ALGEBRAIC L-THEORY

III. TWISTED LAURENT EXTENSIONS

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Introduction

The algebraic definition of the surgery obstruction groups $\begin{cases} L_n^p(\pi) \\ L_n^h(\pi) \\ L_n^s(\pi) \end{cases}$, for surgery on $\begin{cases} \text{open} \\ \text{compact} \\ \text{compact} \end{cases}$ manifolds, over $\begin{cases} - \\ \text{finite} \\ \text{simple} \end{cases}$ Poincaré complexes up to $\begin{cases} \text{proper} \\ - \\ \text{simple} \end{cases}$ homotopy, depends on n(mod 4) and a group ring $Z[\pi]$, together with the involution

$$\bar{\ } : Z[\pi] \longrightarrow Z[\pi] \; ; \; \sum_{g\in\pi} n_g g \longmapsto \sum_{g\in\pi} w(g) n_g g^{-1} \qquad (n_g \in Z)$$

given by a group morphism

$$w : \pi \longrightarrow Z_2 = \{1,-1\}$$

(cf.[10]). For finitely presented groups π it is possible to obtain geometrically direct sum decompositions

$$L_n^h(\pi \times Z) = L_n^h(\pi) \oplus L_{n-1}^p(\pi) \qquad ([3])$$

$$L_n^s(\pi \times Z) = L_n^s(\pi) \oplus L_{n-1}^h(\pi) \qquad ([6])$$

The hamiltonian formalism of [4] allowed a unified approach to the three L-theories, and a purely algebraic description of these decompositions. This was done in parts I. and II. of this paper ([5]), which will be denoted I.,II. . In I. there were defined abelian groups $\begin{cases} U_n(A) \\ V_n(A) \\ W_n(A) \end{cases}$, using quadratic forms on $\begin{cases} \text{f.g.projective} \\ \text{f.g.free} \\ \text{based} \end{cases}$ A-modules, for any associative ring A with 1 and involution and n(mod 4). It was then shown in II. that there are direct sum decompositions

$$V_n(A_z) = V_n(A) \oplus U_{n-1}(A)$$
$$\widetilde{W}_n(A_z) = W_n(A) \oplus V_{n-1}(A)$$

where $A_z = A[z,z^{-1}]$ is the Laurent extension of A, with involution by $z \mapsto z^{-1}$, and $\widetilde{W}_n(A_z)$ differs from $W_n(A_z)$ in at most one element, of order 2.

Here, we shall generalize I. by considering the intermediate L-theories $\begin{cases} U_n^T(A) \\ V_n^R(A) \end{cases}$, defined using quadratic forms on $\begin{cases} \text{f.g.projective} \\ \text{based} \end{cases}$ A-modules such that all the $\begin{cases} \text{projective classes} \\ \text{Whitehead torsions} \end{cases}$ lie in a prescribed subgroup $\begin{cases} T \subseteq \widetilde{K}_0(A) \\ R \subseteq \widetilde{K}_1(A) \end{cases}$. The direct sum decompositions of II. generalize to exact sequences

$$\ldots \to U_n^T(A) \xrightarrow{\bar{\varepsilon}} U_n^{\bar{\varepsilon}T}(A_\alpha) \xrightarrow{B} U_{n-1}^{(1-\alpha)^{-1}T}(A) \xrightarrow{C} U_{n-1}^T(A) \to \ldots \quad \text{(Theorem 5.1)}$$

$$\ldots \to V_n^R(A) \xrightarrow{\bar{\varepsilon}} V_n^{\bar{\varepsilon}R}(A_\alpha) \xrightarrow{B} V_{n-1}^{(1-\alpha)^{-1}R}(A) \xrightarrow{C} V_{n-1}^R(A) \to \ldots \quad \text{(Theorem 5.2)}$$

$$\ldots \to V_n^R(A) \xrightarrow{\bar{\varepsilon}} V_n^{\widetilde{S}}(A_\alpha) \xrightarrow{B} U_{n-1}^T(A) \xrightarrow{C} V_{n-1}^R(A) \to \ldots \quad \text{(Theorem 5.3)}$$

where A_α is the α-twisted Laurent extension of A (assumed to be such that f.g.free A_α-modules have a well-defined rank) for some automorphism α of A, $\bar{\varepsilon}$ is the inclusion of A in A_α, and C is induced by $1-\alpha$.

For $A = Z[\pi]$ it is possible to identify

$$L_n^p(\pi) = U_n(Z[\pi]) = U_n^{\widetilde{K}_0(Z[\pi])}(A)$$
$$L_n^h(\pi) = V_n(Z[\pi]) = V_n^{\widetilde{K}_1(Z[\pi])}(A)$$
$$L_n^s(\pi) = V_n^{\{\pi\}}(Z[\pi]) \ (= W_n(Z[\pi]), \text{ up to 2-torsion }).$$

The special case $R = \{\pi\}$ of Theorem 5.2, with α given by an automorphism $\alpha : \pi \to \pi$ such that $w\alpha = w : \pi \to Z_2$, is the exact sequence

$$\ldots \to L_n^s(\pi) \to L_n^s(\pi \times_\alpha Z) \to L'_{n-1}(\pi) \to L_{n-1}^s(\pi) \to \ldots$$

of the case H = H' = K of Theorem 10 of [1], where a geometric derivation is announced, following on from some earlier work of F.T.Farrell and W.C.Hsiang. The groups $L_n'(\pi)$ are defined as $L_n^s(\pi)$, except that torsions are measured in $Wh\pi/\ker(1-\alpha: Wh\pi \longrightarrow Wh\pi)$ rather than in the Whitehead group $Wh\pi = \tilde{K}_1(Z[\pi])/\{\pi\}$. (Thus, if $\alpha = 1$ torsions are not measured at all, and $L_n'(\pi) = L_n^h(\pi)$). It is Cappell (in [1]) who first used the intermediate L-theories.

I am grateful to Professor C.T.C.Wall for sending a preprint to [10] (which contains an earlier account of the intermediate L-theories), and for suggesting that I generalize II. to the twisted case.

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This part of the paper is divided as follows:

This part can be read independently of the previous parts, taking for granted the proofs of the results quoted from I. and II. .

§1. L-theory

The purpose of this section is to introduce some notation, and to recall those definitions and results from I. which will be needed in this part.

Let A be an associative ring with 1, and with an involution, that is a function

$$\bar{\ }: A \longrightarrow A \ ; \ a \longmapsto \bar{a}$$

such that

$$\text{i)} \quad \overline{(a+b)} = \bar{a} + \bar{b}$$

$$\text{ii)} \quad \overline{(ab)} = \bar{b}.\bar{a}$$

$$\text{iii)} \quad \bar{\bar{a}} = a$$

$$\text{iv)} \quad \bar{1} = 1$$

for all $a,b \in A$.

Let $\mathcal{P}(A)$ be the category of finitely generated (f.g.) projective left A-modules. Denote the class of objects of $\mathcal{P}(A)$ by $|\mathcal{P}(A)|$ by $|\mathcal{P}(A)|$. Given $P,Q \in |\mathcal{P}(A)|$, write $\mathrm{Hom}_A(P,Q)$ for the additive group of morphisms $(f:P \rightarrow Q) \in \mathcal{P}(A)$.

There is defined a contravariant duality functor, by

$$* : \mathcal{P}(A) \longrightarrow \mathcal{P}(A) \ ; \begin{cases} Q \in |\mathcal{P}(A)| \longmapsto \begin{cases} Q^* = \mathrm{Hom}_A(Q,A), \text{ left } A\text{-action by} \\ A \times Q^* \longrightarrow Q^*; (a,f) \longmapsto (x \longmapsto f(x).\bar{a}) \end{cases} \\ f \in \mathrm{Hom}_A(P,Q) \longmapsto (f^*: Q^* \longrightarrow P^*; g \longmapsto (x \longmapsto gf(x))). \end{cases}$$

The natural A-module isomorphisms

$$Q \longrightarrow Q^{**} \ ; \ x \longrightarrow (f \longrightarrow \overline{f(x)}) \qquad (Q \in |\mathcal{P}(A)|)$$

allow an identification

$$** = 1 : \mathcal{P}(A) \longrightarrow \mathcal{P}(A) \ .$$

Let

$$f : A \longrightarrow A'$$

be a morphism of rings with involution (such that $f(1) = 1 \in A'$). Give A' an (A',A)-bimodule structure by

$$A' \times A' \times A \longrightarrow A' \; ; \; (a',x,a) \longmapsto a'.x.f(a) \quad .$$

The induced functor

$$f : \mathcal{P}(A) \longrightarrow \mathcal{P}(A') \; ; \; \begin{cases} P \longmapsto fP = A' \otimes_A P \\ g \in \mathrm{Hom}_A(P,Q) \longmapsto 1 \otimes g \in \mathrm{Hom}_{A'}(fP,fQ) \end{cases}$$

is such that

$$f(A) = A' \in |\mathcal{P}(A')|$$

and

$$*f = f* : \mathcal{P}(A) \longrightarrow \mathcal{P}(A')$$

(up to natural equivalence).

Given $Q \in |\mathcal{P}(A)|$, and $\theta \in \mathrm{Hom}_A(Q,Q^*)$ such that

$$\theta^* = \pm\theta \in \mathrm{Hom}_A(Q,Q^*)$$

(for one of the signs indicated), there is defined a $\pm$hermitian sesquilinear product

$$< \; > : Q \times Q \longrightarrow A \; ; \; (x,y) \longmapsto <x,y> \equiv \theta(x)(y)$$

with

$$\overline{<x,y>} = \pm <y,x> \in A \qquad (x,y \in Q) \; .$$

A <u>$\pm$form</u> (over A) is a pair

$$(Q \in |\mathcal{P}(A)|, \; \varphi \in \mathrm{Hom}_A(Q,Q^*)).$$

We shall be interested only in the $\pm$hermitian products

$$\theta = \varphi \pm \varphi^* : Q \longrightarrow Q^*$$

associated with $\pm$forms (Q,φ).

An <u>equivalence of $\pm$forms</u>

$$f : (Q,\varphi) \longrightarrow (Q',\varphi')$$

(over the same ground ring A) is an isomorphism $f \in \mathrm{Hom}_A(Q,Q')$

such that

$$f^*\varphi' f - \varphi = \gamma \mp \gamma^* \in \mathrm{Hom}_A(Q,Q^*)$$

for some $\mp$form (Q,γ). Then

$$f^*(\varphi' \pm \varphi'^*)f = \varphi \pm \varphi^* \in \mathrm{Hom}_A(Q,Q^*) \quad ,$$

so that equivalences preserve the $\pm$hermitian products associated with $\pm$forms.

The direct sum $\oplus$ in $\mathcal{P}(A)$ generalizes to a sum operation on $\pm$forms: the <u>sum</u> of $\pm$forms is defined by

$$(Q,\varphi) \oplus (Q',\varphi') = (Q \oplus Q', \varphi \oplus \varphi').$$

A $\pm$form is <u>trivial</u> if it is equivalent to the <u>hamiltonian</u> <u>$\pm$form</u>

$$H_{\pm}(P) = (P \oplus P^*, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} : P \oplus P^* \longrightarrow P^* \oplus P = (P \oplus P^*)^*;$$
$$(x,f) \longmapsto ((x',f') \longmapsto f(x')) \ )$$

on some $P \in |\mathcal{P}(A)|$.

L-theory considers $\pm$forms up to equivalence because that is how they arise in even-dimensional surgery obstruction theory. Surgery corresponds to the addition of a trivial $\pm$form (or the inverse operation).

A <u>sublagrangian</u> L of a $\pm$form (Q,φ) is a direct summand L of Q such that

i) $j^*(\varphi \pm \varphi^*) \in \mathrm{Hom}_A(Q,L^*)$ is onto,

ii) $j^*\varphi j = \delta \mp \delta^* \in \mathrm{Hom}_A(L,L^*)$ for some $\mp$form (L,δ),

writing $j \in \mathrm{Hom}_A(L,Q)$ for the inclusion. The <u>annihilator</u> of L in (Q,φ),

$$L^{\perp} = \ker(\ j^*(\varphi \pm \varphi^*) : Q \longrightarrow L^* \ )$$

is then a direct summand of Q (by i)) containing L as a direct summand (by ii)). Restriction of $\varphi \in \mathrm{Hom}_A(Q,Q^*)$ to a direct complement to L in $L^{\perp}$ defines a $\pm$form $(L^{\perp}/L \ ,\hat{\varphi}\ )$ uniquely up to equivalence.

For example, $L \in |\mathcal{P}(A)|$ is a sublagrangian of

$$(Q,\varphi) = H_{\pm}(L) \oplus (P,\theta)$$

for any $\pm$form (P,θ), with

$$(L^{\perp}/L,\hat{\varphi}) = (P,\theta).$$

The converse holds up to equivalence, by the following version of Witt's theorem in the classical theory of quadratic forms.

Theorem 1.1 Let L be a sublagrangian of the $\pm$form (Q,φ). The inclusion

$$j : L \oplus (L^{\perp}/L) \longrightarrow Q$$

extends to an equivalence of $\pm$forms

$$f : H_{\pm}(L) \oplus (L^{\perp}/L,\hat{\varphi}) \longrightarrow (Q,\varphi)$$

uniquely up to composition with the self-equivalences

$$\begin{pmatrix} 1 & \theta \mp \theta^* \\ 0 & 1 \end{pmatrix} \oplus 1 \quad : H_{\pm}(L) \oplus (L^{\perp}/L,\hat{\varphi}) \longrightarrow H_{\pm}(L) \oplus (L^{\perp}/L,\hat{\varphi})$$

given by $\mp$forms (L^*,θ).

[ ]

A sublagrangian L of a $\pm$form (Q,φ) such that

$$L^{\perp} = L$$

is a lagrangian of (Q,φ).

Corollary 1.2 A $\pm$ form is trivial if and only if it admits a lagrangian.

[ ]

A $\pm$formation (over A) , $(Q,\varphi;F,G)$, is a $\pm$form (Q,φ) over A, together with a lagrangian F and a sublagrangian G. An equivalence of $\pm$formations

$$f : (Q,\varphi;F,G) \longrightarrow (Q',\varphi';F',G')$$

is an equivalence of $\pm$forms

$$f : (Q,\varphi) \longrightarrow (Q',\varphi')$$

such that $f(F) = F'$, $f(G) = G'$.

The sum of $\pm$formations is defined by

$(Q,\varphi;F,G)\oplus(Q',\varphi';F',G') = (Q\oplus Q',\varphi\oplus\varphi';F\oplus F',G\oplus G')$.

A stable equivalence of $\pm$formations

$$[f] : (Q,\varphi;F,G) \longrightarrow (Q',\varphi';F',G')$$

is an equivalence of $\pm$formations

$$f : (Q,\varphi;F,G)\oplus(H_{\pm}(P);P,P^*) \longrightarrow (Q',\varphi';F',G')\oplus(H_{\pm}(P');P',P'^*)$$

defined for some $P,P' \in |\mathcal{P}(A)|$.

A $\pm$formation is elementary if it is equivalent to

$$(H_{\pm}(P);P,\Gamma_{(P,\theta)})$$

for some $\mp$form (P,θ), where

$$\Gamma_{(P,\theta)} = \{(x,(\theta\mp\theta^*)x)\in P\oplus P^* \mid x\in P\}$$

is the graph of (P,θ).

L-theory considers $\pm$formations up to stable equivalence because that is how they arise in odd-dimensional surgery obstruction theory. Surgery corresponds to the addition of an elementary $\pm$formation (or the inverse operation).

A hamiltonian complement to a lagrangian L in a $\pm$form (Q,φ) is a lagrangian L' which is a direct complement to L on Q. It follows from Theorem 1.1 that every lagrangian has hamiltonian complements, and that the hamiltonian complements to P^* in $H_{\pm}(P)$ are just the graphs $\Gamma_{(P,\theta)}$ of $\mp$forms (P,θ), for any $P \in |\mathcal{P}(A)|$.

Corollary 1.3 A $\pm$formation $(Q,\varphi;F,G)$ is elementary if and only if G is a lagrangian sharing a hamiltonian complement with F.

[ ]

Given a lagrangian L in a $\pm$form (Q,φ), and a hamiltonian complement L', the A-module isomorphism

$$L' \longrightarrow L^* \ ; \ x \mapsto (y \mapsto (\varphi\pm\varphi^*)(x)(y))$$

will be used to identify L' with L* (in general). This is an abuse of language, as hamiltonian complements are not unique.

§2. Intermediate U-theories

Let I be an abelian monoid. Given a submonoid J of I, define an equivalence relation $\sim_J$ on I by :

$$i \sim_J i' \text{ if there exist } j,j'\in J \text{ such that } i\oplus j = i'\oplus j' \in I \ .$$

Denote the quotient monoid $I/\sim_J$ by $I/\bar{J}$, because it depends only on the stabilization of J in I, the submonoid

$$\bar{J} = \{i\in I \mid i \sim_J 0\}$$

Note that $I/\bar{J}$ is an abelian group if and only if for every $i\in I$ there exists $i'\in I$ such that $i\oplus i'\in J$.

Define the abelian group

$$K_0(A) = K(\mathcal{P}(A))$$

as usual. The reduced group

$$\tilde{K}_0(A) = \operatorname{coker}(K_0(Z) \longrightarrow K_0(A))$$

can be regarded as the quotient monoid

$$\{\text{isomorphism classes in } \mathcal{P}(A)\} \Big/ \{\text{isomorphism classes of f.g.free } A\text{-modules}\}.$$

Duality in $\mathcal{P}(A)$ defines an involution of $K_0(A)$

$$* : K_0(A) \longrightarrow K_0(A);\ [P] \longmapsto [P^*]$$

and similarly for $\tilde{K}_0(A)$.

Theorem 3.2 of I. (the case $T = \tilde{K}_0(A)$) generalizes to

Theorem 2.1 For n(mod 4) let $X_n(A)$ be the abelian monoid of $\begin{cases}\text{equivalence}\\ \text{stable equivalence}\end{cases}$ classes of $\begin{cases}\pm\text{forms}\\ \pm\text{formations}\end{cases}$ over A, if $n = \begin{cases}2i\\ 2i+1\end{cases}$ with $\pm = (-)^i$.

The monoid morphisms

$$\partial : X_n(A) \longrightarrow X_{n-1}(A);\ \begin{cases}(Q,\varphi) \longmapsto (H_{\mp}(Q);Q,\Gamma_{(Q,\varphi)})\\ (Q,\varphi;F,G) \longmapsto (G^{\perp}/G,\hat{\varphi})\end{cases} \quad n = \begin{cases}2i\\ 2i+1\end{cases}$$

are such that $\partial^2 = 0$.

The monoid morphisms

$$\sigma : X_n(A) \longrightarrow \widetilde{K}_0(A) \ ; \begin{cases} (Q,\varphi) \longmapsto [Q] \\ (Q,\varphi;F,G) \longmapsto [G]-[F^*] \end{cases} \qquad n = \begin{cases} 2i \\ 2i+1 \end{cases}$$

define a chain map

$$\sigma : (X_n(A),\partial) \longrightarrow (\widetilde{K}_0(A),\ 1 + (-)^{n+1} *)$$

of chain complexes of abelian monoids.

Given a $*$-invariant subgroup $T \subseteq \widetilde{K}_0(A)$ (that is, $*(T) = T$) define a chain complex of abelian monoids

$$(\overset{T}{X}_n(A),\overset{T}{\partial}) = \sigma^{-1}(T,\ 1 + (-)^{n+1} *) \qquad (n(\bmod 4)) .$$

The subquotient monoids

$$\overset{T}{U}_n(A) = \ker(\overset{T}{\partial} : \overset{T}{X}_n(A) \longrightarrow \overset{T}{X}_{n-1}(A)) / \overline{\operatorname{im}(\overset{T}{\partial} : \overset{T}{X}_{n+1}(A) \longrightarrow \overset{T}{X}_n(A))}$$

are abelian groups.

A 1-preserving morphism of rings with involution

$$f : A \longrightarrow A'$$

induces morphisms of abelian groups

$$f : \overset{T}{U}_n(A) \longrightarrow \overset{T'}{U}_n(A') ; \begin{cases} (Q,\varphi) \longmapsto (A'\otimes_A Q, 1\otimes\varphi) \\ (Q,\varphi;F,G) \longmapsto (A'\otimes_A Q, 1\otimes\varphi; A'\otimes F, A'\otimes G) \end{cases} \quad n=\begin{cases} 2i \\ 2i+1 \end{cases}$$

for any $*$-invariant subgroups $T \subseteq \widetilde{K}_0(A)$, $T' \subseteq \widetilde{K}_0(A')$ such that $f(T) \subseteq T'$.

[ ]

Following I., II. the groups $U_n^{\widetilde{K}_0(A)}(A)$ will be denoted by $U_n(A)$.

A $\begin{cases} \pm \text{ form } (Q,\varphi) \\ \pm\text{formation } (Q,\varphi;F,G) \end{cases}$ is non-singular if $\begin{cases} \varphi\pm\varphi^* \in \operatorname{Hom}_A(Q,Q^*) \text{ is an isomorphism} \\ G \text{ is a lagrangian of } (Q,\varphi) \end{cases}$. Then

$$\overset{T}{U}_n(A) = \begin{cases} \{\text{non-singular } \pm\text{forms} \in \overset{T}{X}_{2i}(A)\} / \overline{\{H_\pm(L) \mid [L]\in T\}} \\ \{\text{non-singular } \pm\text{formations} \in \overset{T}{X}_{2i+1}(A)\} / \overline{\{(H_\pm(P);P,\Gamma_{(P,\theta)}) \mid [P]\in T\}} . \end{cases}$$

Inverses are given by

$$-(Q,\varphi) = (Q,-\varphi) \in U^T_{2i}(A)$$
$$-(Q,\varphi;F,G) = (Q,-\varphi;F^*,G^*) \in U^T_{2i+1}(A) \ .$$

This is clear on noting that the <u>diagonal</u> of a $\pm$form (Q,φ),

$$\Delta_{(Q,\varphi)} = \{ (x,x) \in Q\oplus Q \mid x \in Q \} ,$$

is a $\begin{cases}\text{lagrangian} \\ \text{hamiltonian complement to } L\oplus L^*\end{cases}$ in $(Q\oplus Q,\varphi\oplus -\varphi)$, if (Q,φ) is $\begin{cases}\text{non-singular} \\ \text{trivial, with } L,L^* \text{ any hamiltonian complements in } (Q,\varphi)\end{cases}$.

The sum formula of Lemma 3.3 in I. generalizes to

<u>Lemma 2.2</u> $(Q,\varphi;F,G)\oplus(Q,\varphi;G,H) = (Q,\varphi;F,H) \in U^T_{2i+1}(A)$ if $[F],[G],[H] \in T$.

<u>Proof</u>: The identity

$$\begin{aligned}&(Q,\varphi;F,G)\oplus(Q,\varphi;G,H)\oplus[(Q,-\varphi;G^*,G^*)]\\ &\quad\oplus[(Q\oplus Q,\varphi\oplus-\varphi;F\oplus F^*,H\oplus G^*)\oplus(Q\oplus Q,-\varphi\oplus\varphi;\Delta_{(Q,\varphi)},H^*\oplus G)]\\ &= (Q,\varphi;F,H)\oplus[(Q\oplus Q,\varphi\oplus-\varphi;F\oplus F^*,G\oplus G^*)]\\ &\quad\oplus[(Q\oplus Q,\varphi\oplus-\varphi;G\oplus G^*,H\oplus G^*)\oplus(Q\oplus Q,-\varphi\oplus\varphi;\Delta_{(Q,\varphi)},H^*\oplus G)]\end{aligned}$$

is such that each of the $\pm$formations in square brackets is elementary.

[ ]

Let G be an abelian group with involution

$$* : G \longrightarrow G \ ; \ g \longmapsto g^* \ .$$

The Tate cohomology of this Z_2-action is given by groups

$$H^n(G) = \{ x \in G \mid x^* = (-)^n x \}/\{ y + (-)^n y^* \mid y \in G \}$$

defined for n(mod 2), which are abelian of exponent 2.

The exact sequence of Theorem 4.3 in I. (the case $T = \{0\}$, $T' = \tilde{K}_0(A)$) generalizes to

<u>Theorem 2.3 Given *-invariant subgroups</u> $T \subseteq T' \subseteq \tilde{K}_0(A)$, <u>there is defined an exact sequence of abelian groups</u>

$$\ldots \longrightarrow H^{n+1}(T'/T) \longrightarrow U^T_n(A) \xrightarrow{1} U^{T'}_n(A) \xrightarrow{\sigma} H^n(T'/T) \longrightarrow \ldots$$

<u>where</u> $H^{n+1}(T'/T) \longrightarrow U^T_n(A);[P] \longmapsto \begin{cases} H_\pm(P) \\ (H_\pm(P);P,P)\end{cases}$ <u>if</u> $n = \begin{cases} 2i \\ 2i+1 \end{cases}$.

[ ]

§ 3. Intermediate V-theories

A based A-module, $\underset{\sim}{Q}$, is a f.g.free A-module Q together with a base $\underset{\sim}{q} = (q_1, \ldots, q_n)$, and n is the rank of $\underset{\sim}{q}$. The dual based A-module $\underset{\sim}{Q}^*$ is Q^* with the base $\underset{\sim}{q}^* = (q_1^*, \ldots, q_n^*)$ given by

$$q_i^*(q_j) = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{otherwise} \end{cases} .$$

Identify $\underset{\sim}{Q}^{**}$ with $\underset{\sim}{Q}$.

Define the abelian groups

$$K_1(A) = GL(A)/E(A) \quad , \quad \widetilde{K}_1(A) = \operatorname{coker}(K_1(Z) \longrightarrow K_1(A))$$

as usual, regarding their elements as the torsions $\tau(f:P \to P)$ of automorphisms $(f:P \longrightarrow P) \in \mathcal{P}(A)$. There is defined a duality involution

$$* : K_1(A) \longrightarrow K_1(A) \ ; \quad \tau(f:P \to P) \longmapsto \tau(f^*:P^* \longrightarrow P^*) \ .$$

In dealing with ±forms and ±formations on based A-modules it is more natural to measure torsions not in $\widetilde{K}_1(A)$, but in the slightly larger group $K'(A)$ defined below, which coincides with $\widetilde{K}_1(A)$ if A is such that f.g.free A-modules have a well-defined rank (e.g. $A = Z[\pi]$).

Let $I(A)$ be the abelian monoid of isomorphism classes of triples $(Q,\underset{\sim}{f},\underset{\sim}{g})$, with Q a f.g.free A-module and $\underset{\sim}{f},\underset{\sim}{g}$ two bases of Q (not necessarily of the same rank), under the sum operation

$$(Q,\underset{\sim}{f},\underset{\sim}{g}) \oplus (Q',\underset{\sim}{f}',\underset{\sim}{g}') = (Q \oplus Q',\underset{\sim}{f} \oplus \underset{\sim}{f}',\underset{\sim}{g} \oplus \underset{\sim}{g}') \ .$$

Let $J(A)$ be the submonoid of $I(A)$ generated by the triples of type

i) $(Q,(f_1,\ldots,f_n),(f_1,\ldots,f_{i-1},\delta f_i + a f_j, f_{i+1},\ldots,f_n))$

$(\delta = \pm 1,\ a \in A,\ i \neq j)$

ii) $(Q,\underset{\sim}{f},\underset{\sim}{g}) \oplus (Q,\underset{\sim}{g},\underset{\sim}{h}) \oplus (Q,\underset{\sim}{h},\underset{\sim}{f})$.

The quotient monoid

$$K'(A) = I(A)/\overline{J(A)}$$

is an abelian group in which there is a sum formula

$$(Q,\underset{\sim}{f},\underset{\sim}{g}) \oplus (Q,\underset{\sim}{g},\underset{\sim}{h}) = (Q,\underset{\sim}{f},\underset{\sim}{h}) \in K'(A) \ .$$

It is therefore possible to regard the elements of K'(A) as the torsions

$$\tau(f:\underset{\sim}{P} \longrightarrow \underset{\sim}{Q}) = (Q,\underset{\sim}{q},f(\underset{\sim}{p})) \in K'(A)$$

of isomorphisms $f \in Hom_A(P,Q)$ of based A-modules $\underset{\sim}{P},\underset{\sim}{Q}$.

By the Whitehead lemma, the function

$$\tilde{K}_1(A) \longrightarrow K'(A); \tau(f:P\to P) \longmapsto (P\oplus -P,\underset{\sim}{b},(f\oplus 1)\underset{\sim}{b})$$

is a group morphism, where -P is any projective inverse to P, and $\underset{\sim}{b}$ is any base of $P\oplus -P$. In fact, there is a short exact sequence of abelian groups

$$0 \longrightarrow \tilde{K}_1(A) \longrightarrow K'(A) \longrightarrow \ker(K_0(Z) \longrightarrow K_0(A)) \longrightarrow 0$$

where

$$K'(A) \longrightarrow \ker(K_0(Z) \longrightarrow K_0(A)); (Q,\underset{\sim}{f},\underset{\sim}{g}) \longmapsto [mZ] - [nZ]$$

if $\underset{\sim}{f} = (f_1,\dots,f_m)$, $\underset{\sim}{g} = (g_1,\dots,g_n)$. The duality involution

$$* : K'(A) \longrightarrow K'(A) \; ; \; (Q,\underset{\sim}{f},\underset{\sim}{g}) \longmapsto (Q^*,\underset{\sim}{g}^*,\underset{\sim}{f}^*)$$

agrees with that previously defined on $\tilde{K}_1(A)$, but there is a change of sign in passing to $\ker(K_0(Z) \longrightarrow K_0(A))$.

A <u>based ±form</u> (over A), $(\underset{\sim}{Q},\varphi)$, is a ±form (Q,φ) defined on a based A-module $\underset{\sim}{Q}$. The <u>torsion</u> of $(\underset{\sim}{Q},\varphi)$ is

$$\tau(\underset{\sim}{Q},\varphi) = \begin{cases} \tau(\varphi \pm \varphi^* : \underset{\sim}{Q} \to \underset{\sim}{Q}^*) & \text{if } (Q,\varphi) \text{ is non-singular} \\ 0 & \text{otherwise} \end{cases} \in \tilde{K}_1(A) \quad .$$

Let $S \subseteq K'(A)$ be a $*$-invariant subgroup.

An <u>S-equivalence</u> of based ±forms

$$f:(\underset{\sim}{Q},\varphi) \longrightarrow (\underset{\sim}{Q}',\varphi')$$

is an equivalence of ±forms such that

$$\tau(f:\underset{\sim}{Q} \longrightarrow \underset{\sim}{Q}') \in S. \; .$$

Now $f^*(\varphi' \pm \varphi'^*)f = (\varphi \pm \varphi^*) \in Hom_A(Q,Q^*)$, so that

$$\tau(\underset{\sim}{Q},\varphi) - \tau(\underset{\sim}{Q}',\varphi') = \begin{cases} \tau + \tau^* & \text{if } (Q,\varphi) \text{ is non-singular} \\ 0 & \text{otherwise} \end{cases} \in S \subseteq K'(A)$$

where $\tau = \tau(f:\underset{\sim}{Q} \longrightarrow \underset{\sim}{Q}') \in S$.

Given a free sublagrangian L of a $\pm$form (Q,φ) such that $L^{\perp}/L$ is free, it is possible to extend a base $\underset{\sim}{L}\oplus\underset{\sim}{L^{\perp}/L}$ to one of Q uniquely up to simple changes, using any of the equivalences

$$f : H_{\pm}(L)\oplus(L^{\perp}/L,\hat{\varphi}) \longrightarrow (Q,\varphi)$$

given by Theorem 1.1. Call such a base

$$\underset{\sim}{Q} = f(\underset{\sim}{L}\oplus\underset{\sim}{L}^*\oplus\underset{\sim}{L^{\perp}/L})$$

a <u>subhamiltonian base</u> for (Q,φ), and a <u>hamiltonian base</u> if L is a lagrangian.

A <u>based $\pm$formation</u> $(Q,\varphi;\underset{\sim}{F},\underset{\sim}{G})$ is a $\pm$formation $(Q,\varphi;F,G)$ together with bases $\underset{\sim}{f},\underset{\sim}{g},\underset{\sim}{h}$ for $F,G,G^{\perp}/G$ respectively. The <u>torsion</u> of $(Q,\varphi;\underset{\sim}{F},\underset{\sim}{G})$ is

$$\tau(Q,\varphi;\underset{\sim}{F},\underset{\sim}{G}) = (Q,\underset{\sim}{f}\oplus\underset{\sim}{f}^*,\underset{\sim}{g}\oplus\underset{\sim}{g}^*\oplus\underset{\sim}{h}) \in K'(A)$$

with $\underset{\sim}{f}\oplus\underset{\sim}{f}^*$ any hamiltonian base extending $\underset{\sim}{f}$, and $\underset{\sim}{g}\oplus\underset{\sim}{g}^*\oplus\underset{\sim}{h}$ any subhamiltonian base extending $\underset{\sim}{g}\oplus\underset{\sim}{h}$. As shown above, this definition does not depend on the choice of $\underset{\sim}{f}^*,\underset{\sim}{g}^*$.

As before, let $S\subseteq K'(A)$ be a $*$-invariant subgroup.

An <u>S-equivalence</u> of based $\pm$formations

$$f:(Q,\varphi;\underset{\sim}{F},\underset{\sim}{G}) \longrightarrow (Q',\varphi';\underset{\sim}{F}',\underset{\sim}{G}')$$

is an equivalence of $\pm$formations such that

$$\tau(\underset{\sim}{F}\rightarrow\underset{\sim}{F}'),\quad \tau(\underset{\sim}{G}\rightarrow\underset{\sim}{G}'),\quad \tau(\underset{\sim}{G^{\perp}/G}\rightarrow\underset{\sim}{G'^{\perp}/G'}) \in S .$$

Then

$$\tau(Q',\varphi';\underset{\sim}{F}',\underset{\sim}{G}') - \tau(Q,\varphi;\underset{\sim}{F},\underset{\sim}{G}) = \tau-\tau^* \in S\subseteq K'(A)$$

where $\tau = (\tau(\underset{\sim}{F}\rightarrow\underset{\sim}{F}')-\tau(\underset{\sim}{G}\rightarrow\underset{\sim}{G}')-\tau(\underset{\sim}{G^{\perp}/G}\rightarrow\underset{\sim}{G'^{\perp}/G'})) \in S$.

A <u>stable S-equivalence</u> of based $\pm$formations

$$[f] : (Q,\varphi;\underset{\sim}{F},\underset{\sim}{G}) \longrightarrow (Q',\varphi';\underset{\sim}{F}',\underset{\sim}{G}')$$

is an S-equivalence of based $\pm$formations

$$f:(Q,\varphi;\underset{\sim}{F},\underset{\sim}{G})\oplus(H_{\pm}(P);\underset{\sim}{P},\underset{\sim}{P}^*) \longrightarrow (Q',\varphi';\underset{\sim}{F}',\underset{\sim}{G}')\oplus(H_{\pm}(P');\underset{\sim}{P}',\underset{\sim}{P}'^*)$$

defined for some based A-modules $\underset{\sim}{P},\underset{\sim}{P}'$.

Theorem 2.1 has a based analogue:

Theorem 3.1 For n(mod 4) and a $*$-invariant subgroup $S \subseteq K'(A)$ define the abelian monoid $Y_n^S(A)$ of $\begin{cases} \text{S-equivalence classes} \\ \text{stable S-equivalence classes} \end{cases}$ of $\begin{cases} \text{based } \pm\text{forms} \\ \text{based } \pm\text{formations} \end{cases}$ with torsion in S, with $\pm = (-)^i$ if $n = \begin{cases} 2i \\ 2i+1 \end{cases}$.

The monoid morphisms

$$\partial^S : Y_n^S(A) \longrightarrow Y_{n-1}^S(A); \begin{cases} (\underset{\sim}{Q},\varphi) \longmapsto (H_{\mp}(Q);\underset{\sim}{Q}, \Gamma_{(\underset{\sim}{Q},\varphi)}) \\ (Q,\varphi;\underset{\sim}{F},\underset{\sim}{G}) \longmapsto (\underset{\sim}{G^{\perp}/G},\hat{\varphi}) \end{cases}$$

are such that $(\partial^S)^2 = 0$. The subquotient monoids

$$V_n^S(A) = \ker(\partial^S : Y_n^S(A) \longrightarrow Y_{n-1}^S(A))/\overline{\mathrm{im}(\partial^S : Y_{n+1}^S(A) \longrightarrow Y_n^S(A))}$$

are abelian groups.

A 1-preserving morphism of rings with involution

$$f : A \longrightarrow A'$$

induces morphisms of abelian groups

$$f : V_n^S(A) \longrightarrow V_n^{S'}(A'); \begin{cases} (\underset{\sim}{Q},\varphi) \longmapsto (A' \otimes_A \underset{\sim}{Q}, 1 \otimes \varphi) \\ (Q,\varphi;\underset{\sim}{F},\underset{\sim}{G}) \longrightarrow (A' \otimes_A Q, 1 \otimes \varphi; A' \otimes \underset{\sim}{F}, A' \otimes \underset{\sim}{G}) \end{cases}$$

for any $*$-invariant subgroups $S \subseteq K'(A)$, $S' \subseteq K'(A')$ such that $f(S) \subseteq S'$.

[ ]

Note that

$$V_{2i}^S(A) = \{\text{non-singular based } \pm\text{forms} \in Y_{2i}^S(A)\}/\overline{\{H_{\pm}(m\underset{\sim}{A}) \mid m > 0\}}$$

$$V_{2i+1}^S(A) = \{\text{non-singular based } \pm\text{formations} \in Y_{2i+1}^S(A)\}/\overline{\{(H_{\pm}(P);P, \Gamma_{(\underset{\sim}{P},\theta)}) \mid \tau(\underset{\sim}{P},\theta) \in S\}}$$

Inverses are given by

$$-(\underset{\sim}{Q},\varphi) = (\underset{\sim}{Q},-\varphi) \in V_{2i}^S(A)$$

$$-(Q,\varphi;\underset{\sim}{F},\underset{\sim}{G}) = (Q,-\varphi;\underset{\sim}{F}^*,\underset{\sim}{G}^*) \in V_{2i+1}^S(A) .$$

The sum formula of Lemma 2.2 has a based analogue

Lemma 3.2 $(Q,\varphi;\underset{\sim}{F},\underset{\sim}{G}) \oplus (Q,\varphi;\underset{\sim}{G},\underset{\sim}{H}) = (Q,\varphi;\underset{\sim}{F},\underset{\sim}{H}) \in V_{2i+1}^S(A)$

[ ]

For $S \subseteq \widetilde{K}_1(A)$, this allows the identification of $V^S_{2i+1}(A)$ with the stable unitary group of S-equivalences

$$H_{\pm}(m\underset{\sim}{A}) \longrightarrow H_{\pm}(m\underset{\sim}{A}) \qquad (m > 0)$$

modulo the subgroup generated by those of the type

i) $\begin{pmatrix} f & 0 \\ 0 & f^{*-1} \end{pmatrix}$ where $\tau(f:mA \longrightarrow mA) \in S$

ii) $\begin{pmatrix} 1 & \theta \mp \theta^* \\ 0 & 1 \end{pmatrix}$ for any $\mp$form (mA^*, θ)

iii) $\sigma \oplus \sigma \oplus \ldots \oplus \sigma$ with m copies of

$$\sigma = \begin{pmatrix} 0 & \pm\gamma^{-1} \\ \gamma & 0 \end{pmatrix} : A \oplus A^* \longrightarrow A \oplus A^*$$

where $\gamma : A \longrightarrow A^*; a \longmapsto (b \mapsto b\bar{a})$.

This is the kind of definition adopted for the odd-dimensional L-groups in [9] and [10].

The exact sequence of Theorem 2.3 has a based analogue

<u>Theorem 3.3</u> <u>Given *-invariant subgroups</u> $S \subseteq S' \subsetneq K'(A)$, <u>there is defined an exact sequence of abelian groups</u>

$$\ldots \longrightarrow H^{n+1}(S'/S) \rightarrow V^S_n(A) \xrightarrow{1} V^{S'}_n(A) \xrightarrow{\tau} H^n(S'/S) \longrightarrow \ldots$$

<u>with</u>

$$H^{n+1}(S'/S) \longrightarrow V^S_n(A); (Q, \underset{\sim}{f}, \underset{\sim}{g}) \longmapsto \begin{cases} (\underset{\sim}{Q} \oplus \underset{\approx}{Q}^*, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}) \\ (H_{\pm}(Q); \underset{\sim}{Q}, \underset{\approx}{Q}) \end{cases} \quad \text{if } n = \begin{cases} 2i \\ 2i+1 \end{cases}$$

<u>where</u> $\underset{\sim}{Q}$ <u>is</u> Q <u>with base</u> $\underset{\sim}{f}$, <u>and</u> $\underset{\approx}{Q}$ <u>is</u> Q <u>with base</u> $\underset{\sim}{g}$.

[ ]

This is the exact sequence of Theorem 3 of [10].

Following I.,II. denote the groups $\begin{cases} V^{\widetilde{K}_1(A)}_n(A) \\ V^{\{0\}}_n(A) \end{cases}$ by $\begin{cases} V_n(A) \\ W_n(A) \end{cases}$.

It is possible to identify

$$V^{K'(A)}_n(A) = U^{\{0\}}_n(A)$$

Thus if f.g.free A-modules have a well-defined rank (that is,

$\ker(K_0(Z) \to K_0(A)) = \{0\}$), then

$$U_n^{\{0\}}(A) = V_n(A) \quad .$$

Otherwise, Theorem 3.3 gives exact sequences

$$0 \longrightarrow V_{2i+1}(A) \longrightarrow U_{2i+1}^{\{0\}}(A) \longrightarrow Z_2 \longrightarrow V_{2i}(A) \longrightarrow U_{2i}^{\{0\}}(A) \longrightarrow 0$$

for i(mod 2).

§ 4. K-theory of twisted Laurent extensions

The purpose of this section is to recall those K-theoretic definitions and results from [2], [7] and II. which will be needed in this part.

The Laurent extension of A, A_z, is the ring of polynomials $\sum_{j=-\infty}^{\infty} z^j a_j$ in an indeterminate z and its inverse z^{-1}, with coefficients $a_j \in A$ and $\{ j \in Z \mid a_j \neq 0 \}$ finite. Addition is by

$$(\sum_{j=-\infty}^{\infty} z^j a_j) + (\sum_{k=-\infty}^{\infty} z^k b_k) = (\sum_{l=-\infty}^{\infty} z^l (a_l + b_l)) \in A_z$$

and multiplication by

$$(\sum_{j=-\infty}^{\infty} z^j a_j)(\sum_{k=-\infty}^{\infty} z^k b_k) = \sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} z^{j+k} a_j b_k \in A_z \quad .$$

There is defined an involution on A_z, by

$$\overline{(\sum_{j=-\infty}^{\infty} z^j a_j)} = \sum_{j=-\infty}^{\infty} z^j \bar{a}_{-j} \in A_z \ .$$

Then A_z is an associative ring with 1 and involution, thus satisfying the conditions imposed on A in §1 above.

The functions

$$\bar{\varepsilon} : A \longrightarrow A_z \ ; \ a \longmapsto a$$

$$\varepsilon : A_z \longrightarrow A \ ; \ \sum_{j=-\infty}^{\infty} z^j a_j \longmapsto \sum_{j=-\infty}^{\infty} a_j$$

are 1-preserving morphisms of rings with involution, such that ε splits $\bar{\varepsilon}$,

$$\varepsilon \bar{\varepsilon} = 1 \ : \ A \longrightarrow A \ .$$

Given an automorphism

$$\alpha : A \longrightarrow A$$

(preserving 1 and the involution), define the <u>α-twisted Laurent extension of A</u>, A_α , to be the associative ring with the elements and additive structure of A_z , but multiplication by

$$z^{-1}az = \alpha(a) \in A_\alpha \qquad (a \in A) \ .$$

The involution defined above for A_z is also an involution of A_α . Thus A_α satisfies the conditions imposed on A in §1 . Note that A_z is the special case $A_{1:A\to A}$.

The inclusion

$$\bar{\varepsilon} : A \longrightarrow A_\alpha ; a \longmapsto a$$

is a morphism of rings with involution, though not in general split.

Given $Q \in |\mathcal{P}(A)|$ define $zQ \in |\mathcal{P}(A)|$ by writing z in front of each element of Q, defining addition by

$$zx + zy = z(x+y) \in zQ \qquad (x,y \in Q)$$

and an A-action by

$$A \times zQ \longrightarrow zQ ; \ (a,zx) \longmapsto z\alpha(a)x \ .$$

Then

$$\alpha : K_0(A) \longrightarrow K_0(A) \ ; \ [Q] \longmapsto [zQ] \ .$$

Given $f \in \mathrm{Hom}_A(P,Q)$, define $zf \in \mathrm{Hom}_A(zP,zQ)$ by

$$zf : zP \longrightarrow zQ \ ; \ zx \longmapsto zf(x) \ .$$

Then

$$\alpha : K_1'(A) \longrightarrow K_1'(A) \ ; \ \tau(f:\underset{\sim}{P} \longrightarrow \underset{\sim}{Q}) \longmapsto \tau(zf:z\underset{\sim}{P} \longrightarrow z\underset{\sim}{Q}) \ .$$

Given $Q \in |\mathcal{P}(A)|$ define $Q_\alpha \in |\mathcal{P}(A_\alpha)|$ by extending the action of A on the abelian group

$$Q_\alpha = \sum_{j=-\infty}^{\infty} z^j Q$$

to one of A_α by

$$(z^k a)(z^j x) = z^{j+k}\alpha^j(a)x \in Q_\alpha \quad (a \in A, x \in Q, \ j,k \in \mathbb{Z}) \ .$$

Then

$$\bar{\varepsilon} : K_0(A) \longrightarrow K_0(A_\alpha) ; \ [Q] \longmapsto [Q_\alpha] \ .$$

Given $f \in \mathrm{Hom}_A(P,Q)$ define $f_\alpha \in \mathrm{Hom}_{A_\alpha}(P_\alpha,Q_\alpha)$ by

$$f_\alpha : P_\alpha \longrightarrow Q_\alpha ; \sum_{j=-\infty}^{\infty} z^j x_j \longmapsto \sum_{j=-\infty}^{\infty} z^j f(x_j) .$$

Then

$$\bar{\varepsilon} : K'(A) \longrightarrow K'(A_\alpha) ; \tau(f:\underset{\sim}{P} \to \underset{\sim}{Q}) \mapsto \tau(f_\alpha : \underset{\sim}{P}_\alpha \to \underset{\sim}{Q}_\alpha) .$$

A <u>modular A-base</u> of an A_α-module Q is an A-submodule Q_o of Q such that every $x \in Q$ has a unique expression as

$$x = \sum_{j=-\infty}^{\infty} z^j x_j \qquad (x_j \in Q_o) \qquad .$$

If $Q \in |\mathcal{P}(A_\alpha)|$ has a modular A-base Q_o , then $Q_o \in |\mathcal{P}(A)|$, and it is possible to identify

$$Q = (Q_o)_\alpha .$$

Given $Q_o \in |\mathcal{P}(A)|$ define complementary A-submodules

$$Q_o^+ = \sum_{j=0}^{\infty} z^j Q_o \qquad Q_o^- = \sum_{j=-\infty}^{-1} z^j Q_o$$

in $Q = (Q_o)_\alpha$. If F,G are modular A-bases of Q then

$$z^N F^+ \subseteq G^+$$

for sufficiently large integers $N \geq 0$ .For such N define the A-module

$$B_N(F,G) = z^N F^- \cap G^+ ,$$

and observe that there is a sum formula

$$B_{M+N}(F,H) = z^M B_N(F,G) \oplus B_M(G,H) .$$

This shows that each $B_N(F,G)$ is a f.g.projective A-module, with

$$B_N(F,G) \oplus z^{-N_1} B_{N_1}(G,F) = \sum_{j=-N_1}^{N-1} z^j F \quad ,$$

and also that

$$B : K_1(A_\alpha) \longrightarrow K_0(A); \tau(f:G_\alpha \to G_\alpha) \mapsto [B_N(F,G)] - [\sum_{j=0}^{N-1} z^j F]$$

is a well-defined morphism, where $F = f(G)$.

Recall from §8 of [7] the definition of the group $K(A,\alpha)$. Consider pairs

$$(P \in |\mathcal{P}(A)|, f \in \mathrm{Hom}_A(P,zP) \text{ isomorphism})$$

under the equivalence relation

$$(P,f) \sim (P',f') \text{ if there exists an isomorphism } g \in \mathrm{Hom}_A(P,P')$$
$$\text{such that } \tau(g^{-1} f'^{-1} (zg) f : P \longrightarrow P) = 0 \in K_1(A).$$

Then $K(A,\alpha)$ is the abelian group with one generator $[P,f]$ for each equivalence class of pairs (P,f), under the relations

$$[P,f]\oplus[P',f'] = [P\oplus P',f\oplus f'] \ .$$

Given a based A-module $\underset{\sim}{Q}$, define $[Q,\xi] \in K(A,\alpha)$ by

$$\xi : Q \longrightarrow zQ; \ \sum_{i=1}^{n} a_i q_i \longmapsto \sum_{i=1}^{n} z\alpha(a_i)q_i \qquad (a_i \in A)$$

with $\underset{\sim}{q} = (q_1,\ldots,q_n)$ the given base of Q.

The exact sequence of Theorem 9.2 of [7] can be extended to the right by one term, to give

<u>Lemma 4.1 The sequence of abelian groups</u>

$$K_1(A) \xrightarrow{1-\alpha} K_1(A) \xrightarrow{j} K(A,\alpha) \xrightarrow{p} K_0(A) \xrightarrow{1-\alpha} K_0(A) \xrightarrow{\bar{\varepsilon}} K_0(A_\alpha)$$

<u>is exact, where</u>

$$j : K_1(A) \longrightarrow K(A,\alpha); \tau(f:\underset{\sim}{G}\longrightarrow\underset{\sim}{G}) \longmapsto [G,\xi f] - [G,\xi]$$

$$p : K(A,\alpha) \longrightarrow K_0(A); \ [P,f] \longmapsto [P]$$

<u>Proof</u>: Use the A_α-module isomorphisms

$$Q_\alpha \longrightarrow (zQ)_\alpha \ ; \quad \sum_{j=-\infty}^{\infty} z^j x_j \longmapsto \sum_{j=-\infty}^{\infty} z^{j-1}(zx_j)$$

to identify

$$Q_\alpha = (zQ)_\alpha \in |\mathcal{P}(A_\alpha)| \qquad (Q \in |\mathcal{P}(A)|) \ .$$

It follows that the composite

$$K_0(A) \xrightarrow{1-\alpha} K_0(A) \xrightarrow{\bar{\varepsilon}} K_0(A_\alpha)$$

is zero.

Given $[G] - [F] \in \ker(\bar{\varepsilon}:K_0(A)\longrightarrow K_0(A_\alpha))$, stabilize F and G until there is defined an isomorphism

$$(F_\alpha \longrightarrow G_\alpha) \in \mathcal{P}(A_\alpha) \ .$$

The identity

$$B_{N+1}(F,G) = z^N F \oplus B_N(F,G) = zB_N(F,G) \oplus G$$

shows that

$$[G] - [F] = (1-\alpha)([B_N(F,G)] - [\sum_{j=0}^{N-1} z^j F\ ]) \in \operatorname{im}(1-\alpha : K_0(A)\rightarrow K_0(A)).$$

[ ]

Defining a duality involution

$$*:K(A,\alpha) \longrightarrow K(A,\alpha);[P,f] \longmapsto -[P^*,f^{*-1}] ,$$

note that

$$j* = *j : K_1(A) \longrightarrow K(A,\alpha)$$

$$p* = -*p : K(A,\alpha) \longrightarrow K_0(A) \quad .$$

As in §12 of [7], it is possible to combine the results of [2] and [7] to obtain

<u>Theorem 4.2</u> <u>There is a natural direct sum decomposition</u>

$$K_1(A_\alpha) = K(A,\alpha) \oplus Nil_+(A,\alpha) \oplus Nil_-(A,\alpha)$$

<u>where</u> $Nil_+(A,\alpha) = \{\tau(1+z^{\pm 1}\nu : P_\alpha \to P_\alpha) \mid \nu \in Hom_Z(P,P) \text{ nilpotent}, z\nu \in Hom_A(P,zP)\}$.
<u>The inclusion</u>

$$i : K(A,\alpha) \longrightarrow K_1(A_\alpha) \; ; \; [P,f] \longrightarrow \tau(f_\alpha : P_\alpha \longrightarrow (zP)_\alpha = P_\alpha)$$

<u>is split by</u>

$$q : K_1(A_\alpha) \longrightarrow K(A,\alpha) \; ;$$
$$\tau(f:\underset{\sim}{G}_\alpha \to \underset{\sim}{G}_\alpha) \longmapsto [B_{N+1}(F,G),t] - [\sum_{k=0}^{N} z^k F, \xi]$$

<u>where</u> $\underset{\sim}{F} = f(\underset{\sim}{G})$ <u>and</u>

$$t = 1 \oplus \xi^{N+1} f : B_{N+1}(F,G) = zB_N(F,G) \oplus G \longrightarrow zB_N(F,G) \oplus z^{N+1}F = zB_{N+1}(F,G).$$

<u>The duality involution</u>

$$* : K_1(A_\alpha) \longrightarrow K_1(A_\alpha)$$

<u>is such that</u>

$$i* = *i : K(A,\alpha) \longrightarrow K_1(A_\alpha)$$

$$q* = *q : K_1(A_\alpha) \longrightarrow K(A,\alpha) \quad ,$$

<u>and interchanges</u> $Nil_+(A,\alpha)$, $Nil_-(A,\alpha)$.

<u>In the untwisted case</u>, $\alpha = 1 : A \longrightarrow A$, <u>there are defined morphisms</u>

$$\bar{p} : K_0(A) \longrightarrow K(A,1) \; ; \; [P] \longmapsto [P,z]$$
$$\bar{j} : K(A,1) \longrightarrow K_1(A) \; ; \; [P,f] \longmapsto \tau(z^{-1}f : P \longrightarrow P)$$

<u>such that</u>

$$K_1(A) \underset{\bar{j}}{\overset{j}{\rightleftarrows}} K(A,1) \underset{\bar{p}}{\overset{p}{\rightleftarrows}} K_0(A)$$

is a direct sum system.

[ ]

Note that

$$ij = \vec{\varepsilon} : K_1(A) \longrightarrow K_1(A_\alpha)$$

$$pq = B : K_1(A_\alpha) \longrightarrow K_0(A)$$

with j,p as in Lemma 4.1, and that in the untwisted case

$$i\bar{p} = B : K_0(A) \longrightarrow K_1(A_z) ; \ [P] \longmapsto \tau(z : P_z \to P_z)$$

$$\bar{j}q = (\varepsilon 000) : K_1(A_z) = \bar{\varepsilon}K_1(A) \oplus \bar{B}K_0(A) \oplus Nil_+(A,1) \oplus Nil_-(A,1) \longrightarrow K_1(A)$$

in the untwisted case.

The relation

$$B^* = -{*B} : K_1(A_\alpha) \longrightarrow K_0(A)$$

can be obtained directly, from the A-module isomorphism

$$B_N(F^*,G^*) \longrightarrow B_N(F,G)^* \ ; \ f \mapsto (x \mapsto [f(x)]_0) \ ,$$

where $[a]_0 = a_0 \in A$ if $a = \sum_{j=-\infty}^{\infty} z^j a_j \in A_\alpha$.

Giving Z the identity involution, define a morphism of rings with involution

$$Z_z \longrightarrow A_\alpha \ ; \sum_{j=-\infty}^{\infty} z^j n_j \longmapsto \sum_{j=-\infty}^{\infty} z^j n_j . 1 \quad ,$$

and define reduced groups

$$\tilde{K}(A,\alpha) = \text{coker}(K(Z,1) \longrightarrow K(A,\alpha))$$

$$\tilde{\tilde{K}}_1(A_\alpha) = \text{coker}(K_1(Z_z) \longrightarrow K_1(A_\alpha)) \quad .$$

From now on we shall assume that A_α is such that f.g.free A_α-modules have a well-defined rank .

It follows that A also has this property. Lemma 4.1 gives an exact sequence

$$\tilde{K}_1(A) \xrightarrow{1-\alpha} \tilde{K}_1(A) \xrightarrow{j} \tilde{K}(A,\alpha) \xrightarrow{p} \tilde{K}_0(A) \xrightarrow{1-\alpha} \tilde{K}_0(A) \xrightarrow{\bar{\varepsilon}} \tilde{K}_0(A_\alpha)$$

in the reduced groups. Theorem 4.2 gives a direct sum decomposition

$$\tilde{\tilde{K}}_1(A_\alpha) = \tilde{K}(A,\alpha) \oplus \mathrm{Nil}_+(A,\alpha) \oplus \mathrm{Nil}_-(A,\alpha) \quad .$$

Convention: Given a $*$-invariant subgroup $S \subseteq \tilde{K}(A,\alpha)$ let

$$R = j^{-1}(S) \subseteq \tilde{K}_1(A) \quad , \quad T = p(S) \subseteq \tilde{K}_0(A) \quad .$$

Then $R \subseteq \tilde{K}_1(A)$, $T \subseteq \tilde{K}_0(A)$ are $*$-invariant subgroups.

Theorem 4.3 Given $*$-invariant subgroups $S \subseteq S' \subseteq \tilde{K}(A,\alpha)$, there is defined an exact sequence of Tate cohomology groups

$$\ldots \longrightarrow H^n(R'/R) \xrightarrow{\bar{\varepsilon}} H^n(S'/S) \xrightarrow{B} H^{n-1}(T'/T) \xrightarrow{C} H^{n-1}(R'/R) \rightarrow \ldots$$

with $\bar{\varepsilon}$, B induced by j, p respectively and C the connecting morphism,

$$C : H^n(T'/T) \longrightarrow H^n(R'/R) \;;\; [x] \longmapsto [j^{-1}(y+(-)^n y^*)]$$

for any $y \in S'/S$ such that $p(y) = x \in T'/T$, associated with the short exact sequence

$$0 \rightarrow R'/R \xrightarrow{j} S'/S \xrightarrow{p} T'/T \rightarrow 0 \quad .$$

In the untwisted case $\alpha = 1 : A \rightarrow A$, with

$$S = j(R) \oplus \bar{p}(T) \quad , \quad S' = j(R') \oplus \bar{p}(T') \subseteq \tilde{K}(A,1) = j\tilde{K}_1(A) \oplus \bar{p}\tilde{K}_0(A) ,$$

there is defined a direct sum system

$$H^n(R'/R) \underset{\varepsilon}{\overset{\bar{\varepsilon}}{\rightleftarrows}} H^n(S'/S) \underset{\bar{B}}{\overset{B}{\rightleftarrows}} H^{n-1}(T'/T) \quad .$$

[ ]

§5. L-theory of twisted Laurent extensions

Theorem 5.1 Given a $*$-invariant subgroup $T \subseteq \tilde{K}_0(A)$, there is defined an exact sequence of abelian groups

$$\ldots \longrightarrow U_n^T(A) \xrightarrow{\bar{\varepsilon}} U_n^{\bar{\varepsilon}T}(A_\alpha) \xrightarrow{B} U_{n-1}^{(1-\alpha)^{-1}T}(A) \xrightarrow{C} U_{n-1}^T(A) \longrightarrow \ldots$$

in a natural way.

The exact sequences associated with $*$-invariant subgroups $T \subseteq T' \subseteq \tilde{K}_0(A)$ combine with the exact sequence of Theorem 2.3 and the Tate cohomology of the short exact sequence

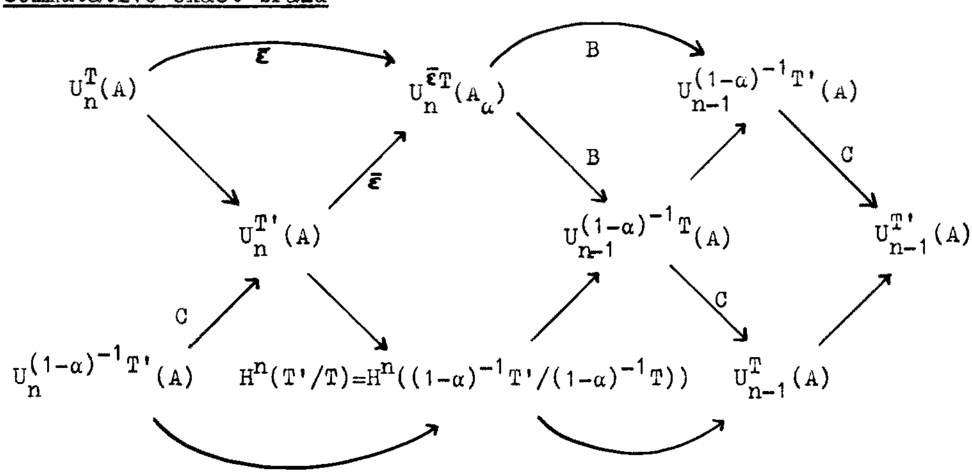
$$0 \rightarrow (1-\alpha)^{-1}T'/(1-\alpha)^{-1}T \xrightarrow{1-\alpha} T'/T \xrightarrow{\bar{\varepsilon}} \bar{\varepsilon}T'/\bar{\varepsilon}T \rightarrow 0 \quad ,$$

to define a commutative diagram

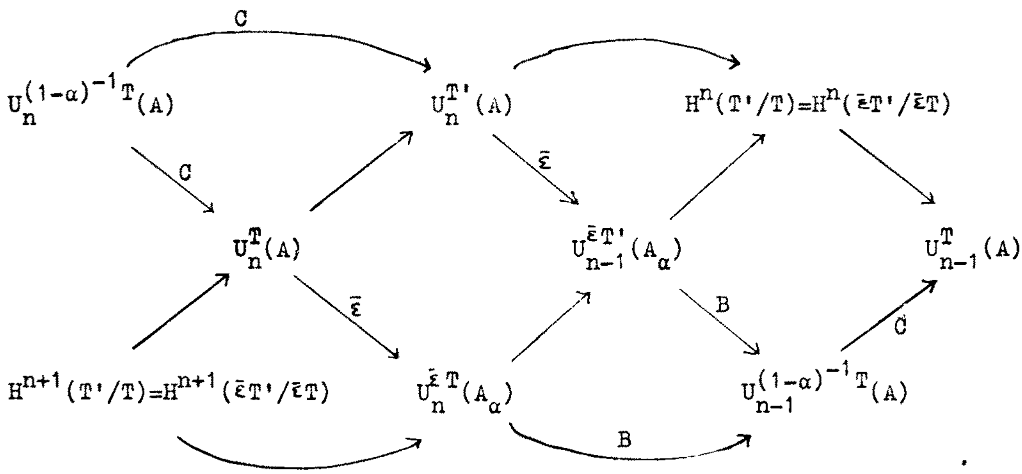
$$\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \vdots & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots \to & & H^{n+1}(T'/T) & \to & H^{n+1}(\bar{\varepsilon}T'/\bar{\varepsilon}T) & \to & H^{n}((1-\alpha)^{-1}T'/(1-\alpha)^{-1}T) & \to & H^{n}(T'/T) & \to \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots \to & & U_n^{T}(A) & \xrightarrow{\bar{\varepsilon}} & U_n^{\bar{\varepsilon}T}(A_\alpha) & \xrightarrow{B} & U_{n-1}^{(1-\alpha)^{-1}T}(A) & \xrightarrow{C} & U_{n-1}^{T}(A) & \to \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots \to & & U_n^{T'}(A) & \xrightarrow{\bar{\varepsilon}} & U_n^{\bar{\varepsilon}T'}(A_\alpha) & \xrightarrow{B} & U_{n-1}^{(1-\alpha)^{-1}T'}(A) & \xrightarrow{C} & U_{n-1}^{T'}(A) & \to \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots \to & & H^{n}(T'/T) & \to & H^{n}(\bar{\varepsilon}T'/\bar{\varepsilon}T) & \to & H^{n-1}((1-\alpha)^{-1}T'/(1-\alpha)^{-1}T) & \to & H^{n-1}(T'/T) & \to \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \vdots & & \vdots & & \vdots & & \vdots & &
\end{array}$$

<u>with exact rows and columns.</u>

<u>If</u> $\bar{\varepsilon}T = \bar{\varepsilon}T' \subseteq \tilde{K}_0(A_\alpha)$, <u>the sequences interlock in a commutative exact braid</u>



If $(1-\alpha)^{-1}T = (1-\alpha)^{-1}T' \subseteq \tilde{K}_0(A)$, *the sequences interlock in a braid*



.

[ ]

(As Wall points out, in a letter of 19th January 1973, these braids are a formal consequence of the larger diagram drawn above.)

Let S_0 be the infinite cyclic subgroup of $\tilde{K}_1(A_\alpha)$ generated by $\tau(\xi:A_\alpha \longrightarrow A_\alpha)$.

Given a $*$-invariant subgroup $R \subseteq \tilde{K}_1(A)$ let

$$\tilde{V}_n^{\bar{\varepsilon}R}(A_\alpha) = V_n^{\bar{\varepsilon}R \oplus S_0}(A_\alpha) \quad ,$$

and denote $V_n^{S_0}(A_\alpha)$ by $\tilde{W}_n(A_\alpha)$. Theorem 3.3 gives an exact sequence

$$0 \longrightarrow V_{2i+1}^{\bar{\varepsilon}R}(A_\alpha) \longrightarrow \tilde{V}_{2i+1}^{\bar{\varepsilon}R}(A_\alpha) \longrightarrow Z_2 \longrightarrow V_{2i}^{\bar{\varepsilon}R}(A_\alpha) \longrightarrow \tilde{V}_{2i}^{\bar{\varepsilon}R}(A_\alpha) \longrightarrow 0$$

for i(mod 2).

By analogy with Theorem 5.1 we have:

Theorem 5.2 *Given a $*$-invariant subgroup* $R \subseteq \tilde{K}_1(A)$ *there is defined an exact sequence of abelian groups*

$$\ldots \longrightarrow V_n^R(A) \xrightarrow{\bar{\varepsilon}} \tilde{V}_n^{\bar{\varepsilon}R}(A_\alpha) \xrightarrow{B} V_{n-1}^{(1-\alpha)^{-1}R}(A) \xrightarrow{C} V_{n-1}^R(A) \longrightarrow \ldots$$

[ ]

with similar naturality and exactness properties.

Given a $*$-invariant subgroup $S \subseteq \tilde{K}(A,\alpha)$, let

$$\tilde{V}_n^S(A_\alpha) = V_n^{\tilde{S}}(A_\alpha)$$

where

$$\tilde{S} = q^{-1}(S) \subseteq \tilde{K}_1(A_\alpha) \quad ,$$

with the projection

$$q : \tilde{K}_1(A_\alpha) \longrightarrow \tilde{K}(A,\alpha)$$

defined as in Theorem 4.2.

The exact sequence of Theorem 3.3 for $\tilde{S} \subseteq \tilde{S}' \subseteq \tilde{K}_1(A_\alpha)$ can be written as

$$\ldots \longrightarrow H^{n+1}(S'/S) \longrightarrow \tilde{V}_n^S(A_\alpha) \longrightarrow \tilde{V}_n^{S'}(A_\alpha) \longrightarrow H^n(S'/S) \longrightarrow \ldots \quad ,$$

using the isomorphism

$$q : \tilde{S}'/\tilde{S} \longrightarrow S'/S$$

to identify

$$H^n(\tilde{S}'/\tilde{S}) = H^n(S'/S) \quad .$$

In particular,

$$\tilde{V}_n^S(A_\alpha) = \begin{cases} V_n(A_\alpha) \\ \tilde{V}_n^{\bar{\varepsilon}R}(A_\alpha) \end{cases} \quad \text{if} \quad S = \begin{cases} \tilde{K}(A,\alpha) \\ j(R) \quad (R \subseteq \tilde{K}_1(A)) . \end{cases}$$

<u>Theorem 5.3 Given a $*$-invariant subgroup</u> $S \subseteq \tilde{K}(A,\alpha)$ <u>there is defined an exact sequence of abelian groups</u>

$$\ldots \longrightarrow V_n^R(A) \xrightarrow{\bar{\varepsilon}} \tilde{V}_n^S(A_\alpha) \xrightarrow{B} U_{n-1}^T(A) \xrightarrow{C} V_{n-1}^R(A) \longrightarrow \ldots$$

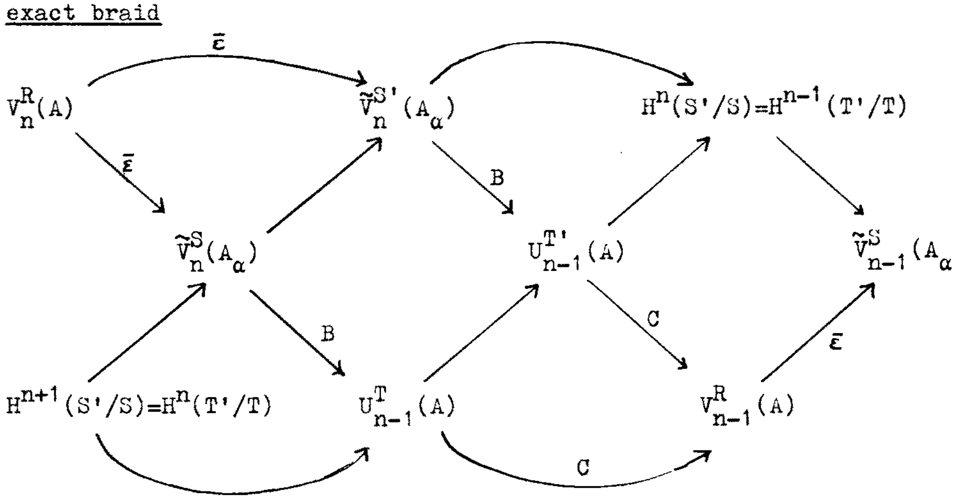
<u>in a natural way, with</u> $R = j^{-1}(S) \subseteq \tilde{K}_1(A)$, $T = p(S) \subseteq \tilde{K}_0(A)$.

<u>The exact sequences associated with $*$-invariant subgroups</u> $S \subseteq S' \subseteq \tilde{K}(A,\alpha)$ <u>and the exact sequences of Theorems 2.3,3.3,4.3 combine, to give a commutative diagram</u>

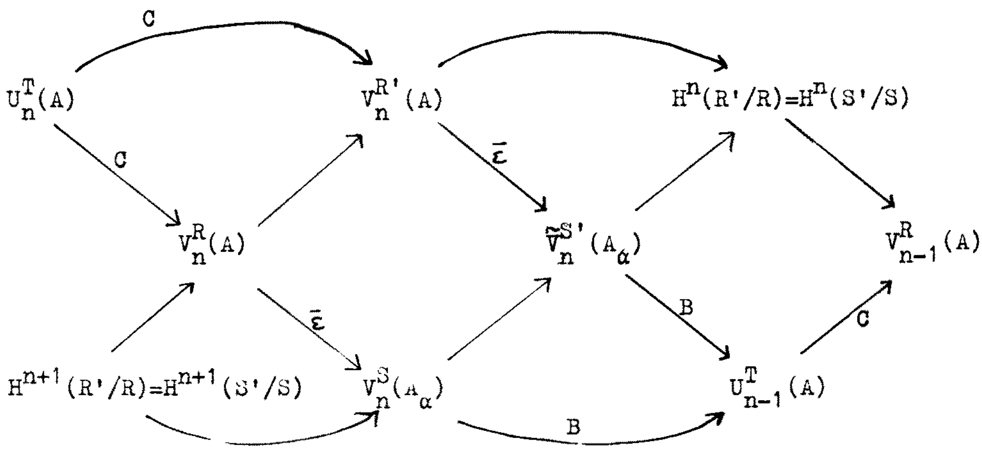
$$
\begin{array}{ccccccccccc}
 & & \vdots & & \vdots & & \vdots & & \vdots & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & H^{n+1}(R'/R) & \xrightarrow{\bar{\varepsilon}} & H^{n+1}(S'/S) & \xrightarrow{B} & H^{n}(T'/T) & \xrightarrow{C} & H^{n}(R'/R) & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & V_n^R(A) & \xrightarrow{\bar{\varepsilon}} & \tilde{V}_n^S(A_\alpha) & \xrightarrow{B} & U_{n-1}^T(A) & \xrightarrow{C} & V_{n-1}^R(A) & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & V_n^{R'}(A) & \xrightarrow{\bar{\varepsilon}} & \tilde{V}_n^{S'}(A_\alpha) & \xrightarrow{B} & U_{n-1}^{T'}(A) & \xrightarrow{C} & V_{n-1}^{R'}(A) & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & H^{n}(R'/R) & \xrightarrow{\bar{\varepsilon}} & H^{n}(S'/S) & \xrightarrow{B} & H^{n-1}(T'/T) & \xrightarrow{C} & H^{n-1}(R'/R) & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \vdots & & \vdots & & \vdots & & \vdots & &
\end{array}
$$

<u>with exact rows and columns</u>.

<u>If</u> $R = R' \subseteq \tilde{K}_1(A)$, <u>the sequences interlock in a commutative exact braid</u>



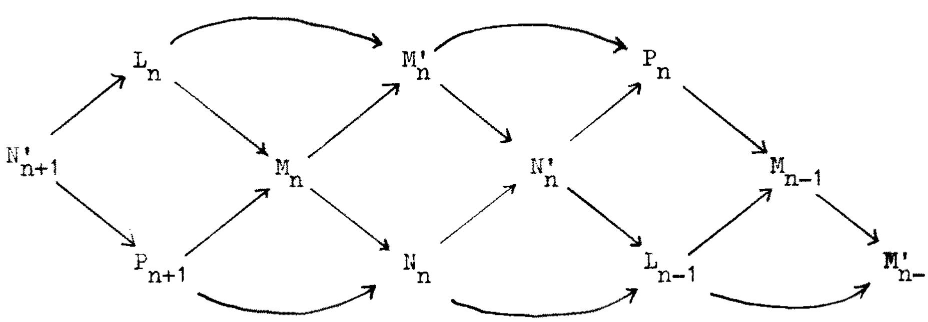
<u>If</u> $T = T' \subseteq \tilde{K}_0(A)$, <u>the sequences interlock in a commutative exact braid</u>



[ ]

In proving the exactness of the sequences of Theorems 5.1,5.2,5.3 (in §6, below) we shall make much use of the following version of Theorem 1 of [8].

<u>Lemma 5.4 Suppose given a commutative diagram of abelian groups and morphisms</u>



<u>such that the sequences</u>

$$P_{n+1} \longrightarrow M_n \longrightarrow M'_n \longrightarrow P_n \longrightarrow M_{n-1} \longrightarrow M'_{n-1}$$

$$N'_{n+1} \longrightarrow P_{n+1} \longrightarrow N_n \longrightarrow N'_n \longrightarrow P_n$$

<u>are exact</u>.

<u>If the composites of successive morphisms in the sequences</u>

$$L_n \longrightarrow M_n \longrightarrow N_n \longrightarrow L_{n-1} \longrightarrow M_{n-1} \qquad (*)$$

$$L_n \longrightarrow M'_n \longrightarrow N'_n \longrightarrow L_{n-1} \longrightarrow M'_{n-1} \qquad (**)$$

<u>are zero, then (*) is exact at</u> M_n (<u>resp.</u> N_n, L_{n-1}) <u>if and only if</u> <u>(**) is exact at</u> M'_n (<u>resp.</u> N'_n, L_{n-1}) .

[ ]

Assuming that the morphisms in the sequences of Theorems 5.1,5.2,5.3 have already been defined, and are such that the composites of successive ones are zero, and that all the braids are indeed commutative, it follows from Lemma 5.4 that the exactness of the sequences for all the coefficient groups T, R, S(but keeping A and α fixed) is related as for (*), (**).

To see this, note first that for any $*$-invariant subgroup $T \subseteq \widetilde{K}_0(A)$ the exactness of the sequences of Theorem 5.1 for T, and $T \cap (1-\alpha)\widetilde{K}_0(A)$ is related (since

$$(1-\alpha)^{-1}T = (1-\alpha)^{-1}(T \cap (1-\alpha)\widetilde{K}_0(A)) \subseteq \widetilde{K}_0(A) \quad) ,$$

as is that for $T \cap (1-\alpha)\widetilde{K}_0(A)$, $\{0\}$ (since

$$\widetilde{\varepsilon}(T \cap (1-\alpha)\widetilde{K}_0(A)) = \{0\} \subseteq \widetilde{K}_0(A_\alpha) \;).$$

Hence the exactness of the sequences for any two $*$-invariant subgroups $T, T' \subseteq \widetilde{K}_0(A)$ is related.

Similar considerations apply to the sequence of Theorem 5.2.

For any $*$-invariant subgroup $S \subseteq \widetilde{K}(A,\alpha)$ the exactness of the sequences of Theorem 5.3 for S , $S + j\widetilde{K}_1(A)$ is related (since

$$p(S) = p(S + j\widetilde{K}_1(A)) \subseteq \widetilde{K}_0(A) \quad) ,$$

as is that for $S + j\widetilde{K}_1(A)$, $\widetilde{K}(A,\alpha)$ (since

$$j^{-1}(S + j\widetilde{K}_1(A)) = j^{-1}(\widetilde{K}(A,\alpha)) = \widetilde{K}_1(A) \quad).$$

Hence the exactness of the sequences for any two $*$-invariant subgroups $S, S' \subseteq \widetilde{K}(A,\alpha)$ is related.

The sequence of Theorem 5.1 for $T = \{0\} \subseteq \tilde{K}_0(A)$

$$\cdots \longrightarrow V_n(A) \xrightarrow{\bar{\varepsilon}} V_n(A_\alpha) \xrightarrow{B} U_{n-1}^{\tilde{K}_0(A)^\alpha}(A) \xrightarrow{C} V_{n-1}(A) \longrightarrow \cdots$$

coincides with that of Theorem 5.3 for $S = \tilde{K}(A,\alpha)$ (or will be seen to do so, once both are defined).

The sequence of Theorem 5.2 for $R = \tilde{K}_1(A)$

$$\cdots \longrightarrow V_n(A) \xrightarrow{\bar{\varepsilon}} \tilde{V}_n^{\bar{\varepsilon}\tilde{K}_1(A)}(A_\alpha) \xrightarrow{B} V_{n-1}(A) \xrightarrow{C} V_{n-1}(A) \longrightarrow \cdots$$

coincides with that for Theorem 5.3 for $S = j\tilde{K}_1(A) \subseteq \tilde{K}(A,\alpha)$.

Hence the exactness of all the sequences is related.

In proving Theorems 5.1,5.2,5.3 (in §6, below) it will be left to the reader to verify that the definitions of the morphisms B, C are sufficiently natural for the commutativity of the diagrams drawn above (implicitly so for 5.2).

§6. Proof of theorems in §5.

Given a $*$-invariant subgroup $T \subseteq \tilde{K}_0(A)$, define

$$B : U_{2i+1}^{\bar{\varepsilon}T}(A_\alpha) \longrightarrow U_{2i}^{(1-\alpha)^{-1}T}(A) ; (Q,\varphi;F,G) \longmapsto (P,\theta)$$

where

$$(P,\theta) = (B_N(F_0 \oplus F_0^*, G_0 \oplus G_0^*), [\varphi]_0) \oplus H_\pm\Big(\sum_{j=0}^{N-1} z^j(-F_0)\Big)$$

for any modular A-bases F_0, G_0 of F, G such that

$$[G_0] - [F_0^*] \in T \subseteq \tilde{K}_0(A) \quad ,$$

with $-F_0$ any projective inverse for F_0 , and F_0^* , G_0^* the dual modular A-bases to F_0, G_0 in any hamiltonian complements F^*, G^* to F, G in (Q,φ) , with

$$[\varphi]_0 : Q \longrightarrow \mathrm{Hom}_A(Q,A) \; ; \; x \mapsto (y \longmapsto [\varphi(x)(y)]_0) \; ,$$

writing $[a]_0$ for $a_0 \in A$ if $a = \sum_{j=-\infty}^{\infty} z^j a_j \in A_\alpha$.

The identity

$$(B_N(F_0 \oplus F_0^*, G_0 \oplus G_0^*), [\varphi]_0) \oplus (z^{-N_1} B_{N_1}(G_0 \oplus G_0^*, F_0 \oplus F_0^*), [\varphi]_0)$$
$$= H\pm\Big(\sum_{j=-N_1}^{N-1} z^j F_0 \Big) \quad \text{(up to equivalence of } \pm\text{forms over A)}$$

shows that (P,θ) is a non-singular $\pm$form. The identity

$$B_{N+1}(F_0 \oplus F_0^*, G_0 \oplus G_0^*) = z^N(F_0 \oplus F_0^*) \oplus B_N(F_0 \oplus F_0^*, G_0 \oplus G_0^*)$$
$$= (G_0 \oplus G_0^*) \oplus z B_N(F_0 \oplus F_0^*, G_0 \oplus G_0^*)$$

shows that

$$(1-\alpha)[P] = [G_0 \oplus G_0^*] - [z^N(F_0 \oplus F_0^*)] + (1-\alpha)\Big[\sum_{j=0}^{N-1} z^j(-F_0 \oplus -F_0^*) \Big]$$
$$= ([G_0] - [F_0^*]) + ([G_0^*] - [F_0]) \in T \subseteq \widetilde{K}_0(A) \quad .$$

Hence $(P,\theta) \in U_{2i}^{(1-\alpha)^{-1}T}(A)$.

For $N \geq 0$ so large that

$$z^N F_0^+ \subseteq (G_0 \oplus G_0^*)^+$$

define a $\pm$form over A

$$(P',\theta') = (E_N(F_0, G_0 \oplus G_0^*)/z^N F_0^+ \ , \ [\varphi]_0) \oplus H\pm\Big(\sum_{j=0}^{N-1} z^j(-F_0)\Big)$$

where

$$E_N(F_0, G_0 \oplus G_0^*) = \{x \in (G_0 \oplus G_0^*)^+ \,|\, [\varphi \pm \varphi^*]_0(x)(z^N F_0^+) = \{0\} \subseteq A \ \} \quad .$$

Increasing N by 1 adds on

$$H\pm(z^N(F_0 \oplus -F_0^*)) = 0 \in U_{2i}^{(1-\alpha)^{-1}T}(A)$$

to (P',θ'), and for N so large that

$$z^N(F_0 \oplus F_0^*)^+ \subseteq (G_0 \oplus G_0^*)^+$$

the $\pm$forms (P,θ) , (P',θ') coincide, as then

$$E_N(F_0, G_0 \oplus G_0^*) = (F \oplus z^N F_0^{*-}) \cap (G_0 \oplus G_0^*)^+ = z^N F_0^+ \oplus P \quad .$$

Hence $(P,\theta) \in U_{2i}^{(1-\alpha)^{-1}T}(A)$ does not depend on the choice of N or of the hamiltonian complement F^* . The choice of G^* can be dealt with similarly.

If $(Q,\varphi;F,G) = 0 \in U_{2i+1}^{\bar{\varepsilon}T}(A_\alpha)$, it may be assumed that

$$(Q,\varphi;F,G) = (H_\pm(L);L,\Gamma_{(L,\lambda)}) \oplus (H_\pm(M);M,M^*)$$

with $[L],[M] \in \bar{\varepsilon}T$. Choosing

$$F_0 = L_0 \oplus M_0 \qquad (\text{with } [L]_0,[M]_0 \in T) \qquad F_0^* = L_0^* \oplus M_0^*$$

$$G_0^* = L_0^* \oplus M_0 \qquad (G_0^*)^* = L_0 \oplus M_0^* \quad (\text{in } Q) \quad ,$$

note that by symmetry of the definition of B with respect to the lagrangians and their hamiltonian complements

$$\begin{aligned} B(Q,\varphi;F,G) &= B(Q,\varphi;F,G^*) \\ &= (B_0(F_0 \oplus F_0^*, G_0 \oplus G_0^*),[\varphi]_0) = 0 \in U_{2i}^{(1-\alpha)^{-1}T}(A) \quad . \end{aligned}$$

It now only remains to verify that the choice of modular A-bases F_0,G_0 for F,G is immaterial to $(P,\theta) \in U_{2i}^{(1-\alpha)^{-1}T}(A)$.

Let F_0',G_0' be some other modular A-bases of F,G such that

$$[G_0'] - [F_0'^*] \in T \ .$$

Choose $N',N'' \geq 0$ so large that

$$z^{N'}(F_0' \oplus F_0'^*)^+ \subseteq (F_0 \oplus F_0^*)^+ \quad , \quad z^{N''}(G_0 \oplus G_0^*)^+ \subseteq (G_0' \oplus G_0'^*)^+ \quad ,$$

and let $M = N + N' + N''$. Then up to equivalence

$$\begin{aligned} &(B_M(F_0' \oplus F_0'^*, G_0' \oplus G_0'^*),[\varphi]_0) \\ &= H_\pm(z^{N+N''}B_{N'}(F_0'^*,F_0^*)) \oplus (z^{N''}B_N(F_0 \oplus F_0^*, G_0 \oplus G_0^*),[\varphi]_0) \oplus H_\pm(B_{N''}(G_0,G_0')) \, , \\ &(z^{N''}B_N(F_0 \oplus F_0^*, G_0 \oplus G_0^*),[\varphi]_0) \oplus H_\pm(\sum_{j=0}^{N''-1} z^j G_0) \\ &= (B_N(F_0 \oplus F_0^*, G_0 \oplus G_0^*),[\varphi]_0) \oplus H_\pm(\sum_{j=0}^{N''-1} z^{j+N} F_0^*) \quad . \end{aligned}$$

Now

$$\begin{aligned} &(1-\alpha)([z^{N+N''}B_{N'}(F_0'^*,F_0^*) \oplus (\sum_{j=0}^{N''-1} z^j(z^N F_0^* \oplus -G_0)) \oplus B_{N''}(G_0,G_0')]) \\ &- (1-\alpha)([\sum_{j=0}^{M-1} z^j F_0'^*] - [\sum_{j=0}^{N-1} z^j F_0^*]) = ([G_0']-[F_0'^*])-([G_0]-[F_0^*]) \in T \subseteq \tilde{K}_0(A) \end{aligned}$$

and so

$$(P',\theta') = (P,\theta) \in U_{2i}^{(1-\alpha)^{-1}T}(A) \quad ,$$

where

$$(P',\theta') = (B_M(F_0' \oplus F_0'^*, G_0' \oplus G_0'^*),[\varphi]_0) \oplus H_\pm(\sum_{j=0}^{M-1} z^j(-F_0'))$$

is defined as (P,θ) but with F_0',G_0',M replacing F_0,G_0,N respectively.

Hence

$$B:U^{\bar{\varepsilon}T}_{2i+1}(A_\alpha) \longrightarrow U^{(1-\alpha)^{-1}T}_{2i}(A);(Q,\varphi;F,G) \longmapsto (P,\theta)$$

is a well-defined morphism.

The composite

$$U^{T}_{2i+1}(A) \xrightarrow{\bar{\varepsilon}} U^{\bar{\varepsilon}T}_{2i+1}(A_\alpha) \xrightarrow{B} U^{(1-\alpha)^{-1}T}_{2i}(A)$$

is zero, sending $(Q,\varphi;F,G) \in U^{T}_{2i+1}(A)$ to

$$B\bar{\varepsilon}(Q,\varphi;F,G) = (B_0(F_0 \oplus F_0^*, G_0 \oplus G_0^*),[\varphi]_0) = 0 \in U^{(1-\alpha)^{-1}T}_{2i}(A) \ .$$

Define

$$C:U^{(1-\alpha)^{-1}T}_{2i}(A) \longrightarrow U^{T}_{2i}(A);(Q,\varphi) \longmapsto (Q,\varphi) \oplus \alpha(Q,-\varphi) \oplus H_{\pm}(-Q) \ .$$

This is well-defined because

$$CH_{\pm}(L) = H_{\pm}(L \oplus zL \oplus -L \oplus -L) = 0 \in U^{T}_{2i}(A) \text{ if } [L] \in (1-\alpha)^{-1}T.$$

The composite

$$U^{(1-\alpha)^{-1}T}_{2i}(A) \xrightarrow{C} U^{T}_{2i}(A) \xrightarrow{\bar{\varepsilon}} U^{\bar{\varepsilon}T}_{2i}(A_\alpha)$$

is zero, sending $(Q,\varphi) \in U^{(1-\alpha)^{-1}T}_{2i}(A)$ to

$$(Q_\alpha,\varphi_\alpha) \oplus (Q_\alpha,-\varphi_\alpha) \oplus H_{\pm}(-Q_\alpha) = H_{\pm}((Q \oplus -Q)_\alpha) = 0 \in U^{\bar{\varepsilon}T}_{2i}(A_\alpha)$$

The composite

$$U^{\bar{\varepsilon}T}_{2i+1}(A_\alpha) \xrightarrow{B} U^{(1-\alpha)^{-1}T}_{2i}(A) \xrightarrow{C} U^{T}_{2i}(A)$$

is zero, as is clear from the identity (valid up to equivalence)

$$\begin{aligned}(B_{N+1}(F_0 \oplus F_0^*, G_0 \oplus G_0^*),[\varphi]_0) &= (B_N(F_0 \oplus F_0^*, G_0 \oplus G_0^*),[\varphi]_0) \oplus H_{\pm}(z^N F_0) \\ &= \alpha(B_N(F_0 \oplus F_0^*, G_0 \oplus G_0^*),[\varphi]_0) \oplus H_{\pm}(G_0).\end{aligned}$$

<u>Lemma 6.1</u> <u>The sequence</u>

$$U^{T}_{2i+1}(A) \xrightarrow{\bar{\varepsilon}} U^{\bar{\varepsilon}T}_{2i+1}(A_\alpha) \xrightarrow{B} U^{(1-\alpha)^{-1}T}_{2i}(A) \xrightarrow{C} U^{T}_{2i}(A) \xrightarrow{\bar{\varepsilon}} U^{\bar{\varepsilon}T}_{2i}(A_\alpha)$$

<u>is exact for all $*$-invariant subgroups</u> $T \subseteq \vec{K}_0(A)$.

<u>Proof</u>: It has already been verified that the composite of successive morphisms in the sequence is zero. As explained in §5, it is therefore sufficient to consider exactness in the special case $T = \{0\} \subseteq \widetilde{K}_0(A)$,

$$V_{2i+1}(A) \xrightarrow{\bar{\varepsilon}} V_{2i+1}(A_\alpha) \xrightarrow{B} U_{2i}^{\tilde{K}_0(A)^\alpha}(A) \xrightarrow{C} V_{2i}(A) \xrightarrow{\bar{\varepsilon}} V_{2i}(A_\alpha)$$

where $\tilde{K}_0(A)^\alpha = \ker(1-\alpha:\tilde{K}_0(A)\longrightarrow\tilde{K}_0(A))$. (This use of Lemma 5.4 anticipates the definition of

$$C:U_{2i+1}^{(1-\alpha)^{-1}T}(A) \longrightarrow U_{2i+1}^{T}(A) \qquad B:U_{2i}^{\bar{\varepsilon}T}(A_\alpha) \longrightarrow U_{2i-1}^{(1-\alpha)^{-1}T}(A)$$

but no extra exactness properties).

Given $(Q,\varphi) \in \ker(\bar{\varepsilon}:V_{2i}(A)\longrightarrow V_{2i}(A_\alpha))$, it may be assumed that

$$\bar{\varepsilon}(Q,\varphi) = \bar{\varepsilon}H_{\pm}(L)$$

for some f.g.free A-module L. Then

$$(P,\theta) = (B_N(L\oplus L^*,Q),[\varphi]_0)$$

is a non-singular $\pm$form over A such that (up to equivalence)

$$(B_{N+1}(L\oplus L^*,Q),[\varphi]_0) = (Q,\varphi)\oplus\alpha(P,\theta) = (P,\theta)\oplus H_{\pm}(z^N L) .$$

Hence

$$(Q,\varphi) = C(P,\theta) \in \operatorname{im}(C:U_{2i}^{\tilde{K}_0(A)^\alpha}(A) \longrightarrow V_{2i}(A)) ,$$

and the sequence is exact at $V_{2i}(A)$.

Given $(Q,\varphi) \in \ker(C:U_{2i}^{\tilde{K}_0(A)^\alpha}(A)\longrightarrow V_{2i}(A))$, it may be assumed that

$$(Q,\varphi)\oplus\alpha(Q,-\varphi)\oplus\alpha H_{\pm}(-Q) = H_{\pm}(L)$$

for some f.g.free A-module L. Then

$$\begin{aligned}(Q,\varphi) &= (B_1(Q\oplus Q\oplus -Q\oplus -Q^*,L\oplus L^*),[\varphi_\alpha]_0)\\ &= B((Q_\alpha\oplus Q_\alpha,\varphi_\alpha\oplus -\varphi_\alpha)\oplus H_{\pm}(-Q_\alpha);\Delta_{(Q_\alpha,\varphi_\alpha)}\oplus -Q_\alpha , L_\alpha)\\ &\in \operatorname{im}(B:V_{2i+1}(A_\alpha) \longrightarrow U_{2i}^{\tilde{K}_0(A)^\alpha}(A)) ,\end{aligned}$$

verifying exactness at $U_{2i}^{\tilde{K}_0(A)^\alpha}(A)$.

Given $(Q,\varphi;F,G) \in \ker(B:V_{2i+1}(A_\alpha) \longrightarrow U_{2i}^{\tilde{K}_0(A)^\alpha}(A))$, it may be assumed that

$$(B_N(F_0\oplus F_0^*,G_0\oplus G_0^*),[\varphi]_0) = H_{\pm}(L) ,$$

with $[L] \in \tilde{K}_0(A)^\alpha$.

Let

$$P_0 = L \oplus L^* = B_N(F_0 \oplus F_0^*, G_0 \oplus G_0^*) \quad \in \quad |\mathcal{P}(A)|$$

and define an A_α-module morphism

$$f : P = (P_0)_\alpha \longrightarrow Q$$

by extending the inclusion of P_0 in Q . Let

$$(P, \psi) = \bar{\varepsilon} H_{\pm}(L)$$

and let

$$\theta : P \longrightarrow P^*$$

be the unique A_α-module morphism such that

$$f^*(\varphi \pm \varphi^*)f = \theta \pm \theta^* \in \mathrm{Hom}_{A_\alpha}(P, P^*) \qquad (\theta - \psi)(P_0) \subseteq \sum_{j=1}^{\infty} z^j P_0^* \quad .$$

Define A_α-module morphisms

$$\eta = \begin{pmatrix} 0 & \mp t_\alpha \\ t_\alpha^* & 0 \end{pmatrix} : P^* = L_\alpha^* \oplus L_\alpha \longrightarrow L_\alpha \oplus L_\alpha^* = P$$

$$\xi = 1 \oplus t_\alpha^{*-1} : P = L_\alpha \oplus L_\alpha^* \longrightarrow L_\alpha \oplus L_\alpha^*$$

for some isomorphism $t \in \mathrm{Hom}_A(L, zL)$.

Then

$$h_1 = 1 \oplus \begin{pmatrix} 1 & \eta \\ 0 & 1 \end{pmatrix} : (Q,\varphi) \oplus H_{\pm}(P) \longrightarrow (Q,\varphi) \oplus H_{\pm}(P)$$

$$h_2 = \begin{pmatrix} 1 & -f\xi & 0 \\ 0 & 1 & 0 \\ \xi^* f^*(\varphi \pm \varphi^*) & -\xi^*\theta\xi & 1 \end{pmatrix} : (Q,\varphi) \oplus H_{\pm}(P) \longrightarrow (Q,\varphi) \oplus H_{\pm}(P)$$

are self-equivalences (over A_α) such that h_1 preserves the lagrangian $F \oplus P$ of $(Q,\varphi) \oplus H_{\pm}(P)$ and h_2 preserves the lagrangian $F \oplus P^*$. It now follows from the sum formula of Lemma 3.2 that

$$(Q', \varphi'; F', G') = ((Q,\varphi) \oplus H_{\pm}(P); h(F \oplus P), G \oplus P) \quad (h = h_1 h_2)$$

is a $\pm$formation over A_α such that

$$\begin{aligned}(Q', \varphi'; F', G') &= ((Q,\varphi) \oplus H_{\pm}(P); F \oplus P, G \oplus P) \\ &= (Q, \varphi; F, G) \in V_{2i+1}(A_\alpha) \ .\end{aligned}$$

Define a modular A-base

$$G_0' = G_0 \oplus P_0$$

for G', giving the hamiltonian complement $G'^* = G^* \oplus P^*$ to G' in (Q',φ') the dual modular A-base

$$G_0'^* = G_0^* \oplus P_0^* \quad .$$

Let

$$Q_0' = G_0' \oplus G_0'^*$$

be the corresponding modular A-base for Q' .

The A-module morphism

$$\nu : Q' \longrightarrow Q' ; \sum_{j=-\infty}^{\infty} z^j x_j \longmapsto \sum_{j=0}^{\infty} z^j x_j \qquad (x_j \in Q_0')$$

is such that

$$\nu(F') \subseteq F'$$

because

$$\nu h(x,y) = \begin{cases} h(x,y) & \text{if } (x,y) \in z^N F_0^+ \oplus P_0^+ \\ h(0,\xi^{-1}\beta(f\xi(y)-x)) & \phantom{\text{if }} (x,y) \in z^N F_0^- \oplus P_0^- \end{cases}$$

where β is the projection

$$\beta = (\ 1 \quad 0\ ) : Q = P_0 \oplus (z^N(F_0 \oplus F_0^*)^+ \oplus (G_0 \oplus G_0^*)^-) \longrightarrow P_0 \quad .$$

It follows that each $x \in F'$ has a unique expression as

$$x = \sum_{j=-\infty}^{\infty} z^j x_j$$

with

$$x_j = z(1-\nu)z^{-1}\nu z^{-j} x \in F' \cap Q_0' \quad ,$$

and so

$$F_0' = F' \cap Q_0'$$

is a modular A-base for F'. (This is precisely the same argument as was used in the untwisted case, in § 2 of II.). Now

$$[F_0'] - [F_0 \oplus P_0] \in \ker(\bar{\varepsilon} : \widetilde{K}_0(A) \to \widetilde{K}_0(A_\alpha)) = \operatorname{im}(1-\alpha : \widetilde{K}_0(A) \to \widetilde{K}_0(A))$$

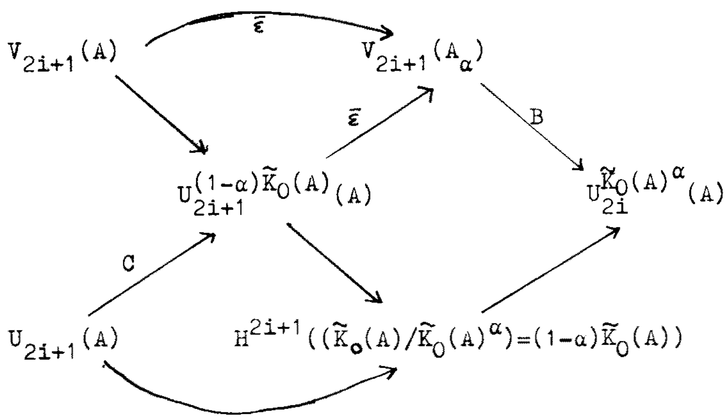
(by the reduced version of Lemma 4.1), so that

$$(Q,\varphi;F,G) = (Q',\varphi';F',G')$$
$$= \bar{\varepsilon}(H_{\pm}(G_0');F_0',G_0') \in \operatorname{im}(\bar{\varepsilon}:U_{2i+1}^{(1-\alpha)\tilde{K}_0(A)}(A) \longrightarrow V_{2i+1}(A_\alpha)) .$$

We have shown that

$$\ker(B:V_{2i+1}(A_\alpha) \longrightarrow U_{2i}^{\tilde{K}_0(A)^\alpha}(A)) \subseteq \operatorname{im}(\bar{\varepsilon}:U_{2i+1}^{(1-\alpha)\tilde{K}_0(A)}(A) \to V_{2i+1}(A_\alpha)) .$$

Chasing round the diagram



(which is part of a braid, and anticipates the definition of

$$C = 1-\alpha \;:\; U_{2i+1}(A) \longrightarrow U_{2i+1}^{(1-\alpha)\tilde{K}_0(A)}(A) \qquad)$$

the exactness of

$$V_{2i+1}(A) \xrightarrow{\bar{\varepsilon}} V_{2i+1}(A_\alpha) \xrightarrow{B} U_{2i}^{\tilde{K}_0(A)^\alpha}(A)$$

follows.

[ ]

In the untwisted case, $\alpha=1:A \longrightarrow A$, Lemma 6.1 gives a short exact sequence

$$0 \longrightarrow U_{2i+1}^{T}(A) \xrightarrow{\bar{\varepsilon}} U_{2i+1}^{\bar{\varepsilon}T}(A_z) \xrightarrow{B} V_{2i}(A) \longrightarrow 0$$

which splits, with B split by

$$\bar{B} : V_{2i}(A) \longrightarrow U_{2i+1}^{\bar{\varepsilon}T}(A_z) \; ;$$
$$(Q,\varphi) \longmapsto (Q_z \oplus Q_z, \varphi_z \oplus -\varphi_z; \Delta_{(Q_z,\varphi_z)}, (z\oplus 1)\Delta_{(Q_z,\varphi_z)}) .$$

Given a $*$-invariant subgroup $R \subseteq \widetilde{K}_1(A)$ define

$$B: \widetilde{V}^{\bar{\varepsilon}R}_{2i+1}(A_\alpha) \longrightarrow V^{(1-\alpha)^{-1}R}_{2i}(A); (Q,\varphi;\underset{\sim}{F},\underset{\sim}{G}) \longmapsto (\underset{\sim}{P},\theta)$$

as follows. Let

$$(P,\theta) = (B_N(F_0 \oplus F_0^*, G_0 \oplus G_0^*), [\varphi]_0)$$

with F_0, G_0 the modular A-bases of F,G generated by the given A_α-bases. Let $\tau_0 \in R$ be such that

$$\tau(Q,\varphi;\underset{\sim}{F},\underset{\sim}{G}) = \bar{\varepsilon}\tau_0 \in \bar{\varepsilon}R \subseteq \widetilde{\widetilde{K}}_1(A_\alpha) \quad (=\mathrm{coker}(K_1(Z_z) \to K_1(A_\alpha)))$$

(so that by the reduced version of the exact sequence of Lemma 4.1 τ_0 is unique up to torsions in $R \cap (1-\alpha)\widetilde{K}_1(A)$). Now

$$[P] = B\bar{\varepsilon}(\tau_0) = 0 \in \widetilde{K}_0(A) \ ,$$

so that for sufficiently large $N \geq 0$ P is a free f.g. A-module. Applying Theorem 4.2, note that

$$\begin{aligned} q\tau(Q,\varphi;\underset{\sim}{F},\underset{\sim}{G}) &= [B_{N+1}(F_0 \oplus F_0^*, G_0 \oplus G_0^*), \\ &\qquad 1 \oplus \xi^{N+1} f : zP \oplus (G_0 \oplus G_0^*) \to zP \oplus z^{N+1}(F_0 \oplus F_0^*)] \\ &= j\tau_0 \in \widetilde{K}(A,\alpha) \end{aligned}$$

with f defined by

$$f(\underset{\sim}{G} \oplus \underset{\sim}{G}^*) = \underset{\sim}{F} \oplus \underset{\sim}{F}^* \ .$$

Choosing any A-base for P, it follows that

$$j\tau(1 : z\underset{\sim}{P} \oplus (\underset{\sim}{G}_0 \oplus \underset{\sim}{G}_0^*) \longrightarrow \underset{\sim}{P} \oplus z^N(\underset{\sim}{F}_0 \oplus \underset{\sim}{F}_0^*)) = j\tau_0 \in \widetilde{K}(A,\alpha) \ ,$$

and so (by Lemma 4.1)

$$\tau(1 : z\underset{\sim}{P} \oplus (\underset{\sim}{G}_0 \oplus \underset{\sim}{G}_0^*) \to \underset{\sim}{P} \oplus z^N(\underset{\sim}{F}_0 \oplus \underset{\sim}{F}_0^*)) - \tau_0 = (1-\alpha)\tau_1 \in \widetilde{K}_1(A)$$

for some $\tau_1 \in \widetilde{K}_1(A)$ which is unique up to torsions in $(1-\alpha)^{-1}R$ (allowing τ_0 to vary). Changing the base of P by τ_1 , we can ensure that

$$\tau(1 : z\underset{\sim}{P} \oplus (\underset{\sim}{G}_0 \oplus \underset{\sim}{G}_0^*) \longrightarrow \underset{\sim}{P} \oplus z^N(\underset{\sim}{F}_0 \oplus \underset{\sim}{F}_0^*)) = \tau_0 \in R \subseteq \widetilde{K}_1(A) \quad (*) \ .$$

Let

$$B(Q,\varphi;\underset{\sim}{F},\underset{\sim}{G}) = (\underset{\sim}{P},\theta)$$

with $\underset{\sim}{P}$ in the preferred class of bases of P (unique up to changes in $(1-\alpha)^{-1}R$) satisfying the condition (*). Then

$$(1-\alpha)\tau(\underset{\sim}{P},\theta) = -(\tau_0+\tau_0{}^*) \in R ,$$

and so we do have an element

$$(\underset{\sim}{P},\theta) \in V_{2i}^{(1-\alpha)^{-1}R}(A)$$

which does not depend on the choice of τ_0 or $\underset{\sim}{P}$. The verification that this does define a morphism

$$B : \tilde{V}_{2i+1}^{\bar{\varepsilon}R}(A_\alpha) \longrightarrow V_{2i}^{(1-\alpha)^{-1}R}(A)$$

is by analogy with that for

$$B : U_{2i+1}^{\bar{\varepsilon}T}(A_\alpha) \longrightarrow U_{2i}^{(1-\alpha)^{-1}T}(A)$$

carried out above, taking into account torsions rather than projective classes.

Define also

$$C : V_{2i}^{(1-\alpha)^{-1}R}(A) \longrightarrow V_{2i}^{R}(A) \ ; \ (\underset{\sim}{Q},\varphi) \longmapsto (\underset{\sim}{Q},\varphi)\oplus \alpha(\underset{\sim}{Q}',-\varphi)$$

where $\underset{\sim}{Q}'$ is Q with the base defined by

$$\underset{\sim}{Q}' = (\varphi\pm\varphi^*)^{-1}(\underset{\sim}{Q}^*) \quad ,$$

so that

$$\tau((\underset{\sim}{Q},\varphi)\oplus\alpha(\underset{\sim}{Q}',-\varphi)) = (1-\alpha)\tau(\underset{\sim}{Q},\varphi) \in R\subseteq\tilde{K}_1(A) \ .$$

Given a $*$-invariant subgroup $S\subseteq\tilde{K}(A,\alpha)$, define

$$B : \tilde{V}_{2i+1}^{S}(A_\alpha) \longrightarrow U_{2i}^{T}(A) \ ; \ (Q,\varphi;\underset{\sim}{F},\underset{\sim}{G}) \longmapsto (B_N(F_0\oplus F_0^*,G_0\oplus G_0^*),[\varphi]_0)$$

with F_0,G_0 the modular A-bases of F,G generated by the given A_α-bases, so that

$$[B_N(F_0\oplus F_0^*,G_0\oplus G_0^*)] = B\tau(Q,\varphi;\underset{\sim}{F},\underset{\sim}{G}) \in T = p(S)\subseteq\tilde{K}_0(A) \ .$$

Define also

$$C : U_{2i}^{T}(A) \longrightarrow V_{2i}^{R}(A) \ ; \ (Q,\varphi) \longmapsto (\underset{\sim}{Q}',\varphi')$$

with

$$(Q',\varphi') = (Q,\varphi)\oplus\alpha(Q,-\varphi)\oplus H_{\pm}(-Q)$$

$$\underset{\sim}{Q}^{\varphi} = (\underset{\sim}{Q\oplus -Q})\oplus(t(\varphi\pm\varphi^*)^{-1}\oplus 1)(\underset{\sim}{Q\oplus -Q})^*$$

for any projective inverse $-Q$ to Q, and any A-base $(\underset{\sim}{Q\oplus -Q})$, where

$t \in \mathrm{Hom}_A(Q,zQ)$ is any isomorphism such that $[Q,t] \in S$ (and is thus unique up to composition with automorphisms of Q with torsion in $j^{-1}(S) = R \subseteq \widetilde{K}_1(A)$). Now

$$\underset{\approx}{Q}'_\alpha = \underbrace{(Q \oplus -Q)}_{\alpha} \oplus ((\varphi_\alpha \pm \varphi_\alpha^*)^{-1} \oplus 1)\underbrace{(Q \oplus -Q)}^*_\alpha$$

is a hamiltonian A_α-base for $\bar{\varepsilon}(Q',\varphi')$ such that

$$\tau(1:\underset{\approx}{Q}'_\alpha \longrightarrow \underset{\sim}{Q}'_\alpha) = i[Q,t] \in \widetilde{\widetilde{K}}_1(A_\alpha) \quad .$$

Applying Theorem 4.2,

$$q\tau(1:\underset{\approx}{Q}'_\alpha \longrightarrow \underset{\sim}{Q}'_\alpha) = [Q,t] \in S \subseteq \widetilde{K}(A,\alpha) \ ,$$

so that

$$\begin{aligned} j\tau(\underset{\sim}{Q}',\varphi') &= q\,\bar{\varepsilon}\tau(\underset{\sim}{Q}',\varphi') \\ &= -([Q,t] + [Q,t]^*) \in S \subseteq \widetilde{K}(A,\alpha) \end{aligned}$$

and

$$\tau(\underset{\sim}{Q}',\varphi') \in j^{-1}(S) = R \subseteq \widetilde{K}_1(A) \ .$$

Thus we do have an element

$$(Q',\varphi') \in V^R_{2i}(A)$$

which does not depend on the choice of $\underbrace{(Q \oplus -Q)}$ or t.

The verification that all the morphisms B, C appearing in the sequences

$$V^R_{2i+1}(A) \xrightarrow{\bar{\varepsilon}} \widetilde{V}^{\bar{\varepsilon}R}_{2i+1}(A_\alpha) \xrightarrow{B} V^{(1-\alpha)^{-1}R}_{2i}(A) \xrightarrow{C} V^R_{2i}(A) \xrightarrow{\bar{\varepsilon}} \widetilde{V}^{\bar{\varepsilon}R}_{2i}(A_\alpha)$$

$$V^R_{2i+1}(A) \xrightarrow{\bar{\varepsilon}} \widetilde{V}^S_{2i+1}(A_\alpha) \xrightarrow{B} U^T_{2i}(A) \xrightarrow{C} V^R_{2i}(A) \xrightarrow{\bar{\varepsilon}} \widetilde{V}^S_{2i}(A_\alpha)$$

are well-defined, and that the composite of successive morphisms is zero, is by analogy with that for the sequence of Lemma 6.1. Exactness follows, by the argument of §5.

In particular, in the untwisted case $\alpha = 1 : A \to A$, with

$$S = j(R) \oplus \bar{p}(T) \subseteq \widetilde{K}(A,1) = j\widetilde{K}_1(A) \oplus \bar{p}\widetilde{K}_0(A)$$

there is defined a split short exact sequence

$$0 \longrightarrow V^R_{2i+1}(A) \xrightarrow{\bar{\varepsilon}} \widetilde{V}^S_{2i+1}(A_z) \xrightarrow{B} U^T_{2i}(A) \longrightarrow 0 \ ,$$

with splitting morphisms

$$\varepsilon:\tilde{V}^{S}_{2i+1}(A_z) \longrightarrow V^{R}_{2i+1}(A)$$

$$\bar{B}:U^{T}_{2i}(A) \longrightarrow \tilde{V}^{S}_{2i+1}(A_z);$$

$$(Q,\varphi) \longrightarrow ((Q_z \oplus Q_z, \varphi_z \oplus -\varphi_z) \oplus H_{\pm}(-Q_z); \underset{\sim}{L}_z, \zeta\underset{\sim}{L}_z)$$

where

$$\zeta = z \oplus 1 : Q_z \oplus (Q_z \oplus -Q_z \oplus -Q_z^*) \longrightarrow Q_z \oplus (Q_z \oplus -Q_z \oplus -Q_z^*)$$

$$\underset{\sim}{L} = \{(x,x,y,0) \in Q_z \oplus Q_z \oplus -Q_z \oplus -Q_z^* \mid (x,y) \in (\underbrace{Q \oplus -Q}) \}$$

for any projective inverse $-Q$ to Q, and any A-base $(\underbrace{Q \oplus -Q})$.

Given a $*$-invariant subgroup $T \subseteq \tilde{K}_0(A)$, define

$$B:U^{\bar{\varepsilon}T}_{2i}(A_\alpha) \longrightarrow U^{(1-\alpha)^{-1}T}_{2i-1}(A);(Q,\varphi) \longmapsto (H_{\mp}(P_N);P_N,B_N(Q_0,\varphi))$$

as follows, where

$$P_N = \sum_{j=0}^{N-1} z^j Q_0 \quad .$$

Choose a modular A-base Q_0 of Q such that

$$[Q_0] \in T \subseteq \tilde{K}_0(A) \quad ,$$

let

$$\nu : Q \oplus Q^* \longrightarrow (Q_0 \oplus Q_0^*)^+ ; \sum_{j=-\infty}^{\infty} z^j x_j \longmapsto \sum_{j=0}^{\infty} z^j x_j \quad (x_j \in (Q_0 \oplus Q_0^*)),$$

and define

$$B_N(Q_0,\varphi) = \{(z^N(1-\nu)z^{-N}x, \nu(\varphi \pm \varphi^*)x) \in P_N \oplus P_N^* \mid x \in B_N((\varphi \pm \varphi^*)^{-1}Q_0^*, Q_0)\}.$$

Then $B_N(Q_0,\varphi)$ is a lagrangian of $H_{\mp}(P_N)$, with hamiltonian complement

$$B_N^*(Q_0,\varphi) = \{(-\nu y, \nu(\varphi \pm \varphi^*)(1-\nu)y) \in P_N \oplus P_N^* \mid y \in B_N(Q_0,(\varphi \pm \varphi^*)^{-1}Q_0^*)\}.$$

The associated $\mp$hermitian product of $H_{\mp}(P_N)$

$$\begin{pmatrix} 0 & 1 \\ \mp 1 & 0 \end{pmatrix} : P_N \oplus P_N^* \longrightarrow P_N^* \oplus P_N = (P_N \oplus P_N^*)^*$$

restricts to the A-module isomorphism

$$B_N^*(Q_0,\varphi) \longrightarrow B_N(Q_0,\varphi)^*;$$

$$(-\nu y, \nu(\varphi \pm \varphi^*)(1-\nu)y) \longmapsto ((z^N(1-\nu)z^{-N}x, \nu(\varphi \pm \varphi^*)x) \mapsto [(\varphi \pm \varphi^*)(y)(x)]_0).$$

Hence

$$[B_N(Q_0,\varphi)] = [B_N((\varphi\pm\varphi^*)^{-1}Q_0^*,Q_0)] \in \tilde{K}_0(A)$$

and

$$\begin{aligned}(1-\alpha)([B_N(Q_0,\varphi)]-[P_N^*]) &= ([Q_0]-[z^N Q_0^*])+([z^N Q_0^*]-[Q_0^*]) \\ &= [Q_0]-[Q_0^*] \in T\subseteq\tilde{K}_0(A) \quad ,\end{aligned}$$

so that we do have an element

$$B(Q,\varphi) = (H_\mp(P_N);P_N,B_N(Q_0,\varphi)) \in U_{2i-1}^{(1-\alpha)^{-1}T}(A) \ .$$

Increasing N by 1, note that

$$B_{N+1}(Q_0,\varphi) = B_N(Q_0,\varphi)\oplus\{(z^{N+1}(1-\nu)z^{-(N+1)}x,(\varphi\pm\varphi^*)x \mid x\in(\varphi\pm\varphi^*)^{-1}(z^N Q_0^*)\} \ .$$

Now $B_N^*(Q_0,\varphi)\oplus z^N Q_0$ is a hamiltonian complement in $H_\mp(P_{N+1})$ to both $B_{N+1}(Q_0,\varphi)$ and $B_N(Q_0,\varphi)\oplus z^N Q_0^*$. Applying the sum formula of Lemma 2.2,

$$\begin{aligned}(H_\mp(P_N);P_N,B_N(Q_0,\varphi)) &= (H_\mp(P_{N+1});P_{N+1},B_N(Q_0,\varphi)\oplus z^N Q_0^*) \\ &= (H_\mp(P_{N+1});P_{N+1},B_N^*(Q_0,\varphi)\oplus z^N Q_0) \\ &= (H_\mp(P_{N+1});P_{N+1},B_{N+1}(Q_0,\varphi)) \\ &\in U_{2i-1}^{(1-\alpha)^{-1}T}(A) \ .\end{aligned}$$

Hence the choice of N is immaterial to $B(Q,\varphi) \in U_{2i-1}^{(1-\alpha)^{-1}T}(A)$.

Let Q_0' be another modular A-base of Q such that

$$[Q_0'] \in T \ ,$$

write

$$P_{N'}' = \sum_{j=0}^{N'-1} z^j Q_0' \quad ,$$

and define

$$\nu' : Q\oplus Q^* \to (Q_0'\oplus Q_0'^*)^+ ; \sum_{j=-\infty}^{\infty} z^j x_j \longmapsto \sum_{j=0}^{\infty} z^j x_j$$

$$(x_j \in (Q_0'\oplus Q_0'^*)) \ .$$

Let $M \geq 0$ be so large that

$$Q_0' \subseteq \sum_{j=-M}^{M} z^j Q_0 \qquad Q_0 \subseteq \sum_{j=-M}^{M} z^j Q_0'$$

Then $N' = N + 2M$ is sufficiently large for $B_{N'}(Q_0',\varphi)$ to be defined, with

$$B_{N'}((\varphi\pm\varphi^*)^{-1}Q_0'^*,Q_0') = (\varphi\pm\varphi^*)^{-1}(z^{M+N}B_M(Q_0'^*,Q_0^*))$$
$$\oplus z^M B_N((\varphi\pm\varphi^*)^{-1}Q_0^*,Q_0)\oplus B_M(Q_0,Q_0')$$

and

$$B_{N'}(Q_0',\varphi) = \{(z^{N'}(1-\nu')z^{-N'}x,(\varphi\pm\varphi^*)x)\,|\,x\in(\varphi\pm\varphi^*)^{-1}(z^{M+N}B_N(Q_0'^*,Q_0^*))\}$$
$$\oplus\{(x,(\varphi\pm\varphi^*)x)\,|\,x\in z^M B_N((\varphi\pm\varphi^*)^{-1}Q_0^*,Q_0)\}$$
$$\oplus\{(x,\nu'(\varphi\pm\varphi^*)x)\,|\,x\in B_M(Q_0,Q_0')\}\subseteq P_{N'}'\oplus P_{N'}'^* \ .$$

Now

$$P_{N'}' = z^{M+N}B_M(Q_0',Q_0)\oplus z^M P_N\oplus B_M(Q_0,Q_0'\ )$$

and

$$z^{M+N}B_M(Q_0',Q_0)\oplus z^M B_N^*(Q_0,\varphi)\oplus B_M(Q_0^*,Q_0'^*)$$

is a hamiltonian complement in $H_{\mp}(P_{N'}')$ to both $B_{N'}(Q_0',\varphi)$ and $z^{M+N}B_M(Q_0'^*,Q_0^*)\oplus z^M B_N(Q_0,\varphi)\oplus B_M(Q_0,Q_0')$. Applying the sum formula of Lemma 2.2,

$$(H_{\mp}(P_{N'}');P_{N'}',B_{N'}(Q_0',\varphi))$$
$$= (H_{\mp}(P_{N'}');P_{N'}',z^{M+N}B_M(Q_0'^*,Q_0^*)\oplus z^M B_N(Q_0,\varphi)\oplus B_M(Q_0,Q_0'))$$
$$= (H_{\mp}(z^{M+N}B_M(Q_0',Q_0));z^{M+N}B_M(Q_0',Q_0),z^{M+N}B_M(Q_0'^*,Q_0^*))$$
$$\oplus\alpha^M(H_{\mp}(P_N);P_N,B_N(Q_0,\varphi))$$
$$\oplus(H_{\mp}(B_M(Q_0,Q_0'));B_M(Q_0,Q_0'),B_M(Q_0^*,Q_0'^*))$$
$$= \alpha^M(H_{\mp}(P_N);P_N,B_N(Q_0,\varphi))\in U_{2i-1}^{(1-\alpha)^{-1}T}(A)\ .$$

But $zB_N^*(Q_0,\varphi)\oplus Q_0$ is a hamiltonian complement to $B_{N+1}(Q_0,\varphi)$ in

$H_{\mp}(P_{N+1})$, so that

$$(H_{\mp}(P_N);P_N,B_N(Q_0,\varphi)) = (H_{\mp}(P_{N+1});P_{N+1},B_{N+1}(Q_0,\varphi))$$

$$= \alpha(H_{\mp}(P_N);P_N,B_N(Q_0,\varphi)) \oplus (H_{\mp}(Q_0);Q_0,Q_0^*)$$

$$= \alpha(H_{\mp}(P_N);P_N,B_N(Q_0,\varphi)) \in U_{2i-1}^{(1-\alpha)^{-1}T}(A) .$$

Hence

$$B(Q,\varphi) = (H_{\mp}(P_N);P_N,B_N(Q_0,\varphi)) \in U_{2i-1}^{(1-\alpha)^{-1}T}(A)$$

does not depend on the choice of modular A-base Q_0 .

Finally, suppose

$$(Q,\varphi) = \bar{\varepsilon}(Q_0,\varphi_0)$$

for some $(Q_0,\varphi_0) \in U_{2i}^T(A)$. Then

$$B(Q,\varphi) = (H_{\mp}(0);0,B_0(Q_0,\varphi)) = 0 \in U_{2i-1}^{(1-\alpha)^{-1}T}(A) .$$

Hence

$$B:U_{2i}^{\bar{\varepsilon}T}(A_\alpha) \longrightarrow U_{2i-1}^{(1-\alpha)^{-1}T}(A);(Q,\varphi) \longmapsto (H_{\mp}(P_N);P_N,B_N(Q_0,\varphi))$$

is well-defined, and such that the composite

$$U_{2i}^T(A) \xrightarrow{\bar{\varepsilon}} U_{2i}^{\bar{\varepsilon}T}(A_\alpha) \xrightarrow{B} U_{2i-1}^{(1-\alpha)^{-1}T}(A)$$

is zero.

The morphism

$$C = 1-\alpha :.U_{2i-1}^{(1-\alpha)^{-1}T}(A) \longrightarrow U_{2i-1}^T(A) ;$$

$$(Q,\varphi;F,G) \longmapsto (Q,\varphi;F,G) \oplus \alpha(Q,-\varphi;F^*,G^*)$$

is clearly well-defined, and such that the composites of successive morphisms in

$$U_{2i}^{\bar{\varepsilon}T}(A_\alpha) \xrightarrow{B} U_{2i-1}^{(1-\alpha)^{-1}T}(A) \xrightarrow{C} U_{2i-1}^T(A) \xrightarrow{\bar{\varepsilon}} U_{2i-1}^{\bar{\varepsilon}T}(A_\alpha)$$

is zero ($CB = 0$ follows from the relation

$$\alpha B(Q,\varphi) = B(Q,\varphi) \in U_{2i-1}^{(1-\alpha)^{-1}T}(A) \quad ((Q,\varphi) \in U_{2i}^{\bar{\varepsilon}T}(A_\alpha))$$

proved above).

Given a $*$-invariant subgroup $R \subseteq \tilde{K}_1(A)$, define

$$B : \tilde{V}_{2i}^{\bar{\varepsilon}R}(A_\alpha) \longrightarrow V_{2i-1}^{(1-\alpha)^{-1}R}(A) \ ; \ (\underset{\sim}{Q},\varphi) \longmapsto (H_{\mp}(P_N);\underset{\sim}{P}_N,\underline{B_N(Q_0,\varphi)})$$

as follows. Let $\underset{\sim}{Q}_0$ be the modular A-base of Q generated by the given A_α-base, with the corresponding A-base. Let $N \geq 0$ be so large that $B_N((\varphi\pm\varphi^*)^{-1}Q_0^*,Q_0)$ is a free A-module. Let $\tau_0 \in R$ be such that

$$\tau(\underset{\sim}{Q},\varphi) = \bar{\varepsilon}\tau_0 \in \bar{\varepsilon}R \subseteq \tilde{\tilde{K}}_1(A_\alpha) \ .$$

Then, working as in the definition of $B : \tilde{V}_{2i+1}^{\bar{\varepsilon}R}(A_\alpha) \longrightarrow V_{2i}^{(1-\alpha)^{-1}R}(A)$, there is a preferred class of A-bases $\underline{B_N((\varphi\pm\varphi^*)^{-1}Q_0^*,Q_0)}$, unique up to changes in $(1-\alpha)^{-1}R$ for varying τ_0, such that

$$\tau(1:z\underline{B_N((\varphi\pm\varphi^*)^{-1}Q_0^*,Q_0)}\oplus\underset{\sim}{Q}_0 \longrightarrow \underline{B_N((\varphi\pm\varphi^*)^{-1}Q_0^*,Q_0)}\oplus(\varphi\pm\varphi^*)^{-1}(z^N\underset{\sim}{Q}_0^*))$$
$$= \tau_0 \in R \subseteq K_1(A) \ .$$

Give $B_N(Q_0,\varphi)$ an A-base by choosing one of these, and setting

$$\underline{B_N(Q_0,\varphi)} = \{(z^N(1-\nu)z^{-N}x,\nu(\varphi\pm\varphi^*)x)\in P_N\oplus P_N^* \mid x\in\underline{B_N((\varphi\pm\varphi^*)^{-1}Q_0^*,Q_0)}\} \ .$$

Let $\begin{cases}\underline{B_{N+1}(Q_0,\varphi)}\\ \underline{\underline{B_{N+1}(Q_0,\varphi)}}\end{cases}$ stand for $B_{N+1}(Q_0,\varphi)$ with the base

$$\begin{cases}\underline{B_{N+1}((\varphi\pm\varphi^*)^{-1}Q_0^*,Q_0)} = z\underline{B_N((\varphi\pm\varphi^*)^{-1}Q_0^*,Q_0)}\oplus\underset{\sim}{Q}_0\\ \underline{\underline{B_{N+1}((\varphi\pm\varphi^*)^{-1}Q_0^*,Q_0)}} = \underline{B_N((\varphi\pm\varphi^*)^{-1}Q_0^*,Q_0)}\oplus(\varphi\pm\varphi^*)^{-1}(z^N\underset{\sim}{Q}_0^*)\end{cases} \ .$$

Using the hamiltonian complements given above (in the definition of $B : U_{2i}^{\bar{\varepsilon}T}(A_\alpha) \longrightarrow U_{2i-1}^{(1-\alpha)^{-1}T}(A)$) it can be shown that

$$(H_{\mp}(P_{N+1});\underset{\sim}{P}_{N+1},\underline{B_{N+1}(Q_0,\varphi)})$$
$$= (H_{\mp}(P_{N+1});\underset{\sim}{P}_{N+1},z\underline{B_N(Q_0,\varphi)}\oplus z^N\underset{\sim}{Q}_0^*)$$
$$= \alpha(H_{\mp}(P_N);\underset{\sim}{P}_N,\underline{B_N(Q_0,\varphi)}) \in V_{2i-1}^{(1-\alpha)^{-1}R}(A)$$

and similarly

$$\begin{aligned}(H_{\mp}(P_{N+1});\underset{\sim}{P}_{N+1},\underset{\approx}{B_{N+1}(Q_0,\varphi)}) &= (H_{\mp}(P_{N+1});\underset{\sim}{P}_{N+1},\underset{\sim}{B_N(Q_0,\varphi)}\oplus z^N\underset{\sim}{Q}_0^*) \\ &= (H_{\mp}(P_N);\underset{\sim}{P}_N,\underset{\sim}{B_N(Q_0,\varphi)}) \in V_{2i-1}^{(1-\alpha)^{-1}R}(A) \quad .\end{aligned}$$

Hence

$$(1-\alpha)\tau(H_{\mp}(P_N);\underset{\sim}{P}_N,\underset{\sim}{B_N(Q_0,\varphi)}) = (\tau_0-\tau_0^*) \in R \ ,$$

and we do have an element

$$B(\underset{\sim}{Q},\varphi) = (H_{\mp}(P_N);\underset{\sim}{P}_N,\underset{\sim}{B_N(Q_0,\varphi)}) \in V_{2i-1}^{(1-\alpha)^{-1}R}(A) \ .$$

Define also

$$C = 1-\alpha \ : \ V_{2i-1}^{(1-\alpha)^{-1}R}(A) \longrightarrow V_{2i-1}^{R}(A);$$

$$(Q,\varphi;\underset{\sim}{F},\underset{\sim}{G}) \longmapsto (Q,\varphi;\underset{\sim}{F},\underset{\sim}{G})\oplus\alpha(Q,-\varphi;\underset{\sim}{F}^*,\underset{\sim}{G}^*) \ .$$

Given a $*$-invariant subgroup $S\subseteq\widetilde{K}(A,\alpha)$ define

$$B:\widetilde{V}_{2i}^{S}(A_\alpha) \longrightarrow U_{2i-1}^{T}(A);(\underset{\sim}{Q},\varphi)\longmapsto(H_{\mp}(P_N);P_N,B_N(Q_0,\varphi))$$

with Q_0 the modular A-base of Q generated by the given A_α-base, so that

$$[B_N(Q_0,\varphi)] = B\tau(\underset{\sim}{Q},\varphi) \in T = p(S)\subseteq\widetilde{K}_0(A) \ .$$

Define also

$$C:U_{2i-1}^{T}(A) \rightarrow V_{2i-1}^{R}(A);(Q,\varphi;F,G) \longmapsto (Q',\varphi';\underset{\sim}{F}',\underset{\sim}{G}')$$

as follows. It may be assumed that F is free and that there is defined an isomorphism $t \in \mathrm{Hom}_A(G,zG)$ such that $[G,t] \in S$ .Let

$$(Q',\varphi';F',G') = (Q,\varphi;F,G)\oplus\alpha(Q,-\varphi;F^*,G^*)$$

for any hamiltonian complements F^*,G^* to F,G . Choosing any base for F, let

$$\underset{\sim}{F}' = \underset{\sim}{F}\oplus z\underset{\sim}{F}^* \qquad \underset{\sim}{G}' = (1\oplus t^{*-1})(\underset{\sim}{G\oplus G^*}) \qquad (\underset{\sim}{G\oplus G^*}) = \underset{\sim}{F}\oplus\underset{\sim}{F}^* \ .$$

Now

$$\begin{aligned}\bar{\varepsilon}\tau(Q',\varphi';\underset{\sim}{F}',\underset{\sim}{G}') &= \tau(Q_\alpha\oplus Q_\alpha,\varphi_\alpha\oplus-\varphi_\alpha;(1\oplus\xi_\alpha)(\underset{\sim}{F}\oplus\underset{\sim}{F}^*)_\alpha,(1\oplus t_\alpha^{*-1})(\underset{\sim}{G\oplus G^*})_\alpha) \\ &= i(*-1)([G,t]-[F^*,\xi]) \in i(S)\subseteq\widetilde{\widetilde{K}}_1(A_\alpha)\end{aligned}$$

(i as in Theorem 4.2).

Hence

$$\tau(Q',\varphi';\underset{\sim}{F}',\underset{\sim}{G}') \in j^{-1}(S) = R \subseteq \widetilde{K}_1(A) \ ,$$

and we do have an element

$$C(Q,\varphi;F,G) = (Q',\varphi';\underset{\sim}{F}',\underset{\sim}{G}') \in V^R_{2i-1}(A) \ .$$

The verification that the morphisms B, C appearing in the sequences

$$V^R_{2i}(A) \xrightarrow{\bar{\varepsilon}} \widetilde{V}^{\bar{\varepsilon}R}_{2i}(A_\alpha) \xrightarrow{B} V^{(1-\alpha)^{-1}R}_{2i-1}(A) \xrightarrow{C} V^R_{2i-1}(A) \xrightarrow{\bar{\varepsilon}} \widetilde{V}^{\bar{\varepsilon}R}_{2i-1}(A_\alpha)$$

$$V^R_{2i}(A) \xrightarrow{\bar{\varepsilon}} \widetilde{V}^S_{2i}(A_\alpha) \xrightarrow{B} U^T_{2i-1}(A) \xrightarrow{C} V^R_{2i-1}(A) \xrightarrow{\bar{\varepsilon}} \widetilde{V}^S_{2i-1}(A_\alpha)$$

are well-defined, and that the composite of successive morphisms is zero, is by analogy with that for the sequence

$$U^T_{2i}(A) \xrightarrow{\bar{\varepsilon}} U^{\bar{\varepsilon}T}_{2i}(A_\alpha) \xrightarrow{B} U^{(1-\alpha)^{-1}T}_{2i-1}(A) \xrightarrow{C} U^T_{2i-1}(A) \xrightarrow{\bar{\varepsilon}} U^{\bar{\varepsilon}T}_{2i-1}(A_\alpha)$$

which was dealt with above.

We can now apply the trick (first used in [4]) of introducing a new Laurent variable to deduce the exactness of these sequences from that of Lemma 6.1.

Note first that for $*$-invariant subgroups

$$S = j(R) \oplus \bar{p}(T) \subseteq \widetilde{K}(A,1) = j\widetilde{K}_1(A) \oplus \bar{p}\widetilde{K}_0(A)$$

there is defined a morphism

$$\bar{B} : U^T_{2i-1}(A) \longrightarrow \widetilde{V}^S_{2i}(A_z) ; (Q,\varphi;F,G) \longmapsto (\underset{\sim}{G}_z \oplus \underset{\sim}{G}^*_z, \begin{pmatrix} \lambda & -z\gamma \\ \delta & (1-z)(\lambda_1 \pm \lambda_1^*) \end{pmatrix})$$

with $\underset{\sim}{G}$ any base for G (which may be assumed to be free), and

$$\begin{pmatrix} \lambda \pm \lambda^* & \gamma \\ \delta & \lambda_1 \pm \lambda_1^* \end{pmatrix} : G \oplus G^* \longrightarrow G^* \oplus G$$

an expression for

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} : F \oplus F^* \longrightarrow F^* \oplus F \ ,$$

for any hamiltonian complements F^*, G^* to F, G in (Q,φ).

It was shown in §3 of II. that this does define a morphism $\overline{B}$, and that

$$V_{2i}^{R}(A) \underset{\varepsilon}{\overset{\overline{\varepsilon}}{\rightleftarrows}} \widetilde{V}_{2i}^{S}(A_z) \underset{\overline{B}}{\overset{B}{\rightleftarrows}} U_{2i-1}^{T}(A)$$

is a direct sum system, if $S = \{0\}$ or $\widetilde{K}(A,1)$. The proof generalizes immediately to any S of type $j(R)\oplus\overline{p}(T)$.

Let z' be an invertible indeterminate over A_α .
Identify $(A_\alpha)_{z'}$ with $(A_{z'})_{\alpha'}$, where

$$\alpha' : A_{z'} \longrightarrow A_{z'} \ ; \ \sum_{j=-\infty}^{\infty} z'^j a_j \longmapsto \sum_{j=-\infty}^{\infty} z'^j \alpha(a_j) \quad ,$$

and write $A_{\alpha,z'}$ for this double Laurent extension of A.

Let $\begin{cases} S_0 \\ S_0' \end{cases}$ be the infinite cyclic subgroup of $\begin{cases} \widetilde{K}_1(A_\alpha) \\ \widetilde{K}_1(A_z) \end{cases}$ generated by $\begin{cases} \tau(\xi : A_\alpha \longrightarrow A_\alpha) \\ \tau(\xi' : A_{z'} \longrightarrow A_{z'}) \end{cases}$, where $\begin{cases} \xi \varepsilon \mathrm{Hom}_{A_\alpha}(A_\alpha, A_\alpha) \\ \xi' \varepsilon \mathrm{Hom}_{A_{z'}}(A_{z'}, A_{z'}) \end{cases}$ is multiplication on the right by $\begin{cases} z \\ z' \end{cases}$. Define

$$\widetilde{\widetilde{W}}_n(A_{\alpha,z'}) = V_n^{\overline{\varepsilon}(z')S_0 \oplus \overline{\varepsilon}(\alpha)S_0'}(A_{\alpha,z'}) \qquad (n(\bmod 4))$$

where $\begin{cases} \overline{\varepsilon}(z') : A_\alpha \longrightarrow A_{\alpha,z'} \\ \overline{\varepsilon}(\alpha) : A_{z'} \longrightarrow A_{\alpha,z'} \end{cases}$ is the inclusion. The preimage of

$$\widetilde{K}(A,1)^{\alpha'} = j\widetilde{K}_1(A)^\alpha \oplus \overline{p}\widetilde{K}_0(A)^\alpha \subseteq \widetilde{K}(A,1)$$

under the projection

$$q : \widetilde{K}_1(A_{z'}) = \overline{\varepsilon}\widetilde{K}_1(A) \oplus \overline{B}K_0(A) \oplus \mathrm{Nil}_+(A,1) \oplus \mathrm{Nil}_-(A,1) \longrightarrow \widetilde{K}(A,1) = j\widetilde{K}_1(A) \oplus \overline{p}\widetilde{K}_0(A)$$

(as defined in Theorem 4.2) is

$$\widetilde{\widetilde{K}(A,1)^{\alpha'}} = \overline{\varepsilon}\widetilde{K}_1(A)^\alpha \oplus \overline{B}(T_0) \oplus \mathrm{Nil}_+(A,1) \oplus \mathrm{Nil}_-(A,1) \subseteq \widetilde{K}_1(A_{z'}) \quad ,$$

where

$$T_0 = (1-\alpha)^{-1}(\mathrm{im}(K_0(Z) \longrightarrow K_0(A)) \subseteq K_0(A) \ .$$

Further,

$$(1-\alpha')^{-1}(S_0') = \overline{\varepsilon}\widetilde{K}_1(A)^\alpha \oplus \overline{B}(T_0) \oplus \mathrm{Nil}_+(A,1)^{\alpha'} \oplus \mathrm{Nil}_-(A,1)^{\alpha'} \subseteq \widetilde{K}_1(A_{z'}),$$

where

$$\mathrm{Nil}_\pm(A,1)^{\alpha'} = \{ \tau \,\varepsilon\, K_1(A_{z'}) \mid \nu \,\varepsilon\, \mathrm{Hom}_A(P,P) \text{ nilpotent} ,$$
$$\tau = \tau(1+\nu z'^{\pm 1} : P_{z'} \rightarrow P_{z'}) = \tau(1+(z\nu)z'^{\pm 1} : (zP)_{z'} \rightarrow (zP)_{z'}) \,\varepsilon\, \widetilde{K}_1(A_{z'})\} .$$

Hence

$$\tilde{V}_n^{\tilde{K}(A,1)^{\alpha'}}(A_{z'}) = V_n^{\widetilde{\tilde{K}(A,1)^{\alpha'}}}(A_{z'}) \qquad \text{(by definition)}$$

$$= V_n^{(1-\alpha')^{-1}(S_0')}(A_{z'}) \quad (= V_n(A_{z'}) \text{ if } \alpha = 1 \ )$$

by the exact sequence of Theorem 3.3 .

All the squares of shape $\begin{array}{ccc}\cdot & \rightarrow & \cdot\\ \downarrow & & \downarrow \\ \cdot & \rightarrow & \cdot\end{array}$, $\begin{array}{ccc}\cdot & \rightarrow & \cdot\\ \uparrow & & \uparrow \\ \cdot & \rightarrow & \cdot\end{array}$ in the diagram

$$\begin{array}{ccccccccc}
V_{2i}(A) & \xrightarrow{\bar{\varepsilon}(\alpha)} & V_{2i}(A_\alpha) & \xrightarrow{B(\alpha)} & U_{2i-1}^{\tilde{K}_0(A)^\alpha}(A) & \xrightarrow{C(\alpha)} & V_{2i-1}(A) & \xrightarrow{\bar{\varepsilon}(\alpha)} & V_{2i-1}(A_\alpha) \\
\bar{B}(z')\downarrow\ \uparrow B(z') & & \bar{B}(z')\downarrow\ \uparrow B(z') & & \bar{B}(z')\downarrow\ \uparrow B(z') & & \bar{B}(z')\downarrow\ \uparrow B(z') & & \bar{B}(z')\downarrow\ \uparrow B(z') \\
\tilde{W}_{2i+1}(A_{z'}) & \xrightarrow{\bar{\varepsilon}(\alpha')} & \tilde{\tilde{W}}_{2i+1}(A_{\alpha,z'}) & \xrightarrow{B(\alpha')} & \tilde{V}_{2i}^{\tilde{K}(A,1)^{\alpha'}}(A_{z'}) & \xrightarrow{C(\alpha')} & \tilde{W}_{2i}(A_{z'}) & \xrightarrow{\bar{\varepsilon}(\alpha')} & \tilde{\tilde{W}}_{2i}(A_{\alpha,z'}) \\
\varepsilon(z')\downarrow\ \uparrow\bar{\varepsilon}(z') & & \varepsilon(z')\downarrow\ \uparrow\bar{\varepsilon}(z') & & \varepsilon(z')\downarrow\ \uparrow\bar{\varepsilon}(z') & & \varepsilon(z')\downarrow\ \uparrow\bar{\varepsilon}(z') & & \varepsilon(z')\downarrow\ \uparrow\bar{\varepsilon}(z') \\
W_{2i+1}(A) & \xrightarrow{\bar{\varepsilon}(\alpha)} & \tilde{W}_{2i+1}(A_\alpha) & \xrightarrow{B(\alpha)} & V_{2i}^{\tilde{K}_1(A)^\alpha}(A) & \xrightarrow{C(\alpha)} & W_{2i}(A) & \xrightarrow{\bar{\varepsilon}(\alpha)} & \tilde{W}_{2i}(A_\alpha)
\end{array}$$

commute, except for those round the shaded area, the columns are direct sum systems, and the rows through $\tilde{W}_{2i+1}(A_{z'}), W_{2i+1}(A)$ are exact (being the special cases $S_0' \subseteq \tilde{K}_1(A_{z'})$, $\{0\} \subseteq \tilde{K}_1(A)$ of the sequence of Theorem 5.2 in the range of dimensions considered in §6). It was shown in Lemma 3.4 of II. that the square

$$\begin{array}{ccc}
V_{2i}(A_\alpha) & \xrightarrow{B(\alpha)} & U_{2i-1}^{\tilde{K}_0(A)^\alpha}(A) \\
\bar{B}(z')\downarrow & & \downarrow\bar{B}(z') \\
\tilde{\tilde{W}}_{2i+1}(A_{\alpha,z'}) & \xrightarrow{B(\alpha')} & \tilde{V}_{2i}^{\tilde{K}(A,1)^{\alpha'}}(A_{z'})
\end{array}$$

skew-commutes for $\alpha = 1$. The proof generalizes immediately to the twisted case (for any α). It follows that both the squares round the shaded area (in the large diagram above) skew-commute, and that the row through $V_{2i}(A)$ is exact as well. But this is the special case $T = \{0\}$ of the sequence of Theorem 5.1 in the range of dimensions not already covered in §6. As explained in §5, this suffices to complete the proof of Theorems 5.1,5.2,5.3.

§7. Lower L-theories

Bass has defined lower K-groups $K_p(A)$ for $p < 0$, with natural split injections

$$\overline{B} : K_p(A) \longrightarrow K_{p+1}(A_z) ,$$

such that

$$K_{p+1}(A_z) = \bar{\varepsilon} K_{p+1}(A) \oplus \overline{B} K_p(A) \oplus Nil_+^{(p)}(A) \oplus Nil_-^{(p)}(A) .$$

There is defined a duality involution

$$* : K_p(A) \longrightarrow K_p(A)$$

for all $p < 0$, with

$$\overline{B}* = -*\overline{B} : K_p(A) \longrightarrow K_{p+1}(A_z)$$

$$*(Nil_{\pm}^{(p)}(A)) = Nil_{\mp}^{(p)}(A) .$$

In II. there were defined "lower L-theories" $L_n^{(p)}(A)$, for $p < 0$ and n(mod 4), by

$$L_n^{(p)}(A) = \ker(\varepsilon : L_{n+1}^{(p+1)}(A_z) \longrightarrow L_{n+1}^{(p+1)}(A))$$

with $L_n^{(0)}(A) = U_n(A)$.

Given a $*$-invariant subgroup $Q \subseteq K_0(A)$ let $\widetilde{Q} \subseteq \widetilde{K}_0(A)$ be the subgroup to which the natural projection $K_0(A) \rightarrow \widetilde{K}_0(A)$ sends Q, and define

$$L_n^Q(A) = U_n^{\widetilde{Q}}(A) \quad (n(\text{mod } 4)).$$

Assuming inductively that $L_n^{Q'}(A_z)$ has already been defined for all $*$-invariant subgroups $Q' \subseteq K_{p+1}(A_z)$, define

$$L_n^Q(A) = \ker(\varepsilon : L_{n+1}^{\bar{\varepsilon} K_{p+1}(A) \oplus \overline{B}Q}(A_z) \longrightarrow L_{n+1}^{K_{p+1}(A)}(A))$$

for $*$-invariant subgroups $Q \subseteq K_p(A)$, $p < 0$.

Theorem 2.3 gives

<u>Theorem 7.1</u> <u>There is defined an exact sequence of abelian groups</u>

$$\ldots \longrightarrow H^{n+1}(Q'/Q) \longrightarrow L_n^Q(A) \longrightarrow L_n^{Q'}(A) \longrightarrow H^n(Q'/Q) \longrightarrow \ldots$$

<u>for $*$-invariant subgroups</u> $Q \subseteq Q' \subseteq K_p(A)$, $p < 0$.

[ ]

In particular, it follows that

$$L_n^Q(A) = \begin{cases} L_n^{(p+1)}(A) \\ L_n^{(p)}(A) \end{cases} \text{if } Q = \begin{cases} \{0\} \subseteq K_p(A) \\ K_p(A) \end{cases} .$$

Theorem 5.1 gives

Theorem 7.2 There is defined an exact sequence of abelian groups

$$\ldots \longrightarrow L_n^Q(A) \xrightarrow{\bar{\varepsilon}} L_n^{\bar{\varepsilon}Q}(A_\alpha) \xrightarrow{B} L_{n-1}^{(1-\alpha)^{-1}Q}(A) \xrightarrow{C} L_{n-1}^Q(A) \longrightarrow \ldots$$

in a natural way, for $*$-invariant subgroups $Q \subseteq K_p(A)$, $p < 0$.

[ ]

A lower L-theoretic analogue of Theorem 5.3 requires a lower K-theoretic analogue of Theorem 4.2. So far, this is only available in the untwisted case:

Theorem 7.3 Let $Q = \bar{\varepsilon}(R) \oplus \bar{B}(S) \subseteq K_{p+1}(A_z)$, for some $*$-invariant subgroups $R \subseteq K_{p+1}(A)$, $S \subseteq K_p(A)$ $(p < 0)$. Then there is defined a direct sum system

$$L_n^R(A) \underset{\varepsilon}{\overset{\bar{\varepsilon}}{\rightleftarrows}} L_n^Q(A_z) \underset{\bar{B}}{\overset{B}{\rightleftarrows}} L_{n-1}^S(A)$$

in a natural way.

[ ]

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