ALGEBRAIC K-THEORY OF SPACES.

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This is an account of foundational material on the algebraic K-theory of spaces functor $X \mapsto A(X)$.

The paper is in three parts which are entitled "Abstract K-theory", "A(X)", and "Relation of A(X) to $Wh^{PL}(X)$", respectively.

The main result of the paper is in the second part. It says that several definitions of $A(X)$ are in fact equivalent to each other, up to homotopy. The proof uses most of the results of the first part. An introduction to this circle of ideas can be obtained from looking at the sections entitled "Review of A(X)" and "Review of algebraic K-theory" in the papers [17] and [18] (these two sections were written with that purpose in mind).

The third part of the paper is devoted to an abstract version of the relation of the A-functor to concordance theory. The content of the *parametrized h-cobordism theorem* in the sense of Hatcher is that PL concordance theory, stabilized with respect to dimension, can be re-expressed in terms of non-manifold data. A detailed account of the translation is given elsewhere [16], in particular the relevant results of Hatcher's are (re-)proved there. The result of the translation (after a dimension shift) is a functor $X \mapsto Wh^{PL}(X)$. It is shown here that there is a map $A(X) \to Wh^{PL}(X)$ and that the homotopy fibre of that map is a homology theory (i.e., that, as a functor of X , the homotopy fibre satisfies the excision property).

The first part of the paper, on which everything else depends, may perhaps look a little frightening because of the abstract language that it uses throughout. This is unfortunate, but there is no way out. It is not the purpose of the abstract language to strive for great generality. The purpose is rather to simplify proofs, and indeed to make some proofs understandable at all. The reader is invited to run the following test: take theorem 2.2.1 (this is about the worst case), translate the complete proof into not using the abstract language, and then try to communicate it to somebody else.

Contents.

1. ABSTRACT K-THEORY.

1.1. Categories with cofibrations, and the language of filtered objects.

A category C is called *pointed* if it is equipped with a distinguished zero object $*$, i.e. an object which is both initial and terminal.

A *category with cofibrations* shall mean a pointed category C together with a subcategory coC satisfying the axioms Cof 1 - Cof 3 below. The feathered arrows '$\rightarrowtail$' will be used to denote the morphisms in coC. Informally the morphisms in coC will simply be referred to as the *cofibrations in* C.

Cof 1. The isomorphisms in C are cofibrations (in particular coC contains all the objects of C).

Cof 2. For every $A \in C$, the arrow $* \to A$ is a cofibration.

Cof 3. Cofibrations admit cobase changes. This means the following two things. If $A \rightarrowtail B$ is a cofibration, and $A \to C$ any arrow, then firstly the pushout $C \cup_A B$ exists in C, and secondly the canonical arrow $C \to C \cup_A B$ is a cofibration again.

Here is some more language. If $A \rightarrowtail B$ is a cofibration then B/A will denote any representative of $* \cup_A B$. We think of it as the quotient of B by A. The canonical map $B \to B/A$ will be referred to as a *quotient map*. The double headed arrows '$\twoheadrightarrow$' are reserved to denote quotient maps. (Note that it is neither asked, nor asserted, that the quotient maps form a category, i.e. that the composite of two quotient maps is always a quotient map again.)

Our usage of the term *cofibration sequence* conforms to the usage in homotopy theory. It refers to a sequence $A \rightarrowtail B \twoheadrightarrow B/A$ where $B \twoheadrightarrow B/A$ is the quotient map associated to $A \rightarrowtail B$.

Beware that we will also be using the term *sequence of cofibrations* which of course refers to a sequence of the type $A_1 \rightarrowtail A_2 \rightarrowtail \dots \rightarrowtail A_n$.

The most important example of a category with cofibrations, for our purposes, is that of the spaces having a given space X as a retract. We will denote this category by $R(X)$. As a technical point, there will be several cases to consider depending on whether *space* means simplicial set, or cell complex, or whatever, and perhaps with a finiteness condition imposed. In any case the term cofibration has essentially its usual meaning here. (As a technical point again, note that the axiom Cof 2 may force us to put a condition on one of the structural maps of an object of $\mathcal{R}(X)$ - the section should be a cofibration).

Another important example, though of less concern to us here, is that of an *exact category* in the sense of Quillen. Any exact category can be considered as a category with cofibrations by choosing a zero object, and declaring the admissible monomorphisms to be the cofibrations. The re-interpretation involves a loss of structure: one ignores that pullbacks used to play a role, too (the base change by admissible epimorphisms).

Since our axioms are so primitive it will not be surprising that they admit examples which are not important at all, and perhaps even embarrassing. Here is a particularly bad case. Consider a category having a zero object and finite colimits. It can be made into a category with cofibrations by declaring *all* morphisms to be cofibrations.

Here is some more language. A functor between categories with cofibrations is called *exact* if it preserves all the relevant structure: it takes $*$ to $*$, cofibrations to cofibrations, and it preserves the pushout diagrams of axiom Cof 3 .

For example, a map $X \to X'$ induces an exact functor $\mathcal{R}(X) \to \mathcal{R}(X')$. On total spaces it is given by pushout of $X \to X'$ with the structural sections.

Another example of an exact functor is the linearization functor (or Hurewicz map) which takes an object of $\mathcal{R}(X)$ to the abelian-group-object in $\mathcal{R}(X)$ which it generates.

There is a concept slightly stronger than that of an *exact inclusion functor* which we will have to consider. We say that C' is a *subcategory with cofibrations of* C if in addition to the exactness of the inclusion functor the following condition is satisfied: an arrow in C' is a cofibration in C' if it is a cofibration in C and the quotient is in C' (up to isomorphism).

An example of a subcategory-with-cofibrations arises if we consider a subcategory of $\mathcal{R}(X)$ defined by a finiteness condition.

Here is a more interesting example. For $n \geqslant 2$ let $\mathcal{R}^n(X)$ denote the full subcategory of $\mathcal{R}(X)$ whose objects are obtainable from X by attaching of n-cells (up to homotopy). It can be considered as a subcategory with cofibrations of $\mathcal{R}(X)$.

In the remainder of the section we will check that certain elementary constructions with categories do not lead one out of the framework of categories with cofibrations. In particular we will be interested in *filtered objects*; that is, sequences of cofibrations. (Despite the fact, exemplified above, that cofibrations need not be monomorphic at all, we shall let ourselves be guided by the more relevant examples to justify using this terminology). The arguments below will not go beyond trivial manipulation with colimits. There is, however, one idea involved. The idea is that the notion of *bifiltered object* (or *lattice*) can be formulated without pullbacks. Namely if the diagram

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \downarrow \\ C & \longrightarrow & D \end{array}$$

is to be a 'lattice' we are inclined to ask this in the form of two conditions: firstly, that all the arrows be cofibrations, and secondly, that the 'images' in D satisfy $\mathrm{Im}(A) \supset \mathrm{Im}(B) \cap \mathrm{Im}(C)$. The latter does not make sense in our context, in general, but we can substitute it with the condition that the arrow $B\cup_A C \to D$ be a cofibration.

For any category $\mathcal{C}$ we let $\mathrm{Ar}\mathcal{C}$ denote the category whose objects are the arrows of $\mathcal{C}$ and whose morphisms are the commutative squares

$$\begin{array}{ccc} \cdot & \longrightarrow & \cdot \\ \downarrow & & \downarrow \\ \cdot & \longrightarrow & \cdot \end{array}$$

in $\mathcal{C}$. If $\mathcal{C}$ is a category with cofibrations then so is $\mathrm{Ar}\mathcal{C}$ in an obvious way: a map is in $\mathrm{coAr}\mathcal{C}$ if and only if the two associated maps in $\mathcal{C}$ are in $\mathrm{co}\mathcal{C}$.

Definition. $F_1\mathcal{C}$ is the full subcategory of $\mathrm{Ar}\mathcal{C}$ whose objects are the cofibrations in $\mathcal{C}$, and $\mathrm{co}F_1\mathcal{C}$ is the class of the maps $(A \rightarrowtail B) \to (A' \rightarrowtail B')$ in $F_1\mathcal{C}$ having the property that both $A \to A'$ and $A'\cup_A B \to B'$ are cofibrations in $\mathcal{C}$.

Lemma 1.1.1. $\mathrm{co}F_1\mathcal{C}$ makes $F_1\mathcal{C}$ a category with cofibrations.

Proof. There are two points that require proof: that $\mathrm{co}F_1\mathcal{C}$ is a category, and that the axiom Cof 3 is satisfied.

As to the first, let $(A \rightarrowtail B) \rightarrowtail (A' \rightarrowtail B')$ and $(A' \rightarrowtail B') \rightarrowtail (A'' \rightarrowtail B'')$ be in $\mathrm{co}F_1\mathcal{C}$. Then $A \rightarrowtail A''$ since $\mathrm{co}\mathcal{C}$ is a category. By assumption about the second map $A''\cup_{A'} B' \rightarrowtail B''$; and by assumption about the first map and by axioms Cof 1 and Cof 3 for $\mathrm{co}\mathcal{C}$, all the following terms are defined and the composed map

$$A''\cup_A B \xrightarrow{\approx} A''\cup_{A'} A'\cup_A B \longrightarrow A''\cup_{A'} A'\cup_A B\cup_{(A'\cup_A B)} B' \xrightarrow{\approx} A''\cup_{A'} B'$$

is also in $\mathrm{co}\mathcal{C}$. Taking the composition of the two maps we obtain that $A''\cup_A B \to B''$ is in $\mathrm{co}\mathcal{C}$, as was to be shown.

As to the second, let $(A \rightarrowtail B) \rightarrowtail (A' \rightarrowtail B')$ and $(A \rightarrowtail B) \to (C \rightarrowtail D)$ be maps in $\mathrm{co}F_1\mathcal{C}$, resp. $F_1\mathcal{C}$. Their pushout exists in $\mathrm{Ar}\mathcal{C}$ by Cof 3 for $\mathcal{C}$ (because $A \rightarrowtail A'$ and $A'\cup_A B \rightarrowtail B'$ implies $B \rightarrowtail B'$) where it is represented by

$$A'\cup_A C \longrightarrow B'\cup_B D .$$

We show below that this is an object of (and consequently also a pushout in) F_1C. We must in addition show that the canonical map $(C \rightarrowtail D) \to (A' \cup_A C \to B' \cup_B D)$ is in coF_1C. This amounts to the two assertions that $C \rightarrowtail A' \cup_A C$, which is clear, and that $(A' \cup_A C) \cup_C D \rightarrowtail (B' \cup_B D)$. The latter map is isomorphic to $A' \cup_A D \to B' \cup_B D$ which in turn is isomorphic to the composed map

$$(A' \cup_A B) \cup_B D \longrightarrow B' \cup_{(A' \cup_A B)} (A' \cup_A B) \cup_B D \xrightarrow{\approx} B' \cup_B D$$

and this is a cofibration since $A' \cup_A B \to B'$ is one. Finally $A' \cup_A C \to (A' \cup_A C) \cup_C D$ is a cofibration since $C \to D$ is one. Composing it with the cofibration $(A' \cup_A C) \cup_C D \to B' \cup_B D$ (above) we obtain the map $A' \cup_A C \to B' \cup_B D$. This proves the postponed claim that the latter map is a cofibration. □

Definition. F_1^+C is the category equivalent to F_1C in which an object consists of an object $A \rightarrowtail B$ of F_1C together with the choice of a quotient B/A; in other words, F_1^+C is the category of cofibration sequences $A \rightarrowtail B \twoheadrightarrow B/A$ in C. It is made into a category with cofibrations by means of the equivalence $F_1^+C \to F_1C$.

<u>Lemma</u> 1.1.2. The three functors $s, t, q\colon F_1^+C \to C$ sending $A \rightarrowtail B \twoheadrightarrow B/A$ to $A, B,$ and B/A, respectively, are exact.

Proof. For s this holds by definition, and for t almost so. The case of q requires proof. We must show that q takes coF_1^+C to coC, and that q preserves the pushout diagrams of axiom Cof 3.

As to the first, if $(A \rightarrowtail B) \to (A' \rightarrowtail B')$ is in coF_1^+C then, by definition, $A' \cup_A B \to B'$ is in coC. Hence so is

$$B/A \xrightarrow{\approx} * \cup_{A'} A' \cup_A B \longrightarrow * \cup_{A'} A' \cup_A B \cup_{(A' \cup_A B)} B' \xrightarrow{\approx} B'/A'$$

as claimed.

As to the second, let such a pushout diagram in F_1^+C be given by the diagram

$$(A \rightarrowtail B \twoheadrightarrow B/A) \longrightarrow (C \rightarrowtail D \twoheadrightarrow D/C)$$

$$(A' \rightarrowtail B' \twoheadrightarrow B'/A') \longrightarrow (A' \cup_A C \rightarrowtail B' \cup_B D \twoheadrightarrow (B' \cup_B D)/(A' \cup_A C)) .$$

Then the assertion means that

$$(B' \cup_B D)/(A' \cup_A C) \quad \text{and} \quad B'/A' \cup_{B/A} D/C$$

are canonically isomorphic. But this is clear from the fact that an iterated colimit may be computed in any way desired provided only that all the colimits involved exist. In particular the two objects at hand are canonically isomorphic because both represent the colimit of the diagram

$$\begin{array}{ccccc} D & \leftarrowtail & C & \rightarrowtail & * \\ \uparrow & & \uparrow & & \uparrow \\ B & \leftarrowtail & A & \rightarrowtail & * \\ \downarrow & & \downarrow & & \downarrow \\ B' & \leftarrowtail & A' & \rightarrowtail & * \end{array}$$

when this colimit is computed in the two obvious ways. □

Definition. F_mC is the category in which an object is a sequence of cofibrations

$$A_0 \rightarrowtail A_1 \rightarrowtail \ldots \rightarrowtail A_m$$

in C, and where a morphism is a natural transformation of diagrams. F_m^+C is the category equivalent to F_mC in which an object consists of one of F_mC together with a choice, for every $0 \leq i < j \leq m$, of a quotient $A_{i,j} = A_j/A_i$.

<u>Lemma</u> 1.1.3. Let $A \to A'$ be a map in F_mC, resp. F_m^+C. Suppose that the maps

$$A_j \longrightarrow A'_j \quad , \qquad A'_j \cup_{A_j} A_{j+1} \longrightarrow A'_{j+1}$$

are cofibrations in C. Then

for every pair $j < k$ the map $A'_j \cup_{A_j} A_k \to A'_k$ is a cofibration, and

for every triple $i < j < k$ the map $A'_{i,j} \cup_{A_{i,j}} A_{i,k} \to A'_{i,k}$ is a cofibration.

Proof. The first results inductively by considering the compositions

$$A'_j \cup_{A_j} A_k \cup_{A_k} A_{k+1} \longrightarrow A'_k \cup_{A_k} A_{k+1} \longrightarrow A'_{k+1}$$

and the second follows from the first by the preceding lemma applied to the cofibration in F_1C,

$$(A'_i \cup_{A_i} A_i \rightarrowtail A'_i) \rightarrowtail (A'_j \cup_{A_j} A_k \rightarrowtail A'_k) \ . \qquad □$$

<u>Proposition</u> 1.1.4. F_mC and F_m^+C are categories with cofibrations in a natural way. The forgetful map $F_m^+C \to F_mC$ is an exact equivalence. The 'subquotient' maps

$$q_j : F_mC \longrightarrow C \quad , \qquad q_{i,j} : F_m^+C \longrightarrow C$$
$$A \longmapsto A_j \qquad\qquad\qquad A \longmapsto A_j/A_i$$

are exact.

In fact, a map in F_mC, resp. F_m^+C, is defined to be a cofibration if it satisfies the hypothesis of lemma 1.1.3, and the assertions of the proposition just summarize the preceding lemmas. □

Iterating the construction one can obtain categories with cofibrations F_nF_mC and $F_n^+F_m^+C$.

<u>Lemma</u> 1.1.5 There are natural isomorphisms of categories with cofibrations

$$F_n F_m C \approx F_m F_n C \ , \qquad F_n^+ F_m^+ C \approx F_m^+ F_n^+ C \ .$$

Proof. It suffices to remark that an object of $F_n F_m C$ can be more symmetrically defined as a rectangular array of squares each of which consists of cofibrations only and satisfies the condition in the definition of a cofibration in $F_1 C$; the point is that the condition is symmetric with respect to *horizontal* and *vertical*. Similarly, a cofibration in $F_n F_m C$, or sequence of such, may be identified to a 3-dimensional diagram satisfying conditions with respect to which none of the three directions is preferred. □

We will want to know that categories with cofibrations reproduce under certain other simple constructions. By the *fibre product* of a pair of functors $f: A \to C$, $g: B \to C$ is meant the category $\Pi(f,g)$ whose objects are the triples

$$(A,c,B) \ , \quad A \in A \ , \quad B \in B \ , \quad c: f(A) \xrightarrow{\approx} g(B) \ ,$$

and where a morphism from (A,c,B) to (A',c',B') is a pair of morphisms (a,b) compatible with the isomorphisms c and c' . In some special cases the fibre product category is equivalent to the pullback category $A \times_C B$; notably this is so if either f or g is a retraction. (If the two are not the same, up to equivalence, the pullback should be regarded as pathological.)

<u>Lemma</u> 1.1.6. If $f: A \to C$ and $g: B \to C$ are exact functors of categories with cofibrations then $\Pi(f,g)$ can be made into a category with cofibrations by letting

$$co(\Pi(f,g)) = \Pi(co(f),co(g)) \ ,$$

and the projection functors from $\Pi(f,g)$ to A and B are exact.

Similarly, if $j \to C_j$, $j \in J$, is a direct system of categories with cofibrations and exact functors then $\varinjlim C_j$ is a category with cofibrations, with

$$co(\varinjlim C_j) = \varinjlim coC_j \ ,$$

and the functors $C_j \to \varinjlim C_j$ are exact. □

Definition and corollary. Let A, B, C be categories with cofibrations and let A and B be subcategories of C in such a way that the inclusion functors are exact. Define $E(A,C,B)$ as the category of the cofibration sequences in C ,

$$A \rightarrowtail C \twoheadrightarrow B \ , \quad A \in A \ , \quad B \in B \ .$$

Then $E(A,C,B)$ is a category with cofibrations, and the projections to A, C, B are exact.

Indeed, $E(A,C,B)$ is the pullback of a diagram $F_1^+ C \longrightarrow C \times C \longleftarrow A \times B$; the pullback is not pathological since the first arrow has a section. □

1.2. Categories with cofibrations and weak equivalences.

Let $\mathcal{C}$ be a category with cofibrations in the sense of section 1.1 (we will from now on drop explicit mentioning of the category of cofibrations $co\mathcal{C}$ from the notation). A *category of weak equivalences* in $\mathcal{C}$ shall mean a subcategory $w\mathcal{C}$ of $\mathcal{C}$ satisfying the following two axioms.

Weq 1. The isomorphisms in $\mathcal{C}$ are contained in $w\mathcal{C}$ (and in particular therefore the category $w\mathcal{C}$ contains all the objects of $\mathcal{C}$).

Weq 2. (*Gluing lemma*). If in the commutative diagram

$$\begin{array}{ccccc} B & \leftarrowtail & A & \longrightarrow & C \\ \downarrow & & \downarrow & & \downarrow \\ B' & \leftarrowtail & A' & \longrightarrow & C' \end{array}$$

the horizontal arrows on the left are cofibrations, and all three vertical arrows are in $w\mathcal{C}$, then the induced map

$$B \cup_A C \longrightarrow B' \cup_{A'} C'$$

is also in $w\mathcal{C}$.

Here are some examples. Any category with cofibrations can be equipped with a category of weak equivalences in at least two ways: the minimal choice is to let $w\mathcal{C}$ be the category of isomorphisms in $\mathcal{C}$, while the maximal choice is to let $w\mathcal{C}$ be equal to $\mathcal{C}$ itself.

To obtain an example of a category of weak equivalences on the category $\mathcal{R}(X)$ (the preceding section) choose a homology theory and define $w\mathcal{R}(X)$ to be the category of those maps which induce isomorphisms of that homology theory.

To obtain another example define $h\mathcal{R}(X)$ to be the category of the *weak homotopy equivalences.*

To obtain yet another example define $s\mathcal{R}(X)$ to be the category of the *simple maps*, i.e. the maps whose point inverses have the shape (or Cech homotopy type) of a point. (We shall consider simple maps in the simplicial setting only in which case the definition simplifies to asking that the point inverses in the geometric realization of the map are contractible.) Neither the fact that $s\mathcal{R}(X)$ is a category nor the gluing lemma are trivial to prove.

The following two further axioms may, or may not, be satisfied by a given category of weak equivalences.

Saturation axiom. If a, b are composable maps in $\mathcal{C}$ and if two of a, b, ab are in $w\mathcal{C}$ then so is the third.

For example the simple maps do not satisfy the saturation axiom. E.g. consider the two maps a, b in $\mathcal{R}(*)$ given by the inclusion of the basepoint in a 1-simplex and by the projection of that 1-simplex to the basepoint, respectively.

Extension axiom. Let

$$\begin{array}{ccccc} A & \rightarrowtail & B & \twoheadrightarrow & B/A \\ \downarrow & & \downarrow & & \downarrow \\ A' & \rightarrowtail & B' & \twoheadrightarrow & B'/A' \end{array}$$

be a map of cofibration sequences. If the arrows $A \to A'$ and $B/A \to B'/A'$ are in $w\mathcal{C}$ then it follows that $B \to B'$ is in $w\mathcal{C}$, too.

For example the weak homotopy equivalences do not satisfy the extension axiom. E.g. consider the diagram in $\mathcal{R}(*)$

$$\begin{array}{ccccc} BZ & \overset{\simeq}{\rightarrowtail} & BZ & \twoheadrightarrow & * \\ \simeq\downarrow & & \downarrow & & \downarrow \\ BZ & \rightarrowtail & BG & \twoheadrightarrow & BG/BZ \end{array}$$

where BZ is the classifying space of the infinite cyclic group and BG the classifying space of a suitable non-abelian group which is normally generated by a subgroup Z, for example a classical knot group.

As the examples show there may be a great profusion of categories of weak equivalences on a given category with cofibrations. Also, we will have occasion to consider a category with cofibrations equipped with *two* categories of weak equivalences at the same time, one finer than the other, and study their interplay. We must therefore exercise some care with the notation, and in general the category of weak equivalences will be explicitly mentioned.

Still there are some situations where there is no danger of confusion. On those occasions we will allow ourselves the abuse of referring to the maps in $w\mathcal{C}$ as the *weak equivalences in* $\mathcal{C}$, and denote them by the decorated arrows '$\xrightarrow{\sim}$'

By a *category with cofibrations and weak equivalences* will be meant a category with cofibrations equipped with one (and only one) category of weak equivalences. A functor between such is called *exact* if it preserves all the relevant structure.

As in the preceding section, the notion of an exact inclusion functor may be sharpened to that of a *subcategory with cofibrations and weak equivalences*.

Finally we note that categories of weak equivalences are inherited by diagram categories. There are lemmas similar to, but easier than, those of the preceding section. We omit their formulation.

1.3. The K-theory of a category with cofibrations and weak equivalences.

Consider the partially ordered set of pairs (i,j) , $0 \leq i \leq j \leq n$, where $(i,j) \leq (i',j')$ if and only if $i \leq i'$ and $j \leq j'$. Regarded as a category it may be identified to the arrow category $Ar[n]$ where as usual $[n]$ denotes the ordered set $(0 < 1 < \ldots < n)$ (considered as a category).

Let C be a category with cofibrations. We consider the functors

$$A: Ar[n] \longrightarrow C$$

$$(i,j) \longmapsto A_{i,j}$$

having the property that for every j ,

$$A_{j,j} = * ,$$

and that for every triple $i \leq j \leq k$, the map

$$A_{i,j} \longrightarrow A_{i,k}$$

is a cofibration, and the diagram

$$\begin{array}{ccc} A_{i,j} & \longrightarrow & A_{i,k} \\ \downarrow & & \downarrow \\ A_{j,j} & \longrightarrow & A_{j,k} \end{array}$$

is a pushout; in other words,

$$A_{i,j} \rightarrowtail A_{i,k} \twoheadrightarrow A_{j,k}$$

is a cofibration sequence. We denote the category of these functors and their natural transformations by S_nC .

To give an object $A \in S_n$ is really the same thing as to give a sequence of cofibrations

$$A_{o,1} \rightarrowtail A_{o,2} \rightarrowtail \ldots \rightarrowtail A_{o,n}$$

together with a choice of subquotients

$$A_{i,j} = A_{o,j}/A_{o,i} .$$

It results that the category S_nC can be identified with one of the categories of filtered objects considered in section 1.1 (namely F^+_{n-1}) and in particular therefore S_nC can be regarded as a category with cofibrations in a natural way.

The definition of S_nC given here has the advantage of making it clear that $[n] \mapsto Ar[n] \mapsto S_nC$ is contravariantly functorial on the category Δ of the ordered sets $[0], [1], \ldots$. We therefore have a simplicial category

$$S.C : \Delta^{op} \longrightarrow (cat)$$
$$[n] \longmapsto S_nC \quad .$$

In fact, we have a *simplicial category with cofibrations*; that is, a simplicial object in the category whose objects are the categories with cofibrations and whose morphisms are the exact functors between those. This results from the lemmas of section 1.1 upon inspection of what the face and degeneracy maps are. For example the face map $d_i : S_nC \to S_{n-1}C$ corresponds, for $i > 0$, to the forgetful map which drops $A_{o,i}$ from the sequence $A_{o,1} \rightarrowtail \ldots \rightarrowtail A_{o,n}$; and for $i = 0$ it corresponds to the map "quotient by $A_{o,1}$ " which replaces that sequence by $A_{1,2} \rightarrowtail \ldots \rightarrowtail A_{1,n}$.

If C is equipped with a category of weak equivalences, wC , then S_nC comes naturally equipped with a category of weak equivalences, wS_nC . By definition here an arrow $A \to A'$ of S_nC is in wS_nC if and only if the arrow $A_{i,j} \to A'_{i,j}$ is in wC for every pair $i \leq j$; or what amounts to the same in view of the assumed gluing lemma, if this is so for $i = 0$. It results that $S.C$ is a *simplicial category with cofibrations and weak equivalences* in this case.

Let us take a look at the simplicial category of weak equivalences

$$wS.C : \Delta^{op} \longrightarrow (cat)$$
$$[n] \longmapsto wS_nC \quad .$$

The category S_oC , and therefore also its subcategory wS_oC , is the trivial category with one object and one morphism. Hence the geometric realization $|wS_oC|$ is the one-point space.

The category S_1C is the category of diagrams

$$* = A_{o,o} \rightarrowtail A_{o,1} \twoheadrightarrow A_{1,1} = *$$

and is thus isomorphic to C . Hence the category of weak equivalences may be identified to wC .

Consider $|wS.C|$, the geometric realization of the simplicial category $wS.C$. The '1-skeleton' in the $S.$-direction is obtained from the '0-skeleton' (which is $|wS_oC|$) by attaching of $|wS_1C| \times |\Delta^1|$ (where $|\Delta^1|$ denotes the topological space 1-*simplex*). It results that the '1-skeleton' is naturally isomorphic to the suspension $S^1 \wedge |wC|$. As a consequence we obtain an inclusion $S^1 \wedge |wC| \to |wS.C|$, and by adjointness therefore an inclusion of $|wC|$ into the loop space of $|wS.C|$,

$$|wC| \longrightarrow \Omega|wS.C|$$

The passage from $|wC|$ to $\Omega|wS.C|$ is reminiscent of the 'group completion' process of Segal [11] (by which it was originally motivated, to some extent). We will have occasion to make an actual comparison later (in section 1.8).

Definition. The *algebraic K-theory* of the category with cofibrations C, with respect to the category of weak equivalences wC, is given by the pointed space

$$\Omega|wS.C| \ .$$

To pursue the analogy with Segal's version of group completion a little further, one can actually describe K-theory as a spectrum rather than just a space. Namely the $S.$-construction extends, by naturality, to simplicial categories with cofibrations and weak equivalences. In particular therefore it applies to $S.C$ to produce a bisimplicial category with cofibrations and weak equivalences, $S.S.C$. Again the construction extends to bisimplicial categories with cofibrations and weak equivalences; and so on. There results a spectrum

$$n \longmapsto |w\underbrace{S. \ \dots \ S.}_{\longleftarrow n \longrightarrow}C|$$

whose structural maps are defined just as the map $|wC| \to \Omega|wS.C|$ above.

It turns out that the spectrum is a Ω-spectrum beyond the first term (the additivity theorem is needed to prove this, below). As the spectrum is connective (the n-th term is (n-1)-connected) an equivalent assertion is that in the sequence

$$|wC| \longrightarrow \Omega|wS.C| \longrightarrow \Omega\Omega|wS.S.C| \longrightarrow \dots$$

all maps except the first are homotopy equivalences. It results that the K-theory of (C,wC) could equivalently be defined as the space

$$\Omega^\infty|wS.^{(\infty)}C| = \lim_{\vec{n}} \Omega^n|wS.^{(n)}C| \ , \qquad wS.^{(n)}C = w\underbrace{S. \ \dots \ S.}_{\longleftarrow n \longrightarrow}C \ .$$

There is another way of making K-theory into a spectrum. Namely the pushout of the cofibrations $* \to A$ induces a sum in C and therefore a composition law in the sense of Segal on wC, $wS.C$, $wS.^{(2)}C$, and so on. As $\Omega|wS.C|$ is 'group-like' Segal's machine produces a connective Ω-spectrum from it. To see that the spectrum is equivalent to the former it suffices to note that the two spectra can be combined into a connective *bi-spectrum*. (A more direct relationship can also be established.)

The definition of K-theory is natural for categories with cofibrations and weak equivalences: an exact functor $F\colon C' \to C$ induces maps $wS.F\colon wS.C' \to wS.C$, etc.

Let a *weak equivalence* of exact functors $F, F'\colon C' \to C$ mean a natural transformation $F \to F'$ having the property that for every $A \in C'$ the map $F(A) \to F'(A)$ is a weak equivalence in C.

Proposition 1.3.1. A weak equivalence from F to F' induces a homotopy between $wS.F$ and $wS.F'$.

Proof. The weak equivalence from F to F' restricts to a natural transformation of the restricted functors $F, F' : wC' \to wC$ and thereby induces a homotopy between these by a well known remark due to Segal [10]. Similarly there is what may be called a simplicial natural transformation from $wS.F$ to $wS.F'$. It gives rise to a homotopy in the same way. □

Let a *cofibration sequence* of exact functors $C' \to C$ mean a sequence of natural transformations $F' \to F \to F''$ having the following two properties: (i) for every $A \in C'$ the sequence $F'(A) \to F(A) \to F''(A)$ is a cofibration sequence, and (ii) for every cofibration $A' \to A$ in C' the square of cofibrations

$$\begin{array}{ccc} F'(A') & \longrightarrow & F'(A) \\ \downarrow & & \downarrow \\ F(A') & \longrightarrow & F(A) \end{array}$$

is *admissible* in the sense that $F(A') \cup_{F'(A')} F'(A) \to F(A)$ is also a cofibration.

Recall the category $E(A,C,B)$ (section 1.1), and let $E(C) = E(C,C,C)$.

<u>Proposition</u> 1.3.2. (*Equivalent formulations of the additivity theorem*). Each of the following four assertions implies all the three others.

(1) The following projection is a homotopy equivalence,

$$\begin{array}{ccc} wS.E(A,C,B) & \longrightarrow & wS.A \times wS.B \\ A \rightarrowtail C \twoheadrightarrow B & \longmapsto & A\ ,\ B\ . \end{array}$$

(2) The following projection is a homotopy equivalence,

$$\begin{array}{ccc} wS.E(C) & \longrightarrow & wS.C \times wS.C \\ A \rightarrowtail C \twoheadrightarrow B & \longmapsto & A\ ,\ B\ . \end{array}$$

(3) The following two maps are homotopic (resp. weakly homotopic),

$$\begin{array}{ccc} wS.E(C) & \longrightarrow & wS.C \\ A \rightarrowtail C \twoheadrightarrow B & \longmapsto & C\ ,\ \text{resp. } A \vee B\ . \end{array}$$

(4) If $F' \to F \to F''$ is a cofibration sequence of exact functors $C' \to C$ then there exists a homotopy

$$|wS.F| \simeq |wS.F'| \vee |wS.F''| \quad (= |wS.(F' \vee F'')|)\ .$$

Proof. (2) is a special case of (1), and (3) is a special case of (4). So it will suffice to show the implications (2) ⇒ (3) ⇒ (4) and (4) ⇒ (1).

Ad (3)⇒(4). To give a cofibration sequence of functors $F' \rightarrowtail F \twoheadrightarrow F''$ from C' to C is equivalent to giving an exact functor $G : C' \to E(C)$, with $F' = sG$, $F = tG$, and $F'' = qG$, where s, t, q are the maps $A \rightarrowtail C \twoheadrightarrow B \longmapsto A, C, B$, respectively (which are exact by proposition 1.1.4). Thus (4) follows from (3) by naturality.

Ad (2)⇒(3). The desired homotopy $|wS.t| \simeq |wS.(svq)|$ is certainly valid upon restriction along the map

$$\begin{array}{ccc} |wS.C| \times |wS.C| & \longrightarrow & |wS.E(C)| \\ A\ ,\ B & \longmapsto & A \rightarrowtail A \vee B \twoheadrightarrow B\ , \end{array}$$

so it will suffice to know that this map is a homotopy equivalence. But the map is a section to the map in (2) and therefore is a homotopy equivalence if that is one.

Ad (4)⇒(1). The map $p\colon wS.E(A,C,B) \to wS.A \times wS.B$ is a retraction, with section σ given by $A,B \longmapsto A \rightarrowtail A \vee B \twoheadrightarrow B$. To show p is a homotopy equivalence it therefore suffices to show that the identity map on $wS.E(A,C,B)$ is homotopic to the map σp. (In fact, it would suffice to know that the two maps are *weakly homotopic*, that is, homotopic upon restriction to any compactum, for that would still imply that the map σ is surjective, and hence bijective, on homotopy groups.) The desired homotopy results from (4) applied to a suitable cofibration sequence of endofunctors on $E(A,C,B)$. The cofibration sequence is shown by the following diagram which depicts the functors (the rows) applied to an object $A \rightarrowtail C \twoheadrightarrow B$,

$$\begin{array}{c} (A \overset{=}{\rightarrowtail} A \twoheadrightarrow *) \\ \downarrow \\ (A \rightarrowtail C \twoheadrightarrow B) \\ \downarrow \\ (* \rightarrowtail B \underset{=}{\twoheadrightarrow} B) \end{array}$$

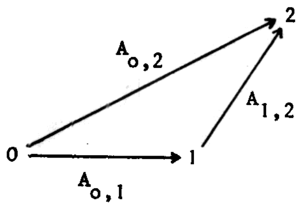
This completes the proof. □

The actual proof of the additivity theorem is rather long and it will be given later (it occupies the next section). We will now convince ourselves that a considerable short cut to the proof is possible if the definition of K-theory is adjusted somewhat. We begin with the

Observation 1.3.3. Let s, t, q denote the maps from $E(C)$ to C given by $A \rightarrowtail C \twoheadrightarrow B \longmapsto A, C, B$, respectively, and let svq denote the sum of s and q. Then the following two composite maps are homotopic,

$$|wE(C)| \underset{svq}{\overset{t}{\rightrightarrows}} |wC| \longrightarrow \Omega|wS.C|\ .$$

This results from an inspection of $|wS.C|_{(2)}$, the '2-skeleton' of $|wS.C|$ in the $S.$-direction. Let us identify wC to wS_1C, as before, and let us identify $wE(C)$ to wS_2C whose objects are the cofibration sequences $A_{o,1} \rightarrowtail A_{o,2} \twoheadrightarrow A_{1,2}$. The face maps from wS_2C to wS_1C then correspond to the three maps s, t, q, respectively, and which is which can be seen from the diagram



Let us consider the canonical map $|wS_2C| \times |\Delta^2| \to |wS.C|_{(2)}$. Regarding the 2-simplex $|\Delta^2|$ as a homotopy from the edge (0,2) to the edge path (0,1)(1,2) we obtain a homotopy from the composite map jt ,

$$|wE(C)| \xrightarrow{t} |wC| \xrightarrow{j} \Omega|wS.C|_{(2)} ,$$

to the *loop product* of the two composite maps js and jq . But in $\Omega|wS.C|$ the loop product is homotopic to the composition law, by a well known fact about loop spaces of H-spaces, whence the observation as stated.

The same consideration shows, more generally,

Observation 1.3.4. For every $n \geqslant 0$ the two composite maps

$$|wS.^{(n)}E(C)| \underset{svq}{\overset{t}{\rightrightarrows}} |wS.^{(n)}C| \longrightarrow \Omega|wS.^{(n+1)}C|$$

are homotopic, where $wS.^{(n)}C = \underbrace{wS. \ldots S.}_{\longleftarrow n \longrightarrow}C$.

Corollary 1.3.5. The additivity theorem (proposition 1.3.2) is valid if the definition of K-theory as $\Omega|wS.C|$ is substituted with $\Omega^\infty|wS.^{(\infty)}C| = \varinjlim \Omega^n|wS.^{(n)}C|$.

Proof. First, proposition 1.3.2 is formal in the sense that it applies to the present definition of K-theory just as well. Second, by the preceding observation the two composite maps

$$\Omega^\infty|wS.^{(\infty)}E(C)| \underset{svq}{\overset{t}{\rightrightarrows}} \Omega^\infty|wS.^{(\infty)}C| \longrightarrow \Omega^\infty|wS.^{(\infty)}C|$$

are weakly homotopic. Since the arrow on the right is an isomorphism this is one of the equivalent formulations of the additivity theorem (proposition 1.3.2). □

Remark. As a consequence of the corollary we could add yet another reformulation of the additivity theorem to the list of proposition 1.3.2. Namely the additivity theorem as stated there implies (section 1.5) that the maps $|wS.^{(n)}C| \to \Omega|wS.^{(n+1)}C|$ are homotopy equivalences for $n \geqslant 1$. Conversely if these maps are homotopy equivalences then so is $\Omega|wS.C| \to \Omega^\infty|wS.^{(\infty)}C|$, and thus the additivity theorem is provided by the corollary.

To conclude this section we describe a modification of the simplicial category $wS.C$ which was suggested by Thomason. It is a simplicial category $wT.C$. By definition wT_nC is a subcategory of the functor category $C^{[n]}$. The objects of wT_nC are the sequences of cofibrations

$$C_o \rightarrowtail C_1 \rightarrowtail \ldots \rightarrowtail C_n$$

and the morphisms are the natural transformations $C \to C'$ satisfying the condition that for every $i \leq j$ the induced map

$$C'_i \cup_{C_i} C_j \longrightarrow C'_j$$

is a map in wC.

$wT.C$ is 'better' than $wS.C$ insofar as it may be regarded as the horizontal nerve of a *bicategory*.

In order to compare the two we have to modify $wT.C$ a little, by including choices. Namely let wT^+_nC be defined just as wT_nC except that in the data of an object we include a choice of quotients $C_{ij} = C_j/C_i$ for every $i \leq j$; the choice is to be arbitrary except if $i = j$ where we insist that $C_{ii} = *$, the basepoint. The forgetful map $wT^+.C \to wT.C$ is an equivalence of categories in each degree, and therefore a homotopy equivalence. The comparison is now made by means of a map of simplicial categories $wT^+.C \to wS.C$ which we show to be a homotopy equivalence. The map is defined as the forgetful map which forgets the C_i and remembers only the subquotients C_{ij}.

To show the map is a homotopy equivalence it suffices to show $wT^+_nC \to wS_nC$ is a homotopy equivalence for every n. For fixed n now wS_nC may be regarded as a retract of wT^+_nC; the section is the map which defines C_i as $C_{o,i}$ (the section is *not* induced by a simplicial map). We show the retraction is a deformation retraction by exhibiting a homotopy explicitly. There is a natural transformation from the identity functor to the composed map $wT^+_nC \to wS_nC \to wT^+_nC$, it is given on an object $C_o \rightarrowtail \ldots \rightarrowtail C_n$ by the quotient map to $C_{o,o} \rightarrowtail \ldots \rightarrowtail C_{o,n}$ which is a map in wT_nC in view of the definition of what this means. The natural transformation gives the desired homotopy.

1.4. The additivity theorem.

The proof of the additivity theorem involves only the cofibration structure, not the weak equivalences. It will therefore be convenient to explicitly concentrate on the cofibrations, a kind of 'separation of variables'.

If $\mathcal{C}$ is a (small) category with cofibrations we let $s_n\mathcal{C} = \mathrm{Ob}(S_n\mathcal{C})$, the set of objects of $S_n\mathcal{C}$, and $s.\mathcal{C}$ the simplicial set $[n] \mapsto s_n\mathcal{C}$.

Lemma 1.4.1. An exact functor of categories with cofibrations $f\colon \mathcal{C} \to \mathcal{C}'$ induces a map $s.f\colon s.\mathcal{C} \to s.\mathcal{C}'$. An isomorphism between two such functors f and f' induces a homotopy between $s.f$ and $s.f'$.

Before proving this we note the following consequence.

Corollary. (1) An exact equivalence of categories with cofibrations $\mathcal{C} \to \mathcal{C}'$ induces a homotopy equivalence $s.\mathcal{C} \to s.\mathcal{C}'$.

(2) Let $\mathcal{C}$ be made into a category with cofibrations and weak equivalences by means of the category $i\mathcal{C}$ of isomorphisms in $\mathcal{C}$. Then there is a homotopy equivalence $s.\mathcal{C} \to iS.\mathcal{C}$.

Indeed, (1) is clear, and (2) results by considering the simplicial object $[m] \mapsto i_mS.\mathcal{C}$, the nerve of $iS.\mathcal{C}$ in the i-direction, and noting that $i_oS.\mathcal{C} = s.\mathcal{C}$ and that the face and degeneracy maps are homotopy equivalences by (1).

Proof of lemma. The first part is clear. To prove the second part we will explicitly write down a simplicial homotopy. This is best done in categorical language. It is quite well known that simplicial objects in a category $\mathcal{D}$ can be regarded as functors $X\colon \Delta^{op} \to \mathcal{D}$, $[n] \mapsto X[n]$; and maps of simplicial objects as natural transformations of such functors. It seems to be less well known that simplicial homotopies can be described in similar fashion. Namely let $\Delta/[1]$ denote the category of objects over $[1]$ in Δ; the objects are the maps $[n] \to [1]$. For any $X\colon \Delta^{op} \to \mathcal{D}$ let X^* denote the composed functor

$$\begin{array}{ccccc} (\Delta/[1])^{op} & \longrightarrow & \Delta^{op} & \xrightarrow{X} & \mathcal{D} \\ ([n] \to [1]) & \longmapsto & [n] & \longmapsto & X[n] . \end{array}$$

Then a *simplicial homotopy of maps from* X *to* Y may be identified with a natural transformation $X^* \to Y^*$.

In the case at hand suppose that a functor isomorphism from f to f' is given and write it as a functor $F: \mathcal{C} \times [1] \to \mathcal{C}'$. The required simplicial homotopy then is the map from $([n]\to[1]) \mapsto s_n\mathcal{C}$ to $([n]\to[1]) \mapsto s_n\mathcal{C}'$ given by

$$(a: [n] \to [1]) \longmapsto ((A: Ar[n] \to \mathcal{C}) \longmapsto (A': Ar[n] \to \mathcal{C}'))$$

where A' is defined as the composition

$$Ar[n] \xrightarrow{(A,a_*)} \mathcal{C} \times Ar[1] \xrightarrow{id \times p} \mathcal{C} \times [1] \xrightarrow{F} \mathcal{C}'$$

and $p: Ar[1] \to [1]$ is given by $(0,0) \mapsto 0$, $(1,1) \mapsto 1$, and $(0,1) \mapsto$ □

Recall the equivalent formulations of the additivity theorem given in proposition 1.3.2. We will now prove one of them.

Theorem 1.4.2. (*Additivity theorem*). Let $\mathcal{C}$ be a category with cofibrations and weak equivalences. Then the following map is a homotopy equivalence,

$$wS.\mathcal{E}(\mathcal{C}) \longrightarrow wS.\mathcal{C} \times wS.\mathcal{C}$$
$$A \rightarrowtail C \twoheadrightarrow B \longmapsto A , B .$$

We deduce this from

<u>Lemma</u> 1.4.3. The map $s.\mathcal{E}(\mathcal{C}) \to s.\mathcal{C} \times s.\mathcal{C}$ is a homotopy equivalence.

The lemma may be regarded as a special case of the theorem, namely the case of the map $iS.\mathcal{E}(\mathcal{C}) \to iS.\mathcal{C} \times iS.\mathcal{C}$, in view of lemma 1.4.1. Conversely,

Proof of theorem from lemma 1.4.3. Define $\mathcal{C}(m,w)$ to be the full subcategory of the functor category $\mathcal{C}^{[m]}$ of those functors which take values in $w\mathcal{C}$. Then $\mathcal{C}(m,w)$ is a subcategory-with-cofibrations of $\mathcal{C}^{[m]}$, and $[m] \mapsto \mathcal{C}(m,w)$ defines a simplicial category with cofibrations. Applying the lemma we obtain that each of the maps $s.\mathcal{E}(\mathcal{C}(m,w)) \to s.\mathcal{C}(m,w) \times s.\mathcal{C}(m,w)$ is a homotopy equivalence. It follows, by the realization lemma, that the map of simplicial objects

$$([m] \mapsto s.\mathcal{E}(\mathcal{C}(m,w))) \longrightarrow ([m] \mapsto s.\mathcal{C}(m,w)) \times ([m] \mapsto s.\mathcal{C}(m,w))$$

is a homotopy equivalence. But this is equivalent to the assertion of the theorem in view of the natural isomorphism of $[m],[n] \mapsto s_n\mathcal{C}(m,w)$ with the bisimplicial set $[m],[n] \mapsto w_mS_n\mathcal{C}$, the nerve of the simplicial category $wS.\mathcal{C}$. □

In the proof of lemma 1.4.3 we will need a version of the fibration criterion, theorem B of Quillen [8], in the framework of simplicial sets. We proceed to formulate this.

Let Δ^n denote the simplicial set *standard* n-*simplex*, $[m] \mapsto Hom_\Delta([m],[n])$. If Y is any simplicial set then its set of n-simplices may be identified with the set of maps $\Delta^n \to Y$ (a case of the Yoneda lemma). Let $f: X \to Y$ be a map of simplicial sets and let y be a n-simplex of Y . Define a simplicial set $f/(n,y)$

as the pullback

$$\begin{array}{ccc} f/(n,y) & \longrightarrow & X \\ \downarrow & & \downarrow f \\ \Delta^n & \xrightarrow{\;y\;} & Y \end{array} \quad .$$

<u>Lemma</u> 1.4.A. If $f/(n,y)$ is contractible for every (n,y) then f is a homotopy equivalence.

<u>Lemma</u> 1.4.B. If for every $a\colon [m] \to [n]$, and every $y \in Y_n$, the induced map from $f/(m,a^*y)$ to $f/(n,y)$ is a homotopy equivalence then for every (n,y) the pullback diagram above is homotopy cartesian.

These two lemmas follow at once from theorems A and B of Quillen [8]. For let $\mathrm{simp}(Y)$ denote the category whose objects are the (n,y) and where a morphism from (n',y') to (n,y) is a morphism $a\colon [n'] \to [n]$ in Δ such that $a^*y = y'$. By applying $\mathrm{simp}(-)$ to everything in sight we obtain a translation of lemmas A and B into cases of theorems A and B, respectively. This uses that $\mathrm{simp}(f/(n,y))$ is naturally isomorphic with $\mathrm{simp}(f)/(n,y)$, the left fibre over (n,y) of the map of categories $\mathrm{simp}(f)$. And it uses further that, if N denotes the nerve functor, there is a natural transformation $N\mathrm{simp}(Y) \to Y$ which is a homotopy equivalence (cf. the end of section 1.6).

Proof of Lemma 1.4.3. We defer till later the proof of the following

<u>Sublemma</u>. The map $f\colon s.E(C) \to s.C$, $A \rightarrowtail C \twoheadrightarrow B \longmapsto A$, satisfies the hypothesis of lemma B above.

Applying lemma B we obtain a certain homotopy cartesian square for each simplex (n,y) of $s.C$. In particular we obtain such a square for the unique 0-simplex $*$ of $s.C$ in which case the homotopy cartesian square may be rewritten as a fibration up to homotopy $f/(0,*) \to s.E(C) \to s.C$. The term $f/(0,*)$ can be identified with $s.E'(C)$ where $E'(C)$ denotes the subcategory with cofibrations of $E(C)$ whose objects are the cofibration sequences $* \rightarrowtail C \twoheadrightarrow B$. As the quotient map in those cofibration sequences is necessarily an isomorphism, $E'(C)$ is equivalent to C , and by lemma 1.4.1 therefore $s.E'(C)$ is homotopy equivalent to $s.C$. We conclude that the sequence

$$\begin{array}{ccccc} s.C & \longrightarrow & s.E(C) & \longrightarrow & s.C \\ & & A \rightarrowtail C \twoheadrightarrow B & \longmapsto & A \\ B & \longmapsto & * \rightarrowtail B \twoheadrightarrow B & & \end{array}$$

is a fibration up to homotopy. There is a map to this fibration sequence from the product fibration sequence. The map is the identity on the fibre and on the base, and on total spaces it is given by the *split cofibration sequences*, i.e. it is the map $S.C \times S.C \to S.E(C)$, $(A,B) \mapsto (A \rightarrowtail A \vee B \twoheadrightarrow B)$. It follows that this map is a homotopy equivalence. The map is a section to the map of lemma 1.4.3, so that map must be a homotopy equivalence, too. □

Proof of sublemma. The assertion is that for every $y \in S_n C$ and $w\colon [m] \to [n]$ in Δ , the map $w_*\colon f/(m,w^*y) \to f/(n,y)$ is a homotopy equivalence.

It will suffice to consider the special case of maps $[0] \to [n]$. For any map $w\colon [m] \to [n]$ can be embedded in some commutative triangle

$$\begin{array}{ccc} [m] & \xrightarrow{\;w\;} & [n] \\ {\scriptstyle u}\nwarrow & & \nearrow{\scriptstyle v} \\ & [0] & \end{array}$$

and if we know that u_* and v_* are both homotopy equivalences then it follows that w_* is a homotopy equivalence, too.

We are thus reduced to proving this: let A' be a n-simplex of $S.C$, for some n , and $*$ the unique 0-simplex of $S.C$. Let $v_i\colon [0] \to [n]$ denote the map which takes 0 to i . Then for every i the map

$$v_{i*} : f/(0,*) \longrightarrow f/(n,A')$$

is a homotopy equivalence.

A m-simplex of $S.E(C)$ may be identified to an object of $E(S_m C)$, that is, a cofibration sequence $A \rightarrowtail C \twoheadrightarrow B$ in the category $S_m C$.

A m-simplex of $f/(n,A')$ now consists of such a m-simplex $A \rightarrowtail C \twoheadrightarrow B$ together with a map $u\colon [m] \to [n]$, and these data are subject to the condition that A is equal to the composite

$$\mathrm{Ar}[m] \xrightarrow{\;u_*\;} \mathrm{Ar}[n] \xrightarrow{\;A'\;} C \quad .$$

The quotient projection $A \rightarrowtail C \twoheadrightarrow B \longmapsto B$ induces a map $p\colon f/(n,A') \to S.C$. It will suffice to show that p is a homotopy equivalence. Indeed, p is left inverse to each of the composed maps

$$S.C \xrightarrow{\;j_*\;} f/(0,*) \xrightarrow{\;v_{i*}\;} f/(n,A') \ ,$$

therefore if p is a homotopy equivalence then so is $v_{i*}j_*$, and hence also v_{i*} , since j_* certainly is a homotopy equivalence, being induced by the equivalence $C \to f/(0,*)$, $B \mapsto (* \rightarrowtail B \xrightarrow{=} B, -)$.

Finally, in order to show p is a homotopy equivalence, it suffices to show that the particular map $v_{n*}j_*p\colon f/(n,A') \to f/(n,A')$ is homotopic to the identity map on $f/(n,A')$. We will construct such a homotopy explicitly.

The homotopy to be constructed will be a lifting of the simplicial homotopy that contracts Δ^n to its last vertex. In categorical language, this simplicial homotopy is given by a map of the composed functor

$$(\Delta/[1])^{op} \longrightarrow \Delta^{op} \longrightarrow (\text{sets})$$
$$([m] \to [1]) \longmapsto [m] \longmapsto \mathrm{Hom}([m],[n])$$

to itself, namely by

$$(v\colon [m] \to [1]) \longmapsto (\ (u\colon [m] \to [n]) \longmapsto (\bar{u}\colon [m] \to [n])\ )$$

where $\bar{u}$ is defined as the composite

$$[m] \xrightarrow{(u,v)} [n] \times [1] \xrightarrow{w} [n]$$

and where $w(j,0) = j$, $w(j,1) = n$.

A lifting of this homotopy to one on $f/(n,A')$ will be a map taking

$$(v\colon [m] \to [1])$$

to

$$(A \rightarrowtail C \twoheadrightarrow B\ ,\ u\colon [m] \to [n]) \longmapsto (\bar{A} \rightarrowtail \bar{C} \twoheadrightarrow \bar{B}\ ,\ \bar{u}\colon [m] \to [n])$$

where $\bar{u}$ is obtained from (v,u) as before and where certain compatibility conditions must be satisfied. In particular $\bar{A}$ must be equal to the composite

$$\mathrm{Ar}[m] \xrightarrow{\bar{u}_*} \mathrm{Ar}[n] \xrightarrow{A'} C$$

and is thus entirely forced.

To see that the rest of the data can be found in the required way we note that for every $j \in [m]$ we have

$$u(j) \leqslant \bar{u}(j)\ .$$

This may be expressed by saying that there is a map of functors

$$(u\colon [m] \to [n]) \longrightarrow (\bar{u}\colon [m] \to [n])\ .$$

Consequently there is also a map of functors

$$(u_*\colon \mathrm{Ar}[m] \to \mathrm{Ar}[n]) \longrightarrow (\bar{u}_*\colon \mathrm{Ar}[m] \to \mathrm{Ar}[n])\ ,$$

and the latter induces a map of the composed functors

$$\mathrm{Ar}[m] \longrightarrow \mathrm{Ar}[n] \longrightarrow C\ ,$$

that is, a map from A to $\bar{A}$ in $S_m C$.

For later reference we record that a map $A \to \bar{A}$ obtained in this fashion is necessarily unique. Indeed, $A \to \bar{A}$ is induced by a map of functors $\mathrm{Ar}[m] \to \mathrm{Ar}[n]$ and the latter map, if it exists at all, is unique because $\mathrm{Ar}[n]$ is a partially ordered set.

We now *define* a cofibration sequence $\bar{A} \rightarrowtail \bar{C} \twoheadrightarrow \bar{B}$ as being obtained from $A \rightarrowtail C \twoheadrightarrow B$ by cobase change, in $S_m C$, with the map $A \to \bar{A}$. Thus

$$\begin{array}{ccccc} A & \rightarrowtail & C & \twoheadrightarrow & B \\ \downarrow & & \downarrow & & \| \\ \bar{A} & \rightarrowtail & \bar{C} & \twoheadrightarrow & \bar{B} \end{array} .$$

The definition involves a choice of pushouts; that is, given $\bar{A} \leftarrow A \rightarrowtail C$ we must complete it to a pushout diagram, with pushout $\bar{C}$, in some definite way. We insist at this point that those choices shall be made in C rather than in $S_m C$. Because of the way pushouts in $S_m C$ are computed (proposition 1.1.4) this gives the required choices in $S_m C$ as well.

We are left to verify that the construction of $\bar{A} \rightarrowtail \bar{C} \twoheadrightarrow \bar{B}$ is compatible with the structure maps of the category $\Delta/[1]$; that is, if in our data we replace $[m]$ by $[m']$ throughout, by means of some map $[m'] \to [m]$, then the structure map in $s.E(C)$ induced by $[m'] \to [m]$ takes the one cofibration sequence to the other.

To see this we review the steps of the construction. The first step was the definition of the map $A \to \bar{A}$. The definition is compatible with structure maps because of the uniqueness property pointed out above.

The second step was the choice of actual pushout diagrams. But this choice was made in C, and an element of $S_m C$ is a certain kind of diagram in C on which the simplicial structure maps operate by omission and/or reduplication of data. So again there is the required compatibility.

With a little extra care we can arrange the choices so that the homotopy starts from the identity map (namely if $A \to \bar{A}$ is an identity map we insist that $C \to \bar{C}$ is also an identity map); and that the image of $v_{n*}j_*$ is fixed under the homotopy (namely if $\bar{A} = *$ we insist that $\bar{C} \to \bar{B}$ is the identity map on $\bar{B}$). We have now constructed the desired homotopy. This completes the proof of the sublemma and hence that of the additivity theorem. □

1.5. Applications of the additivity theorem to relative K-theory, de-looping, and cofinality.

Let $X: \Delta^{op} \to \mathcal{D}$ be a simplicial object in a category $\mathcal{D}$. The associated *path object* PX is defined as the composition of X with the *shift functor* $\Delta \to \Delta$ which takes $[n]$ to $[n+1]$ (by 'sending i to i+1' - this fixes the behaviour on morphisms). The fact that a path space deforms into the subspace of constant paths has the following well known analogue here, e.g. [11], which we record in detail because we need to know the homotopy.

Lemma 1.5.1. PX is simplicially homotopy equivalent to the constant simplicial object $[n] \mapsto X_o$.

Proof. We show there is a simplicial homotopy between the identity on PX and the composite map $PX \to X_o \to PX$ induced from

$$[n] \longmapsto ([n+1] \to [0] \to [n+1])$$
$$0 \longmapsto 0 \quad .$$

The homotopy is given by the natural transformation

$$(a: [n] \to [1]) \longmapsto (\varphi_a^*: X_{n+1} \to X_{n+1})$$

induced from $(a: [n] \to [1]) \mapsto (\varphi_a: [n+1] \to [n+1])$ where $\varphi_a(0) = 0$ and

$$\varphi_a(j+1) = \begin{cases} j+1 & \text{if } a(j) = 1 \\ 0 & \text{if } a(j) = 0 \quad . \end{cases}$$ □

PX comes equipped with a projection $PX \to X$ (it is induced by the 0-face map of X which is not otherwise used in PX) and there is an inclusion of X_1 considered as a constant simplicial object (because $(PX)_o = X_1$). There results a sequence $X_1 \to PX \to X$.

In particular if $\mathcal{C}$ is a category with cofibrations and weak equivalences we obtain a sequence $wS_1\mathcal{C} \to P(wS.\mathcal{C}) \to wS.\mathcal{C}$ which in view of the isomorphism of $wS_1\mathcal{C}$ with $w\mathcal{C}$ we may rewrite as

$$w\mathcal{C} \longrightarrow P(wS.\mathcal{C}) \longrightarrow wS.\mathcal{C} \ .$$

The composite map is constant, and $|P(wS.\mathcal{C})|$ is contractible (for by the preceding lemma it is homotopy equivalent to the one-point space $|wS_o\mathcal{C}|$), so we obtain a map, well defined up to homotopy,

$$|w\mathcal{C}| \longrightarrow \Omega|wS.\mathcal{C}|$$

Lemma 1.5.2. The map can be chosen to agree with the corresponding map in the preceding section.

Proof. From the explicit homotopy of the preceding lemma one actually obtains an explicit choice of the map. This *is* the map in question.

By naturality we can substitute $\mathcal{C}$ with the simplicial category $S.\mathcal{C}$ in the above sequence. We obtain a sequence

$$wS.\mathcal{C} \longrightarrow P(wS.S.\mathcal{C}) \longrightarrow wS.S.\mathcal{C}$$

(where the ' P ' refers to the first $S.$-direction, say).

Proposition 1.5.3. The sequence is a fibration up to homotopy. That is, the map from $|wS.\mathcal{C}|$ to the homotopy fibre of $|P(wS.S.\mathcal{C})| \to |wS.S.\mathcal{C}|$ is a homotopy equivalence.

Proof. This is a special case of proposition 1.5.5 below

Thus $|wS.\mathcal{C}| \to \Omega|wS.S.\mathcal{C}|$ is a homotopy equivalence and more generally therefore, in view of the realization lemma, also the map $|wS.^{(n)}\mathcal{C}| \to \Omega|wS.^{(n+1)}\mathcal{C}|$ for every $n \geq 1$, proving the postponed claim (section 1.3) that the spectrum $n \mapsto |wS.^{(n)}\mathcal{C}|$ is a Ω-spectrum beyond the first term.

We digress to indicate in which way the twice de-looped K-theory $wS.S.\mathcal{C}$ is used in defining *products*; or better, *external pairings* (products are induced from those). The ingredient that one needs is a *bi-exact functor* of categories with cofibrations and weak equivalences. This is a functor $\mathcal{A}\times\mathcal{B} \to \mathcal{C}$, $(A,B) \mapsto A\wedge B$, having the property that for every $A \in \mathcal{A}$ and $B \in \mathcal{B}$ the partial functors $A \wedge ?$ and $? \wedge B$ are exact, and where in addition the following more technical condition must also be satisfied; namely for every pair of cofibrations $A \rightarrowtail A'$ and $B \rightarrowtail B'$ in $\mathcal{A}$ and $\mathcal{B}$, respectively, the induced square of cofibrations in $\mathcal{C}$ must be *admissible* in the sense that the map $A'\wedge B \cup_{A\wedge B} A\wedge B' \to B\wedge B'$ is a cofibration. A bi-exact functor induces a map, of bisimplicial bicategories,

$$wS.\mathcal{A} \times wS.\mathcal{B} \longrightarrow wwS.S.\mathcal{C}$$

which upon passage to geometric realization factors through the smash product

$$|wS.\mathcal{A}| \wedge |wS.\mathcal{B}| \longrightarrow |wwS.S.\mathcal{C}|$$

and in turn induces

$$\Omega|wS.\mathcal{A}| \wedge \Omega|wS.\mathcal{B}| \longrightarrow \Omega\Omega|wwS.S.\mathcal{C}| \ .$$

This is the desired pairing in K-theory in view of the homotopy equivalence of $|wS.\mathcal{C}|$ with $\Omega|wS.S.\mathcal{C}|$, and a (much more innocent) homotopy equivalence of $wS.S.\mathcal{C}$ with $wwS.S.\mathcal{C}$ which we will have occasion later on to consider in detail (the 'swallowing lemma' in section 1.6).

Definition 1.5.4. Let $f\colon A \to B$ be an exact functor of categories with cofibrations and weak equivalences. Then $S.(f:A\to B)$ is the pullback of the diagram

$$S.A \longrightarrow S.B \longleftarrow PS.B \ .$$

Thus for every n we have a pullback diagram

$$\begin{array}{ccc} S_n(f:A\to B) & \longrightarrow & (PS.B)_n = S_{n+1}B \\ \downarrow & & \downarrow \\ S_nA & \longrightarrow & S_nB \end{array} \ .$$

The vertical map on the right has a section (it is not compatible with face maps), so the pullback category is equivalent to the fibre product category and in any case is not pathological. It results (sections 1.1 and 1.2) that $S.(f:A\to B)$ is a simplicial category with cofibrations and weak equivalences in a natural way, and all the maps in the defining diagram (definition 1.5.4) are exact.

Considering B as a simplicial category in a trivial way we have an inclusion $B \to P(S.B)$ whose composition with the projection to $S.B$ is trivial (cf. above). Lifting the inclusion to the pullback, and combining with the other projection, we then obtain a sequence

$$B \longrightarrow S.(f:A\to B) \longrightarrow S.A$$

in which the composed map is trivial. The sequence is formally very similar to the sequence describing the homotopy fibration associated to a map of spaces. The following result says that in fact the sequence serves a similar purpose.

Proposition 1.5.5. The sequence

$$wS.B \longrightarrow wS.S.(f:A\to B) \longrightarrow wS.S.A$$

is a fibration up to homotopy.

Proof. There is a fibration criterion which says that it is enough to show that for every n the sequence $wS.B \to wS.S_n(f:A\to B) \to wS.S_nA$ is a fibration up to homotopy (e.g. since the base term $wS.S_nA$ is connected for every n, the criterion given by lemma 5.2 of [13] will do). Using the additivity theorem we will show that, in fact, the sequence is the same, up to homotopy, as the trivial fibration sequence associated to the product $wS.B \times wS.S_nA$.

Neglecting choices to simplify the notation, we can identify an object of $S_n(f:A\to B)$ to a pair of filtered objects in A and B, respectively, say $A_{o,1} \rightarrowtail \dots \rightarrowtail A_{o,n}$ and $B_o \rightarrowtail B_1 \rightarrowtail \dots \rightarrowtail B_n$, together with an isomorphism of filtered objects,

$$f(A_{o,1}) \rightarrowtail \dots \rightarrowtail f(A_{o,n}) \quad \approx \quad B_1/B_o \rightarrowtail \dots \rightarrowtail B_n/B_o$$

Let $\mathcal{C}'$ denote the subcategory of the objects where all the maps $B_o \to B_1 \to$ ... are identities and all the $A_{o,i}$ are equal to the basepoint; then $\mathcal{C}'$ is isomorphic to $\mathcal{B}$. Let $\mathcal{C}''$ denote the subcategory where B_o is equal to the basepoint; then $\mathcal{C}''$ is isomorphic to $\mathcal{S}_n\mathcal{A}$. There is an obvious cofibration sequence of endofunctors

$$j' \rightarrowtail \mathrm{id} \twoheadrightarrow j''$$

where j' and j'' take values in $\mathcal{C}'$ and $\mathcal{C}''$, respectively. Applying the additivity theorem (in formulation (4) of proposition 1.3.2) we obtain that the identity map on $w\mathcal{S}.\mathcal{S}_n(f:\mathcal{A}\to\mathcal{B})$ is homotopic to the sum of $w\mathcal{S}.j'$ and $w\mathcal{S}.j''$. It results that the map, given by the split cofibration sequences,

$$w\mathcal{S}.\mathcal{B} \times w\mathcal{S}.\mathcal{S}_n\mathcal{A} \longrightarrow w\mathcal{S}.\mathcal{S}_n(f:\mathcal{A}\to\mathcal{B})$$

is a retraction, up to homotopy. On the other hand the map is obviously also a coretraction. It is therefore a homotopy equivalence. We conclude with the remark that the homotopy equivalence can be induced by a map from the product fibration sequence to the sequence in question (i.e. the degree n part of the sequence of the proposition). It follows that the two sequences are the same, up to homotopy. This completes the proof of the proposition. □

In a special situation we can modify the definition of $\mathcal{S}.(f:\mathcal{A}\to\mathcal{B})$ to obtain a variant which is technically a little more convenient. Namely suppose that $\mathcal{A}$ is a *subcategory with cofibrations and weak equivalences* of $\mathcal{B}$ as defined in sections 1.1 and 1.2. Then we define

$$\mathcal{F}_n(\mathcal{B},\mathcal{A})$$

as the category whose objects are the sequences of cofibrations in $\mathcal{B}$,

$$B_o \rightarrowtail B_1 \rightarrowtail \dots \rightarrowtail B_n$$

subject to the condition that for every pair $i \leq j$ the object B_j/B_i is isomorphic to some object of $\mathcal{A}$. There is a forgetful map

$$\mathcal{S}_n(\mathcal{A}\to\mathcal{B}) \longrightarrow \mathcal{F}_n(\mathcal{B},\mathcal{A})$$

(forget choices of quotients B_j/B_i in $\mathcal{A}$). It is an equivalence of categories with cofibrations and weak equivalences. Further the $\mathcal{F}_n(\mathcal{B},\mathcal{A})$ may be assembled to a simplicial category with cofibrations and weak equivalences $\mathcal{F}.(\mathcal{B},\mathcal{A})$. By the realization lemma then the forgetful map

$$w\mathcal{S}.\mathcal{S}.(\mathcal{A}\to\mathcal{B}) \longrightarrow w\mathcal{S}.\mathcal{F}.(\mathcal{B},\mathcal{A})$$

is a homotopy equivalence. Thus $\mathcal{F}.(\mathcal{B},\mathcal{A})$ may be used interchangeably with $\mathcal{S}.(\mathcal{A}\to\mathcal{B})$ if $\mathcal{A}$ is a subcategory with cofibrations and weak equivalences of $\mathcal{B}$.

Corollary 1.5.6. If $A \to B \to C$ are exact functors of categories with cofibrations and weak equivalences then the square

$$\begin{array}{ccc} wS.B & \longrightarrow & wS.S.(A \to B) \\ \downarrow & & \downarrow \\ wS.C & \longrightarrow & wS.S.(A \to C) \end{array}$$

is homotopy cartesian Similarly the square

$$\begin{array}{ccc} wS.B & \longrightarrow & wS.F.(B,A) \\ \downarrow & & \downarrow \\ wS.C & \longrightarrow & wS.F.(C,A) \end{array}$$

is homotopy cartesian if the terms on the right are defined.

Proof. There is a commutative diagram

$$\begin{array}{ccccc} wS.B & \longrightarrow & wS.S.(A \to B) & \longrightarrow & wS.S.A \\ & & \downarrow & & \downarrow \\ wS.C & \longrightarrow & wS.S.(A \to C) & \longrightarrow & wS.S.A \end{array}$$

in which the vertical map on the right is an identity map and where the rows are fibrations up to homotopy, by the preceding proposition. It results that the square on the left is homotopy cartesian.

Concerning the second square, if that is defined, there is a natural transformation between the two squares in which all the maps are homotopy equivalences. The second assertion is just a rewriting of the first. □

Corollary 1.5.7. To an exact functor $B \to C$ there is associated a sequence of the homotopy type of a fibration (with a preferred null-homotopy of the composed map)

$$wS.B \longrightarrow wS.C \longrightarrow wS.S.(B \to C) \ .$$

Indeed, this is the case $A = B$ of corollary 1.5.6 since $wS.S.(A \xrightarrow{=} A)$ is contractible.

Corollary 1.5.8. If C is a retract of B (by exact functors) there is a splitting

$$wS.B \simeq wS.C \times wS.S.(C \to B) \ .$$

Indeed, this is the case of corollary 1.5.6 where the composed map $A \to B \to C$ is an identity map (or more generally, an exact equivalence) since $wS.S.(A \to C)$ is contractible in that case.

Let A be a subcategory with cofibrations and weak equivalences of B. We say that A is *strictly cofinal* in B if for every $B \in B$ there exists a $A \in A$ such that $B \vee A$ is isomorphic to an object of A.

For example the category of free modules over a ring qualifies as strictly cofinal in the category of stably free modules, but not in the category of projective modules.

Proposition 1.5.9. If A is strictly cofinal in B then $wS.A \to wS.B$ is a homotopy equivalence.

Proof. It will be convenient to assume that A is *saturated* in B in the sense that every object of B isomorphic to one of A is actually contained in A. Since A can be enlarged to an equivalent category which is saturated in B and since such an enlargement does not affect any homotopy types, this assumption is not a loss of generality.

By corollary 1.5.7 or 1.5.6 the map $wS.A \to wS.B$ will be a homotopy equivalence if the bisimplicial category $wS.F.(B,A)$ is contractible. By the realization lemma this follows if $wS_nF.(B,A)$ is contractible for every n. We can rewrite

$$wS_nF.(B,A) \approx wF.(S_nB,S_nA) .$$

Assertion 1. If A is strictly cofinal in B then, for every n, S_nA is strictly cofinal in S_nB.

The assertion will be proved later. It reduces us to showing that $wF.(B,A)$ is contractible if A is strictly cofinal in B. By the realization lemma again this follows if the simplicial set $w_mF.(B,A)$, i.e. the degree-m-part of the nerve in the w-direction, is contractible for every m. Let, as before, $B(m,w)$ denote the category of the diagrams $B_o \to B_1 \to \ldots \to B_n$ in B in which the arrows are weak equivalences; and similarly with $A(m,w)$. Let $\mathcal{F}.(B,A)$ denote the simplicial set of objects of $F.(B,A)$. We can rewrite

$$w_mF.(B,A) \approx \mathcal{F}.(B(m,w),A(m,w)) .$$

Assertion 2. If A is strictly cofinal in B then, for every m, $A(m,w)$ is strictly cofinal in $B(m,w)$.

The assertion reduces us to proving

Assertion 3. If A is strictly cofinal in B then $\mathcal{F}.(B,A)$ is contractible.

It remains to prove the assertions.

Proof of assertion 1. Let $B \in \mathcal{S}_n\mathcal{B}$. We think of it as a filtration $B_{o,1} \rightarrowtail B_{o,2} \rightarrowtail \dots \rightarrowtail B_{o,n}$, plus a choice of subquotients $B_{i,j}$. By applying the cofinality hypothesis for $\mathcal{A} \subset \mathcal{B}$ we can find objects $A'_{i,j}$ in $\mathcal{A}$ (not subquotients of a filtration) so that $B_{i,j} \vee A'_{i,j}$ is in $\mathcal{A}$ for every (i,j) . Let A' be the sum of all the $A'_{i,j}$. Then $B_{i,j} \vee A'$ is in $\mathcal{A}$ for every (i,j) . We can define an object A of $\mathcal{S}_n\mathcal{A}$ where, for every $i < j$, $A_{i,j}$ involves at least one summand A' ; briefly, $A_{o,i}$ is the i-fold sum of A' with itself. Then $B \vee A$ is in $\mathcal{S}_n\mathcal{B}$, and all the objects involved in it are in $\mathcal{A}$; it is therefore in $\mathcal{S}_n\mathcal{A}$ in view of the definition of what it means for $\mathcal{A}$ to be a subcategory with cofibrations of $\mathcal{B}$.

Proof of assertion 2. This is similar, but easier.

Proof of assertion 3. A n-simplex of $s.(\mathcal{B},\mathcal{A})$ is a sequence of cofibrations in $\mathcal{B}$, $B_o \rightarrowtail \dots \rightarrowtail B_n$, subject to the condition that every subquotient B_j/B_i is isomorphic to some object of $\mathcal{A}$ (in fact, equal to an object of $\mathcal{A}$, for any choice whatsoever, in view of the assumed fact that $\mathcal{A}$ is saturated in $\mathcal{B}$). We apply the cofinality hypothesis to each of the B_i and then add all the objects of $\mathcal{A}$ obtained. This gives an object A in $\mathcal{A}$ with the property that $B_i \vee A$ is in $\mathcal{A}$ for every i ; the sequence $B_o \vee A \rightarrowtail \dots \rightarrowtail B_n \vee A$ is thus a sequence of cofibrations in $\mathcal{A}$ (since $\mathcal{A}$ is a subcategory with cofibrations of $\mathcal{B}$). We refer to this situation by saying that the object A *moves* the simplex $B_o \rightarrowtail \dots \rightarrowtail B_n$.

More generally, given finitely many simplices, not necessarily of the same dimension, we can find objects as before and add them all up to obtain a single object A which moves every one of these simplices.

The simplicial set $s.(\mathcal{A},\mathcal{A})$ is contractible (it is the nerve of the category of cofibrations in $\mathcal{A}$, which has an initial object). To show $s.(\mathcal{B},\mathcal{A})$ is contractible it suffices therefore to show that the inclusion $s.(\mathcal{A},\mathcal{A}) \to s.(\mathcal{B},\mathcal{A})$ is a homotopy equivalence. This follows if we can show that for every *finite* pair of simplicial subsets $(L,K) \subset (s.(\mathcal{B},\mathcal{A}), s.(\mathcal{A},\mathcal{A}))$ there is a homotopy, of pairs, from the inclusion map to some map with image in $s.(\mathcal{A},\mathcal{A})$.

The simplicial set L has only finitely many non-degenerate simplices. So there is an object $A \in \mathcal{A}$ which moves every one of these simplices. But then A moves every other simplex of L as well.

$s.(\mathcal{B},\mathcal{A})$ is a simplicial subset of the nerve of the category of cofibrations in $\mathcal{B}$. The sum with A induces a natural transformation of that category, and in turn a homotopy of the identity map on $s.(\mathcal{B},\mathcal{A})$. The restriction of that homotopy to L , resp. K , is entirely in $s.(\mathcal{B},\mathcal{A})$, resp. $s.(\mathcal{A},\mathcal{A})$, and the homotopy terminates at a map which takes L into $s.(\mathcal{A},\mathcal{A})$. This gives the required homotopy of pairs. The proof is complete. □

1.6. Cylinder functors, the generic fibration, and the approximation theorem.

Let C be a category with cofibrations and weak equivalences. By a *cylinder functor* on C is meant a functor from $\mathrm{Ar}C$ to the category of diagrams in C taking $f\colon A \to B$ to a diagram

$$\begin{array}{ccccc} A & \xrightarrow{j_1} & T(f) & \xleftarrow{j_2} & B \\ & {\scriptstyle f}\searrow & \downarrow{\scriptstyle p} & \swarrow{\scriptstyle =} & \\ & & B & & \end{array}$$

The functor is required to satisfy the axioms Cyl 1 - Cyl 3 below. The object $T(f)$ will be referred to as the *cylinder of* f, and the maps j_1, j_2, p as the *front inclusion*, *back inclusion*, and *projection*, respectively.

Cyl 1. The front and back inclusions assemble to an exact functor

$$\begin{array}{ccc} \mathrm{Ar}C & \longrightarrow & F_1C \\ (A \xrightarrow[f]{} B) & \longmapsto & (A \vee B \xrightarrow[j_1 \vee j_2]{} T(f)) . \end{array}$$

Cyl 2. $T(* \to A) = A$, for every $A \in C$, and the projection and back inclusion are the identity map on A.

Cyl 3.

(fool's morning song [9], the tune replaces an unnecessary axiom)

Consider, for example, the category $\mathcal{R}(X)$ of the spaces having X as retract. It has a cylinder functor where $T(Y \to Y') = X \cup_{X\times[0,1]} Y\times[0,1] \cup_{Y\times 1} Y'$.

The following axiom may, or may not, be satisfied by a particular category of weak equivalences $w\mathcal{C}$.

Cylinder axiom. The projection $p: T(f) \to B$ is in $w\mathcal{C}$ for every $f: A \to B$ in $\mathcal{C}$.

Note. If in addition to the cylinder axiom $w\mathcal{C}$ also satisfies the saturation axiom (section 1.2) it follows that the back inclusion j_2 is always in $w\mathcal{C}$, and the front inclusion j_1 is in $w\mathcal{C}$ whenever f is.

For example in $\mathcal{R}(X)$ the weak homotopy equivalences and the simple maps satisfy the cylinder axiom while the isomorphisms do not. However the simple maps do not satisfy the saturation axiom, and in fact j_1 and j_2 are not, in general, simple maps.

Lemma 1.6.1. Cylinder functors are inherited by filtered objects. That is, a cylinder functor on $\mathcal{C}$ induces one on $S_n\mathcal{C}$, for every n . If the weak equivalences in $\mathcal{C}$ satisfy the cylinder axiom then so do those in $S_n\mathcal{C}$.

Proof. The required functor on $\mathrm{Ar}S_n\mathcal{C}$ is defined as the induced map

$$\mathrm{Ar}S_n\mathcal{C} \approx S_n\mathrm{Ar}\mathcal{C} \longrightarrow S_n(\text{diagrams in } \mathcal{C}) \approx (\text{diagrams in } S_n\mathcal{C}) .$$

The only non-trivial point to check is the exactness of the functor $\mathrm{Ar}S_n\mathcal{C} \to F_1S_n\mathcal{C}$ of axiom Cyl 1 . But this functor may be identified to the composite

$$\mathrm{Ar}S_n\mathcal{C} \approx S_n\mathrm{Ar}\mathcal{C} \longrightarrow S_nF_1\mathcal{C} \approx F_1S_n\mathcal{C}$$

and hence is exact since $\mathrm{Ar}\mathcal{C} \to F_1\mathcal{C}$ is exact by axiom Cyl 1 in $\mathcal{C}$.

Definition. The *cone functor* $A \mapsto cA$ is defined by

$$cA = T(A \to *) ,$$

and the *suspension functor* is defined as the quotient of the cone by the front inclusion $A \rightarrowtail T(A \to *)$,

$$\Sigma A = cA/A .$$

Proposition 1.6.2. If $\mathcal{C}$ has a cylinder functor and the weak equivalences satisfy the cylinder axiom then the suspension map

$$\Sigma : wS.\mathcal{C} \longrightarrow wS.\mathcal{C}$$

represents a homotopy inverse with respect to the H-space structure on $wS.\mathcal{C}$ given by the sum.

Proof. By the additivity theorem the cofibration sequence of functors $\mathrm{id} \rightarrowtail c \twoheadrightarrow \Sigma$ implies a homotopy of self-maps on $wS.C$, $\mathrm{id} \vee \Sigma \simeq c$. The natural transformation $cA \to *$ is a weak equivalence in view of the assumed cylinder axiom. By lemma 1.3.1 therefore c, and hence $\mathrm{id} \vee \Sigma$, is null-homotopic. □

Define $\overline{w}C$ to be the subcategory of wC of those weak equivalences which are also cofibrations. (This is *not*, in general, a category of weak equivalences in the sense of section 1.2.)

Lemma 1.6.3. If C has a cylinder functor, and the weak equivalences in C satisfy the cylinder axiom and saturation axiom, then the inclusion $\overline{w}C \to wC$ is a homotopy equivalence.

Proof. Calling the inclusion i, it suffices to show by theorem A [8] that for every $B \in wC$ the left fibre i/B is contractible. An object of i/B is a pair (A,f) where $f\colon A \to B$ is a map in $\overline{w}C$. Since the cylinder projection $p\colon T(f) \to B$ is in wC (by the cylinder axiom) we can define a functor $t\colon i/B \to i/B$ by letting $t(A,f) = (T(f),p)$. The front inclusion $j_1\colon A \to T(f)$ and back inclusion $j_2\colon B \to T(f)$ are weak equivalences as well as cofibrations (by the cylinder axiom and saturation axiom), so they define natural transformations to the functor t, one from the identity functor (using that $p\, j_1 = f$) and one from the constant functor with value (B,id_B) (using that $p\, j_2 = \mathrm{id}_B$). It results that t is homotopic to both the identity map on i/B and the trivial map (B,id_B). Hence the latter two are homotopic, and i/B is contractible. □

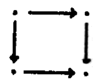
To formulate the next result suppose that C is a category with cofibrations and that C is equipped with *two* categories of weak equivalences, one finer than the other, $vC \subset wC$. Let C^w denote the subcategory with cofibrations of C given by the objects A in C having the property that the map $* \to A$ is in wC. It inherits categories of weak equivalences $vC^w = C^w \cap vC$ and $wC^w = C^w \cap wC$.

Theorem 1.6.4. (*Fibration theorem*). If C has a cylinder functor, and the coarse category of weak equivalences wC satisfies the cylinder axiom, saturation axiom, and extension axiom, then the square

$$\begin{array}{ccc} vS.C^w & \longrightarrow & wS.C^w \quad (\simeq * \\ \downarrow & & \downarrow \\ vS.C & \longrightarrow & wS.C \end{array}$$

is homotopy cartesian, and the upper right term is contractible.

Proof. Define vwC to be the bicategory of the commutative squares



in C in which the vertical and horizontal arrows are in vC and wC, respectively. Considering wC as a bicategory in a trivial way we have an inclusion $wC \to vwC$ which is a homotopy equivalence (lemma 1.6.5 below). There is a map in the other direction. The map exists only after passing to nerves, and diagonalizing (briefly, the map takes each square to its diagonal arrow), but to simplify the notation we will allow ourselves the abuse of writing the map as $vwC \to wC$. The map is left inverse to the former map, hence is a homotopy equivalence itself.

We can similarly define a simplicial bicategory $vwS.C$. By the realization lemma it results from the above that the maps $wS.C \to vwS.C$ and $vwS.C \to wS.C$ are homotopy equivalences as well (again the second map exists only after passing to nerves and diagonalizing the v- and w-directions).

Let $v\bar{w}C$ denote the sub-bicategory of vwC of the squares in which the horizontal arrows are in $\bar{w}C$ rather than just wC. Then the inclusion $v\bar{w}C \to vwC$ is a homotopy equivalence by lemma 1.6.3, which applies in view of the assumed cylinder axiom and saturation axiom. (In detail, by the realization lemma we can reduce to passing to nerves in the v-direction and showing that $v_n\bar{w}C \to v_n wC$ is a homotopy equivalence for every n. The map may be rewritten, in a way we have used before, as $\bar{w}C(v,n) \to wC(v,n)$, and lemma 1.6.3 now applies to the latter). Similarly there is a simplicial bicategory $v\bar{w}S.C$, and the inclusion $v\bar{w}S.C \to vwS.C$ is a homotopy equivalence. (For by the realization lemma we can reduce to showing that $v\bar{w}S_nC \to vwS_nC$ is a homotopy equivalence for every n. As S_nC inherits a cylinder functor from C (lemma 1.6.1) the above considerations apply to it.)

The square of the theorem may be identified to the large square in the following diagram

$$\begin{array}{ccccccc}
vS.C^w & \longrightarrow & v\bar{w}S.C^w & \longrightarrow & vwS.C^w & \longrightarrow & wS.C^w \\
 & & & & & & \\
vS.C & \longrightarrow & v\bar{w}S.C & \longrightarrow & vwS.C & \longrightarrow & wS.C
\end{array}$$

As the preceding discussion shows, the horizontal maps in the middle and on the right are homotopy equivalences. So the square will be homotopy cartesian if and only if the square on the left is. After passing to nerves in the $\bar{w}$-direction we can identify the square on the left to one of the squares of corollary 1.5.6 associated to the categories at hand, namely

$$\begin{array}{ccc} vS.\mathcal{C}^w & \longrightarrow & vS.F.(\mathcal{C}^w,\mathcal{C}^w) \\ \downarrow & & \downarrow \\ vS.\mathcal{C} & \longrightarrow & vS.F.(\mathcal{C},\mathcal{C}^w) \end{array} \;;$$

the point is that a map in $\overline{w}\mathcal{C}$ can be characterized as a cofibration in $\mathcal{C}$ whose quotient is in $\mathcal{C}^w$ (this uses the assumed fact that $w\mathcal{C}$ satisfies the extension axiom). The square is thus homotopy cartesian by corollary 1.5.6.

Finally the simplicial category $wS.\mathcal{C}^w$ is contractible because in each degree it has an initial object. □

The following lemma was used in the preceding argument; cf. [13] for some generalities on *bicategories*.

<u>Lemma</u> 1.6.5. (*Swallowing lemma*). Let $\mathcal{A}$ be a subcategory of $\mathcal{B}$, and $\mathcal{AB}$ the bicategory of the commutative squares with vertical and horizontal arrows in $\mathcal{A}$ and $\mathcal{B}$, respectively. The inclusion $\mathcal{B} \to \mathcal{AB}$ is a homotopy equivalence.

Proof. By the realization lemma it will suffice to take the nerve in the $\mathcal{A}$-direction and show that for every n the map $\mathcal{B} \to \mathcal{A}_n\mathcal{B}$ is a homotopy equivalence. For fixed n we can define a map $\mathcal{A}_n\mathcal{B} \to \mathcal{B}$ by taking the sequence $A_o \to \ldots \to A_n$ to A_o. This is left inverse to the inclusion of $\mathcal{B}$. Composing the other way we obtain the map which takes $A_o \to \ldots \to A_n$ to the appropriate sequence of identity maps on A_o. There is a natural transformation of this map to the identity map; it is given by the diagram

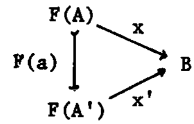
$$\begin{array}{ccccccc} A_o & \xrightarrow{=} & A_o & \xrightarrow{=} & \cdots & \xrightarrow{=} & A_o \\ \downarrow & & \downarrow{\scriptstyle a_1} & & & & \downarrow{\scriptstyle a_n \cdots a_2 a_1} \\ A_o & \xrightarrow[a_1]{} & A_1 & \xrightarrow[a_2]{} & \cdots & \xrightarrow[a_n]{} & A_n \end{array}$$

This shows that $\mathcal{B}$ is a deformation retract of $\mathcal{A}_n\mathcal{B}$. □

In order to formulate the next result it is convenient to introduce the following notion. Let $F\colon \mathcal{A} \to \mathcal{B}$ be an exact functor of categories with cofibrations and weak equivalences. We say it has the *approximation property* if it satisfies the conditions App 1 and App 2 below.

<u>App</u> 1. An arrow in $\mathcal{A}$ is a weak equivalence in $\mathcal{A}$ if (and only if) its image in $\mathcal{B}$ is a weak equivalence in $\mathcal{B}$.

<u>App</u> 2. Given any object A in $\mathcal{A}$ and any map $x: F(A) \to B$ in $\mathcal{B}$ there exist a cofibration $a: A \rightarrowtail A'$ in $\mathcal{A}$ and a weak equivalence $x': F(A') \to B$ in $\mathcal{B}$ so that the following triangle commutes,



<u>Lemma</u> 1.6.6. If $F: \mathcal{A} \to \mathcal{B}$ has the approximation property then so does $S_n\mathcal{A} \to S_n\mathcal{B}$.

Proof. The non-trivial thing to verify is the condition App 2 for the map S_nF . We think of an object of $S_n\mathcal{A}$ as a filtration $A_{o,1} \rightarrowtail A_{o,2} \rightarrowtail \dots \rightarrowtail A_{o,n}$, plus a choice of subquotients. Proceeding by induction on n we suppose we have found already a sequence $A'_{o,1} \rightarrowtail \dots \rightarrowtail A'_{o,n-1}$ together with maps as required. From these data we obtain an object in $\mathcal{A}$,

$$A_{o,n} \cup_{A_{o,n-1}} A'_{o,n-1}$$

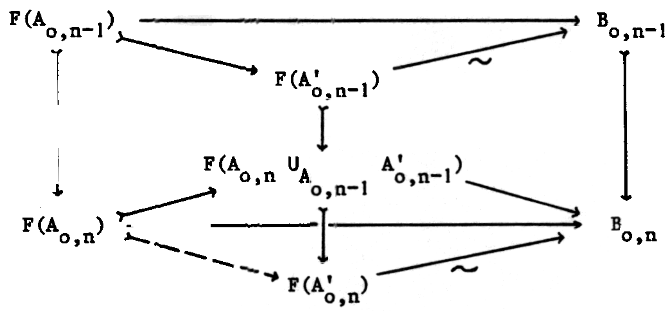
and a map in $\mathcal{B}$,

$$F(A_{o,n} \cup_{A_{o,n-1}} A'_{o,n-1}) \longrightarrow B_{o,n} \quad ,$$

to which the hypothesis App 2 for F may be applied. This gives a cofibration

$$A_{o,n} \cup_{A_{o,n-1}} A'_{o,n-1} \rightarrowtail A'_{o,n}$$

and a weak equivalence $F(A'_{o,n}) \to B_{o,n}$ so that the following diagram commutes (where the broken arrow $A_{o,n} \rightarrowtail A'_{o,n}$ is defined as the composite)



We are done. □

Theorem 1.6.7. (*Approximation theorem*). Let $\mathcal{A}$ and $\mathcal{B}$ be categories with cofibrations and weak equivalences. Suppose the weak equivalences in $\mathcal{A}$ and $\mathcal{B}$ satisfy the saturation axiom. Suppose further that $\mathcal{A}$ has a cylinder functor and the weak equivalences in $\mathcal{A}$ satisfy the cylinder axiom. Let $F: \mathcal{A} \to \mathcal{B}$ be an exact functor. Suppose F has the approximation property. Then the induced maps $w\mathcal{A} \to w\mathcal{B}$ and $wS.\mathcal{A} \to wS.\mathcal{B}$ are homotopy equivalences.

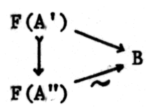
Proof. It will suffice to show that $w\mathcal{A} \to w\mathcal{B}$ is a homotopy equivalence. For this implies, in view of the preceding lemma, that $wS_n\mathcal{A} \to wS_n\mathcal{B}$ is a homotopy equivalence for every n, and hence, by the realization lemma, that $wS.\mathcal{A} \to wS.\mathcal{B}$ is a homotopy equivalence.

The proof that $w\mathcal{A} \to w\mathcal{B}$ is a homotopy equivalence, is quite long. It occupies the rest of this section. Calling the map f, it suffices to show, by theorem A [8], that for every $B \in w\mathcal{B}$ the left fibre f/B is contractible, and this is what we shall prove.

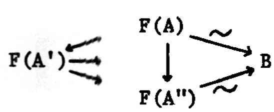
The idea for the proof of contractibility of f/B is in the following observation which says that certain diagrams $\mathcal{D}$ in f/B admit extensions to their cones and are thus contractible in f/B ; by the *cone* on $\mathcal{D}$ is meant here the diagram $\mathcal{D}$ together with an added terminal vertex.

Observation. Let $\mathcal{D}$ be a diagram in f/B. Suppose that *as a diagram in* F/B it extends to the cone (for example, this is the case if the colimit of $\mathcal{D}$ exists in F/B). Then $\mathcal{D} \to f/B$ also extends to the cone.

Indeed, suppose that $\mathcal{D} \to f/B \subset F/B$ extends to the cone. Let the cone point be represented by $(A', F(A') \to B)$ in F/B. Applying the approximation property of F we find a cofibration $A' \rightarrowtail A''$ in $\mathcal{A}$ and a weak equivalence $F(A'') \to B$ in $\mathcal{B}$ so that the triangle



commutes. Then $(A'', F(A'') \to B)$ may be regarded as a terminal vertex to $\mathcal{D}$ in f/B rather than just F/B as we see by checking that certain maps are weak equivalences. Namely let $(A, F(A) \to B)$ represent any vertex of $\mathcal{D}$. Then there is a triangle



in which both of the maps going to B are weak equivalences. Applying the saturation axiom we obtain that $F(A) \to F(A'')$ is a weak equivalence in $\mathcal{B}$. From this we deduce in turn, using property App 1 of F, that $A \to A''$ is a weak equivalence, as required.

For example the empty diagram in f/B has a colimit in F/B provided by the initial object of A. In view of the observation we conclude that f/B is non-empty.

Similarly any discrete two-point-diagram in f/B has a colimit in F/B provided by the sum in A. In view of the observation this shows that f/B is connected.

To show that f/B is contractible it remains to find sufficiently many diagrams to which the observation applies. The sublemma below claims that this can be done. But we must first explain what 'sufficiently many' means in this context.

Let a *non-singular* simplicial set mean one where for every n and every non-degenerate n-simplex, the representing map from Δ^n is an embedding. For example *ordered simplicial complexes* may be regarded as simplicial sets and as such are non-singular.

In order to show the simplicial set $N(f/B)$, the nerve of f/B, is contractible it will suffice to show that for every non-singular X and every map from X to $N(f/B)$, this map is null-homotopic. (E.g. think of X as running through iterated subdivisions of spheres. There are sufficiently many maps from such X to represent all the elements of the homotopy groups of $N(f/B)$. If they are all trivial $N(f/B)$ is thus contractible by the Whitehead theorem).

To any simplicial set Y we can associate its category of simplices $\operatorname{simp}(Y)$, and there is a natural transformation $N(\operatorname{simp}(Y)) \to Y$ (the *last vertex map*) which is a homotopy equivalence (this will be recalled at the end of this section). If Y happens to be the nerve of a category then the natural transformation is the nerve of a map of categories. In particular we have a map $\operatorname{simp}(N(f/B)) \to f/B$.

If Y is non-singular then the category $\operatorname{simp}(Y)$ has a subcategory which is given by the non-degenerate simplices (it is a partially ordered set really). The inclusion $\operatorname{simp}^{n.d.}(Y) \to \operatorname{simp}(Y)$ is a homotopy equivalence (cf. the end of the section).

The map $X \to N(f/B)$ now gives rise to a sequence of maps

$$\operatorname{simp}^{n.d}(X) \xrightarrow{\sim} \operatorname{simp}(X) \longrightarrow \operatorname{simp}(N(f/B)) \xrightarrow{\sim} f/B$$

as well as a diagram

$$\begin{array}{ccc} N\operatorname{simp}(X) & \longrightarrow & N\operatorname{simp}(N(f/B)) \\ \downarrow \wr & & \downarrow \wr \\ X & \longrightarrow & N(f/B) \end{array} .$$

This shows that the map $X \to N(f/B)$ will be null-homotopic as soon as the induced map $\operatorname{simp}^{n.d.}(X) \to f/B$ is. The proof of the theorem has thus been reduced to the

Assertion. Let X be a non-singular finite simplicial set and $q: X \to N(f/B)$ a map. Then the induced map $q_*: \mathrm{simp}^{n.d.}(X) \to f/B$ is null-homotopic.

We prove below

Sublemma. In this situation there exists a functor

$$T_q : \mathrm{simp}^{n.d.}(X) \longrightarrow f/B$$

with the following two properties.

(1) There is a natural transformation from T_q to q_*.

(2) The composite functor

$$\mathrm{simp}^{n.d.}(X) \xrightarrow{T_q} f/B \subset F/B$$

extends to a functor on $s(X)$, the partially ordered set of the simplicial subsets of X.

The sublemma implies the assertion and hence the theorem. For the partially ordered set $s(X)$ has a maximal element, therefore part (2) of the sublemma implies that $\mathrm{simp}^{n.d.}(X) \to F/B$ extends to the cone on $\mathrm{simp}^{n.d.}(X)$. In view of the observation therefore $T_q: \mathrm{simp}^{n.d.}(X) \to f/B$ extends to the cone, too, thus T_q is null-homotopic. By part (1) of the sublemma T_q is homotopic to q_*. It results that q_* is null-homotopic.

Proof of sublemma. In order to define T_q we need the notion of *iterated mapping cylinder*, a notion derived from the cylinder functor on $\mathcal{A}$. Let $A_o \to \ldots \to A_n$ be a sequence of maps in $\mathcal{A}$. We will associate to this sequence the following data

(1) the (iterated) cylinder object $T(A_o \to \ldots \to A_n)$,

(2) a map $\partial_i: T(A_o \to \ldots \to \hat{A}_i \to \ldots \to A_n) \to T(A_o \to \ldots \to A_n)$ for every $0 \leqslant i \leqslant n$, where the hat indicates the omission of A_i from the sequence,

(3) a map $p: T(A_o \to \ldots \to A_n) \to A_n$.

Proceeding inductively we define $T(A_o \to \ldots \to A_n)$ as $T(T(A_o \to \ldots \to A_{n-1}) \to A_n)$, the cylinder of the composed map

$$T(A_o \to \ldots \to A_{n-1}) \xrightarrow{p} A_{n-1} \longrightarrow A_n ,$$

and $p: T(A_o \to \ldots \to A_n) \to A_n$ as the cylinder projection.

The definition of ∂_i requires a case distinction. The map

$$\partial_n : T(A_o \to \ldots \to A_{n-1}) \longrightarrow T(A_o \to \ldots \to A_n)$$

is defined as the front inclusion of the cylinder. If $n = 1$ the map

$$\partial_o : A_1 \longrightarrow T(A_o \to A_1)$$

is the back inclusion. And in general, finally, if $i < n$ and $n > 1$ then the map

$$\partial_i : T(A_o \to \ldots \to \hat{A}_i \to \ldots \to A_n) \longrightarrow T(A_o \to \ldots \to A_n)$$

is defined inductively as $T(\partial_i^!)$ where $\partial_i^!$ is the (vertical) map of diagrams

$$\begin{array}{ccc} T(A_o \to \ldots \to \hat{A}_i \to \ldots \to A_{n-1}) & \longrightarrow & A_n \\ \downarrow \partial_i & & \downarrow \| \\ T(A_o \to \ldots \to A_{n-1}) & \longrightarrow & A_n \end{array}$$

From the particular sequence $A_o \to \ldots \to A_n$ we can obtain a functor

$$\mathrm{simp}^{n.d.}(\Delta^n) \longrightarrow A$$

taking each face of Δ^n to the iterated cylinder of the subsequence indexed by that face. On morphisms the functor is given by the maps ∂_i and their composites. To justify this we must check that the maps ∂_i satisfy the identities for iterated face maps. But for the identities not involving ∂_n this follows inductively from the case $n-1$, and for the identities which do involve ∂_n it follows from the fact that the front inclusion is a natural transformation.

The desired functor T_q is obtained by a slight modification, and generalization, of this construction. Namely let X be a non-singular simplicial set, and q a map from X to the nerve of f/B. Then the image of q on a n-simplex x of X is given by a sequence of weak equivalences in A, over $B \in B$,

$$A_o(x) \longrightarrow \ldots \longrightarrow A_n(x) .$$

Assuming now that x is a non-degenerate n-simplex of X we define $T_q(x)$ to be the iterated cylinder of that sequence, making it an object of f/B by means of the composite map $F(T(A_o(x) \to \ldots \to A_n(x))) \to F(A_n(x)) \to B$ (the first map here is induced from the projection p by the functor F, it is a weak equivalence in view of the assumed cylinder axiom). On morphisms T_q is defined by the maps ∂_i and their iterates (the morphisms are in f/B rather than just F/B in view of the assumed cylinder axiom and saturation axiom). It was checked above that the rule for morphisms is compatible with the identities for iterated face maps. There are no other identities in $\mathrm{simp}^{n.d.}(X)$, so T_q is a functor on it.

The desired natural transformation from T_q to q_* is given by the projection

$$p : T(A_o(x) \to \ldots \to A_n(x)) \xrightarrow{\sim} A_n(x) .$$

This completes the argument for part (1) of the sublemma.

In defining the proposed extension t of the composed functor

$$\mathrm{simp}^{n.d.}(X) \xrightarrow{T_q} f/B \subset F/B$$

we will insist on the following two properties of t

(1) t takes maps in $s(X)$ to cofibrations (as maps in A, after neglect of the

structure maps to B, that is),

(2) $t(X_1 \cup_{X_o} X_2) = t(X_1) \cup_{t(X_o)} t(X_2)$.

Given its restriction to $\mathrm{simp}^{n.d.}(X)$ provided by T_q, the functor t is uniquely determined by these conditions, up to isomorphism.

To establish the existence of t we proceed by induction, assuming in the inductive step that t does exist on the (n-1)-skeleton of X. Our aim is to establish the existence of t on the n-skeleton. There is only one thing that could conceivably go wrong with the inductive step. Namely if x is a n-simplex of X and ∂x its boundary (the union of the proper faces) then $t(\partial x)$ and $t(x)$ are both defined, and a map $t(\partial x) \to t(x)$ is also defined. The problem now is if this map is a cofibration.

Let $\Lambda^n x$ be the n-th horn of x, the union of all the proper faces except $d_n x$; so

$$x = \Lambda^n x \cup_{\partial d_n x} d_n x .$$

Condition (2) above expresses $t(\Lambda^n x)$ in terms of values of t on faces of x. Since a similar formula is valid for the cylinder functor, in view of its exactness, we conclude that

$$t(\Lambda^n x) \approx T(t(\partial d_n x) \to A_n)$$

where A_n denotes the value of t on the n-th vertex of x (and where, for ease of notation, we are ignoring the structure maps of objects in F/B). Applying condition (2) again we obtain that the map $t(\partial x) \to t(x)$ can be identified to the map

$$t(d_n x) \cup_{t(\partial d_n x)} T(t(\partial d_n x)\to A_n) \longrightarrow T(t(d_n x)\to A_n) .$$

That the latter map is a cofibration, is one of the conditions that must be satisfied for the following map in $F_1 A$ to be a cofibration in $F_1 A$,

$$(t(\partial d_n x) \to T(t(\partial d_n x)\to A_n)) \longrightarrow (t(d_n x) \to T(t(d_n x)\to A_n)) ,$$

so it will suffice to know that. The map is the image, with respect to

$$(+) \qquad (A' \to A'') \longmapsto (A' \xrightarrow{j_1} T(A'\to A'')) ,$$

of the following map in ArA,

$$(t(\partial d_n x) \to A_n) \longrightarrow (t(d_n x) \to A_n) ,$$

which is a cofibration in ArA because $t(\partial d_n x) \to t(d_n x)$ is a cofibration by condition (1) above and the inductive hypothesis. We conclude by recalling that a cylinder functor has certain exactness properties, as specified in the axiom Cyl 1. In particular therefore the map (+) preserves cofibrations. This completes the proof of the sublemma and hence also that of the theorem. □

It remains to say a few words, as promised, about the map $Nsimp(Y) \to Y$. In view of the natural isomorphisms $Nsimp(Y) \approx \mathrm{colim}_{simp(Y)}(([n],y) \mapsto Nsimp(\Delta^n))$ and $Y \approx \mathrm{colim}_{simp(Y)}(([n],y) \mapsto \Delta^n)$, the map is fully described once one knows the special case of simplices Δ^n. A m-simplex of $Nsimp(\Delta^n)$ is a sequence of maps in Δ,

$$[n_o] \xrightarrow{a_o} [n_1] \xrightarrow{a_1} \dots \longrightarrow [n_m] \xrightarrow{a_m} [n] ,$$

and one associates to it the m-simplex $b: [m] \to [n]$ in Δ^n given by the last vertices, i.e.

$$b(i) = a_m a_{m-1} \cdots a_i(n_i) .$$

$Nsimp(\Delta^n)$ is contractible since $simp(\Delta^n)$ has a terminal object. Therefore the map $Nsimp(\Delta^n) \to \Delta^n$ is a homotopy equivalence. In view of the gluing lemma it results from this that $Nsimp(Y) \to Y$ is a homotopy equivalence in general (cf. the appendix A to [11]).

Suppose now that Y is the nerve of a category C. Then $simp(NC)$ is the category of pairs $([m],x)$, $x: [m] \to C$, and we can define a natural transformation $simp(NC) \to C$ by $([m],x) \mapsto x(m)$. On passing to nerves this induces the above natural transformation in the case when $C = [n]$, and consequently also in general.

We conclude with

Lemma. If X is non-singular there is a functor $simp(X) \to simp^{n.d.}(X)$ which is left adjoint, and left inverse, to the inclusion functor.

Proof. The functor associates to each simplex of X the unique non-degenerate simplex of which the simplex is a degenerate. It is clear that this works in the special case where X is Δ^n. The general case reduces to this special case in view of the non-singularity of X.

1.7. Spherical objects and cell filtrations

By a *homology theory* on a category with cofibrations $\mathcal{C}$, with values in an abelian category $\mathcal{A}$, will be meant a sequence of functors $H_i: \mathcal{C} \to \mathcal{A}$, $i = 0,1,\ldots$, together with connecting maps $(A \rightarrowtail B) \longmapsto (H_{i+1}(B/A) \to H_i(A))$ such that the long sequence resulting from a cofibration sequence $A \rightarrowtail B \twoheadrightarrow B/A$ is exact and terminates in a surjection $H_o(B) \twoheadrightarrow H_o(B/A)$.

Given such a homology theory, $\mathcal{C}$ may be regarded as a category with cofibrations and weak equivalences where the latter are defined as the maps inducing isomorphisms in homology. The category of weak equivalences will be denoted $w\mathcal{C}$. It satisfies the saturation axiom and extension axiom.

Suppose given a full subcategory $\mathcal{E}$ of the abelian category $\mathcal{A}$ which is closed under the formation of extensions and kernels; that is, if $E' \rightarrowtail E \twoheadrightarrow E''$ is short exact then $E', E'' \in \mathcal{E}$ implies $E \in \mathcal{E}$, and $E, E'' \in \mathcal{E}$ implies $E' \in \mathcal{E}$. For example $\mathcal{A}$ itself will do.

Definition. An object $A \in \mathcal{C}$ is $(H_*,\mathcal{E})$-*spherical of dimension* n if

$$H_i(A) = 0 \quad \text{if} \quad i \neq n , \qquad \text{and} \quad H_n(A) \in \mathcal{E} .$$

With H_* and $\mathcal{E}$ being understood, such an A will also be simply referred to as n-*spherical*.

We denote the category of the n-spherical objects by $\mathcal{C}^n$. It is a subcategory with cofibrations and weak equivalences of $\mathcal{C}$ (section 1.1).

Example. On the category $\mathcal{R}(X)$ of the spaces having X as a retract there is a homology theory with values in the category of $Z[\pi_1 X]$-modules, $H_i(Y,r,s) = H_i(Y,s(X),r^*(Z[\pi_1 X]))$. For $\mathcal{E}$ one can take the category of projective $Z[\pi_1 X]$-modules, or even the subcategory of the stably free ones. The n-spherical objects include the objects (Y,r,s) where Y is obtainable, up to homotopy, by attaching n-cells to X.

We assume that $\mathcal{C}$ has a cylinder functor and that the weak equivalences satisfy the cylinder axiom. Any map $f: A \to B$ then gives rise to a long exact sequence $\ldots \to H_i(A) \to H_i(B) \to H_i(f) \to H_{i-1}(A) \to \ldots$ where

$$H_i(f) = H_i(T(f)/A) .$$

We say the map f is k-*connected* if $H_i(f) = 0$ for $i \leq k$.

The following hypothesis will be needed in the theorem below.

Hypothesis. For every m-connected map $X_m \to Y$ in $\mathcal{C}$ there is a factorization

$$\begin{array}{ccccccccc} X_m & \rightarrowtail & X_{m+1} & \rightarrowtail & \cdots & \rightarrowtail & X_n & \longrightarrow & Y \\ & & \searrow & & & & \searrow & & \\ & & & X_{m+1}/X_m \in \mathcal{C}^{m+1} & & & & X_n/X_{n-1} \in \mathcal{C}^n & \end{array}$$

Recall (proposition 1.6.2) that the suspension induces an exact functor $\Sigma\colon \mathcal{C} \to \mathcal{C}$ and a homotopy equivalence $wS.\mathcal{C} \to wS.\mathcal{C}$. As a consequence if we denote by $\varinjlim_{(\Sigma)} wS.\mathcal{C}$ the direct limit of the system $n \mapsto wS.\mathcal{C}$ in which the maps are given by suspension then

$$wS.\mathcal{C} \longrightarrow \varinjlim_{(\Sigma)} wS.\mathcal{C}$$

is a homotopy equivalence.

The suspension also induces an exact functor $\mathcal{C}^n \to \mathcal{C}^{n+1}$ so we can form $\varinjlim_n \mathcal{C}^n$.

<u>Theorem</u> 1.7.1. The map

$$\varinjlim_n wS.\mathcal{C}^n \longrightarrow \varinjlim_{(\Sigma)} wS.\mathcal{C}$$

is a homotopy equivalence, provided that the hypothesis is satisfied.

The proof of the theorem occupies all of this section. The strategy of the proof is to replace $\mathcal{C}$ by a category of *cell filtrations*, and to study two notions of weak equivalence, as well as their interplay, on that category.

Definition. A *cell filtration* in $\mathcal{C}$ is an eventually stationary sequence of cofibrations

$$* = A_{-1} \rightarrowtail A_o \rightarrowtail \cdots \rightarrowtail A_n \rightarrowtail \cdots$$

such that

$$A_n/A_{n-1} \in \mathcal{C}^n$$

for every n . The object to which the sequence stabilizes is denoted A_∞ .

For example, given any object $A \in \mathcal{C}$ one can find a cell filtration $\{A_i\}$ together with a weak equivalence $A_\infty \to A$. This results from the hypothesis of the theorem applied to the map $* \to A$ in $\mathcal{C}$.

The category of cell filtrations will be denoted $\hat{\mathcal{C}}$. It is a category with cofibrations where, by definition, a map $\{A_i\} \to \{A_i'\}$ is a cofibration if, and only if, for all n the map

$$A_{n-1}' \cup_{A_{n-1}} A_n \longrightarrow A_n'$$

$$A'_{m-1} = A''_{m-1} \rightarrowtail A_m \cup_{A_{m-1}} A''_{m-1} \rightarrowtail A'_m .$$

The associated cofibration sequence

$$A_m/A_{m-1} \rightarrowtail A'_m/A'_{m-1} \twoheadrightarrow A'_m/(A_m \cup_{A_{m-1}} A''_{m-1})$$

has both its 'subobject' and quotient in C^m. Since C^m is extension closed in C we conclude that $A'_m/A'_{m-1} \in C^m$. The lemma is proved. □

Let the fine category of weak equivalences in $\hat{C}$ be defined as the category $v\hat{C}$ of the maps $\{A_i\} \to \{A'_i\}$ having the property that $A_i \to A'_i$ is in wC for every i.

Let $\hat{C}_m$ denote the category of the cell filtrations in dimensions $\leq m$, i.e. the full subcategory of the $\{A_i\}$ in $\hat{C}$ with $A_m = A_\infty$. We consider $\hat{C}_m$ as a subcategory-with-cofibrations-and-weak-equivalences (sections 1.1 and 1.2) of $(\hat{C}, v\hat{C})$.

<u>Lemma</u> 1.7.3. The map

$$\begin{array}{ccc} vS.\hat{C}_m & \longrightarrow & wS.C^o \times wS.C^1 \times \ldots \times wS.C^m \\ (A_o \rightarrowtail A_1 \rightarrowtail . \rightarrowtail A_m) & \longmapsto & A_o \, , \; A_1/A_o \, , \; \ldots \, , \; A_m/A_{m-1} \end{array}$$

is a homotopy equivalence.

Proof.

Let, as usual, $\hat{C}^w$ denote the subcategory of the $\{A_i\}$ in $\hat{C}$ where $* \to \{A_i\}$ is in $w\hat{C}$. Let $\hat{C}^w_m = \hat{C}^w \cap \hat{C}_m$; it is the category of the cell filtrations $(A_o \rightarrowtail . \rightarrowtail A_{m-1} \rightarrowtail A_m)$ having the property that A_m is acyclic. We consider $\hat{C}^w_m$ as a subcategory-with-cofibrations-and-weak-equivalences of $(\hat{C}, v\hat{C})$.

<u>Lemma</u> 1.7.4. If $\{A_i\} \in \hat{C}^w$ then $A_n \in C^n$ for all n.

Proof. Using suitable long exact sequences we obtain

if $k > n$ then $H_k(A_n) \xleftarrow{\approx} H_k(A_{n-1}) \xleftarrow{\approx} \ldots \xleftarrow{\approx} H_k(A_{-1}) = 0$, and

if $k < n$ then $H_k(A_n) \xrightarrow{\approx} H_k(A_{n+1}) \xrightarrow{\approx} \ldots \xrightarrow{\approx} H_k(A_\infty) = 0$, thus $H_k(A_n) = 0$ if $k \neq n$. There is a short exact sequence

$$H_n(A_n) \rightarrowtail H_n(A_n/A_{n-1}) \twoheadrightarrow H_{n-1}(A_{n-1}) .$$

By induction we may assume $H_{n-1}(A_{n-1}) \in \mathcal{E}$, and by definition of a cell filtration we have $H_n(A_n/A_{n-1}) \in \mathcal{E}$. It follows that $H_n(A_n) \in \mathcal{E}$ in view of the assumed fact that the category $\mathcal{E}$ is closed under taking kernels. □

<u>Lemma</u> 1.7.5. The map

$$\begin{array}{ccccc} vS.\hat{C}^w_m & \longrightarrow & wS.C^o \times wS.C^1 \times \ldots \times wS.C^{m-1} \\ (A_o \rightarrowtail A_1 \rightarrowtail . \rightarrowtail A_m) & \longmapsto & A_o , \quad A_1 , \quad \ldots \quad A_{m-1} \end{array}$$

is a homotopy equivalence.

Proof. The map exists by the preceding lemma. To show it is a homotopy equivalence it suffices, by induction, to show that the map p,

$$\begin{array}{ccc} vS.\hat{C}^w_m & \longrightarrow & vS.\hat{C}^w_{m-1} \times wS.C^{m-1} \\ (A_o \rightarrowtail . \rightarrowtail A_m) & \longmapsto & (A_o \rightarrowtail . \rightarrowtail A_{m-2} \rightarrowtail A_m) , \ A_{m-1} , \end{array}$$

is a homotopy equivalence (p exists by the preceding lemma since $H_i(A_m/A_{m-2}) \approx H_{i-1}(A_{m-2})$). We show that the map s in the other direction,

$$(B_o \rightarrowtail . \rightarrowtail B_{m-1}) , B \longmapsto (B_o \rightarrowtail . \rightarrowtail B_{m-2} \rightarrowtail B_{m-1} vB \rightarrowtail B_{m-1} vcB) ,$$

is homotopy inverse to p where, as usual, cB denotes the cone on B.

The composite sp is given by

$$(B_o \rightarrowtail . \rightarrowtail B_{m-1}) , B \longmapsto (B_o \rightarrowtail . \rightarrowtail B_{m-2} \rightarrowtail B_{m-1} vcB) , B_{m-1} vB .$$

There is a natural transformation from the identity map to sp. It is a weak equivalence since both $B_{m-1} \rightarrow B_{m-1} vcB$ and $B \rightarrow B_{m-1} vB$ are weak equivalences. Hence it induces a homotopy (lemma 1.3.1), showing that s is left inverse to p.

To show that s is right inverse to p we construct a homotopy by applying the additivity theorem to a cofibration sequence of maps on $\hat{C}^w_m$. We can write $ps = f'vf''$ where f' and f'' are the self-maps of $\hat{C}^w_m$ taking $(A_o \rightarrowtail . \rightarrowtail A_m)$ to $(* \rightarrow . \rightarrow * \rightarrowtail A_{m-1} \rightarrowtail cA_{m-1})$ and $(A_o \rightarrowtail . \rightarrowtail A_{m-2} \rightarrowtail A_m \stackrel{=}{\rightarrow} A_m)$, respectively. If we could find a cofibration sequence $f' \rightarrowtail f \twoheadrightarrow f''$, where f denotes the identity map on $\hat{C}^w_m$, it would follow by the additivity theorem that there is a homotopy between f and $f'vf''$, and we would be done.

The desired cofibration sequence does not exist directly, but it exists after the maps f and f'' have been modified a little. The modified maps are related to the original maps by chains of weak equivalences.

In a first step we replace the identity map f by a map f_1 taking $(A_o \rightarrowtail . \rightarrowtail A_m)$ to $(A_o \rightarrowtail . \rightarrowtail A_{m-1} \rightarrowtail c(A_m \cup_{A_{m-1}} cA_{m-1}))$. There is a weak equivalence $f \rightarrow f_1$

and we can define a map $f' \to f_1$ now. In a second step we blow up f_1 to a weakly equivalent f_2 so that the map $f' \to f_1$ can be replaced by a cofibration $f' \rightarrowtail f_2$. By definition, f_2 takes $(A_o \rightarrowtail .. \rightarrowtail A_m)$ to

$$(A_o \rightarrowtail \ldots \rightarrowtail A_{m-2} \rightarrowtail TA_{m-1} \rightarrowtail Tc(A_m \cup_{A_{m-1}} cA_{m-1}))$$

where TA is defined as $T(id_A)$, the cylinder of the identity map on A.

Let f_3'' be defined as the quotient f_2/f'. There is a weak equivalence to it from f_2'',

$$(A_o \rightarrowtail \ldots \rightarrowtail A_{m-2} \rightarrowtail TA_{m-1}/A_{m-1} \overset{=}{\to} TA_{m-1}/A_{m-1}) ,$$

the latter maps by weak equivalence to f_1'',

$$(A_o \rightarrowtail \ldots \rightarrowtail A_{m-2} \rightarrowtail TA_m/A_m \overset{=}{\to} TA_m/A_m) ,$$

and, to conclude, we have a weak equivalence $f'' \to f_1''$. We are done. □

Lemma 1.7.6. The map

$$vS.\hat{C}^w_m \times wS.C^m \longrightarrow vS.\hat{C}_m$$

is a homotopy equivalence.

Proof. The map

$$\begin{array}{ccc} wS.C^o \times \ldots \times wS.C^{m-1} & \longrightarrow & vS.\hat{C}^w_m \\ A_o , \ldots , A_{m-1} & \longmapsto & A_o \rightarrowtail cA_o vA_1 \rightarrowtail \ldots \rightarrowtail cA_o v..vcA_{m-2} vA_{m-1} \rightarrowtail cA_o v..vcA_{m-1} \end{array}$$

is a homotopy equivalence. For by composing it with the homotopy equivalence of the preceding lemma we obtain a map induced by a self-map of $C^o \times \ldots \times C^{m-1}$ weakly equivalent to the identity map. As a result it will suffice to show that the composite map

$$(C^o \times \ldots \times C^{m-1}) \times C^m \longrightarrow \hat{C}^w_m \times C^m \longrightarrow \hat{C}_m \longrightarrow C^o \times \ldots \times C^m ,$$

where the right hand map is that of lemma 1.7.3, induces a homotopy equivalence of $wS.C^o \times .. \times wS.C^m$ to itself. The composite map is given by

$$(A_o , \ldots , A_m) \longmapsto (A_o , \Sigma A_o vA_1 , \Sigma A_1 vA_2 , \ldots , \Sigma A_{m-1} vA_m) .$$

This is clearly a homotopy equivalence. □

Lemma 1.7.7. The map

$$\varinjlim_m wS.C^m \times \varinjlim_{(\Sigma)} vS.\hat{C}^w \longrightarrow \varinjlim_{(\Sigma)} vS.\hat{C}$$

(limits by suspension) is a homotopy equivalence.

Proof. The desired homotopy equivalence results by direct limit once it is known that the maps $\varphi_k : \varinjlim_m wS.C^m \times \varinjlim_m vS.\hat{C}^w_{m+k} \longrightarrow \varinjlim_m vS.\hat{C}_{m+k}$ are homotopy equivalences. The case $k = 0$ follows from the preceding lemma by direct limit. We deduce the case $k = 1$ from the case $k = 0$. Namely the two maps

$$\varinjlim_m C^m \xrightarrow[\approx]{\Sigma} \varinjlim_m C^{m+1} \xrightarrow{\psi_o} \varinjlim_m \hat{C}_{m+1} , \qquad \varinjlim_m C^m \xrightarrow{\psi_1} \varinjlim_m \hat{C}_{m+1}$$

are related by a cofibration sequence of functors $\psi_1 \rightarrowtail \theta \twoheadrightarrow \psi_o\Sigma$ where θ is the composite map

$$\varinjlim C^m \longrightarrow \varinjlim \hat{C}^w_{m+1} \longrightarrow \varinjlim \hat{C}_{m+1}$$
$$A \longmapsto (.. \overset{=}{\to} * \rightarrowtail A \rightarrowtail cA \overset{=}{\to} ..)$$

By the additivity theorem there results a homotopy of the induced maps, $\psi_1 \vee \psi_o\Sigma \simeq \theta$, showing that, modulo $\varinjlim \hat{C}^w_{m+1}$, the maps ψ_1 and $\psi_o\Sigma$ are the same up to sign. We conclude that φ_1 is a homotopy equivalence since φ_o is. Similarly it follows that φ_2 is a homotopy equivalence since φ_1 is. And so on. □

Proof of theorem 1.7.1. By the fibration theorem 1.6.4 there is a homotopy cartesian square

$$\begin{array}{ccc} vS.\hat{C}^w & \longrightarrow & vS.\hat{C} \\ \downarrow & & \downarrow \\ wS.\hat{C}^w & \longrightarrow & wS.\hat{C} \end{array} .$$

Suspension induces a self-map of the square, and hence a direct system. Passing to the direct limit we obtain a square which is homotopy cartesian again. It is the large square in the following diagram

$$\begin{array}{ccccc} \varinjlim vS.\hat{C}^w & \longrightarrow & \varinjlim (\, vS.\hat{C}^w \times wS.C^m \,) & \longrightarrow & \varinjlim vS.\hat{C} \\ \downarrow & & \downarrow & & \downarrow \\ \varinjlim wS.\hat{C}^w & \longrightarrow & \varinjlim (\, wS.\hat{C}^w \times wS.C^m \,) & \longrightarrow & \varinjlim wS.\hat{C} \end{array} .$$

By comparing the vertical homotopy fibres we see that the left square in the diagram is also homotopy cartesian. It follows that the square on the right is homotopy cartesian. By the preceding lemma the upper horizontal map in the right hand square is a homotopy equivalence. We conclude that the lower horizontal map is a homotopy equivalence. Discarding the contractible factor $\varinjlim wS.\hat{C}^w$ we obtain the map

$$\varinjlim wS.C^m \longrightarrow \varinjlim wS.\hat{C}$$

which is therefore a homotopy equivalence. In view of the homotopy equivalence

$$\varinjlim wS.\hat{C} \longrightarrow \varinjlim wS.C$$

of lemma 1.7.2 this completes the proof of the theorem. □

1.8. Split cofibrations, and K-theory via group completion.

Let A be a category with *sum* (categorical coproduct), and let A be pointed by an initial object $*$. There is an associated simplicial category

$$\begin{aligned} N.A : \Delta^{op} &\longrightarrow (\text{cat}) \\ [n] &\longmapsto N_nA \ , \end{aligned}$$

the *nerve with respect to the composition law*. By definition N_nA is the category equivalent to A^n in which an object consists of a tuple $A_1,\ldots,A_n$ together with appropriate sum diagrams, one for each subset of $\{1,\ldots,n\}$; these choices are to be compatible, and for the subsets of cardinality ≤ 1 they are to be given by the objects $A_1,\ldots,A_n$ themselves and by the initial object $*$, respectively.

By a *category of weak equivalences* in A will be meant any subcategory wA which contains the isomorphisms and is closed under sum formation; that is, if $A_1 \to A'_1$ and $A_2 \to A'_2$ are in wA then so is $A_1 \vee A_2 \to A'_1 \vee A'_2$.

If A is a *category with sum and weak equivalences* let wN_nA be defined as the subcategory of N_nA whose morphisms are the natural transformations with values in wA . It is a category of weak equivalences in N_nA , and it is equivalent to wA^n by the forgetful map. $N.A$ may be regarded as a *simplicial category with sum and weak equivalences*, and the simplicial category of weak equivalences is

$$\begin{aligned} wN.A : \Delta^{op} &\longrightarrow (\text{cat}) \\ [n] &\longmapsto wN_nA \ . \end{aligned}$$

The construction is a special case of Segal's construction of *Γ-categories* [11]. The present notation has been chosen to conform to that of section 1.3.

Let C be a category with cofibrations and weak equivalences. By neglect of structure C is a category with sum and weak equivalences, $A \vee B = A \cup_* B$. There is a map of simplicial categories

$$wN.C \longrightarrow wS.C \ ,$$

it takes

$$(\ A_1,\ldots,A_n \ , \quad \text{choices}\ )$$

to

$$(\ A_1 \rightarrowtail A_1 \vee A_2 \rightarrowtail \ldots \rightarrowtail A_1 \vee \ldots \vee A_n \ , \quad (\text{fewer})\ \text{choices}\ )\ .$$

The theorem to be formulated below says that the map is a homotopy equivalence in certain cases.

Suppose that $\mathcal{C}$, a category with cofibrations and weak equivalences, has a cylinder functor and that the weak equivalences in $\mathcal{C}$ satisfy the cylinder axiom, saturation axiom, and extension axiom.

Suppose given a sequence of subcategories-with-cofibrations-and-weak-equivalences $\mathcal{C}^n$ in $\mathcal{C}$ subject to the condition that suspension takes $\mathcal{C}^n$ into $\mathcal{C}^{n+1}$ for all n . The example to be kept in mind is that of a sequence of categories of spherical objects in the sense of the preceding section.

Let us say that a cofibration $A \rightarrowtail B$ in $\mathcal{C}^n$ is *splittable up to weak equivalence* if there is a chain of weak equivalences, relative to A , relating $A \rightarrowtail B$ to $A \rightarrowtail B'$ where $B' \approx A \vee B'/A$.

Theorem 1.8.1. The map

$$\varinjlim_n wN.\mathcal{C}^n \longrightarrow \varinjlim_n wS.\mathcal{C}^n$$

is a homotopy equivalence, provided that, for every n , all cofibrations in $\mathcal{C}^n$ are splittable up to weak equivalence.

The proof of the theorem occupies the present section. The argument will be summarized at the end of the section. The splittability condition actually used is slightly weaker than the one formulated here.

For any $X \in \mathcal{C}$ let $\mathcal{C}_X$ denote the *category of the cofibrant objects under* X ; the objects of $\mathcal{C}_X$ are the cofibrations $X \rightarrowtail A$ in $\mathcal{C}$, and the morphisms are the maps $A \to A'$ restricting to the identity map on X . $\mathcal{C}_X$ is a category with sum,

$$(X \rightarrowtail A) \vee (X \rightarrowtail A') \quad = \quad (X \rightarrowtail A \cup_X A') ,$$

and it comes equipped with a category of weak equivalences $w\mathcal{C}_X$, the pre-image of $w\mathcal{C}$ under the projection $\mathcal{C}_X \to \mathcal{C}$, $(X \rightarrowtail A) \mapsto A$.

Let as usual c denote the cone functor derived from the cylinder functor ($cA = T(A \rightarrowtail *)$) and Σ the suspension functor, $\Sigma A = cA/A = cA \cup_A *$.

Lemma 1.8.2. To $X \rightarrowtail A$ in $\mathcal{C}_X$ there is naturally associated a chain of weak equivalences in $\mathcal{C}_{\Sigma X}$,

$$(\Sigma X \rightarrowtail \Sigma A \cup_{\Sigma X} \Sigma A \qquad\qquad (\Sigma X \rightarrowtail \Sigma A \cup_* \Sigma A/\Sigma X$$

Proof. The chain consists of two maps. These are given by the two diagonal arrows in the following diagram

$$
\begin{array}{ccc}
 & \nearrow & \Sigma A \cup_{\Sigma X} \Sigma A \cup_{*} * \\
 & & \uparrow \\
\Sigma A \cup_{\Sigma X} (cA/X \cup_{A/X} cA/cX) & \longrightarrow & \Sigma A \cup_{\Sigma X} \Sigma A \cup_{*} \Sigma A/\Sigma X \\
 & & \downarrow \\
 & \searrow & \Sigma A \cup_{*} \Sigma A/\Sigma X
\end{array}
$$

By definition, the horizontal arrow is given by pushout with the map $A/X \to *$, and the downward vertical arrow is induced by the folding map $\Sigma A \cup_{\Sigma X} \Sigma A \to \Sigma A$. The upper diagonal arrow is a weak equivalence since it is given by pushout with the weak equivalence $cA/cX \to *$. The lower diagonal arrow is a weak equivalence in view of the assumed extension axiom. For by cobase change with the map $\Sigma A \to *$ one obtains from it the weak equivalence $cA/cX \cup_{A/X} cA/cX \to \Sigma A/\Sigma X$. □

Remark. If C happens to be an *additive* category the lemma is true without suspension, one can define a weak equivalence $A \cup_X A \to A \cup_* A/X$ as a map whose restriction to the second A is the sum of the identity $A \to A \cup_* *$ and the projection $A \to * \cup_* A/X$. In the additive case the argument leading to the theorem, and the theorem itself, can thus be simplified. □

If $X \in C^m$ we can form C^m_X . There are maps, of categories with sum and weak equivalences,

$$
q : C^m_X \longrightarrow C^m \qquad\qquad j : C^m \longrightarrow C^m_X
$$
$$
X \rightarrowtail A \longmapsto A/X \qquad\qquad B \longmapsto X \rightarrowtail X \cup_* B
$$

and q is left inverse to j , up to natural isomorphism of $q\,j$ to the identity on C^m .

Proposition 1.8.3. The map

$$
\lim_{\vec{n}} wN.C^{m+n} \longrightarrow \lim_{\vec{n}} wN.C^{m+n}_{\Sigma^n X}
$$

(limits by suspension) is a homotopy equivalence.

Proof. It will suffice to know that for each n the composite $j\,q$ becomes homotopic to the identity upon suspension. The next lemma provides this; upon re-indexing it will suffice to formulate the lemma for the case $n = 0$. □

Lemma 1.8.4. The geometric realizations of the two maps

$$
\Sigma \ , \quad \Sigma\, j\, q \ : \quad wN.C^m_X \longrightarrow wN.C^{m+1}_{\Sigma X}
$$

are homotopic

Proof. The natural transformations of lemma 1.8.1 provide a homotopy between the two maps $wN.C^m_X \to wN.C^{m+1}_{\Sigma X}$ which take $X \rightarrowtail A$ to

$$\Sigma X \rightarrowtail \Sigma A \cup_{\Sigma X} \Sigma A \quad \text{and} \quad \Sigma X \rightarrowtail \Sigma A \cup_* \Sigma A/\Sigma X ,$$

respectively; that is, the maps

$$\Sigma \vee \Sigma \quad \text{and} \quad \Sigma \vee \Sigma jq .$$

The geometric realization of $wN.C^{m+1}_{\Sigma X}$ is an H-space (by $\vee$) which is connected and hence group-like. So we can cancel the left Σ to obtain the desired homotopy. □

The following is the analogue of definition 1.5.4 with the $S.$ construction replaced by the $N.$ construction. In particular the letter P refers to the *simplicial path object* construction whose elementary properties have been recalled in the beginning of section 1.5.

Definition 1.8.5. Let $f: A \to B$ be a map of categories with sum and weak equivalences. Then $N.(f:A\to B)$ is the simplicial category with sum and weak equivalences given by the pullback of the diagram

$$N.A \longrightarrow N.B \longleftarrow PN.B .$$

$N.(f:A\to B)$ represents a *one-sided bar construction* of A acting on B by the sum via f. In fact, notice that in particular for every n there is a pullback diagram

$$\begin{array}{ccc} N_n(f:A\to B) & \longrightarrow & (PN.B)_n = N_{n+1}B \\ \downarrow & & \downarrow \\ N_nA & \longrightarrow & N_nB \end{array}$$

and the vertical map on the right corresponds, under the equivalence of N_mB with the product category B^m, to the projection map $B^{n+1} \to B^n$, the projection away from the first factor; and $N_n(f:A\to B)$ is equivalent to the product category $B\times A^n$.

Considering B as a simplicial category in a trivial way we have a sequence of simplicial categories with sum and weak equivalences

$$B \longrightarrow N.(f:A\to B) \longrightarrow N.A .$$

We would like this sequence to represent a fibration, up to homotopy, of the associated simplicial categories of weak equivalences, but we cannot expect this to be true in general since A need not act invertibly on B. We circumvent the difficulty by introducing another simplicial direction, using either the $S.$ or the $N.$ construction (we need both cases), as follows.

If $f: A \to B$ is a map of categories with cofibrations and weak equivalences then $N.(f:A\to B)$ is a simplicial category with cofibrations and weak equivalences, so we can form $S.N.(f:A\to B)$. Alternatively we could apply the definition 1.8.5 to the map $S.f\colon S.A \to S.B$ to obtain $N.(S.f:S.A\to S.B)$, and the two bisimplicial categories are naturally isomorphic. There is a sequence, of bisimplicial categories with cofibrations and weak equivalences,

$$S.B \longrightarrow S.N.(f:A\to B) \longrightarrow S.N.A \quad ;$$

alternatively we could rewrite it, up to isomorphism, as

$$S.B \longrightarrow N.(S.f:S.A\to S.B) \longrightarrow N.S.A \quad .$$

In general we can apply the $N.$ construction to the simplicial category with sum and weak equivalences $N.(f:A\to B)$ to obtain $N.N.(f:A\to B)$. Alternatively we could apply the definition 1.8.5 to the map $N.f\colon N.A \to N.B$ to obtain $N.(N.f:N.A\to N.B)$, and the two bisimplicial categories are naturally isomorphic (the isomorphism involves a switch of the two $N.$ directions). There is a sequence, of bisimplicial categories with sum and weak equivalences,

$$N.B \longrightarrow N.N.(f:A\to B) \longrightarrow N.N.A \quad ;$$

alternatively we could rewrite it, up to isomorphism, as

$$N.B \longrightarrow N.(N.f:N.A\to N.B) \longrightarrow N.N.A \quad .$$

Lemma 1.8.6. The sequence

$$wN.B \longrightarrow wN.N.(f:A\to B) \longrightarrow wN.N.A$$

is a fibration up to homotopy. Similarly so is the sequence

$$wS.B \longrightarrow wS.N.(f:A\to B) \longrightarrow wS.N.A$$

if that is defined. In either case, if f is an identity map then the middle term $wN.N.(f:A\to B)$, resp. $wS.N.(f:A\to B)$, is contractible.

Proof. This is a special case of a result of Segal [11]. Essentially the same proof results if the argument of proposition 1.5.5 is adapted to the present situation. That is, one observes that (in the second case, say) for every n one has a fibration

$$wS.B \longrightarrow wN_n(S.f:S.A\to S.B) \longrightarrow wN_nS.A$$

namely a product fibration, and one draws the desired conclusion from this, using a suitable fibration criterion for simplicial objects. □

Let $\mathcal{D}$ be a category with cofibrations and weak equivalences. The example to be kept in mind is that of the category $\varinjlim C^n$ of the theorem. Our next result is of a formal nature. It gives a sufficient condition for the conclusion of the theorem to be valid.

Proposition 1.8.7. If for every $X \in \mathcal{D}$ the simplicial category $wN.(j:\mathcal{D} \to \mathcal{D}_X)$ is contractible then the map $wN.\mathcal{D} \to wS.\mathcal{D}$ is a homotopy equivalence.

Proof. Applying lemma 1.8.6 we obtain that the map of the proposition de-loops to $wN.N.\mathcal{D} \to wN.S.\mathcal{D}$, so it will suffice to show that the latter map is a homotopy equivalence. By the realization lemma this follows if for every n the map

$$wN.N_n\mathcal{D} \longrightarrow wN.S_n\mathcal{D}$$

is a homotopy equivalence, and this is what we shall show.

The simplicial category on the left is equivalent to the product $(wN.\mathcal{D})^n$, so our task is to show that the simplicial category on the right is homotopy equivalent to that same product by the subquotient map. In other words, our task is to establish a case of the additivity theorem for the $N.$ construction rather than the $S.$ construction.

By induction it will suffice to show that the map

$$\begin{array}{ccc} wN.S_n\mathcal{D} & \longrightarrow & wN.S_{n-1}\mathcal{D} \times wN.\mathcal{D} \\ (A_1 \rightarrowtail \ldots \rightarrowtail A_n \text{, choices}) & \longmapsto & (A_1 \rightarrowtail \ldots \rightarrowtail A_{n-1} \text{, choices; } A_n/A_{n-1}) \end{array}$$

is a homotopy equivalence. To reduce further we consider the map

$$\begin{array}{rccc} j_n : & \mathcal{D} & \longrightarrow & S_n\mathcal{D} \\ & A & \longmapsto & * \rightarrowtail \ldots \rightarrowtail * \rightarrowtail A \quad . \end{array}$$

By combining these two maps, and using lemma 1.8.6, we obtain a diagram of homotopy fibrations

$$\begin{array}{ccccc} wN.S_n\mathcal{D} & \longrightarrow & wN.N.(j_n:\mathcal{D} \to S_n\mathcal{D}) & \longrightarrow & wN.N.\mathcal{D} \\ \downarrow & & \downarrow & & \downarrow \simeq \\ wN.(S_{n-1}\mathcal{D} \times \mathcal{D}) & \longrightarrow & wN.N.(\mathcal{D} \to S_{n-1}\mathcal{D} \times \mathcal{D}) & \longrightarrow & wN.N.\mathcal{D} \quad . \end{array}$$

So our task of showing that the vertical map on the left is a homotopy equivalence, translates into the task of showing that the vertical map in the middle is one. By the realization lemma this will follow if we can show that

$$wN.(j_n:\mathcal{D} \to S_n\mathcal{D}) \longrightarrow wN.(\mathcal{D} \to S_{n-1}\mathcal{D} \times \mathcal{D})$$

is a homotopy equivalence. Now

$$wN.(\mathcal{D} \to S_{n-1}\mathcal{D} \times \mathcal{D}) \approx wS_{n-1}\mathcal{D} \times wN.(\mathcal{D} \xrightarrow{=} \mathcal{D})$$

and the factor $wN.(\mathcal{D} \to \mathcal{D})$ is contractible. So the proof of the proposition has been reduced to proving the following lemma:

Lemma 1.8.8. If for every $X \in \mathcal{D}$ the simplicial category $wN.(j:\mathcal{D} \to \mathcal{D}_X)$ is contractible then the map $p: wN.(j_n:\mathcal{D} \to S_n\mathcal{D}) \to wS_{n-1}\mathcal{D}$ is a homotopy equivalence.

Proof. There is a variant of theorem A [8] for simplicial categories. A special case, sufficient for the present application, has been described in [13, prop. 6.5] in great detail. A neater, and more general, version may be found in [15, section 4] with a sketch proof. In any case, the criterion says that for every object for the map p to be a homotopy equivalence it suffices that for every object

$$B = (B_1 \rightarrowtail .. \rightarrowtail B_{n-1} \text{ , choices }) \in wS_{n-1}\mathcal{D}$$

the left fibre $(p/B).$ is contractible.

Capitalizing on the special feature that $wS_{n-1}\mathcal{B}$, the target of p , is only a simplicial category in a trivial way, we can re-express $(p/B).$ in terms of left fibres of maps of categories, namely

$$(p/B)_m = p_m/B \text{ .}$$

An object of p_m/B consists of a diagram

$$\begin{array}{ccccccc} A_1 & \rightarrowtail & \cdots & \rightarrowtail & A_{n-1} & \rightarrowtail & A_n \\ \downarrow \wr & & & & \downarrow \wr & & \\ B_1 & \rightarrowtail & \cdots & \rightarrowtail & B_{n-1} & & \end{array}$$

plus a m-tuple of objects in $\mathcal{D}$, plus certain sum diagrams formed from this m-tuple and A_n (plus, as usual, certain other choices).

There is a natural transformation of the identity map on p_m/B , it is given by pushout with the vertical map(s) in the diagram. For varying m the natural transformations are compatible, so they combine to give a homotopy of the identity map of $(p/B).$; namely a deformation retraction into the simplicial subcategory defined by the condition that the vertical map(s) be the identity.

That subcategory is isomorphic to $wN.(j:\mathcal{D} \to \mathcal{D}_X)$ where $X = B_{n-1}$, it is thus contractible by assumption. We are done. □

Let $\mathcal{D}$ be a category with cofibrations and weak equivalences, and $X \in \mathcal{D}$. It turns out that the contractibility of $wN.(\mathcal{D} \to \mathcal{D}_X)$ may be re-expressed in terms of two other conditions which appear to be rather independent of each other.

<u>Proposition</u> 1.8.9. $wN.(\mathcal{D} \to \mathcal{D}_X)$ is contractible if and only if the following two conditions are satisfied:

(1) $wN.(\mathcal{D} \to \mathcal{D}_X)$ is connected,

(2) the map $wN.\mathcal{D} \to wN.\mathcal{D}_X$ is a homotopy equivalence.

Proof. If $wN.(\mathcal{D} \to \mathcal{D}_X)$ is connected it has $wN.N.(\mathcal{D} \to \mathcal{D}_X)$ as a de-loop (by [11] or a variant of lemma 1.8.6). Therefore, provided it is connected, it is contractible

if and only if $wN.N.(\mathcal{D} \to \mathcal{D}_X)$ is contractible. By lemma 1.8.6 we have a dia
of homotopy fibrations

$$\begin{array}{ccccc} wN.\mathcal{D} & \longrightarrow & wN.N.(\mathcal{D} \overset{=}{\to} \mathcal{D}) & \longrightarrow & wN.N.\mathcal{D} \\ \downarrow & & \downarrow & & \| \\ wN.\mathcal{D}_X & \longrightarrow & wN.N.(\mathcal{D} \to \mathcal{D}_X) & \longrightarrow & wN.N.\mathcal{D} \end{array}$$

and the middle term in the upper row is contractible. Therefore $wN.N.(\mathcal{D} \to \mathcal{D}_X$
contractible if and only if the vertical map on the left is a homotopy equival

Proof of theorem 1.8.1. The nerve of the simplicial category $wN.(\mathcal{D} \to \mathcal{D}_X)$ is
simplicial set whose vertices are the objects $X \rightarrowtail A$ in $\mathcal{D}_X$. There are two k
of 1-simplices, corresponding to the morphisms of $w\mathcal{D}_X$ on the one hand, and to
'operation' of the objects of $\mathcal{D}$ on those of $\mathcal{D}_X$ on the other. It results th
the set of connected components is the set of equivalence classes of the $X \rightarrowtail A$
under the equivalence relation generated by

(i) $(X \rightarrowtail A) \sim (X \rightarrowtail A')$ if there is a map $(X \rightarrowtail A) \to (X \rightarrowtail A')$ in $w\mathcal{D}_X$

(ii) $(X \rightarrowtail A) \sim (X \rightarrowtail A \cup_* A'')$ if $A'' \in \mathcal{D}$.

The condition referred to in the theorem, that *cofibrations in* $\mathcal{D}$ *are splittabl*
to weak equivalence, implies that every object of $\mathcal{D}_X$ can be related (in a spec
way, in fact) to the trivial object $X \overset{=}{\rightarrowtail} X$, thus $wN.(\mathcal{D} \to \mathcal{D}_X)$ is connected.

Let $\mathcal{D} = \varinjlim C^n$ now. Then, as just observed, $wN.(\mathcal{D} \to \mathcal{D}_X)$ is connected f
every X, and, by proposition 1.8.3, the map $wN.\mathcal{D} \to wN.\mathcal{D}_X$ is a homotopy equiva
lence. By proposition 1.8.9 these two properties imply that $wN.(\mathcal{D} \to \mathcal{D}_X)$ is con
tractible for every X which in turn, by proposition 1.8.7, implies that

$$wN.\mathcal{D} \longrightarrow wS.\mathcal{D}$$

is a homotopy equivalence, as desired.

1.9. Appendix: Relation with the Q construction.

Let A be an *exact category* in the sense of Quillen [8]. One can make A into a category with cofibrations and weak equivalences by choosing a zero object and by defining the cofibrations and the weak equivalences to be the admissible monomorphisms and the isomorphisms, respectively. So a simplicial category $iS.A$ is defined. It turns out that $iS.A$ is naturally homotopy equivalent to the category QA of Quillen.

To see this we first replace QA by a homotopy equivalent simplicial category $iQ.A$. Namely let iQA be the bicategory of the commutative squares in QA in which the vertical arrows are the isomorphisms (in either A or QA — those are the same). Then QA and iQA are homotopy equivalent (lemma 1.6.5), and we let $iQ.A$ be a partial nerve of iQA , namely the nerve in the Q direction.

Next we replace $iS.A$ by a homotopy equivalent simplicial category $iS^e_.A$. We use the *edgewise subdivision* functor [12] which to any simplicial object $X.$, say $X. : \Delta^{op} \to K$, associates another $X^e_. : \Delta^{op} \to K$, namely the composite

$$X^e_. = X. \, d^{op}$$

where $d: \Delta \to \Delta$ is the *doubling map* which takes $[n]$ to $[2n+1]$ and whose behaviour on maps may be described by saying that it takes

$$(0 < 1 < \ldots < n) \quad \text{to} \quad (n' < \ldots < 1' < 0' < 0 < 1 < \ldots < n) \ .$$

If $X.$ is a simplicial space then the geometric realizations $|X.|$ and $|X^e_.|$ are naturally homeomorphic [12, prop. (A.1)]. Applying this fact to the simplicial space $[n] \mapsto |iS_nA|$ we obtain that $iS.A$ and its edgewise subdivision $iS^e_.A$, or rather their geometric realizations, are homotopy equivalent.

There is a map of simplicial categories

$$iS^e_.A \longrightarrow iQ.A$$

which is an equivalence of categories in each degree, and therefore a homotopy equivalence. The map is best explained by drawing a diagram to illustrate the situation for $n = 3$.

An object of iS^e_3A ($\approx iS_7A$) is a sequence of cofibrations

$$A_{(3',2')} \rightarrowtail A_{(3',1')} \rightarrowtail A_{(3',0')} \rightarrowtail A_{(3',0)} \rightarrowtail A_{(3',1)} \rightarrowtail A_{(3',2)} \rightarrowtail A_{(3',3)}$$

together with a choice of quotients

$$A_{(i,j)} = A_{(3',j)}/A_{(3',i)} \ .$$

By dropping some of the choices while retaining others we can associate to the object the following diagram

$$
\begin{array}{ccccccc}
A_{(3',0)} & \rightarrowtail & A_{(3',1)} & \rightarrowtail & A_{(3',2)} & \rightarrowtail & A_{(3',3)} \\
\downarrow\!\!\!\!\downarrow & & \downarrow\!\!\!\!\downarrow & & \downarrow\!\!\!\!\downarrow & & \\
A_{(2',0)} & \rightarrowtail & A_{(2',1)} & \rightarrowtail & A_{(2',2)} & & \\
\downarrow\!\!\!\!\downarrow & & \downarrow\!\!\!\!\downarrow & & & & \\
A_{(1',0)} & \rightarrowtail & A_{(1',1)} & & & & \\
\downarrow\!\!\!\!\downarrow & & & & & & \\
A_{(0',0)} & & & & & &
\end{array}
$$

The diagram describes a sequence of three composable morphisms in $Q\mathcal{A}$ as well as the different ways in which the actual composition can be performed. In particular the diagram defines an object of $iQ_3\mathcal{A}$. The object in question is not identical to the diagram itself, rather it is an equivalence class of diagrams; two diagrams are considered equivalent if they are isomorphic by an isomorphism which restricts to the identity on each of the diagonal objects $A_{(j',j)}$.

To conclude we note a variant of the homotopy equivalence. Let $s.\mathcal{A}$ denote the simplicial set of objects of $S.\mathcal{A}$. Considering $s.\mathcal{A}$ as a simplicial category in a trivial way we have an inclusion $s.\mathcal{A} \to iS.\mathcal{A}$ which is a homotopy equivalence by lemma 1.4.1. Let $Q.\mathcal{A}$ denote the nerve of the category $Q\mathcal{A}$. Above we have described a map

$$s^e_{.}\mathcal{A} \longrightarrow Q.\mathcal{A} \quad .$$

This map is a homotopy equivalence. For it fits into a diagram

$$
\begin{array}{ccc}
s^e_{.}\mathcal{A} & \longrightarrow & Q.\mathcal{A} \\
\downarrow & & \downarrow \\
iS^e_{.}\mathcal{A} & \longrightarrow & iQ.\mathcal{A}
\end{array}
$$

and we know already that the three other maps in the diagram are homotopy equivalences.

2. THE FUNCTOR A(X) .

2.1. Equivariant homotopy theory, and the definition of A(X) .

Let X be a space. A(X) is defined as the K-theory, in the sense of the preceding chapter, of an equivariant homotopy theory associated to X .

There are several ways of making this precise. The main purpose of this section is to describe a few of those ways in detail and to show that they all lead to the same result, up to homotopy.

The various cases arise from the fact that we want to keep the option of interpreting each of the terms *space*, *equivariant*, and *finite type* in two different ways. Namely we will want to work either with topological spaces or with simplicial sets. We want to use spaces over X on the one hand or spaces with an action of G(X) , the loop group of X , on the other. And finally we want to be free to impose a condition of strict finiteness on the objects of the category or to be content with a condition of finiteness up to homotopy.

We begin with a construction that combines the two equivariant points of view. We will be mainly interested, eventually, in the two special cases where one of G and W below is trivial and the other one is X , resp. a loop group of X .

Let G be a simplicial monoid and W a simplicial set on which G acts (by a monoid is meant an associative semigroup with 1). We define

$$\mathcal{R}(W,G)$$

to be the category of the G-simplicial sets having W as a retract. In detail, the objects of $\mathcal{R}(W,G)$ are the triples (Y,r,s) where Y is a simplicial set with G-action and $s: W \to Y$ and $r: Y \to W$ are G-maps so that $rs = Id_W$, and the morphisms from (Y,r,s) to (Y',r',s') are the G-maps $f: Y \to Y'$ so that $r'f = r$ and $fs = s'$.

If G is the trivial monoid we omit it from the notation. In other words, we let $\mathcal{R}(X)$ denote the category of the simplicial sets having X as a retract.

There are similar constructions in the topological case, and geometric realization induces a functor $\mathcal{R}(W,G) \to \mathcal{R}(|W|,|G|)$.

We define our finite type conditions now. We proceed in the following order:

1. finiteness in the simplicial case,
2. finiteness in the topological case,
3. homotopy finiteness in the topological case,
4. homotopy finiteness in the simplicial case.

1. *Finiteness in the simplicial case.* An object (Y,r,s) of $R(X)$ is called *finite* if the simplicial set Y is generated by the simplices of $s(X)$ together with finitely many other simplices. An equivalent condition is that the geometric realization $|Y|$ is a finite CW complex relative to the subspace $|s(X)|$. The full subcategory of the finite objects is denoted $R_f(X)$.

In the general case of $R(W,G)$ we must combine the finite generation condition with a freeness condition. *Finite generation* of (Y,r,s) means that Y is generated, as a G-simplicial set, by the simplices of $s(W)$ together with finitely many other simplices. *Freeness* means that, for every k, the action of G_k on Y_k is free away from W_k; precisely, the condition is that Y may be obtained from W by *attaching of free G-cells*, that is, by direct limit and the formation of pushouts of diagrams of the kind $Y' \leftarrow \partial\Delta^n \times G \rightarrow \Delta^n \times G$ where Δ^n denotes the simplicial set *n-simplex*, and $\partial\Delta^n$ the simplicial subset *boundary*. We denote $R_f(W,G)$ the full subcategory of $R(W,G)$ given by the objects which are both finitely generated and free; the objects (Y,r,s), in other words, where Y can be obtained from W by attaching of finitely many free G-cells. $R_f(W,G)$ is a category with cofibrations and weak equivalences in the sense of sections 1.1 and 1.2, the cofibrations are the injective maps, and the weak (homotopy) equivalences are the maps $(Y,r,s) \rightarrow (Z,t,u)$ whose underlying maps $Y \rightarrow Z$ are weak homotopy equivalences in the usual sense (that is, induce isomorphisms of homotopy groups upon geometric realization). We denote the category of the weak homotopy equivalences by $hR_f(W,G)$.

2. *Finiteness in the topological case.* Let $|X|$ be a topological space, not necessarily the geometric realization of a simplicial set X. An object (Y,r,s) of $R(|X|)$ is called *finite* if Y is equipped with the structure of a finite CW complex relative to the subspace $s(|X|)$. We let $R_f(|X|)$ denote the category of these objects and their *cellular* maps (it is not, of course, a full subcategory of $R(|X|)$). We consider $R_f(|X|)$ as a category with cofibrations and weak (homotopy) equivalences; by definition, a map in $R_f(|X|)$ is a cofibration if it is isomorphic to a cellular inclusion.

More generally, in the case of $R(|W|,|G|)$, we define $R_f(|W|,|G|)$ to be the category of the finite $|G|$-free CW complexes, relative to $|W|$, and their cellular maps.

3. *Homotopy finiteness in the topological case.* We define $R_{hf}(|W|,|G|)$ as the full subcategory of $R(|W|,|G|)$ given by the (Y,r,s) where (Y,s) has the $|G|$-homotopy type, in the strong sense, of a finite $|G|$-free CW complex relative to $|W|$. This is a category with cofibrations and weak (homotopy) equivalences, where *cofibration* has its usual meaning as a map having the $|G|$-homotopy extension property (after neglect of structural retractions, that is). To see that cobase change by cofibrations does not take one out of the category, i.e. preserves homotopy finiteness, it suffices to note that weak homotopy equivalences have homotopy inverses, after neglect of structural retractions (the Whitehead theorem for $|G|$-free CW complexes).

Remark. On the face of it there are set theoretical difficulties in the construction of K-theory from $R_{hf}(|X|)$. For $hS.R_{hf}(|X|)$ is not a 'small' simplicial category, nor even equivalent to one (in the sense of category theory). Here are a few ways of dealing with this matter, each with its own virtues and drawbacks: (a) one can pick an explicit small category $R'_{hf}(|X|)$ with which to work (for example, have all one's spaces embedded in $|X|\times R^{\infty}$), (b) one may postulate the existence of a universe, in the sense of Grothendieck, work in a fixed one, and check that an enlargement of the universe does not alter the homotopy type, (c) one may regard the notion of a 'large' space as just as legitimate as that of a 'large' category, provided only that certain constructions are avoided (this is the naive version of the preceding). Which one of these or other alternatives to adopt seems a matter of taste. We will not pursue the matter further.

4. *Homotopy finiteness in the simplicial case.* We reduce to the topological case. That is, we define $R_{hf}(W,G)$ as the full subcategory of $R(W,G)$ given by the (Y,r,s) whose geometric realizations are homotopy finite in the sense of the preceding case.

Recall that the *approximation theorem* 1.6.7 describes sufficient conditions for an exact functor $C \to C'$ to induce a homotopy equivalence $hS.C \to hS.C'$.

<u>Proposition</u> 2.1. The approximation theorem applies to the map

$$R_f(W,G) \longrightarrow R_{hf}(W,G) ,$$

resp. its topological analogue.

Proof. The non-trivial thing to verify is the following assertion (the part App 2 of the *approximation property*).

Assertion. Let $(Y,r,s) \in R_f(W,G)$, and let $(Y,r,s) \to (Y',r',s')$ be any map in $R_{hf}(W,G)$. Then the map can be factored as $(Y,r,s) \to (Y_1,r_1,s_1) \to (Y',r',s')$ where $(Y_1,r_1,s_1) \in R_f(W,G)$, the first map is a cofibration in $R_f(W,G)$, and the second map is a weak equivalence in $R_{hf}(W,G)$.

To prove the assertion it will suffice to find a factorization

$$(Y,s) \longrightarrow (Y_1,s_1) \longrightarrow (Y',s') \ .$$

For it is then possible to *define* the structural retraction r_1 as the composite of $Y_1 \to Y'$ with $r'\colon Y' \to W$.

We treat the topological case first. The Whitehead theorem for $|G|$-free CW complexes relative to $|W|$ is available here, so we can find a finite (Y_o,s_o) together with homotopy equivalences $(Y_o,s_o) \to (Y',s')$ and $(Y',s') \to (Y_o,s_o)$, homotopy inverse to each other. Choose a cellular map $(Y,s) \to (Y_o,s_o)$ homotopic to the composition $(Y,s) \to (Y',s') \to (Y_o,s_o)$, and define (Y_1,s_1) as its mapping cylinder. Then there exists a map $(Y_1,s_1) \to (Y',s')$ extending the given maps on (Y,s) and (Y_o,s_o) . This has the required properties.

In the simplicial case we know, by the topological case, that there exists some factorization

$$(|Y|,|s|) \longrightarrow (Y_1,s_1) \longrightarrow (|Y'|,|s'|) \ .$$

We show that, by perturbing (Y_1,s_1) a little, we may lift it back to the simplicial framework.

Proceeding by induction on the cells of Y_1 not in $|Y|$ we suppose that we have found a subcomplex $|Z|$ of Y_1 which does arise by geometric realization, and so that the map $|Z| \to |Y'|$ is a geometric realization, too. To add another one of the cells of Y_1 to $|Z|$, means that we form the pushout of a diagram of the kind

$$|Z| \longleftarrow |\partial\Delta^n| \times |G| \longrightarrow |\Delta^n| \times |G| \ .$$

We use simplicial approximation to rigidify this. Namely let Sd denote the *subdivision functor* for simplicial sets [4], and Sd_k its k-fold iteration. Then if k is large enough one knows [4] that there is a map of simplicial sets,

$$Sd_k\ \partial\Delta^n \longrightarrow Z \ ,$$

whose geometric realization is homotopic to the map

$$|Sd_k\ \partial\Delta^n| \approx |\partial\Delta^n| \times 1 \longrightarrow |Z| \quad , \qquad 1 \in |G| \ ,$$

and, again if k is large enough, the composite map $Sd_k\partial\Delta^n \to Z \to Y'$ extends to $Sd_k\Delta^n$, in the preferred homotopy class. We now define

$$Z' = Z \cup_{Sd_k\partial\Delta^n \times G} Sd_k\Delta^n \times G \ .$$

Then $Z \to Y'$ extends to a map $Z' \to Y'$ in the preferred homotopy class. By the $|G|$-homotopy extension theorem $|Z'|$ in turn may be extended, by induction on the remaining cells, to a $|G|$-CW complex Y_1' mapping to Y_1 by homotopy equivalence. This completes the inductive step, and hence the proof of the proposition. □

<u>Proposition</u> 2. 2 The approximation theorem applies to the geometric realization map

$$R_f(W,G) \longrightarrow R_f(|W|,|G|)$$

Proof. The non-trivial thing to verify is the following assertion.

Assertion. Let $(Y,r,s) \in R_f(W,G)$, and let $(|Y|,|r|,|s|) \to (Y',r',s')$ be any map in $R_f(|W|,|G|)$. Then the map can be factored as

$$(|Y|,|r|,|s|) \longrightarrow (|Y''|,|r''|,|s''|) \longrightarrow (Y',r',s')$$

where the first map is the geometric realization of a cofibration in $R_f(W,G)$, and the second map is a weak equivalence in $R_f(|W|,|G|)$.

As before (the preceding proof) it suffices to find a factorization

$$(|Y|,|s|) \longrightarrow (|Y''|,|s''|) \longrightarrow (Y',s') \ .$$

Define (Y_1,s_1) as the mapping cylinder of $(|Y|,|s|) \to (Y',s')$. Then (Y'',s'') is obtained from (Y_1,s_1) by rigidifying, one after the other, the cells of Y_1 not in $|Y|$. The argument is the same as that in the second part of the preceding proof.

Let G be a simplicial group now, not just monoid, and X a simplicial set. By a *principal* G-*bundle with base* X is meant a free G-simplicial set P together with an isomorphism of X with $P \times^G *$, the simplicial set of orbits.

<u>Lemma</u> 2 There is an equivalence of categories $R(X) \sim R(P,G)$.

Proof. We can define functors between these categories by pullback with $P \to X$ and by the orbit map, respectively. If $(Y,r,s) \in R(X)$ then $(Y \times_X P) \times^G * \approx Y$. And if $(Y',r',s') \in R(P,G)$ then the diagram

$$\begin{array}{ccc} Y' & \longrightarrow & P \\ \downarrow & & \downarrow \\ Y' \times^G * & \longrightarrow & P \times^G * \end{array}$$

is a pullback, thanks to the freeness of the G-action on P and the fact that G is a simplicial group, not just monoid. Hence $Y' \approx (Y' \times^G *) \times_X P$, and the two functors are inverse to each other, up to isomorphism. □

By a *universal* G-*bundle with base* X will be meant a principal bundle whose total space P is contractible (in the weak sense). In this situation it is necessarily the case that G represents the loop space of X , but apart from this restriction one knows that universal bundles exist in great profusion. Specifically there is a functor, due to Kan, which to connected pointed X associates a universal G(X)-bundle where $G(X)$ is a certain free simplicial group, the *loop* *group* of X . Conversely it is also possible, in any of several functorial ways, to associate to a simplicial group G a universal bundle over a *classifying space*.

Given a universal G-bundle over X we can define a functor

$$\begin{array}{ccc} \mathcal{R}(X) & \longrightarrow & \mathcal{R}(*,G) \\ (Y,r,s) & \longmapsto & (\ (Y\times_X P)/(X\times_X P),\bar{r},\bar{s}\ ) \end{array} .$$

The functor respects the notion of finiteness, resp. homotopy finiteness, and it is *exact* (sections 1.1 and 1.2), so it induces a map in K-theory. In a similar way we can also use P to define a map $\mathcal{R}(|X|) \to \mathcal{R}(*,|G|)$.

<u>Proposition</u> 2.1.4. The map $hS.\mathcal{R}_{hf}(X) \to hS.\mathcal{R}_{hf}(*,G)$ is a homotopy equivalence.

Proof. In view of its definition, the map arises as the composite of the equivalence $\mathcal{R}_{hf}(X) \to \mathcal{R}_{hf}(P,G)$ of lemma 2.1.3 with the map $\mathcal{R}_{hf}(P,G) \to \mathcal{R}_{hf}(*,G)$ given by pushout with $P \to *$. It therefore suffices to show that the latter map induces a homotopy equivalence. We show this by providing a homotopy inverse. Consider the map $\mathcal{R}(*,G) \to \mathcal{R}(P,G)$ given by product with P , using the diagonal action of G . The map respects the notion of homotopy finiteness, in view of the contractibility of P , and it is exact, so it induces a map in K-theory. The composite map on $\mathcal{R}(*,G)$ admits a natural transformation to the identity,

$$Y\times P \cup_{*\times P} * \longrightarrow Y ,$$

and the composite map on $\mathcal{R}(P,G)$ admits a natural transformation from the identity,

$$Y \longrightarrow Y\times P \cup_{P\times P} P .$$

In view of the contractibility of P each of these two natural transformations is a weak equivalence. Using proposition 1.3.1 now we are done. □

<u>Theorem</u> 2.1.5. If X is a simplicial set (resp. if G is a simplicial monoid) there is a 2×2 diagram of homotopy equivalences, namely the left one (resp. right one) of the following two squares

$$\begin{array}{ccc} hS.\mathcal{R}_f(X) & \longrightarrow & hS.\mathcal{R}_{hf}(X) \\ \downarrow & & \downarrow \\ hS.\mathcal{R}_f(|X|) & \longrightarrow & hS.\mathcal{R}_{hf}(|X|) \end{array} \qquad \begin{array}{ccc} hS.\mathcal{R}_f(*,G) & \longrightarrow & hS.\mathcal{R}_{hf}(*,G) \\ \downarrow & & \downarrow \\ hS.\mathcal{R}_f(*,|G|) & \longrightarrow & hS.\mathcal{R}_{hf}(*,|G|) \end{array} .$$

If G is a loop group of X , and if a universal G-bundle with base X is given, there is a natural transformation from the left square to the one on the right, and all the arrows in the resulting 2×2×2 diagram are homotopy equivalences.

Proof. This results from propositions 2.1.1, 2.1.2, and 2.1.4. □

Picking one of the choices offered by the theorem we now make the definition

$$A(X) \quad = \quad \Omega|hS.\mathcal{R}_f(X)|$$

if X is a simplicial set.

A map $x\colon X \to X'$ induces $x_*\colon \mathcal{R}(X) \to \mathcal{R}(X')$ by pushout with x, and hence a map in K-theory. In this way $A(X)$ becomes a covariant functor. Below we give an argument to show that this functor is a homotopy functor (proposition 2.1.7).

We have to consider functorial behaviour in a slightly more general situation. Namely let $g\colon G \to G'$ be a group map, and $w\colon W \to W'$ a map under g. These induce a map $(g,w)_*\colon \mathcal{R}(W,G) \to \mathcal{R}(W',G')$ as the composite

$$\mathcal{R}(W,G) \longrightarrow \mathcal{R}(W\times^G G',G') \longrightarrow \mathcal{R}(W',G')$$

where the first map is given by product with G' under G, and the second map by pushout with $W\times^G G' \to W'$.

Let a *map of universal bundles* mean a triple of maps

$$(x,p,g)\ :\ (X,P,G) \longrightarrow (X',P',G')$$

where p is a map under g, and over x. We note that $X\times_{X'}P' \approx P\times^G G'$ in this situation.

<u>Lemma</u> 2.1.6. To such a map there is associated a commutative diagram

$$\begin{array}{ccc}
\mathcal{R}(X) & \xrightarrow{\ x_*\ } & \mathcal{R}(X') \\
\downarrow & & \downarrow \\
\mathcal{R}(*,G) & \xrightarrow[(*,g)_*]{} & \mathcal{R}(*,G')
\end{array} \quad .$$

Proof. This results from the definition of the maps and the commutativity of the diagram

$$\begin{array}{ccccc}
\mathcal{R}(X) & \xrightarrow{\ =\ } & \mathcal{R}(X) & \longrightarrow & \mathcal{R}(X') \\
\downarrow\wr & & \downarrow\wr & & \downarrow\wr \\
\mathcal{R}(P,G) & \xrightarrow{\ \sim\ } & \mathcal{R}(P\times^G G',G') & \longrightarrow & \mathcal{R}(P',G') \\
\downarrow & & \downarrow & & \downarrow \\
\mathcal{R}(*,G) & \longrightarrow & \mathcal{R}(*,G') & \xrightarrow{\ =\ } & \mathcal{R}(*,G')
\end{array}$$

where the arrows $\xrightarrow{\ \sim\ }$ denote equivalences of categories (lemma 2.1.3). □

<u>Proposition</u> 2.1.7. If $x: X \to X'$ is a weak homotopy equivalence then so is the induced map $x_*: A(X) \to A(X')$.

Proof. The functor $X \mapsto hS.\mathcal{R}_f(X)$ commutes with direct limit, and it takes finite disjoint unions to products. As a result it suffices to prove the proposition in the case where X and X' are connected. We may further replace 'h' by 'hf' . Our task is then to show that $x_*: hS.\mathcal{R}_{hf}(X) \to hS.\mathcal{R}_{hf}(X')$ is a homotopy equivalence in that special case.

Choose a universal bundle over X' , say a universal G'-bundle P' . Since $x: X \to X'$ is a weak homotopy equivalence, pullback with it defines a universal G'-bundle $P = X \times_{X'} P'$ over X . There is a map of universal bundles now,

$$(x, pr_2, Id_{G'}) \ : \ (X,P,G') \longrightarrow (X',P',G') \ .$$

Hence (the preceding lemma) there is a commutative diagram

$$\begin{array}{ccc} hS.\mathcal{R}_{hf}(X) & \xrightarrow{x_*} & hS.\mathcal{R}_{hf}(X') \\ \downarrow & & \downarrow \\ hS.\mathcal{R}_{hf}(*,G') & \xrightarrow{=} & hS.\mathcal{R}_{hf}(*,G') \end{array}$$

and the vertical arrows are homotopy equivalences by proposition 2.1.4. It follows that x_* is a homotopy equivalence. □

Remark. For simplicial monoids in general, as opposed to simplicial groups, it does not follow in the same way that $G \mapsto \Omega|hS.\mathcal{R}_f(*,G)|$ is a homotopy functor. The result is still true, however. For example it follows from theorem 2.2.1 below.

2.2. A(X) via spaces of matrices.

Let G be a simplicial monoid. We consider the free pointed $|G|$-CW complex with k $|G|$-cells in dimension n and no other cells; or what is the same thing,

$$\vee^k S^n \wedge |G|_+ ,$$

the half-smash product of $|G|$ with a wedge of k spheres of dimension n.

Let

$$H_k^n(G) = H_{|G|}(\vee^k S^n \wedge |G|_+)$$

denote the simplicial monoid of pointed $|G|$-equivariant (weak) homotopy equivalences, and let $BH_k^n(G)$ denote its classifying space. There are stabilization maps

$$BH_k^n(G) \longrightarrow BH_k^{n+1}(G) \quad , \quad BH_k^n(G) \longrightarrow BH_{k+1}^n(G)$$

given by suspension and by the addition of an identity map, respectively.

The purpose of this section is to show that the K-theory of the preceding section can be re-expressed in terms of the + construction of Quillen, as follows.

Theorem 2.2.1. There is a natural chain of homotopy equivalences

$$\Omega|hS.\mathcal{R}_f(*,G)| \simeq Z \times \lim_{\substack{\rightarrow \\ n,k}} BH_k^n(G)^+ .$$

By combining with theorem 2.1.5 we obtain that, in particular, A(X) may be so re-expressed for connected X,

$$A(X) \simeq Z \times \lim_{\substack{\rightarrow \\ n,k}} BH_k^n(G(X))^+ .$$

This may be regarded as a description of A(X) in terms of *spaces of matrices*, analogous to the definition of the algebraic K-theory of a ring in terms of matrices and the + construction, as follows.

In the case at hand, the 'ring' in question is the *ring up to homotopy*

$$\Omega^\infty S^\infty |G|_+ = \lim_{\substack{\rightarrow \\ n}} Map(S^n, S^n \wedge |G|_+) .$$

Let $M_{k\times k}(\Omega^\infty S^\infty |G|_+)$ denote the product of $k\times k$ copies of this space, considered as a multiplicative H-space by means of matrix multiplication. We denote

$$\widehat{GL}_k(\Omega^\infty S^\infty |G|_+)$$

the sub-H-space of the homotopy-invertible matrices; it is the union of those connected components which are invertible in the monoid of connected components. The point now is simply that

$$\varinjlim_n BH_k^n(G)$$

provides a classifying space for the H-space $\widehat{GL}_k(\Omega^\infty S^\infty|G|_+)$. Indeed, there is a homotopy equivalence of H-spaces

$$\varinjlim_n H_k^n(G) \simeq \widehat{GL}_k(\Omega^\infty S^\infty|G|_+) .$$

It is given, in the limit, by the (n-1)-connected map

$$Map_{|G|}(V^k S^n \wedge |G|_+, V^k S^n \wedge |G|_+) \approx Map(V^k S^n, V^k S^n \wedge |G|_+)$$
$$\longrightarrow Map(V^k S^n, \Pi^k S^n \wedge |G|_+) \approx Map(S^n, S^n \wedge |G|_+)^{k\times k}$$

Proof of theorem. Define $R_k^n(*,G)$ to be the full subcategory of $R_f(*,G)$ given by the objects which are *n-spherical of rank* k . By definition, these are the objects weakly equivalent to

$$* \cup_{\amalg^k \partial\Delta^n \times G} \amalg^k \Delta^n \times G$$

that is, the objects which are in the same connected component, in $hR_f(*,G)$, as that particular object.

It is plausible, and will be shown below (proposition 2.2.5), that there is a natural chain of homotopy equivalences

$$BH_k^n(G) \simeq |hR_k^n(*,G)| .$$

Define $R^n(*,G)$ to be the subcategory of $R_f(*,G)$ of the objects which are n-spherical of unspecified rank; that is, the union of the categories $R_k^n(*,G)$. This is a category with sum and weak equivalences (section 1.8), so the group completion in the sense of Segal is defined; in the language of section 1.8 this is the simplicial category $hN.R^n(*,G)$. By a theorem of Segal [11] there is a homotopy equivalence, well defined up to weak homotopy (homotopy on compacta),

$$\Omega|hN.R^n(*,G)| \simeq Z \times \varinjlim_k |hR_k^n(*,G)|^+ .$$

Combining with the homotopy equivalence above, and passing to the limit with respect to n , we obtain now a homotopy equivalence

$$\varinjlim_n \Omega|hN.R^n(*,G)| \simeq Z \times \varinjlim_{n,k} BH_k^n(G)^+ .$$

This reduces the proof of the theorem to the following proposition.

Proposition 2.2.2. There is a natural chain of homotopy equivalences

$$\varinjlim_n hN.R^n(*,G) \simeq hS.R_f(*,G) .$$

The proof of the proposition is an application of theorems 1.7.1 and 1.8.1. To make these theorems applicable we have to check some things first. Let us define

$$h_*(Y) = \tilde{H}_*(Y \times^G \pi_0 G)$$

for $Y \in R_f(*,G)$, where $\tilde{H}_*$ denotes the reduced integral homology of pointed spaces.

<u>Lemma</u> 2.2.3 If $\tilde{H}_i(Y) = 0$ for $i < m$ then $\tilde{H}_m(Y) \to h_m(Y)$ is an isomorphism

Proof. We give two proofs. The first applies to the special case where G is a simplicial group, not just monoid. In this case $Y \times^G \pi_0 G \approx Y \times^F *$ where F is the connected component of $1 \in G$. Choose a universal F-bundle E and form the associated bundle over $E \times^F *$, i.e. $(Y \times E) \times^F *$. Then $Y \times^F *$ may be identified, up to homotopy, to the quotient $(Y \times E) \times^F * / E \times^F *$, and the lemma results from the Serre spectral sequence of the fibration.

In the general case one notices that the lemma is really a special case of one in the next section (lemma 2.3.4) which concerns simplicial modules over a simplicial ring and whose proof depends on a spectral sequence of Quillen's on (derived) tensor products.

Let $R_f^{(2)}(*,G)$ denote the subcategory of $R_f(*,G)$ of the objects which are 1-connected.

<u>Lemma</u> 2.2.4. The inclusion $hS.R_f^{(2)}(*,G) \to hS.R_f(*,G)$ is a homotopy equivalence.

Proof. Double suspension defines an endomorphism of each of these which is homotopic to the identity map (proposition 1.6.2). On the other hand, double suspension takes $hS.R_f(*,G)$ into $hS.R_f^{(2)}(*,G)$, so it gives a deformation retraction. □

Proof of proposition 2.2.2. The functor $Y \mapsto h_*(Y)$ defines a homology theory on $R_f(*,G)$, in the sense of section 1.7, with values in the category of $Z[\pi_0 G]$-modules.

Restricting attention to 1-connected objects, as we may by lemma 2.2.4, we obtain from lemma 2.2.3 together with the Hurewicz theorem that the weak equivalences are *homologically defined*: a map is a weak equivalence if and only if it induces an isomorphism on h_* .

The objects of $R^n(*,G)$ have the property that $h_i(Y)$ is 0 for $i \neq n$, and free over $Z[\pi_0 G]$ for $i = n$. Conversely they are characterized by this property. To see this it suffices to construct a map from a standard object inducing an isomorphism on h_* . Such a map is obtained by mapping each generating cell $\Delta^n \times 1$, suitably subdivided, so as to represent an appropriate generating element of the module $\pi_n|Y| \approx H_n(Y) \approx h_n(Y)$.

We show next that the hypothesis of section 1.7 is satisfied: if $Y_p \to Y$ is any p-connected map then it is possible to construct a factorization

$$Y_p \longrightarrow Y_{p+1} \longrightarrow \cdots \longrightarrow Y_q \longrightarrow Y$$

where each Y_{n+1} is obtained from Y_n by attaching (n+1)-cells and where the map $Y_q \to Y$ is a weak homotopy equivalence. First, the inductive construction of Y_{n+1} from Y_n is done as follows. The module $h_{n+1}(Y_n \to Y) \approx \pi_{n+1}(|Y_n| \to |Y|)$ is finitely generated over $Z[\pi_o G]$, and each element may be represented by mapping a (suitably subdivided) pair $(\Delta^{n+1}, \partial\Delta^{n+1})$. Picking a generating set, we can use these maps to attach (n+1)-cells to Y_n and to extend the map to Y to the cells. Next, the construction can terminate. For suppose that q is at least as large as the dimension of Y. Then $h_q(Y_{q-1} \to Y)$ is computed from a finitely generated free chain complex which is both (q-1)-connected and q-dimensional. It follows that h_q is the only non-vanishing homology, and that it is stably free. After attaching some more (q-1)-cells to Y_{q-1}, if necessary, we may suppose the homology is actually free, so that in a last step, finally, we can attach q-cells to kill the homology without introducing new homology in the next dimension.

We have verified most of the hypotheses of theorem 1.7.1 now. The one exception is the condition that the category $\mathcal{E}$, in the definition of spherical objects in section 1.7, should be closed under the operation of taking kernels of surjections. Our $\mathcal{E}$ so far is the category of finitely generated free modules over $Z[\pi_o G]$. This does not satisfy the condition, in general, so we must enlarge it. We therefore replace $\mathcal{R}^n(*,G)$ by $\widetilde{\mathcal{R}}^n(*,G)$ which we define as follows. It is the subcategory of $\mathcal{R}_f(*,G)$ of the objects which are n-spherical in the following sense: $h_i(Y)$ is 0 for $i \neq n$, and it is stably free for $i = n$.

Theorem 1.7.1 now applies to give homotopy equivalences

$$\varinjlim_n hS.\widetilde{\mathcal{R}}^n(*,G) \longrightarrow \varinjlim_{(\Sigma)} hS.\mathcal{R}_f(*,G) \longleftarrow hS.\mathcal{R}_f(*,G)$$

(we have used lemma 2.2.4 to suppress the superscript (2) on $\mathcal{R}_f$ again).

It is plain from the preceding discussion, on the other hand, that $\mathcal{R}^n(*,G)$ is *strictly cofinal* in $\widetilde{\mathcal{R}}^n(*,G)$ in the sense of proposition 1.5.9, so the inclusion

$$hS.\mathcal{R}^n(*,G) \longrightarrow hS.\widetilde{\mathcal{R}}^n(*,G)$$

is a homotopy equivalence.

Finally it is also plain that the cofibrations in $\mathcal{R}^n(*,G)$ are *splittable up to weak equivalence* in the sense of theorem 1.8.1, so the map

$$\varinjlim_n hN.\mathcal{R}^n(*,G) \qquad \varinjlim_n hS.\mathcal{R}^n(*,G)$$

is a homotopy equivalence.

The proof of the proposition is now complete. □

Remark. The preceding argument can be varied a little. Namely instead of replacing $\widetilde{R}^n(*,G)$ by $R^n(*,G)$ as we have just done, we could also argue directly that

$$\varinjlim_n hN.\widetilde{R}^n(*,G) \longrightarrow \varinjlim_n hS.\widetilde{R}^n(*,G)$$

is a homotopy equivalence. Segal's theorem used elsewhere in the proof of the theorem then applies in the form of giving a homotopy equivalence

$$\Omega|hN.\widetilde{R}^n(*,G)| \quad \simeq \quad K_o'(Z[\pi_oG]) \times \varinjlim_k |hR_k^n(*,G)|$$

where $K_o'(Z[\pi_oG])$ denotes the subgroup of the class group given by the stably free modules (that subgroup is of course Z again).

The theorem itself can also be varied. Namely the category $R_f(*,G)$ may be enlarged to the category $R_{df}(*,G)$ of the objects *dominated by finite ones* (these are the objects which are retracts, up to homotopy, of finite ones). The theorem then goes through unchanged except that the restricted class group $K_o'(Z[\pi_oG])$ has to be replaced by the full class group $K_o(Z[\pi_oG])$. □

To complete the proof of the theorem we are still left to compare $BH_k^n(G)$ with $hR_k^n(*,G)$.

Let C denote any of the categories $hR_{hf}(*,G)$, $hR_f(*,|G|)$, $hR_{hf}(*,|G|)$. We blow it up to a simplicial category $C.$, $[m] \mapsto C_m$, where C_m is defined as the category whose objects are the same as those of C and whose morphisms are the m-parameter families of morphisms in C . That is, a morphism in C_m from Y to Z is a map

$$Y \longrightarrow Z^{\Delta^m}$$

in C (resp. similarly with Δ^m replaced by $|\Delta^m|$ in the topological case) or, what is the same, a map $Y\times\Delta^m/*\times\Delta^m \to Z$. Considering C as a simplicial category in a trivial way, we have a map $C \to C.$.

If $Y \in C$ we let C_Y , resp. $C._Y$, denote the connected component of C , resp. $C.$, containing Y , and $C.(Y)$ the simplicial subcategory of self-maps of Y in $C.$.

<u>Proposition</u> 2.2.5. In the topological case, the maps

$$C_Y \longrightarrow C._Y \longleftarrow C.(Y)$$

are homotopy equivalences. The same is true in the simplicial case provided that Y satisfies the Kan extension condition.

<u>Corollary</u>. There is a natural chain of homotopy equivalences

$$BH_k^n(G) \quad \simeq \quad |hR_k^n(*,G)| \ .$$

Proof. Let $C = hR_f(*,|G|)$ in the proposition, and $Y = V^k S^n \wedge |G|_+$. Then $|C.(Y)|$ is the same as $BH^n_k(G)$, by definition of the latter, and it is homotopy equivalent to $hR^n_k(*,|G|)$, by application of the proposition. On the other hand, the geometric realization map $hR^n_k(*,G) \to hR^n_k(*,|G|)$ is a homotopy equivalence by proposition 2.1.2.

□

Proof of proposition. By lemma 2.2.6 below, each of the (degeneracy) maps $C \to C_m$ is a homotopy equivalence. It follows (the realization lemma) that $C \to C.$ is a homotopy equivalence. Consequently, $C_Y \to C._Y$ is one, too.

In the topological case, the inclusion $C._Y \leftarrow C.(Y)$ is a homotopy equivalence by lemma 2.2.7 below.

In the simplicial case, that lemma does not apply to $C.$ directly, it only applies to the simplicial subcategory $C!$ of the objects which satisfy the Kan extension condition. It remains to see that the inclusion $C! \to C.$ is a homotopy equivalence. By the first part of the proposition we can reduce to showing that $C' \to C$ is a homotopy equivalence. This follows if we can find a functor $C \to C'$ together with a natural transformation from the identity functor. The desired functor is given by one of the standard devices of forcing the extension condition, namely the process of *filling horns* (which may be arranged in a G-equivariant way). □

Lemma 2.2.6. The map $C \to C_m$ is a homotopy equivalence.

Proof. Call this map j. We define a map $p: C_m \to C$. It is the identity on objects, and it takes a morphism $Y \times \Delta^m / * \times \Delta^m \to Z$ to the map $Y \to Z$ given by restriction to the last vertex of Δ^m. Then pj is the identity map on C. We will show that jp is homotopic to the identity map on C_m.

To construct the homotopy we use an auxiliary functor $F: C_m \to C_m$ which on objects is given by

$$Y \longmapsto Y \times \Delta^1 / * \times \Delta^1 .$$

To define F on morphisms we use the standard contraction of Δ^m, that is, the map $f: \Delta^m \times \Delta^1 \to \Delta^m$ whose restrictions to $\Delta^m \times 0$ and $\Delta^m \times 1$ are the identity map on Δ^m, and the projection of Δ^m into its last vertex, respectively. By definition now F takes a map $Y \times \Delta^m / * \times \Delta^m \to Z$ to the map given by

$$Y \times \Delta^1 \times \Delta^m \approx Y \times \Delta^m \times \Delta^1 \xrightarrow{(a,b)} Z \times \Delta^1$$

(or rather the induced map of quotients) where b is the projection $Y \times \Delta^m \times \Delta^1 \to \Delta^1$, and a is the composite map

$$Y \times (\Delta^m \times \Delta^1) \xrightarrow{Id \times f} Y \times \Delta^m \longrightarrow Z .$$

The point of considering F is that there are natural transformations $Id \to F$

and $jp \to F$. They are induced by the inclusions $Y \to Y\times\Delta^1/*\times\Delta^1$ taking Y to $Y\times 0$ and $Y\times 1$, respectively. In view of these natural transformations, each of the functors Id and jp is homotopic to F. Hence they are homotopic to each other.

□

In order to formulate the next lemma we need a little preparation. Let $C.$ be a simplicial category. We say it is *special* if all the categories C_m have the same objects, and the face and degeneracy maps are the identity on objects. By abuse we can then speak of the objects of $C.$, rather than objects in some fixed degree, and for any two objects Y and Z we have a simplicial set of morphisms, which we denote $C.(Y,Z)$.

As before we let $C.(Y)$ denote the simplicial category of endomorphisms of Y We must carefully distinguish between $C.(Y)$ and $C.(Y,Y)$. For they have different geometric realizations (the geometric realization of the former takes the composition law into account, whereas that of the latter does not).

We will say that two objects Y and Z are *strictly homotopy equivalent* if there exist $f \in C_o(Y,Z)$ and $g \in C_o(Z,Y)$ so that the composite gf is homotopic, in the simplicial set $C.(Y,Y)$, to the identity map on Y, and so that similarly the composite fg is homotopic in $C.(Z,Z)$ to the identity map on Z.

Lemma 2.2.7. Let $C.$ be a special simplicial category in which all objects are strictly homotopy equivalent to each other. Then for every object Y the inclusion $C.(Y) \to C.$ is a homotopy equivalence.

We deduce the lemma from a version of Quillen's theorem A for simplicial categories. In the case of special simplicial categories it takes the following form, cf. [15].

Criterion. Let $F\colon \mathcal{D}. \to C.$ be a map of special simplicial categories. A sufficient condition for F to be a homotopy equivalence is that for every object Z of $C.$ the simplicial category $F./Z : [m] \mapsto F_m/Z$ is contractible.

Proof of lemma. By the criterion applied to the inclusion $F\colon C.(Y) \to C.$ it suffices to show that for every Z the simplicial category $F./Z$ is contractible

Suppose that $f \in C_o(Z,Z')$. It induces a map $f_*\colon F./Z \to F./Z'$,

$$(u \in C_m(Y,Z)) \longmapsto (d^*(f)\, u \in C_m(Y,Z'))$$

where d^* denotes the (degeneracy) map induced by $d\colon [m] \to [0]$

Suppose next that $f_1 \in C_1(Z,Z')$, and let f and f' be its faces in $C_o(Z,Z')$. Then we claim that f_* and f'_* are homotopic. Indeed, a simplicial homotopy from f_* to f'_* is given (cf. the proof of lemma 1.4.1 for a discussion of simplicial homotopies) by the natural transformation which takes $a: [m] \to [1]$ to the map $F_m/Z \to F_m/Z'$,

$$(u \in \mathcal{C}_m(Y,Z)) \longmapsto (a^*(f) u \in \mathcal{C}_m(Y,Z'))$$

By induction we conclude that if f and f'' are in the same connected component of $\mathcal{C}.(Z,Z')$ then they induce homotopic maps $F./Z \to F./Z'$.

In turn we conclude that if Z_o and Z_1 are strictly homotopy equivalent to each other, then $F./Z_o$ and $F./Z_1$ are homotopy equivalent.

Applying the hypothesis of the lemma now we obtain that, for every Z , $F./Z$ is homotopy equivalent to $F./Y$.

But $F./Y$ is the same as $Id_{\mathcal{C}.}/Y : [m] \to Id_{\mathcal{C}_m}/Y$. This is a simplicial object of contractible categories (each has a terminal object). Hence it is contractible. We are done. □

The theme of this section is that much of the material of the preceding two sections can be redone in a 'linearized' setting. This leads to considering a K-theory of simplicial rings, and specifically, to comparing several definitions of it. In the case of discrete rings the K-theory is the same as Quillen's.

There is a natural transformation, *linearization*, from the 'non-linear' to the 'linear' setting. We record the plausible fact that, up to homotopy, the induced map in K-theory does not depend on which particular definition of K-theory is used.

Let R be a simplicial ring (with 1). By a *module* over R is meant a simplicial abelian group A together with a (unital and associative) action of R, that is, a map $A \otimes R \to A$ (degreewise tensor product). We let $M(R)$ denote the category of these modules and their R-linear maps.

A simplicial set Y gives rise to a module $R[Y]$ where $(R[Y])_n = R_n[Y_n]$, the free R_n-module generated by Y_n. By the *attaching of a n-cell* to a module A is meant the formation of a pushout of the kind

$$A \longleftarrow R[\partial\Delta^n] \longrightarrow R[\Delta^n] \ .$$

We say that B is *obtainable from* A *by attaching of cells* if it can be built up by this process together with, perhaps, direct limit; we will also refer to this situation by saying that $A \to B$ is a *free map* (the notion is the same as that of a free map in [6]).

We define $M_f(R)$ to be the full subcategory of the modules which are obtainable from the zero module by attaching of finitely many cells. This is a category with cofibrations (free maps) and weak (homotopy) equivalences.

More generally, we define $M_{hf}(R)$ as the category given by the modules obtainable from 0 by attaching of perhaps infinitely many cells, but homotopy equivalent to some module in $M_f(R)$. Again this is a category with cofibrations and weak equivalences, in the same way.

$M_f(R)$ and $M_{hf}(R)$ give rise to the same K-theory, that is, the map

$$\Omega|hS.M_f(R)| \longrightarrow \Omega|hS.M_{hf}(R)$$

This results from

The approximation theorem applies to the map $M_f(R) \to M_{hf}(R)$

Proof. The argument is the same as that in the first part of the proof of proposition 2.1.1. The point is that the Whitehead theorem is available for objects in $M_f(R)$ or $M_{hf}(R)$ (one just constructs any desired map by induction on the generating simplices $\Delta^n \cdot 1$, $1 \in R$; it is not even necessary to subdivide Δ^n in the process since simplicial abelian groups satisfy the Kan extension condition). □

Let $M_{k\times k}(R)$ denote the simplicial ring of the $k\times k$ matrices in R . We define $\hat{GL}_k(R)$ to be the multiplicative simplicial monoid given by the matrices in $M_{k\times k}(R)$ which are invertible *up to homotopy*. Let $B\hat{GL}_k(R)$ denote the classifying space.

<u>Theorem</u> 2.3.2. There is a natural chain of homotopy equivalences

$$\Omega|hS.M_f(R)| \simeq K'_o(\pi_o R) \times \lim_{\vec{k}} B\hat{GL}_k(R)^+ .$$

Here $K'_o(\pi_o R)$ denotes the subgroup of the class group of the ring $\pi_o R$ given by the free modules (it is cyclic, and in cases of interest it is usually Z).

Remark. There is a variant of the theorem where the category $M_f(R)$ is replaced by the larger category $M_{df}(R)$ of the objects *dominated by finite ones*; that is, the objects which are retracts of such in $M_{hf}(R)$. In that case the restricted class group $K'_o(\pi_o R)$ in the theorem has to be replaced by the full class group $K_o(\pi_o R)$.

Proof of theorem. Define $M^n_k(R)$ to be the full subcategory of $M_f(R)$ given by the objects which are *n-spherical of rank* k ; that is, the objects weakly equivalent to $R[\amalg^k \Delta^n]/R[\amalg^k \partial\Delta^n]$.

It will be shown below (proposition 2.3.5) that there is a natural homotopy equivalence

$$B\hat{GL}_k(R) \simeq |hM^n_k(R)|$$

compatible with suspension (the passage from n to $n+1$ on the right hand side).

Define $M^n(R)$ as the union of the categories $M^n_k(R)$. According to Segal [11] we have a homotopy equivalence

$$\Omega|hN.M^n(R)| \simeq K'_o(\pi_o R) \times \lim_{\vec{k}} |hM^n_k(R)|^+ .$$

Combining with the former homotopy equivalence we obtain one

$$\Omega|hN.M^n(R)| \simeq K'_o(\pi_o R) \times \lim_{\vec{k}} B\hat{GL}_k(R)^+ ,$$

compatible with suspension. The proof of the theorem has thus been reduced to the following proposition.

Proposition 2.3.3. There is a natural chain of homotopy equivalences

$$\varinjlim_{n} hN.M^n(R) \simeq hS.M_f(R) .$$

The proposition is actually true without passage to the limit on the left, but the limit makes for easier quoting of the general results (which were designed for different applications).

The proof is an application of theorems 1.7.1 and 1.8.1. To make these theorems applicable we have to check some things first. Let us define

$$h_*M = \pi_*(M \otimes_R \pi_o R) .$$

Lemma 2.3.4. Let $M \in M_{hf}(R)$. If $\pi_i M = 0$ for $i < n$ then the map $\pi_n M \to h_n M$ is an isomorphism.

Proof. If M and M' are right and left R-modules, respectively, there is a *derived tensor product* $M \overset{L}{\otimes}_R M'$, well defined up to homotopy [6 ,p.6.8]. If the module M happens to be 'free' (in the sense that $0 \to M$ is a *free map* — the objects of $M_{hf}(R)$ have that property, by definition) then the derived tensor product is represented by the actual tensor product $M \otimes_R M'$, by the corollary [6 ,p.6.10]. Therefore the spectral sequence (b) of theorem 6 [6 ,p.6.8] gives, in the case at hand, a first quadrant spectral sequence

$$E^2_{p,q} = \mathrm{Tor}_p^{\pi_* R}(\pi_* M, \pi_o R)_q \Longrightarrow \pi_{p+q}(M \otimes_R \pi_o R)$$

where $\mathrm{Tor}_p(..)_q$ denotes the degree q part of the graded abelian group $\mathrm{Tor}_p(..)$. Now $\pi_i M = 0$ for $i < n$, so $E^2_{p,q} = 0$ for $q < n$, and we obtain an isomorphism $\pi_n(M \otimes_R \pi_o R) \approx E^2_{o,n}$, proving the lemma. □

Proof of proposition. The argument is precisely the same as that of the proof of proposition 2.2.2. Here is a brief account.

The objects of $M^n(R)$ may be characterized by the property that $h_i M$ is 0 for $i \neq n$, and free of finite rank over $\pi_o R$ for $i = n$. Let $\widetilde{M}^n(R)$ be the corresponding category with *free* replaced by *stably free*. Then all the hypotheses of section 1.7 are satisfied, so by theorem 1.7.1 we have homotopy equivalences

$$\varinjlim_{n} hS.\widetilde{M}^n(R) \longrightarrow \varinjlim_{(\Sigma)} hS.M_f(R) \longleftarrow hS.M_f(R) .$$

On the other hand, $M^n(R)$ is strictly cofinal in $\widetilde{M}^n(R)$, so the inclusion

$$hS.M^n(R) \longrightarrow hS.\widetilde{M}^n(R)$$

is a homotopy equivalence by proposition 1.5.9. And finally the cofibrations in $M^n(R)$ are splittable up to weak equivalence, so theorem 1.8.1 applies to show that

$$\varinjlim_{n} hN.M^n(R) \longrightarrow \varinjlim_{n} hS.M^n(R)$$

is a homotopy equivalence. By combining the homotopy equivalences we obtain the proposition. □

To complete the proof of the theorem we are now left to compare $|hM_k^n(R)$ and $B\hat{GL}_k(R)$.

Let us write C instead of $hM_f(R)$, for short. We blow up C to a simplicial category $C.$, $[m] \mapsto C_m$. The objects of C_m are the same as those of C , and the morphisms in C_m are the m-parameter families of morphisms in C . That is, a morphism in C_m from A to B is a map $A[\Delta^m] \approx A\otimes Z[\Delta^m] \to B$. Considering C as a simplicial category in a trivial way we have a map $C \to C.$.

If $A \in C$ we let C_A , resp. $C._A$, denote the connected component of C , resp. $C.$, containing A , and $C.(A)$ the simplicial category of self-maps of A in C.

Proposition 2.3.5. For every $A \in hM_f(R)$ there are homotopy equivalences

$$C_A \longrightarrow C._A \longleftarrow C.(A) .$$

Proof. The argument is similar to that of proposition 2.2.5

Corollary. There is a natural chain of homotopy equivalences, $B\hat{GL}_k(R) \simeq |hM_k^n(R)|$, compatible with suspension.

Proof. Let $A = A_k^n$ denote the module obtained by attaching k n-cells to zero,

$$A_k^n = R[\amalg^k \Delta^n]/R[\amalg^k \partial\Delta^n] .$$

We claim that the simplicial ring of self-maps of A_k^n is homotopy equivalent to $M_{k\times k}(R)$, independently of n . To see this we can reduce, by a direct sum argument, to the special case $k = 1$. Restricting to the generating simplex we then obtain an isomorphism

$$Map_R(A_1^n, A_1^n) \approx Map(\Delta^n/\partial\Delta^n, R[\Delta^n]/R[\partial\Delta^n]) .$$

But it is well known, and easy to prove, that the n-fold loop space of the simplicial abelian group $R[\Delta^n]/R[\partial\Delta^n]$ is R again, up to homotopy. For example consider the *horn* Λ^n , the union of all the faces of Δ^n except the last. Then $R[\Delta^n]/R[\Lambda^n]$ is contractible. Hence the short exact sequence

$$R[\Delta^{n-1}]/R[\partial\Delta^{n-1}] \longrightarrow R[\Delta^n]/R[\Lambda^n] \longrightarrow R[\Delta^n]/R[\partial\Delta^n]$$

gives a looping fibration. It follows from the claim that the simplicial monoid of self-equivalences of A_k^n is homotopy equivalent, as monoid, to $\hat{GL}_k(R)$. Hence $B\hat{GL}_k(R) \simeq |C.(A_k^n)|$. Applying the proposition now we obtain that the latter is homotopy equivalent to $|C_A| = |hM_k^n(R)|$. The corollary results. □

Remark. The theorem includes a description of the Quillen K-theory of a discrete ring in terms of chain complexes over that ring. For if R is discrete then a 'module' in the sense used above is really the same thing as a *simplicial module over* R . In view of the Dold-Kan theorem there is therefore an equivalence (it is given by the normalized chain complex functor) of the category $\mathcal{M}_f(R)$ with a category of chain complexes over R .

Below, in the context of linearization, it will be convenient to know that the foregoing material can be redone topologically rather than simplicially. We record this now.

As a technical point, we will want to know that the geometric realization functor commutes with finite products. Therefore products should be formed in the category of compactly generated spaces. As a result we will restrict ourselves to working in that category. For example, if we mention a topological abelian group it will be tacitly understood that the underlying topological space is compactly generated.

Let $|A|$ be a topological abelian group, not necessarily the geometric realization of a simplicial abelian group A , and $|X|$ a topological space, not necessarily the geometric realization of a simplicial set X either. In this situation we can form $|A|[|X|]$, the topological abelian group freely generated by $|X|$ over $|A|$. The underlying space is the space of linear combinations of the kind

$$a_1x_1 + \ldots + a_kx_k ,$$

subject to a suitable equivalence relation, and topologized accordingly. In detail, one forms

$$\textstyle\coprod_k |A|^k \times |X|^k \,/\sim$$

where the equivalence relation is generated by the rule that for every map of finite sets, $\theta\colon \underline{m} \to \underline{n}$, the two maps

$$|A|^n \times |X|^n \xleftarrow{\;\theta_* \times \mathrm{Id}\;} |A|^m \times |X|^n \xrightarrow{\;\mathrm{Id} \times \theta^*\;} |A|^m \times |X|^m$$

are to be equalized

If, in particular, $|R|$ is a topological ring, and $|X|$ a topological space, we can in this way obtain $|R|[|X|]$, the free $|R|$-module generated by $|X|$. The construction is compatible with geometric realization in the sense that if R is a simplicial ring, and X a simplicial set, then $|R|[|X|] \approx |R[X]|$.

We have the means now of defining the notion of the *attaching of a* n*-cell* to a $|R|$-module M . Namely this is the formation of a pushout of the kind

$$M \longleftarrow |R|[|\partial\Delta^n|] \longrightarrow |R|[|\Delta^n|] .$$

Starting from this notion we can proceed as in section 2.1 to carry over the definitions of $M_f(R)$ and $M_{hf}(R)$ to the topological context to obtain definitions of $M_f(|R|)$ and $M_{hf}(|R|)$.

Proposition 2.3.6. Let R be a simplicial ring. The approximation theorem applies to the geometric realization map $M_f(R) \to M_f(|R|)$

Proof. The argument is similar to that of proposition 2 2

Define $\widehat{GL}_k(|R|)$ as in the simplicial case; that is, it is the simplicial monoid of the homotopy-invertible matrices over $|R|$.

Corollary 2.3.7. Let R be a simplicial ring There is a natural chain of homotopy equivalences

$$\Omega|hS.M_f(|R|)| \simeq K_0'(\pi_0 R) \times \varinjlim_k B\widehat{GL}_k(|R|)^+ ,$$

and the chain is compatible, via geometric realization, to that of theorem 2.3.2

Proof. We consider the chain of maps in theorem 2.3.2 as consisting of three parts. The first part is the chain of maps between $\varinjlim hN.M^n(R)$ and $hS.M_f(R)$ in proposition 2.3.3. The preceding proposition applies to each map in the transformation from this chain to its topological analogue, so these maps are homotopy equivalences. As a result, since the maps in the former chain are homotopy equivalences, it follows that so are those in the latter.

The second part of the chain is Segal's homotopy equivalence of $\Omega|hN.M^n(R)|$ with $K_0'(\pi_0 R) \times \varinjlim |hM_k^n(R)|^+$. This is certainly compatible with its topological analogue.

The third part of the chain, finally, is given by the maps in proposition 2.3.5, resp. its corollary. There is a compatible chain of maps in the topological case, and the maps are homotopy equivalences by the version of proposition 2.3.5 in the topological case. □

Suppose now that G is a simplicial monoid. Let Z be the ring of integers There is an exact functor

$$R(*,G) \longrightarrow M(Z[G])$$
$$Y \longmapsto \tilde{Z}[Y] = Z[Y]/Z[*]$$

and hence an induced map in K-theory, the *linearization map*

$$\Omega|hS.R_f(*,G)| \longrightarrow \Omega|hS.M_f(Z[G])| .$$

On the other hand, the map of *rings up to homotopy* $\Omega^\infty S^\infty|G|_+ \to Z[|G|$ induces, by matrix multiplication, a map of H-spaces

$$\hat{GL}(\Omega^\infty S^\infty|G|_+) \longrightarrow \hat{GL}(Z[|G|]) \ .$$

This is de-loopable to a map of classifying spaces $B\hat{GL}(\Omega^\infty S^\infty|G|_+) \to B\hat{GL}(Z[|G|])$, well defined up to homotopy. Namely the latter is obtained by composing, in the limit with respect to n and k, the map

$$B\,Aut_{|G|}(V^kS^n\wedge|G|_+) \longrightarrow B\,Aut_{Z[|G|]}(\tilde{Z}[V^kS^n\wedge|G|_+])$$

with a homotopy inverse to the homotopy equivalence

$$B\hat{GL}_k(Z[|G|]) \approx B\,Aut_{Z[|G|]}(\tilde{Z}[V^kS^o\wedge|G|_+]) \longrightarrow B\,Aut_{Z[|G|]}(\tilde{Z}[V^kS^n\wedge|G|_+]) \ .$$

We can further compose with an inverse to the homotopy equivalence

$$B\hat{GL}_k(Z[G]) \longrightarrow B\hat{GL}_k(Z[|G|]) \ .$$

<u>Corollary</u> 2.3.8. The linearization map corresponds, under the homotopy equivalences of theorems 2.2.1 and 2.3.2, to the map

$$Z \times B\hat{GL}(\Omega^\infty S^\infty|G|_+)^+ \longrightarrow Z \times B\hat{GL}(Z[G])^+ \ .$$

As indicated in [14], this result can be used to obtain numerical information. For example, as a consequence of the fact that the map $\Omega^\infty S^\infty|G|_+ \to Z[|G|]$ is a rational homotopy equivalence as well as an isomorphism on π_o, it follows that the map of the corollary is a rational homotopy equivalence.

Proof of corollary. This is a matter of checking, similar to the preceding corollary. We regard the chain of homotopy equivalences in theorem 2.2.1 as consisting of three parts. The first part is the chain of maps between $\varinjlim hN.R^n(*,G)$ and $hS.R_f(*,G)$ in proposition 2.2.2. This is compatible, by linearization, to the corresponding chain of maps between $\varinjlim hN.M^n(Z[G])$ and $hS.M_f(Z[G])$ in proposition 2.3.3.

The second part of the chain is Segal's homotopy equivalence of $\Omega|hN.R^n(*,G)|$ with $Z \times \varinjlim |hR^n_k(*,G)|^+$. This is compatible to its linear analogue, the homotopy equivalence between $\Omega|hN.M^n(Z[G])|$ and $Z \times \varinjlim |hM^n_k(Z[G])|^+$.

The third part, finally, is the commutative diagram of homotopy equivalences, with the notation as in proposition 2.2.5, and Y the simplicial version of $V^kS^n\wedge|G|_+$,

$$\begin{array}{ccccc} C_Y & \longrightarrow & C._Y & \leftarrow\!-\,-\,- & \\ \downarrow & & \downarrow & & \vdots \\ C_{|Y|} & \longrightarrow & C._{|Y|} & \longleftarrow & C.(|Y|) \end{array}$$

The notation of the broken arrows here simply means that these arrows are missing

For we have not tried to put anything into the upper right corner. Such a Y would have to satisfy the Kan extension condition (proposition 2.2.5) and it would also have to fit into a sequence of Y's related to each other by some kind of suspension.

At any rate, the diagram is compatible, by linearization, to one

$$\begin{array}{ccccc} C_A & \longrightarrow & C._A & \longleftarrow & C.(A) \\ \\ \check{C}_{|A|} & \longrightarrow & C.\check{}_{|A|} & \longleftarrow & C.(|A|) \end{array}$$

where the upper row is that of proposition 2.3.5, with $A = Z[Y]$, and the lower row is the topological analogue of it. □

To conclude the topic of linearization let us briefly mention that, in the case of $A(X)$, there is a description of the linearization map which uses only spaces over X , not the loop group of X . The map is defined in terms of an exact functor $R(X) \to R^{ab}(X)$ where $R^{ab}(X)$ denotes the category of abelian group objects in $R(X)$.

In particular this means that, for connected X , there is a description of $K(Z[G(X)])$ in terms of $R^{ab}(X)$. To obtain that description, one *defines* a notion of weak equivalence in $R^{ab}(X)$ so that the map $R^{ab}(X) \to R^{ab}(*,G) \approx M(Z[G])$ corresponding to that of proposition 2.1.4, respects *and detects* weak equivalences. The argument of proposition 2.1.4 may then be adapted.

3. THE WHITEHEAD SPACE $Wh^{PL}(X)$, AND ITS RELATION TO $A(X)$.

3.1. Simple maps and the Whitehead space.

A map of simplicial sets is called *simple* if its geometric realization has contractible point inverses. We will admit here that simple maps form a category, that is, that a composite of simple maps is simple again, and that the gluing lemma is valid for simple maps. Proofs of these facts may be found e.g. in [16] where also a few other characterizations of simple maps are given.

If X is a simplicial set we denote by $C(X)$ the category of the cofibrant objects under X ; the objects are the pairs (Y,s) , $s\colon X \rightarrowtail Y$, and the morphisms from (Y,s) to (Y',s') are the maps $f\colon Y \to Y'$ with $fs = s'$.

As before we let $R(X)$ denote the category of the triples (Y,r,s) , $rs = \mathrm{Id}_X$.

In either case, the subscript 'f' will denote the subcategory of the *finite* objects (where Y is generated, as simplicial set, by the simplices of $s(X)$ together with finitely many other simplices) and the superscript 'h' will denote the subcategory of the *homotopically trivial* objects (where $s\colon X \to Y$ is a weak homotopy equivalence). Finally the prefix 's' will denote the subcategory of the *simple maps*.

The category $sC^h_f(X)$ is of interest because of its role in the classification of PL manifolds and their automorphisms [2] [3] [16]; cf. also [15] and especially the proof of proposition 5.5 in that paper.

By the *Whitehead space* (the PL Whitehead space, to be precise) is meant a space whose fundamental group turns out to be the Whitehead group (the Whitehead group of $\pi_1 X$, that is, if X is connected) and which can be obtained from the (classifying space of the) category $sC^h_f(X)$ by de-looping, as follows.

In the language of section 1.8, the category $C^h_f(X)$ may be regarded as a category with sum (gluing at X) and weak equivalences (simple maps). Hence the group completion in the sense of Segal, the simplicial category $sN.C^h_f(X)$, is defined.

<u>Proposition</u> 3.1.1. There is a natural homotopy equivalence

$$|sC^h_f(X)| \simeq \Omega|sN.C^h_f(X)| .$$

Proof. Thanks to Segal [11] one knows that the canonical map from $|sC^h_f(X)|$ to $\Omega|sN.C^h_f(X)|$ is a homotopy equivalence if the H-space $|sC^h_f(X)|$ is *group-like* or, what amounts to the same thing, if the monoid $\pi_o|sC^h_f(X)|$ is a group. But it is well known that this is the case, cf. e.g. [16] for a proof. □

The main goal of this section is to prove the result (theorem 3.1.7 below) that the *sum construction* in $sN.C^h_f(X)$ can be traded for the *cofibration construction*; that is, that 'N.' can be replaced by 'S.' . In order for this replacement to make sense it is necessary to trade 'C' for 'R' first, that is, to impose structural retractions throughout. We also need an auxiliary construction; its purpose is to prevent the homotopy property of the functor $X \mapsto sN.C^h_f(X)$ from being lost upon transition from 'C' to 'R' .

Let F be a functor defined on the category of simplicial sets, with values in a category $\mathcal{B}$, say. We associate to it another functor $\check{F}$, with values in the category of simplicial objects in $\mathcal{B}$,

$$\check{F}(X) = ([n] \mapsto F(X^{\Delta^n}))$$

where X^{Δ^n} denotes the simplicial set of maps $\Delta^n \to X$.

Remark. In cases where the name of the functor is not F but something lengthy, such as for example $sN.C^h_f$, the notation $\check{F}(X)$ would be awkward. We will therefore use instead the notation $F(X^{\Delta^\cdot})$ on such occasions. □

Using the identification of $F(X)$ with $F(X^{\Delta^o})$, and considering objects of $\mathcal{B}$ as simplicial objects in a trivial way, we can define a natural transformation from F to $\check{F}$.

Supposing now that in the receiving category $\mathcal{B}$ it makes sense to speak of weak homotopy equivalences, we will say that the functor F *respects weak homotopy equivalences* if $X \xrightarrow{\sim} X'$ always implies $F(X) \xrightarrow{\sim} F(X')$.

Lemma 3.1.2. If F respects weak homotopy equivalences then the natural transformation $F \to \check{F}$ is a weak homotopy equivalence.

Proof. The (degeneracy) map $X^{\Delta^o} \to X^{\Delta^n}$ is a weak homotopy equivalence and therefore so is $F(X^{\Delta^o}) \to F(X^{\Delta^n})$, by assumption about F . We conclude with the realization lemma. □

Lemma 3.1.3. For any F , the functor $\check{F}$ preserves simplicial homotopies.

Proof. Let $X \to Y^{\Delta^1}$ be a simplicial homotopy. The claim is that one can naturally associate to it a simplicial homotopy of maps $\check{F}(X) \to \check{F}(Y)$. Such a simplicial homotopy may be identified to a natural transformation of functors on the category $\Delta/[1]$,

$$(a\colon [n] \to [1]) \longmapsto (\check{F}(X)_n \to \check{F}(Y)_n$$

The desired map on the right is defined as the composite map

$$F(X^{\Delta^n}) \longrightarrow F(Y^{\Delta^1 \times \Delta^n}) \longrightarrow F(Y^{\Delta^n})$$

where the first and second map are induced, respectively, by the homotopy $X \to Y^{\Delta^1}$ and by the map

$$\Delta^n \xrightarrow{(a_*, Id)} \Delta^1 \times \Delta^n$$

Lemma 3.1.4. Let $F(X) = sR^h_f(X)$. Then the functor $\check{F}$ respects weak homotopy equivalences. Similarly with the functors $sN.R^h_f(X)$ and $sS.R^h_f(X)$.

Proof. By a well known argument (which e.g. may be found in [16]) it suffices to show that $\check{F}(X) \to \check{F}(X')$ is a weak homotopy equivalence if X' is obtained from X by *filling a horn*, that is, if it is the pushout of a diagram $X \leftarrow \Lambda^n_i \to \Delta^n$ where Λ^n_i is the i-*th horn* in Δ^n , the union of all the faces except the i-th. The idea of the following argument is to construct, in this situation, a deformation retraction of $\check{F}(X')$ to $\check{F}(X)$ by using the preceding lemma. Since it is not true, in general, that X is a deformation retract of X' by a simplicial homotopy, we must subdivide first.

Let Sd denote the subdivision functor for simplicial sets, and Sd_k its k-fold iteration. One knows that the subdivision of a simple map is simple again, cf. [16], so we can use Sd_2 , say, to define a map

$$\Phi : \ sR^h_f(X') \longrightarrow sR^h_f(Sd_2X') \ .$$

We compose with the map $f_*\colon sR^h_f(Sd_2X') \to sR^h_f(X')$ induced by pushout with $f\colon Sd_2X' \to X'$ (the composite of the 'last vertex map' $Sd(X'') \to X''$ with itself). The composite map on $sR^h_f(X')$ then is homotopic to the identity. For, it takes (Y,r,s) to

$$Sd_2Y \cup_{Sd_2X'} X' \ ,$$

with the appropriate structure maps, and the desired homotopy is given by the natural transformation to the identity functor induced from $Sd_2Y \to Y$, which is a simple map, cf. [16].

As shown below, $f\colon Sd_2X' \to X'$ is simplicially homotopic, relative to Sd_2X , to a map into X . Applying the preceding lemma we thus obtain a simplicial homotopy of the map $\check{f}_*$. We conclude that there is a map homotopic to the identity on $sR^h_f(X'^{\Delta^\bullet})$, namely $\check{f}_*\Phi$, which is also homotopic to a map into $sR^h_f(X^{\Delta^\bullet})$. The latter homotopy is relative to the 'identity' on $sR^h_f(X^{\Delta^\bullet})$; more precisely, the homotopy is constant on the analogue of the map $\check{f}_*\Phi$ constructed from X instead of X' . So we can draw the desired conclusion that the map $sR^h_f(X^{\Delta^\bullet}) \to sR^h_f(X'^{\Delta^\bullet})$

is a weak homotopy equivalence.

We are left to show that $Sd_2X' \to X'$ is simplicially homotopic, relative to Sd_2X, to a map into X. Since the subdivision functor commutes with pushouts, this reduces to the following special case.

Assertion. The map $Sd_2\Delta^n \to \Delta^n$ is simplicially homotopic, relative to $Sd_2\Lambda^n_i$, to a map into Λ^n_i.

To see this we note that there is a homotopy of maps $|Sd_1\Delta^n| \to |\Delta^n|$ which has all the asserted properties except that it is not quite the geometric realization of a simplicial homotopy; it is only a linear homotopy of *unordered* simplicial complexes. We can get the ordering right by subdividing once more. This gives a simplicial homotopy of maps $Sd_2\Delta^n \to Sd_1\Delta^n$. Composing with the map $Sd_1\Delta^n \to \Delta^n$ we obtain the desired homotopy from it.

The other cases of the lemma are handled similarly. □

<u>Lemma</u> 3.1.5. If X satisfies the Kan condition, the map $sR^h_f(X^{\Delta^\cdot}) \to sC^h_f(X^{\Delta^\cdot})$ a homotopy equivalence.

Proof. We define a simplicial category $[m] \mapsto sR^h_f(X)_m$ in which an object is one of $sC^h_f(X)$, say (Y,y), together with a map $Y\times\Delta^m \to X$ extending the projection $X\times\Delta^m \to X$. Since y is a weak homotopy equivalence, and X satisfies the extension condition, the simplicial set of those objects of $sR^h_f(X)_\cdot$ which arise from any particular (Y,y), is contractible. In other words, the simplicial set of objects of $sR^h_f(X)_\cdot$ maps by homotopy equivalence to the set of objects of $sC^h_f(X)$. Similarly, the simplicial set of morphisms of $sR^h_f(X)_\cdot$ maps by homotopy equivalence to the set of morphisms of $sC^h_f(X)$; and so on. It follows (the realization lemma) that the forgetful map $sR^h_f(X)_\cdot \to sC^h_f(X)$ is a homotopy equivalence.

Next we define a bisimplicial category $[m],[n] \mapsto sR^h_f(X)_{m,n} = sR^h_f(X^{\Delta^n})_m$. In view of the homotopy equivalence just established it follows, by the realization lemma, that the map $sR^h_f(X)_{\cdot\cdot} \to sC^h_f(X^{\Delta^\cdot})$ is a homotopy equivalence. Passing to the diagonal simplicial category of the bisimplicial category on the left (it has the same geometric realization, up to isomorphism) we obtain

$$\operatorname{diag} sR^h_f(X)_{\cdot\cdot} \longrightarrow sC^h_f(X^{\Delta^\cdot}) .$$

The lemma now results by checking that $\operatorname{diag} sR^h_f(X)_{\cdot\cdot}$ contains $sR^h_f(X^{\Delta^\cdot})$ as a deformation retract, and that the map of the lemma is the restriction of the latter homotopy equivalence.

An object of $sR^h_f(X)_{n,n}$ consists of an injective map $X^{\Delta^n} \to Y$ (with a finiteness condition) together with a map $Y\times\Delta^n \to X^{\Delta^n}$ which on $X^{\Delta^n}\times\Delta^n$ restricts to the projection. The object is in the subcategory $sR^h_f(X^{\Delta^n})$ if the map on $Y\times\Delta^n$ itself factors through the projection.

Passing to the adjoint, we can rewrite the map as $Y \to X^{\Delta^n \times \Delta^n}$. The desired simplicial homotopy now is induced by a simplicial deformation retraction of $[n] \mapsto X^{\Delta^n \times \Delta^n}$ to $[n] \mapsto X^{\Delta^n}$. Cf. e.g. [16] for a description of the homotopy.

Lemma 3. 6. If X satisfies the Kan condition, the forgetful map

$$\begin{array}{ccc} sS_n R^h_f(X^{\Delta^\cdot}) & \longrightarrow & sS_{n-1} R^h_f(X^{\Delta^\cdot}) \times sR^h_f(X^{\Delta^\cdot}) \\ (Y_1 \rightarrowtail \ldots \rightarrowtail Y_{n-1} \rightarrowtail Y_n & & Y_1 \rightarrowtail \ldots \rightarrowtail Y_{n-1} \; , \; Y_n/Y_{n-1} \end{array}$$

is a homotopy equivalence.

Proof. Define a category $s\tilde{S}_n R^h_f(X)$ just as $sS_n R^h_f(X)$ except that there is no structural retraction on the object Y_n in the filtration $Y_o \rightarrowtail \ldots \rightarrowtail Y_{n-1} \rightarrowtail Y_n$ There is a forgetful map

$$sS_n R^h_f(X^{\Delta^\cdot}) \longrightarrow s\tilde{S}_n R^h_f(X^{\Delta^\cdot})$$

which forgets the structural retraction in question. This forgetful map is a homotopy equivalence as one sees by a straightforward adaption of the argument of the preceding lemma. Consequently (and in view of the preceding lemma) the assertion of the lemma is equivalent to the assertion that the map

$$s\tilde{S}_n R^h_f(X^{\Delta^\cdot}) \longrightarrow sS_{n-1} R^h_f(X^{\Delta^\cdot}) \times sC^h_f(X^{\Delta^\cdot})$$

is a homotopy equivalence. By the realization lemma this follows if we can show it degreewise, for fixed m . Writing X instead of X^{Δ^m} now, we are reduced to showing that the map

$$s\tilde{S}_n R^h_f(X) \longrightarrow sS_{n-1} R^h_f(X) \times sC^h_f(X)$$

is a homotopy equivalence.

Let us denote the components of this map by p and q , respectively, and the section of the map q by i . In order to show that (p,q) is a homotopy equivalence, it will suffice to show that the sequence

$$sC^h_f(X) \xrightarrow{\ i\ } s\tilde{S}_n R^h_f(X) \xrightarrow{\ p\ } sS_{n-1} R^h_f(X)$$

is a fibration, up to homotopy. We use Quillen's theorem B [8] to prove this. We proceed to show that the theorem applies, in its version for left fibres, to the map p .

Let $(Y_1 \to \ldots \to Y_{n-1})$ be an object of $sS_{n-1} R^h_f(X)$. An object of the category $p/(Y_1 \to \ldots \to Y_{n-1})$ consists of an object $(Y'_o \to \ldots \to Y'_{n-1} \to Y'_n)$ of $s\tilde{S}_n R^h_f(X)$ together with a map g , say, in $sS_{n-1} R^h_f(X)$, the (vertical) transformation

$$\begin{array}{ccc} Y'_o \to & & \to Y'_{n-1} \\ \downarrow & & \downarrow \\ Y_o \to & & \to Y_{n-1} \end{array}$$

Let $p/(Y_o \to .. \to Y_{n-1})'$ denote the subcategory of the objects for which the structural map g is the identity map. It is a deformation retract of $p/(Y_o \to .. \to Y_{n-1})$; in fact, a deformation retraction is given by pushout with g .

On the other hand, $p/(Y_o \to .. \to Y_{n-1})'$ is isomorphic to $sC_f^h(Y_{n-1})$. As shown in [16], the functor $X \mapsto sC_f^h(X)$ respects weak homotopy equivalences. Hence the structural inclusion $X \to Y_{n-1}$ induces a homotopy equivalence $sC_f^h(X) \to sC_f^h(Y_{n-1})$. It results that the maps in $sS_{n-1}R_f^h(X)$ induce homotopy equivalences of the left fibres. Thus theorem B applies, showing that for every $(Y_o \to .. \to Y_{n-1})$ the square

$$\begin{array}{ccc} p/(Y_o \to .. \to Y_{n-1}) & \longrightarrow & s\tilde{S}_n R_f^h(X) \\ \downarrow & & \downarrow \\ Id/(Y_o \to .. \to Y_{n-1}) & \longrightarrow & sS_{n-1} R_f^h(X) \end{array}$$

is homotopy cartesian. In particular this is so for the distinguished object $(X \to .. \to X)$. We saw above that $p/(X \to .. \to X)$ contains as a deformation retract a subcategory isomorphic to $sC_f^h(X)$. Under the horizontal map in the square this subcategory projects to the image of the inclusion map i , and under the vertical map it projects trivially into the contractible category $Id/(X \to .. \to X)$. We obtain that the maps i and p form a homotopy fibration, as claimed. □

Theorem 3.1.7. Let X be a simplicial set. There are homotopy equivalences

$$sN.C_f^h(X) \longrightarrow sN.C_f^h(X^{\Delta^\cdot}) \longleftarrow sN.R_f^h(X^{\Delta^\cdot}) \longrightarrow sS.R_f^h(X^{\Delta^\cdot}) .$$

Proof. It is shown in [16] that the functor $X \mapsto sC_f^h(X)$ respects weak homotopy equivalences. By lemma 3.1.2 therefore the map from $sC_f^h(X)$ to $sC_f^h(X^{\Delta^\cdot})$ is a homotopy equivalence, and consequently also $sN.C_f^h(X) \to sN.C_f^h(X^{\Delta^\cdot})$, in view of the realization lemma. To proceed we choose a weak equivalence $X \to X'$ where X' is a simplicial set satisfying the Kan condition. Then all maps in the transformation of the chain of the theorem to the corresponding chain with X replaced by X' are weak equivalences by lemma 3.1.4. Thus we can reduce to proving the theorem for simplicial sets which actually satisfy the Kan condition. Applying lemmas 3.1.5 and 3.1.6 now to the second and third map, respectively, we obtain that these maps are homotopy equivalences degreewise in the $N.$, resp. $S.$, directions. We conclude with the realization lemma. □

3.2. The homology theory associated to A(*) .

Let F be a functor defined on the category of simplicial sets, with values in some category of spaces. We say F is *excisive* if it satisfies the following two axioms.

(Limit). F commutes with direct limit.

(Excision). If $X_o \to X_1$ is a cofibration, and $X_o \to X_2$ any map, then the square

$$\begin{array}{ccc} F(X_o) & \longrightarrow & F(X_2) \\ \downarrow & & \downarrow \\ F(X_1) & \longrightarrow & F(X_1 \cup_{X_o} X_2) \end{array}$$

is homotopy cartesian.

We say F is a *homological functor* (or a *homology theory*) if, in addition to being excisive, it also satisfies

(Homotopy). If $X \to X'$ is a weak homotopy equivalence then so is $F(X) \to F(X')$.

Recall (the preceding section) that $\check{F}(X) = F(X^{\Delta^\cdot})$ denotes the functor

$$X \longmapsto ([n] \mapsto F(X^{\Delta^n})) .$$

The purpose of this section is to prove the following result.

Theorem 3.2.1. The functor $X \mapsto sS.R_f(X^{\Delta^\cdot})$ is a homology theory.

Addendum 3.2.2. The functor $X \mapsto \Omega|sS.R_f(X^{\Delta^\cdot})|$ may be identified, up to a natural chain of maps, to the homology theory associated to A(*) .

In fact, the chain is given by the maps (of loop spaces of)

$$([n] \mapsto sS.R_f(X^{\Delta^n})) \longleftarrow ([n] \mapsto sS.R_f(X_n)) \longrightarrow ([n] \mapsto hS.R_f(X_n))$$

where $X = ([n] \mapsto X_n)$ and where the first map is induced by the identification $X_n = (X^{\Delta^n})_o$. Each of the three terms is a homology theory. In the first case this is so by the theorem, and in the second and third cases, the terms are the homology theories associated to the Γ-spaces with underlying spaces $sS.R_f(*)$ and $hS.R_f(*)$, respectively (cf. e.g. [13] for a detailed description of the homology theory associated to a (special) Γ-space). Given the fact that the three terms are homology theories, and connected, the proof that the maps are homotopy equivalences can be reduced to checking the case $X = *$. In that case, the first map is an isomorphism, while the second map is the inclusion $sS.R_f(*) \to hS.R_f(*)$. There does not seem to exist a direct proof that the latter map is a homotopy equivalence, but an indirect proof is provided by theorem 3.3.1, below, together with the fact that $sS.R_f^h(*)$ is contractible (which, e.g., follows from proposition 1.3.1).

In order to prove the theorem it will suffice to prove the following two propositions 3.2.3 and 3.2.4.

Proposition 3.2.3. The functor $X \mapsto sS.R_f(X)$ is excisive.

Proof. First, it is clear that the functor commutes with direct limit (up to isomorphism).

Next, suppose that $X_o \to X_1$ is an injective map. Pullback with it defines a map $R_f(X_1) \to R_f(X_o)$ which respects simple maps. The inclusion-induced map $R_f(X_o) \to R_f(X_1)$ also respects simple maps. Composing the two we therefore obtain a subfunctor f of the identity functor on $R_f(X_1)$ which is *exact*, and hence a cofibration sequence of exact functors $f \to \mathrm{Id} \to f'$ where f' is defined as the quotient $f' = \mathrm{Id}/f$. Let $R_f(X_1,X_o)$ be defined as the category of the objects (Y,r,s) in $R_f(X_1)$ having *support away from* X_o; that is, having the property that the pullback

$$X_o \times_{X_1} Y$$

is not bigger than X_o. Then f' takes values in $R_f(X_1,X_o)$, and it restricts to the identity map on that subcategory. Applying the additivity theorem to the cofibration sequence $f \to \mathrm{Id} \to f'$ now, we obtain a homotopy equivalence of $sS.R_f(X_1)$ with the product $sS.R_f(X_o) \times sS.R_f(X_1,X_o)$. In particular, therefore, the sequence

$$sS.R_f(X_o) \longrightarrow sS.R_f(X_1) \longrightarrow sS.R_f(X_1,X_o)$$

is a fibration, up to homotopy.

Applying this consideration in the situation of the excision axiom, we obtain a diagram of homotopy fibrations

$$\begin{array}{ccccc}
sS.R_f(X_o) & \longrightarrow & sS.R_f(X_1) & \longrightarrow & sS.R_f(X_1,X_o) \\
\downarrow & & \downarrow & & \downarrow \\
sS.R_f(X_2) & \longrightarrow & sS.R_f(X_1 \cup_{X_o} X_2) & \longrightarrow & sS.R_f(X_1 \cup_{X_o} X_2, X_2)
\end{array} .$$

The vertical map on the right is an isomorphism (an inverse is induced by pullback). It follows that the square on the left is homotopy cartesian, as asserted by the excision axiom. This completes the proof. □

Proposition 3.2.4. Let F be an excisive functor, and suppose that $F(X)$ is connected for every X. Then the associated functor $\check{F}$ is a homology theory.

The proof will be given at the end of this section. Together with the preparatory material, it occupies the rest of the section.

Remark. The artificial looking connectivity assumption comes from the fact that our proof of the proposition uses the following lemma 3.2.5. Some auxiliary condition, such as connectivity, is definitely needed in that lemma.

Lemma 3.2.5. Let

$$\begin{array}{ccc} W_{..} & \longrightarrow & X_{..} \\ \downarrow & & \downarrow \\ Y_{..} & \longrightarrow & Z_{..} \end{array}$$

be a commutative diagram of bisimplicial sets. Suppose that for every m the diagram of simplicial sets

$$\begin{array}{ccc} W_{m\cdot} & \longrightarrow & X_{m\cdot} \\ \downarrow & & \downarrow \\ Y_{m\cdot} & \longrightarrow & Z_{m\cdot} \end{array}$$

is homotopy cartesian. Suppose further that for every m the simplicial sets $Y_{m\cdot}$ and $Z_{m\cdot}$ are connected. Then the diagram of bisimplicial sets is also homotopy cartesian.

Remark. There are easy examples to show that the connectivity assumption cannot be dropped without replacing it by something else. Here is a particularly bad case. Take any pullback diagram of simplicial sets, and consider it as a diagram of bisimplicial sets in a trivial way. Then in each degree m we have a pullback diagram of sets, and certainly therefore a homotopy cartesian square (of sets !). But it rarely happens, on the other hand, that a pullback diagram of simplicial sets is also homotopy cartesian.

Proof of lemma. We deduce the lemma from a corresponding result for homotopy fibrations which we refer to as the *fibre realization lemma*. A proof may be found in [13]; for convenience we recall the statement here. By a *fibration up to homotopy* is meant here a sequence of maps of 'spaces' of some sort, $X \to Y \to Z$, having the property that, firstly, the composite map $X \to Z$ is a trivial map, with image $*$ say, and, secondly, the map from X to the homotopy fibre of $Y \to Z$ at $*$ is a weak homotopy equivalence. The fibre realization lemma says the following. Let $X_{..} \to Y_{..} \to Z_{..}$ be a sequence of maps of bisimplicial sets so that the composite map $X.. \to Z..$ is a trivial map. Suppose that, for every m, the sequence of maps of simplicial sets $X_m. \to Y_m. \to Z_m.$ is a fibration up to homotopy. Suppose further that for every m the simplicial set $Z_m.$ is connected. Then the sequence of bisimplicial sets, $X.. \to Y.. \to Z..$, is itself a fibration up to homotopy.

The idea for proving the present lemma comes from the fact that a homotopy cartesian square with connected bases can be characterized as a commutative square in which the homotopy fibres of the vertical maps are mapped to each other by homotopy equivalence. Using this one hopes to obtain a translation of the assertion which follows from the fibre realization lemma.

To get the details right, it is convenient to replace homotopy fibres by actual fibres in a systematic way. We need to know that there is a functorial way of turning a map of simplicial sets into a Kan fibration; e.g., the process of *filling horns* [1] will do. Using it we replace, for every m, the square of the lemma by a square

$$\begin{array}{ccc} W'_m. & \longrightarrow & X'_m. \\ \downarrow & & \downarrow \\ Y'_m. & \longrightarrow & Z'_m. \end{array}$$

in which the vertical maps are Kan fibrations. In view of the naturality of the construction, these squares still assemble to a square of bisimplicial sets

$$\begin{array}{ccc} W'.. & \longrightarrow & X'. \\ \downarrow & & \downarrow \\ Y'.. & \longrightarrow & Z'. \end{array}$$

There is a natural transformation from the old square to the new, and the maps $W.. \to W'..$, etc., are homotopy equivalences by the realization lemma. To prove the lemma it will therefore suffice to show that the new square is homotopy cartesian.

Choose any point of $Y'..$ (i.e., a compatible family of points in the $Y'_m.$) as a basepoint; denote it $*$. Let $\mathrm{fibre}(W'_m.\to Y'_m.)_{(*)}$ denote the actual fibre at $*$. Since $W'_m. \to Y'_m.$ is a Kan fibration, it is certainly true that the sequence

$$\mathrm{fibre}(W'_m.\to Y'_m.)_{(*)} \longrightarrow W'_m. \longrightarrow Y'_m.$$

is a fibration up to homotopy, for every m. In view of the fibre realization lemma we deduce from this that the sequence

$$\mathrm{fibre}(W'..\to Y'..)_{(*)} \longrightarrow W'.. \longrightarrow Y'..$$

is also a fibration up to homotopy, where the term on the left denotes the actual fibre again; the point is that $\mathrm{fibre}(W'..\to Y'..)_{(*)} \approx ([m] \mapsto \mathrm{fibre}(W'_m.\to Y'_m.)_{(*)})$.

There are similar fibrations if W' and Y' are replaced by X' and Z'.

We can now complete the proof of the lemma as follows. In view of the assumption of homotopy cartesianness we have, for every m, a homotopy equivalence

$$\text{fibre}(W'_m.\to Y'_m.)_{(*)} \longrightarrow \text{fibre}(X'_m.\to Z'_m.)_{(\text{Im}(*))} \ .$$

By the realization lemma this implies a homotopy equivalence

$$\text{fibre}(W!.\to Y!.)_{(*)} \longrightarrow \text{fibre}(X!.\to Z!.)_{(\text{Im}(*))} \ ,$$

and therefore, in view of the preceding, a homotopy equivalence of the vertical homotopy fibres in the $W!.-X!.-Y!.-Z!.$ square. Thus that square is homotopy cartesian, as was to be shown. □

The lemma enters into the proof of proposition 3.2.4 through the following consequence.

Proposition 3.2.6. Let $[m] \mapsto F_m$ be a simplicial object of functors. Suppose that $F_m(X)$ is connected for every m and every X. Then if the F_m are excisive, it follows that so is F, where $F(X) = ([m] \mapsto F_m(X))$.

Proof. The validity of the limit axiom for F is automatic. The validity of the excision axiom for F follows from its validity for the F_m by application of the preceding lemma. □

For later use we record the following here.

Lemma 3.2.7. Let F^1 and F^2 be excisive functors so that $F^1(X)$ and $F^2(X)$ are connected for every X. Let $F^1 \to F^2$ be a natural transformation. If the natural transformation is a weak equivalence in the cases $X = \Delta^n$, $n = 0, 1, 2, \dots$, then it is a weak equivalence in general.

Proof. By the limit axiom we can reduce to showing that $F^1(X) \to F^2(X)$ is a weak equivalence for finite X. Let X be obtained by attaching a 'last' simplex Δ^n to a simplicial set Y. In other words, choose an isomorphism of X to the pushout in a diagram

$$\begin{array}{ccc} \partial\Delta^n & \longrightarrow & \Delta^n \\ \downarrow & & \downarrow \\ Y & \longrightarrow & Y \cup_{\partial\Delta^n} \Delta^n \ . \end{array}$$

Applying F^1 to the diagram we obtain a homotopy cartesian square, in view of excision, and applying F^2 we obtain another. The map $F^1 \to F^2$ gives a map of the first homotopy cartesian square to the second. Since $F^1(X)$ and $F^2(X)$ are connected we conclude that, in order for $F^1(X) \to F^2(X)$ to be a homotopy equivalence, it suffices that the map is a homotopy equivalence in the other three cases. But in the case of Δ^n this is true by hypothesis, and in the cases of $\partial\Delta^n$ and Y it may be assumed true by induction. □

The crucial step in the proof of proposition 3.2.4 is the construction given in the following two definitions.

Definition 3.2.8. Let X be a simplicial set. Define $[k] \mapsto Cov(X)_k$ to be the simplicial object, in the category of simplicial sets, given by

$$Cov(X)_k = \coprod_{m,n} \Delta^m \times N_k(m,n) \times X_n$$

where $N_k(m,n)$ denotes the set of sequences in Δ,

$$[m] \to [m_1] \to \dots \to [m_{k-1}] \to [n] \qquad (\text{ k arrows}) .$$

To describe the simplicial structure one rewrites $Cov(X)$ as the bisimplicial set where a bisimplex in bidegree (q,k) consists of a sequence

$$[q] \to [m_o] \to [m_1] \to \dots \to [m_{k-1}] \to [m_k]$$

together with an element $x \in X[m_k]$. By definition now the i-th face map with respect to the k-direction is given by omitting $[m_i]$ from the sequence; except if $i = k$ in which case, in addition, the element $x \in X[m_k]$ must be taken to the appropriate element of $X[m_{k-1}]$. The degeneracy maps are given by the insertion of identity maps in the sequence.

Definition 3.2.9. Let F be a functor on the category of simplicial sets. Then

$$F^{\times}(X) = ([k] \mapsto F(Cov(X)_k)) .$$

Considering the simplicial set X as a simplicial object in a trivial way, we can define a natural transformation

$$Cov(X). \longrightarrow X ;$$

by definition, its restriction to $(\Delta^m, [m]\to\dots\to[n], x)$ is the composite map

$$\Delta^m \xrightarrow{([m]\to[n])_*} \Delta^n \xrightarrow{x} X .$$

Lemma 3.2.10. If X is a simplex Δ^p or, more generally, a disjoint union of simplices, then this map is the retraction in a simplicial deformation retraction from the simplicial object $[k] \mapsto Cov(X)_k$ to the trivial simplicial object $[k] \mapsto X$.

Proof. In the case $X = \Delta^p$, the simplicial homotopy is defined as the natural transformation on the category $\Delta/[1]$ taking $a\colon [k] \to [1]$ to the map of $Cov(\Delta^p)_k$ to Δ^p defined in the following way. The map a_* takes the sequence

$$[q] \to [m_o] \to [m_1 \to \dots \to [m_k] \to [p]$$

to the sequence

$$[q] \to [m_o] \to \dots \to [m_{i(a)}] \to [p] \overset{=}{\to} \dots \overset{=}{\to} [p]$$

where $i(a)$ is the largest of the $i \in [k]$ which are in the pre-image of $0 \in [1]$; if a takes $[k]$ entirely into $1 \in [1]$ then the image sequence is

$$[q] \to [p] \overset{=}{\to} \dots \overset{=}{\to} [p] \; .$$

The homotopy is similarly defined in the more general case where X is disjoint union of simplices. □

Considering the objects of the receiving category of the functor F as simplicial objects in a trivial way, we can define a natural transformation

$$F^{\times}(X) \longrightarrow F(X)$$

as the map which in degree k takes $F(Cov(X)_k)$ into $F(X)$ by the map induced from $Cov(X)_k \to X$.

Lemma 3.2.11. In the case where X is a simplex, or a disjoint union of such, the map $F^{\times}(X) \to F(X)$ is a (simplicial) homotopy equivalence.

Proof. The functor $F^{\times}$ has been defined by means of degreewise extension in the k-variable, so it preserves simplicial homotopies in the k-variable. The present lemma thus results from the preceding lemma. □

Remark. It is not difficult to show that $Cov(X). \to X$ is a *weak* homotopy equivalence for all X . On the other hand there seems little reason to suppose, in general, that the natural transformation $F^{\times}(X) \to F(X)$ is a weak equivalence for X which are not just disjoint unions of simplices.

Proposition 3.2.12. Suppose that F(X) is connected for all X , and that F is excisive. Then $F^{\times}(X) \to F(X)$ is a weak homotopy equivalence for all X .

Proof. The functor

$$X \longmapsto Cov(X)_k = \coprod_{m,n} \Delta^m \times N_k(m,n) \times X_I$$

preserves monomorphisms and pushouts. As a result, the functor

$$X \longmapsto F(Cov(X)_k)$$

is excisive since F is. Applying proposition 3.2.6 now we obtain that

$$X \longmapsto ([k] \mapsto F(Cov(X)_k))$$

is an excisive functor, too.

Thus $F^{\times}(X) \to F(X)$ is a map of excisive functors. By lemma 3.2.11 the map is a weak equivalence in the case $X = \Delta^n$. Consequently, by lemma 3.2.7, it is a weak equivalence in general.

□

Proposition 3.2.13. Let G be a functor satisfying that $G(X)$ is connected for all X. Suppose that G commutes with direct limit, and that it takes finite disjoint unions to products (up to homotopy); e.g., suppose that G is excisive. Then the functor $\check{G}^{\times}$ is excisive.

Proof. Let $X = ([j] \mapsto X_j)$. Then the functor $X \mapsto G(X_j)$ is excisive by hypothesis about G. By proposition 3.2.6 therefore the functor $X \mapsto ([j] \mapsto G(X_j))$ is excisive, too. We will show that the latter functor is weakly equivalent to $\check{G}^{\times}$. We show this by constructing an intermediate functor H and relating it to both.

Recalling the definitions

$$F^{\times}(X) = ([k] \mapsto F(Cov(X)_k)) \qquad \text{and} \qquad \check{F}(X) = ([j] \mapsto F(X^{\Delta^j}))$$

we unravel the definition of $\check{G}^{\times}$ as

$$\begin{aligned}\check{G}^{\times}(X) &= ([j] \mapsto ([k] \mapsto G(Cov(X^{\Delta^j})_k))) \\ &= ([j] \mapsto ([k] \mapsto G(\textstyle\coprod_{m,n} \Delta^m \times N_k(m,n) \times (X^{\Delta^j})_n))) \\ &\approx ([k] \mapsto ([j] \mapsto G(\textstyle\coprod_{m,n} \Delta^m \times N_k(m,n) \times (X^{\Delta^n})_j))) .\end{aligned}$$

We define the intermediate functor H by replacing X^{Δ^n} by X in the latter term,

$$H(X) = ([k] \mapsto ([j] \mapsto G(\textstyle\coprod_{m,n} \Delta^m \times N_k(m,n) \times X_j))) .$$

The projection $\Delta^n \to \Delta^0$ induces an inclusion $X \to X^{\Delta^n}$ and hence a map of $H(X)$ to $\check{G}^{\times}(X)$. We claim this map is a homotopy equivalence.

In fact, the map $([j] \mapsto X_j) \to ([j] \mapsto (X^{\Delta^n})_j)$ is a *simplicial* homotopy equivalence. The process of applying functors degreewise preserves simplicial homotopies. Hence the map

$$([j] \mapsto G(\textstyle\coprod_{m,n} \Delta^m \times N_k(m,n) \times X_j)) \longrightarrow ([j] \mapsto G(\textstyle\coprod_{m,n} \Delta^m \times N_k(m,n) \times (X^{\Delta^n})_j))$$

is a (simplicial) homotopy equivalence still. Applying the realization lemma with respect to the k-variable now, we conclude that $H(X) \to \check{G}^{\times}(X)$ is a (weak) homotopy equivalence.

To proceed, we rewrite $H(X)$ as

$$([j] \mapsto ([k] \mapsto G(\textstyle\coprod_{m,n} \Delta^m \times N_k(m,n) \times X_j))) .$$

The map

$$([k] \mapsto G(\textstyle\coprod_{m,n} \Delta^m \times N_k(m,n) \times X_j)) \longrightarrow ([k] \mapsto G(X_j))$$

is a (simplicial) homotopy equivalence by lemma 3.2.11. Applying the realization lemma with respect to the j-variable now we conclude that the map

$$H(X) \longrightarrow (\ [j] \mapsto (\ [k] \mapsto G(X_j)\ )\ )$$

is a (weak) homotopy equivalence. The target of this map is the simplicial object $[j] \to G(X_j)$ considered as a bisimplicial object in a trivial way. We are done. □

Proof of proposition 3.2.4. Recall, the claim is that if F is an excisive functor such that $F(X)$ is connected for every X, then the functor $\check{F}$ is a homology theory.

The main problem is to show that $\check{F}$ is excisive again. To see this we introduce the functor $F^{\times}$ (definition 3.2.9). The natural transformation $F^{\times} \to F$ is a weak homotopy equivalence in the situation at hand (proposition 3.2.12). By the realization lemma it follows that the natural transformation $\check{F}^{\times} \to \check{F}$ is a weak homotopy equivalence as well. Thus we can reduce to showing that the functor $\check{F}^{\times}$ is excisive. This was shown in proposition 3.2.13.

We are left to show now that the functor $\check{F}$ respects weak homotopy equivalences. By a well known argument (which e.g. may be found in [1]) it suffices to show that $\check{F}(X) \to \check{F}(X')$ is a homotopy equivalence if X' is obtained from X by filling a horn, that is, if there is a pushout diagram

$$\begin{array}{ccc} \Lambda^n_i & \longrightarrow & \Delta^n \\ \downarrow & & \downarrow \\ X & \longrightarrow & X' \end{array} .$$

$\check{F}$ applied to this diagram gives a homotopy cartesian square, by excision, so we can reduce further to showing that $\check{F}(\Lambda^n_i) \to \check{F}(\Delta^n)$ is a homotopy equivalence.

Now Δ^n is contractible to its i-th vertex by simplicial homotopy (if $i = 0$ or n, a single homotopy will do; otherwise one needs a chain of two) and the contraction restricts to one of Λ^n_i. Since $\check{F}$ preserves simplicial homotopies (lemma 3.1.3) we conclude that indeed $\check{F}(\Lambda^n_i) \to \check{F}(\Delta^n)$ is a homotopy equivalence. The proof is now complete. □

3.3. The fibration relating $Wh^{PL}(X)$ and $A(X)$.

The fibration arises from the interplay of two notions of weak equivalence on the category $R_f(X)$, where X is a simplicial set. The two notions are given by the *simple maps* on the one hand and by the *weak homotopy equivalences* on the other.

Let the superscript 'h' denote the subcategory of the objects which are homotopically trivial; that is, the (Y,r,s) where s is a weak homotopy equivalence. As before (the preceding two sections) let $R_f(X^{\Delta^\cdot})$ denote the simplicial category $[n] \mapsto R_f(X^{\Delta^n})$.

Theorem 3.3.1. The square

$$\begin{array}{ccc} sS.R_f^h(X^{\Delta^\cdot}) & & hS.R_f^h(X^{\Delta^\cdot}) \\ \downarrow & & \downarrow \\ sS.R_f(X^{\Delta^\cdot}) & \longrightarrow & hS.R_f(X^{\Delta^\cdot}) \end{array}$$

is homotopy cartesian, and the term on the upper right is contractible. The other terms are as follows,

$$\Omega|hS.R_f(X^{\Delta^\cdot})| \simeq A(X) ,$$

$$X \mapsto sS.R_f(X^{\Delta^\cdot}) \quad \text{is a homology theory,}$$

$$sS.R_f^h(X^{\Delta^\cdot}) \simeq Wh^{PL}(X) ,$$

and each of the homotopy equivalences can be described by a natural chain of maps.

Proof. In order to show that the square is homotopy cartesian it will suffice to show, by lemma 3.2.5, that for each n the square with $X^{\Delta^\cdot}$ replaced by X^{Δ^n} is homotopy cartesian. Writing X instead of X^{Δ^n} now we have reduced to showing that the square

$$\begin{array}{ccc} sS.R_f^h(X) & \longrightarrow & hS.R_f^h(X) \\ \downarrow & & \downarrow \\ sS.R_f(X) & \longrightarrow & hS.R_f(X) \end{array}$$

is homotopy cartesian. The desired fact is essentially a special case of theorem 1.6.4. There is a little technical point. Namely the category of weak homotopy equivalences on $R_f(X)$ does not satisfy the *extension axiom* as required for a direct application of theorem 1.6.4. For this reason we compare with the square

$$\begin{array}{ccc} sS.R_f^h(X) & \longrightarrow & hS.R_f^h(X) \\ \downarrow & & \downarrow \\ sS.R_f^{(2)}(X) & \longrightarrow & hS.R_f^{(2)}(X) \end{array}$$

where $R_f^{(2)}(X)$ denotes the subcategory of $R_f(X)$ of the (Y,r,s) where $s: X \to Y$ is a 1-connected map. The weak homotopy equivalences in $R_f^{(2)}(X)$ may alternatively be characterized as the maps inducing isomorphisms in homology (the Whitehead theorem), consequently they do satisfy the extension axiom. Hence theorem 1.6.4 applies to show the latter square is homotopy cartesian. We conclude by noting that the map to the former square is a homotopy equivalence on each of the four corners. In fact, double suspension induces an endomorphism of each of the terms, the endomorphism is homotopic to the identity map (proposition 1.6.2), and it takes $R_f(X)$ into $R_f^{(2)}(X)$.

The upper right term $hS.R_f^h(X^{\Delta^\cdot})$ is contractible since it is a bisimplicial object of categories with initial objects.

The term $hS.R_f(X^{\Delta^\cdot})$ is a de-loop of $A(X)$ since $hS.R_f(X) \to hS.R_f(X^{\Delta^\cdot})$ is a homotopy equivalence (by lemma 3.1.2) in view of the fact that $X \mapsto hS.R_f(X)$ respects weak homotopy equivalences (proposition 2.1.7).

The homotopy equivalence $sS.R_f^h(X^{\Delta^\cdot}) \simeq Wh^{PL}(X)$ is given in theorem 3.1.7.

The fact that $X \mapsto sS.R_f(X^{\Delta^\cdot})$ is a homology theory, finally, is provided by theorem 3.2.1. □

The theorem may be reformulated a little by defining the auxiliary simplicial structure in a slightly different way. Namely define a simplicial category $R_f(X).$ as follows. $R_f(X)_n$ is the subcategory of $R_f(X\times\Delta^n)$ given by the objects (Y,r,s) which have the property that the composite map

$$Y \xrightarrow{r} X \times \Delta^n \xrightarrow{pr_2} \Delta^n$$

is locally fibre homotopy trivial

Proposition 3.3.2. There is a homotopy cartesian square

$$\begin{array}{ccc} sS.R_f^h(X). & \longrightarrow & hS.R_f^h(X). \\ \downarrow & & \downarrow \\ sS.R_f(X). & \longrightarrow & hS.R_f(X). \end{array}$$

and it is homotopy equivalent to the square of the theorem by a natural map.

Proof. The homotopy cartesianness of the square is established in the same way as in the theorem. There is a map from the square of the theorem to that of the proposition. It is induced from the map of simplicial categories $\mathcal{R}_f(X^{\Delta^{\cdot}}) \to \mathcal{R}_f(X)_{\cdot}$ defined as follows. The map in degree n is the composite map

$$\mathcal{R}_f(X^{\Delta^n}) \longrightarrow \mathcal{R}_f(X^{\Delta^n} \times \Delta^n) \longrightarrow \mathcal{R}_f(X \times \Delta^n)$$

where the first map is given by product with Δ^n, and the second map is induced from a map

$$X^{\Delta^n} \times \Delta^n \longrightarrow X \times \Delta^n ,$$

namely the map whose second and first components are the projection map pr_2 and the evaluation map

$$X^{\Delta^n} \times \Delta^n \longrightarrow X ,$$

respectively.

In order to show that the transformation of squares is a homotopy equivalence it suffices, in view of the homotopy cartesianness of the two squares, to show that the map is a homotopy equivalence on three of the four corners.

This is automatic in the case of the upper right corner as both terms are contractible.

It is still easy in the case of the lower right corner. Namely in view of the homotopy equivalence $hS.\mathcal{R}_f(X) \to hS.\mathcal{R}_f(X^{\Delta^{\cdot}})$ (the theorem) it suffices to know that the map $hS.\mathcal{R}_f(X) \to hS.\mathcal{R}_f(X)_{\cdot}$ is a homotopy equivalence. This follows from the fact (by the argument of lemma 2.2.6) that for every n the map $hS.\mathcal{R}_f(X) \to hS.\mathcal{R}_f(X)_n$ is a homotopy equivalence.

As our third case we take that of the upper left corner. That case is less easy. We consider the diagram

$$\begin{array}{ccccccc}
sN.\mathcal{C}^h_f(X) & \longrightarrow & sN.\mathcal{C}^h_f(X^{\Delta^{\cdot}}) & \longleftarrow & sN.\mathcal{R}^h_f(X^{\Delta^{\cdot}}) & \longrightarrow & sS.\mathcal{R}^h_f(X^{\Delta^{\cdot}}) \\
\| & & \downarrow & & \downarrow & & \downarrow \\
sN.\mathcal{C}^h_f(X) & \longrightarrow & sN.\mathcal{C}^h_f(X)_{\cdot} & \longleftarrow & sN.\mathcal{R}^h_f(X)_{\cdot} & \longrightarrow & sS.\mathcal{R}^h_f(X)_{\cdot}
\end{array}$$

where the upper row is the chain of maps of theorem 3.1.7, and the lower row is an analogue of that chain for the other auxiliary simplicial structure. The maps in the upper row are homotopy equivalences (theorem 3.1.7), so it will suffice to know that the maps in the lower row are homotopy equivalences, too. The second and third maps in the chain now are handled as before (lemmas 3.1.5 and 3.1.6). In the case of the first map one can reduce (by the realization lemma) to showing that the map $s\mathcal{C}^h_f(X) \to s\mathcal{C}^h_f(X)_{\cdot}$ is a homotopy equivalence; or in fact, that $s\mathcal{C}^h_f(X) \to s\mathcal{C}^h_f(X)_n$ is, for every n. But this has been proved in [16]. □

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