Meyer functions and the signatures of fibered 4-manifolds

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Abstract. We give a survey on Meyer functions, with emphasis on their application to the signatures of fibered 4-manifolds.

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Contents

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1 Introduction

In this chapter, we give a survey on secondary invariants called *Meyer functions* with emphasis on their application to the signatures of fibered 4-manifolds. These secondary invariants are associated to the vanishing of the primary invariant called the *first MMM class* e_1, the first in a series of characteristic classes of surface bundles [31] [33] [36]. There have been known various representatives of e_1 coming from different geometric contexts, as group 2-cocycles on the mapping class group or differential 2-forms on the moduli space of curves (see [21], especially for the latter). The view point we take here is the signature of surface bundles over surfaces, and we work with the *signature cocycle* τ_g introduced by W. Meyer [30] (and by Turaev [40] independently) a $\mathbb{Z}$-valued 2-cocycle of the mapping class group $\mathcal{M}_g$ of a closed oriented surface of genus g, whose cohomology class is proportional to e_1.

As was shown by Meyer, if $g = 1$ or 2, the cocycle τ_g is the coboundary of a unique $\mathbb{Q}$-valued 1-cochain ϕ_g of $\mathcal{M}_g$. The existence of such a 1-cochain implies that over the rationals, e_1 of a surface bundle with fiber a surface of genus 1 or 2 vanishes. The uniqueness of ϕ_g follows from the fact $H^1(\mathcal{M}_g;\mathbb{Q}) = 0$. These 1-cochains are called the Meyer functions of genus 1 or 2. Meyer [30] extensively studied the case of genus 1 and gave an explicit formula for ϕ_1 which involves the Dedekind sums. In [6], Atiyah reproved Meyer's formula by a quite different method and also showed various number theoretic or differential geometric aspects of ϕ_1.

In §2, we recall basic results of Meyer and Atiyah with a sketch of proof for several assertions. In §3, we mention an application of Meyer functions to localization of the signature of fibered 4-manifolds. This topic has been studied also from algebro-geometric point of view, which we shall mention in §3.1. Recently, various higher genera or higher dimensional analogues of ϕ_1 have been considered and a part of Atiyah's result has been generalized to these generalizations. In §4, we present three examples of these generalizations.

Some conventions about surface bundles follow. Throughout this chapter g is an integer ≥ 1. Let Σ_g be a closed oriented C^∞-surface of genus g. By a Σ_g-*bundle* we mean a smooth fiber bundle $\pi\colon E \to B$ over a C^∞-manifold B with fiber Σ_g such that the fibers are coherently oriented: the tangent bundle along the fibers $T\pi := \{v \in TE; \pi_*(v) = 0\}$ is oriented. The transition functions of such bundles take values in $\mathrm{Diff}^+(\Sigma_g)$, the group of orientation preserving diffeomorphisms of Σ_g endowed with C^∞-topology. The *mapping class group* $\mathcal{M}_g := \pi_0(\mathrm{Diff}^+(\Sigma_g))$ is the group of connected components of $\mathrm{Diff}^+(\Sigma_g)$. In other words, $\mathcal{M}_g$ is the quotient group $\mathrm{Diff}^+(\Sigma_g)/\mathrm{Diff}_0(\Sigma_g)$, where $\mathrm{Diff}_0(\Sigma_g)$ is the group of diffeomorphisms isotopic to the identity.

For a Σ_g-bundle $\pi\colon E \to B$ over a path connectd space B, the associated is (the conjugacy class of) a homomorphism $\chi\colon \pi_1(B) \to \mathcal{M}_g$ called the *monodromy.* This correspondence is defined as the composite

$$\{\Sigma_g\text{−bundles over } B\}/\text{isom} = [B, B\mathrm{Diff}^+(\Sigma_g)] \\ \to \mathrm{Hom}(\pi_1(B), \mathcal{M}_g)/\text{conj.} \tag{1.1}$$

Namely, if $f\colon B \to B\mathrm{Diff}^+(\Sigma_g)$ is a classifying map of $\pi\colon E \to B$, then $\chi = f_*$, the induced map from $\pi_1(B)$ to $\pi_1(B\mathrm{Diff}^+(\Sigma_g)) = \pi_0(\mathrm{Diff}^+(\Sigma_g)) = \mathcal{M}_g$. To be more careful about the base points and to give a more direct description, choose a base point $b_0 \in B$ and fix an orientation preserving diffeomorphism $\varphi\colon \Sigma_g \to \pi^{-1}(b_0)$. Let $\ell\colon [0,1] \to B$ be a based loop. Since $[0,1]$ is contractible, the pull back $\ell^*(E) \to [0,1]$ of $\pi\colon E \to B$ is a trivial Σ_g-bundle. Hence there exist a trivialization $\Phi\colon \Sigma_g \times [0,1] \to \ell^*(E)$ such that $\Phi(x,0) = \varphi(x)$. In this setting, $\chi\colon \pi_1(B, b_0) \to \mathcal{M}_g$ is given by $\chi([\ell]) = [\Phi(x,1)^{-1} \circ \varphi]$. Here our convention is: 1) for any two mapping classes f_1 and f_2, their multiplication $f_1 \circ f_2$ means that f_2 is applied first, 2) for any two homotopy classes of based loops ℓ_1 and ℓ_2, their product $\ell_1 \cdot \ell_2$ means that ℓ_1 is traversed first.

By the result of Earle-Eells [13], if $g \geq 2$ the space $\mathrm{Diff}_0(\Sigma_g)$ is contractible, so the classifying space $B\mathrm{Diff}^+(\Sigma_g)$ is a $K(\mathcal{M}_g, 1)$-space. Hence the map (1.1) is a bijection. If $g = 1$, then $\Sigma_1 = T^2$, the two torus. The embedding $T^2 \hookrightarrow \mathrm{Diff}_0(T^2)$ as parallel translations is a homotopy equivalence, and $\mathcal{M}_1$ is isomorphic to $SL(2;\mathbb{Z})$. Thus we have a fibration $B\mathrm{Diff}^+(T^2) \to BSL(2;\mathbb{Z}) = K(SL(2;\mathbb{Z}), 1)$ with fiber $BT^2 = \mathbb{C}P^\infty \times \mathbb{C}P^\infty$. In particular, by elementary obstruction theory, it follows that if the base space B has a homotopy type of a 1-dimensional CW complex, then the isomorphism class of T^2-bundles over B is also classified well by monodromies: (1.1) is bijective.

2 The signature cocycle and Meyer's theorem

In this section we review the signature cocycle, its variants, and the original version of Meyer functions, i.e., the Meyer function of genus 1 and 2.

2.1 Prehistory

In study of the topology of fiber bundles, a basic question is how the topological invariants of the total space, the base space and the fiber are related. In 50's Chern, Hirzebruch and Serre studied the signature of the total space of a fiber bundle, by an application of the Serre spectral sequence. Recall that the *signature* of a compact oriented manifold M of dimension $4n$ (possibly with boundary), denoted by $\mathrm{Sign}(M)$, is the signature of the intersection form $H_{2n}(M;\mathbb{R}) \times H_{2n}(M;\mathbb{R}) \to \mathbb{R}$, which is a symmetric bilinear form. If the dimension of M is not a multiple of 4, we understand that the signature of M is zero.

Theorem 2.1 (Chern-Hirzebruch-Serre [11]). *Let E and B be closed oriented manifolds and $E \to B$ a fiber bundle with fiber a closed oriented manifold F. We arrange that the orientation of F is compatible with those of E and B. If $\pi_1(B)$ trivially acts on the homology $H_*(F;\mathbb{R})$, then the signature of E is the product of the signatures of B and F:* $\mathrm{Sign}(E) = \mathrm{Sign}(B)\mathrm{Sign}(F)$.

The assumption that $\pi_1(B)$ trivially acts on the homology of the fiber is crucial, and the conclusion of the theorem does not hold in general. Indeed, Atiyah [5] and Kodaira [22] independently constructed an algebraic surface with non-zero signature, which is the total space of a complex analytic family of compact Riemann surfaces over a compact Riemann surface. Their method uses branched covering of algebraic surfaces, and can be used to produce examples such that the genus of the fiber can be taken arbitrarily integers ≥ 4.

One important consequence is that there are non-trivial characteristic classes of surface bundles. In fact, since the signature of a manifold which is the boundary of some manifold is zero, the map

$$\mathrm{Sign}\colon \Omega_2(B\mathrm{Diff}^+(\Sigma_g)) \to \mathbb{Z}, \quad [f] \mapsto \mathrm{Sign}(f^*\xi)$$

is well-defined. Here $\Omega_2(X)$ is the second oriented bordism group of a space X (hence its element is represented by some continuous map f from a closed oriented surface to X) and ξ is a universal Σ_g-bundle over the classifying space $B\mathrm{Diff}^+(\Sigma_g)$. Since $\Omega_2(X)$ is naturally isomorphic to $H_2(X;\mathbb{Z})$, the map Sign becomes an element in $\mathrm{Hom}(H_2(B\mathrm{Diff}^+(\Sigma_g)),\mathbb{Z})$, and the examples by Atiyah and Kodaira shows that the map Sign is non-trivial. Hence $H^2(B\mathrm{Diff}^+(\Sigma_g);\mathbb{Z}) \cong H^2(\mathcal{M}_g;\mathbb{Z})$ is non-trivial and contains an element of infinite order, provided $g \geq 4$. As we recall in the following, Meyer showed that this non-triviality holds when $g \geq 3$.

2.2 The signature cocycle

W. Meyer [29] [30] studied the signature of surface bundles over surfaces and introduced the signature cocycle. The basic idea of Meyer is to decompose the base space into simple pieces: pairs of pants.

Let $\Sigma_{0,n}$ be a compact surface obtained from the two sphere by removing n open disks with embedded disjoint closures. Specifying an orientation of $\Sigma_{0,n}$ and a base point $* \in \mathrm{Int}(\Sigma_{0,n})$, we take n based loops $\ell_1, \dots, \ell_n \in \pi_1(\Sigma_{0,n}, *)$ such that each ℓ_i is freely homotopic to one of the boundaries with the counterclockwise orientation, and the relation $\ell_1 \cdots \ell_n = 1 \in \pi_1(\Sigma_{0,n}, *)$ holds. The group $\pi_1(\Sigma_{0,n}, *)$ is free of rank $n-1$, generated by any $n-1$ of $\ell_1, \dots, \ell_n$. The surface $P = \Sigma_{0,3}$ is called a *pair of pants.*

Given $f_1, \dots, f_{n-1} \in \mathcal{M}_g$, consider a Σ_g-bundle $\pi\colon E(f_1, \dots, f_{n-1}) \to \Sigma_{0,n}$ with $\pi^{-1}(*) = \Sigma_g$ whose monodromy $\chi\colon \pi_1(\Sigma_{0,n}, *) \to \mathcal{M}_g$ sends ℓ_i to f_i $(i = 1, \dots, n-1)$. Since $\Sigma_{0,n}$ is homotopy equivalent to a 1-dimensional CW complex, such a bundle exists and is unique up to isomorphism (see §1). The total space $E(f_1, \dots, f_{n-1})$ is a compact oriented 4-manifold with boundary.

Definition 2.2. The *signature cocycle* $\tau_g\colon \mathcal{M}_g \times \mathcal{M}_g \to \mathbb{Z}$ is defined by

$$\tau_g(f_1, f_2) := \mathrm{Sign}(E(f_1, f_2)), \quad f_1, f_2 \in \mathcal{M}_g.$$

The map τ_g is actually a normalized two cocycle of $\mathcal{M}_g$.

Lemma 2.3. *For $f_1, f_2, f_3 \in \mathcal{M}_g$, we have*

(1) $\tau_g(f_1 f_2, f_3) + \tau_g(f_1, f_2) = \tau_g(f_1, f_2 f_3) + \tau_g(f_2, f_3)$;
(2) $\tau_g(f_1, 1) = \tau_g(1, f_1) = \tau_g(f_1, f_1^{-1}) = 0$;
(3) $\tau_g(f_1^{-1}, f_2^{-1}) = -\tau_g(f_1, f_2)$;
(4) $\tau_g(f_1, f_2) = \tau_g(f_2, f_1)$;
(5) $\tau_g(f_3 f_1 f_3^{-1}, f_3 f_2 f_3^{-1}) = \tau_g(f_1, f_2)$.

sketch of proof. Recall the *Novikov additivity* of the signature. Let M_1 and M_2 be compact oriented manifolds of the same dimension, Y_1 and Y_2 closed and open submanifolds of ∂M_1 and ∂M_2, respectively, and $\varphi\colon Y_1 \to Y_2$ an orientation reversing homeomorphism. Then the signature of the glued manifold $M_1 \cup_\varphi M_2$ is the sum of the signatures of M_1 and M_2.

We only give the proof of (1), the cocycle condition for τ_g. Consider a Σ_g-bundle $\pi\colon E(f_1, f_2, f_3) \to \Sigma_{0,4}$ and let $C_1, C_2 \subset \Sigma_{0,4}$ be essential simple closed curves intersecting each other in two points, such that C_1 cuts $\Sigma_{0,4}$ into two pairs of pants and the boundary of one of the two contains the free homotopy class of ℓ_1 and ℓ_2. According to the decomposition of the base space, the total space $E(f_1, f_2, f_3)$ can be written as a connected sum of $E(f_1 f_2, f_3)$ and $E(f_1, f_2)$. By the Novikov additivity of the signature, we obtain $\mathrm{Sign}(E(f_1, f_2, f_3)) = \tau_g(f_1 f_2, f_3) + \tau_g(f_1, f_2)$. On the other hand cutting along C_2 and arguing similarly, we obtain $\mathrm{Sign}(E(f_1, f_2, f_3)) = \tau_g(f_1, f_2 f_3) + \tau_g(f_2, f_3)$. □

The signature cocycle has a purely algebraic description. We denote by I_n the $n \times n$ identity matrix. The *integral symplectic group* $Sp(2g; \mathbb{Z})$, also called the *Siegel modular group*, is defined by

$$Sp(2g; \mathbb{Z}) := \{A \in GL(2g; \mathbb{Z});\ {}^t A J A = J\},$$

where $J = \begin{pmatrix} 0 & I_g \\ -I_g & 0 \end{pmatrix}$ and I_g is the $g \times g$ identity matrix. Fix a symplectic basis of $H_1(\Sigma_g; \mathbb{Z})$, i.e., elements $A_1, \dots, A_g, B_1, \dots, B_g \in H_1(\Sigma_g; \mathbb{Z})$ whose algebraic itersection numbers satisfy

$$(A_i \cdot B_j) = \delta_{ij}, \quad (A_i \cdot A_j) = (B_i \cdot B_j) = 0.$$

In terms of a symplectic basis, the (left) action of $\mathcal{M}_g$ on $H_1(\Sigma_g;\mathbb{Z})$ is expressed as matrices and we get a (surjective) group homomorphism

$$\rho\colon \mathcal{M}_g \to Sp(2g;\mathbb{Z}). \tag{2.1}$$

Given $A, B \in Sp(2g;\mathbb{Z})$, consider a $\mathbb{R}$-linear space

$$V_{A,B} := \{(x,y) \in \mathbb{R}^{2g} \oplus \mathbb{R}^{2g}; (A^{-1} - I_{2g})x + (B - I_{2g})y = 0\}$$

and a bilinear form $\langle\ ,\ \rangle_{A,B}\colon V_{A,B} \times V_{A,B} \to \mathbb{R}$ defined by

$$\langle (x,y),(x',y')\rangle_{A,B} := {}^t(x+y)J(I_{2g} - B)y'.$$

It turns out that $\langle\ ,\ \rangle_{A,B}$ is symmetric hence its signature $\mathrm{Sign}(V_{A,B}, \langle\ ,\ \rangle_{A,B})$ is defined. We denote by τ_g^{sp} the map $Sp(2g;\mathbb{Z}) \times Sp(2g;\mathbb{Z}) \to \mathbb{Z}, (A,B) \mapsto \mathrm{Sign}(V_{A,B}, \langle\ ,\ \rangle_{A,B})$. Note that τ_g^{sp} is naturally defined on the Lie group $Sp(2g;\mathbb{R})$.

Theorem 2.4 (Meyer [29]). *The signature cocycle on $\mathcal{M}_g$ is the pull-back of τ_g^{sp} on $Sp(2g;\mathbb{Z})$, i.e., for any $f_1, f_2 \in \mathcal{M}_g$, we have*

$$\tau_g(f_1,f_2) := \mathrm{Sign}(V_{\rho(f_1),\rho(f_2)}, \langle\ ,\ \rangle_{\rho(f_1),\rho(f_2)}).$$

sketch of proof. The proof proceeds following the proof of Theorem 2.1. Consider the Serre cohomology spectral sequence of $E(f_1,f_2) \to P$. The E_2 page is $E_2^{p,q} = H^p(P, \partial P; \mathcal{H}^q(\Sigma_g;\mathbb{R}))$, where $\mathcal{H}^q(\Sigma_g;\mathbb{R})$ denotes the local system on P whose stalk at $b \in P$ is the cohomology of $\pi^{-1}(b)$. On the other hand each page E_r is a *Poincaré ring* in the sense of [11], in particular its signature $\mathrm{Sign}(E_r)$ is defined. The proof is done through three steps: (1) to show that $\mathrm{Sign}(E_r) = \mathrm{Sign}(E_{r+1})$, (2) to show that $\mathrm{Sign}(E_\infty) = \mathrm{Sign}(E(f_1,f_2))$, and (3) to show that $\mathrm{Sign}(E_2) = \mathrm{Sign}(V_{\rho(f_1),\rho(f_2)}\langle\ ,\ \rangle_{\rho(f_1),\rho(f_2)})$. To prove the last step, by taking a simplicial decomposition of P, Meyer [29] observed that $E_2^{1,1} = H^1(P,\partial P;\mathcal{H}^1(\Sigma_g;\mathbb{R}))$ is isomorphic to $V_{\rho(f_1),\rho(f_2)}$, and the cup product on the former corresponds to $\langle\ ,\ \rangle_{\rho(f_1),\rho(f_2)}$. □

The signature cocycle is independently introduced by Turaev [40]. He gave another algebraic description for τ_g^{sp} and directly proved that τ_g^{sp} is a normalized two cocycle. He also discusses a relation with the Maslov index. For coincidence of the definition of τ_g^{sp} by Meyer and Turaev, see Endo-Nagami [15] Appendix.

Remark 2.5. Let M be a closed oriented manifold of dimension $4n-2$ and $\pi\colon E \to B$ an oriented M-bundle with B path connected. By mimicking Definition 2.2, i.e., by constructing a M-bundle over P and taking the signature of the total space, we obtain a normalized 2-cocycle $c_M\colon \pi_1(B)\times\pi_1(B)\to\mathbb{Z}$. In another direction, Atiyah [6] introduced the signature cocycle on the Lie group $U(p,q)$, the unitary group of the Hermitian form with signature (p,q). The restriction to $Sp(2g;\mathbb{R})\subset U(p,p)$ is τ_g^{sp}.

2.3 Evaluation of the signature class

The cocycle $\tau_g\in Z^2(\mathcal{M}_g;\mathbb{Z})$ determines a cohomology class $[\tau_g]\in H^2(\mathcal{M}_g;\mathbb{Z})$, which here we call the signature class. We give a combinatorial method to compute the order of $[\tau_g]$. Following Meyer [30], we consider the following slightly general situation: let G be a group and $k\colon G\times G\to\mathbb{Z}$ a normalized 2-cocycle satisfying $z(x,x^{-1})=0$ for any $x\in G$. Suppose a presentation of G is given. Namely G fits into an exact sequence

$$1\to R\to F\overset{\varpi}{\to}G\to 1$$

where F is the free group generated by a set $\{e_i\}_{i\in I}$. Any $x\in F$ can be written as $x=x_1x_2\cdots x_m$, where $x_j\in\{e_i\}\cup\{e_i^{-1}\}$. Define $c\colon F\to\mathbb{Z}$ by

$$c(x):=\sum_{j=1}^{m}z(\varpi(x_1\cdots x_{j-1}),\varpi(x_j)).$$

It follows that c is well-defined and $\delta c=-\varpi^*z$, i.e., $c(xy)=c(x)+c(y)+z(\varpi(x),\varpi(y))$ for $x,y\in F$. Moreover, c is a class function: $c(yxy^{-1})=c(x)$ for $x,y\in F$. The 1-cochain c is involved in a commutative diagram

$$\begin{array}{ccc} H_2(G;\mathbb{Z}) & \xrightarrow{ev([z])} & \mathbb{Z} \\ \cong\uparrow & \nearrow_{c} & \\ R\cap[F,F]/[R,F] & & \end{array}$$

where the vertical isomorphisms is due to Hopf's formula (see [10]) and the upper right arrow is the evaluation map $ev([z])\colon H_2(G;\mathbb{Z})\to\mathbb{Z}$ by $[z]$. For $i\in I$, let $e_i^*\colon F\to\mathbb{Z}$ be the map counting the total exponents of e_i in elements of F.

Proposition 2.6 (Meyer [30]). *For $m\in\mathbb{Z}\backslash\{0\}$, the order of $[z]\in H^2(G;\mathbb{Z})$ divides m if and only if there exists $\{m_i\}_{i\in I}\subset\mathbb{Z}$ such that $mc|_R=\sum_{i\in I}m_ie_i^*|_R$. In particular, if R is the normal closure of a set $\{r_j\}_{j\in J}\subset F$, then $[z]=0\in H^2(G;\mathbb{Q})$ if and only if the liner equation $c(r_j)=\sum_{i\in I}m_ie_i^*(r_j)$, $j\in J$, has a solution $\{m_i\}_{i\in I}\subset\mathbb{Q}$.*

The proof is straightforward, but we briefly mention "if" part. Take $\{m_i\}_{i\in I}$ satisfying the condition. Consider the $(1/n)\mathbb{Z}$-valued 1-cochain $c_1:=c-$

$(1/n)\sum_{i\in I} n_i e_i^*$ of F. Then it turns out that c_1 descends to a 1-cochain $\overline{c}_1\colon G = F/R \to (1/n)\mathbb{Z}$. In fact, for $x \in F$ and $r \in R$, we have

$$\begin{aligned} c_1(xr) &= c(x) + c(r) + \varpi^* z(x,r) - \frac{1}{n}\sum_{i\in I} n_i(e_i^*(x) + e_i^*(r)) \\ &= c(x) - \frac{1}{n}\sum_{i\in I} n_i e_i^*(x) = c_1(x) \end{aligned}$$

(we use $\varpi(r) = 1$). Since ϖ is surjective, it follows that $\delta\overline{c}_1 = -z$.

In a special situation, this criterion becomes simpler. Let $Art(\mathcal{G})$ be a (small) *Artin group* associated to a connected graph $\mathcal{G}$ without loops. This means that $Art(\mathcal{G})$ is generated by the vertex set $\{a_i\}_{i\in I}$ of $\mathcal{G}$, subject to the defining relations $a_i a_j a_i = a_j a_i a_j$ if a_i and a_j are adjacent, and $a_i a_j = a_j a_i$ if not. Further let $\{r_j\}_{j\in J}$ be a set of words in $\{a_i\}_i$. We shall consider the case G is the group obtained by adding relations $r_j = 1$, $j \in J$ to $Art(\mathcal{G})$. Suppose there exists $\{m_i\}_{i\in I} \subset \mathbb{Q}$ satisfying the condition of Proposition 2.6, and let a_k and a_ℓ be adjacent vertices of $\mathcal{G}$. Now we have $r_{k,\ell} := a_k a_\ell a_k a_\ell^{-1} a_k^{-1} a_\ell^{-1} \in R$, and

$$\begin{aligned} c(r_{k,\ell}) &= c(a_k) + c((a_\ell a_k)a_\ell^{-1}(a_\ell a_k)^{-1}) + z(\varpi(a_k), \varpi(a_k)^{-1}) \\ &= c(a_k) + c(a_\ell^{-1}) = 0. \end{aligned}$$

Here we use the condition $z(x, x^{-1}) = 0$ and the fact that c is a class function. On the other hand, we have $\sum_{i\in I} m_i e_i^*(r_{k,\ell}) = m_k - m_\ell$. Therefore we obtain $m_k = m_\ell$. Since $\mathcal{G}$ is connected, we conclude $m_k = m_\ell$ for any $k, \ell \in I$. In summary, we have the following.

Proposition 2.7. *Suppose G is the quotient of an Artin group as above, and let $z \in Z^2(G;\mathbb{Z})$ be a normalized 2-cocycle with $z(x, x^{-1}) = 0$ for any $x \in G$.*

(1) *For $n \in \mathbb{N}$, $n[z] = 0 \in H^2(G;\mathbb{Z})$ if and only if there exist $m \in \mathbb{Z}$ such that $n \cdot c(r_j) = m \cdot \alpha(r_j)$ for all $j \in J$.*

(2) *In the situation of (1), the 1-cochain $\phi\colon G \to (1/n)\mathbb{Z}$ defined by $\phi(\varpi(x)) = -c(x) + (m/n)\alpha(x)$, $x \in F$ is well-defined. Moreover, $\delta\phi = z$.*

Here $\alpha\colon F \to \mathbb{Z}$ is a homomorhism given by $\alpha(a_i) = 1$ for $i \in I$.

For example, the mapping class group admits a presentation as the quotient of an Artin group where the relation $a_i a_j a_i = a_j a_i a_j$ corresponds to the braid relation among two Dehn twists. Thus we can apply this proposition.

2.4 Meyer's theorems

Using the combinatorial criterion in the previous section, Meyer determined the order of the cohomology class $[\tau_g] \in H^2(\mathcal{M}_g;\mathbb{Z})$.

Theorem 2.8 (Meyer [30], Satz 2). *The order of* $[\tau_1]$ *is 3, the order of* $[\tau_2]$ *is 5, and the order of* $[\tau_g]$ *is infinite if* $g \geq 3$.

To settle the case $g = 1$ and 2, Meyer used a classical presentation of $\mathcal{M}_1 \cong SL(2;\mathbb{Z})$ and a presentation of $\mathcal{M}_2$ by Birman-Hilden [8]. For $g \geq 3$, no finite presentation of $\mathcal{M}_g$ was known at that time. Still, using some of the known relations and showing that $[\tau_g]$ is divisible by 4, Meyer proved that the image of $ev([\tau_g])$ is $4\mathbb{Z}$. We remark that by the Hirzebruch signature formula, we have $e_1 = 3[\tau_g] \in H^2(\mathcal{M}_g;\mathbb{Z})$.

Remark 2.9. Nowadays several finite presentations of $\mathcal{M}_g$ for $g \geq 3$ are known. Using one of them, say the one due to Wajnryb [41], one can directly show that the image of $ev([\tau_g])$ is $4\mathbb{Z}$.

The following is an immediate consequence of Theorem 2.8.

Theorem 2.10 (Meyer [30], Satz 3).

(1) *If* $g \leq 2$, *the signature of the total space of any* Σ_g*-bundle over a closed oriented surface is zero.*

(2) *If* $g \geq 3$, *the signature of the total space of a* Σ_g*-bundle over a closed oriented surface is a multiple of* 4. *Conversely, for any* $g \geq 3$ *and* $n \in 4\mathbb{Z}$, *there exist a* Σ_g*-bundle* $E \to B$ *over a closed oriented surface with* $\mathrm{Sign}(E) = n$.

As a consequence of Theorem 2.8, there exist 1-cochains $\phi_1 \colon \mathcal{M}_1 \to (1/3)\mathbb{Z}$ and $\phi_2 \colon \mathcal{M}_2 \to (1/5)\mathbb{Z}$ such that $\delta\phi_1 = \tau_1$ and $\delta\phi_2 = \tau_2$. Here for a 1-cochain $\phi \colon G \to A$ with coefficient in an abelian group A, its *coboundary* $\delta\phi$ is a map from $G \times G$ to A given by $\delta\phi(x, y) = \phi(x) - \phi(xy) + \phi(y)$ (for terminologies of cohomology of groups, see for example, [10]). Thus the condition $\delta\phi_g = \tau_g$ ($g = 1$ or 2) is equivalent to

$$\tau_g(x, y) = \phi_g(x) - \phi_g(xy) + \phi_g(y), \quad x, y \in \mathcal{M}_g. \tag{2.2}$$

Moreover, since $H^1(\mathcal{M}_1;\mathbb{Q}) = H^1(\mathcal{M}_2;\mathbb{Q}) = 0$, such 1-cochains are unique and characterized by (2.2). The 1-cochain ϕ_1 (resp. ϕ_2) is called the *Meyer function of genus 1* (resp. *of genus 2*).

The following lemma can be directly proved by Lemma 2.3 and (2.2).

Lemma 2.11. *The Meyer functions* ϕ_1 *and* ϕ_2 *satisfy the following properties: for* $x, y \in \mathcal{M}_g$ ($g = 1$ *or* 2),

(1) $\phi_g(1) = 0$;

(2) $\phi_g(x^{-1}) = -\phi_g(x)$;

(3) $\phi_g(yxy^{-1}) = \phi_g(x)$.

Consider a surface bundle over a compact oriented surface. Then the values of ϕ_g around a boundary circle (which is well-defined by Lemma 2.11 (3)) is interpreted as signature defects.

Proposition 2.12. *Suppose $g = 1$ or 2 and let $\pi \colon E \to B$ be a Σ_g-bundle over a compact oriented surface B with boundary components ∂B_i, $i \in I$. Then*

$$\mathrm{Sign}(E) = \sum_{i \in I} \phi_g(x_i),$$

where $x_i \in \mathcal{M}_g$ is the monodromy along the boundary component ∂B_i with the counter-clockwise orientation.

sketch of proof. Take a pants decomposition of B. By the Novikov additivity of the signature, $\mathrm{Sign}(E)$ is the sum of the signatures of the components, which is expressed in terms of τ_g. Using (2.2), we obtain the formula. □

Meyer extensively studied the function ϕ_1 and gave its explicit formula. Note that the mapping class group $\mathcal{M}_1$ is isomorphic to $SL(2;\mathbb{Z}) = Sp(2;\mathbb{Z})$ by the homomorphism (2.1). To state his result, let us prepare some notations. The *Rademacher function* [37] is a map $\Psi \colon SL(2;\mathbb{Z}) \to \mathbb{Q}$ defined by

$$\Psi\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \begin{cases} \dfrac{a+d}{c} - 12\mathrm{sign}(c)s(a,c) - 3\mathrm{sign}(c(a+d)) & \text{if } c \neq 0, \\ \dfrac{b}{d} & \text{if } c = 0. \end{cases}$$

Here $\mathrm{sign}(x) \in \{0, \pm 1\}$ is the sign of x if $x \neq 0$, 0 if $x = 0$, and $s(a,c)$ is the *Dedekind sum*

$$s(a,c) := \sum_{k \bmod |c|} \left(\left(\frac{ak}{c}\right)\right)\left(\left(\frac{k}{c}\right)\right)$$

where

$$((x)) = \begin{cases} x - [x] - \dfrac{1}{2} & \text{if } x \in \mathbb{R} \setminus \mathbb{Z}, \\ 0 & \text{if } x \in \mathbb{Z} \end{cases}$$

($[x]$ denotes the integer part of x). Also, for $\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2;\mathbb{Z})$, set $\sigma(\alpha) = \tau_1(\alpha, -1)$, which by a direct computation turns out to be the signature of the symmetric matrix $\begin{pmatrix} -2c & a-d \\ a-d & 2b \end{pmatrix}$.

Theorem 2.13 (Meyer [30], Satz 4). *For any* $\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2;\mathbb{Z})$, *we have*

$$\phi_1(\alpha) = -\frac{1}{3}\Psi(\alpha) + \sigma(\alpha) \cdot \frac{1}{2}(1 + \operatorname{sign}(a+d)).$$

In particular, if $a + d \neq 0, 1, 2$, *then* $\phi_1(\alpha) = -(1/3)\Psi(\alpha)$.

Meyer's proof is based on a certain cocycle identity of Ψ, behind which is the transformation law under $SL(2;\mathbb{Z})$ of the logarithm of the *Dedekind* η*-function*

$$\eta(\tau) = e^{\pi i \tau/12} \prod_{n=1}^{\infty} (1 - e^{2\pi i n \tau}), \quad \tau \in \{z \in \mathbb{C}; \operatorname{Im}(z) > 0\}.$$

Atiyah [6] gave another proof of Theorem 2.13 of more topological nature.

2.5 Atiyah's theorem

Atiyah [6] showed that the value of ϕ_1 on hyperbolic elements coincides with various invariants. Recall that $\alpha \in SL(2;\mathbb{Z})$ is called *hyperbolic* if $|\operatorname{Tr}(\alpha)| > 2$.

Theorem 2.14 (Atiyah [6]). *For a hyperbolic element* $\alpha \in SL(2;\mathbb{Z})$, *the following quantities coincide.*

(1) $\phi_1(\alpha)$;

(2) *Hirzebruch's signature defect* $\delta(\alpha)$;

(3) *the transformation low of the logarithm of the Dedekind* η*-function under* α;

(4) *the logarithmic monodromy of Quillen's determinant line bundle of the mapping torus of* α;

(5) *the value* $L_\alpha(0)$ *of the Shimizu L-function;*

(6) *The Atiyah-Patodi-Singer invariant* $\eta(\alpha)$ *of the mapping torus of* α;

(7) *The adiabatic limit* $\eta^0(\alpha)$.

Since the invariants (6)(7) will appear again in §4, we give a brief explanation of these invariants here. The Atiyah-Patodi-Singer invariant [7], also called the η-invariant, is a spectral invariant of a closed oriented odd dimensional Riemannian manifold (M, g) and is denoted by $\eta(M, g)$ or $\eta(M)$ shortly. Further, let E and B be closed oriented C^∞-manifolds and $\pi \colon E \to B$ a oriented M-bundle with the dimension of E is divisible by 4. Once a metric $g^{E/B}$ on the relative tangent bundle $T(E/B)$, a metric g^B on B, and a connection

∇ on TE are given, the metric on E is given by $g^E := g^{E/B} \oplus \pi^* g^E$ according to the decomposition $TE = T(E/B) \oplus \pi^* TB$ induced from ∇. Then the one parameter family of metrics on E is defined by $g^E_\varepsilon := g^{E/B} \oplus \varepsilon^{-1} \pi^* g^B$, $\varepsilon \in \mathbb{R}_{>0}$. By Bismut-Cheeger [9], it is shown that the limit $\lim_{\varepsilon \to 0} \eta(E, g^E_\varepsilon)$ exists. The limit is called the *adiabatic limit* of the η-invariants and is denoted by $\eta^0(E)$. In Theorem 2.14, a suitable metric is chosen for the mapping torus of α.

In fact, Atiyah also showed the following result, giving an analytic expression of the value of ϕ_1 on any element of $SL(2;\mathbb{Z})$.

Theorem 2.15 (Atiyah [6]). *For $\alpha \in SL(2;\mathbb{Z})$, we have $\phi_1(\alpha) = \eta^0(\alpha)$.*

A generalization of this result to ϕ_2 will be dealt in §4.2.

3 Local signatures

Consider a closed oriented 4-manifold M admitting a fibration $f\colon M \to B$ onto a closed oriented surface B. Under some conditions, the signature of M happens to *localize* to finitely many singular fibers of f. This phenomenon is called the *localization of the signature*, and has been studied from several point of view. In this section we review some of these treatments, and recall an approach using Meyer functions.

3.1 Local signatures and Horikawa index

Let E and B be compact oriented C^∞-manifolds of dimension 4 and 2 respectively, $f\colon E \to B$ a proper surjective C^∞-map having the structure of Σ_g-bundle outside of finitely many points $\{b_i\}_{i\in I} \subset \mathrm{Int}(B)$. We call such a triple (E, f, B) a *fibered 4-manifold* (of genus g). For $b \in B$, we denote by $\mathcal{F}_b$ the fiber germ of f around b. If $b \in B \setminus \{b_i\}_{i\in I}$, $\mathcal{F}_b$ is called a *general fiber*. If $b = b_i$ for some $i \in I$, $\mathcal{F}_b$ is called a *singular fiber*.

Typical examples of fibered 4-manifolds are elliptic surfaces and Lefschetz fibrations. When we work with holomorphic category, then E is a complex surface, B is a Riemann surface, and f is a holomorphic map. In this case if we say, for example, that $f\colon E \to B$ is a hyperelliptic fibration, then general fibers are hyperelliptic Riemann surfaces.

Among the topological invariants of such E, the topological Euler number $\chi(E)$ is easy to compute. For simplicity we assume that E and B are closed, and let $g(B)$ be the genus of B. Let $\Delta_i \subset B$ be a small closed disk with center b_i and we denote $E_i = f^{-1}(\Delta_i)$ and $E_0 = f^{-1}(B \setminus \bigcup_i \mathrm{Int}(\Delta_i))$. Since the topological Euler number is multiplicative in fiber bundles, we have

$\chi(E_0) = (2-2g)(2-2g(B)-|I|)$. Moreover, since f is proper we have $\chi(E_i) = \chi(f^{-1}(b_i))$. Thus

$$\chi(E) = (2-2g)(2-2g(B)) + \sum_{b \in B} \varepsilon(\mathcal{F}_b),$$

where the number $\varepsilon(\mathcal{F}_b) := \chi(f^{-1}(b_i)) - (2-2g)$ is called the *topological Euler contribution*. In short, we can compute $\chi(E)$ by the contributions $\varepsilon(\mathcal{F}_b)$.

On the other hand, the signature of E is not so easy to compute and in general one cannot compute it from the data of singular fiber germs. Nevertheless, under some conditions on the general fibers, it happens that we can assign a rational number $\sigma(\mathcal{F}_b)$ to each fiber $\mathcal{F}_b$ satisfying the following two conditions:

(1) if $\mathcal{F}_b$ is a general fiber, then $\sigma(\mathcal{F}_b) = 0$.

(2) if E is closed, then $\mathrm{Sign}(E) = \sum_{b \in B} \sigma(\mathcal{F}_b)$.

The assignment σ is called a *local signature*, and when such phenomena happens, we say that the signature of E is localized.

The first example of a local signature is the one for fibered 4-manifolds of genus 1 due to Y. Matsumoto [27]. He called such assignment a fractional signature. Later he also gave a local signature for Lefschetz fibrations of genus 2 [28]. In both the examples, he used the Meyer functions ϕ_1 and ϕ_2 to construct a local signature. See the next subsection for details.

In algebro-geometric setting, local signatures are closely related to an invariant of fiber germs which originates in the work of Horikawa [17] [18]. He studied global family of curves of genus 2 $f\colon E \to B$ and defined an invariant $\mathcal{H}(\mathcal{F}_b) \geq 0$ to each fiber germ, and showed the equality

$$K_E^2 = 2\chi(\mathcal{O}_E) - 6 + 6g(B) + \sum_{b \in B} \mathcal{H}(\mathcal{F}_b). \tag{3.1}$$

Here $g(B)$ is the genus of B, K_E^2 is the self intersection number of the canonical bundle of E, and $\chi(\mathcal{O}_E)$ is the Euler characteristic number of the structure sheaf of E. In the geography of complex surfaces of general type, one often studies complex surfaces with the pair of specified numerical invariants $(K_E^2, \chi(\mathcal{O}_E))$. Note that by the Hirzebruch signature formula $\mathrm{Sign}(E) = (1/3)(K_E^2 - 2\chi(E))$ and the Noether formula $\chi(\mathcal{O}_E) = (1/12)(K_E^2 + \chi(E))$, to fix $(K_E^2, \chi(\mathcal{O}_E))$ is equivalent to fix $(\mathrm{Sign}(E), \chi(E))$. The inequality $K_E^2 \geq 2\chi(\mathcal{O}_E) - 6$ is called the *Noether inequality*, a lower bound for the numerical invariants of complex surfaces of general type. Thus $\mathcal{H}(\mathcal{F}_b)$ is regarded as a local contribution of each fiber germ to the distance from the geographical lower bound for $(K_E^2, \chi(\mathcal{O}_E))$. The invariant $\mathcal{H}(\mathcal{F}_b)$ is called the *Horikawa index*.

There are several situations in which the Horikawa index exists. M. Reid [38] defined it for fiber germs of non-hyperelliptic fibrations of genus 3. This is generalized by Konno [24] to Clifford general fibrations of odd genus.

Arakawa and Ashikaga [1] introduced the Horikawa index for hyperelliptic fibrations, which is regarded as a direct generalization of the work of Horikawa. Let $f\colon E \to B$ be a hyperelliptic fibration of genus g with B closed. They introduced an invariant $\mathcal{H}(\mathcal{F}_b) \geq 0$ for each fiber germ satisfying

$$K_{E/B}^2 = \frac{4(g-1)}{g}\chi_f + \sum_{b\in B}\mathcal{H}(\mathcal{F}_b), \tag{3.2}$$

where $K_{E/B}^2 = K_S^2 - 8(g-1)(g(B)-1)$ and $\chi_f = \chi(\mathcal{O}_E) - (g-1)(g(B)-1)$. Moreover, they defined a local signature for hyperelliptic fibrations of genus g by

$$\sigma_g^{\mathrm{alg}}(\mathcal{F}_b) := \frac{1}{2g+1}(g\mathcal{H}(\mathcal{F}_b) - (g+1)\varepsilon(\mathcal{F}_b)). \tag{3.3}$$

Here $\varepsilon(\mathcal{F}_b)$ is the topological Euler contribution as above. That σ_g^{alg} is a local signature follows from (3.2). More generally, if we find a Horikawa index in a class of fibrations (say non-hyperelliptic fibrations of genus 3), then a formula of type (3.3) gives a local signature for such fibrations.

For more detail about local signatures, we refer to the survey articles Ashikaga-Endo [2] and Ashikaga-Konno [3]. We also refer to recent works by Ashikaga-Yoshikawa [4] and Sato [39].

3.2 Matsumoto's formula

For a while we assume g is 1 or 2. Let (E, f, B) be a fibered 4-manifold of genus g. For each $b \in B$, take a small closed disk neighborhood $\Delta \subset B$ of b and consider the restriction of f to $\Delta\setminus\{b\}$. Let $x_b \in \mathcal{M}_g$ be the monodromy of this Σ_g-bundle along the boundary $\partial\Delta$ with the counter-clockwise orientation, and set

$$\sigma_g(\mathcal{F}_b) := \phi_g(x_b) + \mathrm{Sign}(f^{-1}(\Delta)) \in \mathbb{Q}. \tag{3.4}$$

Here ϕ_g is the Meyer function of genus g. Note that although x_b is only defined up to conjugacy, $\phi_g(x_b)$ is well defined by Lemma 2.11 (3).

Proposition 3.1 (Y. Matusmoto [27] [28]). *Let $g = 1$ or 2. The assignment $\sigma_g(\mathcal{F}_b)$ is a local signature for fibered 4-manifolds of genus g.*

Proof. The property (1) is clear since x_b is trivial if $\mathcal{F}_b$ is non-singular. To prove (2), for each i let Δ_i be a small closed disk neighborhood of b_i. By

Proposition 2.12, we have

$$\begin{aligned}\mathrm{Sign}(E) &= \mathrm{Sign}(f^{-1}(B_0)) + \sum_{i\in I}\mathrm{Sign}(f^{-1}(\Delta_i))\\ &= \sum_{i\in I}\phi_g(x_{b_i}) + \sum_{i\in I}\mathrm{Sign}(f^{-1}(\Delta_i) = \sum_{i\in I}\sigma(\mathcal{F}_{b_i}).\end{aligned}$$

□

Matsumoto [27] [28] also gave some computations of his local signatures. Using the Meyer function on the hyperelliptic mapping class group and applying the formula (3.4), Endo [14] introduced a local signature for hyperelliptic fibrations (see §4.1). By Terasoma, it was shown that Endo's local signature and Arakawa-Ashikaga's local signature (3.3) coincide. See [14] Appendix.

The formula (3.4) implies that the local signature is only determined by topological data. But as Konno [23] observed, there exists a topologically non-singular fiber germ of non-hyperelliptic fibrations of genus 3 which has a non-zero Horikawa index. In fact, in the central fiber $f^{-1}(b)$ of Konno's example is a non-singular hyperelliptic curve of genus 3. From the view point of local signatures, this fiber germ should be thought as a singular fiber. A modification of the formula (3.4) for such situations will be explained in §4.3.

4 Variations

In this section we review higher genera analogues and higher dimensional analogues of Meyer's ϕ_1 or ϕ_2. First note that by Theorem 2.8, Meyer functions does not exist on $\mathcal{M}_g$ for $g > 2$. But the signature cocycle happens to be a coboundary when it is pulled back to some group, for example, a subgroup of $\mathcal{M}_g$. The examples in §4.1 and §4.3 are those of this kind. The example in §4.2 is in a situation of Remark 2.5, and can be regard as a generalization of Theorem 2.15.

4.1 Hyperelliptic mapping class group

Let $\iota \in \mathcal{M}_g$ be a *hyperelliptic involution*, i.e., (the class of) an involution of Σ_g acting on $H_1(\Sigma_g;\mathbb{Z})$ as $-\mathrm{id}$. The *hyperelliptic mapping class group* $\mathcal{H}_g$ is the centralizer of ι:

$$\mathcal{H}_g := \{f \in \mathcal{M}_g; f\iota = \iota f\}.$$

Let $\tau_g^H \in Z^2(\mathcal{H}_g;\mathbb{Z})$ be the restriction of τ_g to the subgroup $\mathcal{H}_g \subset \mathcal{M}_g$. Using a finite presentation of $\mathcal{H}_g$ by Birman-Hilden [8] and Proposition 2.6, Endo [14] proved the following theorem.

Theorem 4.1 (Endo [14]). *The order of* $[\tau_g^H] \in H^2(\mathcal{H}_g;\mathbb{Z})$ *is* $2g+1$. *Furthermore, there uniquely exists a function* $\phi_g^H\colon \mathcal{H}_g \to (1/2g+1)\mathbb{Z}$ *such that* $\delta\phi_g^H = \tau_g^H$.

The 1-cochain ϕ_g^H is called the *Meyer function for the hyperelliptic mapping class group of genus* g.

Remark 4.2. The existence and uniqueness of ϕ_g^H also follow from the $\mathbb{Q}$-acyclicity of $\mathcal{H}_g$ which is independently proved by Cohen [12] and Kawazumi [20].

Remark that $\mathcal{H}_g = \mathcal{M}_g$ if $g=1$ or 2. Thus the series ϕ_g^H, $g \geq 3$ could be a higher genus analogue of Meyer's ϕ_1 and ϕ_2. The values of ϕ_g^H on Dehn twists are given as follows ([14] [32]). Let C be an ι-invariant simple closed curve on Σ_g. We denote by t_C the right handed Dehn twist along C, which is an element of $\mathcal{H}_g$. If C is non-separating, then $\phi_g^H(t_C) = (g+1)/2g+1$; if C is separating and separates Σ_g into surfaces of genus h and $g-h$, then $\phi_g^H(t_C) = -4h(g-h)/2g+1$.

A fibered 4-manifold (E,f,B) is called *hyperelliptic* if the monodromy of the Σ_g-bundle over $B\setminus\{b_i\}_i$ can take value in $\mathcal{H}_g$ by a suitable identification of a reference fiber with Σ_g. Replacing ϕ_g with ϕ_g^H in (3.4), Endo [14] introduced a local signature for hyperelliptic fibered 4-manifold.

Morifuji [32] studied geometrical aspects of ϕ_g^H. He showed if $f \in \mathcal{H}_g$ is of finite order, then $\phi_g^H(f)$ equals $\eta(f)$, the η-invariant (see §2.5) of the mapping torus $\Sigma_g \times [0,1]/(x,0) \sim (f(x),1)$. Further, he showed that $\phi_g^H(f) = d_0(f)$ if f belongs to the hyperelliptic Torelli group, where d_0 is the so-called core of the Casson invariant introduced by Morita [34] [35].

4.2 Family of smooth theta divisors

Iida [19] gave a higher dimensional analogue of Meyer's ϕ_2, which he called the *Meyer function for smooth theta divisors*.

Let $\mathfrak{S}_g := \{\tau \in M(g;\mathbb{C});\ {}^t\tau = \tau, \mathrm{Im}(\tau) > 0\}$ be *the Siegel upper half space* of degree g and $f\colon \mathbb{A}_g \to \mathfrak{S}_g$ the universal family of principally polarized Abelian varieties. The fiber of f at $\tau \in \mathfrak{S}_g$ is the complex torus $A_\tau = \mathbb{C}^g/\Lambda_g$, where Λ_g is the lattice spanned by the column vectors of the $g\times 2g$ matrix $(I_g\ \tau)$. We denote $\mathbf{e}(t) = \exp(2\pi\sqrt{-1}t)$. The Riemann theta function

$$\theta(z,\tau) := \sum_{n\in\mathbb{Z}^g} \mathbf{e}\left(\frac{1}{2} n\tau\, {}^t n + n\, {}^t z\right), \quad z \in \mathbb{C}^g,$$

defines a holomorphic section of a certain holomorphic vector bundle on A_τ and its zero locus is called the *theta divisor*. Set

$$\Theta := \{(z,\tau); \tau \in \mathfrak{S}_g, z \in A_\tau, \theta(z,\tau) = 0\}$$

and let $p\colon \Theta \to \mathfrak{S}_g$ be the natural projection. This is the universal family of theta divisors. We denote by Θ_τ the fiber of p at τ. The group $Sp(2g;\mathbb{Z})$, which for simplicity we denote here by Γ_g, naturally acts on $\mathfrak{S}_g$. Iida introduced a Γ_g-action on Θ so that p is Γ_g-equivariant.

The Zariski closed set $\mathcal{N}_g := \{\tau \in \mathfrak{S}_g; \mathrm{Sing}(\Theta_\tau) \neq \emptyset\}$ is called the *Andreotti-Mayer* locus. The group Γ_g acts on the complement $\mathfrak{S}_g^\circ := \mathfrak{S}_g \setminus \mathcal{N}_g$ properly discontinuously. Let $\mathcal{S}_g$ be the orbifold fundamental group of the quotient orbifold $\Gamma_g \backslash \mathfrak{S}_g^\circ$. In other words, $\mathcal{S}_g$ is the fundamental group of the Borel construction $(\mathfrak{S}_g^\circ)_{\Gamma_g} := E\Gamma_g \times_{\Gamma_g} \mathfrak{S}_g^\circ$, where $E\Gamma_g$ is the total space of the classifying space of Γ_g. The group $\mathcal{S}_g$ fits into an exact sequence

$$1 \to \pi_1(\mathfrak{S}_g^\circ) \to \mathcal{S}_g \to \Gamma_g \to 1. \tag{4.1}$$

If $g = 1$, $\mathcal{N}_g = \emptyset$ and $\Gamma_1 \backslash \mathfrak{S}_1^\circ$ is the moduli space of curves of genus 1, hence $\mathcal{S}_1 = \mathcal{M}_1$. By the Torelli theorem, $\Gamma_2 \backslash \mathfrak{S}_2^\circ$ is the moduli space of curves of genus 2 and $\mathcal{S}_2 = \mathcal{M}_g$.

The projection p induces a fiber bundle over $(\mathfrak{S}_g^\circ)_{\Gamma_g}$. The fiber is diffeomorphic to a smooth theta divisor. By the construction given in Remark 2.5, we get the signature cocycle $c_g\colon \mathcal{S}_g \times \mathcal{S}_g \to \mathbb{Z}$. If g is odd, $c_g \equiv 0$ since the real dimension of a smooth theta divisor is $2g-2$. When $g = 2$, c_2 is the pull back of τ_2^{sp} by (4.1). But if $g \geq 3$, this is not the case.

Using adiabatic limits of η-invariants and a certain automorphic form, Iida constructed a 1-cochain of $\mathcal{S}_g$ which cobounds c_g. Suppose g is even. An element $\sigma \in \mathcal{S}_g$ can be written as $\sigma = (\alpha,\gamma)$, where $\alpha\colon [0,1] \to \mathfrak{S}_g^\circ$ is a continuous map with $\alpha(0)$ a specified basepoint of $\mathfrak{S}_g^\circ$ and $\gamma \in \Gamma_g$ such that $\alpha(1) = \gamma \cdot \alpha(0)$. Consider the mapping torus $M_\sigma := [0,1] \times_\alpha \Theta/(0,x) \sim (1,\gamma x)$ and the projection $\pi\colon M_\sigma \to S^1 = [0,1]/0 \sim 1$. He introduced a metric of the relative tangent bundle $T(M_\sigma/S^1)$ and a connection on M_σ. Then the adiabatic limit $\eta^0(M_\sigma)$ is defined (see §2.5). Set

$$\Phi_g(\sigma) := \eta^0(M_\sigma) + \frac{(-1)^{\frac{g}{2}} 2^{g+3}(2^{g+2}-1)}{(g+3)!} B_{\frac{g}{2}+1} \int_{S^1} \alpha^* d^c \log ||\Delta_g(\tau)||.$$

Here $\Delta_g(\tau)$ is a Siegel cusp form of weight $(g+3)g!/2$ with zero divisor $\mathcal{N}_g$ and B_k is the k-th Bernoulli number.

Theorem 4.3 (Iida [19]). *The 1-cochain Φ_g cobounds c_g, i.e.,*

$$c_g(\sigma_1,\sigma_2) = \Phi_g(\sigma_1) - \Phi_g(\sigma_1\sigma_2) + \Phi_g(\sigma_2), \quad \sigma_1,\sigma_2 \in \mathcal{S}_g.$$

It should be remarked that the uniqueness of Φ_g does not hold. In fact, Iida proved that $H^1(\mathcal{S}_g;\mathbb{Z}) = \mathbb{Z}$ for $g \geq 4$ ([19] Theorem 13). The 1-cochain c_g

actually takes values in $\mathbb{Q}$ ([19] Theorem 15). As a special case, Iida obtained an analytic expression of the Meyer function of genus 2.

Corollary 4.4 (Iida [19]). *For $\sigma = (\alpha, \gamma) \in \mathcal{S}_2 = \mathcal{M}_2$, we have*

$$\phi_2(\sigma) = \eta^0(M_\sigma) - \frac{2}{15} \int_{S^1} \alpha^* d^c \log ||\chi_2(\tau)||^2.$$

Here $\chi_2(\tau)$ is a Siegel modular form of weight 5 called the Igusa modular form.

4.3 The Meyer functions for projective varieties

We mention an approach by Kuno [25] [26] to extend Matsumoto's formula (3.4) for generic non-hyperelliptic fibrations of small genera.

Let $X \subsetneq \mathbb{P}_N$ be a smooth projective variety of dimension $n \geq 2$, embedded in a complex projective space of dimension N. The intersection of X and a generic plane in $\mathbb{P}_N$ of codimension $n-1$ is non-singular of dimension 1. Set $k := N - n + 1$ and let $G_k(\mathbb{P}_N)$ be the Grassmann manifold of k-planes of $\mathbb{P}_N$. The set

$$D_X := \{W \in G_k(\mathbb{P}_N); W \text{ meets } X \text{ not transversally } \}$$

is called the k-th *associated subvariety* of X [16]. Over the complement $U^X := G_k(\mathbb{P}_N) \setminus D_X$, there is a family of compact Riemann surfaces $p_X : \mathcal{C}^X \to U^X$ whose fiber at $W \in U^X$ is $X \cap W$. Let g be the genus of the fibers and let $\rho_X : \pi_1(U^X) \to \mathcal{M}_g$ be the monodromy of this family.

Theorem 4.5 (Kuno [26]). *There exists a unique $\mathbb{Q}$-valued 1-cochain $\phi_X : \pi_1(U^X) \to \mathbb{Q}$ whose coboundary equals the pull-back $\rho_X^* \tau_g$.*

The 1-cochain ϕ_X is called the *Meyer function associated to* $X \subset \mathbb{P}_N$. The fundamental group $\pi_1(U^X)$ is *normally* generated by a single element called a *lasso*, which is represented by a loop "going once around D_X". By ρ_X, a lasso is mapped to a Dehn twist. By a certain extension of theory of Lefschetz pencils, the value of ϕ_X on a lasso is given in terms of invariants of X. Under a mild condition on X, it follows that ϕ_X is an unbounded function. As a consequence, we can show that the group $\pi_1(U^X)$ is non-amenable for such X.

As an application, we can define a local signature for generic non-hyperelliptic fibrations of small genera. Let us illustrate this by an example. Let (E, f, B) be a fibered 4-manifold of genus 3, such that the restriction of f to $B \setminus \{b_i\}_{i \in I}$ is a continuous family of Riemann surfaces with fiber non-hyperelliptic. We call such (E, f, B) a *non-hyperelliptic fibration of genus 3*. Note that we assume a fiberwise complex structure on the general fibers, but do not assume a global complex structure. The idea is to construct a certain universal family and to lift the monodromy to the fundamental group of the base space of it.

Hereafter let X be the image of the Veronese embedding $v_4\colon \mathbb{P}_2 \to \mathbb{P}_{14}$ of degree 4. A generic hyperplane section of $\mathbb{P}_{14}$ corresponds to a smooth plane curve of degree 4 in $\mathbb{P}_2$, which is non-hyperelliptic of genus 3. The group $\mathcal{G} = PGL(3)$ naturally acts on $\mathbb{P}_{14}$ preserving D_X. This induces $\mathcal{G}$-actions on $\mathcal{C}_X$ and U^X, making $p_X\colon \mathcal{C}^X \to U^X$ a $\mathcal{G}$-equivariant map. Therefore we have a continuous family of non-hyperelliptic Riemann surfaces of genus 3 over the Borel construction $U^X_{\mathcal{G}} := E\mathcal{G}\times_{\mathcal{G}} U^X$, which we denote by $p_u\colon \mathcal{C}^X_{\mathcal{G}} \to U^X_{\mathcal{G}}$. This family has a certain universal property: if $p\colon E \to B$ is a continuous family of non-hyperelliptic Riemann surfaces of genus 3, then there exist a continuous map $g\colon B \to U^X_{\mathcal{G}}$ such that the fiber product $\mathcal{C}^X \times_g B$ and the original family are *isotopic*. Moreover, such g is unique up to homotopy. The fundamental group $\pi_1(U^X_{\mathcal{G}})$ fits into an exact sequence

$$\pi_1(PGL(3)) \cong \mathbb{Z}/3\mathbb{Z} \to \pi_1(U^X) \to \pi_1(U^X_{\mathcal{G}}) \to 1.$$

From this and the existence of ϕ_X on $\pi_1(U^X)$, we can deduce that there exists a unique $\mathbb{Q}$-valued 1-cochain $\phi_3^{NH}\colon \pi_1(U^X_{\mathcal{G}}) \to \mathcal{Q}$ which cobounds the pull-back of τ_3 by the monodromy $\rho_u\colon \pi_1(U^X_{\mathcal{G}}) \to \mathcal{M}_3$.

Now, let $\mathcal{F}_b$ be a fiber germ of non-hyperelliptic fibration of genus 3. Take a small closed disk Δ with center b, so that there is no singular fiber on $\Delta\setminus\{b\}$. By the universality of p_u, there is a continuous map $g_{\mathcal{F}_b}\colon \Delta\setminus\{b\} \to U^X_{\mathcal{G}}$. Set $x_{\mathcal{F}_b} := (g_{\mathcal{F}_b})_*(\partial\Delta) \in \pi_1(U^X_{\mathcal{G}})$, where we give $\partial\Delta$ the counterclockwise orientation. Note that $x_{\mathcal{F}_b}$ is uniquely determined up to conjugacy. Set

$$\sigma_3^{NH}(\mathcal{F}_b) := \phi_3^{NH}(x_{\mathcal{F}_b}) + \mathrm{Sign}(f^{-1}(\Delta)).$$

By applying the proof of Proposition 3.1, we have the following.

Theorem 4.6 ([25]). *The assignment σ_3^{NH} is a local signature for non-hyperelliptic fibrations of genus 3.*

The formulation of σ_3^{NH} gives a topological interpretation of Konno's example in §3.2. While the monodromy around b is trivial, its lift $x_{\mathcal{F}_b} \in \pi_1(U^X_{\mathcal{G}})$ is non-trivial and contributes to σ_3^{NH}. Similar constructions are possible for generic non-hyperelliptic fibrations of genus 4 and 5. For details, see [26].

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