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The Whitney Trick

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Abstract

The Whitney Trick is a procedure by which submanifolds of a given manifold can, under certain conditions, be isotoped off each other by using an embedded 2-disc. It works well when the ambient manifold is of dimension five or greater. Here, a particularly simple example of its breakdown in dimension four is exhibited.

Keywords: Whitney Trick; Counter-example; Dimension four

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The following arguments apply in each of the piecewise-linear, smooth or topological categories. Suppose M is an orientable manifold containing locally-flat orientable connected submanifolds P and Q, with P and Q of complementary dimension, meeting transversely in precisely two points r and s of opposite sign. Under these conditions, the Whitney Trick is the following procedure (from [3]). The proposition below is used to show the existence of a 2-disc, using which we may construct an isotopy $H: M \times I \to M \times I$ such that $H(-,0)$ is the identity on M, and $(p \circ H(P,1)) \cap (p \circ H(Q,0)) = \emptyset$, where $p: M \times I \to M$ is projection onto the first factor.

Proposition. *Suppose P^p and Q^q are locally flat connected submanifolds of the manifold M^m, that $p+q=m$, and that P and Q intersect transversely. Suppose that $r,s \in P \cap Q$ and that either*

$$p \geqslant 3,\ q \geqslant 3 \quad \textit{and} \quad \pi_1(M) = 0$$

or

$$p = 2,\ q \geqslant 3 \quad \textit{and} \quad \pi_1(M - Q) = 0.$$

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Then there exist arcs α and β in P and Q respectively, running from r to s, which do not run through any other intersections, and a locally-flat 2-disc D^2 embedded in M with $\partial D^2 = \alpha \cup \beta$ and $D^2 \cap (P \cup Q) = \partial D^2$.

This procedure is, for example, a central step in the proof of the h-cobordism theorem, as it is here that we go from a hypothesis of algebraic topology (the signs of the intersection points) to a conclusion of homeomorphism (the existence of an isotopy). It is well known that handle-body theory does not work well in lower dimensions. In particular, Donaldson [1] has shown that, in the smooth case, the h-cobordism theorem is false for cobordisms between 4-manifolds, and as the Whitney Trick is the sole obstruction to proving the h-cobordism theorem in this dimension, it too fails. I exhibit here a very simple example of the breakdown of the Whitney Trick.

Proposition. *The proposition as stated above fails for $p = q = 2$, that is, if $\pi_1(M) = \pi_1(M - Q) = 0$, we cannot necessarily find the required disc.*

Proof. Let $M = S^2 \times S^2$, and $Q = S^2 \times \{*\}$. So $\pi_1(M) = \pi_1(M - Q) = 0$. Note that $[-1, 1] \times S^1$ embeds in S^2 and hence $S^2 \times [-1, 1] \times S^1$ embeds in M. Arrange that $\{*\}$ lies on $\{0\} \times S^1$. Let K be any nontrivial oriented knot of S^1 in S^3, and $\mathrm{rev}\,\overline{K}$ the reverse of its reflection. Then $K + \mathrm{rev}\,\overline{K}$ is nontrivial, as it is a well-known theorem in knot theory that the sum of two nontrivial knots is nontrivial. Now, the construction of $K + \mathrm{rev}\,\overline{K}$ gives us an embedding of S^2 in S^3 which cuts the knot in two points. This S^2 divides S^3 into two 3-balls, an interior one and an exterior one. By removing small 3-balls from the interior and the exterior, away from $K + \mathrm{rev}\,\overline{K}$, we get an embedding of S^1 in $S^2 \times [-1, 1]$ (see Fig. 1), which I shall also call $K + \mathrm{rev}\,\overline{K}$. Let P be the embedding of the torus $(K + \mathrm{rev}\,\overline{K}) \times S^1$ in $S^2 \times [-1, 1] \times S^1$, and so in $S^2 \times S^2$. Thus $P \cap Q$ is two points r and s.

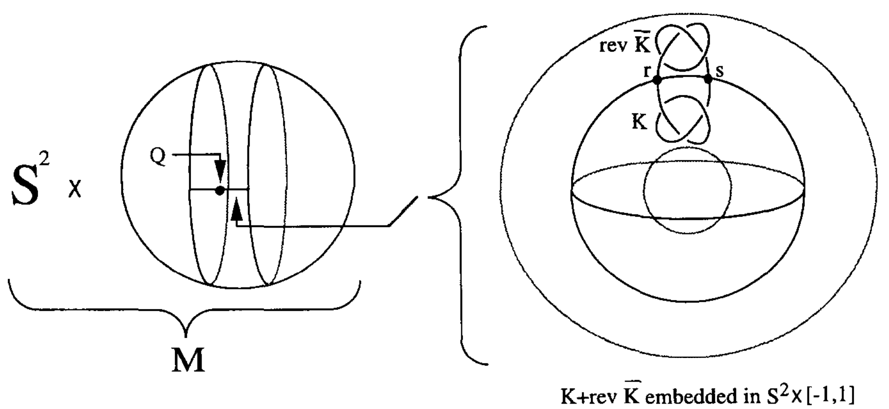

Fig. 1.

I show that there does not exist a map $f: I^2 \to M$ with $f(\partial I^2) = \alpha.\beta$ and $f(\mathrm{int}(I^2)) \cap (P \cup Q) = \emptyset$, where the dot denotes path composition. Assume there is such a map. Consider S^2 embedded in 3-space as the sphere of radius 2, with centre the origin, and let $\psi: S^2 \to [-2, 2]$ be the map that projects elements to their x-coordinates. Define $\phi: S^2 \to [-1, 1]$ as

$$\phi(z) = \begin{cases} \psi(z), & \text{if } \psi(z) \in [-1,1]; \\ 1, & \text{if } \psi(z) \geqslant 1; \\ -1, & \text{if } \psi(z) \leqslant -1. \end{cases}$$

Thus the map $g = \mathrm{id} \times \phi: S^2 \times S^2 \to S^2 \times [-1, 1]$ takes Q to $S^2 \times \{0\}$, and P to $(K + \mathrm{rev}\,\overline{K})$. Now consider $g \circ f: I^2 \to S^2 \times [-1, 1]$. Every point of $\mathrm{int}(I^2)$ misses $K + \mathrm{rev}\,\overline{K}$, which I claim is not possible. For we have a map of a disc into $S^2 \times [-1, 1]$ which sends ∂I^2 to $g(\alpha).\, g(\beta)$, where $g(\alpha)$ goes $n + 1/2$ times round $K + \mathrm{rev}\,\overline{K}$, $n \in \mathbb{Z}$. If we then reflect $S^2 \times [-1, 1]$ in $S^2 \times \{0\}$, we get that $\overline{g(\alpha).\, g(\beta)}$ is homotopically trivial via a homotopy whose interior misses $K + \mathrm{rev}\,\overline{K}$, where $\overline{g(\alpha)}$ is the reflection of $g(\alpha)$ in $S^2 \times \{0\}$. Thus $g(\beta)^{-1}.\, \overline{g(\alpha)}^{-1}$ is similarly homotopically trivial. Using the homotopy of $g(\alpha).\, g(\beta)$ to zero and of $g(\beta)^{-1}.\, \overline{g(\alpha)}^{-1}$ to zero, we can construct a map $h: I^2 \to S^2 \times [-1, 1]$ with $h(\mathrm{int}(I^2))$ missing $K + \mathrm{rev}\,\overline{K}$ and sending ∂I^2 to $g(\alpha).\, \overline{g(\alpha)}^{-1}$, which is a loop going an odd number of times round $K + \mathrm{rev}\,\overline{K}$.

Now the Loop Theorem for 3-manifolds states that if X is a 3-manifold and F is a connected 2-manifold in ∂X with $i_*: \pi_1(F) \to \pi_1(X)$ not injective, then there exists a properly embedded locally-flat 2-disc in X, the boundary of which is homotopically nontrivial in F. Let N be a regular neighborhood of $K + \mathrm{rev}\,\overline{K}$. Let $X = (S^2 \times [-1, 1]) - \mathrm{int}(N)$ and $F = \partial N$. A simple corollary is that the fact that $K + \mathrm{rev}\,\overline{K}$ is knotted implies that $i_*: \pi_1(\partial N) \to \pi_1(X)$ is injective.

In this case we take N sufficiently close to $K + \mathrm{rev}\,\overline{K}$ so that $\partial N \cap h(I^2)$ is a path γ in ∂N representing an odd multiple of a longitudinal element of $\pi_1(\partial N)$, plus perhaps multiples of the meridian, so is nonzero in $\pi_1(\partial N)$. Thus, using the Loop Theorem, γ is nonzero in $\pi_1(X)$. However, by construction, γ is homotopically trivial in X, and so we have a contradiction. □

It is worth noting two things. Firstly, although the Whitney Trick fails here, it is still possible to construct an isotopy of M, which pulls P off Q. For consider the image of P and Q under the projection of $S^2 \times S^2$ onto the second factor. The image of Q is only a point, whereas the image of P is contained in $[-1, 1] \times S^1$. Thus, we can construct an isotopy of M using a rotation of the second S^2 so that these images are disjoint. This will therefore make P and Q disjoint.

Secondly, this simple counter-example assumes a little too much. Here, $\pi_1(M - (P \cup Q)) \neq 0$, as we have effectively given a set of curves that could not be homotoped to a constant. A stronger counter-example would assume that $\pi_1(M - (P \cup Q)) = 0$, but that still the required disc cannot be embedded. The construction and proof of such a result exists in the smooth case, as mentioned earlier ([1], see also [2]) and requires the use of much greater machinery.

References

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