Product and Sum Theorems for Whitehead Torsion

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Introduction

The purpose of this paper is to study the behavior of the Whitehead torsion under the operations of taking products and sums. The product referred to here is the tensor product, and our main theorem on products expresses the torsion of the tensor product $C \otimes C'$ of an acyclic complex C, with a not necessarily acyclic complex C', in terms of the torsion of C and the Euler characteristic of C'. As an application of this result, we obtain an expression for the torsion of the cartesian product of two maps in terms of the torsion of the factors and the Euler characteristic of the spaces involved. In particular, the product of any h-cobordism of dimension $\geqq 5$ with a circle is a trivial h-cobordism. We also make application to the Reidemeister-Franz-de Rham torsion and to the torsion of *special* complexes.

The sum theorem is entirely geometric. If $f\colon X_1 \cup X_2 \to Y_1 \cup Y_2$ is a homotopy equivalence such that each $f_i = f \mid X_i\colon X_i \to Y_i$, $i = 1, 2$, is also a homotopy equivalence, we determine, under suitable circumstances, the torsion of f in terms of the torsion of f_1 and f_2. As a consequence, we prove that the simple homotopy type of the connected sum of two manifolds depends only on the simple homotopy type of the two summands. This result was proved in dimension 3 by Cockcroft [2].

The paper is divided into three sections. The first contains the statements of the results and the last two the proofs of the main theorems.

We would like to express our thanks to the referee for the version of the proof of the product theorem that appears in § 2.

1. Statement of results

In this section, we state the results of the paper, deferring most of the proofs to the last two sections. For details on the notions of Whitehead group and torsion, we refer the reader to Whitehead [9] and Milnor [7]. Our point of view will be that of Milnor as expressed in his excellent set of notes on Whithehead torsion [7].

Let A be an associative ring with the a unit and with the property that any two bases for a free A-module have the same number of elements. (All rings con-

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sidered in this paper will be associative rings with unit satisfying this property.) This will be true, for example, if there is a homomorphism of A into a division ring D which takes the unit of A into the unit of D. Let $\mathrm{GL}(n, A)$ be the group of invertible $n \times n$ A-matrices and $\mathrm{GL}(A) = \bigcup \mathrm{GL}(n,A)$. Define the *reduced Whitehead group* of A, $\bar{K}_1(A)$ to be the quotient $\mathrm{GL}(A)/U$ where U is the subgroup generated by the 1×1 matrix (-1) together with those matrices in $\mathrm{GL}(A)$ which are the identity except for a single non-zero off diagonal entry. It is easily checked that $\bar{K}_1$ is a covariant functor from the category of rings to the category of abelian groups.

Let C be a finitely generated A-complex. (All complexes which we consider will be finitely generated.) We will say C is *based* if each C_q is a free A-module with a preferred basis. If C is based and acyclic ($H_qC = 0$, all q), there is associated to C an element $\tau(C)$ in $\bar{K}_1A$ called the *torsion* of C.

Now, suppose R and R' are rings which are also algebras over the commutative ring A, and let C be a based R-complex, and C' a based R'-complex. Then $C \otimes_A C'$ is a based $R \otimes_A R'$ complex.

PRODUCT THEOREM. *Let C be a based R-complex, and C' a based R'-complex. Then, if C is acyclic, so is $C \otimes_A C'$ and*

$$\tau(C \otimes_A C') = \chi(C') j_* \tau(C)$$

where $\chi(C')$ is the Euler characteristic of C' as an R'-complex and $j_*\colon \bar{K}_1(R) \to \bar{K}_1(R \otimes_A R')$ is induced by the map $r \to r \otimes 1$.

Note that if C and C' are both acyclic, $\tau(C \otimes_A C') = 0$ since $\chi(C') = 0$.

The proof of the product theorem is given in § 2.

Let X be a finite cell complex with fundamental group π, and universal covering space $\hat{X}$. Then $C(\hat{X})$ becomes a based complex over the integral group ring $Z[\pi]$ of π by choosing cells in $\hat{X}$ covering the cells in X. If $\alpha\colon \pi \to \mathrm{O}(n)$ is a real representation, we obtain a homomorphism $\alpha\colon Z[\pi] \to R^{n\times n}$, the ring of $n \times n$ real matrices, and can form $R^{n\times n} \otimes_\pi C(\hat{X})$ which is a based $R^{n\times n}$-complex. If $R^{n\times n} \otimes_\pi C(\hat{X})$ is acyclic, we can define the *R-torsion of X associated with α* to be $\Delta_\alpha(X) = \tau(R^{n\times n} \otimes_\pi C(\hat{X}))$ in the group $\bar{K}_1(R^{n\times n}) \approx R^+$, the multiplicative group of positive reals. It is not difficult to show that $\Delta_\alpha(X)$ does not depend on the choice of cells in $\hat{X}$ which cover the cells in X.

A special case of the following corollary was proved by Milnor [11, Th. B].

COROLLARY 1.1. *Let X and Y be finite cell complexes, $\alpha\colon \pi_1 X \to \mathrm{O}(n)$ and $\beta\colon \pi_1 Y \to \mathrm{O}(m)$ representations. Suppose $\Delta_\alpha(X)$ is defined and*

$$\alpha \otimes \beta : \pi_1 X \times \pi_1 Y \longrightarrow \mathrm{O}(nm)$$

is the tensor product representation. Then $\Delta_{\alpha\otimes\beta}(X \times Y)$ is defined and

$$\Delta_{\alpha\otimes\beta}(X \times Y) = \Delta_\alpha(X)^{m\chi(Y)} .$$

In particular, if $\Delta_\beta(Y)$ *is also defined,* $\Delta_{\alpha\otimes\beta}(X \times Y) = 1$.

PROOF. Let

$$D = (R^{n\times n} \otimes_\pi C(\hat{X})) \otimes (R^{m\times n} \otimes_{\pi'} C(\hat{Y}))$$

and $E = R^{nm\times nm} \otimes_{\pi\times\pi'} (C(\hat{X}) \otimes C(\hat{Y}))$ (where $\pi = \pi_1 X$ and $\pi' = \pi_1 Y$). It is easily seen that E is obtained from D by a change of rings using the homomorphism

$$t\colon R^{n\times n} \otimes R^{m\times m} \longrightarrow R^{nm\times nm} .$$

Thus E is acyclic and $\tau(E) = t_*\tau(D)$, $t_*\colon \bar{K}_1(R^{n\times n} \otimes R^{m\times m}) \to \bar{K}_1(R^{nm\times nm})$. Applying the product theorem, we have $\tau(D) = \chi(Y)j_*\tau(R^{n\times n} \otimes_\pi C(X))$ so

$$\tau(E) = \chi(Y)t_*j_*\tau(R^{n\times n} \otimes_\pi C(X)) .$$

Now, for $x \in \bar{K}_1(R^{n\times n})$, $|\det t_*j_*(x)| = |\det(x)|^m$ so, passing to multiplicative notation, the corollary is proved.

For our next application, we need the notion of a special complex. A finite cell complex X is *special* if its fundamental group is finite abelian and acts trivially on the rational homology of its universal covering space. If N is the kernel of the natural map $Q[\pi] \to Q$, Q the rational numbers, then $NC(\hat{X})$ (rational coefficients throughout) is acyclic (for X special) and we define the torsion $\Delta(X)$ of X to be $\tau(NG(\hat{X}))$ the group $\bar{K}_1(N) \approx N^*$, the group of units in N. (This isomorphism is induced by the map $(a_{ij}) \to |\det(a_{ij})|$.) This element is determined up to multiplication by $\pm g$, $g \in \pi$.

COROLLARY 1.2. *If X and Y are special complexes, so is $X \times Y$ and* $\Delta(X \times Y) = 1$.

PROOF. The fact that $X \times Y$ is special is trivial.

Let $N_1 = \ker Q[\pi_1 X] \to Q$, $N_2 = \ker Q[\pi_1 Y] \to Q$, and

$$N = \ker Q[\pi_1(X \times Y)] \approx Q[\pi_1 X] \otimes Q[\pi_1 Y] \longrightarrow Q .$$

Then N is the direct sum of ideals

$$N = N_1 \otimes Q[\pi_1 Y] \oplus \Sigma_1 \otimes N_2$$

where Σ_1 is the ideal generated by Σg, $g \in \pi_1 X$. Thus

$$\begin{aligned} NC(\hat{X} \times \hat{Y}) &= [(N_1 \otimes Q[\pi_1 Y]) \oplus (\Sigma_1 \otimes N_2)]C(\hat{X} \times \hat{Y}) \\ &\approx N_1C(\hat{X}) \otimes C(\hat{Y}) \oplus \Sigma_1 C(\hat{X}) \otimes N_2 C(\hat{Y}) . \end{aligned}$$

Under these circumstances, it is easily checked that the matrix representing the torsion of $X \times Y$ can be obtained by adding matrices representing the torsions of the two summands. Now, the Euler characteristics of $C(\hat{Y})$ and $\Sigma_1 C(\hat{X})$ are zero since X and Y are special; so by the product theorem, each of the summands has zero torsion and we can choose the identity in each of the groups

$\mathrm{GL}(N_1 \otimes Q[\pi_1 Y])$, $\mathrm{GL}(\Sigma_1 \otimes N_2)$ to represent these torsions. However, the identity in $\mathrm{GL}(N)$ is the sum of these, so the corollary is proved.

Suppose π is a discrete group, $Z[\pi]$ its integral group ring. The *Whitehead group* of π, $\mathrm{Wh}(\pi)$, is defined to be the quotient $\bar{K}_1(Z[\pi])/H$ where H is the subgroup generated by the elements of $\bar{K}_1(Z[\pi])$ represented by the 1×1 matrices (g), $g \in \pi$. This is a covariant functor from the category of discrete groups to the category of abelian groups.

Let $f: X \to Y$ be a homotopy equivalence between the finite simplicial complexes X and Y. Let M_f denote the mapping cylinder of f, and $\hat{M}_f$ its universal covering space. Then $C(\hat{M}_f, \hat{X})$ is a free acyclic $Z[\pi_1 X]$-complex, and we obtain a basis by choosing cells in $\hat{M}_f$ covering the cells in M_f. We define the *torsion of f*, $\tau(f)$, to be the element of $\mathrm{Wh}(\pi_1 Y)$ determined by

$$f_* \tau(C(\hat{M}_f, X)) \in \bar{K}_1(Z[\pi_1 Y]) \; .$$

This torsion does not depend on the choice of cells in $\hat{M}_f$ covering cells in M_f.

COROLLARY 1.3. *Let $f: X \to Y$ and $g: X' \to Y'$ be homotopy equivalences. Then $f \times g: X \times X' \to Y \times Y'$ is also a homotopy equivalence and*

$$\tau(f \times g) = \chi(Y') j_* \tau(f) + \chi(Y) j'_* \tau(g) \; ,$$

where $j_*: \mathrm{Wh}(\pi_1 Y) \to \mathrm{Wh}(\pi_1 Y \times \pi_1 Y')$, $j'_*: \mathrm{Wh}(\pi_1 Y') \to \mathrm{Wh}(\pi_1 Y \times \pi_1 Y')$ are induced by $g \to g \times 1$ and $g' \to 1 \times g'$.

This result is already known in the case that X' is simply connected and g is the identity map. (See [7, p. 35].)

PROOF. Since $f \times g$ can be written as the composite

$$X \times X' \xrightarrow{f \times 1} Y \times X' \xrightarrow{1 \times g} Y \times Y' \; ,$$

we have $\tau(f \times g) = (1 \times g)_* \tau(f \times 1) + \tau(1 \times g)$. Thus, it is enough to prove that $\tau(f \times 1) = \chi(Y') i_* \tau(f)$, where $i_*: \mathrm{Wh}(\pi_1 Y) \to \mathrm{Wh}(\pi_1 Y \times \pi_1 X')$. But that is an immediate consequence of the product theorem.

Remark. It is easily shown that the maps j_* and j'_* define an isomorphism of $\mathrm{Wh}(\pi_1 Y) \oplus \mathrm{Wh}(\pi_1 Y')$ onto a direct summand of $\mathrm{Wh}(\pi_1 Y \times \pi_1 Y')$.

COROLLARY 1.4. *If $f: X \to Y$ is a homotopy equivalence, and $g: S^1 \to S^1$ any homotopy equivalence of the circle, then $f \times g: X \times S^1 \to Y \times S^1$ is a simple homotopy equivalence. (That is $\tau(f \times g) = 0$.)*

This follows from Corollary 1.3 and the facts that $\chi(S^1) = 0$ and $\mathrm{Wh}(Z) = 0$. (See Higman [4].) In fact, since $W(G) = 0$ for any finitely generated free abelian G (see [10]), we can replace S^1 in this corollary by any torus $S^1 \times \cdots \times S^1$.

The following corollary holds for either differentiable or piecewise linear manifolds. (Here $\equiv$ denotes either diffeomorphism or piecewise linear homeomorphism.)

COROLLARY 1.5. *Let* (W, M, M') *be an h-cobordism with* $\dim W \geqq 5$. *Then* $W \times S^1 \equiv M \times S^1 \times [0, 1]$. *In particular* $M \times S^1 \equiv M' \times S^1$.

This corollary is an immediate consequence of Corollary 1.4 and the s-cobordism theorem of Mazur [6], Barden [1], and Stallings [8]. The s-cobordism theorem states that, if (W, M, M') is an h-cobordism, $\dim W \geqq 6$, such that the inclusion map $M \subset W$ has torsion zero, then $W \equiv M \times [0, 1]$.

We now consider the torsion of the sum of two maps.

Let X and Y be finite cell complexes which are the union of subcomplexes $X = X_1 \cup X_2$, $Y = Y_1 \cup Y_2$. Let $f_i: X_i \to Y_i$, $i = 1, 2$, be maps such that

$$f_1 \mid X_1 \cap X_2 = f_2 \mid X_1 \cap X_2$$

and let $f: X \to Y$ be the resulting map. We say that f is the *sum* of f_1 and f_2.

SUM THEOREM. *Let* $f: X \to Y$ *be the sum of maps* $f_1: X_1 \to Y_1$ *and* $f_2: X_2 \to Y_2$. *Suppose* $X_1 \cap X_2$ *is connected and simply connected, and that* f_1, f_2, *and* $f_1 | X_1 \cap X_2$ *are homotopy equivalences. Then* f *is a homotopy equivalence and*

$$\tau(f) = j_{1*}\tau(f_1) + j_{2*}\tau(f_2) ,$$

where $j_i: Y_i \subset Y$, $i = 1, 2$.

We will give the proof of this theorem in § 3.

Remark. Under the hypotheses of the sum theorem, $\pi_1 Y$ is the free product $\pi_1 Y_1 * \pi_1 Y_2$ and the maps j_{1*}, j_{2*} define an isomorphism of $\mathrm{Wh}(\pi_1 Y_1) \oplus \mathrm{Wh}(\pi_1 Y_2)$ onto $\mathrm{Wh}(\pi_1 Y_1 * \pi_1 Y_2)$. This result is due to Stallings (unpublished).

The following corollary holds for either differentiable or piecewise linear manifolds and was proved in dimension 3 by Cockcroft [2].

COROLLARY 1.6. *Let* M_i, N_i *be manifolds of dimension* $n \geqq 3$ *such that* M_i *has the same oriented simple homotopy type as* N_i, $i = 1, 2$. *Then the connected sum* $M_1 \# N_1$ *has the simple homotopy type as* $M_2 \# N_2$.

PROOF. It clearly suffices to prove the corollary for the case $N_1 = N_2 = N$.

Let $f: M_1 \to M_2$ be an oriented simple homotopy equivalence so that $\tau(f) = 0$. By Hopf [5, Satz XIVa] (see also Epstein [3]), we can find a map f_1 homotopic to f with the property that $f_1^{-1}\Delta_2 = \Delta_1$, and $f_1 \mid \Delta_1: \Delta_1 \to \Delta_2$ is a homeomorphism for some pair of n-simplices $\Delta_1 \subset M_1$, $\Delta_2 \subset M_2$.

Let Δ be an n-simplex of N so that $M_1 \# N = (M_1 - \mathrm{int}\,\Delta_1) \cup (N - \mathrm{int}\,\Delta)$ and $M_2 \# N = (M_2 - \mathrm{int}\,\Delta_2) \cup (N - \mathrm{int}\,\Delta)$. Applying the sum theorem, we have

$$\tau(f_1 \# \mathrm{id}) = j_{1*}\tau(f_1') + j_{2*}\tau(\mathrm{id}) = j_{1*}\tau(f_1') ,$$

where $f_1' = f_1 \mid (M_1 - \mathrm{int}\,\Delta_1)$. (Note that $(M_1 - \mathrm{int}\,\Delta_1) \cap (N - \mathrm{int}\,\Delta)$ is an $(n-1)$-

sphere which is simply connected since $n \geqq 3$.) Applying the sum theorem again, we see that $\tau(f_1') = \tau(f_1) = 0$, so $f_1 \# \mathrm{id}$ is a simple homotopy equivalence.

2. Proof of the product theorem

We need the following lemma. (See Whitehead [9, p. 22], or Milnor [11, p. 10].)

LEMMA. *Let $0 \to C' \to C \to C'' \to 0$ be an exact sequence of based, acyclic complexes (with the usual compatibity condition on the bases). Then*

$$\tau(C) = \tau(C') + \tau(C'') .$$

Now fix a based acyclic R-complex C, and define functions f_1 and f_2 which assign to each based R'-complex C' an element of $\bar{K}_1(R \otimes_A R')$ by

$$f_1(C') = \tau(C \otimes_A C')$$
$$f_1(C') = \chi(C')j_*\tau(C) .$$

Supposing C' has the form $0 \to C_p \to C_{p-1} \to \cdots \to C_q \to 0$, we proceed by induction on $p - q$ to show $f_1 = f_2$.

For $p - q = 0$, the result is trivial. Suppose $p - q \geqq 1$, and define complexes B and B' by

$$B : 0 \longrightarrow C_q \longrightarrow 0$$
$$B' : 0 \longrightarrow C_p \longrightarrow C_{p-1} \longrightarrow \cdots \longrightarrow C_{q+1} \longrightarrow 0 .$$

Applying $C \otimes_A$ to the exact sequence

$$0 \longrightarrow B \longrightarrow C' \longrightarrow B' \longrightarrow 0 ,$$

we get an exact sequence

$$0 \longrightarrow C \otimes_A B \longrightarrow C \otimes_A C' \longrightarrow C \otimes_A B' \longrightarrow 0 .$$

By the lemma above,

$$\tau(C \otimes_A C') = \tau(C \otimes_A B) + \tau(C \otimes_A B')$$

which, by induction,

$$= (\chi(B) + \chi(B'))j_*\tau(C)$$
$$= \chi(C')j_*\tau(C) .$$

This completes the proof of the product theorem.

3. Proof of the sum theorem

The fact that f is a homotopy equivalence is not difficult, and the proof is left to the reader. We now compute the torsion of f.

Let $X_1 \cap X_2 = X_0$, and let $f_0 = f \mid X_0$. Then, if M_f is the mapping cylinder of f, we have $M_f = M_{f_1} \cup M_{f_2}$ with $M_{f_1} \cap M_{f_2} = M_{f_0}$.

Let $p\colon \hat{M}_f \to M_f$ be the universal covering for M_f, and consider the short exact sequence of free $Z[\pi_1 X]$ modules

$$0 \longrightarrow C(p^{-1}(M_{f_0}), p^{-1}(X_0))$$
$$\xrightarrow{\varphi} C(p^{-1}(M_{f_1}), p^{-1}(X_1)) \oplus C(p^{-1}(M_{f_2}), p^{-1}(X_2)) \xrightarrow{\psi} C(\hat{M}_f, \hat{M}) \longrightarrow 0\,,$$

where $\varphi(c) = (c, c)$, and $\psi(c_1, c_2) = c_1 - c_2$. It is easy to check that the maps in this sequence are compatible with the natural bases so the torsion of the middle term is the sum of the torsions of the two extreme terms. However, the torsion of $C(p^{-1}(M_{f_0}), p^{-1}X_0)$ is easily seen to be zero since $\pi_1 X_0 = 0$. (Simply choose cells in $p^{-1}(M_{f_0})$ from a single component so that only integers are involved in the boundary operator.) Thus

$$\begin{aligned}\tau(C(\hat{M}_f), C(X)) &= \tau[C(p^{-1}(M_{f_1}), p^{-1}(X_1)) \oplus C(p^{-1}(M_{f_2}), p^{-1}(X_2))]\\ &= \tau[C(p^{-1}(M_{f_1}), p^{-1}(X_1))] + \tau[C(p^{-1}(M_{f_2}), p^{-1}(X_2))]\,.\end{aligned}$$

We compute $\tau[C(p^{-1}(M_{f_1}), p^{-1}(X_1))]$.

Let $p_1\colon \hat{M}_{f_1} \to M_{f_1}$ be the universal covering. Now $p_1^{-1}(M_{f_0})$ is a disjoint union of copies of M_{f_0}, and we form a new space V by attaching to each of these copies of M_{f_0} a copy of M_{f_2}. Clearly V covers M_f and, since $\hat{M}_f$ is universal, we can factor $p\colon \hat{M}_f \to M_f$,

$$\hat{M}_f \xrightarrow{p_2} V \longrightarrow M_f\,,$$

and $p^{-1}(M_{f_1}) = p_1^{-1}(\hat{M}_{f_1})$ is the disjoint union of copies of $\hat{M}_{f_1}$. Thus, if we choose cells in $p^{-1}(M_{f_1})$ from a single component, we see that the matrices involved in the computation of $\tau C(p^{-1}(M_{f_1}), p^{-1}(X_1))$ are exactly the matrices involved in computing $\tau C(\hat{M}_{f_1}, X_1)$. The analogous argument for $\tau C(p^{-1}(M_{f_2}), p^{-1}(X_2))$ completes the proof of the theorem.

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BIBLIOGRAPHY

1. D. BARDEN, The structure of manifolds, Thesis, Cambridge University, 1963.
2. W. H. COCKCROFT, *The simple homotopy type of the composition of 3-manifolds*, Proc. London Math. Soc., 11 (1961), 577-587.
3. D. B. A. EPSTEIN, *The degree of a map*, (to appear).
4. G. HIGMAN, *The units of group rings*, Proc. London Math. Soc., 46 (1940), 231-248.
5. H. HOPF, *Zur Topologie der Abbildungen von Mannigfaltigkeiten*: II, Math. Ann., 102 (1930), 562-623.
6. B. MAZUR, *Regular neighborhoods and the theorems of Smale*, Ann. of Math., 77 (1963), 232-249.
7. J. MILNOR, Whitehead torsion, mimeographed notes, Princeton University, 1964.

8. J. STALLINGS, On infinite processes leading to differentiability in the complement of a point, Differentiability and Combinatorial Topology, (to appear), Princeton University Press.
9. J. H. C. WHITEHEAD, *Simple homotopy types*, Amer. J. Math., 72 (1950), 1-57.
10. H. BASS, A. HELLER, and R. SWAN, The Whitehead group of a polynomial extension. Publ. I.H.E.S., No. 22 (1964), 61-79.
11. J. MILNOR, *Two complexes which are homeomorphic but combinatorially distinct*, Ann. of Math., 74 (1961), 575-590.

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