ANNALS OF MATHEMATICS
Vol. 59, No. 3, May 1954
Printed in U.S.A.

BRANCHED COVERING SPACES AND THE QUADRATIC FORMS OF LINKS

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(Received December 11, 1953)

Following the untimely death in October 1952 of Roger H. Kyle, I have undertaken to publish his mathematical results. These are principally contained in his Princeton Ph.D. thesis [1], an investigation into the imbedding types of Möbius strips and circular rings, and joint work with me on some applications of the free calculus. Following a plan, on which we had agreed, the results of the thesis will be published in two papers. The present paper contains the principal results of Chapters I and IV. Chapters II, III and IV, which will be published in a subsequent paper, are concerned with the cohomology ring of a branched covering, especially of a knot or link.

One of the main results of the thesis was that the Minkowski units of the quadratic form of a knot are completely determined by the linking invariants of the 2-sheeted cyclic branched covering of the knot. This result was obtained independently by D. Puppe [2]. More recently M. Kneser and D. Puppe [3] have established that these linking invariants determine not only the Minkowski units but what is called below the *family* of the quadratic form. In view of these developments it seemed advisable to delete that part of the thesis that has to do with the Minkowski units; it is now clear that the role of these units in knot theory will always be subservient to the role of the form itself.

The beautiful and fundamental result of Kneser and Puppe has brought to completion one aspect of the study of the quadratic form of a knot. Consequently I have rewritten Kyle's work, making use of the Kneser-Puppe result and concentrating on the quadratic form of a link. As will be seen below the passage from knots to links is not trivial for two reasons: (1) the forms may become degenerate, and (2) the local behavior at the prime 2 makes its entrance. No satisfactory complete set of local invariants of a quadratic form at the prime 2 was in existence, and Kyle was studying this problem in the summer of 1952. Although such a set has been given very recently by T. O'Meara [4], I have not attempted to extract from it a complete set of family invariants. In fact the problem is implicitly solved by a result of E. Burger [12], who found a complete set of invariants (of quite a different nature) for a group with linking.

R. H. FOX

1. The quadratic form of a knot

A (polygonal) knot of multiplicity μ is the union of μ disjoint simple closed polygons in the finite part of spherical 3-space $\mathfrak{M}$. A normed regular projection of such a knot consists of a finite number of crossings and arcs (or simple closed curves) and it divides up the plane of projection into a finite number of bounded regions and an unbounded region. Following Reidemeister [5] alternate regions, counting from the unbounded region X_0, are shaded, and each crossing D is thereby assigned an incidence number $\eta(D)$. Let the shaded bounded regions be denoted by $X_1, \cdots, X_n$. The quadratic form of the diagram is then defined [5, 6] to be $f(x_1, \cdots, x_n) = \hat{f}(0, x_1, \cdots, x_n)$ where $\hat{f}(x_0, x_1, \cdots, x_n) = \sum_{i<j} e_{ij}(x_i - x_j)^2$, and $e_{ij} = \sum \eta(D)$, summed over the crossings incident to the two regions X_i and X_j. Clearly $\hat{f}(x_0, x_1, \cdots, x_n) = \sum_{i,j=0}^{n} a_{ij}x_ix_j$, where $a_{ij} = -e_{ij}$ for $i \neq j$, and $a_{ii} = \sum_{j\neq i} e_{ij} = \sum \eta(D)$, summed over the crossings incident to

X_i. Thus to $\hat{f}$ is associated the symmetric integral matrix $\hat{A} = (a_{ij})_{i,j=0,\cdots,n}$; similarly the matrix associated with f is the principal minor $A = (a_{ij})_{i,j=1,\cdots,n}$ of $\hat{A}$.

It is known [5, 3] that the elementary deformations $\Omega_i^{\pm}(i = 1, 2, 3)$ of a projection of a single knot ($\mu = 1$) produce changes in the associated matrix that can be expressed by finite sequences of certain operations $Q_i^{\pm}(i = 1, 2)$ defined as follows:

$$Q_1: A \to T'AT, \text{ with } T \text{ integral and unimodular;}$$

$$Q_2: A \to \begin{pmatrix} A & 0 \\ 0 & \pm 1 \end{pmatrix}.$$

If we consider the elementary deformations of a projection of a multiple knot ($\mu > 1$), the same statement may be made, provided we allow a third operation $Q_3^{\pm}$ defined as follows:

$$Q_3: A \to \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix}.$$

In fact the discussion on p. 27 of [5] of the case there denoted by (Ω. 2β.) assumes that the regions there denoted by Γ_m' and Γ_{m+1}' be distinct, and this condition may not obtain if the knot is multiple. It is, however, easy to verify that (Ω. 2β.) leads to the operation Q_3 if $\Gamma_m' = \Gamma_{m+1}'$.

Furthermore the matrix $\hat{A}$ may be obtained from the matrix A by application of the operations Q_1 and Q_3; this follows easily from the fact that $\sum_{j=0}^{n} a_{ij} = 0 (i = 0, 1, \cdots, n)$.

The above discussion suggests the following definitions: Two (symmetric, integral) matrices A and B will be said to be *related*, or to belong to the same *family*, if one can be obtained from the other by a finite sequence of the operations $Q_i^{\pm}(i = 1, 2, 3)$. They are *closely related*, or belong to the same *immediate family* if one can be obtained from the other by a finite sequence of the operations $Q_i^{\pm}(i = 1, 2)$.

Two quadratic forms belong to the same *family* (*immediate family*) if their associated coefficient matrices belong to the same family (immediate family).

The forms of a given family do not all have the same number of variables; those which do have the same number of variables need not belong to the same genus (for $\begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} A & 0 \\ 0 & -1 \end{pmatrix}$ may not have the same determinant). However forms that belong to the same genus apparently belong to the same family.

LEMMA 1. *Any non-zero symmetric integral matrix may be transformed by* Q_1 *into a matrix of the form* $\begin{pmatrix} B & 0 \\ 0 & 0 \end{pmatrix}$ *where* B *is non-singular*. (Thus every family contains non-singular matrices.)

PROOF. Let A be any symmetric integral matrix. It is well-known that there exist unimodular integral matrices P and Q such that PAQ is diagonal Thus $PAQ = \begin{pmatrix} C & 0 \\ 0 & 0 \end{pmatrix}$ where C is non-singular. Write $Q'P^{-1} = R = \begin{pmatrix} R_{11} & R_{12} \\ R_{21} & R_{22} \end{pmatrix}$. Then

$Q'AQ = RPAQ = \begin{pmatrix} R_{11}C & 0 \\ R_{21}C & 0 \end{pmatrix}$. Since $Q'AQ$ is symmetric, $R_{21}C = 0$; furthermore $R_{11}C$ is non-singular because the order or $R_{11}C$ is equal to the rank of A.

LEMMA 2. *Non-singular matrices are related if and only if they are closely related.*

PROOF. Consider related matrices A and B, not necessarily non-singular, and suppose that $A \xrightarrow{Q_i} B$ $(i = 1, 2, 3)$. By the preceding lemma there exist unimodular matrices P and Q and non-singular matrices C and D such that $P'AP = \begin{pmatrix} C & 0 \\ 0 & 0 \end{pmatrix}$ and $Q'BQ = \begin{pmatrix} D & 0 \\ 0 & 0 \end{pmatrix}$.

$i = 1: B = R'AR$ with R unimodular. Then $\begin{pmatrix} D & 0 \\ 0 & 0 \end{pmatrix} = S' \begin{pmatrix} C & 0 \\ 0 & 0 \end{pmatrix} S$ where $S = P^{-1}RQ$. Since rank B = rank A, order D = order C. Let $S = \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix}$, so that $D = S'_{11}CS_{11}$ and $0 = S'_{11}CS_{12}$. Since D and C are non-singular, S_{11} is non-singular, hence there exists a rational matrix S_{11}^{-1}. Therefore $C^{-1}(S_{11}^{-1})' \cdot 0 = 0 = S_{12}$. Since S is unimodular, it now follows that S_{11} is unimodular.

$i = 2: B = \begin{pmatrix} A & 0 \\ 0 & \pm 1 \end{pmatrix}$. Then

$$\begin{pmatrix} D & 0 \\ 0 & 0 \end{pmatrix} = Q' \begin{pmatrix} (P^{-1})' \begin{pmatrix} C & 0 \\ 0 & 0 \end{pmatrix} P^{-1} & 0 \\ 0 & \pm 1 \end{pmatrix} \cdot Q = \begin{pmatrix} T'CT & 0 & 0 \\ 0 & \pm 1 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

where T is unimodular. Since rank $B = 1 +$ rank A, order $D = 1 +$ order C, hence $D = \begin{pmatrix} T'CT & 0 \\ 0 & \pm 1 \end{pmatrix}$.

$i = 3: B = \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix}$. As before

$$\begin{pmatrix} D & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} T'CT & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

with T unimodular. Since rank B = rank A, order D = order C, hence $D = T'CT$.

In each of the three cases $i = 1, 2, 3$ it has been shown that C and D are closely related. If now A and B are related non-singular matrices there is a sequence of matrices $A = A_1, A_2, \cdots, A_r = B$ such that

$$A_j \xrightarrow{Q_i^{\pm}} A_{j+1}.$$

Writing $P'_jA_jP_j = \begin{pmatrix} C_j & 0 \\ 0 & 0 \end{pmatrix}$, where C_j is non-singular, $P_1 = E, P_r = E$, it follows that $A = C_1$ is closely related to $C_r = B$.

By a *group with linking* or *V-group* is meant [3] a finite abelian group G together with a primitive, bilinear symmetric mapping V of G into the group of rationals mod 1. To a non-singular symmetric integral matrix A there is associated [3] a V-group $G(A^{-1})$ defined as follows: $G(A^{-1})$ is generated by elements $g_1, \cdots, g_n$; a set of defining relations is $\sum_{j=1}^{n} a_{ij}g_j = 0 (i = 1, \cdots, n)$; the linking matrix $(V(g_i, g_j))$ is congruent mod 1 to the rational matrix A^{-1}, i.e. $V(g_i, g_j) \equiv A_{ij}/\det A \pmod 1$.

LEMMA 3. *In order that non-singular matrices A and B be related it is necessary and sufficient that the associated V-groups $G(A^{-1})$ and $G(B^{-1})$ be V-isomorphic.*

PROOF. According to Kneser and Puppe [3], $G(A^{-1})$ and $G(B^{-1})$ are V-isomorphic if and only if A and B are closely related. But Lemma 2 says that A and B are closely related if and only if they are related.

Consider the group Π of a knot K of multiplicity μ, and the subgroup Π_2 defined as follows: an element of Π belongs to Π_2 if and only if it can be represented by a path the sum of whose linking numbers with the components $K_1, \cdots, K_\mu$ of K is an even number. Denote by $\mathfrak{M}_2$ the branched covering of $\mathfrak{M}$ associated with the unbranched covering of $\mathfrak{M} - K$ that belongs to the subgroup Π_2 of Π. It is known [7] that $\mathfrak{M}_2$ is a closed manifold and that it is invariantly associated with the isotopy type of K.

THEOREM. *The torsion group of $\mathfrak{M}_2$ is the V-group defined by the relation matrix B and the linking matrix B^{-1}* (mod 1), *where B is a non-singular matrix that is related to the coefficient matrix of the quadratic form of an arbitrary given projection of the knot.*

PROOF. The argument in Seifert [8] pp. 94–99 apply without change to the case $\mu > 1$ to show that if A is the coefficient matrix of the quadratic form of a projection of K then the homology group of $\mathfrak{M}_2$ is generated by elements $q_1, \cdots, q_n$ satisfying defining relations $\sum a_{ij}q_j \sim 0$, and that there exist 2-chains $Q_1, \cdots, Q_n$ and 1-cycles $q_1^*, \cdots, q_n^*$ such that $q_j^* \sim q_j$, $Q_i \xrightarrow{\partial} \sum a_{ij}q_j$ and $S(Q_i, q_j^*) = \delta_{ij}$. By Lemma 1 there exists a unimodular matrix T and a non-singular matrix B such that $T'AT = \begin{pmatrix} B & 0 \\ 0 & 0 \end{pmatrix}$. Define $y = T^{-1}q$ and $Y = T'Q$. Then $Y = T'Q \xrightarrow{\partial} T'Aq = T'ATy = \begin{pmatrix} B & 0 \\ 0 & 0 \end{pmatrix} y$, and $S(Y, y') = S(T'Q, (T^{-1}q)') = T'S(Q, q')\,(T^{-1})' = T'(T')^{-1} = E$. It follows that B is a relation matrix for the torsion group of $\mathfrak{M}_2$ and that a linking matrix is $B^{-1} \cdot S(Y, y') \equiv B^{-1} \pmod 1$.

Two knots belong to the same *isotopy type* if some isotopy of $\mathfrak{M}$ carries the one into the other; they belong to the same *s-type* [9] if the one can be carried into the other by a semi-linear isotopy. Knots belonging to the same s-type are necessarily in the same isotopy type; thus every isotopy invariant is automatically also an s-invariant. The group of a knot, and hence all the invariants deducible from the group, are obviously isotopy invariants; furthermore the linking invariants of the branched coverings have been shown to be isotopy invariants [8]. Thus lemma 3 and the theorem above show that *the family of the quadratic form of a knot is an isotopy invariant of the knot.* Originally it was known only that the quadratic form of a knot diagram is unaltered by the elementary deformation $\Omega_i^{\pm} (i = 1, 2, 3)$. Recently it has been shown [9] that the normed regular projections of the knots that belong to a given s-type may be obtained from one another by finite sequences of the elementary deformations (and of course, conversely, knots whose normed regular projections are so related must belong to the same s-type). Furthermore there exists a proof that knots belonging to the same s-type belong to the same isotopy type [10]. Thus it was known independently that the family of quadratic forms of a knot is an isotopy invariant. However these proofs are very complicated, and it is therefore worth while to establish directly, as has been done here, that particular given s-invariants, such as the quadratic form family, are indeed isotopy invariants.

2. Invariants of a family

Consider a group G with linking V and denote by $\tau_1, \cdots, \tau_m$ the torsion coefficients of G written in the order in which τ_{r+1} divides τ_r (thus $\tau_1 \geqq \tau_2 \geqq \cdots \geqq \tau_m > 1$). Let p be any prime and write $\tau_r = p^{d_r}\tau_r^*$, where $d_r \geqq 0$ and τ_r^* is prime to p. Thus $d_1 \geqq d_2 \geqq \cdots \geqq d_m > 0$; it is convenient to define $\tau_{m+1} = 1$ and $d_{m+1} = 0$. It is known [11] that, for any r elements $g_1, \cdots, g_r$ of G, the number $D(g_1, \cdots, g_r) = \tau_1 \cdots \tau_r \cdot \det (V(g_i, g_j))_{i,j=1,\cdots,r}$ is an integer, and that this integer is prime to p if $d_r > d_{r+1}$ and $g_1, \cdots, g_r$ are properly chosen. Invariants of the V-group G are defined at the prime p as follows [11]:

$$\chi_r(p) = (D(g_1, \cdots, g_r)/p) \qquad \text{if } p > 2 \text{ and } d_r > d_{r+1},$$
$$\chi_r(4) = (-1/D(g_1, \cdots, g_r)) \qquad \text{if } p = 2 \text{ and } d_r > d_{r+1} + 1,$$
$$\chi_r(8) = (2/D(g_1, \cdots, g_r)) \qquad \text{if } p = 2 \text{ and } d_r > d_{r+1} + 2.$$

Here (N/p), $(-1/N)$, $(2/N)$ are Jacobi symbols; in particular $(-1/N) = (-1)^{(N-1)/2}$ and $(2/N) = (-1)^{(N^2-1)/8}$

It is convenient to modify slightly this definition in the following way: Let

$$D^*(g_1, \cdots, g_r) = p^{d_1+\cdots+d_r} \det (V(g_i, g_j))_{i,j=1,\cdots,r},$$
$$= \frac{1}{\tau_1^* \cdots \tau_r^*} D(g_1, \cdots, g_r),$$

and define

$$\chi_r^*(p) = (D^*(g_1, \cdots, g_r)/p) \qquad \text{if } p > 2 \text{ and } d_r > d_{r+1},$$
$$\chi_r^*(4) = (-1/D^*(g_1, \cdots, g_r)) \qquad \text{if } p = 2 \text{ and } d_r > d_{r+1} + 1,$$
$$\chi_r^*(8) = (2/D^*(g_1, \cdots, g_r)) \qquad \text{if } p = 2 \text{ and } d_r > d_{r+1} + 2,$$

where $((M/N)/p)$, for integral M and N, means the same as (MN/p). Thus

$$\chi_r = (\tau_1^* \cdots \tau_r^*/p)\chi_r^* .$$

If G is the V-group $G(A^{-1})$ associated with a non-singular integral quadratic form $\sum a_{ij}x_ix_j$ the calculations of [11] §3 may be applied with $\mathbf{F} = A$ and $\mathbf{S} = E$. Denoted by ε the sign of the determinant of A, the result of the calculation is

$$D(g_{h_1}, \cdots, g_{h_r}) \equiv \frac{\varepsilon}{\tau_{r+1} \cdots \tau_m} \det (A \between E) h_1, \cdots, h_r \pmod{\tau_r}$$

$$\equiv \frac{\varepsilon}{\tau_{r+1} \cdots \tau_m} A_{h_1,\cdots h_r; h_1,\cdots,h_r} \pmod{\tau_r}.$$

Hence

$$\chi_r^*(p) = (\varepsilon\tau_1^* \cdots \tau_m^*/p)(p^{-d_{r+1}-\cdots-d_m}B_r/p) \qquad \text{if } p > 2 \text{ and } d_r > d_{r+1},$$

$$\chi_r^*(4) = (-1/\varepsilon\tau_1^* \cdots \tau_m^*)(-1/2^{-d_{r+1}-\cdots-d_m}B_r) \qquad \text{if } p = 2 \text{ and } d_r > d_{r+1} + 1,$$

$$\chi_r^*(8) = (2/\varepsilon\tau_1^* \cdots \tau_m^*)(2/2^{-d_{r+1}-\cdots-d_m}B_r) \qquad \text{if } p = 2 \text{ and } d_r > d_{r+1} + 2,$$

where B_r is a properly chosen $(n - r)^{\text{th}}$-order principal minor of A if $r < n$, and $B_n = 1$. Let $\theta_r(p) = (p^{-d_{r+1}-\cdots-d_m}B_r/p)$ for odd prime p, and let $\theta_r(4) = (-1/2^{-d_{r+1}-\cdots-d_m}B_r)$ and $\theta_r(8) = (2/2^{-d_{r+1}-\cdots-d_m}B_r)$. The result obtained may be stated as follows:

THEOREM. *The quantities*

$$\theta_0(p)\theta_r(p) \text{ for } p \text{ an odd prime and } d_r > d_{r+1},$$

$$\theta_0(4)\theta_r(4) \text{ for } p = 2 \text{ and } d_r > d_{r+1} + 1,$$

$$\theta_0(8)\theta_r(8) \text{ for } p = 2 \text{ and } d_r > d_{r+1} + 2,$$

are family invariants of the quadratic form $\sum a_{ij}x_ix_j$.

The residues $\theta_r(p)$, $r = 0$ or $d_r > d_{r+1}$, $\theta_r(4)$, $r = 0$ or $d_r > d_{r+1} + 1$, and $\theta_r(8)$, $r = 0$ or $d_r > d_{r+1} + 2$, may be shown to be invariants of the genus of the form $\sum a_{ij}x_ix_j$. (They are not invariants of the family; in fact $\theta_0(p)$, $\theta_0(4)$ and $\theta_0(8)$ may be altered by the operation Q_2.) They do not depend on a preliminary reduction to canonical form. To find the relation between these invariants and the invariants that are usually associated to a genus of quadratic forms through its canonical forms we proceed as follows:

Since the ring J of integers is imbedded naturally in the ring J_p of p-adic integers ($p \geqq 2$), our integral quadratic form $\sum a_{ij}x_ix_j$ may be considered also as a form with p-adic coefficients. It is known that there is a unimodular matrix T of p-adic integers such that

$$T'AT = p^{d_{r_1}}C_1 \dotplus p^{d_{r_2}}C_2 \dotplus \cdots \dotplus p^{d_r}C_l + C_{l+1},$$

where C_i is a unimodular matrix over J_p of order $r_i - r_{i-1}$ (where $r_0 = 0$ and $r_{l+1} = n - r_1 - \cdots - r_l$). The indices $r_1 < r_2 < \cdots < r_l$ are just those indices r for which $d_r > d_{r+1}$. It is easy to see that

$$\theta_0(p) = (\det C_1 \cdot \det C_2 \cdots \det C_{l+1}/p)$$

$$\theta_{r_i}(p) = (\det C_{i+1}/p) \cdot \theta_{r_{i+1}}(p) \qquad \text{for } i = 1, 2, \cdots, l$$

$$\theta_{r_l}(p) = (\det C_{l+1}/p)$$

with similar results for $\theta_r(4)$ and $\theta_r(8)$.

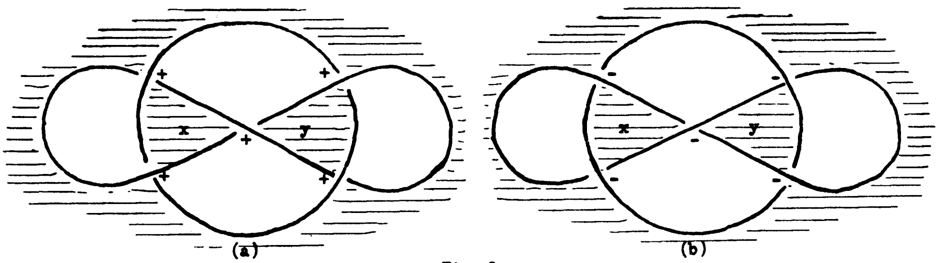

Fig. 1

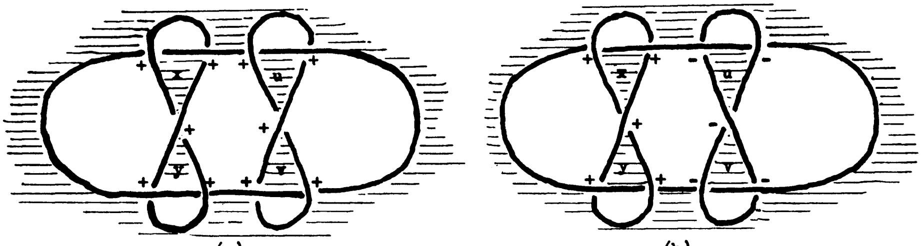

Fig. 2

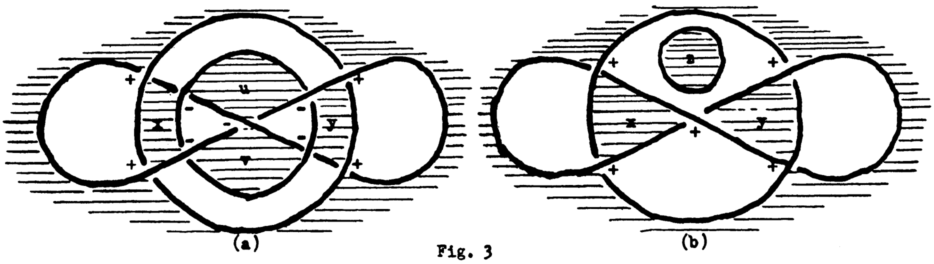

Fig. 3

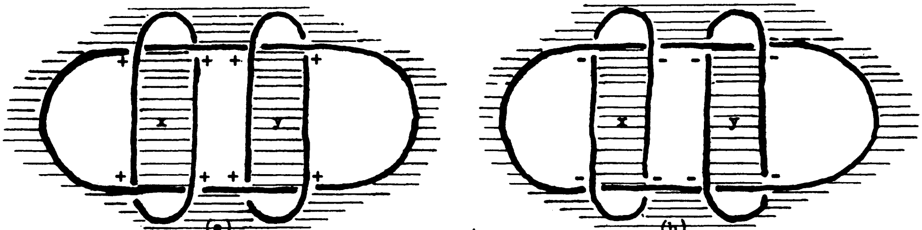

Fig. 4

3. Examples

The first two examples (Figs. 1 and 2) show why it is necessary to consider the prime 2. In fact the quadratic forms of (a) and (b) are indistinguishable at any other prime. Furthermore they can not be distinguished by any of the other known methods of knot theory. The third example (Fig. 3) shows why it is necessary to consider degenerate quadratic forms. The fourth example (Fig. 4) shows that the invariants discussed in §2 are insufficient so that the complete system of invariants defined by Burger [12, 13] may be used instead.

(1a)

$$f = 2x^2 + (x-y)^2 + 2y^2$$

$$A = \begin{pmatrix} 3 & -1 \\ -1 & 3 \end{pmatrix}$$

(1b)

$$f = -2x^2 - (x-y)^2 - 2y^2$$

$$A = \begin{pmatrix} -3 & 1 \\ 1 & -3 \end{pmatrix}$$

$$\tau_1 = 8, \qquad \tau_2 = 1$$

(1a): $\chi_1^*(4) = -1$, $\chi_1^*(8) = -1$

(1b): $\chi_1^*(4) = +1$, $\chi_1^*(8) = -1$

The link is therefore not amphicheiral.

(2a)

$$f = 2x^2 + (x-y)^2 + 2y^2 + 2u^2 + (u-v)^2 + 2v^2$$

$$A = \left(\begin{array}{cc|cc} 3 & -1 & & \\ -1 & 3 & & \\ \hline & & 3 & -1 \\ & & -1 & 3 \end{array}\right)$$

(2b)

$$f = 2x^2 + (x-y)^2 + 2y^2 - 2u^2 - (u-v)^2 - 2v^2$$

$$A = \left(\begin{array}{cc|cc} 3 & -1 & & \\ -1 & 3 & & \\ \hline & & -3 & 1 \\ & & 1 & -3 \end{array}\right)$$

$$\tau_1 = 8, \qquad \tau_2 = 8, \qquad \tau_3 = 1$$

(2a): $\chi_2^*(4) = +1$, $\chi_2^*(8) = +1$

(2b): $\chi_2^*(4) = -1$, $\chi_2^*(8) = +1.$

These two links are therefore inequivalent.

(3a)

$$f = 2x^2 + 2y^2 - (x-u)^2 - (x-v)^2 - (y-u)^2 - (y-v)^2 - (u-v)^2$$

$$A = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & -3 & 1 & 1 \\ 1 & 1 & -3 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix} \sim (-8)$$

(3b)

$$f = 2x^2 + (x-y)^2 + 2y^2$$

$$A = \begin{pmatrix} 3 & -1 & 0 \\ -1 & 3 & 0 \\ 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 3 & -1 \\ -1 & 3 \end{pmatrix}$$

$$\tau_1 = 8, \qquad \tau_2 = 1$$

(3a): $\chi_1^*(4) = -1$, $\chi_1^*(8) = +1$

(3b): $\chi_1^*(4) = -1$, $\chi_1^*(8) = -1.$

These two links are therefore inequivalent. In this example (a) and (b) can be distinguished by their elementary ideals [14]. For (a) we have $\mathfrak{E}_0 = \mathfrak{E}_1 = 0,$

$\mathfrak{E}_2 = (1 - u)\,(1 - v).\ \{1 - u, 1 - v\}$ and for (b) we have $\mathfrak{E}_0 = \mathfrak{E}_1 = 0, \mathfrak{E}_2 = (1 - v)\ \{1 - u, 1 - w\}\ \{1 - v, 1 - uw\}$.

(4a)	(4b)
$f = 4x^2 + 4y^2$	$f = -4x^2 - 4y^2$
$A = \begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix}$	$A = \begin{pmatrix} -4 & 0 \\ 0 & -4 \end{pmatrix}$

$$\tau_1 = 4, \qquad \tau_2 = 4, \qquad \tau_3 = 1$$

$\chi_2^*(4) = +1$	$\chi_2^*(4) = +1.$

Thus these two forms cannot be distinguished by the invariants discussed in §2. Nevertheless they do not belong to the same family. This can be seen by applying the method of [12, 13] as follows:

$A^{-1} \equiv \begin{pmatrix} \frac{1}{4} & 0 \\ 0 & \frac{1}{4} \end{pmatrix}$	$A^{-1} \equiv \begin{pmatrix} -\frac{1}{4} & 0 \\ 0 & -\frac{1}{4} \end{pmatrix}$

$G(A^{-1})$ is the direct sum of two cyclic groups of order 4. The distribution of $V(g, g)$, g ranging over $G(A^{-1})$ is

$\{0\} = 4$	$\{0\} = 4$
$\{\frac{1}{4}\} = 0$	$\{\frac{1}{4}\} = 8$
$\{\frac{1}{2}\} = 4$	$\{\frac{1}{2}\} = 4$
$\{\frac{3}{4}\} = 8$	$\{\frac{3}{4}\} = 0$

Thus the link is not amphicheiral. It is possible to distinguish (a) and (b) by their Alexander polynomials [14]. Orienting these links in such a way that each has linking numbers 2, 0, 2 we find for (a) the polynomial $\Delta(u, v, w) = (1 - v)\,(u + v)\,(v + w)$ and for (b) the polynomial $\Delta(u, v, w) \doteq (1 - v)\,(1 + uw)\,(1 + vw)$.

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BIBLIOGRAPHY

[1] R. H. KYLE. Branched covering spaces and the quadratic forms of knots. Princeton Ph.D. thesis, 1952.

[2] S. D. PUPPE. *Minkowskische Einheiten und Verschlingungsinvarianten von Knoten.* Math. Zeit. 56 (1952) p. 33.

[3] M. KNESER and D. PUPPE. *Quadratische Formen und Verschlingungsinvarianten von Knoten.* Math. Zeit. 58 (1953) p. 376.

[4] T. O'MEARA. *Characterization of quadratic forms over local fields.* Proc. Nat. Acad. Sci., U.S.A., 39 (1953) p. 969.

[5] K. REIDEMEISTER. Knotentheorie. Ergebnisse der Math. 11 (1932).

[6] H. TIETZE. Ein Kapitel Topologie. Hamburger Math. Einzelschriften 36 (1942).

[7] R. H. FOX. *Recent development of knot theory at Princeton.* Proc. Int. Congress, vol. II (1950) p. 453.

[8] H. Seifert. *Die Verschlingungsinvarianten der zyklischen Knotenuberlagerungen* Abh. Math. Sem. Hamburg Univ. 11 (1936) p. 84.
[9] W. Graeub. *Die semilinearen Abbildungen.* Sitz. Ber. Akad. Wiss. Heidelberg 4 (1950) p. 205.
[10] E. Moise. *Affine structures in 3-manifolds VIII.* Ann. of Math. 59 (1954) p. 159–170.
[11] R. C. Blanchfield and R. H. Fox. *Invariants of self-linking.* Ann. of Math. 53 (1951) p. 556.
[12] E. Burger. *Über Gruppen mit Verschlingungen.* J. Reine Angew. Math. 188 (1950) p. 193.
[13] R. H. Fox. Review of [12]. Math. Rev. 13 (1952) p. 204.
[14] R. H. Fox. *Free differential calculus II.* Ann. of Math., 59 (1954) p. 196.