathcal{C}(\sigma, LX)$ is invertible for every object X in $\mathcal{C}$. Thus Σ is closed under taking small coproducts and therefore the quotient functor $\mathcal{C} \to \mathcal{C}[\Sigma^{-1}]$ preserves small coproducts.

3.6. **Notes.** The calculus of fractions for categories has been developed by Gabriel and Zisman in [18] as a tool for homotopy theory.

4. Localization for triangulated categories

4.1. **Triangulated categories.** Let $\mathcal{T}$ be an additive category with an equivalence $S\colon \mathcal{T} \to \mathcal{T}$. A *triangle* in $\mathcal{T}$ is a sequence (α, β, γ) of morphisms

$$X \xrightarrow{\alpha} Y \xrightarrow{\beta} Z \xrightarrow{\gamma} SX,$$

and a morphism between two triangles (α, β, γ) and $(\alpha', \beta', \gamma')$ is a triple (ϕ_1, ϕ_2, ϕ_3) of morphisms in $\mathcal{T}$ making the following diagram commutative.

$$
\begin{array}{ccccccc}
X & \xrightarrow{\alpha} & Y & \xrightarrow{\beta} & Z & \xrightarrow{\gamma} & SX \\
\downarrow{\phi_1} & & \downarrow{\phi_2} & & \downarrow{\phi_3} & & \downarrow{S\phi_1} \\
X' & \xrightarrow{\alpha'} & Y' & \xrightarrow{\beta'} & Z' & \xrightarrow{\gamma'} & SX'
\end{array}
$$

The category $\mathcal{T}$ is called *triangulated* if it is equipped with a set of distinguished triangles (called *exact triangles*) satisfying the following conditions.

(TR1) A triangle isomorphic to an exact triangle is exact. For each object X, the triangle $0 \to X \xrightarrow{\mathrm{id}} X \to 0$ is exact. Each morphism α fits into an exact triangle (α, β, γ).

(TR2) A triangle (α, β, γ) is exact if and only if $(\beta, \gamma, -S\alpha)$ is exact.

(TR3) Given two exact triangles (α, β, γ) and $(\alpha', \beta', \gamma')$, each pair of morphisms ϕ_1 and ϕ_2 satisfying $\phi_2 \circ \alpha = \alpha' \circ \phi_1$ can be completed to a morphism

$$
\begin{array}{ccccccc}
X & \xrightarrow{\alpha} & Y & \xrightarrow{\beta} & Z & \xrightarrow{\gamma} & SX \\
\downarrow{\phi_1} & & \downarrow{\phi_2} & & \downarrow{\phi_3} & & \downarrow{S\phi_1} \\
X' & \xrightarrow{\alpha'} & Y' & \xrightarrow{\beta'} & Z' & \xrightarrow{\gamma'} & SX'
\end{array}
$$

of triangles.

(TR4) Given exact triangles $(\alpha_1, \alpha_2, \alpha_3)$, $(\beta_1, \beta_2, \beta_3)$, and $(\gamma_1, \gamma_2, \gamma_3)$ with $\gamma_1 = \beta_1 \circ \alpha_1$, there exists an exact triangle $(\delta_1, \delta_2, \delta_3)$ making the following diagram commutative.

$$
\begin{array}{ccccccc}
X & \xrightarrow{\alpha_1} & Y & \xrightarrow{\alpha_2} & U & \xrightarrow{\alpha_3} & SX \\
\| & & \downarrow{\beta_1} & & \downarrow{\delta_1} & & \| \\
X & \xrightarrow{\gamma_1} & Z & \xrightarrow{\gamma_2} & V & \xrightarrow{\gamma_3} & SX \\
& & \downarrow{\beta_2} & & \downarrow{\delta_2} & & \downarrow{S\alpha_1} \\
& & W & = & W & \xrightarrow{\beta_3} & SY \\
& & \downarrow{\beta_3} & & \downarrow{\delta_3} & & \\
& & SY & \xrightarrow{S\alpha_2} & SU & &
\end{array}
$$

Recall that an idempotent endomorphism $\phi = \phi^2$ of an object X in an additive category *splits* if there exists a factorization $X \xrightarrow{\pi} Y \xrightarrow{\iota} X$ of ϕ with $\pi \circ \iota = \mathrm{id}\, Y$.

Remark 4.1.1. Suppose a triangulated category $\mathcal{T}$ admits countable coproducts. Then every idempotent endomorphism splits. More precisely, let $\phi\colon X \to X$ be an idempotent

morphism in $\mathcal{T}$, and denote by Y a homotopy colimit of the sequence

$$X \xrightarrow{\phi} X \xrightarrow{\phi} X \xrightarrow{\phi} \cdots .$$

The morphism ϕ factors through the canonical morphism $\pi\colon X \to Y$ via a morphism $\iota\colon Y \to X$, and we have $\pi \circ \iota = \operatorname{id} Y$. Thus ϕ splits; see [33, Proposition 1.6.8] for details.

4.2. **Exact functors.** An *exact* functor $\mathcal{T} \to \mathcal{U}$ between triangulated categories is a pair (F, μ) consisting of a functor $F\colon \mathcal{T} \to \mathcal{U}$ and an isomorphism $\mu\colon F \circ S_{\mathcal{T}} \to S_{\mathcal{U}} \circ F$ such that for every exact triangle $X \xrightarrow{\alpha} Y \xrightarrow{\beta} Z \xrightarrow{\gamma} S_{\mathcal{T}} X$ in $\mathcal{T}$ the triangle

$$FX \xrightarrow{F\alpha} FY \xrightarrow{F\beta} FZ \xrightarrow{\mu X \circ F\gamma} S_{\mathcal{U}}(FX)$$

is exact in $\mathcal{U}$.

We have the following useful lemma.

Lemma 4.2.1. *Let $F\colon \mathcal{T} \to \mathcal{U}$ and $G\colon \mathcal{U} \to \mathcal{T}$ be an adjoint pair of functors between triangulated categories. If one of both functors is exact, then also the other is exact.*

Proof. See [33, Lemma 5.3.6]. $\square$

4.3. **Multiplicative systems.** Let $\mathcal{T}$ be a triangulated category and Σ a set of morphisms which is a multiplicative system. Recall this means that Σ admits a calculus of left and right fractions. Then we say that Σ is *compatible with the triangulation* if

(1) given σ in Σ, the morphism $S^n \sigma$ belongs to Σ for all $n \in \mathbb{Z}$, and
(2) given a morphism (ϕ_1, ϕ_2, ϕ_3) between exact triangles with ϕ_1 and ϕ_2 in Σ, there is also a morphism $(\phi_1, \phi_2, \phi_3')$ with ϕ_3' in Σ.

Lemma 4.3.1. *Let $\mathcal{T}$ be a triangulated category and Σ a multiplicative system of morphisms which is compatible with the triangulation. Then the quotient category $\mathcal{T}[\Sigma^{-1}]$ carries a unique triangulated structure such that the quotient functor $\mathcal{T} \to \mathcal{T}[\Sigma^{-1}]$ is exact.*

Proof. The equivalence $S\colon \mathcal{T} \to \mathcal{T}$ induces a unique equivalence $\mathcal{T}[\Sigma^{-1}] \to \mathcal{T}[\Sigma^{-1}]$ which commutes with the quotient functor $Q\colon \mathcal{T} \to \mathcal{T}[\Sigma^{-1}]$. This follows from the fact that $S\Sigma = \Sigma$. Now take as exact triangles in $\mathcal{T}[\Sigma^{-1}]$ all those isomorphic to images of exact triangles in $\mathcal{T}$. It is straightforward to verify the axioms (TR1) – (TR4); see [48, II.2.2.6]. The functor Q is exact by construction. In particular, we have $Q \circ S_{\mathcal{T}} = S_{\mathcal{T}[\Sigma^{-1}]} \circ Q$. $\square$

4.4. **Cohomological functors.** A functor $H\colon \mathcal{T} \to \mathcal{A}$ from a triangulated category $\mathcal{T}$ to an abelian category $\mathcal{A}$ is *cohomological* if H sends every exact triangle in $\mathcal{T}$ to an exact sequence in $\mathcal{A}$.

Example 4.4.1. For each object X in $\mathcal{T}$, the representable functors $\mathcal{T}(X, -)\colon \mathcal{T} \to \mathrm{Ab}$ and $\mathcal{T}(-, X)\colon \mathcal{T}^{\mathrm{op}} \to \mathrm{Ab}$ into the category Ab of abelian groups are cohomological functors.

Lemma 4.4.2. *Let $H\colon \mathcal{T} \to \mathcal{A}$ be a cohomological functor. Then the set Σ of morphisms σ in $\mathcal{T}$ such that $H(S^n \sigma)$ is invertible for all $n \in \mathbb{Z}$ forms a multiplicative system which is compatible with the triangulation of $\mathcal{T}$.*

Proof. We need to verify that Σ admits a calculus of left and right fractions. In fact, it is sufficient to check conditions (LF1) – (LF3), because then the dual conditions are satisfied as well since the definition of Σ is self-dual.

(LF1): This condition is clear because H is a functor.

(LF2): Let $\alpha\colon X \to Y$ and $\sigma\colon X \to X'$ be morphisms with σ in Σ. We complete α to an exact triangle and apply (TR3) to obtain the following morphism between exact triangles.

$$
\begin{array}{ccccccc}
W & \longrightarrow & X & \xrightarrow{\alpha} & Y & \longrightarrow & SW \\
\| & & \downarrow{\scriptstyle\sigma} & & \downarrow{\scriptstyle\sigma'} & & \| \\
W & \longrightarrow & X' & \xrightarrow{\alpha'} & Y' & \longrightarrow & SW
\end{array}
$$

Then the 5-lemma shows that σ' belongs to Σ.

(LF3): Let $\alpha, \beta\colon X \to Y$ be morphisms in $\mathcal{T}$ and $\sigma\colon X' \to X$ in Σ such that $\alpha \circ \sigma = \beta \circ \sigma$. Complete σ to an exact triangle $X' \xrightarrow{\sigma} X \xrightarrow{\phi} X'' \to SX'$. Then $\alpha - \beta$ factors through ϕ via some morphism $\psi\colon X'' \to Y$. Now complete ψ to an exact triangle $X'' \xrightarrow{\psi} Y \xrightarrow{\tau} Y' \to SX''$. Then τ belongs to Σ and $\tau \circ \alpha = \tau \circ \beta$.

It remains to check that Σ is compatible with the triangulation. Condition (1) is clear from the definition of Σ. For condition (2), observe that given any morphism (ϕ_1, ϕ_2, ϕ_3) between exact triangles with ϕ_1 and ϕ_2 in Σ, we have that ϕ_3 belongs to Σ. This is an immediate consequence of the 5-lemma. $\square$

4.5. Triangulated and thick subcategories.
Let $\mathcal{T}$ be a triangulated category. A non-empty full subcategory $\mathcal{S}$ is a *triangulated subcategory* if the following conditions hold.

(TS1) $S^n X \in \mathcal{S}$ for all $X \in \mathcal{S}$ and $n \in \mathbb{Z}$.

(TS2) Let $X \to Y \to Z \to SX$ be an exact triangle in $\mathcal{T}$. If two objects from $\{X, Y, Z\}$ belong to $\mathcal{S}$, then also the third.

A triangulated subcategory $\mathcal{S}$ is *thick* if in addition the following condition holds.

(TS3) Let $X \xrightarrow{\pi} Y \xrightarrow{\iota} X$ be morphisms in $\mathcal{T}$ such that $\operatorname{id} Y = \pi \circ \iota$. If X belongs to $\mathcal{S}$, then also Y.

Note that a triangulated subcategory $\mathcal{S}$ of $\mathcal{T}$ inherits a canonical triangulated structure from $\mathcal{T}$.

Next observe that a triangulated subcategory $\mathcal{S}$ of $\mathcal{T}$ is thick provided that $\mathcal{S}$ admits countable coproducts. This follows from the fact that in a triangulated category with countable coproducts all idempotent endomorphisms split.

Let $\mathcal{T}$ be a triangulated category and let $F\colon \mathcal{T} \to \mathcal{U}$ be an additive functor. The *kernel* $\operatorname{Ker} F$ of F is by definition the full subcategory of $\mathcal{T}$ which is formed by all objects X such that $FX = 0$. If F is an exact functor into a triangulated category, then $\operatorname{Ker} F$ is a thick subcategory of $\mathcal{T}$. Also, if F is a cohomological functor into an abelian category, then $\bigcap_{n \in \mathbb{Z}} S^n(\operatorname{Ker} F)$ is a thick subcategory of $\mathcal{T}$.

4.6. Verdier localization.
Let $\mathcal{T}$ be a triangulated category. Given a triangulated subcategory $\mathcal{S}$, we denote by $\Sigma(\mathcal{S})$ the set of morphisms $X \to Y$ in $\mathcal{T}$ which fit into an exact triangle $X \to Y \to Z \to SX$ with Z in $\mathcal{S}$.

Lemma 4.6.1. *Let $\mathcal{T}$ be a triangulated category and $\mathcal{S}$ a triangulated subcategory. Then $\Sigma(\mathcal{S})$ is a multiplicative system which is compatible with the triangulation of $\mathcal{T}$.*

Proof. The proof is similar to that of Lemma 4.4.2; see [48, II.2.1.8] for details.   □

The *localization* of $\mathcal{T}$ with respect to a triangulated subcategory $\mathcal{S}$ is by definition the quotient category

$$\mathcal{T}/\mathcal{S} := \mathcal{T}[\Sigma(\mathcal{S})^{-1}]$$

together with the quotient functor $\mathcal{T} \to \mathcal{T}/\mathcal{S}$.

Proposition 4.6.2. *Let $\mathcal{T}$ be a triangulated category and $\mathcal{S}$ a full triangulated subcategory. Then the category $\mathcal{T}/\mathcal{S}$ and the quotient functor $Q\colon \mathcal{T} \to \mathcal{T}/\mathcal{S}$ have the following properties.*

(1) *The category $\mathcal{T}/\mathcal{S}$ carries a unique triangulated structure such that Q is exact.*
(2) *A morphism in $\mathcal{T}$ is annihilated by Q if and only if it factors through an object in $\mathcal{S}$.*
(3) *The kernel $\operatorname{Ker} Q$ is the smallest thick subcategory containing $\mathcal{S}$.*
(4) *Every exact functor $\mathcal{T} \to \mathcal{U}$ annihilating $\mathcal{S}$ factors uniquely through Q via an exact functor $\mathcal{T}/\mathcal{S} \to \mathcal{U}$.*
(5) *Every cohomological functor $\mathcal{T} \to \mathcal{A}$ annihilating $\mathcal{S}$ factors uniquely through Q via a cohomological functor $\mathcal{T}/\mathcal{S} \to \mathcal{A}$.*
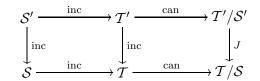
Proof. (1) follows from Lemma 4.3.1.

(2) Let ϕ be a morphism in $\mathcal{T}$. We have $Q\phi = 0$ iff $\sigma \circ \phi = 0$ for some $\sigma \in \Sigma(\mathcal{S})$ iff ϕ factors through some object in $\mathcal{S}$.

(3) Let X be an object in $\mathcal{T}$. Then $QX = 0$ if and only if $Q(\operatorname{id} X) = 0$. Thus part (2) implies that the kernel of Q conists of all direct factors of objects in $\mathcal{S}$.

(4) An exact functor $F\colon \mathcal{T} \to \mathcal{U}$ annihilating $\mathcal{S}$ inverts every morphism in $\Sigma(\mathcal{S})$. Thus there exists a unique functor $\bar{F}\colon \mathcal{T}/\mathcal{S} \to \mathcal{U}$ such that $F = \bar{F} \circ Q$. The functor $\bar{F}$ is exact because an exact triangle Δ in $\mathcal{T}/\mathcal{S}$ is up to isomorphism of the form $Q\Delta'$ for some exact triangle Δ' in $\mathcal{T}$. Thus $\bar{F}\Delta \cong F\Delta'$ is exact.

(5) Analogous to (4).   □

4.7. **Localization of subcategories.** Let $\mathcal{T}$ be a triangulated category with two full triangulated subcategories $\mathcal{T}'$ and $\mathcal{S}$. Then we put $\mathcal{S}' = \mathcal{S} \cap \mathcal{T}'$ and have $\Sigma_{\mathcal{T}'}(\mathcal{S}') = \Sigma_{\mathcal{T}}(\mathcal{S}) \cap \mathcal{T}'$. Thus we can form the following commutative diagram of exact functors

$$\begin{array}{ccccc}
\mathcal{S}' & \xrightarrow{\ \text{inc}\ } & \mathcal{T}' & \xrightarrow{\ \text{can}\ } & \mathcal{T}'/\mathcal{S}' \\
\downarrow{\scriptstyle\text{inc}} & & \downarrow{\scriptstyle\text{inc}} & & \downarrow{\scriptstyle J} \\
\mathcal{S} & \xrightarrow{\ \text{inc}\ } & \mathcal{T} & \xrightarrow{\ \text{can}\ } & \mathcal{T}/\mathcal{S}
\end{array}$$

and ask when the functor J is fully faithful. We have the following criterion.

Lemma 4.7.1. *Let $\mathcal{T}$, $\mathcal{T}'$, $\mathcal{S}$, $\mathcal{S}'$ be as above. Suppose that either*

(1) *every morphism from an object in $\mathcal{S}$ to an object in $\mathcal{T}'$ factors through some object in $\mathcal{S}'$, or*
(2) *every morphism from an object in $\mathcal{T}'$ to an object in $\mathcal{S}$ factors through some object in $\mathcal{S}'$.*

Then the induced functor $J\colon \mathcal{T}'/\mathcal{S}' \to \mathcal{T}/\mathcal{S}$ is fully faithful.

Proof. Suppose that condition (1) holds. We apply the criterion from Lemma 3.4.1. Thus we take a morphism $\sigma\colon Y \to Y'$ from $\Sigma(\mathcal{S})$ with Y in $\mathcal{T}'$ and need to find $\tau\colon Y' \to Y''$ such that $\tau \circ \sigma$ belongs to $\Sigma(\mathcal{S}) \cap \mathcal{T}'$. To this end complete σ to an exact triangle $X \xrightarrow{\phi} Y \xrightarrow{\sigma} Y' \to SX$. Then X belongs to $\mathcal{S}$ and by our assumption we have a factorization $X \xrightarrow{\phi'} Z \xrightarrow{\phi''} Y$ of ϕ with Z in $\mathcal{S}'$. Complete ϕ'' to an exact triangle $Z \xrightarrow{\phi''} Y \xrightarrow{\psi} Y'' \to SZ$. Then (TR3) yields a morphism $\tau\colon Y' \to Y''$ satisfying $\psi = \tau \circ \sigma$. In particular, $\tau \circ \sigma$ lies in $\Sigma(\mathcal{S}) \cap \mathcal{T}'$ since Z belongs to $\mathcal{S}'$. The proof using condition (2) is dual. $\square$

4.8. **Orthogonal subcategories.** Let $\mathcal{T}$ be a triangulated category and $\mathcal{S}$ a triangulated subcategory. Then we define two full subcategories

$$\mathcal{S}^{\perp} = \{Y \in \mathcal{T} \mid \mathcal{T}(X, Y) = 0 \text{ for all } X \in \mathcal{S}\}$$
$$^{\perp}\mathcal{S} = \{X \in \mathcal{T} \mid \mathcal{T}(X, Y) = 0 \text{ for all } Y \in \mathcal{S}\}$$

and call them *orthogonal subcategories* with respect to $\mathcal{S}$. Note that $\mathcal{S}^{\perp}$ and $^{\perp}\mathcal{S}$ are thick subcategories of $\mathcal{T}$.

Lemma 4.8.1. *Let $\mathcal{T}$ be a triangulated category and $\mathcal{S}$ a triangulated subcategory. Then the following are equivalent for an object Y in $\mathcal{T}$.*

(1) *Y belongs to $\mathcal{S}^{\perp}$.*
(2) *Y is $\Sigma(\mathcal{S})$-local, that is, $\mathcal{T}(\sigma, Y)$ is bijective for all σ in $\Sigma(\mathcal{S})$.*
(3) *The quotient functor induces a bijection $\mathcal{T}(X, Y) \to \mathcal{T}/\mathcal{S}(X, Y)$ for all X in $\mathcal{T}$.*

Proof. (1) $\Rightarrow$ (2): Suppose $\mathcal{T}(X, Y) = 0$ for all X in $\mathcal{S}$. Then every σ in $\Sigma(\mathcal{S})$ induces a bijection $\mathcal{T}(\sigma, Y)$ because $\mathcal{T}(-, Y)$ is cohomological. Thus Y is $\Sigma(\mathcal{S})$-local.

(2) $\Rightarrow$ (1): Suppose that Y is $\Sigma(\mathcal{S})$-local. If X belongs to $\mathcal{S}$, then the morphism $\sigma\colon X \to 0$ belongs to $\Sigma(\mathcal{S})$ and induces therefore a bijection $\mathcal{C}(\sigma, Y)$. Thus Y belongs to $\mathcal{S}^{\perp}$.

(2) $\Leftrightarrow$ (3): Apply Lemma 3.2.2. $\square$

4.9. **Bousfield localization.** Let $\mathcal{T}$ be a triangulated category. We wish to study exact localization functors $L\colon \mathcal{T} \to \mathcal{T}$. To be more precise, we assume that L is an exact functor and that L is a localization functor in the sense that there exists a morphism $\eta\colon \operatorname{Id}\mathcal{C} \to L$ with $L\eta\colon L \to L^2$ being invertible and $L\eta = \eta L$. Note that there is an isomorphism $\mu\colon L \circ S \xrightarrow{\sim} S \circ L$ since L is exact, and there exists a unique choice such that $\mu X \circ \eta SX = S\eta X$ for all X in $\mathcal{T}$. This follows from Lemma 2.5.1.

We observe that the kernel of an exact localization functor is a thick subcategory of $\mathcal{T}$. The following fundamental result characterizes the thick subcategories of $\mathcal{T}$ which are of this form.

Proposition 4.9.1. *Let $\mathcal{T}$ be a triangulated category and $\mathcal{S}$ a thick subcategory. Then the following are equivalent.*

(1) *There exists an exact localization functor $L\colon \mathcal{T} \to \mathcal{T}$ with $\operatorname{Ker} L = \mathcal{S}$.*
(2) *The inclusion functor $\mathcal{S} \to \mathcal{T}$ admits a right adjoint.*
(3) *For each X in $\mathcal{T}$ there exists an exact triangle $X' \to X \to X'' \to SX''$ with X' in $\mathcal{S}$ and X'' in $\mathcal{S}^{\perp}$.*
(4) *The quotient functor $\mathcal{T} \to \mathcal{T}/\mathcal{S}$ admits a right adjoint.*

(5) *The composite $\mathcal{S}^{\perp} \xrightarrow{\text{inc}} \mathcal{T} \xrightarrow{\text{can}} \mathcal{T}/\mathcal{S}$ is an equivalence.*

(6) *The inclusion functor $\mathcal{S}^{\perp} \to \mathcal{T}$ admits a left adjoint and $^{\perp}(\mathcal{S}^{\perp}) = \mathcal{S}$.*

Proof. Let $I \colon \mathcal{S} \to \mathcal{T}$ and $J \colon \mathcal{S}^{\perp} \to \mathcal{T}$ denote the inclusions and $Q \colon \mathcal{T} \to \mathcal{T}/\mathcal{S}$ the quotient functor.

(1) $\Rightarrow$ (2): Suppose that $L \colon \mathcal{T} \to \mathcal{T}$ is an exact localization functor with $\operatorname{Ker} L = \mathcal{S}$ and let $\eta \colon \operatorname{Id} \mathcal{T} \to L$ be a morphism such that $L\eta$ is invertible. We obtain a right adjoint $I_{\rho} \colon \mathcal{T} \to \mathcal{S}$ for the inclusion I by completing for each X in $\mathcal{T}$ the morphism ηX to an exact triangle $I_{\rho}X \xrightarrow{\theta X} X \xrightarrow{\eta X} LX \to S(I_{\rho}X)$. Note that $I_{\rho}X$ belongs to $\mathcal{S}$ since $L\eta X$ is invertible. Moreover, $\mathcal{T}(W, \theta X)$ is bijective for all W in $\mathcal{S}$ since $\mathcal{T}(W, LX) = 0$ by Lemma 4.8.1. Here we use that LX is $\Sigma(L)$-local by Proposition 2.5.2 and that $\Sigma(L) = \Sigma(\mathcal{S})$. Thus I_{ρ} provides a right adjoint for I since $\mathcal{T}(W, I_{\rho}X) \cong \mathcal{T}(IW, X)$ for all W in $\mathcal{S}$ and X in $\mathcal{T}$. In particular, we see that the exact triangle defining $I_{\rho}X$ is, up to a unique isomorphism, uniquely determined by X. Therefore I_{ρ} is well defined.

(2) $\Rightarrow$ (3): Suppose that $I_{\rho} \colon \mathcal{T} \to \mathcal{S}$ is a right adjoint of the inclusion I. We fix an object X in $\mathcal{T}$ and complete the adjunction morphism $\theta X \colon I_{\rho}X \to X$ to an exact triangle $I_{\rho}X \xrightarrow{\theta X} X \to X'' \to S(I_{\rho}X)$. Clearly, $I_{\rho}X$ belongs to $\mathcal{S}$. We have $\mathcal{T}(W, X'') = 0$ for all W in $\mathcal{S}$ since $\mathcal{T}(W, \theta X)$ is bijective. Thus X'' belongs to $\mathcal{S}^{\perp}$.

(3) $\Rightarrow$ (4): We apply Proposition 3.2.3 to obtain a right adjoint for the quotient functor Q. To this end fix an object X in $\mathcal{T}$ and an exact triangle $X' \to X \xrightarrow{\eta} X'' \to SX''$ with X' in $\mathcal{S}$ and X'' in $\mathcal{S}^{\perp}$. The morphism η belongs to $\Sigma(\mathcal{S})$ by definition, and the object X'' is $\Sigma(\mathcal{S})$-local by Lemma 4.8.1. Now it follows from Proposition 3.2.3 that Q admits a right adjoint.

(4) $\Rightarrow$ (1): Let $Q_{\rho} \colon \mathcal{T}/\mathcal{S} \to \mathcal{T}$ denote a right adjoint of Q. This functor is fully faithful by Proposition 2.3.1 and exact by Lemma 4.2.1. Thus $L = Q_{\rho} \circ Q$ is an exact functor with $\operatorname{Ker} L = \operatorname{Ker} Q = \mathcal{S}$. Moreover, L is a localization functor by Corollary 2.4.2.

(4) $\Rightarrow$ (5): Let $Q_{\rho} \colon \mathcal{T}/\mathcal{S} \to \mathcal{T}$ denote a right adjoint of Q. The composite $Q \circ J \colon \mathcal{S}^{\perp} \to \mathcal{T}/\mathcal{S}$ is fully faithful by Lemma 4.8.1. Given an object X in $\mathcal{T}/\mathcal{S}$, we have $Q(Q_{\rho}X) \cong X$ by Proposition 2.3.1, and $Q_{\rho}X$ belongs to $\mathcal{S}^{\perp}$, since $\mathcal{T}(W, Q_{\rho}X) \cong \mathcal{T}/\mathcal{S}(QW, X) = 0$ for all W in $\mathcal{S}$. Thus $Q \circ J$ is dense and therefore an equivalence.

(5) $\Rightarrow$ (6): Suppose $Q \circ J \colon \mathcal{S}^{\perp} \to \mathcal{T}/\mathcal{S}$ is an equivalence and let $F \colon \mathcal{T}/\mathcal{S} \to \mathcal{S}^{\perp}$ be a quasi-inverse. We have for all X in $\mathcal{T}$ and Y in $\mathcal{S}^{\perp}$

$$\mathcal{T}(X, JY) \xrightarrow{\sim} \mathcal{T}/\mathcal{S}(QX, QJY) \xrightarrow{\sim} \mathcal{S}^{\perp}(FQX, FQJY) \xrightarrow{\sim} \mathcal{S}^{\perp}(FQX, Y),$$

where the first bijection follows from Lemma 4.8.1 and the others are clear from the choice of F. Thus $F \circ Q$ is a left adjoint for the inclusion J.

It remains to show that $^{\perp}(\mathcal{S}^{\perp}) = \mathcal{S}$. The inclusion $^{\perp}(\mathcal{S}^{\perp}) \supseteq \mathcal{S}$ is clear. Now let X be an object of $^{\perp}(\mathcal{S}^{\perp})$. Then we have

$$\mathcal{T}/\mathcal{S}(QX, QX) \cong \mathcal{S}^{\perp}(FQX, FQX) \cong \mathcal{T}(X, J(FQX)) = 0.$$

Thus $QX = 0$ and therefore X belongs to $\mathcal{S}$.

(6) $\Rightarrow$ (3): Suppose that $J_{\lambda} \colon \mathcal{T} \to \mathcal{S}^{\perp}$ is a left adjoint of the inclusion J. We fix an object X in $\mathcal{T}$ and complete the adjunction morphism $\mu X \colon X \to J_{\lambda}X$ to an exact triangle $X' \to X \xrightarrow{\mu X} J_{\lambda}X \to SX'$. Clearly, $J_{\lambda}X$ belongs to $\mathcal{S}^{\perp}$. We have $\mathcal{T}(X', Y) = 0$ for all Y in $\mathcal{S}^{\perp}$ since $\mathcal{T}(\mu X, Y)$ is bijective. Thus X' belongs to $^{\perp}(\mathcal{S}^{\perp}) = \mathcal{S}$. $\qquad \square$

The following diagram displays the functors which arise from a localization functor $L\colon \mathcal{T} \to \mathcal{T}$. We use the convention that F_ρ denotes a right adjoint of a functor F.

$$\mathcal{S} \; \underset{I=\mathrm{inc}}{\overset{I_\rho}{\longleftrightarrow}} \; \mathcal{T} \; \underset{Q=\mathrm{can}}{\overset{Q_\rho}{\longleftrightarrow}} \; \mathcal{T}/\mathcal{S} \qquad (L = Q_\rho \circ Q \quad \text{and} \quad \Gamma = I \circ I_\rho)$$

4.10. **Acyclic and local objects.** Let $\mathcal{T}$ be a triangulated category and $L\colon \mathcal{T} \to \mathcal{T}$ an exact localization functor. An object X in $\mathcal{T}$ is by definition *L-acyclic* if $LX = 0$. Recall that an object in $\mathcal{T}$ is L-local if and only if it belongs to the essential image $\operatorname{Im} L$ of L; see Proposition 2.5.2. The exactness of L implies that $\mathcal{S} := \operatorname{Ker} L$ is a thick subcategory and that $\Sigma(L) = \Sigma(\mathcal{S})$. Therefore L-local and $\Sigma(\mathcal{S})$-local objects coincide.

The following result says that acyclic and local objects form an orthogonal pair.

Proposition 4.10.1. *Let $L\colon \mathcal{T} \to \mathcal{T}$ be an exact localization functor. Then we have*
$$\operatorname{Ker} L = {}^\perp(\operatorname{Im} L) \quad \text{and} \quad (\operatorname{Ker} L)^\perp = \operatorname{Im} L.$$
More explictly, the following holds.

   (1) *$X \in \mathcal{T}$ is L-acyclic if and only if $\mathcal{T}(X, Y) = 0$ for every L-local object Y.*
   (2) *$Y \in \mathcal{T}$ is L-local if and only if $\mathcal{T}(X, Y) = 0$ for every L-acyclic object X.*

Proof. (1) We write $L = G \circ F$ where F is a functor and G a fully faithful right adjoint; see Corollary 2.4.2. Suppose first we have given objects X, Y such that X is L-acyclic and Y is L-local. Observe that $FX = 0$ since G is faithful. Thus
$$\mathcal{T}(X, Y) \cong \mathcal{T}(X, GFY) \cong \mathcal{T}(FX, FY) = 0.$$

Now suppose that X is an object with $\mathcal{T}(X, Y) = 0$ for all L-local Y. Then
$$\mathcal{T}(FX, FX) \cong \mathcal{T}(X, GFX) = 0$$
and therefore $FX = 0$. Thus X is L-acyclic.

   (2) This is a reformulation of Lemma 4.8.1. $\qquad\qquad\qquad\qquad\qquad\qquad\square$

4.11. **A functorial triangle.** Let $\mathcal{T}$ be a triangulated category and $L\colon \mathcal{T} \to \mathcal{T}$ an exact localization functor. We denote by $\eta\colon \operatorname{Id}\mathcal{T} \to L$ a morphism such that $L\eta$ is invertible. It follows from Proposition 4.9.1 and its proof that we obtain an exact functor $\Gamma\colon \mathcal{T} \to \mathcal{T}$ by completing for each X in $\mathcal{T}$ the morphism ηX to an exact triangle

$$(4.11.1) \qquad\qquad \Gamma X \xrightarrow{\theta X} X \xrightarrow{\eta X} LX \longrightarrow S(\Gamma X).$$
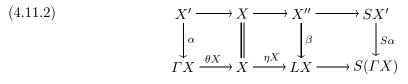
The exactness of Γ follows from Lemma 4.2.1. Observe that ΓX is L-acyclic and that LX is L-local. In fact, the exact triangle (4.11.1) is essentially determined by these properties. This is a consequence of the following basic properties of L and Γ.

Proposition 4.11.1. *The functors $L, \Gamma\colon \mathcal{T} \to \mathcal{T}$ have the following properties.*

   (1) *L induces an equivalence $\mathcal{T}/\operatorname{Ker} L \xrightarrow{\sim} \operatorname{Im} L$.*
   (2) *L induces a left adjoint for the inclusion $\operatorname{Im} L \to \mathcal{T}$.*
   (3) *Γ induces a right adjoint for the inclusion $\operatorname{Ker} L \to \mathcal{T}$.*

Proof. (1) is a reformulation of Corollary 2.5.3, and (2) follows from Corollary 2.4.2. (3) is an immediate consequence of the construction of Γ via Proposition 4.9.1. $\qquad\square$

Proposition 4.11.2. *Let $L\colon \mathcal{T} \to \mathcal{T}$ be an exact localization functor and X an object in $\mathcal{C}$. Given any exact triangle $X' \to X \to X'' \to SX'$ with X' L-acyclic and X'' L-local, there are unique isomorphisms α and β making the following diagram commutative.*

(4.11.2)
$$
\begin{array}{ccccccc}
X' & \longrightarrow & X & \longrightarrow & X'' & \longrightarrow & SX' \\
\downarrow{\scriptstyle\alpha} & & \| & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle S\alpha} \\
\Gamma X & \xrightarrow{\theta X} & X & \xrightarrow{\eta X} & LX & \longrightarrow & S(\Gamma X)
\end{array}
$$

Proof. The morphism θX induces a bijection $\mathcal{T}(X', \theta X)$ since X' is acyclic. Thus $X' \to X$ factors uniquely through θX via a morphism $\alpha\colon X' \to \Gamma X$. An application of (TR3) gives a morphism $\beta\colon X'' \to LX$ making the diagram (4.11.2) commutative. Now apply L to this diagram. Then $L\beta$ is an isomorphism since $LX' = 0 = L\Gamma X$, and $L\beta$ is isomorphic to β since X'' and LX are L-local. Thus β is an isomorphism, and therefore α is an isomorphism. $\qquad\square$

4.12. **Localization versus colocalization.** For exact functors on triangulated categories, we have the following symmetry principle relating localization and colocalization.

Proposition 4.12.1. *Let $\mathcal{T}$ be a triangulated category.*
  (1) *Suppose $L\colon \mathcal{T} \to \mathcal{T}$ is an exact localization functor and $\Gamma\colon \mathcal{T} \to \mathcal{T}$ the functor which is defined in terms of the exact triangle (4.11.1). Then Γ is an exact colocalization functor with $\operatorname{Ker}\Gamma = \operatorname{Im}L$ and $\operatorname{Im}\Gamma = \operatorname{Ker}L$.*
  (2) *Suppose $\Gamma\colon \mathcal{T} \to \mathcal{T}$ is an exact colocalization functor and $L\colon \mathcal{T} \to \mathcal{T}$ the functor which is defined in terms of the exact triangle (4.11.1). Then L is an exact localization functor with $\operatorname{Ker}L = \operatorname{Im}\Gamma$ and $\operatorname{Im}L = \operatorname{Ker}\Gamma$.*

Proof. It suffices to prove (1) because (2) is the dual statement. So let $L\colon \mathcal{T} \to \mathcal{T}$ be an exact localization functor. It follows from the construction of Γ that it is of the form $\Gamma = I \circ I_\rho$ where I_ρ denotes a right adjoint of the fully faithful inclusion $I\colon \operatorname{Ker}L \to \mathcal{T}$. Thus Γ is a colocalization functor by Corollary 2.4.2. The exactness of Γ follows from Lemma 4.2.1, and the identities $\operatorname{Ker}\Gamma = \operatorname{Im}L$ and $\operatorname{Im}\Gamma = \operatorname{Ker}L$ are easily derived from the exact triangle (4.11.1). $\qquad\square$

4.13. **Recollements.** A *recollement* is by definition a diagram of exact functors

(4.13.1)
$$
\mathcal{T}' \; \underset{\underset{I_\lambda}{\longleftarrow}}{\overset{\overset{I_\rho}{\longleftarrow}}{\xrightarrow{I}}} \; \mathcal{T} \; \underset{\underset{Q_\lambda}{\longleftarrow}}{\overset{\overset{Q_\rho}{\longleftarrow}}{\xrightarrow{Q}}} \; \mathcal{T}''
$$

satisfying the following conditions.
  (1) I_λ is a left adjoint and I_ρ a right adjoint of I.
  (2) Q_λ is a left adjoint and Q_ρ a right adjoint of Q.
  (3) $I_\lambda I \cong \operatorname{Id}\mathcal{T}' \cong I_\rho I$ and $QQ_\rho \cong \operatorname{Id}\mathcal{T}'' \cong QQ_\lambda$.
  (4) $\operatorname{Im}I = \operatorname{Ker}Q$.

Note that the isomorphisms in (3) are supposed to be the adjunction morphisms resulting from (1) and (2).

A recollement gives rise to various localization and colocalization functors for $\mathcal{T}$. First observe that the functors I, Q_λ, and Q_ρ are fully faithful; see Proposition 2.3.1. Therefore $Q_\rho Q$ and II_λ are localization functors and $Q_\lambda Q$ and II_ρ are colocalization

functors. This follows from Corollary 2.4.2. Note that the localization functor $L = Q_\rho Q$ has the additional property that the inclusion $\operatorname{Ker} L \to \mathcal{T}$ admits a left adjoint. Moreover, L determines the recollement up to an equivalence.

Proposition 4.13.1. *Let $L\colon \mathcal{T} \to \mathcal{T}$ be an exact localization functor and suppose that the inclusion $\operatorname{Ker} L \to \mathcal{T}$ admits a left adjoint. Then L induces a recollement of the following form.*

$$\operatorname{Ker} L \underset{\xleftarrow{\hspace{2em}}}{\overset{\xrightarrow{\ \text{inc}\ }}{\rightleftarrows}} \mathcal{T} \underset{\xleftarrow{\hspace{2em}}}{\overset{\xleftarrow{\hspace{2em}}}{\rightleftarrows}} \operatorname{Im} L$$

Moreover, any recollement for $\mathcal{T}$ is, up to equivalences, of this form for some exact localization functor $L\colon \mathcal{T} \to \mathcal{T}$.

Proof. We apply Proposition 4.9.1 and its dual assertion. Observe first that any localization functor $L\colon \mathcal{T} \to \mathcal{T}$ induces the following diagram.

$$\operatorname{Ker} L \underset{\xrightarrow{I=\text{inc}}}{\overset{I_\rho=\Gamma}{\longleftarrow}} \mathcal{T} \underset{\xrightarrow{Q=L}}{\overset{Q_\rho=\text{inc}}{\longleftarrow}} \operatorname{Im} L$$

The functor I admits a left adjoint if and only if Q admits a left adjoint. Thus the diagram can be completed to a recollement if and only if the inclusion I admits a left adjoint.

Suppose now there is given a recollement of the form (4.13.1). Then $L = Q_\rho Q$ is a localization functor and the inclusion $\operatorname{Ker} L \to \mathcal{T}$ admits a left adjoint. The functor I induces an equivalence $\mathcal{T}' \xrightarrow{\sim} \operatorname{Ker} L$ and Q_ρ induces an equivalence $\mathcal{T}'' \xrightarrow{\sim} \operatorname{Im} L$. It is straightforward to formulate and check the various compatibilities of these equivalences. $\qquad\square$

As a final remark, let us mention that for any recollement of the form (4.13.1), the functors Q_λ and Q_ρ provide two (in general different) embeddings of $\mathcal{T}''$ into $\mathcal{T}$. If we identify $\mathcal{T}' = \operatorname{Im} I$, then Q_ρ identifies $\mathcal{T}''$ with $(\mathcal{T}')^\perp$ and Q_λ identifies $\mathcal{T}''$ with $^\perp(\mathcal{T}')$; see Proposition 4.10.1.

4.14. **Example: The derived category of a module category.** Let A be an associative ring. We denote by $\mathbf{K}(\operatorname{Mod} A)$ the category of chain complexes of A-modules whose morphisms are the homotopy classes of chain maps. The functor $H^n\colon \mathbf{K}(\operatorname{Mod} A) \to \operatorname{Mod} A$ taking the cohomology of a complex in degree n is cohomological. A morphism ϕ is called *quasi-isomorphism* if $H^n\phi$ is an isomorphism for all $n \in \mathbb{Z}$, and we denote the set of all quasi-isomorphisms by qis. Then

$$\mathbf{D}(A) := \mathbf{D}(\operatorname{Mod} A) := \mathbf{K}(\operatorname{Mod} A)[\operatorname{qis}^{-1}]$$

is by definition the *derived category* of $\operatorname{Mod} A$. The kernel of the quotient functor $Q\colon \mathbf{K}(\operatorname{Mod} A) \to \mathbf{D}(\operatorname{Mod})$ is the full subcategory $\mathbf{K}_{\mathrm{ac}}(\operatorname{Mod} A)$ which is formed by all acyclic complexes. Note that Q admits a left adjoint Q_λ taking each complex to its K-projective resolution and a right adjoint Q_ρ taking each complex to its K-injective resolution. Thus we obtain the following recollement.

$$(4.14.1) \qquad \mathbf{K}_{\mathrm{ac}}(\operatorname{Mod} A) \underset{\xleftarrow{\hspace{2em}}}{\overset{\xrightarrow{\ \text{inc}\ }}{\rightleftarrows}} \mathbf{K}(\operatorname{Mod} A) \underset{\xleftarrow[\ Q_\lambda\ ]{}}{\overset{\xleftarrow[\ Q\ ]{Q_\rho}}{}} \mathbf{D}(\operatorname{Mod} A)$$

It follows that for each pair of chain complexes X, Y the set of morphisms $\mathbf{D}(\operatorname{Mod} A)(X, Y)$ is small, since Q_λ induces a bijection with $\mathbf{K}(\operatorname{Mod} A)(Q_\lambda X, Q_\lambda Y)$. The adjoints of Q are discussed in more detail in Section 5.8.

4.15. **Example: A derived category without small morphism sets.** For any abelian category $\mathcal{A}$, the *derived category* $\mathbf{D}(\mathcal{A})$ is by definition $\mathbf{K}(\mathcal{A})[\text{qis}^{-1}]$. Here, $\mathbf{K}(\mathcal{A})$ denotes the category of chain complexes in $\mathcal{A}$ whose morphisms are the homotopy classes of chain maps, and qis denotes the set of quasi-isomorphisms. Let us identify objects in $\mathcal{A}$ with chain complexes concentrated in degree zero.

We give an example of an abelian category $\mathcal{A}$ and an object X in $\mathcal{A}$ such that the set $\operatorname{Ext}^1_{\mathcal{A}}(X, X) \cong \mathbf{D}(\mathcal{A})(X, SX)$ is not small. This example is taken from Freyd [14, pp. 131] and has been pointed out to me by Neeman.

Let U denote the set of all cardinals of small sets. This set is not small. Consider the free associative $\mathbb{Z}$-algebra $\mathbb{Z}\langle U \rangle$ which is generated by the elements of U. Now let $\mathcal{A} = \operatorname{Mod} A$ denote the category of A-modules, where it is assumed that the underlying set of each module is small. Let $\mathbb{Z}$ denote the trivial A-module, that is, $zu = 0$ for all $z \in \mathbb{Z}$ and $u \in U$. We claim that the set $\operatorname{Ext}^1_{\mathcal{A}}(\mathbb{Z}, \mathbb{Z})$ is not small. To see this, define for each $u \in U$ an A-module $E_u = \mathbb{Z} \oplus \mathbb{Z}$ by

$$(z_1, z_2)x = \begin{cases} (z_2, 0) & \text{if } x = u, \\ (0, 0) & \text{if } x \neq u, \end{cases}$$

where $(z_1, z_2) \in E_u$ and $x \in U$. Then we have short exact sequences $0 \to \mathbb{Z} \xrightarrow{\begin{bmatrix} 1 \\ 0 \end{bmatrix}} E_u \xrightarrow{[0\ 1]} \mathbb{Z} \to 0$ which yield pairwise different elements of $\operatorname{Ext}^1_{\mathcal{A}}(\mathbb{Z}, \mathbb{Z})$ as u runs though the elements in U.

4.16. **Example: The recollement induced by an idempotent.** Recollements can be defined for abelian categories in the same way as for triangulated categories. A typical example arises for any module category from an idempotent element of the underlying ring.

Let A be an associative ring and $e^2 = e \in A$ an idempotent. Then the functor $F \colon \operatorname{Mod} A \to \operatorname{Mod} eAe$ taking a module M to Me and restriction along $p \colon A \to A/AeA$ induce the following recollement.

$$\operatorname{Mod} A/AeA \xleftrightarrow{\quad p_* \quad} \operatorname{Mod} A \xleftrightarrow[\substack{-\otimes_{eAe} eA}]{\substack{\operatorname{Hom}_{eAe}(Ae, -) \\ F}} \operatorname{Mod} eAe$$

Note that we can describe adjoints of F since

$$F = \operatorname{Hom}_A(eA, -) = - \otimes_A Ae.$$

The recollement for $\operatorname{Mod} A$ induces the following recollement of triangulated categories for $\mathbf{D}(A)$.

$$\operatorname{Ker} \mathbf{D}(F) \xleftrightarrow{\quad \text{inc} \quad} \mathbf{D}(A) \xleftrightarrow[\substack{-\otimes^{\mathbf{L}}_{eAe} eA}]{\substack{\mathbf{R}\operatorname{Hom}_{eAe}(Ae, -) \\ \mathbf{D}(F)}} \mathbf{D}(eAe)$$

The functor F is exact and $\mathbf{D}(F)$ takes by definition a complex X to FX. The functor $\mathbf{D}(p_*)\colon \mathbf{D}(A/AeA) \to \mathbf{D}(A)$ identifies $\mathbf{D}(A/AeA)$ with $\operatorname{Ker}\mathbf{D}(F)$ if and only if $\operatorname{Tor}_i^A(A/AeA, A/AeA) = 0$ for all $i > 0$.

4.17. **Notes.** Triangulated categories were introduced independently in algebraic geometry by Verdier in his thèse [48], and in algebraic topology by Puppe [38]. Grothendieck and his school used the formalism of triangulated and derived categories for studying homological properties of abelian categories. Early examples are Grothendieck duality and local cohomology for categories of sheaves. The basic example of a triangulated category from topology is the stable homotopy category.

Localizations of triangulated categories are discussed in Verdier's thèse [48]. In particular, he introduced the localization (or *Verdier quotient*) of a triangulated category with respect to a triangulated subcategory. In the context of stable homotopy theory, it is more common to think of localization functors as endofunctors; see for instance the work of Bousfield [8], which explains the term *Bousfield localization*. The standard reference for recollements is [6]. Resolutions of unbounded complexes were first studied by Spaltenstein in [44]; see also [5].

5. Localization via Brown representability

5.1. **Brown representatbility.** Let $\mathcal{T}$ be a triangulated category and suppose that $\mathcal{T}$ has small coproducts. A *localizing subcategory* of $\mathcal{T}$ is by definition a thick subcategory which is closed under taking small coproducts. A localizing subcategory of $\mathcal{T}$ is *generated* by a fixed set of objects if it is the smallest localizing subcategory of $\mathcal{T}$ which contains this set.

We say that $\mathcal{T}$ is *perfectly generated* by some small set $\mathcal{S}$ of objects of $\mathcal{T}$ provided the following holds.

(PG1) There is no proper localizing subcategory of $\mathcal{T}$ which contains $\mathcal{S}$.

(PG2) Given a family $(X_i \to Y_i)_{i \in I}$ of morphisms in $\mathcal{T}$ such that the induced map $\mathcal{T}(C, X_i) \to \mathcal{T}(C, Y_i)$ is surjective for all $C \in \mathcal{S}$ and $i \in I$, the induced map

$$\mathcal{T}(C, \coprod_i X_i) \longrightarrow \mathcal{T}(C, \coprod_i Y_i)$$

is surjective.

We say that a triangulated category $\mathcal{T}$ with small products is *perfectly cogenerated* if $\mathcal{T}^{\mathrm{op}}$ is perfectly generated.

Theorem 5.1.1. *Let $\mathcal{T}$ be a triangulated category with small coproducts and suppose $\mathcal{T}$ is perfectly generated.*

(1) *A functor $F\colon \mathcal{T}^{\mathrm{op}} \to \mathrm{Ab}$ is cohomological and sends small coproducts in $\mathcal{T}$ to products if and only if $F \cong \mathcal{T}(-, X)$ for some object X in $\mathcal{T}$.*

(2) *An exact functor $\mathcal{T} \to \mathcal{U}$ between triangulated categories preserves small coproducts if and only if it has a right adjoint.*

Proof. For a proof of (1) see [24, Theorem A]. To prove (2), suppose that F preserves small coproducts. Then one defines the right adjoint $G\colon \mathcal{U} \to \mathcal{T}$ by sending an object X in $\mathcal{U}$ to the object in $\mathcal{T}$ representing $\mathcal{U}(F-, X)$. Thus $\mathcal{U}(F-, X) \cong \mathcal{T}(-, GX)$. Conversely, given a right adjoint of F, it is automatic that F preserves small coproducts. $\square$

Remark 5.1.2. (1) In the presence of (PG2), condition (PG1) is equivalent to the following: For an object X in $\mathcal{T}$, we have $X = 0$ if $\mathcal{T}(S^n C, X) = 0$ for all $C \in \mathcal{S}$ and $n \in \mathbb{Z}$.

(2) A perfectly generated triangulated category $\mathcal{T}$ has small products. In fact, Brown representability implies that for any family of objects X_i in $\mathcal{T}$ the functor $\prod_i \mathcal{T}(-, X_i)$ is represented by an object in $\mathcal{T}$.

5.2. **Localization functors via Brown representability.** The existence of localization functors is basically equivalent to the existence of certain right adjoints; see Proposition 4.9.1. We combine this observation with Brown's representability theorem and obtain the following.

Proposition 5.2.1. *Let $\mathcal{T}$ be a triangulated category which admits small coproducts and fix a localizing subcategory $\mathcal{S}$.*

(1) *Suppose $\mathcal{S}$ is perfectly generated. Then there exists an exact localization functor $L \colon \mathcal{T} \to \mathcal{T}$ with $\operatorname{Ker} L = \mathcal{S}$.*

(2) *Suppose $\mathcal{T}$ is perfectly generated. Then there exists an exact localization functor $L \colon \mathcal{T} \to \mathcal{T}$ with $\operatorname{Ker} L = \mathcal{S}$ if and only if the morphisms between any two objects in $\mathcal{T}/\mathcal{S}$ form a small set.*

Proof. The existence of a localization functor L with $\operatorname{Ker} L = \mathcal{S}$ is equivalent to the existence of a right adjoint for the inclusion $\mathcal{S} \to \mathcal{T}$, and it is equivalent to the existence of a right adjoint for the quotient functor $\mathcal{T} \to \mathcal{T}/\mathcal{S}$. Both functors preserve small coproducts since $\mathcal{S}$ is closed under taking small coproducts; see Proposition 3.5.1. Now apply Theorem 5.1.1 for the existence of right adjoints. □

5.3. **Compactly generated triangulated categories.** Let $\mathcal{T}$ be a triangulated category with small coproducts. An object X in $\mathcal{T}$ is called *compact* (or *small*) if every morphism $X \to \coprod_{i \in I} Y_i$ in $\mathcal{T}$ factors through $\coprod_{i \in J} Y_i$ for some finite subset $J \subseteq I$. Note that X is compact if and only if the representable functor $\mathcal{T}(X, -) \colon \mathcal{T} \to \mathrm{Ab}$ preserves small coproducts. The compact objects in $\mathcal{T}$ form a thick subcategory which we denote by $\mathcal{T}^c$.

The triangulated category $\mathcal{T}$ is called *compactly generated* if it is perfectly generated by a small set of compact objects. Observe that condition (PG2) is automatically satisfied if every object in $\mathcal{S}$ is compact.

A compactly generated triangulated category $\mathcal{T}$ is perfectly cogenerated. To see this, let $\mathcal{S}$ be a set of compact generators. Then the objects representing $\operatorname{Hom}_{\mathbb{Z}}(\mathcal{T}(C, -), \mathbb{Q}/\mathbb{Z})$, where C runs through all objects in $\mathcal{S}$, form a set of perfect cogenerators for $\mathcal{T}$.

The following proposition is a reformulation of Brown representability for compactly generated triangulated categories.

Proposition 5.3.1. *Let $F \colon \mathcal{T} \to \mathcal{U}$ be an exact functor between triangulated categories. Suppose that $\mathcal{T}$ has small coproducts and that $\mathcal{T}$ is compactly generated.*

(1) *The functor F admits a right adjoint if and only if F preserves small coproducts.*

(2) *The functor F admits a left adjoint if and only if F preserves small products.*

5.4. **Right adjoint functors preserving coproducts.** The following lemma provides a characterization of the fact that a right adjoint functor preserves small coproducts. This will be useful in the context of compactly generated categories.

Lemma 5.4.1. *Let* $F\colon \mathcal{T} \to \mathcal{U}$ *be an exact functor between triangulated categories which has a right adjoint* G.

(1) *If* G *preserves small coproducts, then* F *preserves compactness.*
(2) *If* F *preserves compactness and* $\mathcal{T}$ *is generated by compact objects, then* G *preserves small coproducts.*

Proof. Let X be an object in $\mathcal{T}$ and $(Y_i)_{i \in I}$ a family of objects in $\mathcal{U}$.
(1) We have

$$(5.4.1) \qquad \mathcal{U}(FX, \coprod_i Y_i) \cong \mathcal{T}(X, G(\coprod_i Y_i)) \cong \mathcal{T}(X, \coprod_i GY_i).$$

If X is compact, then the isomorphism shows that a morphism $FX \to \coprod_i Y_i$ factors through a finite coproduct. It follows that FX is compact.

(2) Let X be compact. Then the canonical morphism $\phi\colon \coprod_i GY_i \to G(\coprod_i Y_i)$ induces an isomorphism

$$\mathcal{T}(X, \coprod_i GY_i) \cong \coprod_i \mathcal{T}(X, GY_i) \cong \coprod_i \mathcal{U}(FX, Y_i) \cong \mathcal{U}(FX, \coprod_i Y_i) \cong \mathcal{T}(X, G(\coprod_i Y_i)),$$

where the last isomorphism uses that FX is compact. It is easily checked that the objects X' in $\mathcal{T}$ such that $\mathcal{T}(X', \phi)$ is an isomorphism form a localizing subcategory of $\mathcal{T}$. Thus ϕ is an isomorphism because the compact objects generate $\mathcal{T}$. □

5.5. **Localization functors preserving coproducts.** The following result provides a characterization of the fact that an exact localization functor L preserves small coproducts; in that case one calls L *smashing*. The example given below explains this terminology.

Proposition 5.5.1. *Let* $\mathcal{T}$ *be a category with small coproducts and* $L\colon \mathcal{T} \to \mathcal{T}$ *an exact localization functor. Then the following are equivalent.*

(1) *The functor* $L\colon \mathcal{T} \to \mathcal{T}$ *preserves small coproducts.*
(2) *The colocalization functor* $\Gamma\colon \mathcal{T} \to \mathcal{T}$ *with* $\operatorname{Ker} \Gamma = \operatorname{Im} L$ *preserves small coproducts.*
(3) *The right adjoint of the inclusion functor* $\operatorname{Ker} L \to \mathcal{T}$ *preserves small coproducts.*
(4) *The right adjoint of the quotient functor* $\mathcal{T} \to \mathcal{T}/\operatorname{Ker} L$ *preserves small coproducts.*
(5) *The subcategory* $\operatorname{Im} L$ *of all* L-*local objects is closed under taking small coproducts.*

If $\mathcal{T}$ *is perfectly generated, in addition the following is equivalent.*

(6) *There exists a recollement of the following form.*

$$(5.5.1) \qquad \operatorname{Im} L \overset{\xleftarrow{\quad}}{\underset{\xleftarrow{\quad}}{\xrightarrow{\ \text{inc}\ }}} \mathcal{T} \overset{\xleftarrow{\quad}}{\underset{\xleftarrow{\quad}}{\xrightarrow{\quad\quad}}} \operatorname{Ker} L$$

Proof. (1) $\Leftrightarrow$ (4) $\Leftrightarrow$ (5) follows from Proposition 2.7.1.

(1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) is easily deduced from the functorial triangle (4.11.1) relating L and Γ.

(5) $\Leftrightarrow$ (6): Assume that $\mathcal{T}$ is perfectly generated. Then we can apply Brown's representability theorem and consider the sequence

$$\operatorname{Im} L \xrightarrow{\ I\ } \mathcal{T} \xrightarrow{\ Q\ } \operatorname{Ker} L$$

where I denotes the inclusion and Q a right adjoint of the inclusion $\operatorname{Ker} L \to \mathcal{T}$. Note that Q induces an equivalence $\mathcal{T}/\operatorname{Im} L \xrightarrow{\sim} \operatorname{Ker} L$; see Propositions 4.11.1 and 4.12.1. The functors I and Q have left adjoints. Thus the pair (I, Q) gives rise to a recollement if and only if I and Q both admit right adjoints. It follows from Proposition 4.9.1 that this happens if and only if Q admits a right adjoint. Now Brown's representability theorem implies that this is equivalent to the fact that Q preserves small coproducts. And Proposition 3.5.1 shows that Q preserves small coproducts if and only if $\operatorname{Im} L$ is closed under taking small coproducts. This finishes the proof. $\square$

Remark 5.5.2. (1) The implication (6) $\Rightarrow$ (5) holds without any extra assumption on $\mathcal{T}$.

(2) Suppose an exact localization functor $L \colon \mathcal{T} \to \mathcal{T}$ preserves small coproducts. If $\mathcal{T}$ is compactly generated, then $\operatorname{Im} L$ is compactly generated. This follows from Lemma 5.4.1, because the left adjoint of the inclusion $\operatorname{Im} L \to \mathcal{T}$ sends the compact generators of $\mathcal{T}$ to compact generators for $\operatorname{Im} L$. A similar argument shows that $\operatorname{Im} L$ is perfectly generated provided that $\mathcal{T}$ is perfectly generated.

Example 5.5.3. Let $\mathcal{S}$ be the stable homotopy category of spectra and $\wedge$ its smash product. Then an exact localization functor $L \colon \mathcal{S} \to \mathcal{S}$ preserves small coproducts if and only if L is of the form $L = - \wedge E$ for some spectrum E. We sketch the argument. Let S denote the sphere spectrum. There exists a natural morphism $\eta X \colon X \wedge LS \to LX$ for each X in $\mathcal{S}$. Suppose that L preserves small coproducts. Then the subcategory of objects X in $\mathcal{S}$ such that ηX is invertible contains S and is closed under forming small coproduts and exact triangles. Thus $L = - \wedge E$ for $E = LS$.
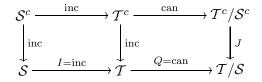
Let $L \colon \mathcal{T} \to \mathcal{T}$ be an exact localization functor which induces a recollement of the form (5.5.1). Then the sequence $\operatorname{Ker} L \to \mathcal{T} \to \operatorname{Im} L$ of left adjoint functors induces a sequence

$$(\operatorname{Ker} L)^c \longrightarrow \mathcal{T}^c \longrightarrow (\operatorname{Im} L)^c$$

of exact functors, by Lemma 5.4.1. This sequence is of some interest. The functor $(\operatorname{Ker} L)^c \to \mathcal{T}^c$ is fully faithful and identifies $(\operatorname{Ker} L)^c$ with a thick subcategory of $\mathcal{T}^c$, whereas the functor $\mathcal{T}^c \to (\operatorname{Im} L)^c$ shares some formal properties with a quotient functor. A typical example arises from finite localization; see Theorem 5.6.1. However, there are examples where $\mathcal{T}$ is compactly generated but $0 = (\operatorname{Ker} L)^c \subseteq \operatorname{Ker} L \neq 0$; see [25] for details.

5.6. **Finite localization.** A common type of localization for triangulated categories is finite localization. Here, we explain the basic result and refer to our discussion of well generated categories for a more general approach and further details.

Let $\mathcal{T}$ be a compactly generated triangulated category and suppose we have given a subcategory $\mathcal{S}' \subseteq \mathcal{T}^c$. Let $\mathcal{S}$ denote the localizing subcategory generated by $\mathcal{S}'$. Then $\mathcal{S}$ is compactly generated and therefore the inclusion functor $\mathcal{S} \to \mathcal{T}$ admits a right adjoint by Brown's representability theorem. In particular, we have a localization functor $L \colon \mathcal{T} \to \mathcal{T}$ with $\operatorname{Ker} L = \mathcal{S}$ and the morphisms between any pair of objects in $\mathcal{T}/\mathcal{S}$ form a small set; see Proposition 5.2.1. We observe that the compact objects of $\mathcal{S}$ identify with the smallest thick subcategory of $\mathcal{T}^c$ containing $\mathcal{S}'$. This follows from

Corollary 7.2.2. Thus we obtain the following commutative diagram of exact functors.

$$
\begin{array}{ccccc}
\mathcal{S}^c & \xrightarrow{\text{inc}} & \mathcal{T}^c & \xrightarrow{\text{can}} & \mathcal{T}^c/\mathcal{S}^c \\
\downarrow{\text{inc}} & & \downarrow{\text{inc}} & & \downarrow{J} \\
\mathcal{S} & \xrightarrow{I=\text{inc}} & \mathcal{T} & \xrightarrow{Q=\text{can}} & \mathcal{T}/\mathcal{S}
\end{array}
$$

Theorem 5.6.1. *Let $\mathcal{T}$ and $\mathcal{S}$ be as above. Then the quotient category $\mathcal{T}/\mathcal{S}$ is compactly generated. The induced exact functor $J\colon \mathcal{T}^c/\mathcal{S}^c \to \mathcal{T}/\mathcal{S}$ is fully faithful and the category $(\mathcal{T}/\mathcal{S})^c$ of compact objects equals the full subcategory consisting of all direct factors of objects in the image of J. Moreover, the inclusion $\mathcal{S}^\perp \to \mathcal{T}$ induces the following recollement.*

$$
\mathcal{S}^\perp \; \underset{\xrightarrow{\hspace{1.2cm}\text{inc}\hspace{1.2cm}}}{\overset{\xleftarrow{\hspace{2.4cm}}}{\xleftarrow{\hspace{2.4cm}}}} \; \mathcal{T} \; \underset{\xrightarrow{\hspace{2.4cm}}}{\overset{\xleftarrow{\hspace{2.4cm}}}{\xleftarrow{\hspace{2.4cm}}}} \; \mathcal{S}
$$

Proof. The inclusion I preserves compactness and therefore the right adjoint I_ρ preserves small coproducts by Lemma 5.4.1. Thus Q_ρ preserve small coproducts by Proposition 5.5.1, and therefore Q preserves compactness, again by Lemma 5.4.1. It follows that J induces a functor $\mathcal{T}^c/\mathcal{S}^c \to (\mathcal{T}/\mathcal{S})^c$. In particular, Q sends a set of compact generators of $\mathcal{T}$ to a set of compact generators for $\mathcal{T}/\mathcal{S}$.

Next we apply Lemma 4.7.1 to show that J is fully faithful. For this, one needs to check that every morphism from a compact object in $\mathcal{T}$ to an object in $\mathcal{S}$ factors through some object in $\mathcal{S}^c$. This follows from Theorem 7.2.1. The image of J is a full triangulated subcategory of $\mathcal{T}^c$ which generates $\mathcal{T}/\mathcal{S}$. Another application of Corollary 7.2.2 shows that every compact object of $\mathcal{T}/\mathcal{S}$ is a direct factor of some object in the image of J.
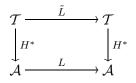
Let $L\colon \mathcal{T} \to \mathcal{T}$ denote the localization functor with $\operatorname{Ker} L = \mathcal{S}$. Then $\mathcal{S}^\perp$ equals the full subcategory of L-local objects. This subcategory is closed under small coproducts since $\mathcal{S}$ is generated by compact objects. Thus the existence of the recollement follows from Proposition 5.5.1. $\qquad\square$

5.7. Cohomological localization via localization of graded modules.
Let $\mathcal{T}$ be a triangulated category which admits small coproducts. Suppose that $\mathcal{T}$ is generated by a small set of compact objects. We fix a graded[1] ring Λ and a graded cohomological functor

$$
H^*\colon \mathcal{T} \longrightarrow \mathcal{A}
$$

into the category $\mathcal{A}$ of graded Λ-modules. Thus H^* is a functor which sends each exact triangle in $\mathcal{T}$ to an exact sequence in $\mathcal{A}$, and we have an isomorphism $H^* \circ S \cong T \circ H^*$ where T denotes the shift functor for $\mathcal{A}$. In addition, we assume that H^* preserves small products and coproducts.

Theorem 5.7.1. *Let $L\colon \mathcal{A} \to \mathcal{A}$ be an exact localization functor for the category $\mathcal{A}$ of graded Λ-modules. Then there exists an exact localization functor $\tilde{L}\colon \mathcal{T} \to \mathcal{T}$ such that the following square commutes up to a natural isomorphism.*

$$
\begin{array}{ccc}
\mathcal{T} & \xrightarrow{\tilde{L}} & \mathcal{T} \\
\downarrow{H^*} & & \downarrow{H^*} \\
\mathcal{A} & \xrightarrow{L} & \mathcal{A}
\end{array}
$$

---

[1]All graded rings and modules are graded over $\mathbb{Z}$. Morphisms between graded modules are degree zero maps.

More precisely, the adjunction morphisms $\mathrm{Id}\,\mathcal{A} \to L$ *and* $\mathrm{Id}\,\mathcal{T} \to \tilde{L}$ *induce for each* X
in $\mathcal{T}$ *the following isomorphisms.*

$$H^*\tilde{L}X \xrightarrow{\sim} L(H^*\tilde{L}X) = LH^*(\tilde{L}X) \xleftarrow{\sim} LH^*(X)$$

An object X *in* $\mathcal{T}$ *is* $\tilde{L}$*-acyclic if and only if* H^*X *is* L*-acyclic. If an object* X *in* $\mathcal{T}$ *is* $\tilde{L}$*-local, then* H^*X *is* L*-local. The converse holds, provided that* H^* *reflects isomorphisms.*

Proof. We recall that $\mathcal{T}$ is perfectly cogenerated because it is compactly generated. Thus Brown's representability theorem provides a compact object C in $\mathcal{T}$ such that

$$H^*X \cong \mathcal{T}(C, X)^* := \coprod_{i \in \mathbb{Z}} \mathcal{T}(C, S^i Y) \quad \text{for all} \quad X \in \mathcal{T}.$$

Now consider the essential image $\mathrm{Im}\,L$ of L which equals the full subcategory formed by all L-local objects in $\mathcal{A}$. Because L is exact, this subcategory is *coherent*, that is, for any exact sequence $X_1 \to X_2 \to X_3 \to X_4 \to X_5$ with $X_1, X_2, X_4, X_5 \in \mathcal{A}$, we have $X_3 \in \mathcal{A}$. This is an immediate consequence of the 5-lemma. In addition, $\mathrm{Im}\,L$ is closed under taking small products. The L-local objects form an abelian Grothendieck category and therefore $\mathrm{Im}\,L$ admits an injective cogenerator, say I; see [16]. Using again Brown's representability theorem, there exists $\tilde{I}$ in $\mathcal{T}$ such that

(5.7.1) $\mathcal{A}(H^*-, I) \cong \mathcal{T}(-, \tilde{I})$ and therefore $\mathcal{A}(H^*-, I)^* \cong \mathcal{T}(-, \tilde{I})^*.$

Now consider the subcategory $\mathcal{V}$ of $\mathcal{T}$ which is formed by all objects X in $\mathcal{T}$ such that H^*X is L-local. This is a triangulated subcategory which is closed under taking small products. Observe that $\tilde{I}$ belongs to $\mathcal{V}$. To prove this, take a free presentation

$$F_1 \longrightarrow F_0 \longrightarrow H^*C \longrightarrow 0$$

over Λ and apply $\mathcal{A}(-, I)^*$ to it. Using the isomorphism (5.7.1), we see that $H^*\tilde{I}$ belongs to $\mathrm{Im}\,L$ because $\mathrm{Im}\,L$ is coherent and closed under taking small products.

Now let $\mathcal{U}$ denote the smallest triangulated subcategory of $\mathcal{T}$ containing $\tilde{I}$ and closed under taking small products. Observe that $\mathcal{U} \subseteq \mathcal{V}$. We claim that $\mathcal{U}$ is perfectly cogenerated by $\tilde{I}$. Thus, given a family of morphisms $\phi_i \colon X_i \to Y_i$ in $\mathcal{U}$ such that $\mathcal{T}(Y_i, \tilde{I}) \to \mathcal{T}(X_i, \tilde{I})$ is surjective for all i, we need to show that $\mathcal{T}(\prod_i Y_i, \tilde{I}) \to \mathcal{T}(\prod_i X_i, \tilde{I})$ is surjective. We argue as follows. If $\mathcal{T}(Y_i, \tilde{I}) \to \mathcal{T}(X_i, \tilde{I})$ is surjective, then the isomorphism (5.7.1) implies that $H^*\phi_i$ is a monomorphism since I is an injective cogenerator for $\mathrm{Im}\,L$. Thus the product $\prod_i \phi_i \colon \prod_i X_i \to \prod_i Y_i$ induces a monomorphism $H^* \prod_i \phi_i = \prod_i H^*\phi_i$ and therefore $\mathcal{T}(\prod_i \phi_i, \tilde{I})$ is surjective. We conclude from Brown's representability theorem that the inclusion functor $G \colon \mathcal{U} \to \mathcal{T}$ has a left adjoint $F \colon \mathcal{T} \to \mathcal{U}$. Thus $\tilde{L} = G \circ F$ is a localization functor by Corollary 2.4.2.

Next we show that an object $X \in \mathcal{T}$ is $\tilde{L}$-acyclic if and only if H^*X is L-acyclic. This follows from Proposition 4.10.1 and the isomorphism (5.7.1), because we have

$$\tilde{L}X = 0 \iff \mathcal{T}(X, \tilde{I}) = 0 \iff \mathcal{A}(H^*X, I) = 0 \iff LH^*X = 0.$$

Now denote by $\eta\colon \operatorname{Id}\mathcal{A} \to L$ and $\tilde{\eta}\colon \operatorname{Id}\mathcal{T} \to \tilde{L}$ the adjunction morphisms and consider the following commutative square.

(5.7.2)
$$\begin{array}{ccc}
H^*X & \xrightarrow{\;\eta H^*X\;} & LH^*X \\
{\scriptstyle H^*\tilde{\eta}X}\big\downarrow & & \big\downarrow{\scriptstyle LH^*\tilde{\eta}X} \\
H^*\tilde{L}X & \xrightarrow{\;\eta H^*\tilde{L}X\;} & LH^*\tilde{L}X
\end{array}$$

We claim that $LH^*\tilde{\eta}X$ and $\eta H^*\tilde{L}X$ are invertible for each X in $\mathcal{T}$. The morphism $\tilde{\eta}X$ induces an exact triangle

$$X' \to X \xrightarrow{\tilde{\eta}X} \tilde{L}X \to SX'$$

with $\tilde{L}X' = 0 = \tilde{L}SX'$. Applying the cohomological functor LH^*, we see that $LH^*\tilde{\eta}X$ is an isomorphism, since $LH^*X' = 0 = LH^*SX'$. Thus $LH^*\tilde{\eta}$ is invertible. The morphism $\eta H^*\tilde{L}X$ is invertible because $H^*\tilde{L}X$ is L-local. This follows from the fact that $\tilde{L}X$ belongs to $\mathcal{U}$.

The commutative square (5.7.2) implies that $H^*\tilde{\eta}X$ is invertible if and only if ηH^*X is invertible. Thus if X is $\tilde{L}$-local, then H^*X is L-local. The converse holds if H^* reflects isomorphisms. $\qquad\square$

Remark 5.7.2. (1) The localization functor $\tilde{L}$ is essentially uniquely determined by H^* and L, because $\operatorname{Ker}\tilde{L} = \operatorname{Ker}LH^*$.

(2) Suppose that C is a generator of $\mathcal{T}$. If L preserves small coproducts, then it follows that $\tilde{L}$ preserves small coproducts. In fact, the assumption implies that $H^*\tilde{L}$ preserves small coproducts, since $LH^* \cong H^*\tilde{L}$. But H^* reflects isomorphisms because C is a generator of $\mathcal{T}$. Thus $\tilde{L}$ preserves small coproducts.

5.8. **Example: Resolutions of chain complexes.** Let A be an associative ring. Then the derived category $\mathbf{D}(A)$ of unbounded chain complexes of modules over A is compactly generated. A compact generator is the ring A, viewed as a complex concentrated in degree zero. Let us be more precise, because we want to give an explicit construction of $\mathbf{D}(A)$ which implies that the morphisms between any two objects in $\mathbf{D}(A)$ form a small set. Moreover, we combine Brown representability with Proposition 4.9.1 to provide descriptions of the adjoints Q_λ and Q_ρ of the quotient functor $Q\colon \mathbf{K}(\operatorname{Mod}A) \to \mathbf{D}(A)$ which appear in the recollement (4.14.1).

Denote by $\operatorname{Loc}A$ the localizing subcategory of $\mathbf{K}(\operatorname{Mod}A)$ which is generated by A. Then $\operatorname{Loc}A$ is a compactly generated triangulated category and $(\operatorname{Loc}A)^\perp = \mathbf{K}_{\mathrm{ac}}(\operatorname{Mod}A)$ since

$$\mathbf{K}(\operatorname{Mod}A)(A, S^n X) \cong H^n X.$$

Brown representability provides a right adjoint for the inclusion $\operatorname{Loc}A \to \mathbf{K}(\operatorname{Mod}A)$ and therefore the composite $F\colon \operatorname{Loc}A \xrightarrow{\mathrm{inc}} \mathbf{K}(\operatorname{Mod}A) \xrightarrow{\mathrm{can}} \mathbf{D}(A)$ is an equivalence by Proposition 4.9.1. The right adjoint of the inclusion $\operatorname{Loc}A \to \mathbf{K}(\operatorname{Mod}A)$ annihilates the acyclic complexes and induces therefore a functor $\mathbf{D}(A) \to \operatorname{Loc}A$ (which is a quasi-inverse for F). The composite with the inclusion $\operatorname{Loc}A \to \mathbf{K}(\operatorname{Mod}A)$ is the left adjoint Q_λ of Q and takes a complex to its *K-projective resolution*.

Now fix an injective cogenerator I for the category of A-modules, for instance $I = \operatorname{Hom}_{\mathbb{Z}}(A, \mathbb{Q}/\mathbb{Z})$. We denote by $\operatorname{Coloc}I$ the smallest thick subcategory of $\mathbf{K}(\operatorname{Mod}A)$

closed under small products and containing I. Then I is a perfect cogenerator for $\operatorname{Coloc} I$ and $^{\perp}(\operatorname{Coloc} I) = \mathbf{K}_{\mathrm{ac}}(\operatorname{Mod} A)$ since

$$\mathbf{K}(\operatorname{Mod} A)(S^n X, I) \cong \operatorname{Hom}_A(H^n X, I).$$

Brown representability provides a left adjoint for the inclusion $\operatorname{Coloc} I \to \mathbf{K}(\operatorname{Mod} A)$ and therefore the composite $G \colon \operatorname{Coloc} I \xrightarrow{\mathrm{inc}} \mathbf{K}(\operatorname{Mod} A) \xrightarrow{\mathrm{can}} \mathbf{D}(A)$ is an equivalence by Proposition 4.9.1. The left adjoint of the inclusion $\operatorname{Coloc} I \to \mathbf{K}(\operatorname{Mod} A)$ annihilates the acyclic complexes and induces therefore a functor $\mathbf{D}(A) \to \operatorname{Coloc} I$ (which is a quasi-inverse for G). The composition with the inclusion $\operatorname{Coloc} I \to \mathbf{K}(\operatorname{Mod} A)$ is the right adjoint Q_ρ of Q and takes a complex to its *K-injective resolution*.

5.9. **Example: Homological epimorphisms.** Let $f \colon A \to B$ be a ring homomorphism and $f_* \colon \mathbf{D}(B) \to \mathbf{D}(A)$ the functor given by restriction of scalars. Clearly, f_* preserves small products and coproducts. Thus Brown representability implies the existence of left and right adjoints for f_* since $\mathbf{D}(A)$ is compactly generated. For instance, the left adjoint is the derived tensor functor $- \otimes_A^{\mathbf{L}} B \colon \mathbf{D}(A) \to \mathbf{D}(B)$ which preserves compactness.

The functor f_* is fully faithful if and only if $(f_* -) \otimes_A^{\mathbf{L}} B \cong \operatorname{Id} \mathbf{D}(B)$ iff $B \otimes_A B \cong B$ and $\operatorname{Tor}_i^A(B, B) = 0$ for all $i > 0$. In that case f is called *homological epimorphism* and the exact functor $L \colon \mathbf{D}(A) \to \mathbf{D}(A)$ sending X to $f_*(X \otimes_A^{\mathbf{L}} B)$ is a localization functor.

Take for instance a commutative ring A and let $f \colon A \to S^{-1}A = B$ be the localization with respect to a multiplicatively closed subset $S \subseteq A$. Then the induced exact localization functor $L \colon \mathbf{D}(A) \to \mathbf{D}(A)$ takes a chain complex X to $S^{-1}X$. Note that L preserves small coproducts. In particular, L gives rise to the following recollement.

$$\mathbf{D}(B) \underset{-\otimes_A^{\mathbf{L}} B}{\overset{\mathbf{R}\operatorname{Hom}_A(B,-)}{\underset{\xleftarrow{\hspace{1cm}}}{\overset{\xleftarrow{\hspace{1cm}}}{\xrightarrow{\quad f_* \quad}}}}} \mathbf{D}(A) \underset{\xleftarrow{\hspace{1cm}}}{\overset{\xleftarrow{\hspace{1cm}}}{\xrightarrow{\hspace{1.5cm}}}} \mathcal{U}$$

The triangulated category $\mathcal{U}$ is equivalent to the kernel of $- \otimes_A^{\mathbf{L}} B$, and one can show that $\operatorname{Ker}(- \otimes_A^{\mathbf{L}} B)$ is the localizing subcategory of $\mathbf{D}(A)$ generated by the complexes of the form

$$\cdots \to 0 \to A \xrightarrow{x} A \to 0 \to \cdots \quad (x \in S).$$

5.10. **Notes.** The Brown representability theorem in homotopy theory is due to Brown [9]. Generalizations of the Brown representability theorem for triangulated categories can be found in work of Franke [13], Keller [21], and Neeman [31, 33]. The version used here is taken from [24]. The finite localization theorem for compactly generated triangulated categories is due to Neeman [30]; it is based on previous work of Bousfield, Ravenel, Thomason-Trobaugh, Yao, and others. The cohomological localization functors commuting with localization functors of graded modules have been used to set up local cohomology functors in [7].

6. Well generated triangulated categories

6.1. **Regular cardinals.** A cardinal α is called *regular* if α is not the sum of fewer than α cardinals, all smaller than α. For example, $\aleph_0$ is regular because the sum of finitely many finite cardinals is finite. Also, the successor κ^+ of every infinite cardinal κ

is regular. In particular, there are arbitrarily large regular cardinals. For more details on regular cardinals, see for instance [26].

6.2. **Localizing subcategories.** Let $\mathcal{T}$ be a triangulated category and α a regular cardinal. A coproduct in $\mathcal{T}$ is called α-*coproduct* if it has less than α factors. A full subcategory of $\mathcal{T}$ is called α-*localizing* if it is a thick subcategory and closed under taking α-coproducts. Given a subcategory $\mathcal{S} \subseteq \mathcal{T}$, we denote by $\mathrm{Loc}_\alpha\, \mathcal{S}$ the smallest α-localizing subcategory of $\mathcal{T}$ which contains $\mathcal{S}$. Note that $\mathrm{Loc}_\alpha\, \mathcal{S}$ is small provided that $\mathcal{S}$ is small.

A full subcategory of $\mathcal{T}$ is called *localizing* if it is a thick subcategory and closed under taking small coproducts. The smallest localizing subcategory containing a subcategory $\mathcal{S} \subseteq \mathcal{T}$ is $\mathrm{Loc}\, \mathcal{S} = \bigcup_\alpha \mathrm{Loc}_\alpha\, \mathcal{S}$ where α runs through all regular cardinals. We call $\mathrm{Loc}\, \mathcal{S}$ the localizing subcategory generated by $\mathcal{S}$.

6.3. **Well generated triangulated categories.** Let $\mathcal{T}$ be a triangulated category which admits small coproducts and fix a regular cardinal α. An object X in $\mathcal{T}$ is called α-*small* if every morphism $X \to \coprod_{i \in I} Y_i$ in $\mathcal{T}$ factors through $\coprod_{i \in J} Y_i$ for some subset $J \subseteq I$ with $\mathrm{card}\, J < \alpha$. The triangulated category $\mathcal{T}$ is called α-*well generated* if it is perfectly generated by a small set of α-small objects, and $\mathcal{T}$ is called *well generated* if it is β-well generated for some regular cardinal β.

Suppose $\mathcal{T}$ is α-well generated by a small set $\mathcal{S}$ of α-small objects. Given any regular cardinal $\beta \geq \alpha$, we denote by $\mathcal{T}^\beta$ the β-localizing subcategory $\mathrm{Loc}_\beta\, \mathcal{S}$ generated by $\mathcal{S}$ and call the objects of $\mathcal{T}^\beta$ β-*compact*. Choosing a representative for each isomorphism class, one can show that the β-compact objects form a small set of β-small perfect generators for $\mathcal{T}$. Moreover, $\mathcal{T}^\beta$ does not depend on the choice of $\mathcal{S}$. For a proof we refer to [23, Lemma 5]; see also Proposition 6.8.1 and Remark 6.10.2. Note that $\mathcal{T} = \bigcup_\beta \mathcal{T}^\beta$, where β runs through all regular cardinals greater or equal than α, because $\bigcup_\beta \mathcal{T}^\beta$ is a triangulated subcategory containing $\mathcal{S}$ and closed under small coproducts.

Remark 6.3.1. The α-small objects of $\mathcal{T}$ form an α-localizing subcategory.

Example 6.3.2. A triangulated category $\mathcal{T}$ is $\aleph_0$-well generated if and only if $\mathcal{T}$ is compactly generated. In that case $\mathcal{T}^{\aleph_0} = \mathcal{T}^c$.

Example 6.3.3. Let $\mathcal{A}$ be the category of sheaves of abelian groups on a non-compact, connected manifold of dimension at least 1. Then the derived category $\mathbf{D}(\mathcal{A})$ of unbounded chain complexes is well generated, but the only compact object in $\mathbf{D}(\mathcal{A})$ is the zero object; see [34]. For more examples of well generated but not compactly generated triangulated categories, see [32].

6.4. **Filtered categories.** Let α be a regular cardinal. A category $\mathcal{C}$ is called α-*filtered* if the following holds.

(FIL1) There exists an object in $\mathcal{C}$.
(FIL2) For every family $(X_i)_{i \in I}$ of fewer than α objects there exists an object X with morphisms $X_i \to X$ for all i.
(FIL3) For every family $(\phi_i \colon X \to Y)_{i \in I}$ of fewer than α morphisms there exists a morphism $\psi \colon Y \to Z$ with $\psi \phi_i = \psi \phi_j$ for all i and j.

One drops the cardinal α and calls $\mathcal{C}$ *filtered* in case it is $\aleph_0$-filtered.

Given a functor $F\colon \mathcal{C} \to \mathcal{D}$, we use the term α-*filtered colimit* for the colimit $\underset{X\in\mathcal{C}}{\mathrm{colim}}\, FX$ provided that $\mathcal{C}$ is a small α-filtered category.

Lemma 6.4.1. *Let $i\colon \mathcal{C}' \to \mathcal{C}$ be a fully faithful functor with $\mathcal{C}$ a small α-filtered category. Suppose that i is cofinal in the sense that for any $X \in \mathcal{C}$ there is an object $Y \in \mathcal{C}'$ and a morphism $X \to iY$. Then $\mathcal{C}'$ is a small α-filtered category, and for any functor $F\colon \mathcal{C} \to \mathcal{D}$ into a category which admits α-filtered colimits, the natural morphism*

$$\underset{Y\in\mathcal{C}'}{\mathrm{colim}}\, F(iY) \longrightarrow \underset{X\in\mathcal{C}}{\mathrm{colim}}\, FX$$

is an isomorphism.

Proof. See [19, Proposition 8.1.3]. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

A full subcategory $\mathcal{C}'$ of a small α-filtered category $\mathcal{C}$ is called *cofinal* if for any $X \in \mathcal{C}$ there is an object $Y \in \mathcal{C}'$ and a morphism $X \to Y$.

6.5. **Comma categories.** Let $\mathcal{T}$ be a triangulated category which admits small coproducts and fix a full subcategory $\mathcal{S}$. Given an object X in $\mathcal{T}$, let $\mathcal{S}/X$ denote the category whose objects are pairs (C, μ) with $C \in \mathcal{S}$ and $\mu \in \mathcal{T}(C, X)$. The morphisms $(C, \mu) \to (C', \mu')$ are the morphisms $\gamma\colon C \to C'$ in $\mathcal{T}$ making the following diagram commutative.

$$
\begin{array}{ccc}
C & \xrightarrow{\ \gamma\ } & C' \\
& {\scriptstyle \mu}\searrow \quad \swarrow{\scriptstyle \mu'} & \\
& X &
\end{array}
$$

Analogously, one defines for a morphism $\phi\colon X \to X'$ in $\mathcal{T}$ the category $\mathcal{S}/\phi$ whose objects are commuting squares in $\mathcal{T}$ of the form

$$
\begin{array}{ccc}
C & \xrightarrow{\ \gamma\ } & C' \\
{\scriptstyle \mu}\downarrow & & \downarrow{\scriptstyle \mu'} \\
X & \xrightarrow{\ \phi\ } & X'
\end{array}
$$

with $C, C' \in \mathcal{S}$.

Lemma 6.5.1. *Let α be a regular cardinal and $\mathcal{S}$ an α-localizing subcategory of $\mathcal{T}$. Then the categories $\mathcal{S}/X$ and $\mathcal{S}/\phi$ are α-filtered for each object X and each morphism ϕ in $\mathcal{T}$.*

Proof. Straightforward. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

6.6. **The comma category of an exact triangle.** Let $\mathcal{T}$ be a triangulated category. We consider the category of pairs (ϕ_1, ϕ_2) of composable morphisms $X_1 \xrightarrow{\phi_1} X_2 \xrightarrow{\phi_2} X_3$ in $\mathcal{T}$. A morphism $\mu\colon (\phi_1, \phi_2) \to (\phi_1', \phi_2')$ is a triple $\mu = (\mu_1, \mu_2, \mu_3)$ of morphisms in $\mathcal{T}$ making the following diagram commutative.
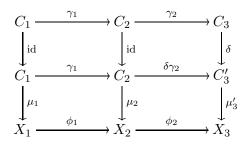
$$
\begin{array}{ccccc}
X_1 & \xrightarrow{\ \phi_1\ } & X_2 & \xrightarrow{\ \phi_2\ } & X_3 \\
{\scriptstyle \mu_1}\downarrow & & {\scriptstyle \mu_2}\downarrow & & \downarrow{\scriptstyle \mu_3} \\
X_1' & \xrightarrow{\ \phi_1'\ } & X_2' & \xrightarrow{\ \phi_2'\ } & X_3'
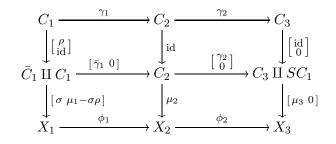\end{array}
$$

A pair (ϕ_1, ϕ_2) of composable morphisms is called *exact* if it fits into an exact triangle $X_1 \xrightarrow{\phi_1} X_2 \xrightarrow{\phi_2} X_3 \xrightarrow{\phi_3} SX_1$.

Lemma 6.6.1. *Let* $\mu\colon (\gamma_1, \gamma_2) \to (\phi_1, \phi_2)$ *be a morphism between pairs of composable morphisms and suppose that* (ϕ_1, ϕ_2) *is exact. Then* μ *factors through an exact pair of composable morphisms which belong to the smallest full triangulated subcategory containing* γ_1 *and* γ_2.

Proof. We proceed in two steps. The first step provides a factorization of μ through a pair (γ_1', γ_2') of composable morphisms such that $\gamma_2'\gamma_1' = 0$. To achieve this, complete γ_1 to an exact triangle $C_1 \xrightarrow{\gamma_1} C_2 \xrightarrow{\bar{\gamma}_2} \bar{C}_3 \to SC_1$. Note that $\phi_2\mu_2$ factors through $\bar{\gamma}_2$. Now complete $\left[\begin{smallmatrix} \gamma_2 \\ \bar{\gamma}_2 \end{smallmatrix} \right]$ to an exact triangle $C_2 \xrightarrow{\left[\begin{smallmatrix} \gamma_2 \\ \bar{\gamma}_2 \end{smallmatrix} \right]} C_3 \amalg \bar{C}_3 \xrightarrow{[\delta\ \bar{\delta}]} C_3' \to SC_2$ and observe that μ_3 factors through δ via a morphism $\mu_3'\colon C_3' \to X_3$. Thus we obtain the following factorization of μ with $(\delta\gamma_2)\gamma_1 = -\bar{\delta}\bar{\gamma}_2\gamma_1 = 0$.

For the second step we may assume that $\gamma_2\gamma_1 = 0$. We complete γ_2 to an exact triangle $\bar{C}_1 \xrightarrow{\bar{\gamma}_1} C_2 \xrightarrow{\gamma_2} C_3 \to S\bar{C}_1$. Clearly, γ_1 factors through $\bar{\gamma}_1$ via a morphism $\rho\colon C_1 \to \bar{C}_1$ and $\mu_2\bar{\gamma}_1$ factors through ϕ_1 via a morphism $\sigma\colon \bar{C}_1 \to X_1$. Thus we obtain the following factorization of μ
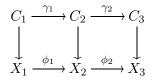

where the middle row fits into an exact triangle. $\qquad\square$

The following statement is a reformulation of the previous one in terms of cofinal subcategories.

Proposition 6.6.2. *Let* $\mathcal{T}$ *be a triangulated category and* $\mathcal{S}$ *a full triangulated subcategory. Suppose that* $X_1 \xrightarrow{\phi_1} X_2 \xrightarrow{\phi_2} X_3 \xrightarrow{\phi_3} SX_1$ *is an exact triangle in* $\mathcal{T}$ *and denote by* $\mathcal{S}/(\phi_1, \phi_2)$ *the category whose objects are the commutative diagrams in* $\mathcal{T}$ *of the following*

form.

$$C_1 \xrightarrow{\gamma_1} C_2 \xrightarrow{\gamma_2} C_3$$
$$\downarrow \qquad \downarrow \qquad \downarrow$$
$$X_1 \xrightarrow{\phi_1} X_2 \xrightarrow{\phi_2} X_3$$

such each C_i belongs to $\mathcal{S}$. Then the full subcategory formed by the diagrams such that there exists an exact triangle $C_1 \xrightarrow{\gamma_1} C_2 \xrightarrow{\gamma_2} C_3 \xrightarrow{\gamma_3} SC_1$ is a cofinal subcategory of $\mathcal{S}/(\phi_1, \phi_2)$.

6.7. **A Kan extension.** Let $\mathcal{T}$ be a triangulated category with small coproducts and $\mathcal{S}$ a small full subcategory. Suppose that the objects of $\mathcal{S}$ are α-small and that $\mathcal{S}$ is closed under α-coproducts. We denote by $\mathrm{Add}_\alpha(\mathcal{S}^{\mathrm{op}}, \mathrm{Ab})$ the category of α-product preserving functors $\mathcal{S}^{\mathrm{op}} \to \mathrm{Ab}$. This is a locally presentable abelian category in the sense of [17] and we refer to the Appendix B for basic facts on locally presentable categories. Depending on the choice of $\mathcal{S}$, we can think of $\mathrm{Add}_\alpha(\mathcal{S}^{\mathrm{op}}, \mathrm{Ab})$ as a locally presentable approximation of the triangulated category $\mathcal{T}$. In order to make this precise, we need to introduce various functors.

Let $h_{\mathcal{T}} \colon \mathcal{T} \to A(\mathcal{T})$ denote the abelianization of $\mathcal{T}$; see Appendix A. Sometimes we write $\widehat{\mathcal{T}}$ instead of $A(\mathcal{T})$. The inclusion functor $f \colon \mathcal{S} \to \mathcal{T}$ induces a functor

$$f_* \colon A(\mathcal{T}) \longrightarrow \mathrm{Add}_\alpha(\mathcal{S}^{\mathrm{op}}, \mathrm{Ab}), \quad X \mapsto A(\mathcal{T})((h_{\mathcal{T}} \circ f)-, X),$$

and we observe that the composite

$$\mathcal{T} \xrightarrow{h_{\mathcal{T}}} A(\mathcal{T}) \xrightarrow{f_*} \mathrm{Add}_\alpha(\mathcal{S}^{\mathrm{op}}, \mathrm{Ab})$$

is the restricted Yoneda functor sending each $X \in \mathcal{T}$ to $\mathcal{T}(-, X)|_{\mathcal{S}}$.

The next proposition discusses a left adjoint for f_*. To this end, we denote for any category $\mathcal{C}$ by $h_{\mathcal{C}}$ the Yoneda functor sending X in $\mathcal{C}$ to $\mathcal{C}(-, X)$.

Proposition 6.7.1. *The functor f_* admits a left adjoint f^* which makes the following diagram commutative.*

$$\begin{array}{ccc} \mathcal{S} & \xrightarrow{h_{\mathcal{S}}} & \mathrm{Add}_\alpha(\mathcal{S}^{\mathrm{op}}, \mathrm{Ab}) \\ \downarrow{\scriptstyle f=\mathrm{inc}} & & \downarrow{\scriptstyle f^*} \\ \mathcal{T} & \xrightarrow{h_{\mathcal{T}}} & A(\mathcal{T}) \end{array}$$

Moreover, the functor f^* has the following properties.

- (1) *f^* is fully faithful and identifies $\mathrm{Add}_\alpha(\mathcal{S}^{\mathrm{op}}, \mathrm{Ab})$ with the full subcategory formed by all colimits of objects in $\{\mathcal{T}(-, X) \mid X \in \mathcal{S}\}$.*
- (2) *f_* preserves small coproducts if and only if (PG2) holds for $\mathcal{S}$.*
- (3) *Suppose that $\mathcal{S}$ is a triangulated subcategory of $\mathcal{T}$. Then for X in $\mathcal{T}$, the adjunction morphism $f^* f_*(h_{\mathcal{T}} X) \to h_{\mathcal{T}} X$ identifies with the canonical morphism*

$$\underset{(C,\mu) \in \mathcal{S}/X}{\underrightarrow{\mathrm{colim}}} h_{\mathcal{T}} C \longrightarrow h_{\mathcal{T}} X.$$

Proof. The functor f^* is constructed as a left Kan extension. To explain this, it is convenient to identify $\mathrm{Add}_\alpha(\mathcal{S}^{\mathrm{op}}, \mathrm{Ab})$ with the category $\mathrm{Lex}_\alpha(\widehat{\mathcal{S}}^{\mathrm{op}}, \mathrm{Ab})$ of left exact functors

$\widehat{\mathcal{S}}^{\mathrm{op}} \to \mathrm{Ab}$ which preserve α-products. To be more precise, the Yoneda functor $h\colon \mathcal{S} \to \widehat{\mathcal{S}}$ induces an equivalence

$$\mathrm{Lex}_\alpha(\widehat{\mathcal{S}}^{\mathrm{op}}, \mathrm{Ab}) \xrightarrow{\sim} \mathrm{Add}_\alpha(\mathcal{S}^{\mathrm{op}}, \mathrm{Ab}), \quad F \mapsto F \circ h,$$

because every additive functor $\mathcal{S}^{\mathrm{op}} \to \mathrm{Ab}$ extends uniquely to a left exact functor $\widehat{\mathcal{S}}^{\mathrm{op}} \to \mathrm{Ab}$; see Lemma A.1.

Using this identification, the existence of a fully faithful left adjoint $\mathrm{Lex}_\alpha(\widehat{\mathcal{S}}^{\mathrm{op}}, \mathrm{Ab}) \to A(\mathcal{T})$ for f_* and its basic properties follow from Lemma B.6, because the inclusion $f\colon \mathcal{S} \to \mathcal{T}$ induces a fully faithful and right exact functor $\widehat{f}\colon \widehat{\mathcal{S}} \to \widehat{\mathcal{T}} = A(\mathcal{T})$. This functor preserves α-coproducts and identifies $\widehat{\mathcal{S}}$ with a full subcategory of α-presentable objects, since the objects from $\mathcal{S}$ are α-small in $\mathcal{T}$.

(2) Let $\Sigma = \Sigma(f_*)$ denote the set of morphisms of $A(\mathcal{T})$ which f_* makes invertible. It follows from Proposition 2.3.1 that f_* induces an equivalence

$$A(\mathcal{T})[\Sigma^{-1}] \xrightarrow{\sim} \mathrm{Lex}_\alpha(\widehat{\mathcal{S}}^{\mathrm{op}}, \mathrm{Ab}),$$

and therefore f_* preserves small coproducts if and only if Σ is closed under taking small coproducts, by Proposition 3.5.1. It is not hard to check that f_* is exact, and therefore a morphism in $A(\mathcal{T})$ belongs to Σ if and only if its kernel and cokernel are annihilated by f_*. Now observe that an object F in $A(\mathcal{T})$ with presentation $\mathcal{T}(-, X) \to \mathcal{T}(-, Y) \to F \to 0$ is annihilated by f_* if and only if $\mathcal{T}(C, X) \to \mathcal{T}(C, Y)$ is surjective for all $C \in \mathcal{S}$. It follows hat f_* preserves small coproducts if and only if (PG2) holds for $\mathcal{S}$.

(3) Let $F = f_*(h_\mathcal{T} X) = \mathcal{T}(-, X)|_\mathcal{S}$. Then Lemma B.7 implies that $F = \varinjlim_{(C,\mu)\in \mathcal{S}/X} h_\mathcal{S} C$,

since $\mathcal{S}/F = \mathcal{S}/X$. Thus $f^* F = \varinjlim_{(C,\mu)\in \mathcal{S}/X} h_\mathcal{T} C$. $\square$

Corollary 6.7.2. *Let $\mathcal{T}$ be a triangulated category with small coproducts. Suppose $\mathcal{T}$ is α-well generated and denote by $\mathcal{T}^\alpha$ the full subcategory formed by all α-compact objects. Then the functor $\mathcal{T} \to A(\mathcal{T})$ taking an object X to*

$$\varinjlim_{(C,\mu)\in \mathcal{T}^\alpha/X} \mathcal{T}(-, C)$$

preserves small coproducts.

6.8. **A criterion for well generatedness.** Let $\mathcal{T}$ be a triangulated category which admits small coproducts. The following result provides a useful criterion for $\mathcal{T}$ to be well generated in terms of cohomological functors into locally presentable abelian categories.

Proposition 6.8.1. *Let $\mathcal{T}$ be a triangulated category with small coproducts and α a regular cardinal. Let $\mathcal{S}_0$ be a small set of objects and denote by $\mathcal{S}$ the full subcategory formed by all α-coproducts of objects in $\mathcal{S}_0$. Then the following are equivalent.*

(1) *The objects of $\mathcal{S}_0$ are α-small and (PG2) holds for $\mathcal{S}_0$.*
(2) *The objects of $\mathcal{S}$ are α-small and (PG2) holds for $\mathcal{S}$.*
(3) *The functor $H\colon \mathcal{T} \to \mathrm{Add}_\alpha(\mathcal{S}^{\mathrm{op}}, \mathrm{Ab})$ taking X to $\mathcal{T}(-, X)|_\mathcal{S}$ preserves small coproducts.*

Proof. It is clear that (1) and (2) are equivalent, and it follows from Proposition 6.7.1 that (2) implies (3). To prove that (3) implies (2), assume that H preserves small coproducts. Let $\phi\colon X \to \coprod_{i\in I} Y_i$ be a morphism in $\mathcal{T}$ with $X \in \mathcal{S}$. Write $\coprod_{i\in I} Y_i =$

$\underset{J \subseteq I}{\underrightarrow{\operatorname{colim}}} Y_J$ as α-filtered colimit of coproducts $Y_J = \coprod_{i \in J} Y_i$ with card $J < \alpha$. Then we have

$$\underset{J \subseteq I}{\underrightarrow{\operatorname{colim}}} \mathcal{T}(X, Y_J) \cong \underset{J \subseteq I}{\underrightarrow{\operatorname{colim}}} \operatorname{Hom}_{\mathcal{S}}(\mathcal{S}(-, X), HY_J)$$

$$\cong \operatorname{Hom}_{\mathcal{S}}(\mathcal{S}(-, X), \underset{J \subseteq I}{\underrightarrow{\operatorname{colim}}} HY_J)$$

$$\cong \operatorname{Hom}_{\mathcal{S}}(\mathcal{S}(-, X), \coprod_{i \in I} HY_i)$$

$$\cong \operatorname{Hom}_{\mathcal{S}}(\mathcal{S}(-, X), H(\coprod_{i \in I} Y_i))$$

$$\cong \mathcal{T}(X, \coprod_{i \in I} Y_i).$$

Thus ϕ factors through some Y_J, and it follows that X is α-small. Now Proposition 6.7.1 implies that (PG2) holds for $\mathcal{S}$. $\square$

6.9. **Cohomological functors via filtered colimits.** The following theorem shows that cohomological functors on well generated triangulated categories can be computed via filtered colimits. This generalizes a fact which is well known for compactly generated triangulated categories. We say that an abelian category has *exact α-filtered colimits* provided that every α-filtered colimit of exact sequences is exact.

Theorem 6.9.1. *Let $\mathcal{T}$ be a triangulated category with small coproducts. Suppose $\mathcal{T}$ is α-well generated and denote by $\mathcal{T}^{\alpha}$ the full subcategory formed by all α-compact objects. Let $\mathcal{A}$ be an abelian category which has small coproducts and exact α-filtered colimits. If $H \colon \mathcal{T} \to \mathcal{A}$ is a cohomological functor which preserves small coproducts, then we have for X in $\mathcal{T}$ a natural isomorphism*

$$(6.9.1) \qquad \qquad \underset{(C, \mu) \in \mathcal{T}^{\alpha}/X}{\underrightarrow{\operatorname{colim}}} HC \xrightarrow{\sim} HX.$$

Proof. The left hand term of (6.9.1) defines a functor $\tilde{H} \colon \mathcal{T} \to \mathcal{A}$ and we need to show that the canonical morphism $\tilde{H} \to H$ is invertible.

First observe that $\tilde{H}$ is cohomological. This is a consequence of Proposition 6.6.2 and Lemma 6.4.1, because for any exact triangle $X_1 \to X_2 \to X_3 \to SX_1$ in $\mathcal{T}$, the sequence $\tilde{H}X_1 \to \tilde{H}X_2 \to \tilde{H}X_3$ can be written as α-filtered colimit of exact sequences in $\mathcal{A}$.

Next we claim that $\tilde{H}$ preserves small coproducts. To this end consider the exact functor $\bar{H} \colon A(\mathcal{T}) \to \mathcal{A}$ which extends H; see Lemma A.2. Note that $\bar{H}$ preserve small coproducts because H has this property. We have for X in $\mathcal{T}$

$$\tilde{H}X = \underset{(C, \mu) \in \mathcal{T}^{\alpha}/X}{\underrightarrow{\operatorname{colim}}} \bar{H}\big(\mathcal{T}(-, C)\big) \cong \bar{H}\big(\underset{(C, \mu) \in \mathcal{T}^{\alpha}/X}{\underrightarrow{\operatorname{colim}}} \mathcal{T}(-, C)\big).$$

Now the assertion follows from Corollary 6.7.2.

To complete the proof, consider the full subcategory $\mathcal{T}'$ consisting of those objects X in $\mathcal{T}$ such that the morphism $\tilde{H}X \to HX$ is an isomorphism. Clearly, $\mathcal{T}'$ is a triangulated subcategory since both functors are cohomological, it is closed under taking small coproducts since they are preserved by both functors, and it contains $\mathcal{T}^{\alpha}$. Thus $\mathcal{T}' = \mathcal{T}$. $\square$

Remark 6.9.2. For an alternative proof of the fact that $\tilde{H}$ is cohomological, one uses Lemma B.5.
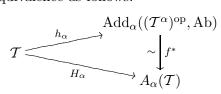
6.10. **A universal property.** Let $\mathcal{T}$ be a triangulated category which admits small coproducts and is α-well generated. We denote by $A_\alpha(\mathcal{T})$ the full subcategory of $A(\mathcal{T})$ which is formed by all colimits of objects $\mathcal{T}(-, X)$ with X in $\mathcal{T}^\alpha$. Observe that $A_\alpha(\mathcal{T})$ is a locally presentable abelian category with exact α-filtered colimits. This follows from Proposition 6.7.1 and the discussion in Appendix B, because $A_\alpha(\mathcal{T})$ can be identified with a category of left exact functors.

We have two functors

$$H_\alpha \colon \mathcal{T} \longrightarrow A_\alpha(\mathcal{T}), \quad X \mapsto \underset{(C,\mu)\in\mathcal{T}^\alpha/X}{\underrightarrow{\operatorname{colim}}} \mathcal{T}(-, C),$$

$$h_\alpha \colon \mathcal{T} \longrightarrow \operatorname{Add}_\alpha((\mathcal{T}^\alpha)^{\mathrm{op}}, \mathrm{Ab}), \quad X \mapsto \mathcal{T}(-, X)|_{\mathcal{T}^\alpha},$$

which are related by an equivalence as follows.

$$
\begin{array}{ccc}
 & & \operatorname{Add}_\alpha((\mathcal{T}^\alpha)^{\mathrm{op}}, \mathrm{Ab}) \\
 & \overset{h_\alpha}{\nearrow} & \\
\mathcal{T} & & \sim \downarrow f^* \\
 & \underset{H_\alpha}{\searrow} & \\
 & & A_\alpha(\mathcal{T})
\end{array}
$$

The functor f^* is induced by the inclusion $f \colon \mathcal{T}^\alpha \to \mathcal{T}$ and discussed in Proposition 6.7.1. In particular, there it is shown that $f^*(h_\alpha X) = f^* f_*(h_\mathcal{T} X) = H_\alpha X$ for all X in $\mathcal{T}$.

Proposition 6.10.1. *The functor $H_\alpha \colon \mathcal{T} \to A_\alpha(\mathcal{T})$ has the following universal property.*

(1) *The functor H_α is a cohomological functor to an abelian category with small coproducts and exact α-filtered colimits and H_α preserves small coproducts.*

(2) *Given a cohomological functor $H \colon \mathcal{T} \to \mathcal{A}$ to an abelian category with small coproducts and exact α-filtered colimits such that H preserves small coproducts, there exists an essentially unique exact functor $\bar{H} \colon A_\alpha(\mathcal{T}) \to \mathcal{A}$ which preserves small coproducts and satisfies $H = \bar{H} \circ H_\alpha$.*

Proof. (1) It is clear that h_α is cohomological and it follows from Proposition 6.7.1 that h_α preserves small coproducts.

(2) Given $H \colon \mathcal{T} \to \mathcal{A}$, we denote by $\tilde{H} \colon A(\mathcal{T}) \to \mathcal{A}$ the exact functor which extends H, and we define $\bar{H} \colon A_\alpha(\mathcal{T}) \to \mathcal{A}$ by sending each X to $\tilde{H}X$. The following commutative diagram illustrates this construction.

$$
\begin{array}{ccccc}
\mathcal{T}^\alpha & \overset{h_{\mathcal{T}^\alpha}}{\longrightarrow} & A(\mathcal{T}^\alpha) & \overset{h_{A(\mathcal{T}^\alpha)}}{\longrightarrow} & A_\alpha(\mathcal{T}) \\
{\scriptstyle f=\mathrm{inc}}\downarrow & & {\scriptstyle A(f)}\downarrow & & {\scriptstyle f^*=\mathrm{inc}}\downarrow \\
\mathcal{T} & \overset{h_\mathcal{T}}{\longrightarrow} & A(\mathcal{T}) & =\!=\!=\!=\!= & A(\mathcal{T}) \\
\Vert & & {\scriptstyle \tilde{H}}\downarrow & & \\
\mathcal{T} & \overset{H}{\longrightarrow} & \mathcal{A} & &
\end{array}
$$

Let us check the properties of $\bar{H}$. The functor $\bar{H}$ preserves small coproducts since $\tilde{H}$ has this property. The functor $\bar{H}$ is exact when restricted to $A(\mathcal{T}^\alpha)$. Thus it follows from Lemma B.5 that $\bar{H}$ is exact. The equality $H = \bar{H} \circ H_\alpha$ is a consequence of Theorem 6.9.1

since both functors coincide on $\mathcal{T}^\alpha$. Suppose now there is a second functor $A_\alpha(\mathcal{T}) \to \mathcal{A}$ having the properties of $\bar{H}$. Then both functors agree on $\mathcal{P} = \{\mathcal{T}(-, X) \mid X \in \mathcal{T}^\alpha\}$ and therefore on all of $A_\alpha(\mathcal{T})$ since each object in $A_\alpha(\mathcal{T})$ is a colimit of objects in $\mathcal{P}$ and both functors preserve colimits. $\qquad\square$

Remark 6.10.2. The universal property can be used to show that the category $\mathcal{T}^\alpha$ of α-compact objects does not depend on the choice of a perfectly generating set for $\mathcal{T}$. More precisely, if $\mathcal{T}$ is α-well generated, then two α-localizing subcategories coincide if each contains a small set of α-small perfect generators. This follows from the fact that the functor H_α identifies the α-compact objects with the α-presentable projective objects of $A_\alpha(\mathcal{T})$.

6.11. **Notes.** Well generated triangulated were introduced and studied by Neeman in his book [33] as a natural generalization of compactly generated triangulated categories. For an alternative approach which simplifies the definition, see [23]. More recently, well generated categories with specific models have been studied; see [37, 47] for work involving algebraic models via differential graded categories, and [20] for topological models. In [43], Rosický used combinatorial models and showed that there exist universal cohomological functors into locally presentable categories which are full. Interesting consequences of this fact are discussed in [35]. The description of the universal cohomological functors in terms of filtered colimits seems to be new.

7. LOCALIZATION FOR WELL GENERATED CATEGORIES

7.1. **Cohomological localization.** The following theorem shows that cohomological functors on well generated triangulated categories induce localization functors. This generalizes a fact which is well known for compactly generated triangulated categories.

Theorem 7.1.1. *Let $\mathcal{T}$ be a triangulated category with small coproducts which is well generated. Let $H\colon \mathcal{T} \to \mathcal{A}$ be a cohomological functor into an abelian category which has small coproducts and exact α-filtered colimits for some regular cardinal α. Suppose also that H preserves small coproducts. Then there exists an exact localization functor $L\colon \mathcal{T} \to \mathcal{T}$ such that for each object X we have $LX = 0$ if and only if $H(S^n X) = 0$ for all $n \in \mathbb{Z}$.*

Proof. We may assume that $\mathcal{T}$ is α-well generated. Let $\Sigma = \Sigma(H)$ denote the set of morphisms σ in $\mathcal{T}$ such $H\sigma$ is invertible. Next we assume that $S\Sigma = \Sigma$. Otherwise, we replace $\mathcal{A}$ by a countable product $\mathcal{A}^\mathbb{Z}$ of copies of $\mathcal{A}$ and H by $(HS^n)_{n\in\mathbb{Z}}$. Then Σ admits a calculus of right fractions by Lemma 4.4.2, and we apply the criterion of Lemma 3.3.1 to show that the morphisms between any two objects in $\mathcal{T}[\Sigma^{-1}]$ form a small set. The existence of a localization functor $L\colon \mathcal{T} \to \mathcal{T}$ with $\operatorname{Ker} L = \operatorname{Ker} H$ then follows from Proposition 5.2.1.

Thus we need to specify for each object Y of $\mathcal{T}$ a small set of objects $S(Y, \Sigma)$ such that for every morphism $X \to Y$ in Σ, there exist a morphism $X' \to X$ in Σ with X' in $S(Y, \Sigma)$. Suppose that Y belongs to $\mathcal{T}^\kappa$. We define by induction $\kappa_{-1} = \kappa + \alpha$ and

$$\kappa_n = \sup\{\operatorname{card} \mathcal{T}^\alpha/U \mid U \in \mathcal{T}^{\kappa_{n-1}}\}^+ + \kappa_{n-1} \quad \text{for} \quad n \geq 0.$$
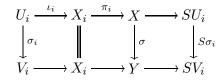
Then we put $S(Y, \Sigma) = \mathcal{T}^{\bar{\kappa}}$ with $\bar{\kappa} = (\sum_{n \geq 0} \kappa_n)^+$.

Now fix $\sigma\colon X \to Y$ in Σ. The morphism $X' \to X$ in Σ with X' in $S(Y, \Sigma)$ is constructed as follows. The canonical morphism $\pi\colon \coprod_{(C,\mu)\in\mathcal{T}^\alpha/X} C \to X$ induces an

epimorphism $H\pi$ by Theorem 6.9.1. We can choose $\mathcal{C} \subseteq \mathcal{T}^\alpha/X$ with $\operatorname{card}\mathcal{C} \leq \operatorname{card}\mathcal{T}^\alpha/Y$ such that $\pi_0\colon X_0 = \coprod_{(C,\mu)\in\mathcal{C}} C \to X$ induces an epimorphism $H\pi_0$ since $H\sigma$ is invertible. More precisely, we call two objects (C,μ) and (C',μ') of $\mathcal{T}^\alpha/X$ equivalent if $\sigma\mu = \sigma\mu'$, and we choose as objects of $\mathcal{C}$ precisely one representative for each equivalence class.

Suppose we have already constructed $\pi_i\colon X_i \to X$ with X_i in $\mathcal{T}^{\kappa_i}$ for some $i \geq 0$. Then we form the following commutative diagram with exact rows.

$$\begin{array}{ccccccc}
U_i & \xrightarrow{\iota_i} & X_i & \xrightarrow{\pi_i} & X & \longrightarrow & SU_i \\
\downarrow{\scriptstyle\sigma_i} & & \| & & \downarrow{\scriptstyle\sigma} & & \downarrow{\scriptstyle S\sigma_i} \\
V_i & \longrightarrow & X_i & \longrightarrow & Y & \longrightarrow & SV_i
\end{array}$$

Note that $H\sigma_i$ is invertible. Thus we can choose $\mathcal{C}_i \subseteq \mathcal{T}^\alpha/U_i$ with

$$\operatorname{card}\mathcal{C}_i \leq \operatorname{card}\mathcal{T}^\alpha/V_i \leq \operatorname{card}\mathcal{T}^\alpha/X_i + \operatorname{card}\mathcal{T}^\alpha/Y < \kappa_{i+1}$$

such that $\xi_i\colon \coprod_{(C,\mu)\in\mathcal{C}_i} C \to U_i$ induces an epimorphism $H\xi_i$. Now complete $\iota_i \circ \xi_i$ to an exact triangle and define $\pi_{i+1}\colon X_{i+1} \to X$ by the commutativity of the following diagram.

$$\coprod_{(C,\mu)\in\mathcal{C}_i} C \xrightarrow{\iota_i \circ \xi_i} X_i \xrightarrow{\phi_i} X_{i+1} \longrightarrow S\big(\coprod_{(C,\mu)\in\mathcal{C}_i} C\big)$$
$$\pi_i \searrow \quad \swarrow \pi_{i+1}$$
$$X$$

Observe that X_{i+1} belongs to $\mathcal{T}^{\kappa_{i+1}}$ and that $\operatorname{Ker} H\pi_i = \operatorname{Ker} H\phi_i$. The ϕ_i induce an exact triangle

$$(7.1.1) \qquad \coprod_{i\in\mathbb{N}} X_i \xrightarrow{(\mathrm{id}-\phi_i)} \coprod_{i\in\mathbb{N}} X_i \xrightarrow{\psi} X' \longrightarrow S\big(\coprod_{i\in\mathbb{N}} X_i\big)$$

such that X' belongs to $S(Y,\Sigma)$ and the morphism $(\pi_i)\colon \coprod_{i\in\mathbb{N}} X_i \to X$ factors through ψ via a morphism $\tau\colon X' \to X$. We claim that $H\tau$ is invertible. In fact, the lemma below implies that the π_i induce the following exact sequence
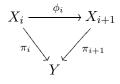
$$0 \longrightarrow \coprod_{i\in\mathbb{N}} HX_i \xrightarrow{(\mathrm{id}-H\phi_i)} \coprod_{i\in\mathbb{N}} HX_i \xrightarrow{(H\pi_i)} HX \longrightarrow 0.$$

On the other hand, the exact triangle (7.1.1) induces the exact sequence

$$H\big(\coprod_{i\in\mathbb{N}} X_i\big) \xrightarrow{H(\mathrm{id}-\phi_i)} H\big(\coprod_{i\in\mathbb{N}} X_i\big) \xrightarrow{H\psi} HX' \longrightarrow HS\big(\coprod_{i\in\mathbb{N}} X_i\big),$$

and a comparison shows that $H\tau$ is invertible. Here, we use again that H preserves small coproducts, and this completes the proof. $\square$
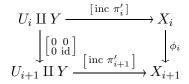
Lemma 7.1.2. *Let $\mathcal{A}$ be an abelian category which admits countable coproducts. Then a sequence of epimorphisms $(\pi_i)_{i\in\mathbb{N}}$*

$$X_i \xrightarrow{\phi_i} X_{i+1}$$
$$\pi_i \searrow \quad \swarrow \pi_{i+1}$$
$$Y$$

satisfying $\pi_i = \pi_{i+1} \circ \phi_i$ *and* $\operatorname{Ker}\pi_i = \operatorname{Ker}\phi_i$ *for all* i *induces an exact sequence*

$$0 \longrightarrow \coprod_{i \in \mathbb{N}} X_i \xrightarrow{(\operatorname{id} - \phi_i)} \coprod_{i \in \mathbb{N}} X_i \xrightarrow{(\pi_i)} Y \longrightarrow 0.$$

Proof. The assumption $U_i := \operatorname{Ker}\pi_i = \operatorname{Ker}\phi_i$ implies that there exists a morphism $\pi_i' \colon Y \to X_i$ with $\pi_i \pi_i' = \operatorname{id} Y$ and $\phi_i \pi_i' = \pi_{i+1}'$ for all $i \geq 1$. Thus we have a sequence of commuting squares

$$\begin{array}{ccc}
U_i \amalg Y & \xrightarrow{\ [\operatorname{inc}\ \pi_i']\ } & X_i \\
\left[\begin{smallmatrix} 0 & 0 \\ 0 & \operatorname{id} \end{smallmatrix}\right]\Big\downarrow & & \Big\downarrow \phi_i \\
U_{i+1} \amalg Y & \xrightarrow[\ [\operatorname{inc}\ \pi_{i+1}']\ ]{} & X_{i+1}
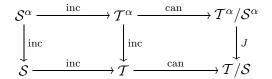\end{array}$$

where the horizontal maps are isomorphisms. Taking colimits on both sides, the assertion follows. $\qquad\square$

7.2. Localization with respect to a small set of objects.

Let $\mathcal{T}$ be a well generated triangulated category and $\mathcal{S}$ a localizing subcategory which is generated by a small set of objects. The following result says that $\mathcal{S}$ and $\mathcal{T}/\mathcal{S}$ are both well generated and that the filtration $\mathcal{T} = \bigcup_\alpha \mathcal{T}^\alpha$ via α-compact objects induces canonical filtrations

$$\mathcal{S} = \bigcup_\alpha (\mathcal{S} \cap \mathcal{T}^\alpha) \quad \text{and} \quad \mathcal{T}/\mathcal{S} = \bigcup_\alpha \mathcal{T}^\alpha/(\mathcal{S} \cap \mathcal{T}^\alpha).$$
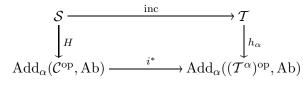
Theorem 7.2.1. *Let $\mathcal{T}$ be a well generated triangulated category and $\mathcal{S}$ a localizing subcategory which is generated by a small set of objects. Fix a regular cardinal α such that $\mathcal{T}$ is α-well generated and $\mathcal{S}$ is generated by α-compact objects.*

(1) *An object X in $\mathcal{T}$ belongs to $\mathcal{S}$ if and only if every morphism $C \to X$ from an object C in $\mathcal{T}^\alpha$ factors through some object in $\mathcal{S} \cap \mathcal{T}^\alpha$.*

(2) *The localizing subcategory $\mathcal{S}$ and the quotient category $\mathcal{T}/\mathcal{S}$ are α-well generated.*

(3) *We have $\mathcal{S}^\alpha = \mathcal{S} \cap \mathcal{T}^\alpha$ and a commutative diagram of exact functors*

$$\begin{array}{ccccc}
\mathcal{S}^\alpha & \xrightarrow{\ \operatorname{inc}\ } & \mathcal{T}^\alpha & \xrightarrow{\ \operatorname{can}\ } & \mathcal{T}^\alpha/\mathcal{S}^\alpha \\
\Big\downarrow{\scriptstyle\operatorname{inc}} & & \Big\downarrow{\scriptstyle\operatorname{inc}} & & \Big\downarrow{\scriptstyle J} \\
\mathcal{S} & \xrightarrow[\ \operatorname{inc}\ ]{} & \mathcal{T} & \xrightarrow[\ \operatorname{can}\ ]{} & \mathcal{T}/\mathcal{S}
\end{array}$$

such that J is fully faithful. Moreover, J induces a functor $\mathcal{T}^\alpha/\mathcal{S}^\alpha \to (\mathcal{T}/\mathcal{S})^\alpha$ such that every object of $(\mathcal{T}/\mathcal{S})^\alpha$ is a direct factor of an object in the image of J. This functor is an equivalence if $\alpha > \aleph_0$.

Proof. Let $\mathcal{C} = \mathcal{S} \cap \mathcal{T}^\alpha$. Then the inclusion $i \colon \mathcal{C} \to \mathcal{T}^\alpha$ induces a fully faithful and exact functor $i^* \colon \operatorname{Add}_\alpha(\mathcal{C}^{\mathrm{op}}, \operatorname{Ab}) \to \operatorname{Add}_\alpha((\mathcal{T}^\alpha)^{\mathrm{op}}, \operatorname{Ab})$ which is left adjoint to the functor i_* taking F to $F \circ i$; see Lemma B.8. Note that the image $\operatorname{Im} i^*$ of i^* is closed under small coproducts. We consider the restricted Yoneda functor $h_\alpha \colon \mathcal{T} \to \operatorname{Add}_\alpha((\mathcal{T}^\alpha)^{\mathrm{op}}, \operatorname{Ab})$ taking X to $\mathcal{T}(-, X)|_{\mathcal{T}^\alpha}$ and observe that $h_\alpha^{-1}(\operatorname{Im} i^*)$ is a localizing subcategory of $\mathcal{T}$

containing $\mathcal{C}$. Thus we obtain a functor H making the following diagram commutative.

$$\begin{array}{ccc}
\mathcal{S} & \xrightarrow{\ \text{inc}\ } & \mathcal{T} \\
\downarrow{\scriptstyle H} & & \downarrow{\scriptstyle h_\alpha} \\
\mathrm{Add}_\alpha(\mathcal{C}^{\mathrm{op}}, \mathrm{Ab}) & \xrightarrow{\ i^*\ } & \mathrm{Add}_\alpha((\mathcal{T}^\alpha)^{\mathrm{op}}, \mathrm{Ab})
\end{array}$$

Let us compare H with the restricted Yoneda functor

$$H' \colon \mathcal{S} \longrightarrow \mathrm{Add}_\alpha(\mathcal{C}^{\mathrm{op}}, \mathrm{Ab}), \quad X \mapsto \mathcal{S}(-, X)|_{\mathcal{C}}.$$

In fact, we have an isomorphism

$$H \xrightarrow{\sim} i_* \circ i^* \circ H = i_* \circ h_\alpha|_{\mathcal{S}} = H'$$
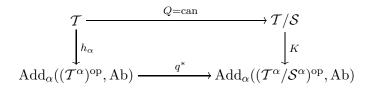
and H' preserves small coproducts since h_α does. It follows from Proposition 6.8.1 that $\mathcal{C}$ provides a small set of α-small perfect generators for $\mathcal{S}$. Thus $\mathcal{S}$ is α-well generated and $\mathcal{S}^\alpha = \mathrm{Loc}_\alpha \mathcal{C} = \mathcal{S} \cap \mathcal{T}^\alpha$.

Next we apply Proposition 5.2.1 and obtain a localization functor $L \colon \mathcal{T} \to \mathcal{T}$ with $\mathrm{Ker}\, L = \mathcal{S}$. We use L to show that $\mathcal{S} = h_\alpha^{-1}(\mathrm{Im}\, i^*)$. We know already that $\mathcal{S} \subseteq h_\alpha^{-1}(\mathrm{Im}\, i^*)$. Now let X be an object in $h_\alpha^{-1}(\mathrm{Im}\, i^*)$ and consider the exact triangle $\Gamma X \to X \to LX \to S(\Gamma X)$. Then $\mathcal{T}(C, LX) = 0$ for all $C \in \mathcal{C}$ and therefore $i_* h_\alpha LX = 0$. On the other hand, $h_\alpha LX = i^* F$ for some functor F and therefore $0 = i_* h_\alpha LX = i_* i^* F \cong F$. Thus $LX = 0$ and therefore X belongs to $\mathcal{S}$. This shows $\mathcal{S} = h_\alpha^{-1}(\mathrm{Im}\, i^*)$.

Now we prove (1) and use the description of the essential image of i^* from Lemma B.8. We have for an object X in $\mathcal{T}$ that X belongs to $\mathcal{S}$ iff $h_\alpha X$ belongs to $\mathrm{Im}\, i^*$ iff every morphism $\mathcal{T}^\alpha(-, C) \to h_\alpha X$ with $C \in \mathcal{T}^\alpha$ factors through $\mathcal{T}^\alpha(-, C')$ for some $C' \in \mathcal{C}$ iff every morphism $C \to X$ with $C \in \mathcal{T}^\alpha$ factors through some $C' \in \mathcal{C}$.

An immediate consequence of (1) is the fact that J is fully faithful. This follows from Lemma 4.7.1.

Now consider the quotient functor $q \colon \mathcal{T}^\alpha \to \mathcal{T}^\alpha/\mathcal{S}^\alpha$. This induces an exact functor $q^* \colon \mathrm{Add}_\alpha((\mathcal{T}^\alpha)^{\mathrm{op}}, \mathrm{Ab}) \to \mathrm{Add}_\alpha((\mathcal{T}^\alpha/\mathcal{S}^\alpha)^{\mathrm{op}}, \mathrm{Ab})$ which is left adjoint to the fully faithful functor q_* taking F to $F \circ q$; see Lemma B.8. Clearly, $q^* \circ h_\alpha$ annihilates $\mathcal{S}$ and induces therefore a functor K making the following diagram commutative.

$$\begin{array}{ccc}
\mathcal{T} & \xrightarrow{\ Q = \mathrm{can}\ } & \mathcal{T}/\mathcal{S} \\
\downarrow{\scriptstyle h_\alpha} & & \downarrow{\scriptstyle K} \\
\mathrm{Add}_\alpha((\mathcal{T}^\alpha)^{\mathrm{op}}, \mathrm{Ab}) & \xrightarrow{\ q^*\ } & \mathrm{Add}_\alpha((\mathcal{T}^\alpha/\mathcal{S}^\alpha)^{\mathrm{op}}, \mathrm{Ab})
\end{array}$$

Note that Q admits a right adjoint which we denote by Q_ρ. We identify $\mathcal{T}^\alpha/\mathcal{S}^\alpha$ via J with a full triangulated subcategory of $\mathcal{T}/\mathcal{S}$ and consider the restricted Yoneda functor

$$K' \colon \mathcal{T}/\mathcal{S} \longrightarrow \mathrm{Add}_\alpha((\mathcal{T}^\alpha/\mathcal{S}^\alpha)^{\mathrm{op}}, \mathrm{Ab}), \quad X \mapsto \mathcal{T}/\mathcal{S}(-, X)|_{\mathcal{T}^\alpha/\mathcal{S}^\alpha}.$$

Adjointness gives the following isomorphism

$$\mathcal{T}/\mathcal{S}(JqC, X) = \mathcal{T}/\mathcal{S}(QC, X) \cong \mathcal{T}(C, Q_\rho X)$$

for all $C \in \mathcal{T}^\alpha$ and $X \in \mathcal{T}/\mathcal{S}$. Thus we have an isomorphism

$$K \xleftarrow{\sim} K \circ Q \circ Q_\rho = q^* \circ h_\alpha \circ Q_\rho \cong q^* \circ q_* \circ K' \xrightarrow{\sim} K'$$

and K' preserves small coproducts since h_α does. It follows from Proposition 6.8.1 that $\mathcal{T}^\alpha/\mathcal{S}^\alpha$ provides a small set of α-small perfect generators for $\mathcal{T}/\mathcal{S}$. Thus $\mathcal{T}/\mathcal{S}$ is α-well generated and $(\mathcal{T}/\mathcal{S})^\alpha = \mathrm{Loc}_\alpha(\mathcal{T}^\alpha/\mathcal{S}^\alpha)$. $\square$

Corollary 7.2.2. *Let $\mathcal{T}$ be an α-well generated triangulated category and $\mathcal{S}$ a localizing subcategory generated by a small set $\mathcal{S}_0$ of α-compact objects. Then $\mathcal{S}$ is α-well generated and $\mathcal{S}^\alpha$ equals the α-localizing subcategory generated by $\mathcal{S}_0$.*

Proof. In the preceding proof of Theorem 7.2.1, we can choose for $\mathcal{C}$ instead of $\mathcal{S} \cap \mathcal{T}^\alpha$ the α-localizing subcategory of $\mathcal{T}$ which is generated by $\mathcal{S}_0$. Then the proof shows that $\mathcal{C}$ provides a small set of α-small perfect generators for $\mathcal{S}$. Thus we have $\mathcal{S}^\alpha = \mathcal{C}$ by definition. $\square$

The localization with respect to a localizing subcategory generated by a small set of objects can be interpreted in various ways. The following remark provides some indication.
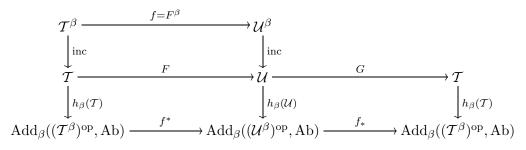
Remark 7.2.3. (1) Let $\mathcal{T}$ be a well generated triangulated category and ϕ a morphism in $\mathcal{T}$. Then there exists a universal exact localization functor $L: \mathcal{T} \to \mathcal{T}$ inverting ϕ. To see this, complete ϕ to an exact triangle $X \xrightarrow{\phi} Y \to Z \to SX$ and let L be the localization functor such that $\mathrm{Ker}\, L$ equals the localizing subcategory generated by Z. Conversely, any exact localization functor $L: \mathcal{T} \to \mathcal{T}$ is the universal exact localization functor inverting some morphism ϕ provided that $\mathrm{Ker}\, L$ is generated by a small set $\mathcal{S}_0$ of objects. To see this, take $\phi: 0 \to \coprod_{X \in \mathcal{S}_0} X$.

(2) Let $\mathcal{T}$ be a triangulated category and $L: \mathcal{T} \to \mathcal{T}$ an exact localization functor such that $\mathcal{S} = \mathrm{Ker}\, L$ is generated by a single object W. Then the first morphism $\Gamma X \to X$ from the functorial triangle $\Gamma X \to X \to LX \to S(\Gamma X)$ is called *cellularization* and the second morphism $X \to LX$ is called *nullification* with respect to W. The objects in $\mathcal{S}$ are *built from W*.

7.3. Functors between well generated categories.

We consider functors between well generated triangulated categories which are exact and preserve small coproducts. The following result shows that such functors are controlled by their restriction to the subcategory of α-compact objects for some regular cardinal α.

Proposition 7.3.1. *Let $F: \mathcal{T} \to \mathcal{U}$ be an exact functor between α-well generated triangulated categories. Suppose that F preserves small coproducts and let G be a right adjoint.*

(1) *There exists a regular cardinal $\beta_0 \geq \alpha$ such that F preserves β_0-compactness. In that case F preserves β-compactness for all regular $\beta \geq \beta_0$.*

(2) *Given a regular cardinal $\beta \geq \beta_0$, the restriction $f: \mathcal{T}^\beta \to \mathcal{U}^\beta$ of F induces the following diagram of functors which commute up to natural isomorphisms.*

Proof. (1) Choose $\beta_0 \geq \alpha$ such that $F(\mathcal{T}^\alpha) \subseteq \mathcal{U}^{\beta_0}$. Then we get for $\beta \geq \beta_0$

$$F(\mathcal{T}^\beta) = F(\mathrm{Loc}_\beta \, \mathcal{T}^\alpha) \subseteq \mathrm{Loc}_\beta \, F(\mathcal{T}^\alpha) \subseteq \mathrm{Loc}_\beta \, \mathcal{U}^{\beta_0} = \mathcal{U}^\beta.$$

(2) We apply Theorem 6.9.1 to show that $h_\beta(\mathcal{U}) \circ F \cong f^* \circ h_\beta(\mathcal{T})$. In fact, it follows from Proposition 6.10.1 and Lemma B.8 that both composites are cohomological functors, preserve small coproducts, and agree on $\mathcal{T}^\beta$.

The isomorphism $h_\beta(\mathcal{T}) \circ G \cong f_* \circ h_\beta(\mathcal{U})$ follows from the adjointness of F and G, since $\mathcal{T}(C, GX) \cong \mathcal{U}(fC, X)$ for every $C \in \mathcal{T}^\beta$ and $X \in \mathcal{U}$. $\qquad \square$

7.4. **The kernel of a functor between well generated categories.** We show that the class of well generated triangulated categories is closed under taking kernels of exact functors which preserve small coproducts.

Theorem 7.4.1. *Let* $F \colon \mathcal{T} \to \mathcal{U}$ *be an exact functor between* α-*well generated triangulated categories and suppose that* F *preserves small coproducts. Let* $\mathcal{S} = \mathrm{Ker}\, F$ *and choose a regular cardinal* $\beta \geq \alpha$ *such that* F *preserves* β-*compactness.*
   (1) *An object* X *in* $\mathcal{T}$ *belongs to* $\mathcal{S}$ *if and only if every morphism* $C \to X$ *with* $C \in \mathcal{T}^\beta$ *factors through a morphism* $\gamma \colon C \to C'$ *in* $\mathcal{T}^\beta$ *satisfying* $F\gamma = 0$.
   (2) *Suppose* $\beta > \aleph_0$. *Then* $\mathcal{S}$ *is* β-*well generated and* $\mathcal{S}^\beta = \mathcal{S} \cap \mathcal{T}^\beta$.

Proof. Let $f \colon \mathcal{T}^\beta \to \mathcal{U}^\beta$ be the restriction of F and denote by $\mathfrak{I}$ the set of morphisms in $\mathcal{T}^\beta$ which are annihilated by F.

(1) Let X be an object in $\mathcal{T}$. Then it follows from Proposition 7.3.1 that $FX = 0$ if and only if $f^* h_\beta X = 0$. Now Lemma B.8 implies that $f^* h_\beta X = 0$ iff each morphism $C \to X$ with $C \in \mathcal{T}^\beta$ factors through some morphism $C \to C'$ in $\mathfrak{I}$.

(2) Let $\mathcal{S}'$ denote the localizing subcategory of $\mathcal{T}$ which is generated by all homotopy colimits of sequences

$$C_0 \longrightarrow C_1 \longrightarrow C_2 \longrightarrow \cdots$$

of morphisms in $\mathfrak{I}$. We claim that $\mathcal{S}' = \mathrm{Ker}\, F$. Clearly, we have $\mathcal{S}' \subseteq \mathrm{Ker}\, F$. Now fix an object $X \in \mathrm{Ker}\, F$. We have seen in (1) that each morphism $\mu \colon C \to X$ with $C \in \mathcal{T}^\beta$ factors through some morphism $C \to C'$ in $\mathfrak{I}$. We obtain by induction a sequence

$$C = C_0 \xrightarrow{\gamma_0} C_1 \xrightarrow{\gamma_1} C_2 \xrightarrow{\gamma_2} \cdots$$

of morphisms in $\mathfrak{I}$ such that μ factors through each finite composite $\gamma_i \ldots \gamma_0$. Thus μ factors through the homotopy colimit of this sequence and therefore through an object of $\mathcal{S}' \cap \mathcal{T}^\beta$. Here one uses that $\beta > \aleph_0$. We conclude from Theorem 7.2.1 that X belongs to $\mathcal{S}'$. Moreover, we conclude from this theorem that $\mathcal{S}'$ is β-well generated. $\qquad \square$

Remark 7.4.2. It is necessary to assume in part (2) of the preceding theorem that $\beta > \aleph_0$. For example, there exists a ring A with Jacobson radical $\mathfrak{r}$ such that the functor $F = - \otimes_A^{\mathbf{L}} A/\mathfrak{r} \colon \mathbf{D}(A) \to \mathbf{D}(A/\mathfrak{r})$ satisfies $\mathcal{S} = \mathrm{Ker}\, F \neq 0$ but $\mathcal{S} \cap \mathbf{D}(A)^c = 0$; see [22].

Observe that Theorem 7.4.1 provides a partial answer to the telescope conjecture for compactly generated categories. This conjecture claims that the kernel of a localization functor $L \colon \mathcal{T} \to \mathcal{T}$ is generated by compact objects provided that L preserves small coproducts. Part (1) implies that $\mathcal{S} = \mathrm{Ker}\, L$ is generated by morphisms between compact objects, and part (2) says that $\mathcal{S}$ is generated by $\aleph_1$-compact objects. I am grateful to Amnon Neeman for explaining to me how to deduce (2) from (1). The following corollary makes the connection with the telescope conjecture more precise; just put $\alpha = \aleph_0$.

Corollary 7.4.3. *Let $L\colon \mathcal{T} \to \mathcal{T}$ be an exact localization functor which preserves small coproducts. Suppose that $\mathcal{T}$ is α-well generated and let $\beta \geq \max(\alpha, \aleph_1)$. Then $\mathcal{S} = \operatorname{Ker} L$ is β-well generated and $\mathcal{S}^\beta = \mathcal{S} \cap \mathcal{T}^\beta$.*

Proof. Let $L\colon \mathcal{T} \to \mathcal{T}$ be a localization functor which preserves small coproducts. Write $L = G \circ F$ as the composite of a quotient functor $F\colon \mathcal{T} \to \mathcal{U}$ and a fully faithful right adjoint G, where $\mathcal{U} = \mathcal{T}/\mathcal{S}$ and $\mathcal{S} = \operatorname{Ker} L$. Then G preserves small coproducts by Proposition 5.5.1. The isomorphism (5.4.1) from the proof of Lemma 5.4.1 shows that F preserves α-smallness and sends a set of perfect generators of $\mathcal{T}$ to a set of perfect generators of $\mathcal{U}$. In particular, F preserves β-compactness for all regular $\beta \geq \alpha$. Now apply Theorem 7.4.1. $\qquad\square$

7.5. The kernel of a cohomological functor on a well generated category.

The following result says that kernels of cohomological functors from well generated triangulated categories into locally presentable abelian categories are well generated. The argument is basically the same as that for kernels of exact functors between well generated triangulated categories.

Theorem 7.5.1. *Let $H\colon \mathcal{T} \to \mathcal{A}$ be a cohomological functor from a well generated triangulated category into a locally presentable abelian category and suppose that H preserves small coproducts. Let $\mathcal{S}$ denote the localizing subcategory of $\mathcal{T}$ consisting of all objects X such that $H(S^n X) = 0$ for all $n \in \mathbb{Z}$. Then $\mathcal{S}$ is a well generated triangulated category.*

Proof. Replacing H by $(HS^n)_{n\in\mathbb{Z}}$, we may assume that $\mathcal{S} = \operatorname{Ker} H$. Choose a regular cardinal α such that $\mathcal{T}$ is α-well generated and $\mathcal{A}$ is locally α-presentable. Then we have $H(\mathcal{T}^\alpha) \subseteq \mathcal{A}^\beta$ for some regular cardinal β and we assume $\beta \geq \max(\alpha, \aleph_1)$. The description of H in Theorem 6.9.1 shows that H restricts to a functor $h\colon \mathcal{T}^\beta \to \mathcal{A}^\beta$, and we denote by $\bar{h}\colon A(\mathcal{T}^\beta) \to \mathcal{A}^\beta$ the induced exact functor. Then we obtain the following functor

$$h^*\colon \operatorname{Add}_\beta((\mathcal{T}^\beta)^{\mathrm{op}}, \mathrm{Ab}) \xrightarrow{\sim} \operatorname{Lex}_\beta(A(\mathcal{T}^\beta)^{\mathrm{op}}, \mathrm{Ab}) \xrightarrow{\bar{h}^*} \operatorname{Lex}_\beta((\mathcal{A}^\beta)^{\mathrm{op}}, \mathrm{Ab}) \xrightarrow{\sim} \mathcal{A}$$

where the first equivalence follows from Lemma B.1 and the second equivalence follows from Lemma B.6. The functor $\bar{h}^*$ is a left Kan extension; it takes a filtered colimit
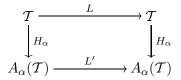
$$F = \operatorname*{colim}_{(C,\mu)\in A(\mathcal{T}^\beta)/F} A(\mathcal{T}^\beta)(-, C) \quad \text{to} \quad \operatorname*{colim}_{(C,\mu)\in A(\mathcal{T}^\beta)/F} \mathcal{A}^\beta(-, \bar{h}C).$$

Note that h^* is exact and preserves small coproducts. This follows from Lemma B.5 and the fact that $\bar{h}^*$ is left adjoint to the restriction functor $\bar{h}_*$.

The composite $h^* \circ h_\beta \colon \mathcal{T} \to \mathcal{A}$ coincides with H on $\mathcal{T}^\beta$ and therefore $h^* \circ h_\beta \cong H$ by Theorem 6.9.1. In particular, we have for each X in $\mathcal{T}$ that $HX = 0$ if and only if $h^*(h_\beta X) = 0$. Now we use the same argument as in the proof of Theorem 7.4.1 and show that $\operatorname{Ker} H$ is generated by all homotopy colimits of countable sequences of morphisms in $\mathcal{T}^\beta$ which are annihilated by H. $\qquad\square$

7.6. Localization of well generated categories versus abelian localization.

We demonstrate the interplay between triangulated and abelian localization. To this end recall from Proposition 6.10.1 that we have for each well generated category $\mathcal{T}$ a universal cohomological functor $H_\alpha\colon \mathcal{T} \to A_\alpha(\mathcal{T})$ into a locally α-presentable abelian category. We show that each exact localization functor for $\mathcal{T}$ can be extended to an exact localization functor for $A_\alpha(\mathcal{T})$ for some regular cardinal α.

Theorem 7.6.1. *Let $\mathcal{T}$ be a well generated triangulated category and $L\colon \mathcal{T} \to \mathcal{T}$ an exact localization functor. Suppose that $\operatorname{Ker} L$ is well generated. Then there exists a regular cardinal α and an exact localization functor $L'\colon A_\alpha(\mathcal{T}) \to A_\alpha(\mathcal{T})$ such that the following square commutes up to a natural isomorphism.*

$$
\begin{array}{ccc}
\mathcal{T} & \xrightarrow{\ \ L\ \ } & \mathcal{T} \\
{\scriptstyle H_\alpha}\big\downarrow & & \big\downarrow{\scriptstyle H_\alpha} \\
A_\alpha(\mathcal{T}) & \xrightarrow{\ \ L'\ \ } & A_\alpha(\mathcal{T})
\end{array}
$$

More precisely, the adjunction morphisms $\operatorname{Id} \mathcal{T} \to L$ and $\operatorname{Id} A_\alpha(\mathcal{T}) \to L'$ induce for each X in $\mathcal{T}$ the following isomorphisms.

$$
H_\alpha LX \xrightarrow{\ \sim\ } L'(H_\alpha LX) = L'H_\alpha(LX) \xleftarrow{\ \sim\ } L'H_\alpha(X)
$$

An object X in $\mathcal{T}$ is L-acyclic if and only if $H_\alpha X$ is L'-acyclic, and X is L-local if and only if $H_\alpha X$ is L'-local.

Proof. Choose a regular cardinal $\alpha > \aleph_0$ such that $\mathcal{T}$ is α-well generated and $\mathcal{S} = \operatorname{Ker} L$ is generated by α-compact objects. Let $\mathcal{U} = \mathcal{T}/\mathcal{S}$ and write $L = G \circ F$ as the composite of the quotient functor $F\colon \mathcal{T} \to \mathcal{U}$ with its right adjoint $G\colon \mathcal{U} \to \mathcal{T}$.

Now identify $A_\alpha(\mathcal{T}) = \operatorname{Add}_\alpha((\mathcal{T}^\alpha)^{\mathrm{op}}, \mathrm{Ab})$ and $A_\alpha(\mathcal{U}) = \operatorname{Add}_\alpha((\mathcal{U}^\alpha)^{\mathrm{op}}, \mathrm{Ab})$. The induced functor $f\colon \mathcal{T}^\alpha \to \mathcal{U}^\alpha$ equals, up to an equivalence, the quotient functor $\mathcal{T}^\alpha \to \mathcal{T}^\alpha/\mathcal{S}^\alpha$, by Theorem 7.2.1. From f we obtain a pair of adjoint functors f^* and f_* by Lemma B.8. Both functors are exact and the right adjoint f_* is fully faithful. Thus we obtain an exact localization functor $L' = f_* \circ f^*$ for $A_\alpha(\mathcal{T})$ by Corollary 2.4.2. The commutativity $H_\alpha \circ L \cong L' \circ H_\alpha$ and the assertions about acyclic and local objects then follow from Proposition 7.3.1. $\square$

7.7. Example: The derived category of an abelian Grothendieck category.

Let $\mathcal{A}$ be an abelian Grothendieck category. Then the derived category $\mathbf{D}(\mathcal{A})$ of unbounded chain complexes is a well generated triangulated category. Let us sketch an argument.
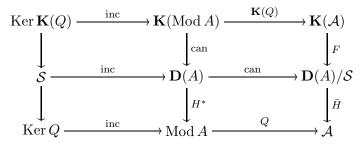
The Popescu-Gabriel theorem says that for each generator G of $\mathcal{A}$, the functor $T = \mathcal{A}(G, -)\colon \mathcal{A} \to \operatorname{Mod} A$ (where $A = \mathcal{A}(G, G)$ denotes the endomorphism ring of G) is fully faithful and admits an exact left adjoint, say Q; see [45, Theorem X.4.1]. Consider the cohomological functor $H\colon \mathbf{D}(A) \to \mathcal{A}$ taking a complex X to $Q(\coprod_{n \in \mathbb{Z}} H^n X)$. Then an application of Theorem 7.5.1 shows that $\mathcal{S} = \operatorname{Ker} H$ is well generated, and therefore $\mathbf{D}(A)/\mathcal{S}$ is well generated by Theorem 7.2.1.

Next observe that $\mathbf{K}(Q)$ induces an equivalence

$$
\mathbf{K}(\operatorname{Mod} A)/(\operatorname{Ker} \mathbf{K}(Q)) \xrightarrow{\ \sim\ } \mathbf{K}(\mathcal{A})
$$

since $\mathbf{K}(Q)$ has $\mathbf{K}(T)$ as a fully faithful right adjoint. Moreover, the cohomology of each object in the kernel of $\mathbf{K}(Q)$ lies in the kernel of Q. Thus we obtain the following

commutative diagram.

$$
\begin{array}{ccccc}
\operatorname{Ker}\mathbf{K}(Q) & \xrightarrow{\ \text{inc}\ } & \mathbf{K}(\operatorname{Mod}A) & \xrightarrow{\ \mathbf{K}(Q)\ } & \mathbf{K}(\mathcal{A}) \\
\downarrow & & \downarrow{\scriptstyle\text{can}} & & \downarrow{\scriptstyle F} \\
\mathcal{S} & \xrightarrow{\ \text{inc}\ } & \mathbf{D}(A) & \xrightarrow{\ \text{can}\ } & \mathbf{D}(A)/\mathcal{S} \\
\downarrow & & \downarrow{\scriptstyle H^*} & & \downarrow{\scriptstyle \bar{H}} \\
\operatorname{Ker}Q & \xrightarrow{\ \text{inc}\ } & \operatorname{Mod}A & \xrightarrow{\ Q\ } & \mathcal{A}
\end{array}
$$

It is easily checked that the kernel of F consists of all acyclic complexes. Thus F induces an equivalence $\mathbf{D}(\mathcal{A}) \xrightarrow{\sim} \mathbf{D}(A)/\mathcal{S}$.

7.8. **Notes.** Given a triangulated category $\mathcal{T}$, there are two basic questions when one studies exact localization functors $\mathcal{T} \to \mathcal{T}$. One can ask for the existence of a localization functor with some prescribed kernel, and one can ask for a classification, or at least some structural results, for the set of all localization functors on $\mathcal{T}$. Well generated categories provide a suitable setting for some partial answers.

The fact that cohomological functors induce localization functors is well known for compactly generated triangulated categories [28], but the result seems to be new for well generated categories. The localization theorem which describes the localization with respect to a small set of objects is due to Neeman [33]. The example of the derived category of an abelian Grothendieck category is discussed in [3, 34]. The description of the kernel of an exact functor between well generated categories seems to be new. A motivation for this is the telescope conjecture which is due to Bousfield and Ravenel [8, 40] and originally formulated for the stable homotopy category of CW-spectra.

It is interesting to note that the existence of localization functors depends to some extent on axioms from set theory; see for instance [11, 10].

8. Epilogue: Beyond well generatedness

Well generated triangulated categories were introduced by Neeman as a class of triangulated categories which includes all compactly generated categories and behaves well with respect to localization. We have discussed in Sections 6 and 7 most of the basic properties of well generated categories but the picture is still not complete because some important questions remain open. For instance, given a well generated triangulated category $\mathcal{T}$, we do not know when a localizing subcategory arises as the kernel of a localization functor and when it is generated by a small set of objects. Also, one might ask when the set of all localizing subcategories is small. Another aspect is Brown representability. We do know that every cohomological functor $\mathcal{T}^{\mathrm{op}} \to \mathrm{Ab}$ preserving small products is representable, but what about covariant functors $\mathcal{T} \to \mathrm{Ab}$? It seems that one obtains more insight by studying the universal cohomological functors $\mathcal{T} \to A_\alpha(\mathcal{T})$; in particular we need to know when they are full; see [43, 35] for some recent work in this direction.

Instead of answering these open questions, let us be adventurous and move a little bit beyond the class of well generated categories. In fact, there are natural examples of triangulated categories which are not well generated. Such examples arise from additive categories by taking their homotopy category of chain complexes. More precisely, let $\mathcal{A}$

be an additive category and suppose that $\mathcal{A}$ admits small coproducts. We denote by $\mathbf{K}(\mathcal{A})$ the category of chain complexes in $\mathcal{A}$ whose morphisms are the homotopy classes of chain maps. Take for instance the category $\mathcal{A} = \mathrm{Ab}$ of abelian groups. Then one can show that $\mathbf{K}(\mathrm{Ab})$ is not well generated; see [33]. In fact, more is true. The category $\mathbf{K}(\mathrm{Ab})$ admits no small set of generators, that is, any localizing subcategory generated by a small set of objects is a proper subcategory. However, it is not difficult to show that any localizing subcategory generated by a small set of objects is well generated. So we may think of $\mathbf{K}(\mathrm{Ab})$ as *locally well generated*. In fact, discussions with Jan Šťovíček suggest that $\mathbf{K}(\mathcal{A})$ is locally well generated whenever $\mathcal{A}$ is locally finitely presented; see [46]. Recall that $\mathcal{A}$ is *locally finitely presented* if $\mathcal{A}$ has filtered colimits and there exists a small set of finitely presented objects $\mathcal{A}_0$ such that every object can be written as the filtered colimit of objects in $\mathcal{A}_0$. On the other hand, $\mathbf{K}(\mathcal{A})$ is only generated by a small set of objects if $\mathcal{A} = \mathrm{Add}\,\mathcal{A}_0$ for some small set of objects $\mathcal{A}_0$. Here, $\mathrm{Add}\,\mathcal{A}_0$ denotes the smallest subcategory of $\mathcal{A}$ which contains $\mathcal{A}_0$ and is closed under taking small coproducts and direct summands. We refer to [46] for further details.

Appendix A. The abelianization of a triangulated category

Let $\mathcal{C}$ be an additive category. We consider functors $F\colon \mathcal{C}^{\mathrm{op}} \to \mathrm{Ab}$ into the category of abelian groups and call a sequence $F' \to F \to F''$ of functors *exact* if the induced sequence $F'X \to FX \to F''X$ of abelian groups is exact for all X in $\mathcal{C}$. A functor F is said to be *coherent* if there exists an exact sequence (called *presentation*)

$$\mathcal{C}(-, X) \longrightarrow \mathcal{C}(-, Y) \longrightarrow F \longrightarrow 0.$$

The morphisms between two coherent functors form a small set by Yoneda's lemma, and the coherent functors $\mathcal{C}^{\mathrm{op}} \to \mathrm{Ab}$ form an additive category with cokernels. We denote this category by $\widehat{\mathcal{C}}$.

A basic tool is the fully faithful *Yoneda functor* $h_{\mathcal{C}}\colon \mathcal{C} \to \widehat{\mathcal{C}}$ which sends an object X to $\mathcal{C}(-, X)$. One might think of this functor as the completion of $\mathcal{C}$ with respect to the formation of finite colimits. To formulate some further properties, we recall that a morphism $X \to Y$ is a *weak kernel* for a morphism $Y \to Z$ if the induced sequence $\mathcal{C}(-, X) \to \mathcal{C}(-, Y) \to \mathcal{C}(-, Z)$ is exact.

Lemma A.1. *Let $\mathcal{C}$ be an additive category.*

(1) *Given an additive functor $H\colon \mathcal{C} \to \mathcal{A}$ to an additive category which admits cokernels, there is (up to a unique isomorphism) a unique right exact functor $\bar{H}\colon \widehat{\mathcal{C}} \to \mathcal{A}$ such that $H = \bar{H} \circ h_{\mathcal{C}}$.*
(2) *If $\mathcal{C}$ has weak kernels, then $\widehat{\mathcal{C}}$ is an abelian category.*
(3) *If $\mathcal{C}$ has small coproducts, then $\widehat{\mathcal{C}}$ has small coproducts and the Yoneda functor preserves small coproducts.*

Proof. (1) Extend H to $\bar{H}$ by sending F in $\widehat{\mathcal{C}}$ with presentation

$$\mathcal{C}(-, X) \xrightarrow{(-,\phi)} \mathcal{C}(-, Y) \longrightarrow F \longrightarrow 0$$

to the cokernel of $H\phi$.

(2) The category $\widehat{\mathcal{C}}$ has cokernels, and it is therefore sufficient to show that $\widehat{\mathcal{C}}$ has kernels. To this end fix a morphism $F_1 \to F_2$ with the following presentation.

$$
\begin{array}{ccccccc}
\mathcal{C}(-, X_1) & \longrightarrow & \mathcal{C}(-, Y_1) & \longrightarrow & F_1 & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
\mathcal{C}(-, X_2) & \longrightarrow & \mathcal{C}(-, Y_2) & \longrightarrow & F_2 & \longrightarrow & 0
\end{array}
$$

We construct the kernel $F_0 \to F_1$ by specifying the following presentation.

$$
\begin{array}{ccccccc}
\mathcal{C}(-, X_0) & \longrightarrow & \mathcal{C}(-, Y_0) & \longrightarrow & F_0 & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
\mathcal{C}(-, X_1) & \longrightarrow & \mathcal{C}(-, Y_1) & \longrightarrow & F_1 & \longrightarrow & 0
\end{array}
$$

First the morphism $Y_0 \to Y_1$ is obtained from the weak kernel sequence

$$ Y_0 \longrightarrow X_2 \amalg Y_1 \longrightarrow Y_2. $$

Then the morphisms $X_0 \to X_1$ and $X_0 \to Y_0$ are obtained from the weak kernel sequence

$$ X_0 \longrightarrow X_1 \amalg Y_0 \longrightarrow Y_1. $$

(3) For every family of functors F_i having a presentation

$$ \mathcal{C}(-, X_i) \xrightarrow{(-, \phi_i)} \mathcal{C}(-, Y_i) \longrightarrow F_i \longrightarrow 0, $$

the coproduct $F = \coprod_i F_i$ has a presentation

$$ \mathcal{C}\left(-, \coprod_i X_i\right) \xrightarrow{(-, \amalg \phi_i)} \mathcal{C}\left(-, \coprod_i Y_i\right) \longrightarrow F \longrightarrow 0. $$

Thus coproducts in $\widehat{\mathcal{C}}$ are not computed pointwise. $\qquad\square$

The assigment $\mathcal{C} \mapsto \widehat{\mathcal{C}}$ is functorial in the following weak sense. Given a functor $F: \mathcal{C} \to \mathcal{D}$, there is (up to a unique isomorphism) a unique right exact functor $\widehat{F}: \widehat{\mathcal{C}} \to \widehat{\mathcal{D}}$ extending the composite $h_\mathcal{D} \circ F: \mathcal{C} \to \widehat{\mathcal{D}}$.

Now let $\mathcal{T}$ be a triangulated category. Then we write $A(\mathcal{T}) = \widehat{\mathcal{T}}$ and call this category the *abelianization* of $\mathcal{T}$, because the Yoneda functor $\mathcal{T} \to A(\mathcal{T})$ is the universal cohomological functor for $\mathcal{T}$.

Lemma A.2. *Let $\mathcal{T}$ be a triangulated category. Then the category $A(\mathcal{T})$ is abelian and the Yoneda functor $h_\mathcal{T}: \mathcal{T} \to A(\mathcal{T})$ is cohomological.*

(1) *Given a cohomological functor $H: \mathcal{T} \to \mathcal{A}$ to an abelian category, there is (up to a unique isomorphism) a unique exact functor $\bar{H}: A(\mathcal{T}) \to \mathcal{A}$ such that $H = \bar{H} \circ h_\mathcal{T}$.*

(2) *Given an exact functor $F: \mathcal{T} \to \mathcal{T}'$ between triangulated categories, there is (up to a unique isomorphism) a unique exact functor $A(F): A(\mathcal{T}) \to A(\mathcal{T}')$ such that $h_{\mathcal{T}'} \circ F = A(F) \circ h_\mathcal{T}$.*

Proof. The category $\mathcal{T}$ has weak kernels and therefore $A(\mathcal{T})$ is abelian. Note that the weak kernel of a morphism $Y \to Z$ is obtained by completing the morphism to an exact triangle $X \to Y \to Z \to SX$.

(1) Let $H\colon \mathcal{T} \to \mathcal{A}$ be a cohomological functor and let $\bar{H}\colon A(\mathcal{T}) \to \mathcal{A}$ be the right exact functor extending H which exists by Lemma A.1. Then $\bar{H}$ is exact because H is cohomological.

(2) Let $F\colon \mathcal{T} \to \mathcal{T}'$ be exact. Then $H = h_{\mathcal{T}'} \circ F$ is a cohomological functor and we let $A(F) = \bar{H}$ be the exact functor which extends H. $\square$

The assignment $\mathcal{T} \mapsto A(\mathcal{T})$ from triangulated categories to abelian categories preserves various properties of exact functors between triangulated categories. Let us mention some of them.

Lemma A.3. *Let $F\colon \mathcal{T} \to \mathcal{T}'$ and $G\colon \mathcal{T}' \to \mathcal{T}$ be exact functors between triangulated categories.*

(1) *F is fully faithful if and only if $A(F)$ is fully faithful.*
(2) *If F induces an equivalence $\mathcal{T}/\operatorname{Ker} F \xrightarrow{\sim} \mathcal{T}'$, then $A(F)$ induces an equivalence $A(\mathcal{T})/(\operatorname{Ker} A(F)) \xrightarrow{\sim} A(\mathcal{T}')$.*
(3) *F preserves small (co)products if and only if $A(F)$ preserves small (co)products.*
(4) *F is left adjoint to G if and only if $A(F)$ is left adjoint to $A(G)$.*

Proof. Straightforward. $\square$

Notes. The abelianization of a triangulated category appears in Verdier's thèse [48] and in Freyd's work on the stable homotopy category [15]. Note that their construction is slightly different from the one given here, which is based on coherent functors in the sense of Auslander [4].

Appendix B. Locally presentable abelian categories

Fix a regular cardinal α and a small additive category $\mathcal{C}$ which admits α-coproducts. We denote by $\operatorname{Add}(\mathcal{C}^{\mathrm{op}}, \mathrm{Ab})$ the category of additive functors $\mathcal{C}^{\mathrm{op}} \to \mathrm{Ab}$ into the category of abelian groups. This is an abelian category which admits small (co)products. In fact, (co)kernels and (co)products are computed pointwise in Ab. Given functors F and G in $\operatorname{Add}(\mathcal{C}^{\mathrm{op}}, \mathrm{Ab})$, we write $\operatorname{Hom}_{\mathcal{C}}(F, G)$ for the set of morphisms $F \to G$. The most important objects in $\operatorname{Add}(\mathcal{C}^{\mathrm{op}}, \mathrm{Ab})$ are the *representable functors* $\mathcal{C}(-, X)$ with $X \in \mathcal{C}$. Recall that Yoneda's lemma provides a bijection

$$\operatorname{Hom}_{\mathcal{C}}(\mathcal{C}(-, X), F) \xrightarrow{\sim} FX$$

for all $F\colon \mathcal{C}^{\mathrm{op}} \to \mathrm{Ab}$ and $X \in \mathcal{C}$.

We denote by $\operatorname{Add}_{\alpha}(\mathcal{C}^{\mathrm{op}}, \mathrm{Ab})$ the full subcategory of $\operatorname{Add}(\mathcal{C}^{\mathrm{op}}, \mathrm{Ab})$ which is formed by all functors preserving α-products. This is an exact abelian subcategory, because kernels and cokernels of morphism between α-product preserving functors preserve α-products. In particular, $\operatorname{Add}_{\alpha}(\mathcal{C}^{\mathrm{op}}, \mathrm{Ab})$ is an abelian category.

Now suppose that $\mathcal{C}$ admits cokernels. Then $\operatorname{Lex}_{\alpha}(\mathcal{C}^{\mathrm{op}}, \mathrm{Ab})$ denotes the full subcategory of $\operatorname{Add}(\mathcal{C}^{\mathrm{op}}, \mathrm{Ab})$ which is formed by all left exact functors preserving α-products. This category is locally presentable in the sense of Gabriel and Ulmer and we refer to [17, §5] for an extensive treatment. In this appendix we collect some basic facts.

First observe that α-filtered colimits in $\operatorname{Lex}_{\alpha}(\mathcal{C}^{\mathrm{op}}, \mathrm{Ab})$ are computed pointwise. This follows from the fact that in Ab taking α-filtered colimits commutes with taking α-limits; see [17, Satz 5.12]. In particular, $\operatorname{Lex}_{\alpha}(\mathcal{C}^{\mathrm{op}}, \mathrm{Ab})$ has small coproducts because every small coproduct is the α-filtered colimit of its subcoproducts with less than α factors.

Next we show that one can identify $\mathrm{Add}_\alpha(\mathcal{C}^{\mathrm{op}}, \mathrm{Ab})$ with a category of left exact functors. To this end consider the Yoneda functor $h_\mathcal{C} \colon \mathcal{C} \to \widehat{\mathcal{C}}$ taking X to $\mathcal{C}(-, X)$.

Lemma B.1. *Let $\mathcal{C}$ be a small additive category with α-coproducts. Then the Yoneda functor induces an equivalence*

$$\mathrm{Lex}_\alpha(\widehat{\mathcal{C}}^{\mathrm{op}}, \mathrm{Ab}) \xrightarrow{\ \sim\ } \mathrm{Add}_\alpha(\mathcal{C}^{\mathrm{op}}, \mathrm{Ab})$$

by taking a functor F to $F \circ h_\mathcal{C}$.

Proof. Use that every additive functor $\mathcal{C}^{\mathrm{op}} \to \mathrm{Ab}$ extends uniquely to a left exact functor $\widehat{\mathcal{C}}^{\mathrm{op}} \to \mathrm{Ab}$; see Lemma A.1. $\qquad\square$

From now on we assume that $\mathcal{C}$ admits α-coproducts and cokernels. Given any additive functor $F \colon \mathcal{C}^{\mathrm{op}} \to \mathrm{Ab}$, we consider the category $\mathcal{C}/F$ whose objects are pairs (C, μ) consisting of an object $C \in \mathcal{C}$ and an element $\mu \in FC$. A morphism $(C, \mu) \to (C', \mu')$ is a morphism $\phi \colon C \to C'$ such that $F\phi(\mu') = \mu$.

Lemma B.2. *Let $F \colon \mathcal{C}^{\mathrm{op}} \to \mathrm{Ab}$ be an additive functor.*

(1) *The canonical morphism*

$$\varinjlim_{(C,\mu) \in \mathcal{C}/F} \mathcal{C}(-, C) \longrightarrow F$$

in $\mathrm{Add}(\mathcal{C}^{\mathrm{op}}, \mathrm{Ab})$ is an isomorphism.

(2) *The functor F belongs to $\mathrm{Lex}_\alpha(\mathcal{C}^{\mathrm{op}}, \mathrm{Ab})$ if and only if the category $\mathcal{C}/F$ is α-filtered.*

Proof. (1) is easy. For (2), see [17, Satz 5.3]. $\qquad\square$

The representable functors in $\mathrm{Lex}_\alpha(\mathcal{C}^{\mathrm{op}}, \mathrm{Ab})$ share the following finiteness property. Recall that an object X from an additive category $\mathcal{A}$ with α-filtered colimits is α-*presentable* if the representable functor $\mathcal{A}(X, -) \colon \mathcal{A} \to \mathrm{Ab}$ preserves α-filtered colimits. Next observe that the inclusion $\mathrm{Lex}_\alpha(\mathcal{C}^{\mathrm{op}}, \mathrm{Ab}) \to \mathrm{Add}(\mathcal{C}^{\mathrm{op}}, \mathrm{Ab})$ preserves α-filtered colimits. This follows from the fact that in Ab taking α-filtered colimits commutes with taking α-limits. This has the following consequence.

Lemma B.3. *For each X in $\mathcal{C}$, the representable functor $\mathcal{C}(-, X)$ is an α-presentable object of $\mathrm{Lex}_\alpha(\mathcal{C}^{\mathrm{op}}, \mathrm{Ab})$.*

Proof. Combine Yoneda's lemma with the fact that the inclusion $\mathrm{Lex}_\alpha(\mathcal{C}^{\mathrm{op}}, \mathrm{Ab}) \to \mathrm{Add}(\mathcal{C}^{\mathrm{op}}, \mathrm{Ab})$ preserves α-filtered colimits. $\qquad\square$

There is a general result for the category $\mathrm{Lex}_\alpha(\mathcal{C}^{\mathrm{op}}, \mathrm{Ab})$ which says that taking α-filtered colimits commutes with taking α-limits; see [17, Korollar 7.12]. Here we need the following special case.

Lemma B.4. *Suppose the category $\mathrm{Lex}_\alpha(\mathcal{C}^{\mathrm{op}}, \mathrm{Ab})$ is abelian. Then an α-filtered colimit of exact sequences is again exact.*

Proof. We need to show that taking α-filtered colimits commutes with taking kernels and cokernels. A cokernel is nothing but a colimit and therefore taking colimits and cokernels commute. The statement about kernels follows from the fact that the inclusion $\mathrm{Lex}_\alpha(\mathcal{C}^{\mathrm{op}}, \mathrm{Ab}) \to \mathrm{Add}(\mathcal{C}^{\mathrm{op}}, \mathrm{Ab})$ preserves kernels and α-filtered colimits. Thus we can compute kernels and α-filtered colimits in $\mathrm{Add}(\mathcal{C}^{\mathrm{op}}, \mathrm{Ab})$ and therefore in the category

Ab of abelian groups. In Ab it is well known that taking kernels and filtered colimits commute. $\square$

Lemma B.5. *Suppose that $\mathcal{C}$ is abelian. Then $\mathrm{Lex}_\alpha(\mathcal{C}^{\mathrm{op}}, \mathrm{Ab})$ is abelian and the Yoneda functor $h_\mathcal{C}\colon \mathcal{C} \to \mathrm{Lex}_\alpha(\mathcal{C}^{\mathrm{op}}, \mathrm{Ab})$ is exact. Given an abelian category $\mathcal{A}$ which admits small coproducts and exact α-filtered colimits, and given a functor $F\colon \mathrm{Lex}_\alpha(\mathcal{C}^{\mathrm{op}}, \mathrm{Ab}) \to \mathcal{A}$ preserving α-filtered colimits, we have that F is exact if and only if $F \circ h_\mathcal{C}$ is exact.*

Proof. We use the analogue of Lemma B.2 for morphisms which says that each morphism ϕ in $\mathrm{Lex}_\alpha(\mathcal{C}^{\mathrm{op}}, \mathrm{Ab})$ can be written as α-filtered colimit $\phi = \underset{i\in\mathcal{C}/\phi}{\mathrm{colim}}\, \phi_i$ of morphisms between representable functors. Thus one computes

$$\mathrm{Coker}\,\phi = \underset{i\in\mathcal{C}/\phi}{\mathrm{colim}}\,\mathrm{Coker}\,\phi_i \quad \text{and} \quad \mathrm{Ker}\,\phi = \underset{i\in\mathcal{C}/\phi}{\mathrm{colim}}\,\mathrm{Ker}\,\phi_i,$$

and we see that $\mathrm{Lex}_\alpha(\mathcal{C}^{\mathrm{op}}, \mathrm{Ab})$ is abelian; see Lemma B.4. The formula for kernels and cokernels shows that each exact sequence can be written as α-filtered colimit of exact sequences in the image of the Yoneda embedding. The criterion for the exactness of a functor $\mathrm{Lex}_\alpha(\mathcal{C}^{\mathrm{op}}, \mathrm{Ab}) \to \mathcal{A}$ is an immediate consequence. $\square$

Let $\mathcal{A}$ be a cocomplete additive category. We denote by $\mathcal{A}^\alpha$ the full subcategory which is formed by all α-presentable objects. Following [17], the category $\mathcal{A}$ is called *locally α-presentable* if $\mathcal{A}^\alpha$ is small and each object is an α-filtered colimit of α-presentable objects. We call $\mathcal{A}$ *locally presentable* if it is locally β-presentable for some cardinal β. Note that we have for each locally presentable category $\mathcal{A}$ a filtration $\mathcal{A} = \bigcup_\beta \mathcal{A}^\beta$ where β runs through all regular cardinals. We have already seen that $\mathrm{Lex}_\alpha(\mathcal{C}^{\mathrm{op}}, \mathrm{Ab})$ is locally α-presentable, and the next lemma implies that, up to an equivalence, all locally α-presentable categories are of this form.

Let $f\colon \mathcal{C} \to \mathcal{A}$ be a fully faithful and right exact functor into a cocomplete additive category. Suppose that f preserves α-coproducts and that each object in the image of f is α-presentable. Then f induces the functor

$$f_*\colon \mathcal{A} \longrightarrow \mathrm{Lex}_\alpha(\mathcal{C}^{\mathrm{op}}, \mathrm{Ab}), \quad X \mapsto \mathcal{A}(f-, X),$$

and the following lemma discusses its left adjoint.

Lemma B.6. *There is a fully faithful functor $f^*\colon \mathrm{Lex}_\alpha(\mathcal{C}^{\mathrm{op}}, \mathrm{Ab}) \to \mathcal{A}$ which sends each representable functor $\mathcal{C}(-, X)$ to fX and identifies $\mathrm{Lex}_\alpha(\mathcal{C}^{\mathrm{op}}, \mathrm{Ab})$ with the full subcategory of $\mathcal{A}$ formed by all colimits of objects in the image of f. The functor f^* is a left adjoint of f_*.*

Proof. The functor is the left Kan extension of f; it takes $F = \underset{(C,\mu)\in\mathcal{C}/F}{\mathrm{colim}}\, \mathcal{C}(-, C)$ in $\mathrm{Lex}_\alpha(\mathcal{C}^{\mathrm{op}}, \mathrm{Ab})$ to $\underset{(C,\mu)\in\mathcal{C}/F}{\mathrm{colim}}\, fC$ in $\mathcal{A}$. We refer to [17, Satz 7.8] for details. $\square$

Suppose now that $\mathcal{C}$ is a triangulated category. The following lemma characterizes the cohomological functors $\mathcal{C}^{\mathrm{op}} \to \mathrm{Ab}$.

Lemma B.7. *Let $\mathcal{C}$ be a small triangulated category and suppose $\mathcal{C}$ admits α-coproducts. For a functor F in $\mathrm{Add}_\alpha(\mathcal{C}^{\mathrm{op}}, \mathrm{Ab})$ the following are equivalent.*

(1) *The category $\mathcal{C}/F$ is α-filtered.*

(2) F is an α-filtered colimit of representable functors.

(3) F is a cohomological functor.

Proof. The implications $(1) \Rightarrow (2) \Rightarrow (3)$ are clear. So we prove $(3) \Rightarrow (1)$. It is convenient to identify $\mathrm{Add}_\alpha(\mathcal{C}^{\mathrm{op}}, \mathrm{Ab})$ with $\mathrm{Lex}_\alpha(\widehat{\mathcal{C}}^{\mathrm{op}}, \mathrm{Ab})$ and this identifies F with the left exact functor $\bar{F} \colon \widehat{\mathcal{C}} \to \mathrm{Ab}$ which extends F. In fact, $\bar{F}$ is exact since F is cohomological, by Lemma A.2. Now write $\bar{F}$ as α-filtered colimit of representable functors $\bar{F} = \underset{(M,\nu) \in \widehat{\mathcal{C}}/\bar{F}}{\mathrm{colim}} \ \widehat{\mathcal{C}}(-, M)$; see Lemma B.2. The exactness of $\bar{F}$ implies that the representable functors $\mathcal{C}(-, C)$ with $C \in \mathcal{C}$ form a full subcategory of $\widehat{\mathcal{C}}/\bar{F}$ which is cofinal. We identify this subcategory with $\mathcal{C}/F$ and conclude from Lemma 6.4.1 that $\mathcal{C}/F$ is α-filtered. $\qquad\square$

Next we discuss the functoriality of the assignment $\mathcal{C} \mapsto \mathrm{Add}_\alpha(\mathcal{C}^{\mathrm{op}}, \mathrm{Ab})$.

Lemma B.8. *Let $f \colon \mathcal{C} \to \mathcal{D}$ be an exact functor between small triangulated categories which admit α-coproducts. Suppose that f preserves α-coproducts. Then the restriction functor*

$$f_* \colon \mathrm{Add}_\alpha(\mathcal{D}^{\mathrm{op}}, \mathrm{Ab}) \longrightarrow \mathrm{Add}_\alpha(\mathcal{C}^{\mathrm{op}}, \mathrm{Ab}), \quad F \mapsto F \circ f,$$

has a left adjoint f^* which sends $\mathcal{C}(-, X)$ to $\mathcal{D}(-, fX)$ for all X in $\mathcal{C}$. Moreover, the following holds.

(1) *The functors f_* and f^* are exact.*

(2) *Suppose f induces an equivalence $\mathcal{C}/\mathrm{Ker} f \xrightarrow{\sim} \mathcal{D}$. Then f_* is fully faithful.*

(3) *Suppose f is fully faithful. Then f^* is fully faithful. Moreover, a cohomological functor $F \colon \mathcal{D}^{\mathrm{op}} \to \mathrm{Ab}$ is in the essential image of f^* if and only if every morphism $\mathcal{D}(-, D) \to F$ factors through $\mathcal{D}(-, fC)$ for some object C in $\mathcal{C}$.*

(4) *A cohomological functor $F \colon \mathcal{C}^{\mathrm{op}} \to \mathrm{Ab}$ belongs to the kernel of f^* if and only if every morphism $\mathcal{C}(-, C) \to F$ factors through a morphism $\mathcal{C}(-, \gamma) \colon \mathcal{C}(-, C) \to \mathcal{C}(-, C')$ such that $f\gamma = 0$.*

Proof. The left adjoint of f_* is the left Kan extension. We can describe it explicitly if we identify $\mathrm{Add}_\alpha(\mathcal{C}^{\mathrm{op}}, \mathrm{Ab})$ with $\mathrm{Lex}_\alpha(\widehat{\mathcal{C}}^{\mathrm{op}}, \mathrm{Ab})$; see Lemma B.1. Given a functor F in $\mathrm{Lex}_\alpha(\widehat{\mathcal{C}}^{\mathrm{op}}, \mathrm{Ab})$ written as α-filtered colimit $F = \underset{(C,\mu) \in \widehat{\mathcal{C}}/F}{\mathrm{colim}} \ \widehat{\mathcal{C}}(-, C)$ of representable functors, we put

$$f^* F = \underset{(C,\mu) \in \widehat{\mathcal{C}}/F}{\mathrm{colim}} \ \widehat{\mathcal{D}}(-, \widehat{f}C).$$

Thus f^* makes the following diagram commutative.

$$
\begin{array}{ccccccc}
\mathcal{C} & \xrightarrow{h_\mathcal{C}} & \widehat{\mathcal{C}} & \xrightarrow{h_{\widehat{\mathcal{C}}}} & \mathrm{Lex}_\alpha(\widehat{\mathcal{C}}^{\mathrm{op}}, \mathrm{Ab}) & \xrightarrow{=} & \mathrm{Add}_\alpha(\mathcal{C}^{\mathrm{op}}, \mathrm{Ab}) \\
\downarrow{\scriptstyle f} & & \downarrow{\scriptstyle \widehat{f}} & & \downarrow{\scriptstyle f^*} & & \downarrow{\scriptstyle f^*} \\
\mathcal{D} & \xrightarrow{h_\mathcal{D}} & \widehat{\mathcal{D}} & \xrightarrow{h_{\widehat{\mathcal{D}}}} & \mathrm{Lex}_\alpha(\widehat{\mathcal{D}}^{\mathrm{op}}, \mathrm{Ab}) & \xrightarrow{=} & \mathrm{Add}_\alpha(\mathcal{D}^{\mathrm{op}}, \mathrm{Ab})
\end{array}
$$

We check that f^* is a left adjoint for f_*. For a representable functor $F = \widehat{\mathcal{C}}(-, X)$ we have

$$\mathrm{Hom}_{\widehat{\mathcal{D}}}(f^*\widehat{\mathcal{C}}(-, X), G) = \mathrm{Hom}_{\widehat{\mathcal{D}}}(\widehat{\mathcal{D}}(-, \widehat{f}X), G) \cong G(\widehat{f}X)$$
$$= f_*G(X) \cong \mathrm{Hom}_{\widehat{\mathcal{C}}}(\widehat{\mathcal{C}}(-, X), f_*G)$$

for all G in $\mathrm{Lex}_\alpha(\widehat{\mathcal{D}}^{\mathrm{op}}, \mathrm{Ab})$. Clearly, this isomorphism extends to every colimit of representable functors.

(1) The exactness of f_* is clear because a sequence $F' \to F \to F''$ in $\mathrm{Add}_\alpha(\mathcal{C}^{\mathrm{op}}, \mathrm{Ab})$ is exact if and only if $F'X \to FX \to F''X$ is exact for all X in $\mathcal{C}$. For the exactness of f^* we identify again $\mathrm{Add}(\mathcal{C}^{\mathrm{op}}, \mathrm{Ab})$ with $\mathrm{Lex}_\alpha(\widehat{\mathcal{C}}^{\mathrm{op}}, \mathrm{Ab})$ and apply Lemma B.5. Thus we need to check that the composition of f^* with the Yoneda functor $h_{\widehat{\mathcal{C}}}$ is exact. But we have that $f^* \circ h_{\widehat{\mathcal{C}}} = h_{\widehat{\mathcal{D}}} \circ \widehat{f}$, and now the exactness follows from that of f. Finally, we use the fact that taking α-filtered colimits in $\mathrm{Add}_\alpha(\mathcal{D}^{\mathrm{op}}, \mathrm{Ab})$ is exact by Lemma B.4.

(2) It is well known that for any epimorphism $f \colon \mathcal{C} \to \mathcal{D}$ of additive categories inducing a bijection $\mathrm{Ob}\,\mathcal{C} \to \mathrm{Ob}\,\mathcal{D}$, the restriction functor $\mathrm{Add}(\mathcal{D}^{\mathrm{op}}, \mathrm{Ab}) \to \mathrm{Add}(\mathcal{C}^{\mathrm{op}}, \mathrm{Ab})$ is fully faithful; see [29, Corollary 5.2]. Given a triangulated subcategory $\mathcal{C}' \subseteq \mathcal{C}$, the quotient functor $\mathcal{C} \to \mathcal{C}/\mathcal{C}'$ is an epimorphism. Thus the assertion follows since $\mathrm{Add}_\alpha(\mathcal{C}^{\mathrm{op}}, \mathrm{Ab})$ is a full subcategory of $\mathrm{Add}(\mathcal{C}^{\mathrm{op}}, \mathrm{Ab})$.

(3) We keep our identification $\mathrm{Add}_\alpha(\mathcal{C}^{\mathrm{op}}, \mathrm{Ab}) = \mathrm{Lex}_\alpha(\widehat{\mathcal{C}}^{\mathrm{op}}, \mathrm{Ab})$ and consider the adjunction morphism $\eta \colon \mathrm{Id} \to f_* \circ f^*$. We claim that η is an isomorphism. Because f is fully faithful, ηF is an isomorphism for each representable functor $F = \widehat{\mathcal{C}}(-, X)$. It follows that ηF is an isomorphism for all F since f^* and f_* both preserve α-filtered colimits and each F can be expressed as α-filtered colimit of representable functors. Now Proposition 2.3.1 implies that f^* is fully faithful.

Let F be a cohomological functor in $\mathrm{Add}_\alpha(\mathcal{D}^{\mathrm{op}}, \mathrm{Ab})$ and apply Lemma B.7 to write the functor as α-filtered colimit $F = \varinjlim_{(D,\mu) \in \mathcal{D}/F} \mathcal{D}(-, D)$ of representable functors. Suppose first that every morphism $\mathcal{D}(-, D) \to F$ factors through $\mathcal{D}(-, fC)$ for some $C \in \mathcal{C}$. Then $\mathrm{Im}\, f/F$ is a cofinal subcategory of $\mathcal{D}/F$ and therefore $F = \varinjlim_{(D,\mu) \in \mathrm{Im}\, f/F} \mathcal{D}(-, D)$ by Lemma 6.4.1. Thus F belongs to the essential image of f^* since $\mathcal{D}(-, fC) = f^*\mathcal{C}(-, C)$ for all $C \in \mathcal{C}$ and the essential image is closed under taking colimits. Now suppose that F belongs to the essential image of f^*. Then $F = f^*G \cong f^*f_*f^*G = f^*f_*F$ for some G. The functor f_*F is cohomological and therefore $f_*F = \varinjlim_{(C,\mu) \in \mathcal{C}/f_*F} \mathcal{C}(-, C)$, again by Lemma B.7. Thus $F \cong \varinjlim_{(C,\mu) \in \mathcal{C}/f_*F} \mathcal{D}(-, fC)$ and we use Lemma B.3 to conclude that each morphism $\mathcal{D}(-, D) \to F$ factors through $\mathcal{D}(-, fC)$ for some $(C, \mu) \in \mathcal{C}/f_*F$.

(4) Let F be a cohomological functor in $\mathrm{Add}_\alpha(\mathcal{C}^{\mathrm{op}}, \mathrm{Ab})$ and apply Lemma B.7 to write the functor as α-filtered colimit $F = \varinjlim_{(C,\mu) \in \mathcal{C}/F} \mathcal{C}(-, C)$ of representable functors. Now $f^*F = \varinjlim_{(C,\mu) \in \mathcal{C}/F} \mathcal{D}(-, fC) = 0$ if and only if for each $D \in \mathcal{D}$, we have $\varinjlim_{(C,\mu) \in \mathcal{C}/F} \mathcal{D}(D, fC) = 0$. This happens iff for each $(C, \mu) \in \mathcal{C}/F$, we find a morphism $\gamma \colon C \to C'$ in $\mathcal{C}/F$ inducing a map $\mathcal{D}(fC, fC) \to \mathcal{D}(fC, fC')$ which annihilates the

identity morphism. But this means that $f\gamma = 0$ and that $\mu\colon \mathcal{C}(-, C) \to F$ factors through $\mathcal{C}(-, \gamma)$. □

Notes. Locally presentable categories were introduced and studied by Gabriel and Ulmer in [17]; see [1] for a modern treatment. In [33], Neeman initiated the use of locally presentable abelian categories for studying triangulated categories.

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