Vol. 58, No. 1 DUKE MATHEMATICAL JOURNAL © February 1989

HOMOLOGICAL PROPERTIES OF PERIODIC HOMEOMORPHISMS OF 4-MANIFOLDS

SŁAWOMIR KWASIK AND REINHARD SCHULTZ

Given a mapping f from a space X into itself, it is often possible to obtain significant information about f from the algebraic endomorphisms induced by f on the homology and cohomology of X. For example, if X is a compact polyhedron or topological manifold, then the Lefschetz fixed-point theorem relates the existence of fixed points for f to a function of the eigenvalues of the rational homology or cohomology self-maps defined by f (i.e., the *Lefschetz number*; compare [G-H]). Frequently, some natural assumptions on f and X allow one to retrieve much more information about f than in the general case. In particular, if X is a compact differentiable manifold and f is a diffeomorphism such that $f^N = 1_X$ for some X (in other words, if f is *periodic*), then the Lefschetz number of f equals the Euler characteristic $\chi(F)$ or the set of points F left fixed by f (compare [Kob]). Furthermore, if $f \neq 1$ but $f^p = 1$ for some prime p, then the action of f on the homology groups $H_k(X; \mathbf{Z})$ makes the latter into $\mathbf{Z}[\mathbf{Z}_p]$-modules, and results of R. Swan [Sw1] imply strong restrictions on these modules. For example, if X is an $(n-1)$-connected $2n$-manifold ($n \geqslant 2$) and $\xi \in \mathbb{C}$ is a primitive pth root of 1, then the $\mathbf{Z}[\xi]$-module

$$H_n(X; \mathbf{Z}) \otimes_\Lambda \mathbf{Z}[\xi]$$

(where $\Lambda = \mathbf{Z}[\mathbf{Z}_p]$) is projective and determines the zero element of the projective class group $\tilde{K}_0(\mathbf{Z}[\xi]) \cong \tilde{K}_0(\mathbf{Z}[\mathbf{Z}_p])$.

In [E2] A. Edmonds considers the extent to which such relationships hold for periodic homeomorphisms that are not necessarily smooth. The results in [E2] lead naturally to several basic conjectures and problems that are formulated throughout the article. In this paper we answer three of these questions.

Our first result is a Lefschetz formula for periodic homeomorphisms of 4-manifolds:

THEOREM 1. *Let M^4 be a closed 4-manifold, and let $f\colon M^4 \to M^4$ be a periodic homeomorphism. Then the fixed set F of f has finitely generated Čech cohomology, and the Lefschetz number of f is equal to the (Čech) Euler characteristic $\chi(F)$.*

This result does not generalize to periodic homeomorphisms on n-manifolds for $n > 4$ if the period of f is divisible by distinct primes. Classes of examples are discussed at the end of section 1.

Received February 1, 1988. Second author partially supported by NSF Grant MCS 86-02543 and the Max-Planck-Institut für Mathematik in Bonn.

Our other results deal with $\mathbf{Z}_p$-actions on closed simply connected 4-manifolds (p is an odd prime). In this case the cohomology group $H_2(M; \mathbf{Z})$ is isomorphic as a $\mathbf{Z}[\mathbf{Z}_p]$-module to a direct sum of

(i) copies of $\mathbf{Z}$ with trivial $\mathbf{Z}_p$-action,
(ii) ideals in the domain $\mathbf{Z}[\xi]$ with the $\mathbf{Z}[\mathbf{Z}_p]$-module structure defined by sending the generator $t \in \mathbf{Z}_p$ to ξ, and
(iii) inverse images of the ideals in (ii) under the surjection $\mathbf{Z}_p \to \mathbf{Z}[\xi]$.

As noted in [E2], section 1, this decomposition follows from purely algebraic considerations. This decomposition is not unique, but each summand of type (ii) or (iii) defines an element of the projective class group $\tilde{K}_0(\mathbf{Z}[\xi]) \cong \tilde{K}_0\mathbf{Z}[\mathbf{Z}_p])$, and the *sum* of these elements depends only on the $\mathbf{Z}[\mathbf{Z}_p]$-module $H_2(M; \mathbf{Z})$. This sum is called the *ideal class invariant* of the action in [E2]; we shall denote this class $\alpha(M, f)$. The previously mentioned results of Swan [Sw1] (see also Weintraub [Wtb] and Illman [Ill]) state that $\alpha(M, f) = 0$ if f is smooth.

Examples of continuous periodic maps with $\alpha(M, f) \neq 0$ have been described by D. Ruberman and S. Weinberger; the actions in question have isolated fixed points, and at one of these points the action is not locally equivalent to a linear representation. In [E2] Edmonds asks if $\alpha(M, f)$ can be nonzero for a locally linear action (i.e., locally smooth in the sense of [Bre1]), and he further asks for a description of which classes in $\tilde{K}_0(\mathbf{Z}[\xi])$ are realizable as $\alpha(M, f)$ for suitable M and f. The answers to both questions are as good as one can expect.

THEOREM 2. *If M is a closed simply connected 4-manifold and $f\colon M^4 \to M^4$ is a locally linear homeomorphism with (odd) prime period, then $\alpha(M, f) = 0$.*

THEOREM 3. *A class $u \in \tilde{K}_0(\mathbf{Z}[\xi])$ is the ideal class invariant for some $\alpha(M, f)$ if and only if u is invariant under the automorphism of $\tilde{K}_0(\mathbf{Z}[\xi])$ induced by a canonical conjugation involution.*

Note. The conjugation involution on $\tilde{K}_0(\mathbf{Z}[\xi])$ is defined as follows: Given a finitely generated projective module P, let $P^* = \mathrm{Hom}_{\mathbf{Z}}(P, \mathbf{Z})$ with left multiplication by ξ equal to $\mathrm{Hom}_{\mathbf{Z}}(L(\xi^{-1}, P), \mathbf{Z})$, where $L(v, P)$ is left multiplication by $v \in \mathbf{Z}[\xi]$. Elementary arguments show that this map passes to an involution of the group $\tilde{K}_0(\mathbf{Z}[\xi])$.

In many respects four-dimensional topology is a curious mixture of influences from lower and higher dimensions, and the proofs of Theorems 1–3 all reflect this principle. Cohomological fixed-point theory and the geometrization principle for generalized manifolds of dimension $\leqslant 2$ yield very strong conclusions on the fixed-point sets of group actions on 4-manifolds. Theorem 1 follows from these restrictions and results of K. S. Brown [Bro1, Bro2]; the details are presented in section 1. In contrast, Theorem 2 follows from the existence of closed tubular neighborhoods for locally flat embeddings of surfaces in 4-manifolds, and the homotopy finiteness of compact bounded topological manifolds, both of which are analogs of corresponding results in higher dimensions. Finally, Theorem 3 follows from four-dimensional topological surgery theory.

1. The Lefschetz formula. Results of K. S. Brown [Bro1, Bro2] state that a periodic homeomorphism f of a compact manifold M will satisfy a Lefschetz-type formula if for each positive integer m less than the period of f, the fixed-point set of f^m has finitely generated integral Čech cohomology (see [Bro2], page 104; also see the footnote in [Bro1], page 233). We claim *this condition automatically holds for periodic homeomorphisms on closed* 4-*manifolds.*

First of all, M^4 has the homotopy type of a finite complex by the results of Kirby and Siebenmann [K-S]. Let N denote the period of f, and view f as the generator of a continuous $\mathbf{Z}_N$-action on M^4. For each prime divisor p of N, let F_p be the fixed-point set of $\mathbf{Z}_p$. By Smith theory F_p is a finite disjoint union of mod p cohomology manifolds $F_{p,1} \cup \cdots \cup F_{p,k}$. Furthermore, the cohomological dimensions of the $F_{p,i}$ are at most 3, and codim $F_{p,i} = 3$ only if $p = 2$ and $\mathbf{Z}_2$ acts locally orientation-reversingly near points of $F_{p,i}$. Since cohomology manifolds are the same as (topological or smooth) manifolds in dimension $\leqslant 2$ (compare [Wi], chapter 9), it follows that each $F_{p,i}$ is a manifold unless $p = 2$ and the action of $\mathbf{Z}_2$ is locally orientation-reversing.

Now let r be an arbitrary proper divisor of N, and consider the fixed-point set F_r of $\mathbf{Z}_r \subseteq \mathbf{Z}_N$. Suppose first that r is not a power of 2, and let p be an odd prime dividing r. It is immediate that F_r is contained in F_p; furthermore, if some component $F_{p,j}$ contains a point of F_r, then $F_{p,j}$ is $\mathbf{Z}_r$-invariant. By the geometrization principle for group actions on low-dimensional manifolds ([E1], section 1), the action of $\mathbf{Z}_{r/p}$ on $F_{p,j}$ obtained in this manner is equivalent to a smooth action; and therefore if $F_r \cap F_{p,j}$ is nonempty, it is a finite union of closed manifolds. Therefore F_r is also a finite union of closed manifolds if r is not a power of 2. Suppose now that $r \geqslant 4$ is a power of 2. Then the fixed-point set of $\mathbf{Z}_r$ is a finite union of cohomology manifolds by Smith theory. If F_r denotes the fixed-point set of $\mathbf{Z}_r$ and $F_2 = \bigcup F_{2,j}$ (as in the case $p > 2$), then F_r is contained in F_2. Furthermore, if some component $F_{2,j}$ contains a point of F_r, then $\mathbf{Z}_r$ maps $F_{2,j}$ into itself, and the action of $\mathbf{Z}_2$ is locally orientation-preserving at $F_{2,j}$ (look at some point $x \in F_r \cap F_{2,j}$ and notice that $\mathbf{Z}_4$ acts on $H^*(M, M - \{x\})$). Thus F_r is contained in a union of components $F_{2,j}$ with cohomological dimension at most 2. The preceding argument now implies that F_r is a finite union of closed manifolds.

To summarize the preceding paragraphs, we have verified that *each subset F_r has finitely generated Čech cohomology except perhaps if $p = 2$. Furthermore, $F_2 = F_2' \cup F_2''$, where F_2' is a finite disjoint union of closed manifolds and F_2'' is a mod 2 cohomology* 3-*manifold.* By Brown's results the Lefschetz formula will hold if F_2'' has finitely generated integral Čech cohomology. However, this is a consequence of the compactness of F_2'' and the following two facts:

(i) If V^n is an unbounded (not necessarily compact) manifold with an involution T and the fixed-point set of T is a mod 2 cohomology $(n - 1)$-manifold, then the fixed-point set is an *integral* cohomology manifold.

(ii) If X is a compact integral cohomology manifold, then the Čech homology $\check{H}^*(X; \mathbf{Z})$ is finitely generated in each dimension and zero above $\mathrm{codim}_{\mathbf{Z}}(X)$.

The first assertion is proved in [Bre1], Theorem 7.8, and [Ya], Lemma 1.6. The second assertion is essentially proved in [Fl], section 3; the analogous result for field coefficients is given by [Wi], Corollary 3.2, page 181. Strictly speaking, [Fl] deals with compact Čech homology $\check{H}_*(X; \mathbf{R}/\mathbf{Z})$, but the results on cohomology follow from [Fl] and the Pontryagin duality relationship:

$$\check{H}^k(X; \mathbf{Z}) \cong \mathrm{Hom}_C(\check{H}_k(X; \mathbf{R}/\mathbf{Z}), \mathbf{R}/\mathbf{Z})$$

(here "Hom_C" denotes continuous homomorphisms). Note that the *clc* conditions of [Fl] are contained in the definition of cohomology manifold (see [Bor]) and the equivalence of Čech and sheaf cohomology for the paracompact spaces under consideration (compare [Sw2], page 108, lines 3–7; the terminology is explained on pages 33 and 34 of ([Sw2]).

As noted previously, this completes the proof of the Lefschetz formula for periodic homeomorphisms on closed 4-manifolds.

Remarks. 1. As noted in the introduction, Theorem 1 does not generalize to higher dimensions if the period N of f is divisible by two distinct primes. If $N = 4k + 2$, specific examples of this sort are constructed in [Bre2], pages 276–279; for these examples the fixed-point set is a countably infinite union of circles. Variants of this construction and subsequent results give many additional examples. For example, it is possible to construct actions for which the fixed-point set is a countably infinite union of circles and a finite or countably infinite set of points. Also, the method of construction extends to all periods N that are not prime powers, because the examples of [Bre2] are built from smooth $\mathbf{Z}_{4k+2}$-actions on S^5 with fixed-point set equal to two circles, and subsequent results of E. V. Stein [St] and R. Fintushel and P. Pao [F-P] show that such actions exist whenever N is not a prime power. By suspending the initial smooth actions of $\mathbf{Z}_N$ on S^5 and by substituting these into the construction, one can obtain even wilder topological actions of $\mathbf{Z}_N$ on S^6.

2. If N is not a prime power, it is also possible to construct $\mathbf{Z}_N$-actions such that the fixed-point set is a finite complex but the Lefschetz number of a generator is not equal to the Euler characteristic of the fixed-point set. For example, if $n \geqslant 8$ one can take a periodic self-map of S^n given by the one-point compactification of a diffeomorphism of $\mathbf{R}^n$ with period N and no fixed points (compare [Bre2], chapter 1, and the comments in [E2]). It would be very enlightening to know the minimum dimension for which there are examples of $\mathbf{Z}_N$-actions such that the fixed-point sets have finitely generated integral Čech cohomology but the topological Lefschetz formula fails.

2. Ideal class invariants. Throughout this section p will denote an odd prime; all the results admit formal generalizations to the prime 2, but the ideal class group vanishes in this case, and consequently the conclusions are meaningless.

We begin with the proof of Theorem 2. Let M be a closed, 1-connected, locally linear $\mathbf{Z}_p$-manifold of dimension 4. The object is to prove that the ideal class invariant

$$\alpha(M; \mathbf{Z}_p) \in \tilde{K}_0(\mathbf{Z}[\xi])$$

is trivial. The first step is to formulate a simple generalization of Swan's results:

(2.1) *If M has the $\mathbf{Z}_p$-homotopy type of a finite $\mathbf{Z}_p$-CW complex, then $\alpha = 0$.*

The proof in [Sw1] and [Wtb] extends almost verbatim from finite simplicial complexes with $\mathbf{Z}_p$-actions to finite $\mathbf{Z}_p$-CW complexes.

Thus Theorem 2 will follow from (2.1) and the next assertion.

PROPOSITION 2.2. *If M^4 is a closed, connected 4-manifold with a locally linear $\mathbf{Z}_p$-action, then $(M, \mathbf{Z}_p)$ has the $\mathbf{Z}_p$-homotopy type of a finite $\mathbf{Z}_p$-CW complex.*

Proof. The fixed-point set F splits as a disjoint union $F_0 \cup F_2$, where F_0 is a finite isolated set and F_2 is a (not necessarily connected) 2-manifold. Since $\mathbf{Z}_p$ acts locally linearly, there are closed linear disks about each point of F_0; these can be chosen to be pairwise disjoint and also disjoint from F_2. Set R_0 equal to a union of such disks, and let $M_1 = M - \text{Int}(R_0)$. If $X_1 = M_1/\mathbf{Z}_p$, then X_1 is a 4-manifold with boundary $\partial R_0/\mathbf{Z}_p$, and by local linearity F_2 is a locally flat two-dimensional submanifold of $\text{Int}(X_1)$. Therefore the four-dimensional version of the thin h-cobordism theorem (see [Fr1], [Fr2], [Q]) implies that F_2 has a linear tubular neighborhood in Int X_1 (compare [K-V]). Let $\bar{R}_2 \subseteq \text{Int } X_1$ be a closed linear tubular neighborhood of F_2, and let $R_2 \subseteq \text{Int } M_1$ be the inverse image of $\bar{R}_2$ in M_1. Set $M_2 = M_1 - \text{Int}(R_2)$. Then M_2 is a bounded compact Z_p-manifold with a free $\mathbf{Z}_p$-action, and both R_0 and R_2 are compact *smooth* $\mathbf{Z}_p$-manifolds. But $R_0 \cup R_2$ is $\mathbf{Z}_p$-homotopy equivalent to $F_0 \cup F_2$, and by [K-S] the free $\mathbf{Z}_p$-manifolds $\partial(R_0 \cup R_2) \cong \partial M_2$ and M_2 have the $\mathbf{Z}_p$-homotopy types of finite $\mathbf{Z}_p$-CW complexes. Since M is equivalent to the pushout of

$$\begin{array}{c} \partial(R_0 \cup R_2) \cong \partial M_2 \to M_2 \\ \downarrow \\ R_0 \cup R_2 \end{array}$$

and all the objects under consideration have finite $\mathbf{Z}_p$-homotopy type, the $\mathbf{Z}_p$-manifold M^4 must also have this property. □

Finally, we determine which elements of $\tilde{K}_0(\mathbf{Z}[\xi])$ can be realized as ideal class invariants. Theorem 3 claims that *a class is realizable if and only if it is self-conjugate.*

The following lemma proves half of Theorem 3.

PROPOSITION 2.3. *Let p be an odd prime, and suppose that $\mathbf{Z}_p$ acts on the closed simply connected 4-manifold M^4. Then $H^2(M; \mathbf{Z}) \otimes_\Lambda \mathbf{Z}[\xi]$ and $\text{Hom}_{\mathbf{z}}(H^2(M; \mathbf{Z}) \otimes_\Lambda \mathbf{Z}[\xi], \mathbf{Z})$ are isomorphic $\mathbf{Z}[\xi]$-modules.*

Notation. As usual, Λ denotes the group ring $\mathbf{Z}[\mathbf{Z}_p]$, and the $\mathbf{Z}[\xi]$-module structure on $\mathrm{Hom}_{\mathbf{Z}}(B; \mathbf{Z})$ is defined by taking left multiplication by ξ to be $\mathrm{Hom}_{\mathbf{Z}}(\xi^{-1}, \mathbf{Z})$.

The cup product pairing

$$\lambda\colon H^2(M; \mathbf{Z}) \otimes H^2(M; \mathbf{Z}) \to H^4(M; \mathbf{Z}) = \mathbf{Z}$$

defines an isomorphism of abelian groups

$$\lambda'\colon H^2(M; \mathbf{Z}) \to \mathrm{Hom}_{\mathbf{Z}}(H^2(M; \mathbf{Z}), \mathbf{Z}),$$

and the identity $\lambda(ta \otimes b) = \lambda(a \otimes t^{-1}b)$ for $t \in \mathbf{Z}_p$ implies that λ' is an isomorphism of $\mathbf{Z}[\mathbf{Z}_p]$-modules. If we tensor λ' with $\mathbf{Z}[\xi]$ over $\mathbf{Z}[\mathbf{Z}_p]$, then we obtain an isomorphism λ'' from $H^2(M; \mathbf{Z}) \otimes_\Lambda \mathbf{Z}[\xi]$ to $\mathrm{Hom}_{\mathbf{Z}}(H^2(M; \mathbf{Z}), \mathbf{Z}) \otimes_\Lambda \mathbf{Z}[\xi]$.

Given the module $A = H^2(M; \mathbf{Z})$, define $h_A = f_A g_A$ to be the following composite:

$$\begin{aligned}\mathrm{Hom}_{\mathbf{Z}}(A \otimes_\Lambda \mathbf{Z}[\xi], \mathbf{Z}) &\xrightarrow{g_A} \mathrm{Hom}_{\mathbf{Z}}(A, \mathbf{Z})\\ &\cong \mathrm{Hom}_{\mathbf{Z}}(A, \mathbf{Z}) \otimes_\Lambda \Lambda \xrightarrow{f_A} \mathrm{Hom}_{\mathbf{Z}}(A, \mathbf{Z}) \otimes_\Lambda \mathbf{Z}[\xi].\end{aligned}$$

This definition is of course valid for an arbitrary Λ-module A and is functorial in A. We claim *that h_A is an isomorphism if $A = H^2(M; \mathbf{Z})$*. This will imply that $h\lambda''$ defines an isomorphism from $A = H^2(M; \mathbf{Z}) \otimes_\Lambda \mathbf{Z}[\xi]$ to $\mathrm{Hom}_{\mathbf{Z}}(A; \mathbf{Z})$.

Since $H^2(M; \mathbf{Z})$ is a direct sum of indecomposables isomorphic to (i) $\mathbf{Z}$ with trivial $\mathbf{Z}_p$-action, (ii) $\beta =$ an ideal in $\mathbf{Z}[\xi]$, or (iii) $P =$ a finitely generated projective Λ-module, it suffices to prove that h_A is bijective for modules of these three types. Each case can be verified by a direct elementary argument. □

The proof of the other half of Theorem 3 has two steps. The following result yields the required examples.

PROPOSITION 2.4. *Let G be a finite group that acts freely and linearly on S^3, and let $\mathscr{A} \subseteq \tilde{K}_0(\mathbf{Z}[G])$ be the set of all stable classes of projective modules P that represent elements of the projective Wall group $L_0^p(\mathbf{Z}[G], 1)$. Let $0 \neq u \in \mathscr{A}$ be given. Then there is an orientation-preserving topological G-action on some simply connected manifold M^4 with the following properties*:

(i) *G has exactly two fixed points.*
(ii) *G acts freely on the complement of the fixed-point set.*
(iii) *G acts locally linearly at exactly one of the fixed points.*
(iv) *The cohomology group $H^2(M; \mathbf{Z})$ is a projective $\mathbf{Z}[G]$-module representing u.*

Results of this type have been known for some time (for example, this sort of construction was used by Ruberman and Weinberger; compare also [K-Sch]).

By construction the set $\mathscr{A}$ is a subgroup of the set S of self-conjugate elements in $\tilde{K}_0(\mathbf{Z}[G])$; a Hermitian form on a projective module P defines an isomorphism

$P \cong P^*$. Furthermore, $\mathscr{A}$ contains all classes represented by elements of the type $Q \oplus Q^*$ (take the standard kernel form), and therefore $S/\mathscr{A}$ has exponent at most 2. Of course, if $S/\mathscr{A} = 0$, then every self-conjugate class can be realized. In particular, Theorem 3 will follow from 2.4 if $S = \mathscr{A}$ for $G = \mathbf{Z}_p$. This is the second step in the proof.

PROPOSITION 2.5. *In the notation of* 2.4, *if* $G =$ *is a finite cyclic group, then* $\mathscr{A}$ *is the set of all self-conjugate elements.*

Proof. The Ranicki sequence relating L^h and L^p contains a segment of the following form:

$$\cdots \to L_0^p(\mathbf{Z}[G]) \xrightarrow{\chi} S/S' \to L_3^h(\mathbf{Z}[G]) \to L_3^p(\mathbf{Z}[G]) \to \cdots$$

Here $S' =$ all classes determined by modules of type $Q \oplus Q^*$ and χ is given by the projective class group invariant of a representative for an element of L_0^p. Therefore $\mathscr{A} = S$ if χ is surjective. By exactness this holds if and only if the forgetful homomorphism $L_3^h(\mathbf{Z}[G]) \to L_3^p(\mathbf{Z}[G])$ is a monomorphism. But the latter is true by a result of A. Bak (see [Ba], Theorem 8). □

Proof of 2.4. Let $V_0 = S^3/G$ be the orbit space of a free linear action of G on S^3, let Ω be the corresponding four-dimensional G-module, and let $D(\Omega)$ denote the unit disk in Ω. Given an element $\alpha \in L_0^p(\mathbf{Z}[G])$, view α as an element of $L_1^h(\mathbf{Z}[G \times \mathbf{Z}])$ by the Ranicki splitting

$$L_1^h(\mathbf{Z}[G \times Z]) \cong L_1^h(\mathbf{Z}[G]) \oplus L_0^p(\mathbf{Z}[G])$$

(compare [P-R]), and construct a topological normal cobordism W from the identity on $S^1 \times V_0$ to some homotopy equivalence $f_1: N \to S^1 \times V_0$ with surgery obstruction α. Let $F: W \to S^1 \times V_0 \times [0, 1]$ be the associated normal map. Make F transverse to $\{1\} \times V_0 \times [0, 1]$ without changing $F|\partial_0 W =$ identity. The transverse inverse image of $\{1\} \times V_0 \times [0, 1]$ will then define a cobordism W_0 from V_0 to some manifold N_0; by construction, W_0 is bicollared in W and N_0 is bicollared in N. Define $N_1 = N - \text{Int}(N_0 \times [-1, 1])$; it is immediate that $\partial N_1 \cong N_0 \amalg -N_0$.

The usual techniques of surgery can be applied inside W and N to yield a transverse map homotopic to F such that $(K(G, 1), W_0)$ and $(K(G, 1), N_0)$ are 2- and 1-connected, respectively.

The infinite cyclic covering of N is homeomorphic to a union of submanifolds $N_{1,j}$ indexed over all $j \in \mathbf{Z}$, with each $N_{1,j}$ homeomorphic to N_1 and $N_{1,j} \cap N_{1,k} = \partial_+ N_{1,j} = \partial_- N_{1,k}$ if $k = j + 1$, but $N_{1,j} \cap N_{1,k} = \varnothing$ if $|j - k| \geqslant 2$. Let U_+ be the union of all $N_{1,j}$'s for $j \geqslant 0$ and U_- the union of all $N_{1,j}$'s for $j < 0$; by construction $U_+ \cap U \cong N_0$ is the boundary of both U_+ and U_-. Form the open 4-manifold

$$X = W_0 \cup_{N_0} U_+.$$

As in [P-R], pages 246–248, there is a proper degree-1 normal map

$$\psi\colon (X, V_0) \to (V_0 \times [0, \infty), V_0 \times \{0\})$$

that is a homeomorphism on the boundary and has proper surgery obstruction equal to $\alpha \in L_0^p(\mathbf{Z}[G])$.

If we choose W_0 and N_0 so that (W_0, V_0) is 2-connected and (N_0, V_0) is 1-connected, then the map ψ becomes 2-connected. If $\tilde{X}$ denotes the universal covering of X, this means that $H^2(\tilde{X}, S(\Omega))$ is isomorphic to a representative for α as a $\mathbf{Z}[G]$-module.

The results of [Fr1] imply that the one-point compactification of $\tilde{X}$ is a manifold; the free G-action on $\tilde{X}$ by covering transformations extends uniquely to a semifree G-action on the one-point compactification $\tilde{X}^\bullet$. Let

$$M = D(\Omega) \cup_{S(\Omega)} \tilde{X}^\bullet,$$

and take the G-action on M given by the preceding action on $\tilde{X}^\bullet$ and the linear action on $D(\Omega)$. It is immediate that M is free on the complement of the fixed set.

By excision $H^2(M - \{\infty\}, D(\Omega))$ is isomorphic to $H^2(\tilde{X}, S(\Omega))$, and this is in fact an isomorphism of $\mathbf{Z}[G]$-modules. Since $D(\Omega)$ is contractible and $H^i(M, M - \{\infty\}) = 0$ if $i \neq 4$, it follows that $H^2(M)$ is isomorphic to $H^2(\tilde{X}, S(\Omega))$ as a $\mathbf{Z}[G]$-module. Therefore $H^2(M)$ is a projective $\mathbf{Z}[G]$-module, and the results of [P-R], pages 246–248, imply that the projective class of $H^2(M)$ represents the image of α in $\mathscr{A}/S'$ (notation as in the proof of 2.5). Thus we see that for every $\beta \in \mathscr{A}/S'$ there is an action of the desired type such that $H^2(M)$ is projective and represents β.

We now claim that every element in $\mathscr{A}$ can be realized. By the preceding paragraph, every element in $\mathscr{A}$ may be written as a sum $\beta_0 + \gamma$, where β_0 can be realized by the preceding construction and $\gamma \in S'$. If $\gamma \in S'$, then γ has the form $\delta + \delta^*$ for some $\delta \in \tilde{K}_0(\mathbf{Z}[G])$. Let $B\colon \mathrm{Wh}(G \times \mathbf{Z}) \to \tilde{K}_0(\mathbf{Z}[G])$ be the canonical splitting map (see [R] for a thorough discussion of splittings of $\mathrm{Wh}(G \times \mathbf{Z})$). Choose $\Delta \in \mathrm{Wh}(G \times \mathbf{Z})$. Choose $\Delta \in \mathrm{Wh}(G \times \mathbf{Z})$ so that $B(\Delta) = \delta$. Since β is realized by the preceding question, let V_0, N, W_0, N_0, etc., be as before. By [Fr1] and [Fr2] there is an h-cobordism $(W'; N, N')$ with Whitehead torsion equal to Δ. Furthermore, the transverse homotopy equivalence $f_1\colon N \to S^1 \times V_0$ extends to an equivalence $F'\colon W' \to S^1 \times V_0$ that may again be assumed transverse along $\{1\} \times V_0$. Let $W_0' \subseteq W$ and $N_0' \subseteq N'$ denote the transverse inverse images, and split $N' = (N_0' \times [-1, 1]) \cup N_1'$ as before; we may assume $(K(G, 1), W_0')$ and $(K(G, 1), N_0')$ are 2- and 1-connected for the same reasons as before.

If we write the infinite cyclic covering of N_1' as before, we once again have a splitting of this covering space as $U_+' \cup U_-'$. Suppose that we form the open manifold $X' = W_0 \cup W_0' \cup U_+'$. The geometric interpretation of the splitting map B implies that the Siebenmann invariants $\sigma(X')$, $\sigma(X) \in \tilde{K}_0(\mathbf{Z}[G])$ for the ends of X' and X are related by the formula

$$\sigma(X') - \sigma(X) = B(\tau(N \to N')),$$

where τ denotes Whitehead torsion (compare [R], section 3). But the duality formula for the Whitehead torsion of the h-cobordism $(W'; N, N')$ implies that the torsion of $N \to N'$ is simply $\Delta + \Delta'$. Thus $\sigma(X') = \sigma(X) + (\delta + \delta')$. If we form M' from X' as we formed M from X, we then see that if $H^2(M; \mathbf{Z})$ represents $\beta_0 \in \tilde{K}_0(\mathbf{Z}[G])$, then $H^2(M'; \mathbf{Z})$ represents $\beta_0 + \delta + \delta^*$.

Therefore we know that every coset of S' in $\mathscr{A}/S'$ contains a realizable projective class and that every element in a coset is representable if one element is. Together these imply that every element of $\mathscr{A}$ is representable. This proves all of 2.4 except for the assertion that $\mathbf{Z}_p$ is not locally linear at the second fixed point if $H^2(M; \mathbf{Z})$ determines a nonzero element of $\tilde{K}_0(\mathbf{Z}[G])$. But by construction the class of $H^2(M; \mathbf{Z})$ in the projective class group is the Siebenmann invariant for the end of X', and if the action were locally linear, then the end would be collared and the Siebenmann invariant would vanish [Sieb]. □

References

[Ba] A. Bak, "The computation of surgery groups of finite groups with abelian 2-hyperelementary subgroups," in *Algebraic K-Theory, Evanston 1976*, Lecture Notes in Math. **551**, Springer, Berlin, 1976, pp. 384–409.

[Bor] A. Borel, *Seminar on Transformation Groups*, Ann. of Math. Stud. **46**, Princeton University Press, Princeton, 1960.

[Bre1] G. Bredon, *Orientation in generalized manifolds and applications to the theory of transformation groups*, Michigan Math. J. **7** (1960), 37–64.

[Bre2] ———, *Introduction to Compact Transformation Groups*, Pure and Applied Math. **46**, Academic Press, New York, 1972.

[Bro1] K. S. Brown, *Euler characteristics of discrete groups and G-spaces*, Invent. Math. **27** (1974), 229–264.

[Bro2] ———, *Complete Euler characteristics and fixed point theory*, J. Pure Appl. Algebra **24** (1982), 103–121.

[E1] A. Edmonds, "Transformation groups and low-dimensional manifolds," in *Group Actions on Manifolds (Conference Proceedings, University of Colorado, 1983)*, Contemp. Math. **36**, Amer. Math. Soc., Providence, RI, 1985, pp. 341–368.

[E2] ———, *Aspects of group actions on four-manifolds*, preprint, Indiana University, 1987.

[F-P] R. Fintushel and P. Pao, *Circle actions on homotopy spheres with codimension 4 fixed point set*, Pacific J. Math. **109** (1983), 349–362.

[Fl] E. E. Floyd, *Closed coverings in Čech homology theory*, Trans. Amer. Math. Soc. **84** (1957), 319–337.

[Fr1] M. Freedman, *The topology of four-dimensional manifolds*, J. Differential Geom. **17** (1982), 357–453.

[Fr2] ———, "The disk theorem for 4-dimensional manifolds," in *Proc. Internat. Cong. Math. (Warsaw, 1983), Vol. 1*, PWN-Polish Scientific Publishers, Warsaw, and North-Holland, Amsterdam, 1984, pp. 647–663.

[G-H] M. J. Greenberg and J. R. Harper, *Algebraic Topology: A First Course*, Benjamin-Cummings, Reading, MA, 1981.

[Ill] S. Illman, *Smooth equivariant triangulation of G-manifolds for G a finite group*, Math. Ann. **233** (1978), 199–220.

[K-S] R. Kirby and L. Siebenmann, *Foundational Essays on Topological Manifolds: Smoothings, Embeddings, and Triangulations*, Ann. of Math. Stud. **88**, Princeton University Press, Princeton, 1977.

[Kob] S. Kobayashi, *Fixed points of isometries*, Nagoya Math. J. **13** (1958), 63–68.

[K-Sch] S. KWASIK AND R. SCHULTZ, *Desuspension of group actions and the Ribbon theorem*, Topology, to appear.

[K-V] S. KWASIK AND P. VOGEL, *Asymmetric four-dimensional manifolds*, Duke Math. J. **53** (1986), 759–764.

[P-R] E. K. PEDERSEN AND A. A. RANICKI, *Projective surgery theory*, Topology **19** (1980), 239–254.

[Q] F. QUINN, *Ends of maps III: Dimensions* 4 *and* 5, J. Differential Geom. **17** (1982), 503–521.

[R] A. A. RANICKI, "Algebraic and geometric splittings of the K- and L-groups of polynomial extensions," in *Transformation Groups, Poznań 1985, Proceedings*, Lecture Notes in Math. **1216**, Springer, New York, 1986, pp. 321–363.

[Sieb] L. SIEBENMANN, *The obstruction to finding a boundary for an open manifold of dimension greater than five*, Ph.D. thesis, Princeton University, 1965.

[St] E. V. STEIN, *On the orbit types in a circle action*, Proc. Amer. Math. Soc. **66** (1977), 143–147.

[Sw1] R. G. SWAN, *The Grothendieck ring of a finite group*, Topology **2** (1963), 85–110.

[Sw2] ———, *Theory of Sheaves*, University of Chicago Press, Chicago, 1964.

[Wtb] S. WEINTRAUB, *Topological realization of equivariant intersection forms*, Pacific J. Math. **73** (1977), 257–280.

[Wi] R. L. WILDER, *Topology of Manifolds*, Amer. Math. Soc. Colloq. Publ. **32**, Amer. Math. Soc., New York, 1949.

[Ya] C. T. YANG, *Transformation groups on a homological manifold*, Trans. Amer. Math. Soc. **87** (1958), 261–283.

KWASIK: DEPARTMENT OF MATHEMATICS, TULANE UNIVERSITY, NEW ORLEANS, LOUISIANA 70118
SCHULTZ: DEPARTMENT OF MATHEMATICS, PURDUE UNIVERSITY, WEST LAFAYETTE, INDIANA 47907