SOME THEOREMS ON TOPOLOGICAL MANIFOLDS

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We list here some theorems about (metrizable) topological manifolds, most of which were announced in [7], [8], with proofs to appear in [9], [10], [11]. In addition, see [12], [13], [15], [6].

First is the theorem on existence and uniqueness of PL (manifold) structures, which settles the triangulation problem and the Hauptvermutung for manifolds. (Cohomology will be Cech cohomology throughout.)

<u>Theorem 1</u>: Let Q^q be a q-dimensional topological manifold and let C be a closed subset of Q. Suppose that a neighborhood of C has a PL structure Σ_0. Let $q \geq 6$, or $q \geq 5$ if $\partial Q \subset C$.
If $H^4(Q, C; Z_2) = 0$, then Q has a PL structure Σ which agrees with Σ_0 near C. Given the PL structure Σ, then the PL structures (up to isotopy rel C) on Q which agree with Σ near C are in one to one correspondence with the elements of $H^3(Q, C; Z_2)$.

<u>Definition</u>: Let Σ and Θ be two PL structures on Q. Then Σ and Θ are said to be equivalent (up to homotopy, or up to isotopy) if there exists a PL homeomorphism $f: Q_\Sigma \to Q_\Theta$ (which is homotopic, or isotopic, to the identity). In the relative case, $\Sigma = \Theta$ near C and the homotopy, or isotopy, fixes a neighbourhood of C, and respects both ∂Q and $Q-C$.

Sullivan proved the following theorem on uniqueness up to homotopy [21], [1], [17].

<u>Theorem 2</u>: For the data of Theorem 1 suppose $H^4(Q, C; Z)$ has no 2-torsion. Suppose also that Q is compact and C is a codimension 1 (locally flat) submanifold and suppose that for each connected component Q' of $Q-C$ either
(i) $\partial Q'$ is connected and $\pi_1 \partial Q' \cong \pi_1 Q'$ by inclusion, or
(ii) $\partial Q' = \emptyset$ and $\pi_1 Q' = 0$.
Then any two PL structures on Q which agree near C are homotopic (rel C).

The two uniqueness theorems are related as follows: let $\beta: H^3(Q, C; Z_2) \to H^4(Q, C; Z)$ be the Bockstein homomorphism coming from the exact sequence of coefficients $0 \to Z \to Z \to Z_2 \to 0$ (which is the sequence $0 \to \pi_4(G/PL) \to \pi_4(G/TOP) \to \pi_3(TOP/PL) \to 0$).
Assume the conditions of Theorem 2. Then if Q has PL structures Σ and Θ corresponding to $[\Sigma]$, $[\Theta] \in H^3(Q, C; Z_2)$, then Σ and Θ are equivalent (rel C) up to homotopy iff $\beta([\Sigma]) = \beta([\Theta])$.

This fails however for $S^3 \times S^1 \times S^1$ which has two PL structures up to isomorphism homotopy or isotopy.

The uniqueness half of Theorem 1 fails for 3-manifolds, where PL structures unique up to isotopy (Moise). But it is a reasonable conjecture that Theorem 1 holds whenever $\dim Q \neq 3 \neq \dim \partial Q$.

Theorem 1 is elucidated by two more detailed theorems 3 and 4 below, which also deal with DIFF (smooth C^∞) manifold structures.

Let $CAT(q)$ be the semi-simplicial group of CAT automorphisms of R^q fixing the origin. Here CAT = DIFF, PL or TOP. Recall that $DIFF(q) \simeq Gl(q) \simeq O(q)$, [14].

Theorem 3. [7], [13], [15], [11]. Let $r \geq 5$ throughout. Then the stabilization map $s: \pi_k(TOP(r), PL(r)) \to \pi_k(TOP(r+1), PL(r+1))$ is an isomorphism. Further $\pi_k(TOP(r), PL(r)) = \pi_k(TOP, PL)$ is zero if $k \neq 3$ and is Z_2 if $k = 3$. For DIFF in place of PL, the stabilization s is bijective if $k \leq r+1$, and surjective if $k = r+2$ (using Cerf's pseudo-isotopy result).

Thus, for example, the classifying map $f: Q \to B_{TOP(q)}$ for the tangent q-microbundle τQ of Q lifts to $B_{PL(q)}$ if and only if an obstruction in $H^4(Q; \pi_3(TOP(q), PL(q))) = H^4(Q; Z_2)$ is zero. By the classification theorem below, this is the obstruction of Theorem 1 to imposing a PL manifold structure on Q.

Let $\gamma: B_{CAT(q)} \to B_{TOP(q)}$ be the map of classifying spaces for CAT and TOP q-microbundles that corresponds to forgetting CAT structure. Form a triangle

$$\begin{array}{ccc} B_{CAT(q)} & \xrightarrow{\gamma} & B_{TOP(q)} \\ {\scriptstyle i}\searrow & & \nearrow{\scriptstyle \gamma'} \\ & B'_{CAT(q)} & \end{array}$$

where i is a homotopy equivalence and γ' is a Hurewicz fibration. For example $B'_{CAT(q)}$ can be the space of maps $(B_{TOP(q)}, B_{CAT(q)})^{([0,1], 0)}$.

Consider a TOP q-manifold Q, a closed subset $C \subset Q$ and a CAT structure Σ_0 on a neighbourhood of C. Define $Cat(Q \text{ rel } C; \Sigma_0)$ to be the (semi-simplicial, Kan) complex of which a typical d-simplex is a CAT structure Γ on $\Delta^d \times Q$ such that projection onto Δ^d is a CAT submersion *) and $\Gamma = \Delta^d \times \Sigma_0$ near $\Delta^d \times C$. If $C = \emptyset$, we write $Cat(Q)$.

Similarly, let $Lift(f \text{ rel } C, f_0)$ be the (semi-simplicial, Kan) complex of lifts of $f: Q \to B_{TOP(q)}$ to maps $f': Q \to B'_{CAT(q)}$, where f' agrees near C with a fixed lifting f_0 of f near C induced by Σ_0.

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*) A map $f: X \to Y$ of CAT manifolds is a CAT submersion if $f^{-1}(y)$ is a CAT submanifold of X for each y in Y and for each point $x \in X$ there exists an open neighbourhood V of $f(x)$ in Y, an open neighbourhood U of x in the submanifold $f^{-1}f(x)$ and a CAT isomorphism ϕ of $U \times V$ onto an open neighbourhood of x in X such that $f\phi(u, v) = v$ for each (u, v) in $U \times V$.

Theorem 4. (Classification Theorem) [10]. There is a natural map (defined up to homotopy) $f: \mathrm{Cat}\,(Q \text{ rel } C, \Sigma_0) \to \mathrm{Lift}\,(f \text{ rel } C, f_0)$ that is a homotopy equivalence if $q \neq 4$ and $\partial Q \subset C$.

If $\partial Q \not\subset C$ but $q \geq 6$, Lift $(f \text{ rel } C, f_0)$ must be replaced by a new more complicated complex that takes account of liftings of the map $\partial Q \to B_{TOP(q-1)}$ classifying $\tau\partial Q$; also q must be ≥ 6. But happily, for $q \geq 6$, the new complex has the same π_0, and in case CAT = PL it is actually equivalent to Lift $(f \text{ rel } C, f_0)$ by Theorem 3. Lashof [12], Morlet [15], and C. Rourke have also proved versions of the classification theorem. We discuss ingredients of the proof below.

Let Γ be a CAT structure on $I \times Q$ and let $0 \times \Sigma$ be its restriction to $0 \times Q$. Suppose $\Gamma = I \times \Sigma$ near $I \times C$.

Theorem 5. (Concordance implies isotopy) [9]. Let $q \geq 6$, or $q \geq 5$ if $\partial Q \subset C$. There exists an isotopy $h_t: I \times Q \to I \times Q$, $t \in [0, 1]$, fixing $0 \times Q$ and a neighbourhood of $I \times C$, such that h_0 = identity and $h_1: I \times Q_\Sigma \to (I \times Q)_\Gamma$ is a CAT isomorphism.
Furthermore h_t can be chosen arbitrarily close to the identity.

Theorem 6. (Sliced concordance implies isotopy) [9]. Suppose $q \neq 4$, and $q \neq 5$ if $\partial Q \not\subset C$. Also suppose the projection $(I \times Q)_\Gamma \to I$ is a CAT submersion. Then h_t exists as in Theorem 5. Furthermore $h_t(s \times Q) = s \times Q$ for all $s \in I$. This assertion holds if, more generally, a pair (Δ^d, Λ) replaces the pair $(I, 0)$, where Δ^d is the standard d-simplex and Λ is a contractible subcomplex such that $\Gamma \mid \Lambda \times Q = \Lambda \times \Sigma$.

Theorem 6 readily implies the sliced concordance extension theorem:

Theorem 7. Let Q' be an open subset of Q. Suppose $q \neq 4$, and $q \neq 5$ if $\partial Q \neq \emptyset$. Then the restriction map $\mathrm{Cat}(Q) \to \mathrm{Cat}(Q')$ is a Kan fibration.

Theorems 5 and 6 are established first for $\partial Q = \emptyset$, by decomposing Q_Σ into small handles and then applying inductively a version where Q is an open k-handle $R^k \times R^n$, $q = k+n$, and $C = (R^k - \mathrm{int}\, B^k) \times R^n$. This version differs in that h_1 needs to be a CAT imbedding only near $I \times (R^k \times B^n)$ and the smallness condition is replaced by the condition that h_t fix all points outside a compactum (which is independent of t).

The handle version of Theorem 5 is most efficiently established by use of a variant of the Main Diagram of [7] (see also [9], [6]) which uses only the s-cobordism theorem. In case CAT = PL (but not DIFF), the Alexander isotopy device (invalid for DIFF) and the s-cobordism theorem can be used quite directly in the Main Diagram to strengthen Theorem 5 to read: Given the data of Theorem 5, the semi-simplicial space of concordances of the given structure Σ on Q, rel $I \times C$, is contractible (a d-simplex is a PL structure Γ on $\Delta^d \times I \times Q$ such that projection on Δ^d is a submersion, $\Gamma \,|\Delta^d \times 0 \times Q = \Delta^d \times 0 \times \Sigma$, and Γ equals $\Delta^d \times I \times \Sigma$ near $\Delta^d \times I \times C$).

For the handle version of Theorem 6, first note that the (many parameter) TOP isotopy extension theorem will deduce it from the statement that $(\Delta^d \times B^k \times R^n)_\Gamma$ is CAT isomorphic to $\Delta^d \times (B^k \times R^n)_\Sigma$ by a map respecting projection to Δ^d. This statement follows from a similar isomorphism of $(\Delta^d \times B^k \times (R^n - 0))_\Gamma$ with $\Delta^d \times (B^k \times (R^n - 0))_\Sigma$ which is established via a useful technical lemma.

<u>Lemma</u>. (This lemma will hold for CAT equal TOP as well as PL or DIFF.) Consider a CAT manifold E with two ends equipped with

a) a CAT submersion $p: E \to \Delta^d$ (the leaves $F_u \equiv p^{-1}(u)$, $u \in \Delta^d$, are CAT manifolds, possibly with boundary),

b) a proper continuous $\pi: E \to R$ such that for each pair of integers a, b with $a < b$ the preimage $F_u(a, b) \equiv (\pi \mid F_u)^{-1}(a, b) \equiv F_u \cap \pi^{-1}(a, b)$ of the open interval (a, b) of real numbers is a CAT product of a compact manifold with R, or at least has the following engulfing property:

(*) $F_u(a, b)$ has two ends ε_-, ε_+ and if U_-, U_+ are given open neighborhoods of ε_-, ε_+, then there exists a CAT self-isomorphism h of $F_u(a, b)$ fixing points outside some compactum such that $h(U_-) \cup U_+ = F_u(a, b)$.

Then $p: E \to \Delta^d$ is a CAT product bundle.

To prove this one can apply the d-parameter CAT isotopy extension theorem and an elementary engulfing argument to prove a "global, sliced" version of (*) namely the version with E in place of $F_u(a, b)$, where h is supposed to respect each F_u. Then glue together the ends of E as in [18] to deduce the result from the known fact that every proper CAT submersion is a CAT bundle map.

We remark that Theorems 4, 6, 7 shun dimension 4 precisely because for CAT = DIFF or PL we are unable to verify (*) given that $F_u(a, b)$ is topologically $S^3 \times R$.

The classification Theorem now follows from the immersion theory machine [2]. Theorem 7 is the key tool; with it, the machine works easily, establishing the homotopy equivalence handle by handle, once we observe that the classification theorem holds for zero-handles. But this amounts to observing that the complex Cat $(R^q, 0)$ of CAT manifold structures on R^q respecting the origin is identical to the complex of CAT microbundle structures on the trivial R^q bundle over a point, which in turn is TOP(q)/CAT(q). This proof of Theorem 1 can be regarded as a semi-simplicial version (with improvements) of the handle-by-handle argument sketched in [7].

A very different method of proving theorem 1 involves a stable version of the classification theorem.

By Milnor's argument in [14] a lifting $Q \to B_{CAT}$ of the stable classifying map $Q \to B_{TOP}$ of τ_Q gives a CAT structure on $Q \times R^z$ for some z; and concordance classes of such liftings correspond to concordance classes of CAT structures on $Q \times R^z$. Application of the concordance-implies-isotopy Theorem (Theorem 5) and the Product Structure Theorem below finish this proof of Theorem 1.

Theorem 8. (Product Structure Theorem). Let $q \geq 6$ or $q = 5$ if $\partial Q \subset C$, and let Σ_C be a CAT structure near C. Let Θ be a CAT structure on $Q \times R^s$ which agrees with $\Sigma_0 \times R^s$ near $C \times R^s$. Then Q has a CAT structure Σ, extending Σ_0 near C, so that $\Sigma \times R^s$ is concordant to Θ modulo $C \times R^s$.

Moreover, there is an ε-isotopy $h_t: Q_\Sigma \times R^1 \to (Q \times R^s)_\Theta$ with h_0 = identity, h_1 CAT, and h_t = identity near $C \times R^s$. Here $\varepsilon: Q \times R^s \to R_+$ is a given continuous function. Note that the theorem fails for closed 3-manifolds; e.g. $S^3 \times R^2$ has two PL structures but S^3 has only one.

The Product Structure Theorem is equivalent to the Concordance-implies-isotopy Theorem plus the Annulus Theorem [5]; the equivalence is not too hard to prove [9]. The classical PL-DIFF Product Structure Theorem (Cairns-Hirsch Theorem) [3] follows easily from the TOP-CAT versions of the Product Structure Theorem and Concordance-implies-isotopy Theorem. By using the strengthened Concordance-implies-isotopy Theorem in addition, we can also recover the PL-DIFF version of Concordance-implies-isotopy [4], [16].

To be sure, we land up with the same restrictions to high dimensions that we have for the TOP-CAT versions, whereas in fact no dimension restrictions are necessary [4]. The Product Structure Theorem is particularly significant because of Theorems 9, 10, and 11 below which follow easily, (see [11], [6]).

Theorem 9. Let M^m be a TOP manifold. Then M has a well defined simple homotopy type (infinite if M is non-compact [19]) which agrees with the usual definition if M is PL or is a handlebody. This implies that the Whitehead torsion of a homeomorphism is zero [7].

Theorem 10. (Transversality). Let $\xi^n : E(\xi^n) \to X$ be TOP R^n bundle over a topological space X and let $f: M^m \to E(\xi^n)$ be a continuous function. Then if $m \neq 4$ $m-n \neq 4$ and $\partial M = \emptyset$, f is homotopic to a map f_1 which is transverse to the zero-section of ξ. This means f_1^{-1}(0-section) is an $(m-n)$-manifold P (possibly empty) with a normal TOP R^n bundle ν such that the restriction of f_1 to a neighborhood of P gives a (micro-)bundle map $\nu \to \xi$. If f is transverse near a closed set $C \subset M$, then the homotopy can equal f near C. (A version with $\partial M \neq \emptyset$ results).

Theorem 11. Every closed TOP manifold M^m of dimension $m \geq 6$ admits a Morse function $f: M \to R$; that is, f is locally of the form $-x_1^2 - \ldots - x_\lambda^2 + x_{\lambda+1}^2 + \ldots + x_m^2$. Equivalently M is a TOP handlebody. If M^m, $m \geq 6$, is compact with non-empty boundary ∂M and V, V' are given compact $(m-1)$-submanifolds of ∂M such that $\partial M - \text{int}(V \cup V') \approx \partial V \times [0, 1]$, then there exists a Morse function $f: M \to R$ such that $f^{-1}(0) = V$, $f^{-1}(1) = V'$ on $\partial M - \text{int}(V \cup V')$ f is projection $\partial V \times [0, 1] \to [0, 1] \subset R$, and all critical points lie in $M - \partial M$.

On the other hand, in dimension 4 or 5 (or both) there exists a manifold which is not a handlebody, and thus has no Morse function.

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