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# The mapping class group and the Meyer function for plane curves

**Yusuke Kuno**

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**Abstract** For each $d \geq 2$, *the mapping class group for plane curves of degree d* will be defined and it is proved that there exists uniquely *the Meyer function* on this group. In the case of $d = 4$, using our Meyer function, we can define the local signature for four-dimensional fiber spaces whose general fibers are non-hyperelliptic compact Riemann surfaces of genus 3. Some computations of our local signature will be given.

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## 0 Introduction

Let $\Sigma_g$ be a closed oriented $C^\infty$-surface of genus $g \geq 0$ and let $\Gamma_g$ be the mapping class group of $\Sigma_g$, namely the group of all isotopy classes of orientation preserving diffeomorphisms of $\Sigma_g$.

In [12] Meyer discovered and studied a cocycle $\tau_g \colon \Gamma_g \times \Gamma_g \to \mathbb{Z}$. For the sake of the reader a brief definition of $\tau_g$ will be given in Appendix. This cocycle is called *Meyer's signature cocycle*. In his paper Meyer showed that the cohomology class $[\tau_g] \in H^2(\Gamma_g; \mathbb{Z})$ is torsion for $g = 1, 2$ and has infinite order for $g \geq 3$, and gave an explicit formula for the unique $\mathbb{Q}$-valued 1-cochain of $\Gamma_1$ cobounding $\tau_1$ using the Rademacher function ([12], p. 259 Satz 4). Since the hyperelliptic mapping class group $\Gamma_g^H$, a subgroup of $\Gamma_g$, was shown to be $\mathbb{Q}$-acyclic by Cohen [5] and Kawazumi [9] independently, it was known to specialists that there exists the unique 1-cochain

Y. Kuno (✉)
Graduate School of Mathematical Sciences, The University of Tokyo,
3-8-1 Komaba Meguro-ku, Tokyo 153-0041, Japan
e-mail: kunotti@ms.u-tokyo.ac.jp

of $\Gamma_g^H$ cobounding $\tau_g$ restricted to $\Gamma_g^H$. In [7] Endo directly showed the existence and the uniqueness of such a 1-cochain $\phi_g^H : \Gamma_g^H \to \frac{1}{2g+1}\mathbb{Z}$ using a finite presentation of $\Gamma_g^H$ by Birman and Hilden [3]. He also defined the local signature for hyperelliptic fibrations using $\phi_g^H$, and studied the geometry of hyperelliptic fibrations; for example, he derived a signature formula for such fibrations over a closed surface. His formula originates from Matsumoto [11,Theorem 3.3] where genus 2 fibrations are discussed. For the study of the function $\phi_g^H$, see also Morifuji's paper [13].

The purpose of the present paper is to give another interesting example of these phenomena; *the Meyer function on the mapping class group for plane curves*.

For $d \geq 2$ a group $\Pi(d)$ and a homomorphism $\rho \colon \Pi(d) \to \Gamma_g$,where $g = \frac{1}{2}(d-1)(d-2)$, will be constructed. The group $\Pi(d)$ can be considered as the fundamental group of the classifying space for isotopy classes of continuous families of non-singular plane curves of degree $d$; the precise meaning of this statement will be given in Theorem 6.1 later.

The main results of this paper are Theorems 4.1 and 4.2. As a consequence of them it follows that the pull back $\rho^*[\tau_g]$ vanishes in the rational cohomology $H^2(\Pi(d);\mathbb{Q})$ and there exists the unique 1-cochain $\phi^d : \Pi(d) \to \mathbb{Q}$ such that $\delta\phi^d = \rho^*\tau_g$. $\phi^d$ will be called *the Meyer function for plane curves of degree $d$*.

This is similar to the case of $\Gamma_1$, $\Gamma_2$, and $\Gamma_g^H$, but we remark that the homomorphism $\rho$ seems no more injective nor surjective. In fact, for $d = 4$ we will see in Proposition 6.3 that $\rho$ is surjective but has non-trivial kernels. In this sense our result is different from the works of Meyer and Endo where *subgroups* of $\Gamma_g$ are considered.

While they did explicit computations of $\tau_g$ for certain relators of the mapping class groups to prove the vanishing of $[\tau_g]$, our method depends on the vanishing of $[\tau_g]$ pulled back to the cohomology of a fundamental group of the complement of a hypersurface in a complex vector space, which will be stated in Proposition 3.1 and proved using the definition of Meyer's signature cocycle and the standard argument in differential topology; the way from Proposition 3.1 to the vanishing of $[\tau_g]$ pulled back to $H^2(\Pi(d);\mathbb{Q})$ are elementary. Since this needs no explicit computations of $\tau_g$, we believe that our method has its own meaning to grasp the conceptual reason of the vanishing of $[\tau_g]$ and can be applied to other cases in the future.

Our study of the vanishing of $\rho^*[\tau_g] \in H^2(\Pi(d);\mathbb{Q})$ has a connection with localization of the signature of four-dimensional fiber spaces, that is a recent hot topic studied in various fields such as topology, algebraic geometry, and complex analysis (see [1,2]).

As an application of our study, especially $d = 4$, we define the local signature for the set of all fiber germs of four-dimensional fiber spaces whose general fibers are non-hyperelliptic compact Riemann surfaces of genus 3 by using our 1-cochain $\phi^4$ of $\Pi(4)$. The fact that any non-hyperelliptic compact Riemann surface of genus 3 can be realized as a smooth quartic curve in $\mathbb{P}^2$ by the canonical embedding, is crucial.

In this case of non-hyperelliptic family of genus 3, Ashikaga and Konno [2] and Yoshikawa [15] have already defined local signature independently. The definition of [2] is algebro geometric and that of [15] is complex analytic. We compute some examples of values of our local signature, defined by topological way, and observe that they coincide with those computed in [2,15].

## 1 Definitions

Throughout this paper, $d$ denotes a fixed integer $\geq 2$. Let $V^d$ be the complex vector space of homogeneous polynomials of degree $d$ in the determinates $x$,$y$, and $z$, and let $\mathbb{P}(d) = \mathbb{P}(V^d)$ be the projectivization of $V^d$. By taking the set of monomials $\{x^{\ell(k)}y^{m(k)}z^{n(k)}\}_{k=0}^{N}$ of degree $d$, where $N = \frac{1}{2}(d+2)(d+1) - 1$, each element of $V^d$ can be uniquely written as the form

$$\Phi = \sum_{k=0}^{N} a_k x^{\ell(k)} y^{m(k)} z^{n(k)},$$

where $a_k \in \mathbb{C}$. We denote the corresponding homogeneous coordinates of $\mathbb{P}(d)$ by $[a_0 : a_1 : \cdots : a_N]$. Each element $a \in \mathbb{P}(d)$ determines an algebraic curve $C_a \subset \mathbb{P}^2$ of degree $d$. Later we also denote by $C_F$ the algebraic curve defined by $F \in V^d \backslash \{0\}$. We believe this use of notation does not confuse the reader. Let $D$ be the set of points $a \in \mathbb{P}(d)$ such that the corresponding curve $C_a$ is singular. $D$ is called *the discriminant locus* and is well-known to be irreducible and of codimension 1. For a proof, see also the remark after the proof of Proposition 2.1 in this paper.

There is an action of $GL(3;\mathbb{C})$ on $V^d$ given by

$$(A \cdot F)(x, y, z) = F\left((x, y, z) \cdot {}^t A^{-1}\right),$$

where $A \in GL(3;\mathbb{C})$ and $F \in V^d$. Here ${}^t A$ is the transpose of the matrix $A$. This action induces the action of $PGL(3)$ on $\mathbb{P}(d)$, $D$, and $\mathbb{P}(d)\backslash D$.

Let $EPGL(3) \to BPGL(3)$ be the universal principal $PGL(3)$ bundle. We denote by $\Pi(d)$ the fundamental group of the Borel construction $(\mathbb{P}(d)\backslash D)_{PGL(3)} = EPGL(3) \times_{PGL(3)} (\mathbb{P}(d)\backslash D)$ and call this group *the mapping class group for plane curves of degree $d$*.

For $(e, a) \in EPGL(3) \times (\mathbb{P}(d)\backslash D)$, we denote by $[e, a]$ the element of $(\mathbb{P}(d)\backslash D)_{PGL(3)}$ represented by $(e, a)$. This notation concerning Borel construction will be used several times.

Let $\bar{\mathcal{F}}$ (respectively, $\mathcal{F}$) be the hypersurface in $\mathbb{P}(d) \times \mathbb{P}^2$ (respectively, $(\mathbb{P}(d)\backslash D) \times \mathbb{P}^2$) defined as the zero set of $\Phi$ considered as a bi-homogeneous polynomial in $a_0, \ldots, a_N$ and $x, y, z$. Then the restriction of the first projection $p\colon \mathcal{F} \to \mathbb{P}(d)\backslash D$ is a family of non-singular plane curves of degree $d$ whose fiber over $a \in \mathbb{P}(d)\backslash D$ is $C_a$. Since the diagonal action of $PGL(3)$ on $\mathbb{P}(d) \times \mathbb{P}^2$ preserves $\mathcal{F}$ and $p$ is $PGL(3)$-equivariant, we have a family of Riemann surfaces $p_u\colon \mathcal{F}_{PGL(3)} \to (\mathbb{P}(d)\backslash D)_{PGL(3)}$. We denote the topological monodromy (see Appendix) of this family by $\rho\colon \Pi(d) \to \Gamma_g$, where $g = \frac{1}{2}(d-1)(d-2)$. Note that the genus of a non-singular plane curve of degree $d$ is given by $\frac{1}{2}(d-1)(d-2)$.

In Sect. 4 we will prove that the rational cohomology class $\rho^*[\tau_g] \in H^2(\Pi(d);\mathbb{Q})$ vanishes and compute the abelianization of $\Pi(d)$. In Sect. 6 we will prove that the space $(\mathbb{P}(d)\backslash D)_{PGL(3)}$ is the classifying space of the set of all isotopy classes of continuous families of non-singular plane curves of degree $d$.

## 2 The discriminant locus

In this section we investigate the discriminant locus $D$, which also can be described in terms of dual variety as follows. For generality of dual variety, see [8] or [10]. Let $\mathbb{P}(d)^\vee$ be the dual projective space of $\mathbb{P}(d)$, i.e., the space of all hyperplanes of $\mathbb{P}(d)$. We denote by $[\alpha^0 : \alpha^1 : \cdots : \alpha^N]$ the homogeneous coordinates of $\mathbb{P}(d)^\vee$ corresponding to the homogeneous coordinates $[a_0 : a_1 : \cdots : a_N]$ of $\mathbb{P}(d)$; $\alpha = [\alpha^0 : \alpha^1 : \cdots : \alpha^N]$ is the hypersurface of $\mathbb{P}(d)$ defined by

$$\alpha^0 a_0 + \alpha^1 a_1 + \cdots + \alpha^N a_N = 0.$$

The Veronese embedding $v : \mathbb{P}^2 \to \mathbb{P}(d)^\vee$ is defined by

$$v([x : y : z]) = \left[x^{\ell(0)} y^{m(0)} z^{n(0)} : \cdots : x^{\ell(N)} y^{m(N)} z^{n(N)}\right].$$

Since the dual of $\mathbb{P}(d)^\vee$ is canonically isomorphic to $\mathbb{P}(d)$, each element $a \in \mathbb{P}(d)$ determines the hypersurface of $\mathbb{P}(d)^\vee$ which we denote by $H_a$. We set

$$\mathcal{X}' := \left\{(a, \alpha) \in \mathbb{P}(d) \times \mathbb{P}(d)^\vee \ ; \ \alpha \in v(\mathbb{P}^2) \text{ and } H_a \text{ is tangent to } v(\mathbb{P}^2) \text{ at } \alpha\right\}.$$

Then the image of $\mathcal{X}'$ by the first projection is just $D$, i.e., $D$ is the dual variety of $v(\mathbb{P}^2)$.

Let $\mathcal{X}$ be the analytic subset of $\mathbb{P}(d) \times \mathbb{P}^2$ defined by the equations

$$\Phi = \Phi_x = \Phi_y = \Phi_z = 0,$$

where $\Phi_x$ is the partial derivative of $\Phi$ with respect to $x$, etc. Thus if $(a, p)$ is a point of $\mathcal{X}$, then $a$ is a point of $D$ and $p$ is a singular point of $C_a$. Then we see that $\mathcal{X} \to \mathcal{X}'$, $(a, p) \mapsto (a, v(p))$ is an isomorphism. $\mathcal{X}'$ has the structure of fiber bundle over $v(\mathbb{P}^2)$ whose fiber over $\alpha \in v(\mathbb{P}^2)$ is the set of all hyperplanes in $\mathcal{X}'$ tangent to $v(\mathbb{P}^2)$ at $\alpha$, which is isomorphic to a $(N-3)$-dimensional projective space. From this point of view it is clear that $\mathcal{X}$ is non-singular (see also [8], p. 30), but for later consideration we give here an alternative proof using coordinate description.

**Proposition 2.1** *$\mathcal{X}$ is non-singular.*

*Proof* Let $(a^0, [x_0 : y_0 : z_0])$ be a point of $\mathcal{X}$. We will show $\mathcal{X}$ is non-singular at this point. Since the action of $PGL(3)$ on $\mathbb{P}(d) \times \mathbb{P}^2$ preserves $\mathcal{X}$, we may assume that $[x_0 : y_0 : z_0] = [0 : 0 : 1]$. Take a polynomial representative $F \in V^d$ of $a^0$, then the coefficient of $z^d$ of $F$ is zero because $[0 : 0 : 1] \in C_{a^0}$. Moreover, $F$ cannot be written as the form

$$F = (\alpha x + \beta y) z^{d-1},$$

where $(\alpha, \beta) \neq (0, 0)$ because $[0 : 0 : 1]$ is a singular point of $C_{a^0}$. Therefore there is a monomial $x^{\ell(k)} y^{m(k)} z^{n(k)}$ which is different from $z^d$, $xz^{d-1}$, and $yz^{d-1}$ such that the coefficient of $x^{\ell(k)}y^{m(k)}z^{n(k)}$ of $F$ is not zero. By a rearrangement of indices we may assume that $k=0$ and $a_1, a_2$ , and $a_3$ correspond to monomials $z^d$, $xz^{d-1}$, and $yz^{d-1}$, respectively. Then setting $a_0=1$ and $z=1$, we have an inhomogeneous coordinates $(a_1,\dots,a_N,x,y)$ of $\mathbb{P}(d)\times\mathbb{P}^2$ near $(a^0,[0\colon 0\colon 1])$. In this local coordinate system $\mathcal{X}$ is defined by the equations

$$\Psi=\Psi_x=\Psi_y=0,$$

where $\Psi=\Phi(1,a_1,\dots,a_N,x,y,1)$.

Now the Jacobian matrix of $(\Psi,\Psi_x,\Psi_y)$ at $(a^0,[0\colon 0\colon 1])$ is

$$\begin{aligned} J &= \begin{pmatrix} \Psi_{a_1} & \Psi_{a_2} & \Psi_{a_3} & \cdots & \Psi_x & \Psi_y \\ \Psi_{x,a_1} & \Psi_{x,a_2} & \Psi_{x,a_3} & \cdots & \Psi_{xx} & \Psi_{xy} \\ \Psi_{y,a_1} & \Psi_{y,a_2} & \Psi_{y,a_3} & \cdots & \Psi_{yx} & \Psi_{yy} \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & \Psi_{xx} & \Psi_{xy} \\ 0 & 0 & 1 & \cdots & 0 & \Psi_{yx} & \Psi_{yy} \end{pmatrix}, \end{aligned}$$

we see that the rank of $J$ is 3. This shows that $\mathcal{X}$ is non-singular at $(a^0,[0\colon 0\colon 1])$. □

Let $\pi\colon\mathcal{X}\to D\subset\mathbb{P}(d)$ be the first projection. The above proof shows that $(a^0,[0\colon 0\colon 1])$ is a regular point of $\pi$ if and only if

$$\det\begin{pmatrix} \Psi_{xx} & \Psi_{xy} \\ \Psi_{yx} & \Psi_{yy} \end{pmatrix}\neq 0$$

at $(a^0,[0\colon 0\colon 1])$. By an argument like the Morse lemma, we can take a coordinate system $(X,Y)$ of $\mathbb{P}^2$ centered at $[0\colon 0\colon 1]$ such that $C_{a^0}$ is locally given by the equation $X^2+Y^2=0$. Thus $[0\colon 0\colon 1]$ is a nodal singularity. This holds for other points of $\mathcal{X}$; $(a,p)\in\mathcal{X}$ is a regular point of $\pi$ if and only if $p$ is a nodal singularity of $C_a$.

Let $E$ be the union of singular points of $D$ and the $\pi$-image of critical points of $\pi$. $E$ is a proper analytic subset of $D$ by Sard's theorem.

Here we give a short proof that $D$ is irreducible and of codimension 1. At first, $\mathcal{X}\cong\mathcal{X}'$ is non-singular and connected hence irreducible. Therefore, $D=\pi(\mathcal{X})$ is also irreducible. On the other hand $D$ is at most $N-1$-dimensional because $D$ is a proper analytic subset of $\mathbb{P}(d)$. Let $a$ be a point of $D\backslash E$ and take a point $(a,p)$ in the fiber $\pi^{-1}(a)$. Then $D$ is smooth around $a$ and the differential of $\pi$ at $(a,p)$ is of maximal rank $N-1$. This shows $D$ is indeed $N-1$-dimensional. Note that $E$ is at most $N-2$-dimensional.

In the next lemma we shall describe the hyperplane of $\mathbb{P}(d)$ tangent to $D$ at a point in $D\backslash E$.

**Lemma 2.2** *Let $(a^0,[x_0\colon y_0\colon z_0])$ be a point of $\mathcal{X}$ and suppose that $a^0\in D\backslash E$. Then the hyperplane $T_{a^0}$ tangent to $D$ at $a^0$ is given by*

$$T_{a^0} = \left\{ [\xi_0 : \xi_1 : \cdots : \xi_N] \in \mathbb{P}(d) \; ; \; \sum_{k=0}^{N} \xi_k x_0^{\ell(k)} y_0^{m(k)} z_0^{n(k)} = 0 \right\}.$$

*Moreover,* $[x_0 : y_0 : z_0]$ *is the unique singular point of* $C_{a^0}$.

*Proof* To prove the first part, we may assume $a_0^0 = z_0 = 1$ and take an inhomogeneous coordinate system $(a_1, \ldots, a_N, x, y)$ of $\mathbb{P}(d) \times \mathbb{P}^2$ near $(a^0, [x_0 : y_0 : 1])$. Since $a^0$ is a non-singular point of $D$ and $(a^0, [x_0 : y_0 : 1])$ is a regular point of $\pi$, we have $T_{a^0} D = \tilde{\pi}_*(T_{(a^0,[x_0 : y_0 : 1])}\mathcal{X})$, where $\tilde{\pi}_* : T_{(a^0,[x_0 : y_0 : 1])}(\mathbb{P}(d) \times \mathbb{P}^2) \to T_{a^0}\mathbb{P}(d)$ is the differential of the first projection $\tilde{\pi} : \mathbb{P}(d) \times \mathbb{P}^2 \to \mathbb{P}(d)$ and we regard $T_{(a^0,[x_0 : y_0 : 1])}\mathcal{X}$ (respectively, $T_{a^0} D$) as the subspace of $T_{(a^0,[x_0 : y_0 : 1])}(\mathbb{P}(d) \times \mathbb{P}^2)$ (respectively, $T_{a^0}\mathbb{P}(d)$).

Now the Jacobian matrix $J$ appeared in the proof of Proposition 2.1 has the form

$$J = \begin{pmatrix} x_0^{\ell(1)} y_0^{m(1)} & \cdots & x_0^{\ell(N)} y_0^{m(N)} & 0 & 0 \\ * & \cdots & * & \Psi_{xx} & \Psi_{xy} \\ * & \cdots & * & \Psi_{yx} & \Psi_{yy} \end{pmatrix}$$

at $(a^0, [x_0 : y_0 : 1])$. The rank of this matrix is 3, because $\det \begin{pmatrix} \Psi_{xx} & \Psi_{xy} \\ \Psi_{yx} & \Psi_{yy} \end{pmatrix} \neq 0$ at $(a^0, [x_0 : y_0 : 1])$ by $a^0 \notin E$ and there is an index $i$ such that $x_0^{\ell(i)} y_0^{m(i)} \neq 0$.

Therefore

$$T_{(a^0,[x_0 : y_0 : 1])}\mathcal{X} = \left\{ \sum_{k=1}^{N} \xi_k \frac{\partial}{\partial a_k} + \xi_{N+1} \frac{\partial}{\partial x} + \xi_{N+2} \frac{\partial}{\partial y} \; ; \; J \begin{pmatrix} \xi_1 \\ \vdots \\ \xi_{N+2} \end{pmatrix} = 0 \right\}$$

and

$$T_{a^0} D = \tilde{\pi}_*(T_{(a^0,[x_0 : y_0 : 1])}\mathcal{X}) = \left\{ \sum_{k=1}^{N} \xi_k \frac{\partial}{\partial a_k} \; ; \; \sum_{k=1}^{N} \xi_k x_0^{\ell(k)} y_0^{m(k)} = 0 \right\}.$$

Interpreting this equation in terms of homogeneous coordinates of $\mathbb{P}(d)$, we obtain the desired description of $T_{a^0}$. The latter statement of the lemma follows from the form of $T_{a^0}$ just proved and the injectivity of the Veronese embedding. □

Combining the remark after the proof of Proposition 2.1, we can say more about the curve $C_{a^0}$:

**Lemma 2.3** *Let* $a^0 \in D \backslash E$ *and* $[x_0 : y_0 : z_0]$ *be as in Lemma 2.2. Then* $[x_0 : y_0 : z_0]$ *is a nodal singularity of* $C_{a^0}$, *and* $C_{a^0}$ *is irreducible except for* $d = 2$. *Thus if* $d \geq 3$ *the topological type of* $C_{a^0}$ *is Lefschetz singular fiber of type I, that is obtained by pinching a non-separating simple closed curve on* $\Sigma_g$ *into a point.*

*Proof* We only have to show the irreducibility of $C_{a^0}$ for $d \geq 3$. If $C_{a^0}$ is reducible it has two irreducible components $C_1$ and $C_2$ with degrees $d_1$ and $d_2$, and they intersect transversely at one point. We have $d_1 d_2 = 1$ by Bézout's theorem, but this contradicts to $d_1 + d_2 = d \geq 3$. □

The projective space $\mathbb{P}(d)$ can be regarded as the set of all complex lines through the origin in $V^d$. Let $\widetilde{D}$ (respectively, $\widetilde{E}$) be the union of all lines in $D$ (respectively, $E$). In the coordinate system $(a_0, \ldots, a_N)$ of $V^d$, the tangent space of $\widetilde{D}$ at $F \in \widetilde{D} \backslash \widetilde{E}$ is given by

$$T_F \widetilde{D} = \left\{ \sum_{k=0}^{N} \xi_k \frac{\partial}{\partial a_k} \; ; \; \sum_{k=0}^{N} \xi_k x_0^{\ell(k)} y_0^{m(k)} z_0^{n(k)} = 0 \right\},$$

where $[x_0 \colon y_0 \colon z_0]$ is the singular point of $C_F$. This follows from Lemma 2.2.

We shall prove a useful lemma which will be used in the next two sections. Let $\widetilde{\mathcal{F}}$ be the family of algebraic curves over $V^d \backslash \{0\}$ defined as in the case of $\bar{\mathcal{F}}$ over $\mathbb{P}(d)$.

**Lemma 2.4** *Let $B$ be a $C^\infty$-manifold of dimension $s \geq 2$ and $j \colon B \to V^d$ a $C^\infty$-map such that $j(B) \subset V^d \backslash \widetilde{E}$ and $j$ is transverse to $\widetilde{D}$. Then the total space $j^* \widetilde{\mathcal{F}}$ of the pull back of the family $\widetilde{\mathcal{F}}$ by $j$ is a $C^\infty$-manifold.*

*Proof* $j^* \widetilde{\mathcal{F}}$ is given by

$$j^* \widetilde{\mathcal{F}} = \left\{ (b, p) \in B \times \mathbb{P}^2 \; ; \; \Phi(j(b), p) = 0 \right\}$$

and it is easy to see that if $(b^0, p_0) \in j^* \widetilde{\mathcal{F}}$ and $p_0$ is a smooth point of $C_{j(b^0)}$ then $j^* \widetilde{\mathcal{F}}$ is smooth at $(b^0, p_0)$.

Suppose $(b^0, p_0) \in j^* \widetilde{\mathcal{F}}$ and $p_0 = [x_0 \colon y_0 \colon z_0]$ is the singular point of $C_{j(b^0)}$. Note that we have $j(b^0) \in \widetilde{D} \backslash \widetilde{E}$. Let $(j_0, j_1, \ldots, j_N)$ denote the $N + 1$-tuples of smooth functions on $B$ determined by $j$ and the coordinate system $(a_0, a_1, \ldots, a_N)$ of $V^d$. By the assumption of transversality and the description of $T_{j(b^0)} \widetilde{D}$ given above, we can choose a suitable local coordinate system $(b_1, \ldots, b_s)$ of $B$ around $b_0$ such that complex numbers

$$\sum_{k=0}^{N} \frac{\partial j_k}{\partial b_1}(b^0) x_0^{\ell(k)} y_0^{m(k)} z_0^{n(k)} \text{ and } \sum_{k=0}^{N} \frac{\partial j_k}{\partial b_2}(b^0) x_0^{\ell(k)} y_0^{m(k)} z_0^{n(k)}$$

are linearly independent over the real numbers. From this we can conclude that $j^* \widetilde{\mathcal{F}}$ is smooth at $(b^0, p_0)$. This completes the proof. □

We remark that in holomorphic category one can say more; if $B$ is a complex manifold of complex dimension $\geq 1$ and $j$ is holomorphic, $j^* \widetilde{\mathcal{F}}$ has a complex structure as a hypersurface in $B \times \mathbb{P}^2$.

## 3 The 1-cochain of $\pi_1(V^d\backslash\widetilde{D})$

Let $\chi_1\colon \pi_1(V^d\backslash\widetilde{D}) \to \pi_1(\mathbb{P}(d)\backslash D)$ be the homomorphism induced by the projection map $V^d\backslash\widetilde{D} \to \mathbb{P}(d)\backslash D$ and let $\chi_2\colon \pi_1(\mathbb{P}(d)\backslash D) \to \Pi(d)$ be the homomorphism induced by the inclusion map $\mathbb{P}(d)\backslash D \to (\mathbb{P}(d)\backslash D)_{PGL(3)}, a \mapsto [e_0, a]$ where $e_0$ is the base point of $EPGL(3)$. We set $\chi := \chi_2 \circ \chi_1$ and $\tilde{\rho} := \rho \circ \chi$. Then $\tilde{\rho}\colon \pi_1(V^d\backslash\widetilde{D}) \to \Gamma_g$ is the topological monodromy of the family over $V^d\backslash\widetilde{D}$ defined as in the case of $\mathcal{F} \to \mathbb{P}(d)\backslash D$.

In this section, we shall construct a 1-cochain $c\colon \pi_1(V^d\backslash\widetilde{D}) \to \mathbb{Z}$ and prove that $\delta c = \tilde{\rho}^*\tau_g$. The key is that $V^d\backslash\widetilde{E}$ is 2-connected, which follows from the fact that the complex codimension of $\widetilde{E}$ in $V^d$ is $\geq 2$. All of the spaces that we consider in this section as well as all of the maps are based.

We regard the circle $S^1$ as the boundary of the unit disk $D^2$ in $\mathbb{R}^2$. $D^2$ has the natural orientation induced by that of $\mathbb{R}^2$ and this induces the orientation of $S^1$ by counter clockwise manner. Let $\ell\colon S^1 \to V^d\backslash\widetilde{D}$ be a $C^\infty$-map. Since $V^d\backslash\widetilde{E}$ is simply connected we can extend $\ell$ to a $C^\infty$-map $\tilde{\ell}\colon D^2 \to V^d\backslash\widetilde{E}$. We may assume that $\tilde{\ell}$ is transverse to $\widetilde{D}$. By Lemma 2.4 $\tilde{\ell}^*\widetilde{\mathcal{F}}$ is a compact four-dimensional $C^\infty$-manifold with boundary and has the natural orientation induced by the orientation of $D^2$ and that of the fibers, which have the natural orientations as compact Riemann surfaces. Set

$$c([\ell]) := \mathrm{Sign}(\tilde{\ell}^*\widetilde{\mathcal{F}}),$$

where $[\ell]$ denotes the element of $\pi_1(V^d\backslash\widetilde{D})$ represented by $\ell$ and the right hand side is the signature of $\tilde{\ell}^*\widetilde{\mathcal{F}}$.

**Proposition 3.1** *The above definition of $c$ is well defined and $\delta c = \tilde{\rho}^*\tau_g$, i.e., $c$ is a cobounding cochain for $\tilde{\rho}^*\tau_g$.*

*Proof* We first show that $c$ is well defined. Let $\ell_0$ and $\ell_1$ are $C^\infty$-maps from $S^1$ to $V^d\backslash\widetilde{D}$, and suppose that they represent the same element of $\pi_1(V^d\backslash\widetilde{D})$. Then there exists a $C^\infty$-homotopy $H\colon S^1 \times [0, 1] \to V^d\backslash\widetilde{D}$ such that $H(\cdot, 0) = \ell_0$ and $H(\cdot, 1) = \ell_1$.

Regard the 2-sphere $S^2$ as the annulus $S^1 \times [0, 1]$ with two copies of $D^2$ attached along its two boundary circles $S^1 \times \{0\}$ and $S^1 \times \{1\}$. We denote by $D_0^2$ one of copies of $D^2$ attached to $S^1 \times \{0\}$ and $D_1^2$ the other. Using some extensions $\tilde{\ell}_i\colon D_i^2 \to V^d\backslash\widetilde{E}$ of $\ell_i$ for $i = 0$ and 1, $H$ extends to a $C^\infty$-map $\widetilde{H}\colon S^2 \to V^d\backslash\widetilde{E}$. We introduce the orientation of $S^2$ so that the inclusion $D_0^2 \hookrightarrow S^2$ is orientation preserving. Thus the other inclusion $D_1^2 \hookrightarrow S^2$ is orientation reversing.

Since $\pi_2(V^d\backslash\widetilde{E}) = 0$, we can extend $\widetilde{H}$ to a $C^\infty$-map $\bar{H}\colon D^3 \to V^d\backslash\widetilde{E}$ transverse to $\widetilde{D}\backslash\widetilde{E}$. Then $\bar{H}^*\widetilde{\mathcal{F}}$ is a $C^\infty$-manifold with boundary $\widetilde{H}^*\widetilde{\mathcal{F}}$. Since the signature of the boundary of a manifold is zero, we have by the Novikov additivity of the signature

$$\mathrm{Sign}(\tilde{\ell_0}^*\widetilde{\mathcal{F}}) - \mathrm{Sign}(\tilde{\ell_1}^*\widetilde{\mathcal{F}}) = 0,$$

so $c$ is well defined.

We next show the latter part. Let $\ell_0$ and $\ell_1$ be $C^\infty$-maps from $S^1$ to $V^d \backslash \widetilde{D}$. We will show

$$c([\ell_0]) + c([\ell_1]) - c([\ell_0][\ell_1]) = \tilde{\rho}^* \tau_g([\ell_0], [\ell_1]). \tag{1}$$

Let $P$ denote the pair of pants; this is the 2-sphere $S^2$ with the interior of the three disjoint closed disks removed. We also choose two of three boundary components of $P$ and denote them by $S_0^1$ and $S_1^1$, respectively. $S_0^1$ and $S_1^1$ have the natural orientations induced by that of $P$ and can be naturally identified with $S^1$. Since $P$ has the homotopy type of the bouquet $S^1 \vee S^1$, we can construct a $C^\infty$-map $L\colon P \to V^d \backslash \widetilde{D}$ such that the restriction of $L$ to $S_0^1$ (respectively, $S_1^1$) are exactly same as $\ell_0$ (respectively, $\ell_1$).

We notice that the restriction of $L$ to the remaining boundary component of $P$ with the natural orientation is homotopic to the inverse of the composition loop $\ell_0 \cdot \ell_1$. We also have $\mathrm{Sign}(L^* \widetilde{\mathcal{F}}) = -\tilde{\rho}^* \tau_g([\ell_0], [\ell_1])$ by the definition of Meyer's signature cocycle $\tau_g$. Using some extensions $\tilde{\ell_i}$ of $\ell_i$ for $i = 0$ and 1, and an extension $\widetilde{\ell_0 \cdot \ell_1}$ of $\ell_0 \cdot \ell_1$, $L$ extends to a $C^\infty$-map $\widetilde{L}\colon S^2 \to V^d \backslash \widetilde{E}$. Moreover $\widetilde{L}$ extends to a map $\bar{L}\colon D^3 \to V^d \backslash \widetilde{E}$ transverse to $\widetilde{D}$. We have $\mathrm{Sign}(\widetilde{L}^* \widetilde{\mathcal{F}}) = 0$ since $\widetilde{L}^* \mathcal{F}$ is the boundary of $\bar{L}^* \mathcal{F}$ hence we obtain by the Novikov additivity

$$0 = \mathrm{Sign}\left(\widetilde{L}^* \widetilde{\mathcal{F}}\right) = \mathrm{Sign}\left(\tilde{\ell_0}^* \widetilde{\mathcal{F}}\right) + \mathrm{Sign}\left(\tilde{\ell_1}^* \widetilde{\mathcal{F}}\right) - \mathrm{Sign}\left(\widetilde{\ell_0 \cdot \ell_1}^* \widetilde{\mathcal{F}}\right) + \mathrm{Sign}\left(L^* \widetilde{\mathcal{F}}\right),$$

that is just the Eq. (1). □

## 4 Main theorems

In this section we shall state and prove the main results of this paper. In Sect. 1 we defined the group $\Pi(d)$ and the homomorphism $\rho\colon \Pi(d) \to \Gamma_g$.

**Theorem 4.1** $\rho^*[\tau_g] = 0 \in H^2(\Pi(d); \mathbb{Q})$.

**Theorem 4.2** *The first homology group of* $\Pi(d)$ *is given as follows*:

$$H_1(\Pi(d); \mathbb{Z}) = \begin{cases} \mathbb{Z}/3(d-1)^2\mathbb{Z} & \text{if } d \equiv 0 \bmod 3, \\ \mathbb{Z}/(d-1)^2\mathbb{Z} & \text{if } d \equiv 1 \text{ or } 2 \bmod 3. \end{cases}$$

*In particular, we have* $H^1(\Pi(d); \mathbb{Q}) = 0$.

As an immediate consequence of these theorems, it follows that there exists the unique 1-cochain $\phi^d\colon \Pi(d) \to \mathbb{Q}$ such that $\delta\phi^d = \rho^* \tau_g$. We will call $\phi^d$ *the Meyer function for plane curves of degree* $d$.

The rest of this section will be devoted to the proof of these theorems. In Proposition 3.1 we have showed that $\tilde{\rho}^*[\tau_g] = 0 \in H^2(\pi_1(V^d \backslash \widetilde{D}); \mathbb{Z})$. Thus Theorem 4.1 follows from the following:

**Lemma 4.3** *The homomorphism*

$$\chi^* : H^2(\Pi(d); \mathbb{Q}) \to H^2(\pi_1(V^d \backslash \widetilde{D}); \mathbb{Q})$$

*induced by* $\chi$*, introduced in Sect. 3, is injective.*

*Proof* Recall that $\chi$ is the composition of $\chi_1$ and $\chi_2$. We first consider $\chi_1$. Let $\xi \in H^2(\mathbb{P}(d); \mathbb{Q})$ denote the first Chern class of the principal $\mathbb{C}^*$ bundle $V^d \backslash \{0\} \to \mathbb{P}(d)$. Then the restriction of $\xi$ to $\mathbb{P}(d)\backslash D$ is zero, for the first Chern class $c_1([D]) \in H^2(\mathbb{P}(d); \mathbb{Q})$ of the line bundle $[D]$ determined by the divisor $D$ of $\mathbb{P}(d)$ is a multiple of $\xi$ and of course the restriction of $c_1([D])$ to $\mathbb{P}(d)\backslash D$ is zero.

By the Gysin sequence

$$H^0(\mathbb{P}(d)\backslash D; \mathbb{Q}) \overset{\cup\xi}{\to} H^2(\mathbb{P}(d)\backslash D; \mathbb{Q}) \to H^2(V^d \backslash \widetilde{D}; \mathbb{Q})$$

of the principal $\mathbb{C}^*$ bundle $V^d \backslash \widetilde{D} \to \mathbb{P}(d)\backslash D$ we see that $H^2(\mathbb{P}(d)\backslash D; \mathbb{Q}) \to H^2(V^d \backslash \widetilde{D}; \mathbb{Q})$ is injective. Therefore

$$\chi_1^* : H^2(\pi_1(\mathbb{P}(d)\backslash D); \mathbb{Q}) \to H^2(\pi_1(V^d \backslash \widetilde{D}); \mathbb{Q})$$

is also injective.

We next consider $\chi_2$. By the homotopy exact sequence of the $\mathbb{P}(d)\backslash D$ bundle $(\mathbb{P}(d)\backslash D)_{PGL(3)} \to BPGL(3)$, $[e, a] \mapsto \varpi(e)$ where $\varpi$ denotes the projection map $EPGL(3) \to BPGL(3)$, we have an exact sequence

$$\mathbb{Z}/3\mathbb{Z} \cong \pi_2(BPGL(3)) \to \pi_1(\mathbb{P}(d)\backslash D) \overset{\chi_2}{\to} \Pi(d) \to 1. \quad (2)$$

This implies that

$$\chi_2^* : H^2(\Pi(d); \mathbb{Q}) \to H^2(\pi_1(\mathbb{P}(d)\backslash D); \mathbb{Q})$$

is isomorphic. Since $\chi^* = \chi_1^* \circ \chi_2^*$, the lemma follows. □

We next proceed to Theorem 4.2. In the following we consider (co)homology with coefficients in $\mathbb{Z}$. We need the following two lemmas:

**Lemma 4.4** *Let* $a^0 \in \mathbb{P}(d)\backslash D$ *and denote by* $\mathbb{P}$ *the set of all projective lines in* $\mathbb{P}(d)$ *through* $a^0$*. Then there exist a non-empty Zariski open subset* $U \subset \mathbb{P}$ *such that each element of* $U$ *does not meet* $E$ *and is transverse to* $D$*.*

*Proof* Consider the projection with center $a^0$

$$f : D \to \mathbb{P}, \quad f(a) = \text{the line through } a^0 \text{ and } a.$$

Note that for $a \in D\backslash E$, $f$ is critical at $a$ if and only if $f(a)$ is contained in the hyperplane $T_a$ appeared in Lemma 2.2, namely $f(a)$ is not transverse to $D$ at $a$.

$\mathbb{P}$ is a $(N-1)$-dimensional projective space and $f(E)$ is a proper algebraic set in $\mathbb{P}$ since $\dim E \leq N-2$. Let $K$ denote the set of all critical values of $f \circ \pi : \mathcal{X} \to \mathbb{P}$. $K$ contains all critical values of $f|_{D\setminus E}$ since $\pi|_{\pi^{-1}(D\setminus E)} : \pi^{-1}(D\setminus E) \to D\setminus E$ is biholomorphic by Lemma 2.2, and is algebraic and proper because $K$ is nowhere dense in $\mathbb{P}$ by Sard's theorem. Therefore if we set

$$U := \mathbb{P}\setminus(f(E) \cup K),$$

$U$ has the desired property. □

**Lemma 4.5** *Let $a^0$ and $U$ be as in Lemma 4.4. For each $Q \in U$ the invariants of the complex surface $M = \{(a, p) \in Q \times \mathbb{P}^2\ ;\ \ p \in C_a\}$ is given as follows*:

$$c_1^2(M) = -d^2 + 9, \quad c_2(M) = d^2 + 3, \quad \mathrm{Sign}(M) = 1 - d^2.$$

*Proof* Since $Q \cong \mathbb{P}^1$ we can regard $M$ as a smooth hypersurface in $\mathbb{P}^1 \times \mathbb{P}^2$ determined by a $(1, d)$ homogeneous polynomial. For $i = 1$ and 2, respectively, we denote by $\xi_i \in H^2(\mathbb{P}^1 \times \mathbb{P}^2; \mathbb{Z})$ the pull back of the first Chern class of $\mathcal{O}(1)$ by $H^2(\mathbb{P}^i; \mathbb{Z}) \to H^2(\mathbb{P}^1 \times \mathbb{P}^2; \mathbb{Z})$ induced by the projection $\mathbb{P}^1 \times \mathbb{P}^2 \to \mathbb{P}^i$. Here $\mathcal{O}(1)$ denotes the dual of the tautological line bundle over $\mathbb{P}^i$. The first Chern class of the line bundle $[M]$ determined by the divisor $M$ of $\mathbb{P}^1 \times \mathbb{P}^2$ is $c_1([M]) = \xi_1 + d\xi_2$. Therefore by the adjunction formula, the first Chern class of $M$ is

$$\begin{aligned} c_1(M) &= \Big(c_1(\mathbb{P}^1 \times \mathbb{P}^2) - c_1([M])\Big)\,|_M \\ &= (2\xi_1 + 3\xi_2 - (\xi_1 + d\xi_2))\,|_M \\ &= (\xi_1 + (3-d)\xi_2)\,|_M. \end{aligned}$$

Then the Chern number $c_1^2(M)$ is computed as follows:

$$\begin{aligned} c_1^2(M) &= \Big\langle c_1(M)^2, \mu_M \Big\rangle \\ &= \Big\langle (\xi_1 + (3-d)\xi_2)^2 c_1([M]), \mu_{\mathbb{P}^1\times\mathbb{P}^2} \Big\rangle \\ &= \Big\langle (\xi_1 + (3-d)\xi_2)^2 (\xi_1 + d\xi_2), \mu_{\mathbb{P}^1\times\mathbb{P}^2} \Big\rangle \\ &= \Big\langle (-d^2 + 9)\xi_1\xi_2^2, \mu_{\mathbb{P}^1\times\mathbb{P}^2} \Big\rangle \\ &= -d^2 + 9. \end{aligned}$$

Here $\mu_M$ (respectively, $\mu_{\mathbb{P}^1\times\mathbb{P}^2}$) denotes the fundamental homology class of $M$ (respectively, $\mathbb{P}^1 \times \mathbb{P}^2$) and $\langle -, - \rangle$ denotes the Kronecker pairing between cohomology and homology. We next compute $c_2(M)$. Again by the adjunction formula, the second

Chern class of $M$ is

$$
\begin{aligned}
c_2(M) &= c_2(\mathbb{P}^1 \times \mathbb{P}^2)|_M - c_1(M) \cdot c_1([M])|_M \\
&= \Big(3\xi_2^2 + 6\xi_1\xi_2 - (\xi_1 + (3-d)\xi_2)(\xi_1 + d\xi_2)\Big)\,|_M \\
&= \Big(3\xi_1\xi_2 + (d^2 - 3d + 3)\xi_2^2\Big)\,|_M,
\end{aligned}
$$

and the Chern number which will also be denoted by $c_2(M)$ is

$$
\begin{aligned}
c_2(M) &= \langle c_2(M), \mu_M \rangle \\
&= \Big\langle (3\xi_1\xi_2 + (d^2 - 3d + 3)\xi_2^2) c_1([M]), \mu_{\mathbb{P}^1 \times \mathbb{P}^2} \Big\rangle \\
&= \Big\langle (3\xi_1\xi_2 + (d^2 - 3d + 3)\xi_2^2)(\xi_1 + d\xi_2), \mu_{\mathbb{P}^1 \times \mathbb{P}^2} \Big\rangle \\
&= \Big\langle (d^2 + 3)\xi_1\xi_2^2, \mu_{\mathbb{P}^1 \times \mathbb{P}^2} \Big\rangle \\
&= d^2 + 3.
\end{aligned}
$$

Finally by the Hirzebruch signature theorem we have $\mathrm{Sign}(M) = \frac{1}{3}(c_1^2(M) - 2c_2(M)) = 1 - d^2$. □

Let $a^0$ and $Q \in U$ be as in Lemma 4.5. Using the above two lemmas we can compute the first homology group of $\pi_1(\mathbb{P}(d)\backslash D)$:

**Proposition 4.6** $H_1(\pi_1(\mathbb{P}(d)\backslash D); \mathbb{Z}) = \mathbb{Z}/3(d-1)^2\mathbb{Z}$.

*Proof* The first projection $g\colon M \to Q$ is a family of algebraic curves, whose all singular fibers are of type I by Lemma 2.3. Since the Euler contribution (see [4], p. 118, (11.4) Proposition) of a singular fiber of type I is $+1$, the number of singular fibers of $g\colon M \to Q \cong \mathbb{P}^1$ is

$$
c_2(M) - 2(2 - 2g) = d^2 + 3 - 2\left(2 - 2 \cdot \frac{1}{2}(d-1)(d-2)\right) = 3(d-1)^2. \quad (3)
$$

Now consider the following commutative diagram:

$$
\begin{array}{ccccccc}
\mathbb{Z} \cong H_2(\mathbb{P}(d)) & \longrightarrow & H_2(\mathbb{P}(d), \mathbb{P}(d)\backslash D) & \longrightarrow & H_1(\mathbb{P}(d)\backslash D) & \longrightarrow & 0 \\
\cong \downarrow & & \downarrow \cong & & & & \\
H^{2N-2}(\mathbb{P}(d)) & \xrightarrow{\iota^*} & H^{2N-2}(D) \cong \mathbb{Z}. & & & &
\end{array}
$$

Here the vertical isomorphisms are Poincaré duality and the first horizontal sequence is a part of the homology exact sequence of the pair $(\mathbb{P}(d), \mathbb{P}(d)\backslash D)$, and $\iota^*$ is induced by the inclusion $D \hookrightarrow \mathbb{P}(d)$. For a generator of $H_2(\mathbb{P}(d))$ we can choose $[Q]$. We can conclude this generator is mapped to $3(d-1)^2$ times a generator of $H^{2N-2}(D)$ in the above diagram, because (3) shows that $Q$ and $D$ intersect transversally in $3(d-1)^2$

points. This completes the proof, since $H_1(\mathbb{P}(d)\backslash D) \cong H_1(\pi_1(\mathbb{P}(d)\backslash D);\mathbb{Z})$ is isomorphic to the cokernel of $H_2(\mathbb{P}(d)) \to H_2(\mathbb{P}(d), \mathbb{P}(d)\backslash D)$. □

Now we start the proof of Theorem 4.2. Let $F_0 \in V^d\backslash\widetilde{D}$ be a base point and $a^0$ the image of $F_0$ under the map $V^d\backslash\widetilde{D} \to \mathbb{P}(d)\backslash D$. We consider the maps $\lambda\colon GL(3;\mathbb{C}) \to V^d\backslash\widetilde{D}$, $A \mapsto A\cdot F_0$ and $\bar{\lambda}\colon PGL(3) \to \mathbb{P}(d)\backslash D$, $\bar{A} \mapsto \bar{A}\cdot a^0$. Since the isomorphism $\pi_2(BPGL(3)) \cong \pi_1(PGL(3))$ induced by the homotopy exact sequence of the universal $PGL(3)$ bundle $EPGL(3) \to BPGL(3)$ is compatible with (2) and $\bar{\lambda}_*\colon \pi_1(PGL(3)) \to \pi_1(\mathbb{P}(d)\backslash D)$, we have an exact sequence of group homology

$$\mathbb{Z}/3\mathbb{Z}\cong H_1(\pi_1(PGL(3))) \overset{\bar{\lambda}_*}{\to} H_1(\pi_1(\mathbb{P}(d)\backslash D))\cong\mathbb{Z}/3(d-1)^2\mathbb{Z} \overset{\chi_{2*}}{\to} H_1(\Pi(d))\to 0.$$

Therefore we must compute the map $\bar{\lambda}_*$ to determine $H_1(\Pi(d))$.

For this purpose, we consider the following exact sequence

$$\mathbb{Z} \cong H_1(\mathbb{C}^*) \to H_1(V^d\backslash\widetilde{D}) \to H_1(\mathbb{P}(d)\backslash D) \to 0$$

induced by a part of the homotopy exact sequence of the principal $\mathbb{C}^*$ bundle $V^d\backslash\widetilde{D} \to \mathbb{P}(d)\backslash D$. We have $H_1(V^d\backslash\widetilde{D}) \cong \mathbb{Z}$ (see [6], chap. 4, Corollary (1.4)). Let $\gamma$ be the generator of $H_1(\mathbb{C}^*)$ represented by the loop $\gamma(t) = e^{2\pi\sqrt{-1}t}$, $0 \le t \le 1$. By Proposition 4.6 we see that the image of $\gamma$, which is represented by the loop $t \mapsto e^{2\pi\sqrt{-1}t}\cdot F_0$, is $3(d-1)^2$ times a generator of $H_1(V^d\backslash\widetilde{D})$. On the other hand the loop

$$t \mapsto \begin{pmatrix} e^{2\pi\sqrt{-1}t} & 0 & 0 \\ 0 & e^{2\pi\sqrt{-1}t} & 0 \\ 0 & 0 & e^{2\pi\sqrt{-1}t} \end{pmatrix}, \quad 0 \le t \le 1$$

in $GL(3;\mathbb{C})$, representing three times a generator of $H_1(GL(3;\mathbb{C})) \cong \mathbb{Z}$, is mapped to the loop $t \mapsto (e^{2\pi\sqrt{-1}t})^{-d}\cdot F_0$ by $\lambda$. Hence in the commutative diagram

$$\begin{array}{ccc} \mathbb{Z} \cong H_1(GL(3;\mathbb{C})) & \xrightarrow{\lambda_*} & H_1(V^d\backslash\widetilde{D}) \\ \downarrow & & \downarrow \\ \mathbb{Z}/3\mathbb{Z} \cong H_1(PGL(3)) & \xrightarrow{\bar{\lambda}_*} & H_1(\mathbb{P}(d)\backslash D) \end{array}$$

we have $\lambda_*(1) = \pm d(d-1)^2 \in \mathbb{Z} \cong H_1(V^d\backslash\widetilde{D})$ so we can conclude $\bar{\lambda}_*(1 \bmod 3) = \pm d(d-1)^2 \bmod 3(d-1)^2$. This completes the proof of Theorem 4.2.

## 5 The value of the Meyer function

By Proposition 4.6, we have $H^1(\pi_1(\mathbb{P}(d)\backslash D);\mathbb{Q}) = 0$. Therefore $\bar{\phi}^d := \phi^d \circ \chi_2$ is the unique 1-cochain of $\pi_1(\mathbb{P}(d)\backslash D)$ satisfying $(\rho\circ\chi_2)^*\tau_g = \delta\bar{\phi}^d$. In this section we will compute the value of $\bar{\phi}^d$ on a special element in $\pi_1(\mathbb{P}(d)\backslash D)$ so called *lasso*.

We first explain what a lasso is. Let $M$ be a connected complex manifold of dimension $m$ and $N$ an irreducible hypersurface of $M$. Then the inclusion $M\backslash N \hookrightarrow M$ induces the following exact sequence:

$$1 \to \langle\sigma\rangle \to \pi_1(M\backslash N) \to \pi_1(M) \to 1.$$

Here $\langle\sigma\rangle$ denotes the normal closure of an element $\sigma$ of $\pi_1(M\backslash N)$, which is described in the following. Let $p$ be a non-singular point of $N$ and $(z_1, \dots, z_m)$ a local coordinate system of $M$ around $p$ such that $N$ is defined by $z_1 = 0$. For a sufficiently small $\varepsilon > 0$, consider a loop defined in this coordinate system by

$$[0, 1] \to M\backslash N, \ t \mapsto (\varepsilon e^{2\pi\sqrt{-1}t}, 0, \dots, 0)$$

based at $q = (\varepsilon, 0, \dots, 0)$. Joining this loop with a path from the base point of $M\backslash N$ to $q$, we get an element $\sigma$ of $\pi_1(M\backslash N)$. Since $N$ is irreducible, the conjugacy class of $\sigma$ in $\pi_1(M\backslash N)$ is independent of choices of $p$ and a local coordinate system. Each element of this conjugacy class is called *a lasso* around $N$.

Returning to $\pi_1(\mathbb{P}(d)\backslash D)$, $D$ is an irreducible hypersurface of $\mathbb{P}(d)$. Let $\sigma^d \in \pi_1(\mathbb{P}(d)\backslash D)$ be a lasso around $D$. Since $\bar{\phi}^d$ is a class function (see Lemma 8.2 in Appendix), the values of $\bar{\phi}^d$ on the conjugacy class of $\sigma^d$ is constant.

**Proposition 5.1** *For $d \geq 3$,*

$$\bar{\phi}^d(\sigma^d) = -\frac{d+1}{3(d-1)}.$$

*Proof* Choose $a^0$ and $Q \in U$ as in Lemma 4.5. In the proof of Proposition 4.6 we see that $Q$ meets $D$ transversely in $3(d-1)^2$ points. Let $Q \cap D = \{q_1, \dots, q_{3(d-1)^2}\}$ and let $D_i$ $(i = 1, \dots, 3(d-1)^2)$ be a small closed 2-disk in $Q$ such that $q_i \in \operatorname{Int} D_i$ and $D_i \cap D_j = \emptyset$ for $i \neq j$. We fix a base point of $Q_0 := Q\backslash \bigcup_{i=1}^{3(d-1)^2} \operatorname{Int} D_i$ and for each $i = 1, \dots, 3(d-1)^2$, choose a based loop $\sigma_i$ in $Q_0$ such that $\sigma_i$ is free homotopic to the loop traveling once the boundary $\partial D_i$ by counter clockwise manner. Note that regarded as an element in $\pi_1(\mathbb{P}(d)\backslash D)$, $\sigma_i$ is a lasso around $D$ hence we have $\bar{\phi}^d(\sigma_i) = \bar{\phi}^d(\sigma^d)$.

Let $g \colon M \to Q$ be as in the proof of Proposition 4.6 and set $M_0 := g^{-1}(Q_0)$ and $M_i := g^{-1}(D_i)$, $i = 1, \dots, 3(d-1)^2$. By Meyer's signature formula ([12] Satz 1) and the equation $(\rho \circ \chi_2)^* \tau_g = \delta\bar{\phi}^d$, we obtain

$$\operatorname{Sign}(M_0) = \sum_{i=1}^{3(d-1)^2} \bar{\phi}^d(\sigma_i) = 3(d-1)^2 \bar{\phi}^d(\sigma^d).$$

Since the topological type of $g^{-1}(q_i)$ is Lefschetz singular fiber of type I, we have $\operatorname{Sign}(M_i) = 0$. We compute by the Novikov additivity and Lemma 4.5

$$1 - d^2 = \mathrm{Sign}(M) = \mathrm{Sign}(M_0) + \sum_{i=1}^{3(d-1)^2} \mathrm{Sign}(M_i) = 3(d-1)^2 \bar{\phi}^d(\sigma^d).$$

This completes the proof. □

In the rest of this section we consider the remaining case $d = 2$. Since $V^2$ is the set of quadratic forms each element of $V^2$ can be expressed by a $3 \times 3$ symmetric matrix $S$. In this view point $V^2 \backslash \widetilde{D}$ is the space of non-singular symmetric matrices and the action of $GL(3; \mathbb{C})$ on $V^2 \backslash \widetilde{D}$ is given by

$$A \cdot S = {}^tA^{-1} \cdot S \cdot A^{-1}, \quad A \in GL(3; \mathbb{C}).$$

Since this action is transitive and the isotropy group of the unit matrix is the complex orthogonal group $O_3(\mathbb{C}) = \{A \in GL(3; \mathbb{C}) \ ; \ {}^tA \cdot A = I\}$, we have

$$V^2 \backslash \widetilde{D} \cong GL(3; \mathbb{C}) / O_3(\mathbb{C}).$$

Also we have

$$\mathbb{P}(2) \backslash D \cong PGL(3) / SO_3(\mathbb{C}),$$

where $SO_3(\mathbb{C}) = \{A \in O_3(\mathbb{C}) \ ; \ \det A = 1\}$ is regarded as a subgroup of $PGL(3)$ by the injection $SO_3(\mathbb{C}) \hookrightarrow PGL(3)$ induced by the projection $GL(3; \mathbb{C}) \to PGL(3)$. Therefore, we obtain

$$\begin{aligned}(\mathbb{P}(2) \backslash D)_{PGL(3)} &= EPGL(3) \times_{PGL(3)} (\mathbb{P}(2) \backslash D) \\ &\cong EPGL(3)/SO_3(\mathbb{C}) = BSO_3(\mathbb{C}) \simeq BSO_3.\end{aligned}$$

The last homotopy equivalence holds because the natural inclusion $SO_3 \hookrightarrow SO_3(\mathbb{C})$ is homotopy equivalence. In particular, we have

$$\Pi(2) \cong \pi_1(BSO_3) = 1.$$

## 6 The universal property of $(\mathbb{P}(d) \backslash D)_{PGL(3)}$

In this section we will show the universal property of the space $(\mathbb{P}(d) \backslash D)_{PGL(3)}$. In the latter part of the section, we consider the case $d = 4$ more detail; in particular, we prove that $\rho \colon \Pi(4) \to \Gamma_3$ is surjective.

We first make some definitions. Let $\iota \colon X \to P$ be a continuous map and $h \colon P \to B$ a $\mathbb{P}^2$ bundle whose structure group is $PGL(3)$. We call $\xi = (X, \iota, P, h, B)$ *a family of non-singular plane curves of degree* $d$ if

1. $p := h \circ \iota \colon X \to B$ is a continuous family of compact Riemann surfaces of genus $g = \frac{1}{2}(d-1)(d-2)$, and
2. for each $b \in B$, the restriction $\iota|_{X_b} \colon X_b \to P_b$ is a holomorphic embedding where $X_b = p^{-1}(b)$ and $P_b = h^{-1}(b)$.

For each $b \in B$, the image $\iota(X_b) \subset P_b \cong \mathbb{P}^2$ is a non-singular plane curve of degree $d$. Two such families $\xi_i = (X^i, \iota_i, P^i, h_i, B)$, $i = 0, 1$, are called *isotopic* if there exists a family of non-singular curves of degree $d$ over $B \times [0, 1]$, denoted by $\tilde{\xi} = (\widetilde{X}, \tilde{\iota}, \widetilde{P}, \tilde{h}, B \times [0, 1])$, such that for $i = 0, 1$, the restriction of $\tilde{\xi}$ to $B \times \{i\}$ is isomorphic to $\xi_i$, i.e., for $i = 0, 1$, there exists a homeomorphism $\Psi_i \colon P^i \to \widetilde{P}|_{B\times\{i\}}$ and $\psi_i \colon X^i \to \widetilde{X}|_{B\times\{i\}}$ such that the diagram

$$\begin{array}{ccc} X^i & \xrightarrow{\psi_i} & \widetilde{X}|_{B\times\{i\}} \\ \downarrow{\scriptstyle \iota_i} & & \downarrow{\scriptstyle \tilde{\iota}} \\ P^i & \xrightarrow{\Psi_i} & \widetilde{P}|_{B\times\{i\}} \\ \downarrow{\scriptstyle h_i} & & \downarrow{\scriptstyle \tilde{h}} \\ B & \longrightarrow & B \times \{i\}, \end{array}$$

where the last horizontal arrow is the homeomorphism $B \to B \times \{i\}$ given by $b \mapsto (b, i)$, commutes and $\Psi_i$ (respectively, $\psi_i$) maps each fiber $P^i_b$ (respectively, $X^i_b$) onto $\tilde{h}^{-1}(b, i)$ (respectively, $(\tilde{h} \circ \tilde{\iota})^{-1}(b, i)$) biholomorphically.

For a given space $B$, we denote by $\mathcal{PC}_d(B)$ the set of all isotopy classes of families of non-singular plane curves of degree $d$ over $B$. $\mathcal{PC}_d(\bullet)$ is contravariant; for a given continuous map $f \colon B' \to B$ we have a natural map $\mathcal{PC}_d(B) \to \mathcal{PC}_d(B')$ which assigns the isotopy class of $\xi$ the isotopy class of the pull back of $\xi$ by $f$, which will be denoted by $f^*\xi$. In fact, the isotopy class of $f^*\xi$ is uniquely determined by the homotopy class $[f] \in [B', B]$.

Among such families of non-singular plane curves of degree $d$, there is a universal one. Consider the inclusion map $\mathcal{F} \hookrightarrow (\mathbb{P}(d)\backslash D) \times \mathbb{P}^2$ and the first projection $(\mathbb{P}(d)\backslash D) \times \mathbb{P}^2 \to \mathbb{P}(d)\backslash D$. For simplicity, we write $Y$ instead of $(\mathbb{P}(d)\backslash D) \times \mathbb{P}^2$. Since these maps are $PGL(3)$-equivariant, we obtain

$$\iota_u \colon \mathcal{F}_{PGL(3)} \to Y_{PGL(3)}$$

and

$$h_u \colon Y_{PGL(3)} \to (\mathbb{P}(d)\backslash D)_{PGL(3)}.$$

The map $p_u := h_u \circ \iota_u$ is the same as the map defined in Sect. 1 and

$$\xi_u := \left(\mathcal{F}_{PGL(3)}, \iota_u, Y_{PGL(3)}, h_u, (\mathbb{P}(d)\backslash D)_{PGL(3)}\right)$$

is a family of non-singular plane curves of degree $d$. The next theorem says that $(\mathbb{P}(d)\backslash D)_{PGL(3)}$ is the classifying space for the functor $\mathcal{PC}_d(\bullet)$ and $\xi_u$ is the universal family.

**Theorem 6.1** *For any space $B$, the map*

$$\eta\colon [B, (\mathbb{P}(d)\backslash D)_{PGL(3)}] \to \mathcal{PC}_d(B)$$

*which assigns the homotopy class of $f\colon B \to (\mathbb{P}(d)\backslash D)_{PGL(3)}$ the isotopy class of the pull back $f^*\xi_u$, is bijective.*

In the following we shall construct the inverse of $\eta$.

Let $\xi = (X, \iota, P, h, B)$ be given. We divide the argument in three steps.

*Step 1* We first consider the case when $P$ is trivial: suppose that $P = B \times \mathbb{P}^2$. Then for each $b \in B$, $\iota(X_b) \subset \{b\} \times \mathbb{P}^2 \cong \mathbb{P}^2$ is a non-singular plane curve of degree $d$, so the defining equation of $\iota(X_b)$ in $\mathbb{P}^2$ is uniquely determined as an element of $\mathbb{P}(d)\backslash D$. Denoting it by $\mathrm{Eq}(b)$, we obtain a map

$$\mathrm{Eq}\colon B \to \mathbb{P}(d)\backslash D.$$

**Lemma 6.2** *The map* Eq *is continuous.*

*Proof* Regard $\mathbb{P}^2$ as the set of all complex lines through the origin in $\mathbb{C}^3$. Then the holomorphic line bundle $\mathcal{O}(d)$ over $\mathbb{P}^2$ is given by

$$\mathcal{O}(d) = \mathcal{O}(1)^{\otimes d} = \bigcup_{\ell \in \mathbb{P}^2} \mathrm{Hom}(\ell, \mathbb{C})^{\otimes d}.$$

Let $p_2\colon B \times \mathbb{P}^2 \to \mathbb{P}^2$ be the second projection and consider the pull back $L := (p_2 \circ \iota)^*\mathcal{O}(d)$. $L \to X$ is a continuous family over $B$ of holomorphic vector bundles. Now $H^0(\mathbb{P}^2; \mathcal{O}(d))$ is canonically isomorphic to $V^d$ and for each $b \in B$ there is the natural homomorphism

$$\sigma_b\colon V^d \cong H^0(\mathbb{P}^2; \mathcal{O}(d)) \to H^0\left(\iota(X_b); \mathcal{O}(d)|_{\iota(X_b)}\right) \cong H^0(X_b; L_b),$$

where $L_b$ is the restriction of $L$ to $X_b$. Combining all $\sigma_b$, $b \in B$ together, we obtain a homomorphism of vector bundles

$$\sigma : B \times V^d \to \bigcup_{b \in B} H^0(X_b; L_b).$$

We see that for each $b \in B$, $\sigma_b$ is surjective and its kernel is one-dimensional generated by the defining equation of $\iota(X_b)$, i.e., $\mathrm{Eq}(b) = \ker\sigma_b$. This shows that Eq is continuous. □

We also define a $PGL(3)$-equivariant continuous map $\Psi\colon B \times PGL(3) \to \mathbb{P}(d)\backslash D$ by

$$\Psi(b, g) = g \cdot \mathrm{Eq}(b).$$

Here we regard $B \times PGL(3)$ as the trivial principal $PGL(3)$ bundle with *left* $PGL(3)$ action.

*Step 2* We next consider the general case $\xi = (X, \iota, P, h, B)$. Let $\{U_i\}_{i \in I}$ be an open covering of $B$ trivializing $h\colon P \to B$: There is an isomorphism $\varphi_i\colon h^{-1}(U_i) \to U_i \times \mathbb{P}^2$ for each $i$ and a system of transition functions $g_{ij}\colon U_i \cap U_j \to PGL(3)$ for each $(i, j)$ satisfying $U_i \cap U_j \neq \emptyset$, such that

$$(\varphi_i \circ \varphi_j^{-1})(b, p) = (b, g_{ij}(b) \cdot p), \quad b \in U_i \cap U_j, \quad p \in \mathbb{P}^2.$$

As in Step 1, we have a continuous map $\mathrm{Eq}^i\colon U_i \to \mathbb{P}(d)\backslash D$ and a $PGL(3)$-equivariant map $\Psi^i\colon U_i \times PGL(3) \to \mathbb{P}(d)\backslash D$ for each $i$. Let $Q(\xi)$ be a principal $PGL(3)$ bundle over $B$ associated to $h\colon P \to B$: namely $Q(\xi)$ is constructed from the disjoint union $\coprod_{i \in I} U_i \times PGL(3)$ by identifying $(b, g) \in U_i \times PGL(3)$ with $(b, g \cdot g_{ij}(b)) \in U_j \times PGL(3)$ where $b \in U_i \cap U_j$. We have $g_{ij}(b) \cdot \mathrm{Eq}^j(b) = \mathrm{Eq}^i(b)$ for $b \in U_i \cap U_j$ because $g \cdot C_a = C_{g \cdot a}$ for $g \in PGL(3)$, $a \in \mathbb{P}(d)\backslash D$. Therefore piecing all $\Psi^i$, $i \in I$ together, we obtain a $PGL(3)$ equivariant map $\Psi\colon Q(\xi) \to \mathbb{P}(d)\backslash D$ and a continuous map

$$\Psi_{PGL(3)}\colon Q(\xi)_{PGL(3)} \to (\mathbb{P}(d)\backslash D)_{PGL(3)}.$$

Note that $Q(\xi)$ and $\Psi$ are determined up to isomorphism over $B$.

*Step 3* The natural map

$$T\colon Q(\xi)_{PGL(3)} = EPGL(3) \times_{PGL(3)} Q(\xi) \to PGL(3)\backslash Q(\xi) \cong B$$

is a homotopy equivalence because this is an $EPGL(3)$ bundle. Taking a homotopy inverse map $\zeta\colon B \to Q(\xi)_{PGL(3)}$ of $T$, we set

$$\theta([\xi]) := [\Psi_{PGL(3)} \circ \zeta].$$

Here $[\xi]$ denotes the element of $\mathcal{PC}_d(B)$ represented by $\xi$ and $[\Psi_{PGL(3)} \circ \zeta]$ denotes the element of $[B, (\mathbb{P}(d)\backslash D)_{PGL(3)}]$ represented by $\Psi_{PGL(3)} \circ \zeta$. It is easy to see that

$$\theta\colon \mathcal{PC}_d(B) \to [B, (\mathbb{P}(d)\backslash D)_{PGL(3)}]$$

is well defined.

Before starting the proof of Theorem 6.1, we describe the above construction applied to the family $\xi_u$. In the below, $e \in EPGL(3)$, $g, h \in PGL(3)$ and $a \in \mathbb{P}(d)\backslash D$. We can write

$$Q(\xi_u) \cong ((\mathbb{P}(d)\backslash D) \times PGL(3))_{PGL(3)} \tag{4}$$

where the action of $PGL(3)$ on $(\mathbb{P}(d)\backslash D) \times PGL(3)$ is diagonal, i.e.,

$$g \cdot (a, h) = (g \cdot a, g \cdot h),$$

and the left action of $PGL(3)$ on the right hand side of (4) is given by

$$g \cdot [e, (a, h)] = [e, (a, h \cdot g^{-1})].$$

The $PGL(3)$-equivariant map $\Psi_u \colon Q(\xi_u) \to \mathbb{P}(d) \backslash D$ defined as in Step 2 is given by

$$\Psi_u([e, (a, g)]) = g^{-1} \cdot a,$$

and moreover, the induced map $Q(\xi_u)_{PGL(3)} \to (\mathbb{P}(d) \backslash D)_{PGL(3)}$ has a section $s_u$ given by

$$s_u([e, a]) = [e, [e, (a, 1)]].$$

*Proof of Theorem 6.1* We first prove $\eta \circ \theta = id_{\mathcal{PC}_d(B)}$. Let $\xi = (X, \iota, P, h, B)$ be given. By construction, there is the canonical isomorphism $T^*\xi \to \Psi^*_{PGL(3)}\xi_u$ as families of non-singular plane curves of degree $d$ over $Q(\xi)_{PGL(3)}$. Thus we have

$$(\Psi_{PGL(3)} \circ \zeta)^*\xi_u = \zeta^*\Psi^*_{PGL(3)}\xi_u = \zeta^*T^*\xi = (T \circ \zeta)^*\xi.$$

Since $T \circ \zeta$ is homotopic to the identity map of $E$, this shows $\eta \circ \theta = id_{\mathcal{PC}_d(B)}$.

We next show $\theta \circ \eta = id_{[B,(\mathbb{P}(d)\backslash D)_{PGL(3)}]}$. Let $f \colon B \to (\mathbb{P}(d) \backslash D)_{PGL(3)}$ be a continuous map.

Starting from the family $f^*\xi$ and tracing the construction of $\theta$, we construct the map

$$\Psi_{PGL(3)} \colon Q(f^*\xi_u)_{PGL(3)} \to (\mathbb{P}(d) \backslash D)_{PGL(3)}.$$

$Q(f^*\xi_u)_{PGL(3)}$ is naturally isomorphic to the pull back of $Q(\xi_u)_{PGL(3)} \to (\mathbb{P}(d) \backslash D)_{PGL(3)}$ by $f$. Thus pulling back the section $s_u$, we obtain a map $\zeta' := f^*s_u \colon B \to Q(f^*\xi_u)_{PGL(3)}$ such that $T \circ \zeta' = id_B$ and $\Psi_{PGL(3)} \circ \zeta' = f$. Then $\zeta'$ is a homotopy inverse of $T$ and $\theta \circ \eta([f]) = [\Psi_{PGL(3)} \circ \zeta'] = [f]$, so we obtain $\theta \circ \eta = id_{[B,(\mathbb{P}(d)\backslash D)_{PGL(3)}]}$. □

We call any representative of $\theta([\xi])$ *the classifying map for the family* $\xi$.

For $d = 4$, we do not have to consider $\mathbb{P}^2$ bundles. Recall that a non-hyperelliptic Riemann surface $C$ of genus 3 can be realized as a non-singular plane curve of degree 4 (=plane quartic) by the canonical embedding. This means that the canonical map

$$\iota_C \colon C \to \mathbb{P}\left(H^0(C; K_C)^\vee\right)$$

where $H^0(C; K_C)$ is the space of holomorphic 1-forms on $C$, is an embedding and if we identify $\mathbb{P}(H^0(C; K_C)^\vee)$ with $\mathbb{P}^2$ by a choice of a basis of $H^0(C; K_C)$, the image of $C$ is a non-singular plane curve of degree 4. The defining equation of the image is uniquely determined as an element of $\mathbb{P}(4) \backslash D$.

Let $p\colon X \to B$ be a continuous family of compact Riemann surfaces of genus 3. We call it *a non-hyperelliptic family of genus 3* if the complex structure of each fiber $p^{-1}(b), b \in B$ is non-hyperelliptic. Two such families $p_i: X_i \to B$ $(i = 0, 1)$ are called isotopic if there exists a non-hyperelliptic family of genus 3 over $B \times [0, 1]$ such that for $i = 0, 1$, its restriction to $B \times \{i\}$ is isomorphic to $p_i: X_i \to B$ as continuous family of Riemann surfaces over $B \cong B \times \{i\}$.

For a given space $B$, we denote by $\mathcal{NH}_3(B)$ the set of all isotopy classes of non-hyperelliptic families of genus 3 over $B$. Then the forgetful functor

$$\mathcal{PC}_4(\bullet) \to \mathcal{NH}_3(\bullet) \tag{5}$$

defined by an obvious manner, is bijective. For, let $p\colon X \to B$ be a given non-hyperelliptic family of genus 3. Set $\Lambda_X := \bigcup_{b\in B} H^0(p^{-1}(b); K_b)$, where $H^0(p^{-1}(b); K_b)$ denotes the space of holomorphic 1-forms on $p^{-1}(b)$. This has the structure of complex vector bundle over $B$. Projectivising the dual of $\Lambda_X$, we obtain a $\mathbb{P}^2$ bundle

$$h'\colon P' = \bigcup_{b\in B} \mathbb{P}\left(H^0(p^{-1}(b); K_b)^\vee\right) \to B,$$

and piecing the fiberwise canonical maps $\iota_{X_b},\ b \in B$ together, we get a map $\iota\colon X \to P'$. Then we obtain an element $\xi = (X, \iota, P', h', B) \in PC_4(\bullet)$. This correspondence gives the inverse of (5).

We continue the consideration of the case $d = 4$. We next prove that:

**Proposition 6.3** *The homomorphism $\rho\colon \Pi(4) \to \Gamma_3$ is surjective.*

Combining this with Theorem 4.1, which implies that $\rho^*\colon H^2(\Gamma_3; \mathbb{Q}) \to H^2(\Pi(4); \mathbb{Q})$ is not injective, we see that the order of the kernel of $\rho$ is infinite.

*Proof of Proposition 6.3* Let $\mathcal{T}_3$ be the Teichmüller space of compact Riemann surfaces of genus 3 and $H_3$ the hyperelliptic locus of $\mathcal{T}_3$; namely the set of marked Riemann surfaces whose complex structure is hyperelliptic. $H_3$ is a complex analytic closed submanifold of codimension 1 with infinitely many components (see [14], pp. 259–260). In particular, $\mathcal{T}_3\backslash H_3$ is path connected.

We recall; there is a holomorphic family $\pi: V_3 \to \mathcal{T}_3$ called the universal Teichmüller curve, whose fiber over the marked Riemann surface $[f, C]$ is isomorphic to $C$; the mapping class group $\Gamma_3$ acts on $\mathcal{T}_3$ and $V_3$, and $\pi$ is equivariant with respect to these actions; it is well known that the quotient space $\Gamma_3\backslash\mathcal{T}_3$ is the Riemann moduli space. Since the action of $\Gamma_3$ on $\mathcal{T}_3$ preserves $H_3$, $\Gamma_3$ also acts on $\mathcal{T}_3\backslash H_3$ and its inverse image by $\pi$. Restricting $\pi$ to $\mathcal{T}_3\backslash H_3$ and taking the Borel construction, we obtain a non-hyperelliptic family of genus 3 over $(\mathcal{T}_3\backslash H_3)_{\Gamma_3}$.

It is not difficult to see that this family also have the universal property which the family $p_u$ over $(\mathbb{P}(4)\backslash D)_{PGL(3)}$ has. Therefore, $(\mathcal{T}_3\backslash H_3)_{\Gamma_3}$ is homotopy equivalent to $(\mathbb{P}(4)\backslash D)_{PGL(3)}$ hence its fundamental group is isomorphic to $\Pi(4)$.

By the homotopy exact sequence of the $\mathcal{T}_3\backslash H_3$ bundle $(\mathcal{T}_3\backslash H_3)_{\Gamma_3} \to B\Gamma_3 = K(\Gamma_3, 1)$ we obtain an exact sequence

$$\Pi(4) \cong \pi_1((\mathcal{T}_3\backslash H_3)_{\Gamma_3}) \overset{\rho'}{\to} \pi_1(B\Gamma_3) = \Gamma_3 \to \pi_0(\mathcal{T}_3\backslash H_3).$$

We notice that the homomorphism $\rho'$ just coincides with the topological monodromy over $(\mathcal{T}_3\backslash H_3)_{\Gamma_3}$, and $\pi_0(\mathcal{T}_3\backslash H_3)$ is one point. This shows $\rho'$ is surjective, so $\rho$ is. □

## 7 Local signature for four-dimensional non-hyperelliptic fibration of genus 3

As an application, we will define the local signature for the set of all fiber germs of four-dimensional fiber spaces whose general fibers are non-hyperelliptic Riemann surfaces of genus 3, using the Meyer function $\phi^4$. This local signature is used to derive a signature formula for a class of four-dimensional fiber spaces, whose general fibers are non-hyperelliptic Riemann surfaces of genus 3.

Let $\Delta$ be a closed oriented 2-disk and $p$ its center. A 4-tuple $\mathcal{F} = (E, \pi, \Delta, p)$ is called *a fiber germ of non-hyperelliptic family of genus 3* if

1. $E$ is a $C^\infty$ manifold of dimension 4 and $\pi : E \to \Delta$ is a $C^\infty$ map,
2. the restriction of $\pi$ to $\Delta\backslash\{p\}$ is a non-hyperelliptic family of genus 3.

Note that $E$ has the natural orientation and compact, hence its signature $\mathrm{Sign}(E)$ is defined. Two such germs $(E, \pi, \Delta, p)$ and $(E', \pi', \Delta', p')$ are called *equivalent* if there exist a smaller disk $\Delta_0 \subset \Delta$ (respectively, $\Delta'_0 \subset \Delta'$) whose center is $p$ (respectively, $p'$), and there exist orientation preserving diffeomorphisms $\varphi\colon (\Delta_0, p) \to (\Delta'_0, p')$ and $\tilde{\varphi}\colon \pi^{-1}(\Delta_0) \to \pi'^{-1}(\Delta'_0)$ such that $\varphi \circ \pi = \pi' \circ \tilde{\varphi}$ and

$$\tilde{\varphi}|_{\pi^{-1}(\Delta_0\backslash\{p\})}\colon \pi^{-1}(\Delta_0\backslash\{p\}) \to \pi'^{-1}(\Delta'_0\backslash\{p'\})$$

maps each fiber biholomorphically.

Let $\mathcal{NH}_3$ denote the set of all equivalence classes of such 4-tuples. We denote the element of $\mathcal{NH}_3$ also by $\mathcal{F} = (E, \pi, \Delta, p)$. For $\mathcal{F} = (E, \pi, \Delta, p) \in \mathcal{NH}_3$, $\gamma$ denotes the element of $\pi_1(\Delta\backslash\{p\})$ traveling once the boundary $\partial\Delta$ by counter clockwise manner. We denote by $\mathcal{F}^0$ the restriction of $\pi : E \to \Delta$ to $\Delta\backslash\{p\}$. $\mathcal{F}^0$ is a non-hyperelliptic family of genus 3 and can be considered as an element of $\mathcal{PC}_4(\Delta\backslash\{p\})$ in view of (5).

**Definition 7.1** Define $\mathrm{loc.sig}^{\mathcal{Q}}\colon \mathcal{NH}_3 \to \mathbb{Q}$ by

$$\mathrm{loc.sig}^{\mathcal{Q}}(\mathcal{F}) := \phi^4(\theta(\mathcal{F}^0)_*(\gamma)) + \mathrm{Sign}(E).$$

Here, $\theta(\mathcal{F}^0)_*$ is the homomorphism from $\pi_1(\Delta\backslash\{p\})$ to $\Pi(4)$ induced by the classifying map $\theta(\mathcal{F}^0)$ for $\mathcal{F}^0$. It is assumed that suitable base points of $\Delta\backslash\{p\}$ and $(\mathbb{P}(4)\backslash D)_{PGL(3)}$ are chosen. Since $\phi^4$ is a class function, we don't have to care about base point so we omit it.

We call a triple $(E, \pi, B)$ *a four-dimensional non-hyperelliptic fibration of genus* 3 if

1. $E$ (respectively, $B$) is a closed oriented $C^\infty$-manifold of dimension 4 (respectively, 2) and $\pi : E \to B$ is a $C^\infty$-map,
2. there exist finitely many points $b_1, \ldots, b_n \in B$ such that the restriction of $\pi$ to $B\backslash\{b_1, \ldots, b_n\}$ is a non-hyperelliptic family of genus 3.

For $i = 1, \dots, n$, we obtain an element of $\mathcal{NH}_3$ by restricting $\pi$ to a small closed disk neighborhood of $b_i$. we denote it by $\mathcal{F}_i$. Then, we obtain

**Theorem 7.2** (The signature formula) *Let* $(E, \pi, B)$ *be a four-dimensional non-hyperelliptic fibration of genus* 3. *Then*

$$Sign(E) = \sum_{i=1}^{n} \mathrm{loc.sig}^{\mathcal{Q}}(\mathcal{F}_i).$$

*Proof* For $i = 1, \dots, n$, take a small closed 2-disk $D_i$ around $b_i$ so that they do not intersect each other. Then $\mathcal{F}_i = (\pi^{-1}(D_i), \pi, D_i, b_i)$. We denote by $\mathcal{F}_i^0$ the restriction of $\pi$ to $D_i \backslash \{b_i\}$ and set $B_0 := B \backslash \bigcup_{i=1}^{n} \mathrm{Int} D_i$. By Meyer's signature formula, we get

$$\mathrm{Sign}(\pi^{-1}(B_0)) = \sum_{i=1}^{n} \phi^4 \left( \theta(\mathcal{F}_i^0)_*(\gamma) \right).$$

Using the Novikov additivity, we compute

$$\begin{aligned} \mathrm{Sign}(E) &= \mathrm{Sign}(\pi^{-1}(B_0)) + \sum_{i=1}^{n} \mathrm{Sign}(\pi^{-1}(D_i)) \\ &= \sum_{i=1}^{n} \phi^4(\theta(\mathcal{F}_i^0)_*(\gamma)) + \sum_{i=1}^{n} \mathrm{Sign}(\pi^{-1}(D_i)) \\ &= \sum_{i=1}^{n} \mathrm{loc.sig}^{\mathcal{Q}}(\mathcal{F}_i). \end{aligned}$$

□

**Corollary 7.3** *Let* $g \colon E \to B$ *be a non-hyperelliptic family of genus 3 over a closed oriented surface* $B$. *Then* $Sign(E) = 0$.

We compute some examples. Comparing the following computations with those in [2,15], we see that their values coincide.

**Singular fiber of type I** Let $\Delta \subset \mathbb{P}(4)$ be a closed 2-disk intersecting with $D$ only in its center $p \in \Delta$ transversely. Let $\pi_I \colon E_I \to \Delta$ be the restriction of $\bar{\mathcal{F}} \to \mathbb{P}(4)$ to $\Delta$. Then $E_I$ is smooth by Lemma 2.4 and $\mathcal{F}_I = (E_I, \pi_I, \Delta, p)$ is a fiber germ of non-hyperelliptic family of genus 3. By Lemma 2.3 the topological type of $\pi_I^{-1}(p)$ is Lefschetz singular fiber of type I, therefore we also call $\mathcal{F}_I \in \mathcal{NH}_3$ *a singular fiber germ of type I*. The signature of $E_I$ is 0 and by definition, the inclusion $\Delta \backslash \{p\} \hookrightarrow \mathbb{P}(4) \backslash D \hookrightarrow (\mathbb{P}(4) \backslash D)_{PGL(3)}$ is the classifying map for $\mathcal{F}_I^0$ and the boundary of $\Delta$ is a lasso about $D$. Therefore, by Proposition 5.1, we have

**Proposition 7.4**

$$\mathrm{loc.sig}^{\mathcal{Q}}(\mathcal{F}_I) = -\frac{5}{9}.$$

**Hyperelliptic fiber** Let $F \in V^4 \setminus \{0\}$ be a polynomial such that $C_F$ intersects with the non-singular conic $C\colon\ yz - x^2 = 0$ in eight points, and let $\Delta$ be a small closed 2-disk around $0 \in \mathbb{C}$ with the complex coordinate $s$. Let $S_F$ be the hypersurface in $\Delta \times \mathbb{P}^2$ defined by the equation

$$(yz - x^2)^2 + s^2 F(x, y, z) = 0.$$

$S_F$ is singular along $C' = \{0\} \times C$. Blowing up $\Delta \times \mathbb{P}^2$ along $C$, let $\widetilde{S_F}$ be the proper transform of $S_F$ and $\pi\colon \widetilde{S_F} \to \Delta$ the composition of $\widetilde{S_F} \to S_F$ and the first projection $S_F \to \Delta$. Then $\widetilde{S_F}$ is non-singular and the exceptional divisor $\pi^{-1}(0)$ is a non-singular hyperelliptic curve of genus 3 with a natural projection onto $C' \cong \mathbb{P}^1$, which is a double cover.

Choose $\Delta$ small enough so that the singular fiber of $\pi$ is $\pi^{-1}(0)$ only. Set $\mathcal{F}_{\mathrm{h}} = (\widetilde{S_F}, \pi, \Delta, 0)$ and call this fiber germ *a hyperelliptic germ*. Let $\ell_{\mathrm{h}}$ be the corresponding loop in $\mathbb{P}(4) \setminus D$ defined by

$$\ell_{\mathrm{h}}(t) = (yz - x^2)^2 + \left(\varepsilon e^{2\pi\sqrt{-1}t}\right)^2 F(x, y, z), \quad 0 \le t \le 1,$$

where $\varepsilon$ is the radius of $\Delta$.

**Proposition 7.5**

$$\mathrm{loc.sig}^{\mathcal{Q}}(\mathcal{F}_{\mathrm{h}}) = \bar{\phi}^4([\ell_{\mathrm{h}}]) = \frac{4}{9}.$$

*Proof* We first note that $\mathrm{loc.sig}^{\mathcal{Q}}(\mathcal{F}_{\mathrm{h}}) = \bar{\phi}^4([\ell_{\mathrm{h}}])$ since a hyperelliptic germ is topologically trivial.

The set $W$ of all polynomials in $V^4$ such that the corresponding curve intersects with $C$ in eight points is a non-empty Zariski open subset of $V^4$. Since $[\ell_{\mathrm{h}}]$ and $\mathrm{Sign}(\widetilde{S_F})$ does not change under any small perturbation of $F$ in $V^4$, it suffices to show the proposition for a particular element of $W$. But by the same reason as in Lemma 4.4, there is actually an element $F \in W$ such that the map

$$\mathbb{P}^1 \to \mathbb{P}(4),\ [w_0\colon w_1] \mapsto w_0^2(yz - x^2)^2 + w_1^2 F(x, y, z),$$

does not meet $E$ and is transverse to $D$, except at $[w_0\colon w_1] = [1\colon 0]$. Then for this choice of $F$, the complex surface $S$ in $\mathbb{P}^1 \times \mathbb{P}^2$ defined by the equation

$$w_0^2(yz - x^2)^2 + w_1^2 F(x, y, z) = 0,$$

has singularities only along the conic $\{[1\colon 0]\} \times C$. After blowing up $\mathbb{P}^1 \times \mathbb{P}^2$ along this conic, we obtain the proper transform $\widetilde{S}$ of $S$. By the choice of $F$, $\widetilde{S}$ is non-singular. The composition of $\widetilde{S} \to S$ and the first projection $S \to \mathbb{P}^1$ is a family of algebraic curves whose all singular fiber germs are singular fiber germ of type I except the fiber germ around $[1\colon 0]$, and the fiber germ around $[1\colon 0]$ is a hyperelliptic germ. The invariants of $\widetilde{S}$ are computed as: $c_1^2(\widetilde{S}) = -6,\ c_2(\widetilde{S}) = 18$, and $\mathrm{Sign}(\widetilde{S}) = -14$.

Now the number of singular fiber germs of type I is equal to the total Euler contribution

$$18 - 2(2 - 2 \cdot 3) = 26.$$

Note that a hyperelliptic germ, which is topologically trivial, does not contribute to the Euler number. By Theorem 7.2 and Proposition 7.4, we have

$$-14 = -\frac{5}{9} \times 26 + \mathrm{loc.sig}^{\mathcal{Q}}(\mathcal{F}_{\mathrm{h}}),$$

hence $\mathrm{loc.sig}^{\mathcal{Q}}(\mathcal{F}_{\mathrm{h}}) = \frac{4}{9}$. □

**Singular fiber of type II** Let $\Delta$ be as in the previous example, and let $S$ be the surface in $\Delta \times \mathbb{P}^2$ defined by

$$z^3x + y^2x^2 + y^4 + s^6x^4 = 0.$$

$S$ has an isolated singularity at $p_0 = (0, [1: 0: 0])$ so called a singularity of type $\widetilde{E}_8$. The inverse image $C_2$ of $0 \in \Delta$ by the first projection $p_1: S \to \Delta$ is a curve of geometric genus 2 with one cusp singularity.

Let $\varpi : \widetilde{S} \to S$ be the minimal resolution of the singularity of $S$ at $p_0$. Then the exceptional curve is a non-singular elliptic curve $C_1$ with self intersection number $-1$. If $\Delta$ is small enough, $\mathcal{F}_{II} = (\widetilde{S}, p_1 \circ \varpi, \Delta, 0)$ is a fiber germ of non-hyperelliptic family of genus 3. The topological type of the singular fiber $(p_1 \circ \varpi)^{-1}(0)$ is obtained by the disjoint union of $C_1$ and $C_2$ by identifying a point of $C_1$ with the cusp singularity of $C_2$, that is, Lefschetz singular fiber of type II. We call $\mathcal{F}_{II}$ *a singular fiber germ of type II*.

Let $\ell_{II}$ be the corresponding loop in $\mathbb{P}(4)\backslash D$ defined by

$$\ell_{II}(t) = z^3x + y^2x^2 + y^4 + (\varepsilon e^{2\pi\sqrt{-1}t})^6x^4, \quad 0 \le t \le 1.$$

**Proposition 7.6**

$$\mathrm{loc.sig}^{\mathcal{Q}}(\mathcal{F}_{II}) = \frac{1}{3}, \qquad \bar{\phi}^4([\ell_{II}]) = \frac{4}{3}.$$

*Proof* In this case $\bar{\phi}^4([\ell_{II}]) = \mathrm{loc.sig}^{\mathcal{Q}}(\mathcal{F}_{II}) + 1$ because the intersection form of $\widetilde{S}$ is given by $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ hence $\mathrm{Sign}(\widetilde{S}) = -1$.

We perturb $S$ slightly by adding a higher term about $s$; consider the surface in $\Delta \times \mathbb{P}^2$ defined by

$$z^3x + y^2x^2 + y^4 + s^6x^4 + s^mF(x, y, z) = 0,$$

where $m$ is an integer $\ge 7$ and $F$ is a polynomial in $V^4$. The singularity of this surface remains at the origin and is still of type $\widetilde{E}_8$. Taking the minimal resolution of this singularity and taking $\Delta$ to be smaller if needed, we obtain a new fiber germ $\mathcal{F}'_{II}$ and a new loop $\ell'_{II}$ in $\mathbb{P}(4)\backslash D$. This perturbation does not influence the value of $\phi^4$ and the topology of the fiber neighborhood of the singular fiber. So it suffices to compute $\operatorname{loc.sig}^{\mathcal{Q}}(\mathcal{F}'_{II})$.

Let $S'$ be the complex surface in $\mathbb{P}^1 \times \mathbb{P}^2$ defined by the equation

$$w_0^m(z^3x + y^2x^2 + y^4) + w_0^{m-6}w_1^6x^4 + w_1^m F(x, y, z) = 0,$$

and let $\widetilde{S'} \to S'$ be the minimal resolution of the singularity of $S'$ at $p_0 = ([1: 0], [1: 0: 0])$. If a generic $F$ is chosen, then $\widetilde{S'}$ is non-singular and the singular fiber germs of the family of algebraic curves $\widetilde{S'} \to S' \to \mathbb{P}^1$ are all of type I except the fiber germ around $[1: 0]$, and the fiber germ around $[1: 0]$ is $\mathcal{F}'_{II}$. The invariants of $\widetilde{S'}$ are computed as: $c_1^2(\widetilde{S'}) = 9m - 17$, $c_2(\widetilde{S'}) = 27m - 19$, and $\operatorname{Sign}(\widetilde{S'}) = -15m + 7$.

Now the number of singular fiber germs of type I is equal to

$$27m - 19 - 2(2 - 2 \cdot 3) - 1 = 27m - 12.$$

This time, the fiber over $[1: 0]$ do contribute to the Euler number.

By the signature formula,

$$-15m + 7 = -\frac{5}{9} \times (27m - 12) + \operatorname{loc.sig}^{\mathcal{Q}}(\mathcal{F}'_{II}).$$

Thus we obtain $\operatorname{loc.sig}^{\mathcal{Q}}(\mathcal{F}_{II}) = \operatorname{loc.sig}^{\mathcal{Q}}(\mathcal{F}'_{II}) = \frac{1}{3}$. □

## Appendix

In this appendix we give a definition of Meyer's signature cocycle in the form used in the present paper and review its properties. For details, see Meyer's original paper [12].

We first explain the topological monodromy of surface bundles. Let $\pi : E \to B$ be an oriented $\Sigma_g$ bundle whose structure group is the group of all orientation preserving diffeomorphisms of $\Sigma_g$. Choose a base point $b_0 \in B$ and fix an identification $\phi\colon \Sigma_g \overset{\cong}{\to} \pi^{-1}(b_0)$. For each based loop $\ell\colon [0, 1] \to B$ the pull back $\ell^*(E) \to [0, 1]$ of $\pi : E \to B$ by $\ell$ is trivial. Hence there exist a trivialization $\Phi\colon \Sigma_g \times [0, 1] \to \ell^*(E)$ such that $\Phi(x, 0) = \phi(x)$. By assigning the isotopy class of $\Phi(x, 1)^{-1} \circ \phi$ to the homotopy class of $\ell$, we obtain a map $\chi : \pi_1(B, b_0) = \pi_1(B) \to \Gamma_g$. This map becomes a homomorphism under the conventions; (1) for any two mapping classes $f_1$ and $f_2$, the multiplication $f_1 \circ f_2$ means that $f_2$ is applied first, (2) for any two homotopy classes of based loops $\ell_1$ and $\ell_2$, their product $\ell_1 \cdot \ell_2$ means that $\ell_1$ is traversed first. $\chi$ is called *the topological monodromy of* $\pi : E \to B$ and determined up to inner automorphisms of $\Gamma_g$.

Let $P$ denote the pair of pants, i.e., $P = S^2 \backslash \bigcup_{i=1}^3 \operatorname{Int} D_i$ where $D_i$, $i = 1, 2$, and 3 are the three disjoint closed disks in the 2-sphere $S^2$. Choose a base point $p_0 \in \operatorname{Int} P$ and fix a based loop $\ell_1$ and $\ell_2$ such that $\ell_i$ is free homotopic to the loop traveling once the boundary $\partial D_i$ by counter clockwise manner ($i = 1, 2$). For $(f_1, f_2) \in \Gamma_g \times \Gamma_g$, we can construct an oriented $\Sigma_g$ bundle $E(f_1, f_2)$ over $P$ such that the topological monodromy $\chi : \pi_1(P) \to \Gamma_g$ sends $[\ell_i]$ to $f_i$ for $i = 1, 2$. (If $g \geq 2$, the isomorphism class of this bundle is unique.) $E(f_1, f_2)$ is a compact $C^\infty$-manifold of dimension 4 and has the natural orientation induced by the orientation of $P$ and that of the fibers. Then the signature of $E(f_1, f_2)$ is defined and we set

$$\tau_g(f_1, f_2) := -\mathrm{Sign}(E(f_1, f_2)).$$

This turns out to be well defined even when $g = 1$, and $\tau_g : \Gamma_g \times \Gamma_g \to \mathbb{Z}$ is called *Meyer's signature cocycle*. The basic properties of $\tau_g$ are

(1) $\tau_g(f_1 f_2, f_3) + \tau_g(f_1, f_2) = \tau_g(f_1, f_2 f_3) + \tau_g(f_2, f_3)$;
(2) $\tau_g(f_1, 1) = \tau_g(1, f_1) = \tau_g(f_1, f_1^{-1}) = 0$;
(3) $\tau_g(f_1^{-1}, f_2^{-1}) = -\tau_g(f_1, f_2)$;
(4) $\tau_g(f_1, f_2) = \tau_g(f_2, f_1)$;
(5) $\tau_g(f_3 f_1 f_3^{-1}, f_3 f_2 f_3^{-1}) = \tau_g(f_1, f_2)$,

where $f_1$, $f_2$, and $f_3$ are elements of $\Gamma_g$.

For an oriented $\Sigma_g$ bundle $\pi : E \to B$ and a choice of base point $b_0$ of $B$, we obtain a 2-cocycle $\chi^*\tau_g$ of $\pi_1(B) = \pi_1(B, b_0)$ by pulling back $\tau_g$ by the topological monodromy $\chi : \pi_1(B) \to \Gamma_g$. Although $\chi$ is determined only up to conjugacy, $\chi^*\tau_g$ is uniquely determined by the property (5) of $\tau_g$ above. Moreover, $\chi^*\tau_g$ does not depend on the choice of base point of $B$ in the following sense: suppose $b_0' \in B$ and $b_0$ are in the same path component of $B$ then under any isomorphism $\pi_1(B, b_0) \cong \pi_1(B, b_0')$ using a path from $b_0$ to $b_0'$, two cocycles of $\pi_1(B, b_0)$ and $\pi_1(B, b_0')$ defined as the pull back of $\tau_g$ by topological monodromies, correspond to each other.

Let $G$ be a group and $\varphi : G \to \Gamma_g$ a homomorphism.

**Definition 8.1** A $\mathbb{Q}$-valued 1-cochain $\phi: G \to \mathbb{Q}$ is called a Meyer function with respect to the pull back $\varphi^*\tau_g$ of $\tau_g$ by $\varphi$ if it satisfies $\delta\phi = \varphi^*\tau_g$, i.e., $\phi$ cobounds the 2-cocycle $\varphi^*\tau_g$.

If a Meyer function exists on $G$, the cohomology class $\varphi^*[\tau_g] \in H^2(G; \mathbb{Z})$ is torsion. The following properties of $\phi$ are easily derived by the above properties of $\tau_g$ (see also [7, Proposition 3.1]).

**Lemma 8.2** *If $\phi$ is a Meyer function with respect to $\varphi^*\tau_g$, we have*

(1) $\phi(xy) = \phi(x) + \phi(y) - \varphi^*\tau_g(x, y)$;
(2) $\phi(1) = 0$;
(3) $\phi(x^{-1}) = -\phi(x)$;
(4) $\phi(yxy^{-1}) = \phi(x)$,

*where $x, y \in G$.*

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