ANNALS OF MATHEMATICS
Vol. 69, No. 3, May, 1959
Printed in Japan

BRANCHED COVERING SPACES AND THE QUADRATIC FORMS OF LINKS, II

BY R. H. KYLE

(Received May 7, 1958)

This paper consists of Chapters II, III and IV of the Princeton Ph. D. thesis [6] of the late Roger H. Kyle. (The principal results of the other chapters, I and V, were published in [7].) In making this revision I was fortunate in having the invaluable assistance of Dr. Hale Trotter.

R. H. FOX

1. Introduction

Self-linking in the homology groups of an oriented manifold was introduced by Seifert [12] and applied to the cyclic coverings of knots [13]. More or less complete systems of invariants of self-linking were defined by Seifert [12], Burger [2] and Blanchfield and Fox [1]. It was shown by Puppe [9], Kneser and Puppe [8], and Kyle [6] that the so-called quadratic form of a knot or link is determined by the self-linking in the homology groups of the second cyclic covering.

It is the object of this paper to dualize this theory. The dualization of self-linking is carried out in § 2; homology groups are replaced by cohomology groups and self-linking in the homology groups by a product operation $\smile_\delta$ in the cohomology groups. More precisely self-linking in the torsion subgroups of the homology groups is replaced by a dual operation in the torsion subgroups of the cohomology groups.

The dualized theory is, of course, more general, in the sense that it applies to arbitrary complexes and not just to oriented manifolds. It is also more algebraic in nature. The calculations can be made from the incidence matrices of a regular cell-complex, and do not require the geometric determination of intersections.

Calculation of $\smile_\delta$ makes use of a chain approximation to the diagonal map and is indicated in more detail for the case of special interest here, the case of a 3-dimensional manifold.

The topological theory of branched covering spaces has been put on a solid foundation in [5], and in § 3 results of [5] are put into a form suitable for the calculations to follow.

In § 4 a method for the calculation of $\smile_\delta$ in the cyclic branched coverings of a tame knot or link is given. In § 5 it is shown that Seifert's results on the second cyclic branched covering can be deduced from this.

2. Cohomological generalization of self-linking

The exact sequence of coefficient groups,

$$0 \longrightarrow I \xrightarrow{\ i\ } R \xrightarrow{\ \eta\ } R/I \longrightarrow 0\ ,$$

gives rise to an exact sequence of cohomology groups of a space X

$$(1) \quad \longrightarrow H^q(X;I) \xrightarrow{\ i^*\ } H^q(X;R) \xrightarrow{\ \eta^*\ } H^q(X;R/I) \xrightarrow{\ \delta^*\ } H^{q+1}(X;I) \longrightarrow$$

where i^* and η^* are the maps induced by i and η respectively and δ^* is the Bockstein coboundary operator [4, exercise 3, p. 158]. There is a natural pairing of I and R/I to R/I, which induces a cup-product

$$\smile\ : H^p(X;R/I) \otimes H^q(X;I) \quad \longrightarrow H^{p+q}(X;R/I)\ .$$

We define a new product

$$\smile_\delta : H^p(X;R/I) \otimes H^q(X;R/I) \longrightarrow H^{p+q+1}(X;R/I)$$

by setting $u \smile_\delta v = u \smile \delta^* v$.

Any element of $H^p(X;R/I)$ can be represented by an element of $Z^p(X_R, X_I)$, the group of rational cochains of X whose coboundaries are integral cochains. If $\bar{u} \in Z^p(X_R, X_I)$, $\bar{v} \in Z^q(X_R, X_I)$ are representatives for $u \in H^p(X;R/I)$ and $v \in H^q(X;R/I)$ then $\bar{u} \smile \delta\bar{v}$ is a representative for $u \smile_\delta v$, where $\bar{u} \smile \delta\bar{v}$ is the ordinary cup-product of rational cochains. We have

$$\delta(\bar{u} \smile \bar{v}) = \delta\bar{u} \smile \bar{v} + (-1)^p \bar{u} \smile \delta\bar{v}$$

so that

$$\bar{u} \smile \delta\bar{v} \sim (-1)^{p+1}\delta\bar{u} \smile \bar{v} \sim (-1)^{p+1}(-1)^{(p+1)q}\bar{v} \smile \delta\bar{u} = (-1)^{(p+1)(q+1)}\bar{v} \smile \delta\bar{u}.$$

Returning to cohomology classes we obtain the commutation rule. *If* $u \in H^p(X;R/I)$, $v \in H^q(X;R/I)$ *then* $u \smile_\delta v = (-1)^{(p+1)(q+1)} v \smile_\delta u$. Similarly, supposing $\bar{w}$ to be a representative of $w \in H^r(X;R/I)$, we have

$$\bar{u} \smile \delta(\bar{v} \smile \delta\bar{w}) = \bar{u} \smile (\delta\bar{v} \smile \delta\bar{w}) \sim (\bar{u} \smile \delta\bar{v}) \smile \delta\bar{w}$$

since the ordinary cup-product is associative modulo coboundaries. Hence, *the operation* $\smile_\delta$ *is associative.*

Since $u \smile_\delta v = 0$ if either u or v belongs to the kernel of δ^*, there is induced a pairing of $\delta^*(H^p(X;R/I))$ and $\delta^*(H^q(X;R/I))$ to $H^{p+q+1}(X;R/I)$. Since the sequence (1) is exact, the image of δ^* is the kernel of i^*, which is precisely the torsion subgroup $T^{p+1}(X;I) \subset H^{p+1}(X;I)$. Thus we obtain a pairing

$$T^{p+1}(X;I) \otimes T^{q+1}(X;I) \longrightarrow H^{p+q+1}(X;R/I)\ ,$$

which may be calculated as follows. If $\bar{u}$, $\bar{v}$ are integral cocycles representing $u \in T^{p+1}(X; I)$, $v \in T^{q+1}(X; I)$ respectively, and $\bar{u}' \in C^p(X; I)$ is such that $\delta\bar{u}' = m\bar{u}$, then $m^{-1}(\bar{u}' \smile \bar{v})$ represents $u \cdot v$.

The product $\smile_\delta$ forms a generalization of the linking of cycles in manifolds. Let X be an oriented n-dimensional manifold, with fundamental cycle M. Then if $p+q+1 = n$, $\smile_\delta$ defines a pairing of $H^p(X; R/I)$ and $H^q(X; R/I)$ to R/I, given by $L(u, v) \equiv (u \smile_\delta v)(M) (\text{mod } 1)$. The pairing induced between $T^{p+1}(X; I)$ and $T^{q+1}(X; I)$ is given by $L(u, v) = m^{-1}(\bar{u}' \smile \bar{v})(M)$ where $\bar{u}$, $\bar{v}$, $\bar{u}'$, m are as in the preceding paragraph. Passing to a dual subdivision gives a pairing of the groups of torsion cycles $T_{n-p-1}(X; I) = T_q(X; I)$ and $T_{n-q-1}(X; I) = T_p(X; I)$ to R/I. In this pairing $L(u^*, v^*) \equiv m^{-1}S(u', v^*)$, where u^*, v^* are torsion cycles, u' is a chain with $\partial u' = mu^*$ and $S(u', v^*)$ is the Kronecker index of the intersection of the chains u' and v^*. This, however, is precisely the pairing defined by the self-linking in a manifold [14, p. 278].

The pairing can be computed in terms of the incidence matrices of the manifold and a chain approximation to the diagonal map. In the application to covering spaces of links treated in Section 3, the space X is an orientable 3-dimensional manifold and $\smile_\delta$ defines a symmetric pairing of $H^1(X; R/I)$ with itself to R/I. We shall describe the computation of the induced pairing of $T^2(X; I)$ with itself to R/I. First obtain bases a_i, b_i, c_i and f_i, g_i, e_i for the chain-groups $C_1(X)$ and $C_2(X)$ so that the matrix of the boundary operator $\partial : C_2(X) \to C_1(X)$ takes the form

$$\begin{array}{c|ccc|} & f_i & g_i & e_i \\ \hline a_i & E & 0 & 0 \\ b_i & 0 & A & 0 \\ c_i & 0 & 0 & 0 \end{array}$$

where E is an identity matrix and A is a non-singular $n \times n$ matrix. The transpose of this matrix describes the coboundary operator $\delta : C^1(X) \to C^2(X)$ with respect to the dual basis for the cochain groups. Let $x_1, \cdots, x_n$ be the basis elements of $C^2(X)$ dual to the basis elements $g_1, \cdots, g_n$ of $C_2(X)$ and let $y_1, \cdots, y_n$ in $C^1(X)$ be dual to $b_1, \cdots, b_n$. Then $T^2(X; I)$ has a presentation as an abelian group with generators $x_1, \cdots, x_n$ and relations $\delta y_i = \sum_j a_{ji}x_j = 0$ where $\| a_{ij} \|$ is the matrix A. With rational coefficients, $\delta^{-1}x_i = \sum_j (A^{-1})_{ji}y_j$.

Now let M be the fundamental 3-cycle of X. Its image under a chain approximation to the diagonal is some element of

$$C_0(X) \otimes C_3(X) + C_1(X) \otimes C_2(X) + C_2(X) \otimes C_1(X) + C_3(X) \otimes C_0(X) .$$

Let the $n \times n$ matrix D be defined by taking d_{ij} equal to the coefficient of

$b_i \otimes g_j$ in the expression for the image of M in terms of the chosen basis. Then the value of $y_i \smile x_j$ on M is d_{ij}, and we have

$$L(x_i, x_j) \equiv (\delta^{-1}x_i \smile x_j)(M) \qquad (\text{mod } 1)$$
$$\equiv \textstyle\sum_k (A^{-1})_{ik}\, d_{kj} \qquad (\text{mod } 1)\ .$$

This result may be summarized as:

THEOREM. *The group $T^2(X; I)$ with self-pairing to R/I is given by*

$$\{x_i : \textstyle\sum_j a_{ji}x_j = 0\}\ ,\quad \| L(x_i, x_j) \| \equiv A^{-1}D \quad (\text{mod } 1)\ .$$

3. Branched coverings of a regular cell-complex

Let Z be a connected, locally finite, regular cell-complex [10] and let L be any subcomplex of Z which is such that, for every (open) cell τ of Z, the intersection $S(\tau)$ of $Z - L$ with the star of τ is non-vacuous and connected. This condition is satisfied, for example, if Z is an n-dimensional combinatorial manifold and L is a subcomplex of dimensoin $\leqq n - 2$, hence, in particular, if Z is the 3-sphere and L is a knot or link. Select a point q of $Z - L$ for base point.

To each subgroup H of $G = \pi(Z - L)$ there belongs an unbranched covering X of $Z - L$, and to this is uniquely associated by completion a covering Y of Z. Every covering space of Z that is non-singular over $Z - L$ is obtained in this way and hence belongs to a subgroup of G. The argument on pp. 251-2 of [5] shows[1] that Y is a locally finite regular cell-complex and $e : Y \to Z$ is a cell covering if the index of branching $j(y)$ is finite for each point of y, i.e., if the following condition is satisfied:

(*) *for each path β in $Z - L$ from q to $S(\tau)$, the number of right cosets of H represented by loops of the form $\beta\gamma\beta^{-1}$ where γ is any loop in $S(\tau)$ based at $\beta(1)$, is finite.*

In order to put the incidence relations of [5, p. 252] into a more useful form, we select, corresponding to each cell τ of Z, a path β_τ in $Z-L$ from $\beta_\tau(0) = q$ to a point $\beta_\tau(1)$ of $S(\tau)$. Furthermore if $\tau < \tau^*$, so that $S(\tau) \supset S(\tau^*)$, we select a path $\beta_{\tau^*\tau}$ in $S(\tau)$ from $\beta_{\tau^*}(1)$ to $\beta_\tau(1)$, and denote by $g_{\tau^*\tau}$ the element of G represented by the loop $\beta_{\tau^*}\beta_{\tau^*\tau}\beta_\tau^{-1}$.

For any cell τ of Z, denote by F_τ the subgroup of G that consists of the elements represented by loops of the form $\beta_\tau\gamma\beta_\tau^{-1}$ where γ is a loop in $S(\tau)$ based at $\beta_\tau(1)$. Note that if $\tau \notin L$ F_τ is trivial, and the choice of $\beta_{\tau^*\tau}$ is immaterial, for in that case every β is contractible in $S(\tau)$.

[1] The hypothesis of the theorem p. 251 that Z be barycentrically subdivided is only used to deduce that Y is a *simplicial* complex. (Cf. the parenthetical remark in the fifth paragraph of p. 252.) If Z is not barycentrically subdivided Y will still be a regular cell-complex.

If σ_i is any cell of Y that lies over τ, join the base point $p \in e^{-1}(q)$ by a path α_{σ_i} in X to the point of σ_i that lies over $\beta_\tau(1)$. The loop $e(\alpha_{\sigma_i}) \cdot \beta_\tau^{-1}$ determines an element g_{σ_i} of G. A different choice of the path α_{σ_i} determines a possibly different element g_{σ_i} but always one that belongs to the same double coset of (H, F_τ). Furthermore if σ_i and σ_j are different cells of Y lying over τ, the elements g_{σ_i} and g_{σ_j} belong to different double cosets of (H, F_τ). Since every double coset of (H, F_τ) is determined by some cell of $e^{-1}(\tau)$ we may index the cells of $e^{-1}(\tau)$ by these double cosets. (If $\tau \notin L$ this reduces to the well-known indexing by the right cosets of H.) If now paths α_{σ_i} are selected for each σ_i, the elements g_{σ_i} constitute a set of representatives of the double cosets of (H, F_τ).

The incidence relations given on p. 252 of [5] amount to this: If $\tau < \tau^*$, then $\sigma_i < \sigma_j^*$ iff $S(\sigma_i) \supset S(\sigma_j^*)$.

But $S(\sigma_i) \supset S(\sigma_j^*)$ iff g_{σ_i} and $g_{\sigma_j^*} g_{\tau^*\tau}$ belong to the same double coset of (H, F_τ). Thus we get the following incidence criterion:

($\cdot$) *If $\tau < \tau^*$, then $\sigma_i < \sigma_j^*$ iff $Hg_{\sigma_i}F_\tau = Hg_{\sigma_j^*}g_{\tau^*\tau}F_\tau$.*

Finally we remark that the condition (*) that we imposed on H can be reformulated as follows:

(*) *Each double coset HgF_σ contains only a finite number of the right cosets of H.*

4. Branched coverings of a link

We suppose the given link L to lie between two horizontal planes in such a way that when it is projected orthogonally on the planes, the resulting diagram is regular [11]. (For purposes of geometrical description, we consider S^3 as euclidean 3-spaces plus a point at infinity.) We also assume that this projection of the link is connected, and that at every double-point the four regions coming together are distinct. (The first of these conditions is necessary in order for our construction to yield a *cell-decomposition*; the second is necessary to make the resulting cell-complex regular.) It is always possible to modify the link so that these conditions are satisfied, without changing its type.

Let C be the (self-intersecting) cylindrical surface consisting of all the vertical line-segments contained between the two planes and passing through L. The top of C is clearly the vertical projection of L on the upper plane, while the bottom of C is the projection of L on the lower plane. Both of these projections have the same diagram. We label the vertices of the diagram $\bar{p}_k$, the edges $\bar{e}_j$, and the regions $\bar{X}_i$. Corresponding to $\bar{p}_k$ there is a line of self-intersection of C which cuts L in two points. We label the upper point p_k and the lower one q_k. Let v be a point above the upper plane, and w a point below the lower plane. These two points, together with the points p_k and q_k will be the vertices of our cell-decomposition. With the k^{th} double point we also associate three 1-cells, a_k, b_k, and c_k. The segment a_k consists of the vertical line joining p_k to its projection on the upper plane plus the line joining this projection to v. Similarly, b_k consists of the vertical line segment joining q_k to its projection on the lower plane plus the join of the latter to w, while c_k is the vertical segment joining p_k and q_k. The points p_k and q_k divide L into segments e_j, which correspond to the edges $\bar{e}_j$ of the diagram, and these are also taken as 1-cells of the decomposition. Corresponding to each edge $\bar{e}_j$ of the diagram there are two 2-cells of the decomposition. The 2-cell A_j consists of the part of C lying over e_j plus the join of the projection of e_j on the upper plane with the vertex v. Similarly, B_j consists of the part of C lying below e_j plus the join of the lower projection of e_j with w. The cells so far described form a two-dimensional complex which separates S^3 into a number of 3-cells X_i, corresponding to the regions $\bar{X}_i$ of the diagram, and these complete the cell-decomposition. (Figure 1 shows a part of C near a self-intersection.)

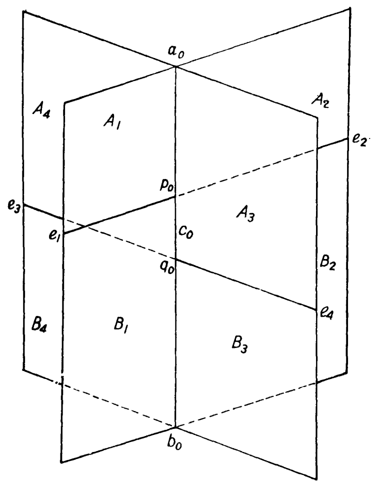

Fig. 1

We color the infinite region of the diagram black, and color the remaining regions alternately black and white in the usual way [11]. Each 3-cell of the decomposition is to be given the same color as its corresponding region. We call the infinite region $\bar{X}_0$. The corresponding 3-cell is the one containing the point at infinity in S^3.

The cells are oriented as follows. All 3-cells have the orientation of a right-handed screw. The 2-cells are oriented coherently with the *black* 3-cells (and hence anti-coherently with the white 3-cells). The 1-cells e_j are oriented so that e_j is coherent with A_j and anti-coherent with B_j. The other 1-cells are oriented from top to bottom, so that $\partial(a_k + b_k + c_k) = w - v$. In the diagram we orient the regions with an anti-clockwise indicatrix (as seen from above) and orient the edges $\bar{e}_j$ to agree with the orientations of the e_j under projection. Thus the edges $\bar{e}_j$ are oriented coherently with the black regions of the diagram.

We introduce the following notations and conventions. For each vertex of the diagram we select one of the two black regions adjacent to it and say that the vertex in question *belongs* to the selected region. (This selection is arbitrary, although we will find it convenient in Section 4 to suppose that all the vertices adjacent to the infinite region belong to it.) Then $\overline{X}_{1,k}$ denotes the black region to which $\overline{p}_k$ belongs, while $\overline{X}_{2,k}$, $\overline{X}_{3,k}$ and $\overline{X}_{4,k}$ denote the other regions adjacent to $\overline{p}_k$ as shown in Figure 2.

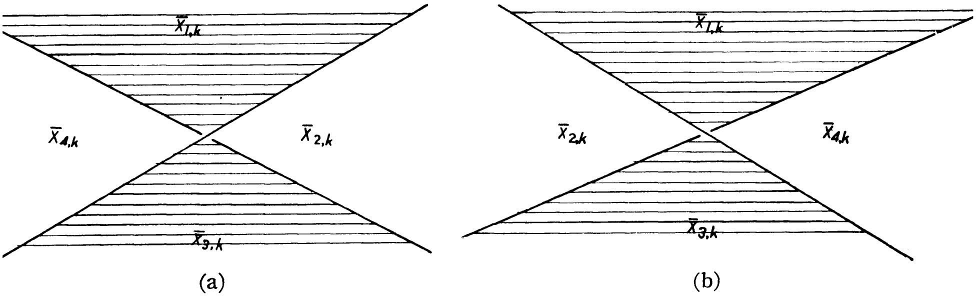

Fig. 2

We define $\varepsilon_k = +1$ or -1 according as $\overline{p}_k$ is of type (a) or (b) (see Figure 2). For convenience we also define $\delta_k = \frac{1}{2}(1 + \varepsilon_k)$.

Let $\overline{X}'_j$ be the black region having e_j on its boundary, and let $\overline{X}''_j$ be the white region having $\overline{e}_j$ on its boundary. Let $j+$ and $j-$ be the labels of the double-points at the ends of $\overline{e}_j$ so that $\partial\overline{e}_j = \overline{p}_{j+} - \overline{p}_{j-}$. We extend this notation in the obvious way so that, for example, a_{j+}, b_{j+}, c_{j+} are the one-cells associated with $\overline{p}_{j+}$, and $\overline{X}_{1,j+}$ is the black region to which $\overline{p}_{j+}$ belongs. (In the situation shown in Figure 1, $a_0 = a_{1+} = a_{3+} = a_{2-} = a_{4-}$, $p_0 = p_{1+} = p_{3+} = p_{2-} = p_{4-}$, etc.) It should also be understood that X'_j, $X_{1,k}$, etc., denote the 3-cells corresponding to the regions $\overline{X}'_j$, $\overline{X}_{1,k}$, etc. Finally, let $J(i)$ be the set of values of j for which $\overline{e}_j$ is on the boundary of $\overline{X}_i$.

With these notations, the boundary relations in the cell-complex we have constructed are :

$$\begin{aligned}
\partial X_i &= \textstyle\sum_{j\in J(i)} (A_j + B_j) \qquad \text{if } X_i \text{ is black.}\\
&= -\textstyle\sum_{j\in J(i)} (A_j + B_j) \qquad \text{if } X_i \text{ is white.}\\
\partial A_j &= e_j - a_{j+} + a_{j-} - \delta_{j+}c_{j+} + (1 - \delta_{j-})c_{j-}\\
\partial B_j &= -e_j - b_{j+} + b_{j-} - (1 - \delta_{j+})c_{j+} + \delta_{j-}c_{j-}\\
\partial a_k &= p_k - v, \quad \partial b_k = w - q_k, \quad \partial c_k = q_k - p_k\\
\partial e_j &= (1 - \delta_{j+})p_{j+} + \delta_{j+}q_{j+} - \delta_{j-}p_{j-} - (1 - \delta_{j-})q_{j-} .
\end{aligned}$$

We now apply the procedures of Section 2 to obtain a cell-decomposition of the branched covering space Y corresponding to a subgroup H of the fundamental group of $S^3 - L$.

We shall use the Dehn presentation [3, p. 154] to describe $\pi_1(S^3 - L)$. We take the vertex v as base-point. Corresponding to each region of the projection there is a generator x_i which is represented by a path running from v to w through the interior of X_i and returning to v through the interior of X_0. It is clear that x_0 is trivial, but we retain it among the generators for the sake of symmetry, and adjoin the relation $x_0 = 1$. There is also a relation for each vertex of the diagram. The relation at the k^{th} vertex is $x_{1,k}x_{2,k}^{-1}x_{3,k}x_{4,k}^{-1} = 1$.

For each cell τ we now select a path from v to a point in $S(\tau)$. For each X_i let the selected path run directly from v to an interior point of X_i. For A_j, B_j, e_j, we take the path already selected in X'; for a_k, b_k, c_k, p_k, q_k, we take the path selected in $X_{1,k}$, and for w we take the path selected in X_0,

The subgroups F_τ are now determined. As remarked in Section 2, F_τ is trivial for the cells which are not in L. For every $\tau \in L$, F_τ is infinite cyclic; F_{e_j} is generated by $x\, x_j''^{-1}$, F_{p_k} by $x_{1,k}x_{2,k}^{-1} = x_{4,k}x_{3,k}^{-1}$, and F_{q_k} by $x_{4,k}^{-1}x_{1,k} = x_{2,k}^{-1}x_{3,k}$.

In writing the boundary relations in the covering space we shall use the following notation. For any group element g we write σg for the cell over σ corresponding to the coset H_gF_σ, and in case $g = 1$, simply write σ. Then if σ is not in the singular set it suffices to give the formulas for $\partial\sigma$, since $\partial(\sigma g)$ is then given by $(\partial\sigma)g$. If σ is in the singular set, the fact that F_σ is in general not a normal subgroup causes this formalism to break down. The boundary operator can of course be calculated in any particular case from the rules laid down in Section 2, but we shall not give a general explicit formula. For the two-sheeted covering discussed in detail in Section 5 there is only one cell over each singular cell and there is no difficulty. The other boundary relations are:

$$\begin{aligned}
\partial X_i &= \textstyle\sum_{j\in J(i)} (A_j + B_j) && \text{if } X_i \text{ is black}\\
&= -\textstyle\sum_{j\in J(i)} (A_j + B_jx_j''x_j'^{-1}) && \text{if } X_i \text{ is white}\\
\partial A_j &= e_j - a_{j+} + a_{j-} - \delta_{j+}c_{j+}g_j' + (1-\delta_{j-})c_{j-}g_j'' \\
\partial B_j &= -e_j - b_{j+}x_j'x_{1,j+}^{-1} + b_{j-}x_j'x_{1,j-}^{-1} - (1-\delta_{j+})c_{j+}g_j' + \delta_{j-}c_{j-}g_j''
\end{aligned}$$

where

$$\begin{aligned}
g_j' &= 1 && \text{if } X_j' = X_{1,j+}\\
&= x_j'x_{4,j+}^{-1} && \text{if } X_j' = X_{3,j+}
\end{aligned}$$

and

$$\begin{aligned}
g_j'' &= 1 && \text{if } X_j' = X_{1,j-}\\
&= x_j'x_{4,j-}^{-1} && \text{if } X_j' = X_{3,j-}
\end{aligned}$$

$$\partial a_k = p_k - v\ , \qquad \partial c_k = q_k - p_k\ , \qquad \partial b_k = wx_{1,k} - q_k\ .$$

We now define a map D of the chain complex of M into its tensor product with itself such that for each cell σ, $D(\sigma)$ is in the subcomplex generated by $\sigma \otimes \sigma$. Since M is a regular cell-complex, D is a chain approximation to the diagonal map [10] of M into $M \times M$. We shall explicitly describe such a map for the complex Y, rather than for the covering complex M. For $\sigma \in Y$, $D(\sigma)$ is a sum of products of cells in $\bar{\sigma}$. Then for $\sigma' \in M$ lying over $\sigma \in Y$, $D(\sigma')$ is defined as the corresponding sum of products of the cells of $\bar{\sigma}'$ lying over the cells of $\bar{\sigma}$.

For any vertex u, we define $D(u) = u \otimes u$, and for any 1-cell g, with $\partial g = u - u'$, we define $D(g) = u \otimes g + g \otimes u'$. For each 2-cell F, pick a vertex u and let $g_1, g_2, \cdots, g_n$ be the edges of F, starting from u and going around the boundary of F in the direction coherent with the orientation of F. Let η_i be the incidence number of g_i with F and define $\gamma_i = \sum_{j=1}^{i} \eta_j g_j - \frac{1}{2}(1 - \eta_j)g_j$. Then we define

$$D(F) = u \otimes F + F \otimes u + \textstyle\sum_{i=1}^{n} \eta_i \gamma_i \otimes g_i\ .$$

This definition of course depends on the choice of u. We take $u = v$ for every A_j and $u = w$ for every B_j.

The definition of $D(X_i)$ depends upon whether X_i is black or white. In either case, choose one vertex of $\bar{X}_i$ to be $\bar{p}_{i0}$ and number the vertices and edges around in anti-clockwise order $\bar{p}_{i0}, \bar{e}_{i1}, \bar{p}_{i1}, \cdots, \bar{e}_{in}, \bar{p}_{in} = \bar{p}_{i0}$. Define

$$\begin{aligned} C_m &= \varepsilon_m \textstyle\sum_{r=1}^{m} (A_r + B_r) + \delta_m(B_{m+1} - A_m) && \text{if } X_i \text{ is black,} \quad m = 1, 2, \cdots, n \\ &= \varepsilon_m \textstyle\sum_{r=1}^{m} (A_r + B_r) + \delta_m(A_{m+1} - B_m) && \text{if } X_i \text{ is white,} \quad m = 1, 2, \cdots, n \\ E_m &= -\textstyle\sum_{r=1}^{m} (A_r + B_r) + A_m && \text{if } X_i \text{ is black,} \\ &= \textstyle\sum_{r=1}^{m} (A_r + B_r) - B_m && \text{if } X_i \text{ is white.} \end{aligned}$$

Then if X_i is black, we take

$$\begin{aligned} D(X_i) = v \otimes X_i + X_i \otimes w + (a_0 + c_0 + b_0) \otimes \textstyle\sum_{m=1}^{n} B_m \\ + \textstyle\sum_{m=1}^{n} \{-A_m \otimes a_m + B_{m+1} \otimes b_m + C_m \otimes c_m + E_m \otimes e_m\} \\ - \textstyle\sum_{m=1}^{n} (A_m + B_m) \otimes (b_0 + \delta_0 c_0) \end{aligned}$$

and if X_i is white

$$\begin{aligned} D(X_i) = v \otimes X_i + X_i \otimes w - (a_0 + c_0 + b_0) \otimes \textstyle\sum_{m=1}^{n} B_m \\ + \textstyle\sum_{m=1}^{n} \{A_{m+1} \otimes a_m - B_m \otimes b_m + C_m \otimes c_m - E_m \otimes e_m\} \\ + \textstyle\sum_{m=1}^{n} (A_m + B_m) \otimes (b_0 + (1 - \delta_0)c_0)\ . \end{aligned}$$

In the foregoing formulas, A_m, a_m, etc. are to be understood to stand for A_m, a_{im}, etc. Strictly speaking, we should also write n_i rather than n, since the number of terms in the sum depends upon i. It is a matter of straightforward, though tedious, computation to check that these definitions actually yield a chain map (that is, one which commutes with the boundary operator).

We are interested in the terms of the form (1-cell) $\otimes$ (2-cell) occurring in $D(M)$ where M is the fundamental cycle of Y (which we now assume to be compact). Denoting these by $D'(M)$ we have

$$D'(M) = \sum_{i,g} \pm (a_{i0}g + b_{i0}gr_i + c_{i0}gs_i) \otimes \sum_{j \in J(i)} B_j g t_j$$

where i runs through the regions of the projection of the link and g runs through a set of representatives for the right cosets of H, the group of the covering space. The sign is to be taken as plus or minus according as $\overline{X}_i$ is black or white. The group elements r_i, s_i and t_j are determined as follows. If X_i is black $t_j = 1$, while $r_i = s_i = 1$ if $\overline{p}_{i0}$ belongs to X_i. Otherwise $r_i = x_i\bar{x}_{1,i0}$ and $s_i = x_i\bar{x}_{4,i0}$. If X_i is white $t_j = x_i\bar{x}'_j$ and $r_i = x_i\bar{x}_{1,i0}$, while $s_i = x_i\bar{x}_{1,i0}$ if $X_i = X_{2,i0}$ and $s_i = 1$ if $X_i = X_{4,i0}$.

5. The two-sheeted branched covering

We consider the branched covering of $S^3 - L$ whose corresponding subgroup H consists of those elements of $\pi_1(S^3 - L)$ represented by loops whose total linking number with L is even (cf. [7]). Then $x_i \in H$ if and only if X_i is black. Of course, H has only two right cosets. Furthermore, for every $\tau \in L$, F_τ is generated by an element of g which links L once, so that there is only one double coset HgF_τ. Hence, each cell of $S^3 - L$ has two cells over it in the covering and to each cell of L there corresponds a single cell in the covering. We shall denote the cells v, v^*, w, w^*, a_k, a_k^*, etc. The boundary relations are

$$\begin{aligned}
\partial X_i &= \textstyle\sum_{j \in J(i)} (A_j + B_j),\ \partial X_i^* = \sum_{j \in J(i)} (A_j^* + B_j^*) \ \text{ if } \overline{X}_i \text{ is black}\\
\partial X_i &= -\textstyle\sum_{j \in J(i)} (A_j + B_j^*),\ \partial X_i^* = -\sum_{j \in J(i)} (A_j^* + B_j) \ \text{ if } \overline{X}_i \text{ is white}\\
\partial A_j &= e_j - a_{j+} + a_{j-} - \delta_{j+}(\eta_{j+}c_{j+} + (1 - \eta_{j+})c_{j+}^*)\\
&\qquad + (1 - \delta_{j-})(\eta_{j-}c_{j-} + (1 - \eta_{j-})c_{j-}^*)\\
\partial A_j^* &= e_j - a_{j+}^* + a_{j-}^* - \delta_{j+}(\eta_{j+}c_{j+}^* + (1 - \eta_{j+})c_{j+})\\
&\qquad + (1 - \delta_{j-})(\eta_{j-}c_{j-}^* + (1 - \eta_{j-})c_{j-})\\
\partial B_j &= -e_j - b_{j+} + b_{j-} - (1 - \delta_{j+})(\eta_{j+}c_{j+} + (1 - \eta_{j+})c_{j+}^*)\\
&\qquad + \delta_{j-}(\eta_{j-}c_{j-} + (1 - \eta_{j-})c_{j-}^*)\\
\partial B_j^* &= -e_j - b_{j+}^* + b_{j-}^* - (1 - \delta_{j+})(\eta_{j+}c_{j+}^* + (1 - \eta_{j+})c_{j+})\\
&\qquad + \delta_{j-}(\eta_{j-}c_{j-}^* + (1 - \eta_{j-})c_{j-})
\end{aligned}$$

where η_{j+} is 1 if $\bar{p}_{j+}$ belongs to $\bar{X}'_j$, and 0 if it does not, and η_{j-} is 1 if $\bar{p}_{j-}$ belongs to $\bar{X}'_j$ and 0 if it does not.

$$\partial e_j = (1 - \delta_{j+})p_{j+} + \delta_{j+}q_{j+} - \delta_{j-}p_{j-} - (1 - \delta_{j-})q_{j-}$$

$$\partial a_k = p_k - v\ , \qquad \partial b_k = w - q_k\ , \qquad \partial c_k = q_k - p_k$$

$$\partial a_k^* = p_k - v^*\ , \qquad \partial b_k^* = w^* - q_k\ , \qquad \partial c_k^* = q_k - p_k\ .$$

We now proceed to find bases for the one and two dimensional chains which reduce the matrix of the boundary operator to a simpler form. We choose a vertex $\bar{p}_0$, adjacent to $\bar{X}_0$, and then take a maximal tree consisting of the 1-cells a_j, b_j, c_0, a_0^* and b_0^*. In conjunction with the maximal tree, every 1-cell determines a unique cycle, which we denote by the same letter as the cell. If the cell is in the tree it determines the zero cycle ; the non-zero cycles together with the cells of the tree form a basis for the 1-chains. Let C be the subgroup of chains generated by all the c_k, c_k^*. We write $f \equiv g (\text{mod } C)$ if f and g are 1-chains such that $f - g \in C$.

The chains A_j, $A_j + B_j + A_j^* + B_j^*$, $A_j^* + B_j$ and $A_j + B_j$ form a basis for the chains. For each j, $\partial A_j \equiv e_j (\text{mod } C)$ and hence the replacement of e_j by ∂A_j is an allowable change of basis.

Let T be a maximal tree in the projection of L. Then for each vertex $\bar{p}_k \neq \bar{p}_0$ there is a unique 1-chain $\bar{D}_k$ lying in T such that $\partial \bar{D}_k = \bar{p}_k - \bar{p}_0$, and for every region $\bar{X}_i$ there is a 1-chain $\bar{Z}_i = \partial \bar{X}_i$. It is easy to see that the chains $\bar{D}_k$, $\bar{Z}_i$ $(i \neq 0)$ form a basis for the 1-chains of the projection. Define

$$D'_k = \sum \eta_j(A_j + B_j + A_j^* + B_j^*), \quad D''_k = \sum \eta_j(A_j^* + B_j)$$

where the sum is taken over those j for which $\bar{e}_j \in \bar{D}_k$ and $\eta_j = \pm 1$ is the coefficient with which $\bar{e}_j$ appears in $\bar{D}_k$. Similarly let

$$Z'_i = \sum \eta_j(A_j + B_j + A_j^* + B_j^*)$$

and $Z''_i = \sum \eta_j(A_j^* + B_j)$ where the sum is taken over those j for which $\bar{e}_j \in \bar{Z}_i$ and η_j is the coefficient of $\bar{e}_j$ in $\bar{Z}_i$. Then the replacement of the $(A_j + B_j + A_j^* + B_j^*)$ and $(A_j^* + B_j)$ by the D'_k, D''_k, Z'_k and Z''_k is an allowable change of basis for the 2-chains. Now $\partial D'_k \equiv a_k^* + b_k^* (\text{mod } C)$ and $\partial D''_k \equiv a_k^* (\text{mod } C)$, so we may change the basis for the 1-chains by replacing the a_k^*, b_k^* by $\partial D'_k$ and $\partial D''_k$. For any i, $\partial Z'_i = \partial\partial(X_i + X_i^*) = 0$, while if X_i is white $\partial Z''_i = \partial\partial X_i^* = 0$. On the other hand, if $\bar{X}_i$ is black, $\partial Z''_i$ will not be zero in general (although $\partial Z''_i \equiv 0 (\text{mod } C)$). We shall consider it in more detail below.

The 2-chains $(A_j + B_j)$ are still to be considered. We group them according to the black region with which $\bar{e}_j$ is incident. For each region X_i, a vertex $\bar{p}_{i0}$ was picked out in the course of defining the chain approximation to the diagonal map. As before, we suppose the edges and vertices to be numbered $\bar{p}_{i0}, \bar{e}_{i1}, \bar{p}_{i1}, \cdots, \bar{e}_{n}, \bar{p}_{in} = \bar{p}_{i0}$. For each i define $E_{im} = \sum_{r=1}^{m}(A_{ir} + B_{ir})$, $m = 1, 2, \cdots, n_i$. Define y_{ir} to be c_{ir} if $\bar{p}_{ir}$ belongs to $\bar{X}_i$ and c_{ir}^* if it does not. We write y_i in place of y_{i0}. Note that choices have been made so that $y_0 = c_0 = 0$. Then $\partial E_{im} = y_i - y_{im}$ for $m = 1, 2, \cdots, n_i - 1$, and $\partial E_{in_i} = 0$. The y_i $(i \neq 0)$ and the ∂E_{im} $(m < n_i)$ then form a basis for C.

In terms of the new basis,

$$\partial Z_i'' = \sum \varepsilon_{ik}(y_i - y_{h(k)}) + \partial E_i''$$

where the sum is over those k for which $\bar{p}_k$ is on the boundary of $\bar{X}_i$. Here $h(k)$ is the label of the other black region adjacent to $\bar{p}_{ik}$ and E_i'' is a linear combination of the basis elements E_{jm}. Thus if we make a final change of basis and replace Z_i'' by $Y_i = Z_i'' - E_i''$, we have $\partial Y_i = \sum \varepsilon_k(y_i - y_{h(k)})$. If we now write the bases in an appropriate order, beginning with the y_i and the Y_i, the matrix of the boundary operator assumes the form

$$\begin{bmatrix} A & 0 & 0 \\ 0 & E & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

where E is an identity matrix and A is a square matrix with a row and column for each black region in the link projection except $\bar{X}_0$. Each diagonal element a_{ii} is equal to $\sum \varepsilon_k$ where the sum is over all vertices of $\bar{X}_i$, while, when $i \neq j$, $a_{ij} = -\sum \varepsilon_k$ with the sum taken over the vertices which $\bar{X}_i$ and $\bar{X}_j$ have in common. Thus A is the matrix of the quadratic form of the link [11].

We must now express $D^1(M)$ in terms of the new basis and pick out the terms $y_i \otimes Y_j$. Let $\bar{Y}$ be the sub-group of 2-chains spanned by all the basis elements except the Y_i. Then since

$$B_j^* = (A_j + A_j^* + B_j + B_j^*) - A_j^* - (A_j + B_j)\ ,$$

$B_j^* \equiv 0 \pmod{\bar{Y}}$. Also, if X_i is white $\sum_{j\in J(i)} B_j \equiv 0 \pmod{\bar{Y}}$ and consequently

$$D^1(M) \equiv \sum_i (a_{i0}+b_{i0}+\eta_i c_{i0}+(1-\eta_i)c_{i0}^*-a_0-b_0-c_0) \otimes \sum_{j\in J(i)} B_j \pmod{\bar{Y}}$$

where $\eta_i = 1$ or 0 according to whether $\bar{p}_{i0}$ belongs to $\bar{X}_i$ or not, and the sum is taken over the black regions only. Hence

$$D^1(M) \equiv \textstyle\sum_i y_i \otimes Y_i \pmod{\bar{Y}}$$

and the matrix D has the form

$$\begin{bmatrix} E & * & * \\ * & * & * \\ * & * & * \end{bmatrix}$$

where E is a unit matrix of the same size as A.

If A is singular, then by Lemma 1 of [6] there exists an integral unimodular matrix T such that $T'AT = \begin{bmatrix} B & 0 \\ 0 & 0 \end{bmatrix}$ where B is non-singular. In the terminology of [6], B is *related* to A. If we define new bases y'_i, Y'_i for the chain groups by the equations $y_i = \sum_i t_{ji} y'_i$ and $Y'_i = \sum_j t_{ji} Y_j$ then the matrices for ∂ and D become

$$\partial : \begin{bmatrix} \begin{matrix} B & 0 \\ 0 & 0 \end{matrix} & 0 & 0 \\ 0 & E & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad D : \begin{bmatrix} \begin{matrix} E & 0 \\ 0 & E \end{matrix} & * & * \\ * & * & * \\ * & * & * \end{bmatrix}.$$

Applying the theorem of Section 1 we have

THEOREM. *The torsion sub-group of the two-dimensional cohomology group of the two-sheeted cyclic branched covering of a link has a presentation* $\{x_i : \sum b_{ij} x_j = 0\}$ *with pairing* $\| L(x_i, x_j) \| \equiv B^{-1} \pmod 1$, *where* $B = \| b_{ij} \|$ *is a non-singular matrix related to the matrix of the quadratic form of the link.*

REFERENCES

1. R. C BLANCHFIELD and R. H. FOX, *Invariants of self-linking*, Ann. of Math., 53 (1951), 556-564.
2. E. BURGER, *Über Gruppen mit Verschlingungen*, J. Reine Angew. Math., 188 (1950), 193.
3. M. DEHN, *Über die Topologie des dreidimensionalen Raumes*, Math. Ann., 69 (1910), 137-168.
4. S. EILENBERG and N. E. STEENROD, Foundations of Algebraic Topology, Princeton, 1952.
5. R. H. FOX, Covering spaces with singularities, in Algebraic Geometry and Topology, Princeton, 1957 (pp. 243-257).
6. R. H. KYLE, Branched covering spaces and the quadratic forms of knots. Princeton Ph. D. thesis, 1952.
7. ———, *Branched covering spaces and the quadratic forms of links*, Ann. of Math., 59 (1954), 539-548.
8. M. KNESER and D. PUPPE, *Quadratische Formen und Verschlingungsinvarianten von Knoten*, Math. Zeit., 58 (1953), 376.

9. D. PUPPE, *Minkowskische Einheiten und Verschlingungsinvarianten von Knoten*, Math. Zeit., 56 (1952), 33.
10. S. LEFSCHETZ, Algebraic topology, Amer. Math. Soc. Colloquium Publication 27, 1942.
11. K. REIDEMEISTER, Knotentheorie, Ergebnisse der Math. 1, Berlin, 1932.
12. H. SEIFERT, *Verschlingungsinvarianten*, Preuss. Akad. Wiss. Sitzungsber. Berlin 26 (1933), 84-101.
13. ———, *Die Verschlingungsinvarianten der zyklischen Knotenüberlagerungen*, Abh. Math. Sem. Hamburg Univ., 11 (1936), 84.
14. ——— and W. THRELFALL, Lehrbuch der Topologie, Leipzig, 1934.