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Higher K-theory of group-rings of virtually infinite cyclic groups

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Abstract. F.T. Farrell and L.E. Jones conjectured in [7] that Algebraic K-theory of virtually cyclic subgroups V should constitute 'building blocks' for the Algebraic K-theory of an arbitrary group G. In [6], they obtained some results on lower K-theory of V. In this paper, we obtain results on higher K-theory of virtually infinite cyclic groups V in the two cases: (i) when V admits an epimorphism (with finite kernel) to the infinite cyclic group (see 2.1 and 2.2(a),(b)) and (ii) when V admits an epimorphism (with finite kernel) to the infinite dihedral group (see 3.1, 3.2, 3.3).

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0. Introduction

In [7], Farrell-Jones conjectured that Algebraic K-theory of an arbitrary group G can be "computed" in terms of virtually cyclic subgroups of G. So it becomes essential to understand the K-theory of virtually cyclic subgroups as possible building blocks for the understanding the K-theory of arbitrary group. Recall that a group is virtually cyclic if it is either finite or virtually infinite cyclic, i.e., contains a finite index subgroup which is infinite cyclic. More precisely, virtually infinite cyclic groups V are of two types, namely,

1) The group V that admits an epimorphism (with finite kernel G) to the infinite cyclic group $T =< t >$, i.e., V is the semi-direct product $G \rtimes_\alpha T$ where $\alpha : G \longrightarrow G$ is an automorphism and the action of T is given by $tgt^{-1} = \alpha(g)$ for all $g \in G$.
2) The group V which admits an epimorphism (with finite kernel) to the infinite dihedral group D_∞, i.e., $V = G_0 *_H G_1$ where the groups G_i, $i = 0, 1$, and H are finite and $[G_i : H] = 2$.

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In [6], Farrell-Jones studied lower K-theory of virtually infinite cyclic groups with copious references on work already done for lower K-groups of finite groups. In this paper, we focus attention on higher K-theory of virtually cyclic groups.

Let R be the ring of integers in a number field F, Λ any R-order in a semi-simple F-algebra Σ. In [10], [11], A. Kuku proved that for all $n \geq 1$, $K_n(\Lambda)$ and $G_n(\Lambda)$ are finitely generated Abelian groups and hence that for any finite group G, $K_n(RG)$ and $G_n(RG)$ are finitely generated. One consequence of this result is that for all $n \geq 1$, if C is a finitely generated free Abelian group or monoid, then $G_n(\Lambda[C])$ are also finitely generated (using the fundamental Theorem for G-theory). However we can not draw the same conclusion for $K_n(\Lambda[C])$ since for a ring A, it is known that all the $NK_n(A)$ are not finitely generated unless they are zero ([16], Proposition 4.1).

We now briefly review the results in this paper.

In §1 we set the stage by proving theorems 1.1 and 1.6 which constitute generalizations of theorems 1.2 and 1.5 of [6]. Here, we prove the results for an arbitrary R-order Λ in a semi-simple F-algebra Σ (where R is the ring of integers in a number field F) rather than for the special case $\Lambda = \mathbb{Z}G$ (G finite group) treated in [6].

In §2, we prove that if R is the ring of integers in a number field F and Λ an R-order in a semi-simple F-algebra Σ, α an automorphism of Λ, then for all $n \geq 0$, $NK_n(\Lambda, \alpha)$ is s-torsion for some positive integer s and that the torsion free rank of $K_n(\Lambda_\alpha[t])$ is equal to the torsion free rank of $K_n(\Lambda)$ which is finite by ([10], [12]). When $V = G \rtimes_\alpha T$ is a virtually infinite cyclic group of the first type, we show that for all $n \geq 0$, $G_n(RV)$ is a finitely generated Abelian group and that for all $n < -1$, $K_n(RV) = 0$. We also show that for all $n \geq 0$, $NK_n(RV)$ is $|G|$-torsion.

For a virtually infinite cyclic group $V = G_0 *_H G_1$ of the second type, a triple $(\mathbb{Z}H; \mathbb{Z}[G_0 - H], \mathbb{Z}[G_1 - H])$ arises as a special case of a triple $\mathbf{R} = (R; B, C)$ where R is a ring with identity, and B, C are R-bimodules (see §3). In the case $(\mathbb{Z}H; \mathbb{Z}[G_0 - H], \mathbb{Z}[G_1 - H])$, the $\mathbb{Z}H$-bimodule $\mathbb{Z}[G_i - H]$ is isomorphic to $\mathbb{Z}H$ as a left $\mathbb{Z}H$-module but the right action is twisted by an automorphism of H. We are thus inspired to consider the general case $\mathbf{R} = (R; R^\alpha, R^\beta)$ of R being a unital ring, α,β automorphisms $R \longrightarrow R$, R^α(resp. R^β) the $R - R$-bimodule which is R as a left R-module but with right multiplication given by $a \cdot r = a\alpha(r)$(resp. $b \cdot r = b\beta(r)$).

If $\mathcal{T}$ is the category of triples $\mathbf{R} = (R; B, C)$, then, there exists a functor

$$\rho : \mathcal{T} \longrightarrow Rings, \qquad \rho(\mathbf{R}) = R_\rho$$

(see §3 and [3]) and an augmentation map

$$\epsilon : R_\rho \longrightarrow \begin{pmatrix} R & 0 \\ 0 & R \end{pmatrix}.$$

Then the Nil-groups associated to $\mathbf{R}$ are defined for all $n \in \mathbb{Z}$ as $NK_n(\mathbf{R}) =$ kernel of the maps induced by ϵ on the K_n-groups (see §3). We prove the important result that R_ρ is a twisted polynomial ring over $\begin{pmatrix} R & 0 \\ 0 & R \end{pmatrix}$. We also prove that when R is a regular ring, $NK_n(R; R^\alpha, R^\beta) = 0$ for all $n \in \mathbb{Z}$ and that if R is quasi-regular, then $NK_n(R; R^\alpha, R^\beta) = 0$ for all $n \le 0$. When $n \le 1$, the statement above was proved in [4] using isomorphism between $NK_n(R; R^\alpha, R^\beta)$ and Waldhausen's groups $\widetilde{\mathrm{Nil}}^W_{n-1}(R; R^\alpha, R^\beta)$ and the fact that $\widetilde{\mathrm{Nil}}^W_{n-1}(R; R^\alpha, R^\beta)$ vanishes for regular rings R [4]. In effect, we have given here another proof of the vanishing of lower Waldhausen's groups $\widetilde{\mathrm{Nil}}^W_{n-1}(R; R^\alpha, R^\beta)$ based on the isomorphism of $NK_n(R; R^\alpha, R^\beta)$ and $\widetilde{\mathrm{Nil}}^W_{n-1}(R; R^\alpha, R^\beta)$ for $n \le 1$. We then prove that if $V = G_0 *_H G_1$, where $G_i, i = 0, 1$, and H are finite and $[G_i : H] = 2$, then the Nil-groups $NK_n(\mathbb{Z}H; \mathbb{Z}[G_0 - H], \mathbb{Z}[G_1 - H])$ are $|H|$-torsion.

Notes on Notations

For an exact category $\mathcal{C}$ we write $K_n(\mathcal{C})$ for the Quillen higher K-theory $\pi_{n+1}(BQ\mathcal{C})$ for $n \ge 0$ see [13].

If A is any ring with identity, we write, for $n \ge 0$, $K_n(A) = K_n(\mathbf{P}(A))$ where $\mathbf{P}(A)$ is the exact category of finitely generated projective modules over A and when A is Noetherian, we write $G_n(A)$ for $K_n(\mathbf{M}(A))$ where $\mathbf{M}(A)$ is the exact category of finitely generated A-modules.

We write $T =< t >$ for the infinite cyclic group, and T^r for the free Abelian group of rank r. If α is an automorphism of a ring A, we shall write $A_\alpha[T]$ for the α-twisted Laurent series ring, i.e., $A_\alpha[T] = A[T]$ additively and multiplication given by $(rt^i)(st^j) = r\alpha^{-i}(s)t^{i+j}$. $A_\alpha[t]$:=subgroup of $A_\alpha[T]$ generated by A and t, that is, $A_\alpha[t]$ is the twisted polynomial ring. We define, for any $n \ge 0$, $NK_n(A, \alpha)$ =ker$(K_n(A_\alpha[t]) \xrightarrow{\epsilon^*} K_n(A))$ where ϵ^* is induced by the augmentation $\epsilon : A_\alpha[t] \longrightarrow A, t \mapsto 0$. If α is the identity automorphism, then $NK_n(A)$ =ker$((K_n A[t]) \longrightarrow K_n(A))$.

1. Some preliminary results

Let R be the ring of integers in a number field F, Λ an R-order in a semi-simple F-algebra Σ and $\alpha : \Lambda \longrightarrow \Lambda$ is an R-automorphism. Then α extends to F-automorphism on Σ. Suppose that Γ is a maximal element in the set of all α-invariant R-orders in Σ containing Λ. Let max(Γ) denote the set of all two-sided maximal ideals in Γ and max$_\alpha(\Gamma)$ the set of all two-sided maximal α-invariant ideals in Γ. Recall that a Λ-lattice in Σ is a Λ-Λ submodule of Σ which generates Σ as a F-vector space. The aim of this section is to prove Theorem 1.1 and 1.6 below which constitute generalizations of 1.2 and 1.5 of [6]. The proof follows that of [6] rather closely and some details are omitted.

Theorem 1.1. *The set of all two-sided, α-invariant, Γ-lattices in Σ is a free Abelian group under multiplication and has* $\max_\alpha(\Gamma)$ *as a basis.*

Proof. Let $\mathfrak{a}$ be a two-sided, α-invariant, Γ-lattice in Σ. Then $\{x \in \Sigma | x\mathfrak{a} \subseteq \mathfrak{a}\}$ is an α-invariant R-order containing Γ. Hence, it must be equal to Γ by the maximality of Γ. Similarly $\{x \in \Sigma | \mathfrak{a}x \subseteq \mathfrak{a}\} = \Gamma$.

Now let $\mathfrak{a} \subseteq \Gamma$ be a two-sided, α-invariant, Γ-lattice in Σ. Then $B = \Gamma/\mathfrak{a}$ is a finite ring and, hence, Artinian. So rad$B$ is a nilpotent, α-invariant, two-sided ideal in B and $B/\text{rad}B$ is semi-simple ring. Hence $B/\text{rad}B$ decomposes as a direct sum of simple rings B_i, i.e.,

$$B/\text{rad}B = B_1 \oplus B_2 \oplus \cdots \oplus B_n, \qquad \text{(I)}$$

and $\alpha : B/\text{rad}B \longrightarrow B/\text{rad}B$ induces a permutation of the factors B_i, i.e.,

$$\alpha(B_i) = B_{\hat{\alpha}(i)} \qquad \text{(II)}$$

where $\hat{\alpha}$ is a permutation of $\{1, 2, \ldots, n\}$. So, $\mathfrak{a} \in \max_\alpha(\Gamma)$ if and only if both rad$(\Gamma/\mathfrak{a})= 0$ and $\hat{\alpha}$ is a cyclic permutation. Hence, $\mathfrak{a} \notin \max_\alpha(\Gamma)$ if and only if there exist a pair of two-sided, α-invariant, Γ-lattices $\mathfrak{b}$ and $\mathfrak{c}$ satisfying the following three properties:

(i) both $\mathfrak{b}$ and $\mathfrak{c}$ properly contain $\mathfrak{a}$;
(ii) $\mathfrak{b}$ and $\mathfrak{c}$ are both contained in Γ; (III)
(iii) $\mathfrak{b}\mathfrak{c} \subseteq \mathfrak{a}$.

Hence, just as in [6], we deduce the following fact:

If $\mathfrak{a}$ is a two-sided, α-invariant, Γ-lattice, then $\mathfrak{a}$ contains a (finite) product of elements from $\max_\alpha(\Gamma)$. (IV)

If $\mathfrak{a}$ be a two-sided, α-invariant, Γ-lattice, then we write

$$\bar{\mathfrak{a}} = \{x \in \Sigma | x\mathfrak{a} \subseteq \Gamma\} \qquad \text{(V)}$$

which is also a two-sided, α-invariant, Γ-lattice. We now prove the following

Lemma 1.2. *If* $\mathfrak{p} \in \max_\alpha(\Gamma)$*, then*

$$\bar{\mathfrak{p}} \neq \Gamma. \qquad \text{(VI)}$$

Proof. Choose a positive integer $s \in \mathbb{Z}$ such that $s\Gamma \subset \mathfrak{p}$. Applying (i)–(iii) with $\mathfrak{a} = s\Gamma$ we can find elements $\mathfrak{p}_i \in \max_\alpha(\Gamma)$ such that

$$\mathfrak{p}_1\mathfrak{p}_2 \cdots \mathfrak{p}_n \subset s\Gamma.$$

Let us assume that n is the smallest possible integer with this property. Using the characterization of $\max_\alpha(\Gamma)$ given above (III and IV), we see that some $\mathfrak{p}_i$

must be contained in $\mathfrak{p}$ since $\mathfrak{p} \in \max_\alpha(\Gamma)$. And since $\mathfrak{p}_i \in \max_\alpha(\Gamma)$, $\mathfrak{p} = \mathfrak{p}_i$. We can therefore write $\mathfrak{a}\mathfrak{p}\mathfrak{b} \subset s\Gamma$ where $\mathfrak{a} = \mathfrak{p}_1 \cdots \mathfrak{p}_{i-1}$ and $\mathfrak{b} = \mathfrak{p}_{i+1} \cdots \mathfrak{p}_n$. Thus $\frac{1}{s}\mathfrak{a}\mathfrak{p}\mathfrak{b} \subset \Gamma$ and further $(\frac{1}{s}\mathfrak{b}\mathfrak{a}\mathfrak{p})\mathfrak{b} \subset \mathfrak{b}$. So we have $(\frac{1}{s}\mathfrak{b}\mathfrak{a}\mathfrak{p}) \subset \Gamma$. Hence $\frac{1}{s}\mathfrak{b}\mathfrak{a} \subset \bar{\mathfrak{p}}$ by the definition of $\bar{\mathfrak{p}}$. Since $\mathfrak{b}\mathfrak{a}$ is a product of $n-1$ elements in $\max_\alpha(\Gamma)$, the minimality of n implies that $\mathfrak{b}\mathfrak{a} \not\subset s\Gamma$, so $\frac{1}{s}\mathfrak{b}\mathfrak{a} \not\subset \Gamma$. Thus $\bar{\mathfrak{p}} \neq \Gamma$.

Lemma 1.3.

$$\text{If } \mathfrak{p} \in \max_\alpha(\Gamma), \text{ then } \bar{\mathfrak{p}}\mathfrak{p} = \Gamma = \mathfrak{p}\bar{\mathfrak{p}}.$$

Proof. Similar to that in step 4, page 21 of [6].

Lemma 1.4.

$$\text{If } \mathfrak{p}_1, \mathfrak{p}_2 \in \max_\alpha(\Gamma), \text{ then } \mathfrak{p}_1\mathfrak{p}_2 = \mathfrak{p}_2\mathfrak{p}_1.$$

Proof. Similar to that of step 5 in [6].

Note that the proof in ([1], p.158) is easily adapted to yield the following conclusion that

A two-sided, α-invariant, Γ-lattice $\mathfrak{a} \subseteq \Gamma$ is uniquely, up to order, a product of elements of $\max_\alpha(\Gamma)$ and we can finish the proof as in [1], p.158. □

Corollary 1.5. *If every element in* $\max_\alpha(\Gamma)$ *is a right projective* Γ*-module, then every element in* $\max(\Gamma)$ *is also a right projective* Γ*-module and consequently* Γ *is a hereditary ring.*

Proof. It is the same as the proof of Corollary 1.6 in [6].

Theorem 1.6. *Let R be the ring of integers in a number field F, Λ any R-order in a semi-simple F-algebra Σ. If $\alpha : \Lambda \to \Lambda$ is an R-automorphism, then there exists an R-order $\Gamma \subset \Sigma$ such that*

1) $\Lambda \subset \Gamma$,

2) Γ is α-invariant, and

3) Γ is a (right) regular ring. In fact, Γ is a (right) hereditary ring.

Proof. Let $\mathcal{S}$ be the set consisting of all α-invariant R-orders M of Σ which contains Λ. Then $\mathcal{S}$ is not empty since $\Lambda \in \mathcal{S}$. Choose Γ to be any maximal member of $\mathcal{S}$. Such a member exists by Zorn's Lemma. Note that this R-order Γ satisfies properties 1) and 2) by definition, and is clearly a right Noetherian ring. Hence, it suffices to show that Γ is a right hereditary ring; i.e., that every right Γ-module is either projective or has a length 2 resolution by projective right Γ-modules. To do this, it suffices to show that every maximal two-sided ideal in Γ is a projective right Γ-module. Let $\max(\Gamma)$ denote the set of all two-sided maximal ideals in Γ, and $\max_\alpha(\Gamma)$ the set of all maximal members among the two-sided α-invariant proper ideals in Γ. Note that if $\mathfrak{a} \in \max_\alpha(\Gamma)$, then $\Gamma/\mathfrak{a}$ is a finite ring. To see this, first observe that $\Gamma/\mathfrak{a}$ is finitely generated as an Abelian group under addition. If it were not finite, then there would exist a prime $p \in \mathbb{Z}$ such that the multiplies of p in $\Gamma/\mathfrak{a}$ would form a proper two-sided α-invariant proper ideals in $\Gamma/\mathfrak{a}$. But this would contradict the maximality of $\mathfrak{a}$. Also $\Gamma/\mathfrak{a}$ is a (right) Artinan ring since it is a finite ring. But rad$(\Gamma/\mathfrak{a})$ is an α-invariant two-sided ideal in $\Gamma/\mathfrak{a}$. So the maximality of $\mathfrak{a}$ again shows that rad$(\Gamma/\mathfrak{a})= 0$. Hence, $\Gamma/\mathfrak{a}$ is a semi-simple ring. We finally remark that $\mathfrak{a}$ is a two-sided Γ-lattice in Σ since $\mathfrak{a}$ has finite index in the lattice Γ.

By Corollary 1.5, it suffices to show that every element in $\mathfrak{p} \in \max_\alpha(\Gamma)$ is a right projective Γ-module. Let $\mathfrak{q}$ be the inverse of $\mathfrak{p}$ given by Theorem 1.1; i.e., $\mathfrak{q}$ is a two-sided Γ-lattice in Σ which is α-invariant and satisfies the equations $\mathfrak{p}\mathfrak{q} = \mathfrak{q}\mathfrak{p} = \Gamma$. Consequently, there exist elements $a_1, a_2, \dots, a_n \in \mathfrak{p}$ and $b_1, b_2, \dots, b_n \in \mathfrak{q}$ such that

$$a_1b_1 + a_2b_2 \cdots + a_nb_n = 1.$$

Now define (right) Γ-module homomorphisms $f : \mathfrak{p} \longrightarrow \Gamma^n$ and $g : \Gamma^n \longrightarrow \mathfrak{p}$ by

$$f(x) = (b_1x, b_2x, \dots, b_nx) \qquad g(y_1, y_2, \dots, y_n) = a_1y_1 + a_2y_2 + \cdots + a_ny_n$$

where $x \in \mathfrak{p}$ and $y = (y_1, y_2, \dots, y_n) \in \Gamma^n$. Note that the composite $g \circ f = \mathrm{id}_{\mathfrak{p}}$ Consequently, $\mathfrak{p}$ is a direct summand of Γ^n which shows that $\mathfrak{p}$ is a projective right Γ-module. □

2. *K*-theory for the first type of virtually infinite cyclic groups

The aim of this section is to prove Theorem 2.2 below. However we start by proving the following result which we shall need to prove 2.2.

Theorem 2.1. *Let A be a Noetherian ring and α an automorphism of A, $A_\alpha[t]$ the twisted polynomial ring. Then*

(1) $G_n(A_\alpha[t]) \cong G_n(A)$ for all $n \geq 0$.

(2) There exists a long exact sequence

$$\cdots \longrightarrow G_n(A) \xrightarrow{1-\alpha_*} G_n(A) \longrightarrow G_n(A_\alpha[t, t^{-1}]) \longrightarrow G_{n-1}(A) \longrightarrow \cdots.$$

Proof. 1) follows directly from Theorem 2.18 of (cf. [8], p.194). To prove the long exact sequence in 2), we denote by $\mathcal{A} = \mathbb{M}(A_\alpha[t])$ the category consisting of finitely generated $A_\alpha[t]$-modules. Consider the Serre subcategory $\mathcal{B}$ of $\mathcal{A} = \mathbb{M}(A_\alpha[t])$ which consists of modules $M \in \mathrm{obj}\mathcal{A}$ on which t is nilpotent, i.e.,

$$\mathrm{obj}\mathcal{B} = \{M \in \mathrm{obj}\mathcal{A} |\text{ there exists an } m \geq 0 \text{ such that } Mt^m = 0\}.$$

Applying the localization theorem to the pair ($\mathcal{A}$,$\mathcal{B}$) we obtain a long exact sequence

$$\cdots \longrightarrow K_{n+1}(\mathcal{A}/\mathcal{B}) \xrightarrow{1-\alpha_*} K_n(\mathcal{B}) \longrightarrow K_n(\mathcal{A}) \longrightarrow K_n(\mathcal{A}/\mathcal{B}) \longrightarrow \cdots.$$

By definition and 1) $K_n(\mathcal{A}) = G_n(A_\alpha[t]) \cong G_n(A)$. We will prove that

$$K_n(\mathcal{B}) \cong G_n(A)$$

and

$$K_n(\mathcal{A}/\mathcal{B}) \cong G_n(A_\alpha[T]) := G_n(A_\alpha[t, t^{-1}])$$

for all $n \geq 0$. At first $\mathbb{M}(A) \subseteq \mathbb{M}(A_\alpha[t]/tA_\alpha[t]) \subseteq \mathcal{B}$ (Note that although $t \notin$center$(A_\alpha[t])$, we have $tA_\alpha[t] = A_\alpha[t]t \lhd A_\alpha[t]$). Using the Devissage theorem one gets

$$K_n(\mathcal{B}) \cong K_n(\mathbb{M}(A_\alpha[t]/tA_\alpha[t])).$$

But

$$0 \longrightarrow tA_\alpha[t] \longrightarrow A_\alpha[t] \xrightarrow{t=0} A \longrightarrow 0$$

is an exact sequence of homomorphisms of rings. So, we have

$$A_\alpha[t]/tA_\alpha[t] \cong A$$

as rings, and so $K_n(\mathbb{M}(A_\alpha[t]/tA_\alpha[t])) \cong G_n(A)$. Thus

$$K_n(\mathcal{B}) \cong K_n(\mathbb{M}(A_\alpha[t]/tA_\alpha[t])) \cong G_n(A).$$

Next we prove that

$$K_n(\mathcal{A}/\mathcal{B}) \cong G_n(A_\alpha[T]).$$

Since $A_\alpha[T]$ is a direct limit of free $A_\alpha[t]$-modules $A_\alpha[t]t^{-n}$, it is a flat $A_\alpha[t]$-module and this implies that $- \otimes_{A_\alpha[t]} A_\alpha[T]$ is an exact functor from $\mathcal{A}$ to $\mathbb{M}(A_\alpha[T])$ and further induces an exact functor

$$F: \quad \mathcal{A}/\mathcal{B} \longrightarrow \mathbb{M}(A_\alpha[T]).$$

We now prove that F is an equivalence. For any $M \in$obj$\mathbb{M}(A_\alpha[T])$, pick a generating set $\{x_1, x_2, \cdots, x_l\}$ of the finitely generated $A_\alpha[T]$-module M. Let

$$M_1 = \sum_{i=1}^{l} x_i A_\alpha[t].$$

Then $M_1 \in \mathcal{A}$ and $M_1 \otimes_{A_\alpha[t]} A_\alpha[T] \cong M$. The exact sequence of $A_\alpha[t]$-modules

$$0 \longrightarrow M_1 \longrightarrow M \longrightarrow M/M_1 \longrightarrow 0,$$

induces an exact sequence

$$0 \longrightarrow M_1 \otimes_{A_\alpha[t]} A_\alpha[T] \longrightarrow M \otimes_{A_\alpha[t]} A_\alpha[T] \longrightarrow M/M_1 \otimes_{A_\alpha[t]} A_\alpha[T] \longrightarrow 0.$$

Since $\{x_1, x_2, \cdots, x_l\}$ is a generating set for the $A_\alpha[T]$-module M, then for any $x \in M$, there exist $f_i \in A_\alpha[T] (i = 1, \cdots, l)$ such that $x = \sum_{i=1}^{l} x_i f_i$. Thus, there exists $n \geq 0$ such that $mt^n = \sum_{i=1}^{l} m_i(f_i t^n) \in M_1$, i.e., M/M_1 is t-torsion. Thus $M/M_1 \otimes_{A_\alpha[t]} A_\alpha[T] = 0$. It follows that

$$0 \longrightarrow M_1 \otimes_{A_\alpha[t]} A_\alpha[T] \longrightarrow M \otimes_{A_\alpha[t]} A_\alpha[T] \longrightarrow 0 \quad \text{is exact.}$$

that is:

$$M_1 \otimes_{A_\alpha[t]} A_\alpha[T] \cong M \otimes_{A_\alpha[t]} A_\alpha[T].$$

For any $M, N \in \mathcal{A}$, we have, by definition

$$\mathrm{Hom}_{\mathcal{A}/\mathcal{B}}(M, N) = \lim_{\rightarrow} \mathrm{Hom}_{A_\alpha[t]}(M', N/N')$$

where M/M' and N' are t-torsion. One gets easily:

$$\mathrm{Hom}_{\mathcal{A}/\mathcal{B}}(M, N) = \lim_{\rightarrow} \mathrm{Hom}_{A_\alpha[t]}(M', N/N') = \lim_{\rightarrow} \mathrm{Hom}_{A_\alpha[t]}(M', N/N_t),$$

where $N_t = \{x \in N |$ there exists an $m \geq 0$ such that $xt^m = 0\}$. Define a map

$$\phi: \quad \mathrm{Hom}_{\mathcal{A}/\mathcal{B}}(M, N) \longrightarrow \mathrm{Hom}_{\mathcal{B}}(M \otimes_{A_\alpha[t]} A_\alpha[T], N \otimes_{A_\alpha[t]} A_\alpha[T]).$$

For any $f \in \mathrm{Hom}_{\mathcal{B}}(M \otimes_{A_\alpha[t]} A_\alpha[T], N \otimes_{A_\alpha[t]} A_\alpha[T])$, since M is a finitely generated $A_\alpha[T]$-module, there exists an $m \geq 0$ such that

$$f(Mt^m \otimes 1) \subseteq N \otimes 1.$$

We can define $Mt^m \xrightarrow{\sigma} N/N_t$, $xt^m \mapsto n$ if $f(xt^m \otimes 1) = n \otimes 1$. This is well defined and σ maps to f under ϕ.

If $\sigma \in \mathrm{Hom}_{A_\alpha[t]}(M', N/N_t)$ is such that its image in $\mathrm{Hom}_{\mathcal{B}}(M \otimes_{A_\alpha[t]} A_\alpha[T], N \otimes_{A_\alpha[t]} A_\alpha[T])$ is zero, then $\sigma \otimes 1 = 0$ implies that $\sigma m \otimes 1 = (\sigma \otimes 1)(m \otimes 1) = 0$ in $N/N_t \otimes_{A_\alpha[t]} A_\alpha[T]$. Hence $\sigma(m) = 0$, and so,

$$\mathrm{Hom}_{\mathcal{A}/\mathcal{B}}(M, N) \cong \mathrm{Hom}_{\mathcal{B}}(M \otimes_{A_\alpha[t]} A_\alpha[T], N \otimes_{A_\alpha[t]} A_\alpha[T])$$

that is

$$\mathcal{A}/\mathcal{B} \cong \mathbb{M}(A_\alpha[T]).$$

Hence $K_n(\mathcal{A}/\mathcal{B}) \cong K_n(\mathbb{M}(A_\alpha[T]))$ completing the proof of 2).

Theorem 2.2. *Let R be the ring of integers in a number field F, Λ any R-order in a semi-simple F-algebra Σ, α an automorphism of Λ. Then*

(a) For all $n \geq 0$

- *(i) $NK_n(\Lambda, \alpha)$ is s-torsion for some positive integer s. Hence the torsion free rank of $K_n(\Lambda_\alpha[t])$ is the torsion free rank of $K_n(\Lambda)$ and is finite. If $n \geq 2$, then the torsion free rank of $K_n(\Lambda_\alpha[t])$ is equal to the torsion free rank of $K_n(\Sigma)$.*
- *(ii) If G is a finite group of order r, then $NK_n(RG, \alpha)$ is r-torsion, where α is the automorphism of RG induced by that of G.*

(b) Let $V = G \rtimes_\alpha T$ be the semi-direct product of a finite group G of order r with an infinite cyclic group $T = \langle t \rangle$ with respect to the automorphism $\alpha : G \longrightarrow G : \quad g \mapsto tgt^{-1}$. Then

(i) $K_n(RV) = 0$ for all $n < -1$.

(ii) The inclusion $RG \to RV$ induces an epimorphism $K_{-1}(RG) \to K_{-1}(RV)$. Hence $K_{-1}(RV)$ is finitely generated Abelian group.

(iii) For all $n \geq 0$, $G_n(RV)$ is a finitely generated Abelian group.

(iv) $NK_n(RV)$ is r-torsion for all $n \geq 0$.

Proof. (a)(i) By Theorem 1.6, we can choose an α-invariant R-order Γ in Σ which contains Λ and is regular. First note that since every R-order is a $\mathbb{Z}$-order, there is a non-zero integer s such that $\Lambda \subseteq \Gamma \subseteq \Lambda(1/s)$, where $A(1/s)$ denote $A \otimes \mathbb{Z}(1/s)$ for an Abelian group A. Put $\underline{q} = s\Gamma$. Then we have a Cartesian square

$$\begin{array}{ccc} \Lambda & \longrightarrow & \Gamma \\ \downarrow & & \downarrow \\ \Lambda/\underline{q} & \longrightarrow & \Gamma/\underline{q} \end{array} . \tag{I}$$

Since α induces automorphisms of all the four rings in the square (I) (cf. [6]), we have another Cartesian square

$$\begin{array}{ccc} \Lambda_\alpha[t] & \longrightarrow & \Gamma_\alpha[t] \\ \downarrow & & \downarrow \\ (\Lambda/\underline{q})_\alpha[t] & \longrightarrow & (\Gamma/\underline{q})_\alpha[t] \end{array} . \tag{II}$$

Note that both $(\Lambda/\underline{q})_\alpha[t]$ and $(\Gamma/\underline{q})_\alpha[t]$ are $\mathbb{Z}/s\mathbb{Z}$-algebra, and so, it follows from (II) that we have a long exact Mayer-Vietoris sequence (cf. [16] or [2])

$$\begin{aligned} \cdots &\to K_{n+1}((\Gamma/\underline{q})_\alpha[t])(1/s) \to K_n(\Lambda_\alpha[t])(1/s) \\ &\to K_n((\Lambda/\underline{q})_\alpha[t])(1/s) \oplus K_n(\Gamma_\alpha[t])(1/s) \\ &\to K_n((\Gamma/\underline{q})_\alpha[t])(1/s) \to K_{n-1}(\Lambda_\alpha[t])(1/s) \to \cdots . \end{aligned} \tag{III}$$

Since we also have a long exact Mayer-Vietoris sequence

$$\begin{aligned} \cdots &\to K_{n+1}(\Gamma/\underline{q})(1/s) \to K_n(\Lambda)(1/s) \to K_n(\Lambda/\underline{q})(1/s) \oplus K_n(\Gamma)(1/s) \\ &\to K_n(\Gamma/\underline{q})(1/s) \to K_{n-1}(\Lambda)(1/s) \to \cdots , \end{aligned} \tag{IV}$$

then, by mapping sequence (III) to sequence (IV) and taking kernels, we obtain another long exact Mayer-Vietoris sequence

$$\begin{aligned} \cdots &\to NK_{n+1}(\Gamma/\underline{q}, \alpha)(1/s) \to NK_n(\Lambda, \alpha)(1/s) \\ &\to NK_n(\Lambda/\underline{q}, \alpha)(1/s) \oplus NK_n(\Gamma, \alpha)(1/s) \to \\ &\to NK_n(\Gamma/\underline{q}, \alpha)(1/s) \to NK_{n-1}(\Lambda, \alpha)(1/s) \to \cdots . \end{aligned}$$

However, by [6] $\Gamma_\alpha[t]$ is regular since Γ is. So $NK_n(\Gamma, \alpha) = 0$ by Theorem 2.1 (i) since $K_n(\Gamma_\alpha[t]) = G_n(\Gamma_\alpha[t]) = G_n(\Gamma) = K_n(\Gamma)$. Both $\Lambda/\underline{q}$ and $\Gamma/\underline{q}$ are finite, hence quasi-regular. They are also $\mathbb{Z}/s\mathbb{Z}$-algebra, and so it follows that $(\Lambda/\underline{q})_\alpha[t]$ and $(\Gamma/\underline{q})_\alpha[t]$ are also quasi-regular and $\mathbb{Z}/s\mathbb{Z}$-algebra. We now prove that for a finite $\mathbb{Z}/s\mathbb{Z}$-algebra A, $NK_n(A,\alpha)$ is s-torsion. Since A is finite, its Jacobson radical $J(A)$ is nilpotent, and by Corollary 5.4 of [15] the relative K-groups $K_n(A, J(A))$ are s-torsion for any $n \geq 0$. This implies that

$$K_n(A)(1/s) \cong K_n(A/J(A))(1/s)$$

from the relative K-theory long exact sequence tensored with $\mathbb{Z}\left(\frac{1}{s}\right)$. Similarly, one gets

$$K_n(A_\alpha[x])(1/s) \cong K_n((A/J(A))_\alpha[x])(1/s).$$

However, $A/J(A)$ is regular and so,

$$K_n((A/J(A))_\alpha[x])(1/s) \cong K_n(A/J(A))(1/s) \quad \text{by 2.1(1)}.$$

Hence, we have

$$K_n(A_\alpha[x])(1/s) \cong K_n(A)(1/s).$$

From the finiteness of A one gets $K_n(A)$ is finite (cf. [10]). Hence both $K_n(A_\alpha[x])(1/s)$ and $K_n(A)(1/s)$ have the same cardinality. From the exact sequence

$$0 \longrightarrow NK_n(A,\alpha) \longrightarrow K_n(A_\alpha[x]) \longrightarrow K_n(A) \longrightarrow 0 \ \text{ tensored with } \ \mathbb{Z}\left(\frac{1}{s}\right),$$

we obtain the exact sequence

$$0 \longrightarrow NK_n(A,\alpha)(1/s) \longrightarrow K_n(A_\alpha[x])(1/s) \longrightarrow K_n(A)(1/s) \longrightarrow 0.$$

Hence $NK_n(A,\alpha)(1/s)$ is zero since $K_n(A_\alpha[x])(1/s)$ and $K_n(A)(1/s)$ are isomorphic. Hence, both $NK_{n+1}(\Gamma/\underline{q},\alpha)(1/s)$ and $NK_n(\Lambda/\underline{q},\alpha)(1/s)$ are zero, and so, $NK_n(\Lambda,\alpha)$ is s-torsion.

Since $K_n(\Lambda_\alpha[t]) = K_n(\Lambda) \oplus NK_n(\Lambda,\alpha)$ and $NK_n(\Lambda,\alpha)$ is torsion, the torsion free rank of $K_n(\Lambda_\alpha[t])$ is the torsion free rank of $K_n(\Lambda)$. By [9] the torsion free rank of $K_n(\Lambda)$ is finite and if $n \geq 2$ the torsion free rank of $K_n(\Sigma)$ is the torsion free rank of $K_n(\Lambda)$ (see [12]).

(a)(ii) is a direct consequence of (a)(i) since if $|G| = r$, and we take $\Lambda = RG$ then $r\Gamma \subseteq \Lambda$.

(b)(i) By Theorem 1.6, there exists an α-invariant regular ring Γ in FG which contains RG. Then for the integer $s = |G|$, $RG \subseteq \Gamma \subseteq RG(1/s)$. Put $\underline{q} = s\Gamma$. Then we have a Cartesian square

$$\begin{array}{ccc} RG & \longrightarrow & \Gamma \\ \downarrow & & \downarrow \\ RG/\underline{q} & \longrightarrow & \Gamma/\underline{q} \end{array} . \tag{VI}$$

Since α induces automorphisms of all the four rings in the square (VI), we have another Cartesian square

$$\begin{array}{ccc} RV & \longrightarrow & \Gamma_\alpha[T] \\ \downarrow & & \downarrow \\ (RG/\underline{q})_\alpha[T] & \longrightarrow & (\Gamma/\underline{q})_\alpha[T] \end{array} \quad . \tag{VII}$$

see [6].

Since lower K-theory has excision property, it follows from [6] that we have lower K-theory exact sequence

$$\begin{aligned} &K_0(RV) \to K_0((RG/\underline{q})_\alpha[T]) \oplus K_0(\Gamma_\alpha[T]) \to K_0((\Gamma/\underline{q})_\alpha[T]) \to \\ &\longrightarrow K_{-1}(RV) \to K_{-1}((RG/\underline{q})_\alpha[T]) \oplus K_{-1}(\Gamma_\alpha[T]) \to \cdots . \end{aligned} \tag{VIII}$$

However, $(\Gamma/\underline{q})_\alpha[T]$ and $(RG/\underline{q})_\alpha[T]$ are quasi-regular since, $\Gamma/\underline{q}$ and $RG/\underline{q}$ are quasi-regular (see [6]). Also, $\Gamma_\alpha[T]$ is regular, since Γ is regular (see [6]).

Hence

$$K_i(\Gamma_\alpha[T]) = K_i((RG/\underline{q})_\alpha[T]) = K_i((\Gamma/\underline{q})_\alpha[T]) = 0$$

for all $i \leq -1$. Thus $K_i(RV) = 0$ for all $i < -1$, from the exact sequence (VIII).

(b)(ii) The proof of b(ii) is similar to the proof of a similar statement for $\mathbb{Z}V$ in [6] Corollary 1.3 and is omitted.

(b)(iii) Is a direct consequence of Theorem 2.1(2) since $G_n(RG)$ is finitely generated for all $n \geq 1$ (see [11]).

(b)(iv) By [6], 1.3.2, we have a Cartesian square

$$\begin{array}{ccc} RV & \longrightarrow & \Gamma_\alpha[T] \\ \downarrow & & \downarrow \\ (RG/\underline{q})_\alpha[T] & \longrightarrow & (\Gamma/\underline{q})_\alpha[T] \end{array}$$

where $\Gamma_\alpha[T]$ is regular and $(RG/\underline{q})_\alpha[T]$, $(\Gamma/\underline{q})_\alpha[T]$ are quasi-regular (see [6] 1.1 and 1.41).

Moreover, since r annihilates $RG/\underline{q}$ and $\Gamma/\underline{q}$ it also annihilates $(RG/\underline{q})_\alpha[T]$, $(\Gamma/\underline{q})_\alpha[T]$ since for $A = RG/\underline{q}$, or $\Gamma/\underline{q}$, $A_\alpha[T]$ is a direct limit of free $A_\alpha[t]$-module $A_\alpha[t]t^{-n}$. Hence by [16], Corollary 3.3(d), $NK_n(A_\alpha[T])$ is r-torsion. (Note that [16], Corollary 3.3(d) is valid when p is any integer r – we confirmed this from the author C. Weibel.)

We also have by [2] and [16] a long exact Mayer-Vietoris sequence

$$\begin{aligned} \ldots \longrightarrow NK_{n+1}(\Gamma/\underline{q})_\alpha[T]\left(\tfrac{1}{r}\right) \longrightarrow NK_n(RV)\left(\tfrac{1}{r}\right) \longrightarrow NK_n(RG/\underline{q})_\alpha[T]\left(\tfrac{1}{r}\right) \\ \oplus NK_n(\Gamma_\alpha[T]\left(\tfrac{1}{r}\right) \longrightarrow NK_n(\Gamma/\underline{q})_\alpha[T]\left(\tfrac{1}{r}\right) \longrightarrow \ldots \end{aligned} \tag{IX}$$

But $NK_n(\Gamma_\alpha[T]) = 0$ since $\Gamma_\alpha[T]$ is regular. Hence we have $NK_n(RV)\left(\frac{1}{r}\right) = 0$ from IX, and so, $NK_n(RV)$ is r-torsion.

3. Nil-groups for the second type of virtually infinite cyclic groups

The algebraic structure of the groups in the second class is more complicated. We recall that a group V in the second class has the form $V = G_0 *_H G_1$ where the groups $G_i, i = 0, 1$, and H are finite and $[G_i : H] = 2$. We will show that the Nil-groups in this case are torsion, too. At first we recall the definition of Nil-groups in this case.

Let $\mathcal{T}$ be the category of triples $\mathbf{R} = (R; B, C)$, where B and C are R-bimodules. A morphism in $\mathcal{T}$ is a triple

$$(\phi; f, g) : (R; B, C) \longrightarrow (S; D, E)$$

where $\phi : R \longrightarrow S$ is a ring homomorphism and both $f : B \longrightarrow D$ and $g : C \longrightarrow E$ are $R - S$-bimodule homomorphisms. There is a functor ρ from the category $\mathcal{T}$ to the category $Rings$ defined by

$$\rho(\mathbf{R}) = R_\rho = \begin{pmatrix} T_R(C \otimes_R B) & C \otimes_R T_R(B \otimes_R C) \\ B \otimes_R T_R(C \otimes_R B) & T_R(B \otimes_R C) \end{pmatrix}$$

where $T_R(B \otimes_R C)$(resp. $T_R(C \otimes_R B)$) is the tensor algebra of $B \otimes_R C$(resp. $C \otimes_R B$) and $\rho(\mathbf{R})$ is the ring with multiplication given as matrix multiplication and each entry by concatenation. There is a natural augmentation map(cf. [3])

$$\epsilon : R_\rho \longrightarrow \begin{pmatrix} R & 0 \\ 0 & R \end{pmatrix}.$$

The Nil-group $\mathrm{NK}_n(\mathbf{R})$ is defined to be the kernel of the map induced by ϵ on K_n-groups.

We now formulate the Nil-groups of interest. Let V be a group in the second class of the form $V = G_0 *_H G_1$ where the groups $G_i, i = 0, 1$, and H are finite and $[G_i : H] = 2$. Considering $G_i - H$ as the right coset of H in G_i which is different from H, the free $\mathbb{Z}$-module $\mathbb{Z}[G_i - H]$ with basis $G_i - H$ is a $\mathbb{Z}H$-bimodule which is isomorphic to $\mathbb{Z}H$ as a left $\mathbb{Z}H$-module, but the right action is twisted by an automorphism of $\mathbb{Z}H$ induced by an automorphism of H. Then the Waldhausen's Nil-groups are defined to be $NK_n(\mathbb{Z}H; \mathbb{Z}[G_0 - H], \mathbb{Z}[G_1 - H])$ using the triple $(\mathbb{Z}H; \mathbb{Z}[G_0 - H], \mathbb{Z}[G_1 - H])$. This inspires us to consider the following general case. Let R be a ring with identity and $\alpha : R \longrightarrow R$ a ring automorphism. We denote by R^α the $R - R$-bimodule which is R as a left R-module but with right multiplication given by $a \cdot r = a\alpha(r)$. For any automorphisms α and β of R, we consider the triple $\mathbf{R} = (R; R^\alpha, R^\beta)$. We will prove that $\rho(\mathbf{R})$ is in fact a twisted polynomial ring and this is important for later use.

Theorem 3.1. *Suppose that α and β are automorphisms of R. For the triple $\mathbf{R} = (R; R^\alpha, R^\beta)$, let R_ρ be the ring $\rho(\mathbf{R})$, and let γ be a ring automorphism of $\begin{pmatrix} R & 0 \\ 0 & R \end{pmatrix}$ defined by*

$$\gamma : \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \longmapsto \begin{pmatrix} \beta(b) & 0 \\ 0 & \alpha(a) \end{pmatrix}.$$

Denote by 1_α*(resp.* 1_β*) the generator of* R^α*(resp.* R^β*) corresponding to* 1*. Then, there is a ring isomorphism*

$$\mu : R_\rho \longrightarrow \begin{pmatrix} R & 0 \\ 0 & R \end{pmatrix}_\gamma [x],$$

defined by mapping an element

$$\sum_{i\geq 0} \begin{pmatrix} a_i & 0 \\ 0 & b_i \end{pmatrix} \begin{pmatrix} (1_\beta \otimes 1_\alpha)^i & 0 \\ 0 & (1_\alpha \otimes 1_\beta)^i \end{pmatrix}$$
$$+ \sum_{i\geq 0} \begin{pmatrix} a_i' & 0 \\ 0 & b_i' \end{pmatrix} \begin{pmatrix} 0 & 1_\beta \otimes (1_\alpha \otimes 1_\beta)^i \\ 1_\alpha \otimes (1_\beta \otimes 1_\alpha)^i & 0 \end{pmatrix}$$

to an element

$$\sum_{i\geq 0} \begin{pmatrix} a_i & 0 \\ 0 & b_i \end{pmatrix} x^{2i} + \sum_{i\geq 0} \begin{pmatrix} a_i' & 0 \\ 0 & b_i' \end{pmatrix} x^{2i+1}.$$

Proof. By definition, each element of R_ρ can be written uniquely as

$$\sum_{i\geq 0} \begin{pmatrix} a_i & 0 \\ 0 & b_i \end{pmatrix} \begin{pmatrix} (1_\beta \otimes 1_\alpha)^i & 0 \\ 0 & (1_\alpha \otimes 1_\beta)^i \end{pmatrix}$$
$$+ \sum_{i\geq 0} \begin{pmatrix} a_i' & 0 \\ 0 & b_i' \end{pmatrix} \begin{pmatrix} 0 & 1_\beta \otimes (1_\alpha \otimes 1_\beta)^i \\ 1_\alpha \otimes (1_\beta \otimes 1_\alpha)^i & 0 \end{pmatrix}.$$

It is easy to see that μ is an isomorphism from the additive group of R_ρ to $\begin{pmatrix} R & 0 \\ 0 & R \end{pmatrix}_r [x]$. To complete the proof, we only need to check that μ preserves any product of two elements such as:

$$u_i = \begin{pmatrix} a_i & 0 \\ 0 & b_i \end{pmatrix} \begin{pmatrix} (1_\beta \otimes 1_\alpha)^i & 0 \\ 0 & (1_\alpha \otimes 1_\beta)^i \end{pmatrix}, \quad i \geq 0,$$

and

$$v_j = \begin{pmatrix} a_j' & 0 \\ 0 & b_j' \end{pmatrix} \begin{pmatrix} 0 & 1_\beta \otimes (1_\alpha \otimes 1_\beta)^j \\ 1_\alpha \otimes (1_\beta \otimes 1_\alpha)^j & 0 \end{pmatrix}, \quad j \geq 0.$$

We check these case by case. Note that for any $a, b \in R$, $1_\alpha \cdot a = \alpha(a)1_\alpha$ and $1_\beta \cdot b = \beta(b)1_\beta$, thus

$$\begin{pmatrix} (1_\beta \otimes 1_\alpha)^i & 0 \\ 0 & (1_\alpha \otimes 1_\beta)^i \end{pmatrix} \begin{pmatrix} a_j & 0 \\ 0 & b_j \end{pmatrix}$$

$$= \begin{pmatrix} (\beta\alpha)^i(a_j) & 0 \\ 0 & (\alpha\beta)^i(b_j) \end{pmatrix} \begin{pmatrix} (1_\beta \otimes 1_\alpha)^i & 0 \\ 0 & (1_\alpha \otimes 1_\beta)^i \end{pmatrix}.$$

Note that the following equations hold in R_ρ:

$$\begin{aligned} (1_\beta \otimes 1_\alpha)^i (1_\beta \otimes 1_\alpha)^j &= (1_\beta \otimes 1_\alpha)^{i+j} \\ (1_\alpha \otimes 1_\beta)^i (1_\alpha \otimes 1_\beta)^j &= (1_\alpha \otimes 1_\beta)^{i+j} \\ (1_\beta \otimes 1_\alpha)^i (1_\beta \otimes (1_\alpha \otimes 1_\beta)^j) &= 1_\beta \otimes (1_\alpha \otimes 1_\beta)^{i+j} \\ (1_\alpha \otimes 1_\beta)^i (1_\alpha \otimes (1_\beta \otimes 1_\alpha)^j) &= 1_\alpha \otimes (1_\beta \otimes 1_\alpha)^{i+j}. \end{aligned}$$

This implies that $\mu(u_i u_j) = \mu(u_i)\mu(u_j)$ and $\mu(u_i v_j) = \mu(u_i)\mu(v_j)$. Similarly, we have

$$\begin{pmatrix} 0 & 1_\beta \otimes (1_\alpha \otimes 1_\beta)^i \\ 1_\alpha \otimes (1_\beta \otimes 1_\alpha)^i & 0 \end{pmatrix} \begin{pmatrix} a'_j & 0 \\ 0 & b'_j \end{pmatrix}$$

$$= \begin{pmatrix} (\beta\alpha)^i \beta(b'_j) & 0 \\ 0 & (\alpha\beta)^i \alpha(a'_j) \end{pmatrix} \begin{pmatrix} 0 & 1_\beta \otimes (1_\alpha \otimes 1_\beta)^i \\ 1_\alpha \otimes (1_\beta \otimes 1_\alpha)^i & 0 \end{pmatrix}.$$

We have also equations in R_ρ:

$$\begin{aligned} (1_\beta \otimes (1_\alpha \otimes 1_\beta)^i)(1_\alpha \otimes 1_\beta)^j &= 1_\beta \otimes (1_\alpha \otimes 1_\beta)^{i+j} \\ (1_\alpha \otimes (1_\beta \otimes 1_\alpha)^i)(1_\beta \otimes 1_\alpha)^j &= 1_\alpha \otimes (1_\beta \otimes 1_\alpha)^{i+j} \\ (1_\beta \otimes (1_\alpha \otimes 1_\beta)^i)(1_\alpha \otimes (1_\beta \otimes 1_\alpha)^j) &= (1_\beta \otimes 1_\alpha)^{i+j+1} \\ (1_\alpha \otimes (1_\beta \otimes 1_\alpha)^i)(1_\beta \otimes (1_\alpha \otimes 1_\beta)^j) &= (1_\alpha \otimes 1_\beta)^{i+j+1}. \end{aligned}$$

It follows that $\mu(v_j u_i) = \mu(v_j)\mu(u_i)$ and $\mu(v_i v_j) = \mu(v_i)\mu(v_j)$. Hence μ is an isomorphism. □

From Theorem 3.1 above, we obtain the following important result.

Theorem 3.2. *If R is regular, then $NK_n(R; R^\alpha, R^\beta) = 0$ for all $n \in \mathbb{Z}$. If R is quasi-regular then $NK_n(R; R^\alpha, R^\beta) = 0$ for all $n \leq 0$.*

Proof. Since R is regular then $\begin{pmatrix} R & 0 \\ 0 & R \end{pmatrix}$ is regular too. By Theorem 3.1, R_ρ is a twisted polynomial ring over $\begin{pmatrix} R & 0 \\ 0 & R \end{pmatrix}$, and so it is regular [6]. By the fundamental Theorem of Algebraic K-Theory it follows that $NK_n(R; R^\alpha, R^\beta) = 0$ for all $n \in \mathbb{Z}$ (see [13]). If R is quasi-regular, then R_ρ is quasi-regular also [6]. By the fundamental Theorem of Algebraic K-Theory for lower K-theory $NK_n(R; R^\alpha, R^\beta) = 0$ for any $n \leq 0$.

Remark. When $n \leq 1$, the result above is proved in [4] using isomorphism between $NK_n(R; R^\alpha, R^\beta)$ and Waldhausen's groups $\widetilde{\mathrm{Nil}}^W_{n-1}(R; R^\alpha, R^\beta)$ and the fact that $\widetilde{\mathrm{Nil}}^W_{n-1}(R; R^\alpha, R^\beta)$ vanishes for regular rings R [4]. In effect, we have given here another proof of the vanishing of lower Waldhausen's groups $\widetilde{\mathrm{Nil}}^{W}_{n-1}(R; R^{\alpha}, R^{\beta})$ based on the isomorphism of $NK_n(R; R^{\alpha}, R^{\beta})$ and $\widetilde{\mathrm{Nil}}^{W}_{n-1}(R; R^{\alpha}, R^{\beta})$ for $n \leq 1$. $\square$

Now, we specialize to the case that $R = \mathbb{Z}H$, the group ring of a finite group H of order h. Let α and β be automorphisms of R induced by automorphisms of H. Choose a hereditary order Γ as in Theorem 1.6. Then we can define triples in $\mathcal{T}$,

$$\mathbf{R} = (R; R^{\alpha}, R^{\beta}),$$

$$\Gamma = (\Gamma; \Gamma^{\alpha}, \Gamma^{\beta}),$$

$$\mathbf{R}/h\Gamma = (R/h\Gamma; (R/h\Gamma)^{\alpha}, (R/h\Gamma)^{\beta}),$$

$$\Gamma/h\Gamma = (\Gamma/h\Gamma; (\Gamma/h\Gamma)^{\alpha}, (\Gamma/h\Gamma)^{\beta}).$$

The triples determine twisted polynomial rings

$$\begin{gathered} R_\rho \text{ corresponding to } \mathbf{R}, \\ \Gamma_\rho \text{ corresponding to } \Gamma, \\ (R/h\Gamma)_\rho \text{ corresponding to } \mathbf{R}/h\Gamma, \\ (\Gamma/h\Gamma)_\rho \text{ corresponding to } \Gamma/h\Gamma. \end{gathered} \tag{1}$$

Hence there is a Cartesian square

$$\begin{array}{ccc} RG & \longrightarrow & \Gamma \\ \downarrow & & \downarrow \\ RG/\underline{q} & \longrightarrow & \Gamma/\underline{q} \end{array} . \tag{2}$$

which implies that the square

$$\begin{array}{ccc} \begin{pmatrix} R & 0 \\ 0 & R \end{pmatrix} & \longrightarrow & \begin{pmatrix} \Gamma & 0 \\ 0 & \Gamma \end{pmatrix} \\ \downarrow & & \downarrow \\ \begin{pmatrix} R/h\Gamma & 0 \\ 0 & R/h\Gamma \end{pmatrix} & \longrightarrow & \begin{pmatrix} \Gamma/h\Gamma & 0 \\ 0 & \Gamma/h\Gamma \end{pmatrix} \end{array} \tag{3}$$

is a Cartesian square. By Theorem 3.1, we have the following Cartesian square

$$\begin{array}{ccc} R_\rho & \longrightarrow & \Gamma_\rho \\ \downarrow & & \downarrow \\ (R/h\Gamma)_\rho & \longrightarrow & (\Gamma/h\Gamma)_\rho \end{array} . \tag{4}$$

Theorem 3.3. *Let V be a virtually infinite cylic group in the second class having the form $V = G_0 *_H G_1$ where the groups $G_i, i = 0, 1$, and H are finite and $[G_i : H] = 2$. Then the Nil-groups*

$$NK_n(\mathbb{Z}H; \mathbb{Z}[G_0 - H], \mathbb{Z}[G_1 - H])$$

defined by the triple $(\mathbb{Z}H; \mathbb{Z}[G_0 - H], \mathbb{Z}[G_1 - H])$ are $|H|$-torsion when $n \geq 0$ and 0 when $n \leq -1$.

Proof. The proof is similar to that of Theorem 2.2(a)(i) using the Cartesian squares (2) and (3) above instead of (I) and (II) used in the proof of Theorem 2.2(a)(i). Details are left to the reader.

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