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THE "FUNDAMENTAL THEOREM" FOR THE ALGEBRAIC K-THEORY OF SPACES. III. THE NIL-TERM

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ABSTRACT. In this paper we identify the "nil-terms" for Waldhausen's algebraic K-theory of spaces functor as the reduced K-theory of a category of equivariant spaces equipped with a homotopically nilpotent endomorphism.

1. Introduction

This is the third in a series of papers which concerns the decomposition

$$A^{fd}(X \times S^1) \;\simeq\; A^{fd}(X) \times \mathcal{B}A^{fd}(X) \times N_- A^{fd}(X) \times N_+ A^{fd}(X)\,.$$

Here, $A^{fd}(X)$ is Waldhausen's algebraic K-theory of the space X and $\mathcal{B}A^{fd}(X)$ is a certain nonconnective delooping of it. The remaining factors on the right, called "nil-terms", are homotopy equivalent [H$_+$], [H$_+$2]. They have not been given a K-theoretic description thus far.

In this installment, we will identify the nil-terms as a shifted copy of the reduced K-theory of a category whose objects are equivariant spaces equipped with a homotopically nilpotent endomorphism.

Let X be a connected based space. Let $G.$ denote the Kan loop group of the total singular complex of X, and let G denote the geometric realization of $G.$. Then the classifying space BG has the weak homotopy type of X.

Define a category $\mathrm{nil}(X)$ in which an *object* consists of a pair

$$(Y, f)$$

such that Y is a based space with G-action and $f\colon Y \to Y$ is an equivariant map which is *homotopically nilpotent* under composition. Additionally, we assume that Y admits the structure of a based G-cell complex in which the action of G is free away from the basepoint. A *morphism* $(Y, f) \to (Z, g)$ is a based G-map $e\colon Y \to Z$ such that $g \circ e = e \circ f$.

There is a full subcategory $\mathrm{nil}_{fd}(X)$ of $\mathrm{nil}(X)$ whose objects are those Y which are *finitely dominated* in the sense that Y is a retract up to homotopy of an object which is built up from a point by attaching a finite number of free G-cells. A morphism of $\mathrm{nil}_{fd}(X)$ is a *weak equivalence* if and only if its underlying map of topological spaces is a weak homotopy equivalence. It is a *cofibration* if its underlying map of spaces is obtained up to isomorphism by attaching free G-cells.

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With the above structure, it turns out that $\mathrm{nil}_{fd}(X)$ is a category with cofibrations and weak equivalences. It therefore has a K-theory, which is denoted $K^{fd}(\mathrm{nil}(X))$.

The forgetful functor $(Y, f) \mapsto Y$ gives rise to a map on K-theories

$$K^{fd}(\mathrm{nil}(X)) \to A^{fd}(X)\,.$$

Here we are using the model for $A^{fd}(X)$ given by the algebraic K-theory of the category of finitely dominated based G-spaces ([W, §2.1], [H_+, 1.5]). Let

$$\widetilde{K}^{fd}(\mathrm{nil}(X))$$

denote the homotopy fiber of the map $K^{fd}(\mathrm{nil}(X)) \to A^{fd}(X)$.

Our main result establishes the other half of the "fundamental theorem" for $A^{fd}(X)$:

Main Theorem. *There is a homotopy equivalence of functors*

$$\widetilde{K}^{fd}(\mathrm{nil}(X)) \quad \simeq \quad \Omega N_+ A^{fd}(X)\,.$$

Remark. The above result is used in the paper [GKM], where it is shown that the homotopy groups of $N_+A^{fd}(X)$ are either trivial or infinitely generated. Another result of that paper determines the p-complete homotopy type of $N_+A^{fd}(*)$ in degrees $\le 4p - 7$, for p an odd prime.

2. Preliminaries

In what follows, we assume that the reader is familiar with the material of [H_+].

The spaces in this paper are to be given the compactly generated topology. Products are taken in the compactly generated sense. Let $M.$ be a simplicial monoid, and let $M = |M.|$ denote its geometric realization. Let $\mathbb{T}(M)$ denote the category of based (left) M-spaces and based M-maps. We say that a based morphism $Y \to Z$ of $\mathbb{T}(M)$ is *weak equivalence* if (and only if) it is a weak homotopy equivalence of underlying topological spaces. Similarly, we say that is a *fibration* if it is a Serre fibration after forgetting actions. A morphism is a cofibration if and only if it satisfies the left lifting property with respect to the acyclic fibrations (i.e., those fibrations which are weak equivalences). Then $\mathbb{T}(M)$ is a Quillen model category (see, e.g., [VS]).

Then every object of $\mathbb{T}(M)$ is fibrant, and the cofibrant objects are precisely the retracts of those objects which are built up from a point by *cell attachments,* where the cell of dimension n is given by

$$D^n \times M$$

with action defined by left translation.

Recall from [H_+] that $\mathbb{C}(M)$ denotes the full subcategory of $\mathbb{T}(M)$ consisting of the cofibrant objects. Then $\mathbb{C}(M)$ is a *category with cofibrations and weak equivalences* in the sense of Waldhausen [W]. For objects Y and Z of $\mathbb{C}(M)$, we let

$$[Y, Z]$$

denote the homotopy classes of morphisms in $\mathbb{C}_{fd}(G)$, i.e., the based equivariant homotopy classes.

We next recall the various finiteness notions. An object of $\mathbb{C}(M)$ is *finite* if it is built up from a point by finitely many cell attachments (up to isomorphism). An object of $\mathbb{C}(M)$ is said to be *homotopy finite* if there exists a weak equivalence to a finite object. An object of $\mathbb{C}(M)$ is said to be *finitely dominated* if it is a retract of a homotopy finite object. Let $\mathbb{C}_{fd}(M)$ denote the full subcategory of $\mathbb{C}(M)$ whose objects are finitely dominated.

We let $h\mathbb{C}_{fd}(M)$ denote the subcategory of $\mathbb{C}_{fd}(M)$ defined by the weak equivalences. Then the associated K-theory space is given by

$$A^{fd}(*;M) \quad := \quad \Omega|h\mathcal{S}.\mathbb{C}_{fd}(M)|\,,$$

where the right side is the based loop space of the geometric realization of Waldhausen's $\mathcal{S}.$-construction of $\mathbb{C}_{fd}(M)$ ([W, p. 330]). If M is the realization of a simplicial group, then $A^{fd}(*;M)$ is one of the definitions of $A^{fd}(BM)$ (cf. [W, p. 379], [H_+, 1.6]).

The category $\mathrm{nil}_{fd}(X)$ has *objects* specified by pairs (Y,f) with $Y \in \mathbb{C}_{fd}(G)$ and object $f\colon Y \to Y$ a morphism which is homotopically nilpotent under composition, i.e., the associated homotopy class

$$[f] \in [Y,Y]$$

is nilpotent in the sense that some iterate $[f^{\circ k}] = [f]^{\circ k}$ is trivial.

A *morphism* $(Y,f) \to (Z,g)$ of $\mathrm{nil}_{fd}(X)$ is a map $e\colon Y \to Z$ such that $g\circ e = e\circ f$. A *cofibration* of $\mathrm{nil}_{fd}(X)$ is a morphism $(Y,f) \to (Z,g)$ such that $Y \to Z$ is a cofibration of $\mathbb{C}_{fd}(G)$. A *weak equivalence* is a morphism whose underlying map of spaces is a weak homotopy equivalence.

Lemma 2.1. *With respect to the above conventions,* $\mathrm{nil}_{fd}(X)$ *is a category with cofibrations and weak equivalences.*

Proof. The nontrivial thing to be verified is that the cobase change axiom holds. Given a diagram

$$(B,f_1) \leftarrow (A,f_0) \rightarrowtail (C,f_2)$$

we define the pushout to be $(B\cup_A C, f)$, where f denotes $f_1 \cup_{f_0} f_2$. Choose a positive integer k such that $[f_i]^{\circ k}$ is trivial, for $i = 0,1,2$. It will be sufficient to check that $[f]$ is nilpotent. Let us rename $g_i = f_i^{\circ k}$ and $g = f^{\circ k}$. Then, using the model structure, one has a Barratt-Puppe cofiber sequence

$$B \vee C \xrightarrow{\ \ j\ \ } B\cup_A C \xrightarrow{\delta} \Sigma A$$

in $\mathbb{T}(M)$, where $\vee$ means wedge and Σ is suspension. Consequently, there is an exact sequence of pointed sets

$$[\Sigma A, B\cup_A C] \xrightarrow{\ \ \delta^*\ \ } [B\cup_A C, B\cup_A C] \xrightarrow{\ \ j^*\ \ } [B\vee C, B\cup_A C]\,.$$

Then

$$j^*([g]) = [g\circ j] = [g_1 \vee g_2] = 0\,,$$

so there is a homotopy class

$$\gamma \in [\Sigma A, B\cup_A C]$$

such that $[g] = \delta^*(\gamma) = \gamma\circ[\delta]$. Then

$$[g]^{\circ 2} = \gamma\circ[\delta]\circ\gamma\circ[\delta]$$

is trivial because $[\delta]\circ\gamma\circ[\delta] = [\delta]\circ[g]$ coincides with $[\Sigma g_0]\circ[\delta]$, and $[\Sigma g_0]$ is trivial. $\square$

3. Another look at the projective line

Let $\mathbb{N}_-$ denote the monoid of negative integers with generator t^{-1} and $\mathbb{N}_+$ denote the monoid of positive integers with generator t. Let G be the realization of a simplicial group $G.$.

Recall that the *mapping telescope* of an object $Y_+ \in \mathbb{C}_{fd}(G \times \mathbb{N}_+)$ is the object $Y_+(t^{-1}) \in \mathbb{C}_{fd}(G \times \mathbb{Z})$ defined by taking the categorical colimit of the sequence

$$\cdots \xrightarrow{\ t\ } Y_+ \xrightarrow{\ t\ } Y_+ \xrightarrow{\ t\ } \cdots .$$

Similarly, if $Y_- \in \mathbb{C}_{fd}(G \times \mathbb{N}_-)$ is an object, we have a mapping telescope $Y_-(t)$ given by the colimit of

$$\cdots \xrightarrow{\ t^{-1}\ } Y_+ \xrightarrow{\ t^{-1}\ } Y_+ \xrightarrow{\ t^{-1}\ } \cdots .$$

Define $\mathbb{D}_{fd}(G \times \mathbb{Z})$ to be the category whose *objects* are diagrams

$$Y_- \to Y \leftarrow Y_+$$

in which $Y_- \in \mathbb{C}_{fd}(G \times \mathbb{N}_-)$, $Y \in \mathbb{C}_{fd}(G \times \mathbb{Z})$ and $Y_+ \in \mathbb{C}_{fd}(G \times \mathbb{N}_+)$, and where the maps $Y_- \to Y$ and $Y_+ \to Y$ are required to be based and equivariant. Moreover, the induced morphisms

$$Y_-(t) \to Y(t) \cong Y \quad \text{and} \quad Y_+(t^{-1}) \to Y(t^{-1}) \cong Y$$

are required to be cofibrations of $\mathbb{C}_{fd}(G \times \mathbb{Z})$. We take the liberty of specifying the object as a diagram or as a triple (Y_-, Y, Y_+).

A *morphism* $(Y_-, Y, Y_+) \to (Z_-, Z, Z_+)$ of $\mathbb{D}_{fd}(G \times \mathbb{Z})$ is a morphism $Y_- \to Z_-$, a morphism $Y \to Z$ and a morphism $Y_+ \to Z_+$ so that the evident diagram commutes. A *cofibration* is a morphism $(Y_-, Y, Y_+) \to (Z_-, Z, Z_+)$ in which

- each of the maps
$$Y_- \to Z_-, \quad Y_+ \to Z_+ \quad \text{and} \quad Y \to Z$$
is a cofibration (of $\mathbb{C}_{fd}(G \times \mathbb{N}_-)$, $\mathbb{C}_{fd}(G \times \mathbb{N}_+)$ resp. $\mathbb{C}_{fd}(G \times \mathbb{Z})$), and
- the induced maps
$$Y \cup_{Y_-(t)} Z_-(t) \to Z \quad \text{and} \quad Y \cup_{Y_+(t^{-1})} Z_+(t^{-1}) \to Z$$
are cofibrations of $\mathbb{C}_{fd}(G \times \mathbb{Z})$.

The *projective line* $\mathbb{P}_{fd}(G)$ of [H+] is given by the full subcategory of $\mathbb{D}_{fd}(G \times \mathbb{Z})$ whose objects (Y_-, Y, Y_+) satisfy an auxiliary condition, viz., that the induced maps $Y_-(t) \to Y$ and $Y_+(t^{-1}) \to Y$ are weak homotopy equivalences. A *cofibration* is a morphism which is a cofibration of $\mathbb{D}_{fd}(G \times \mathbb{Z})$. A *weak equivalence* is a morphism in which $Y_- \to Z_-$, $Y \to Z$ and $Y_+ \to Z_+$ are weak homotopy equivalences of spaces.

Let $\mathbb{D}_{fd}(G \times \mathbb{N}_-) \subset \mathbb{D}_{fd}(G \times \mathbb{Z})$ denote the full subcategory whose objects (Y_-, Y, Y_+) satisfy the condition that $Y_-(t) \to Y$ is a weak equivalence. Similarly, define $\mathbb{D}_{fd}(G \times \mathbb{N}_+)$ to be the full subcategory whose objects (Y_-, Y, Y_+) satisfy the condition that $Y_+(t^{-1}) \to Y$ is a weak equivalence.

A morphism $(Y_-, Y, Y_+) \to (Z_-, Z, Z_+)$ of $\mathbb{D}_{fd}(G \times \mathbb{N}_+)$ is a *weak equivalence* if the map $Y_+ \to Z_+$ is a weak homotopy equivalence. It is a *cofibration* if it is so when considered in $\mathbb{D}_{fd}(G \times \mathbb{Z})$.

Let $\mathbb{P}_{fd}^{h_{\mathbb{N}_+}}(G) \subset \mathbb{P}_{fd}(G)$ denote the full subcategory with objects (Y_-, Y, Y_+) such that Y_+ is acyclic.

Proposition 3.1. *There is a homotopy fiber sequence*

$$\Omega|h\mathcal{S}.\mathbb{P}_{fd}^{h_{\mathbb{N}_+}}(G)| \to \Omega|h\mathcal{S}.\mathbb{P}_{fd}(G)| \to \Omega|h\mathcal{S}.\mathbb{D}_{fd}(G \times \mathbb{N}_+)| \,.$$

Proof. Define a coarser notion of weak equivalence on the projective line by specifying a morphism $(Y_-, Y, Y_+) \to (Z_-, Z, Z_+)$ to be an $h_{\mathbb{N}_+}$*-equivalence* if (and only if) the map $Y_+ \to Z_+$ is a weak equivalence. Application of the *fibration theorem* [W, 1.6.5] shows that the sequence

$$\Omega|h\mathcal{S}.\mathbb{P}_{fd}^{h_{\mathbb{N}_+}}(G)| \to \Omega|h\mathcal{S}.\mathbb{P}_{fd}(G)| \to \Omega|h_{\mathbb{N}_+}\mathcal{S}.\mathbb{P}_{fd}(G)|$$

is a fibration up to homotopy.

Let $\mathbb{P}_{fd}(G) \to \mathbb{D}_{fd}(G \times \mathbb{N}_+)$ denote the inclusion functor. By [H$_+$, §4] we have that the induced map

$$|h_{\mathbb{N}_+}\mathcal{S}.\mathbb{P}_{fd}(G)| \to |h\mathcal{S}.\mathbb{D}_{fd}(G \times \mathbb{N}_+)|$$

induces an isomorphism on homotopy groups in degrees > 1. Hence, the homotopy fiber of the induced map of loop spaces

$$\Omega|h_{\mathbb{N}_+}\mathcal{S}.\mathbb{P}_{fd}(G)| \to \Omega|h\mathcal{S}.\mathbb{D}_{fd}(G \times \mathbb{N}_+)|$$

is homotopically trivial.

It follows that the homotopy fiber of the map

$$\Omega|h\mathcal{S}.\mathbb{P}_{fd}(G)| \to \Omega|h\mathcal{S}.\mathbb{D}_{fd}(G \times \mathbb{N}_+)|$$

is identified with the homotopy fiber of the map

$$\Omega|h\mathcal{S}.\mathbb{P}_{fd}(G)| \to \Omega|h_{\mathbb{N}_+}\mathcal{S}.\mathbb{P}_{fd}(G)| \,.$$

The result follows. □

4. The "characteristic sequence"

Let $(Y, f) \in \mathrm{nil}_{fd}(X)$ be an object, and let $Y \otimes \mathbb{N}_- \in \mathbb{C}_{fd}(G)$ be the object given by

$$(Y \times \mathbb{N}_-)/(* \times \mathbb{N}_-) \,.$$

Then f induces a self-map of $Y \otimes \mathbb{N}_-$ which is given by $(y, r) \mapsto (f(y), r)$. We will denote this self-map also by f.

Let Y_f be the *homotopy coequalizer* of the pair of maps

$$Y \otimes \mathbb{N}_- \ \underset{t^{-1}}{\overset{f}{\rightrightarrows}} \ Y \otimes \mathbb{N}_- \,,$$

where t^{-1} denotes the map $(y, r) \mapsto (y, r-1)$. (Recall that the homotopy coequalizer of a pair of morphisms $\alpha, \beta \colon U \to V$ is defined to be the quotient of the disjoint union $V \amalg (U \times [0,1])$ which is given by identifying $(u, 0)$ with $\alpha(u)$, $(u, 1)$ with $\beta(u)$ and $* \times [0,1]$ with the basepoint of V.)

If we give Y the structure of a based $(G{\times}\mathbb{N}_-)$-space by letting $\mathbb{N}_-$ act by means of f, then we also have a $(G{\times}\mathbb{N}_-)$-equivariant map

$$\pi_f \colon Y \otimes \mathbb{N}_- \to Y$$

which is given by $(y, r) \mapsto f^{-r}(y)$. Then π_f coequalizes f and t^{-1}, so by the universal property of the homotopy coequalizer, there is an induced map

$$Y_f \to Y \,,$$

which is $(G{\times}\mathbb{N}_-)$-equivariant.

Lemma 4.1. *The map $Y_f \to Y$ induces an isomorphism in reduced singular homology.*

Proof. Let $p\colon S^1 \to S^1 \vee S^1$ be the pinch map, and let $\rho\colon S^1 \to S^1$ be the reflection map. Then the composite

$$S^1 \xrightarrow{p} S^1 \vee S^1 \xrightarrow{\mathrm{id}\vee\rho} S^1 \vee S^1$$

will be denoted by $(1, -1)$.

The homotopy coequalizer induces a homotopy cofiber sequence

$$\Sigma(Y \otimes \mathbb{N}_-) \xrightarrow{t^{-1}-f} \Sigma(Y \otimes \mathbb{N}_-) \to \Sigma Y_f$$

where the first map is defined to be the composite

$$\Sigma(Y \otimes \mathbb{N}_-) \xrightarrow{(1,-1)\wedge \mathrm{id}} \Sigma(Y \otimes \mathbb{N}_-) \vee \Sigma(Y \otimes \mathbb{N}_-) \xrightarrow{t^{-1}\vee f} \Sigma(Y \otimes \mathbb{N}_-)\,.$$

Taking reduced singular chains, we get an induced homotopy cofiber sequence of chain complexes

$$C_*(Y) \otimes \mathbb{Z}[t^{-1}] \xrightarrow{t_*^{-1}-f_*} C_*(Y) \otimes \mathbb{Z}[t^{-1}] \longrightarrow C_*(Y_f)\,. \tag{1}$$

(Recall that a sequence $A \xrightarrow{i} B \xrightarrow{j} C$ of chain complexes is a homotopy cofiber sequence when the composite $j \circ i\colon A \to C$ is equipped with a null homotopy such that the induced map from the mapping cone $T_{i\circ j}$ to C is a quasi-isomorphism.)

Now, for any $\mathbb{Z}$-module M equipped with a self-map $f\colon M \to M$, we have an exact sequence of $\mathbb{Z}[t^{-1}]$-modules

$$0 \longrightarrow M \otimes \mathbb{Z}[t^{-1}] \xrightarrow{t^{-1}-f} M \otimes \mathbb{Z}[t^{-1}] \longrightarrow M_f \longrightarrow 0 \tag{2}$$

in which M_f denotes M considered as a $\mathbb{Z}[t^{-1}]$-module where t^{-1} acts via f (see [B, p. 630]). This implies that the sequence (1) becomes exact when $C_*(Y_f)$ is replaced by $C_*(Y)$ by means of the chain map $C_*(Y_f) \to C_*(Y)$ which is induced by the map $Y_f \to Y$. Consequently, the five lemma implies that the chain map $C_*(Y_f) \to C_*(Y)$ is a quasi-isomorphism. □

Remark 4.2. The sequence (1) is a chain complex version of the so-called, "characteristic sequence" (2) of the module M. Consequently, it is not inappropriate to think of the homotopy coequalizer diagram

$$Y \otimes \mathbb{N}_- \underset{t^{-1}}{\overset{f}{\rightrightarrows}} Y \otimes \mathbb{N}_- \longrightarrow Y_f$$

as a kind of nonlinear version of the characteristic sequence (of the object Y).

Preliminary identification of $K(\mathrm{nil}_{fd}(X))$. Define an exact functor

$$\mathrm{nil}_{fd}(X) \xrightarrow{\Phi} \mathbb{P}_{fd}^{h_{\mathbb{N}_+}}(G)$$

by

$$(Y, f) \quad \mapsto \quad (Y_f, Y_f(t), *)\,,$$

where Y_f is defined above.

In the other direction, define an exact functor

$$\mathbb{P}_{fd}^{h_{\mathbb{N}_+}}(G) \xrightarrow{\ \Psi\ } \mathrm{nil}_{fd}(X)$$

by

$$(Y_-, Y, Y_+) \quad \mapsto \quad (Y_-, t^{-1})\,.$$

To see that Ψ is well-defined, let (Y_-, Y, Y_+) be an object of $\mathbb{P}_{fd}^{h_{\mathbb{N}_+}}(G)$. Then Y_+ and Y are acyclic. Hence Y_- has an acyclic mapping telescope. This implies that there exists a $k \in \mathbb{N}_-$ such that $t^k \colon Y_- \to Y_-$ is (equivariantly) null homotopic (this follows for finite objects by the “small object” argument, and hence for finitely dominated ones since a retract of a null homotopic morphism is again null homotopic; compare [H_+, p. 40 bottom]).

Let Z denote the quotient

$$Y_-/t^k(Y_-)$$

considered as an object of $\mathbb{C}(G)$. Then Z is finitely dominated. This is a consequence of a cell-by-cell induction when Y_- is a finite object of $\mathbb{C}(G \times \mathbb{N}_-)$. It is true for homotopy finite objects because the functor $Y_+ \mapsto Y_+/t^k(Y_-)$ preserves weak equivalences. It is therefore also true when Y_- is finitely dominated since this functor also preserves retracts (cf. [H_+, p. 41 top]).

Since t^k is G-equivariantly null homotopic, the identity map $Y_- \to Y_-$ factors through Z up to homotopy. It follows that Y_- is also finitely dominated when considered as an object of $\mathbb{C}(G)$. This shows that (Y_-, t^{-1}) is an object of $\mathrm{nil}_{fd}(X)$.

Lemma 4.3. *The functors Ψ and Φ induce mutually inverse homotopy equivalences on K-theory.*

Proof. The composite $\Psi \circ \Phi$ is given by

$$(Y, f) \quad \mapsto \quad (Y_f, t^{-1})$$

and Lemma 4.1 implies that there is a morphism $(Y_f, t^{-1}) \to (Y, f)$ which is a weak equivalence after taking a suitable number of suspensions. Since suspension induces a homotopy equivalence on the level of K-theory [W, 1.6.2], it follows that $\Psi \circ \Phi$ induces a homotopy equivalence.

The composite $\Phi \circ \Psi$ is given by

$$(Y_-, Y, Y_+) \quad \mapsto \quad (Y_-, Y_-(t), *)\,.$$

This admits an evident equivalence to the identity functor. Consequently $\Phi \circ \Psi$ induces a map which is homotopic to the identity on the level of K-theory. $\square$

5. Proof of the main theorem

By Lemma 4.3, we have a homotopy equivalence,

$$\Omega|h\mathcal{S}.\mathrm{nil}_{fd}(X)| \quad \simeq \quad \Omega|h\mathcal{S}.\mathbb{P}_{fd}^{h_{\mathbb{N}_+}}(G)|\,.$$

Plugging this into Proposition 3.1, we obtain a homotopy fiber sequence

$$\Omega|h\mathcal{S}.\mathrm{nil}_{fd}(X)| \to \Omega|h\mathcal{S}.\mathbb{P}_{fd}(G)| \to \Omega|h\mathcal{S}.\mathbb{D}_{fd}(G \times \mathbb{N}_+)|\,.$$

Let $\epsilon \colon \Omega|h\mathcal{S}.\mathbb{D}_{fd}(G \times \mathbb{N}_+)| \to \Omega|h\mathcal{S}.\mathbb{C}_{fd}(G)|$ denote the *augmentation* map of [H_+, 7.1], which is induced by

$$(Y_-, Y, Y_+) \quad \mapsto \quad Y/\mathbb{Z}\,,$$

where $Y/\mathbb{Z}$ denotes the orbit space under the $\mathbb{Z}$-action. Recall that the *nil-term* $N_+A^{fd}(X)$ was defined to be the homotopy fiber of ϵ. Similarly, ϵ restricts to a map on $\Omega|h\mathcal{S}.\mathbb{P}_{fd}(G)|$. Denote the homotopy fiber of this restriction by $\Omega|h\mathcal{S}.\mathbb{P}_{fd}(G)|^\epsilon$. Consequently, we have an induced homotopy fiber sequence

$$\Omega|h\mathcal{S}.\mathrm{nil}_{fd}(X)| \to \Omega|h\mathcal{S}.\mathbb{P}_{fd}(G)|^\epsilon \to N_+A^{fd}(X)\,.$$

In was shown in [H$_+$, 7.6] that the second of these maps,

$$\Omega|h\mathcal{S}.\mathbb{P}_{fd}(G)|^\epsilon \to N_+A^{fd}(X),$$

is null homotopic. Moreover, it was shown in [H$_+$, 7.5] that there is a homotopy equivalence

$$\Omega|h\mathcal{S}.\mathbb{P}_{fd}(G)|^\epsilon \quad \simeq \quad \Omega|h\mathcal{S}.\mathbb{C}_{fd}(G)|$$

induced by the *global sections* functor $\Gamma\colon \mathbb{P}_{fd}(G) \to \mathbb{C}_{fd}(G)$ defined by

$$(Y_-, Y, Y_+) \quad \mapsto \quad CY_- \cup Y \cup CY_+\,,$$

where CY_- denotes the cone on Y_-.

Assembling this information, we have a homotopy fiber sequence

$$\Omega|h\mathcal{S}.\mathrm{nil}_{fd}(X)| \xrightarrow{\ \alpha\ } \Omega|h\mathcal{S}.\mathbb{C}_{fd}(G)| \xrightarrow{\ \beta\ } N_+A^{fd}(X) \tag{3}$$

where α is induced by the functor $(Z, f) \mapsto \Sigma Z$. Since the suspension functor $\Sigma\colon \mathbb{C}_{fd}(G) \to \mathbb{C}_{fd}(G)$ induces a homotopy equivalence (by [W, 1.6.2]), we see that the homotopy fiber of α is homotopy equivalent to the homotopy fiber of the map α' which is induced by the forgetful map $(Z, f) \mapsto Z$.

On the one hand, the homotopy fiber of α' is $\widetilde{K}^{fd}(\mathrm{nil}(X))$, by definition. On the other hand, the homotopy fiber sequence (3) implies that the homotopy fiber of α is homotopy equivalent to $\Omega N_+A^{fd}(X)$. We conclude that there is a homotopy equivalence

$$\widetilde{K}^{fd}(\mathrm{nil}(X)) \quad \simeq \quad \Omega N_+A^{fd}(X)\,.$$

This completes the proof of the theorem.

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