Splitting formulas for certain Waldhausen Nil-groups

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Abstract

We provide splitting formulas for certain Waldhausen Nil-groups. We focus on Waldhausen Nil-groups associated to *acylindrical* amalgamations $\Gamma = G_1 *_H G_2$ of the groups G_1 and G_2 over a common subgroup H. For these amalgamations, we explain how, provided that G_1, G_2 and Γ satisfy the Farrell–Jones isomorphism conjecture, the Waldhausen Nil-groups $\mathrm{Nil}_*^W(RH; R[G_1 - H], R[G_2 - H])$ can be expressed as a direct sum of Nil-groups associated to a specific collection of virtually cyclic subgroups of Γ. A special case covered by our theorem is the case of arbitrary amalgamations over a finite group H.

1. *Introduction*

Waldhausen's Nil-groups were introduced in the two seminal papers [**35**, **36**]. The motivation behind these Nil-groups originated from a desire to have a Mayer–Vietoris type sequence in algebraic K-theory. More precisely, if a group $\Gamma = G_1 *_H G_2$ splits as an amalgamation of two groups G_1 and G_2 over a common subgroup H, then one can ask how the algebraic K-theory of the group ring $R\Gamma$ is related to the algebraic K-theory of the integral group rings RG_1, RG_2, and RH. Motivated by the corresponding question in homology (or cohomology), one might expect a Mayer–Vietoris type exact sequence:

$$\cdots \longrightarrow K_{i+1}(R\Gamma) \longrightarrow K_i(RH) \longrightarrow K_i(RG_1) \oplus K_i(RG_2) \longrightarrow K_i(R\Gamma) \longrightarrow \cdots.$$

A major result in [**35**, **36**] was the realization that the Mayer–Vietoris sequence above holds, provided that one inserts suitable 'error terms', which are the Waldhausen Nil-groups associated to the amalgamation $\Gamma = G_1 *_H G_2$. In general, associated to any ring S (such as RH), and any pair of flat S-bimodules M_1 and M_2 (such as the $R[G_i - H]$), Waldhausen defines Nil-groups $\mathrm{Nil}_*^W(S; M_1, M_2)$. The Waldhausen Nil-groups $\mathrm{Nil}_*^W(RH; R[G_1 - H], R[G_2 - H])$ are the 'error terms' mentioned above.

Another context in which these Nil-groups make an appearance has to do with the *reduction to finites*. To explain this we recall the existence of a generalized equivariant homology theory, having the property that, for any group Γ, one has an isomorphism:

$$H_n^\Gamma(*; \mathbb{K}R^{-\infty}) \cong K_n(R\Gamma).$$

The term appearing on the left-hand side is the homology of the Γ-space consisting of a point $*$ with the trivial Γ-action. Now, for any Γ-space X, the obvious map $X \to *$ is clearly Γ-equivariant, and hence induces an *assembly map* homomorphism:

$$H_n^\Gamma(X; \mathbb{K}R^{-\infty}) \longrightarrow H_n^\Gamma(*; \mathbb{K}R^{-\infty}) \cong K_n(R\Gamma).$$

The Farrell–Jones isomorphism conjecture [**15**] asserts that, if $X = E_{\mathcal{VC}}\Gamma$ is a model for the classifying space for Γ-actions with isotropy in the family of virtually cyclic subgroups, then

the homomorphism described above is actually an isomorphism. Explicit models for $E_{\mathcal{VC}}\Gamma$ are known for only a few classes of groups: virtually cyclic groups, crystallographic groups (Alves and Ontaneda [**1**] and Connolly, Fehrman, and Hartglass [**9**]), hyperbolic groups (Lück [**25**] and Juan-Pineda and Leary [**19**]), and relatively hyperbolic groups (Lafont and Ortiz [**23**]). In contrast, classifying spaces for *proper* actions, denoted by $E_{\mathcal{FIN}}\Gamma$, are known for many classes of groups. Now, for any group Γ, one always has a unique (up to Γ-equivariant homotopy) map $E_{\mathcal{FIN}}\Gamma \to E_{\mathcal{VC}}\Gamma$, which induces a well-defined relative assembly map:

$$H_n^{\Gamma}(E_{\mathcal{FIN}}\Gamma; \mathbb{K}R^{-\infty}) \longrightarrow H_n^{\Gamma}(E_{\mathcal{VC}}\Gamma; \mathbb{K}R^{-\infty}).$$

In view of the Farrell–Jones isomorphism conjecture, it is reasonable to ask whether this latter map is itself an isomorphism. Bartels [**4**] has shown that the relative assembly map is split injective for arbitrary groups Γ, and arbitrary rings R. If the relative assembly map discussed above is actually an isomorphism, then we say that Γ satisfies the *reduction to finites.*

Let us now specialize to the case where $R = \mathbb{Z}$, that is, we will be focusing on integral group rings. In this situation, the obstruction to the relative assembly map being an isomorphism lies in the Nil-groups associated to the various infinite virtually cyclic subgroups of Γ. More precisely, if every infinite virtually cyclic subgroup $V \leqslant \Gamma$ satisfies the reduction to finites, then the entire group Γ satisfies the reduction to finites (see [**15**, Theorem A.10]). Now, for a virtually cyclic group V, the *failure* of the reduction to finites can be measured by the cokernel of the relative assembly map. Let us recall that infinite virtually cyclic groups V come in two flavors (see [**32**]):

- groups that surject onto the infinite dihedral group $D_\infty = \mathbb{Z}_2 * \mathbb{Z}_2$, and hence can be decomposed as $V = A *_C B$, with A, B, and C finite, and C of index two in the groups A and B;
- groups that do *not* surject onto D_∞ that can always be written in the form $V = F \rtimes_\alpha \mathbb{Z}$, where F is a finite group and $\alpha \in \mathrm{Aut}(F)$.

In the case where V surjects onto D_∞, the cokernel of the relative assembly map coincides with the Waldhausen Nil-group associated to the splitting $V = A *_C B$. In the case where V does not surject onto D_∞, the cokernel of the relative assembly map consists of two copies of the Farrell Nil-group associated to $V = F \rtimes_\alpha \mathbb{Z}$, denoted by $NK_*(\mathbb{Z}F, \alpha)$.

We now have two contexts in which Waldhausen Nil-groups make an appearance:

(1) they measure failure of the Mayer–Vietoris sequence in algebraic K-theory;
(2) they contain obstructions for groups to satisfy the reduction to finites.

Having motivated our interest in these groups, we can now state our main theorem.

MAIN THEOREM.   *Let $\Gamma = G_1 *_H G_2$ be an acylindrical amalgamation, and assume that the Farrell–Jones isomorphism conjecture holds for the groups Γ, G_1, and G_2. Denote by $\mathcal{V}$ a collection of subgroups of Γ consisting of one representative from each conjugacy class of subgroups V satisfying:*

(1) *V is virtually cyclic;*
(2) *V is not conjugate to a subgroup of G_1 or G_2;*
(3) *V is maximal with respect to subgroups satisfying* (1) *and* (2).

Then for arbitrary rings R we have the following isomorphisms:

$$\mathrm{Nil}_*^W(RH; R[G_1 - H], R[G_2 - H]) \cong \bigoplus_{V \in \mathcal{V}} H_*^V(E_{\mathcal{FIN}}V \longrightarrow *; \mathbb{K}R^{-\infty}),$$

where $H_*^V(E_{\mathcal{FIN}}V \to *; \mathbb{K}R^{-\infty})$ are the cokernels of the relative assembly maps associated to the virtually cyclic subgroups $V \in \mathcal{V}$.

REMARK 1. An amalgamation $\Gamma = G_1 *_H G_2$ is said to be *acylindrical* if there exists an integer k such that, for every path η of length k in the Bass–Serre tree T associated to the splitting of Γ, the stabilizer of η is finite. The notion of an acylindrical amalgamation was first formulated by Sela [33] in relation to his work on the accessibility problem for finitely generated groups. We use a generalization of Sela's original definition that is due to Delzant [13]. More generally, given a family $\mathcal{C}$ of subgroups, Delzant calls an amalgamation $(k,\mathcal{C})$-acylindrical provided that, for the associated action of Γ on its Bass–Serre tree, the stabilizer of any path of length k lies in $\mathcal{C}$. The version that we are using corresponds to the case where $\mathcal{C} = \mathcal{FIN}$, the family of finite subgroups of Γ.

Let us now give some examples of acylindrical amalgamations. Observe that, if the amalgamating subgroup H is finite, then the amalgamation is automatically acylindrical (with $k = 1$), as every edge will have a finite stabilizer. If the amalgamating subgroup H has the property that $|gHg^{-1} \cap H| < \infty$ for every $g \in G_i - H$ (for either i), then the amalgamation is acylindrical (with $k = 3$). In particular, we note that this is satisfied for any acylindrical amalgamation where H is malnormal in either G_i. We refer the reader to Remark 9 for a more thorough discussion of this notion, and to Corollary 4 for a concrete application.

REMARK 2. The cokernels $H_*^{V_i}(E_{\mathcal{FIN}}V_i \to *; \mathbb{K}R^{-\infty})$ are the familiar Waldhausen or (two copies of the) Farrell Nil-groups, according to whether the virtually cyclic group V_i surjects onto D_∞ or not. Note that every virtually cyclic subgroup V that maps onto D_∞ contains a canonical index two subgroup V' that does *not* map onto D_∞ (the pre-image of the obvious index two subgroup $\mathbb{Z} \triangleleft D_\infty$). Recent independent work by various authors (Davis [11], Davis, Khan, and Ranicki [12], as well as a paper in preparation by Davis, Quinn, and Reich) has established that the Waldhausen Nil-group of V is isomorphic to the Farrell Nil-group of V'.

REMARK 3. From the computational viewpoint, the Main Theorem combined with Remark 2 completely reduces (modulo the isomorphism conjecture) the computation of Waldhausen Nil-groups associated to acylindrical amalgamations to that of Farrell Nil-groups.

REMARK 4. Consider the simple case of a *free product* $\Gamma = G_1 * G_2$. In this situation, the group Γ is known to be (strongly) relatively hyperbolic, relative to the subgroups G_1 and G_2. Assuming the Farrell–Jones isomorphism conjecture for Γ, previous work of the authors [24, Corollary 3.3] yields the following expression for $K_n(R\Gamma)$:

$$H_n^\Gamma(E_{\mathcal{FIN}}\,\Gamma) \oplus \left(\bigoplus_{i=1,2} H_n^{G_i}(E_{\mathcal{FIN}}G_i \longrightarrow E_{\mathcal{VC}}G_i) \right) \oplus \left(\bigoplus_{V \in \mathcal{V}} H_n^V(E_{\mathcal{FIN}}V \longrightarrow *) \right),$$

where $\mathcal{V}$ is the collection of virtually cyclic subgroups mentioned in our Main Theorem (we omitted the coefficients $\mathbb{K}R^{-\infty}$ to simplify notation). Now, morally speaking, the Waldhausen Nil-group is the portion of the K-theory of Γ that does *not come from the K-theory of the factors G_i*. In the expression above, it is clear that the second term is determined by the K-theory of the factors. Furthermore, recalling the well-known fact that every finite subgroup of Γ has to be conjugate into one of the G_i, one sees that the first term also comes from the K-theory of the factors. Hence the Waldhausen Nil-group should correspond to the last term in the expression above. Our Main Theorem came about from trying to make this heuristic precise.

2. *Proof of Main Theorem*

Given the group $\Gamma = G_1 *_H G_2$ satisfying the hypotheses of our theorem, let us form the family $\mathcal{F}$ of subgroups of Γ consisting of all virtually cyclic subgroups that can be conjugated into either $G_1 \leqslant \Gamma$ or $G_2 \leqslant \Gamma$. Observe that we have a containment of families $\mathcal{F} \subset \mathcal{VC}$ of subgroups of Γ, which in turn induces an assembly map:

$$\rho : H_n^\Gamma(E_{\mathcal{F}}\,\Gamma; \mathbb{K}R^{-\infty}) \to H_n^\Gamma(E_{\mathcal{VC}}\,\Gamma; \mathbb{K}R^{-\infty}).$$

Our proof will focus on analyzing the map ρ, and, in particular, on gaining an understanding of the cokernel of that map. Let us start by describing the Waldhausen Nil-group as the cokernel of a suitable assembly map. Corresponding to the splitting $\Gamma = G_1 *_H G_2$, we have a simplicial action of Γ on the corresponding Bass–Serre tree T (see [**34**]). From the natural Γ-equivariant map $T \to *$, we get an assembly map:

$$\rho' : H_n^\Gamma(T; \mathbb{K}R^{-\infty}) \longrightarrow H_n^\Gamma(*; \mathbb{K}R^{-\infty}) \cong K_*(R\,\Gamma).$$

We begin with the following important fact.

FACT.   The map ρ' is split injective, and

$$\mathrm{coker}(\rho') \cong \mathrm{Nil}_*^W(RH; R[G_1 - H], R[G_2 - H]).$$

A proof of the Fact can be found in Davis [**11**, Lemma 7] (see also Remark 5 at the end of this section). In view of this result, we are merely trying to identify the cokernel of the map ρ'. The first step is to relate the cokernel of ρ' with the cokernel of the map ρ.

CLAIM 1.   The map ρ is split injective, and there is a canonical isomorphism

$$\mathrm{coker}(\rho) \cong \mathrm{coker}(\rho').$$

Proof.   We observe that we have four families of subgroups that we are dealing with: the three that we have looked at so far are $\mathcal{VC}$, $\mathcal{ALL}$, and the family $\mathcal{F}$ that we introduced at the beginning of our proof (consisting of virtually cyclic subgroups conjugate into one of the G_i); in addition, there is the family $\mathcal{G}$ consisting of *all* subgroups of Γ that can be conjugated into either G_1 or G_2. Now observe that we have containments of families $\mathcal{F} \subset \mathcal{G} \subset \mathcal{ALL}$, and $\mathcal{F} \subset \mathcal{VC} \subset \mathcal{ALL}$. Furthermore, we have that T is a model for $E_{\mathcal{G}}\,\Gamma$. This yields the following commutative diagram:

$$
\begin{array}{ccc}
H_*^\Gamma(E_{\mathcal{F}}\,\Gamma; \mathbb{K}R^{-\infty}) & \xrightarrow{\ \ \rho\ \ } & H_*^\Gamma(E_{\mathcal{VC}}\,\Gamma; \mathbb{K}R^{-\infty}) \\
\downarrow & & \downarrow{\scriptstyle \cong} \\
H_*^\Gamma(T; \mathbb{K}R^{-\infty}) & \xrightarrow[\ \ \rho'\ \ ]{} & H_*^\Gamma(*; \mathbb{K}R^{-\infty})
\end{array}
$$

where all the maps are relative assembly maps corresponding to the inclusions of the various families of subgroups. Note that the horizontal maps are precisely the ones that we are trying to relate. Now recall that we are assuming that Γ satisfies the Farrell–Jones isomorphism conjecture. This immediately implies that the second vertical map is an isomorphism, as indicated in the commutative diagram. So, in order to identify the cokernels of the two horizontal maps, we are left with showing that the first vertical map is also an isomorphism.

The first vertical map is a relative assembly map, corresponding to the inclusion of the families $\mathcal{F} \subset \mathcal{G}$ of subgroups of Γ. In order to show that the relative assembly map is an isomorphism, one merely needs to establish that, for every maximal subgroup $H \in \mathcal{G} - \mathcal{F}$, the corresponding relative assembly map induced by the inclusions of families $\mathcal{F}(H) \subset \mathcal{G}(H)$ of subgroups of H

is an isomorphism (see [**15**, Theorem A.10; **26**, Theorem 2.3]). But observe that the maximal subgroups in $\mathcal{G} - \mathcal{F}$ are precisely the (conjugates of) the subgroups $G_i \leqslant \Gamma$. Furthermore, for these subgroups, we have that $\mathcal{G}(G_i) = \mathcal{ALL}(G_i)$, and that $\mathcal{F}(G_i) = \mathcal{VC}(G_i)$. Hence the relative assembly maps that we require to be isomorphisms are exactly those induced by $E_{\mathcal{VC}}G_i \to E_{\mathcal{ALL}}G_i \cong *$, that is, those that arise in the Farrell–Jones isomorphism conjecture. Since we are assuming that the isomorphism conjecture holds for the groups G_1 and G_2, we conclude that the first vertical map is indeed an isomorphism, completing the proof of the claim. $\qquad\square$

At this point, combining Claim 1 with the Fact, we have an identification:

$$\operatorname{coker}(\rho) \cong \operatorname{coker}(\rho') \cong \operatorname{Nil}_*^W(RH; R[G_1 - H], R[G_2 - H]).$$

In order to complete the proof, we now focus entirely on studying the map ρ, with the goal of showing that one can express its cokernel as a direct sum of the desired Nil-groups associated with the virtually cyclic subgroups $V \in \mathcal{V}$. We remind the reader that ρ is the relative assembly map induced by the map $E_{\mathcal{F}}\Gamma \to E_{\mathcal{VC}}\Gamma$, where $\mathcal{F}$ is the family of subgroups consisting of all virtually cyclic subgroups of Γ that can be conjugated into either G_i.

In order to analyze this relative assembly map, we will need to make use of some properties of the Γ-action on the Bass–Serre tree. Particularly, we would like to understand the behavior of virtually cyclic subgroups $V \in \mathcal{VC} - \mathcal{F}$. The specific result that we will require is contained in our following claim.

CLAIM 2. In the case of an acylindrical amalgamation, the stabilizer of any geodesic γ in the Bass–Serre tree T is a virtually cyclic subgroup of Γ. Furthermore, every virtually cyclic subgroup $V \leqslant \Gamma$ satisfying $V \in \mathcal{VC} - \mathcal{F}$ stabilizes a unique geodesic $\gamma \subset T$.

Proof. Let us start by recalling some basic facts concerning the action of Γ on the Bass–Serre tree T corresponding to the amalgamation $\Gamma = G_1 *_H G_2$:
- the action is without inversions, that is, if an element stabilizes an edge e, then it automatically preserves the chosen orientation of e;
- the stabilizer of any vertex $v \in T$ is isomorphic to a conjugate of G_1 or G_2;
- the stabilizer of any edge $e \subset T$ is isomorphic to a conjugate of H;
- any finite subgroup of Γ fixes a vertex in T.

The first three statements above are built into the definition of the Bass–Serre tree (see [**34**]), while the last statement is a well-known general fact about group actions on trees. We remind the reader that a *geodesic* in a tree T will be a subcomplex simplicially isomorphic to $\mathbb{R}$, with the standard simplicial structure (that is, vertices at the integers, and edges between).

To show the first statement in our claim, we note that $\operatorname{Stab}_\Gamma(\gamma)$ clearly fits into a short exact sequence:

$$0 \longrightarrow \operatorname{Fix}_\Gamma(\gamma) \longrightarrow \operatorname{Stab}_\Gamma(\gamma) \longrightarrow \operatorname{Simp}_{\Gamma,\gamma}(\mathbb{R}) \longrightarrow 0,$$

where $\operatorname{Fix}_\Gamma(\gamma)$ is the subgroup fixing γ pointwise, and $\operatorname{Simp}_{\Gamma,\gamma}(\mathbb{R})$ is the induced simplicial action on $\mathbb{R}$ (obtained by simplicially identifying γ with $\mathbb{R}$). Note that the group of simplicial automorphisms of $\mathbb{R}$ is D_∞, the infinite dihedral group. In particular, we see that $\operatorname{Simp}_{\Gamma,\gamma}(\mathbb{R})$ is virtually cyclic (in fact, it is isomorphic to either the trivial group, $\mathbb{Z}_2, \mathbb{Z}$, or D_∞).

Next we observe that $\operatorname{Fix}_\Gamma(\gamma)$ is finite. To see this, we recall that the amalgamation $\Gamma = G_1 *_H G_2$ was assumed to be *acylindrical*, which means that there exists an integer $k \geqslant 1$ with the property that the stabilizer $\operatorname{Stab}_\Gamma(\eta) \leqslant \Gamma$ of any combinatorial path $\eta \subset T$ of length at least k is finite. Since $\gamma \subset T$ is a geodesic, it contains combinatorial subpaths η of arbitrarily long length (in particular, length at least k). The obvious containment $\operatorname{Fix}_\Gamma(\gamma) \leqslant \operatorname{Stab}_\Gamma(\eta)$ now completes the argument for the first statement in our claim.

For the second statement, we note that Γ acts simplicially on the Bass–Serre tree, and hence the given virtually cyclic subgroup $V \in \mathcal{VC} - \mathcal{F}$ likewise inherits an action on T. Since $V \notin \mathcal{F}$, we have that the V-action on T has no globally fixed point, and hence cannot be finite. In particular, V must be an *infinite* virtually cyclic subgroup, and hence contains a finite index infinite cyclic normal subgroup $V' \triangleleft V$, with V' generated by $g \in V$. We now claim that g cannot fix any vertex in T.

Assume, by way of contradiction, that there exists a vertex v fixed by g (and hence by V'). Let $T' \subset T$ be the subset consisting of points that are fixed by V'. Note that T' is non-empty (since $v \in T'$), and is a subtree of T (since Γ acts simplicially on T). Furthermore, observe that the group $F := V/V'$ inherits a simplicial action on T'. But note that F is finite, and hence the F-action on T' has a fixed vertex $w \in T' \subset T$. But this immediately implies that the original group V fixes w; a contradiction as $V \notin \mathcal{F}$.

Now establishing that V stabilizes a geodesic is a straightforward application of standard techniques in the geometry of group actions on trees (applied to T). For the convenience of the reader, we give a quick outline of the argument. For an arbitrary element g of infinite order in V, one can look at the associated displacement function on T, which is the distance from v to $g \cdot v$. The previous paragraph established that this function is strictly positive. One then considers the set $\mathrm{Min}(g)$ of points in T that minimize the displacement function, and call this minimal value μ_g. It is easy to see that:

- $\mathrm{Min}(g)$ contains a geodesic γ (take a vertex $v \in \mathrm{Min}(g)$, and consider $\gamma := \bigcup_{i \in \mathbb{Z}} g^i \cdot \eta$, where η is the geodesic segment from v to $g \cdot v$; note that such a non-trivial segment exists by the previous paragraph);
- in fact, $\gamma = \mathrm{Min}(g)$ (any point at distance $r > 0$ from γ will be displaced $2r + \mu_g > \mu_g$, and so cannot lie in $\mathrm{Min}(g)$);
- for any non-zero integer i, we have $\mathrm{Min}(g) = \mathrm{Min}(g^i)$ (any point at distance $r > 0$ from γ will be displaced $2r + |i| \cdot \mu_g > |i| \cdot \mu_g$, while points on γ will clearly only be displaced $|i| \cdot \mu_g$ by the element g^i);
- for any two elements g and h of infinite order in V, we have $\mathrm{Min}(g) = \mathrm{Min}(h)$ (two such elements have a common power, and then apply the previous statement).

From the observations above, we see that every single element in V of infinite order stabilizes the exact same geodesic $\gamma \subset T$.

Hence the only elements that we might have to worry about are elements $h \in V$ of *finite order*. For these, we just note that V contains $V' \triangleleft V$, a finite index cyclic normal subgroup generated by an element g of infinite order. We have a natural morphism from $H = \langle h \rangle$ to $\mathrm{Aut}(V') \cong \mathbb{Z}/2$. In particular, we have that $hgh^{-1} = g^{\pm 1}$ and hence, for any vertex $v \in \mathrm{Min}(g)$, we have the obvious equalities:

$$d(g \cdot hv, hv) = d(h^{-1}gh \cdot v, v) = d(g^{\pm 1} \cdot v, v) = \mu_{g^{\pm 1}} = \mu_g.$$

Since hv is minimally displaced by g, it must also lie on $\gamma = \mathrm{Min}(g)$. This deals with elements of finite order, and hence completes the verification that γ is V-invariant. Finally, from the fact that V has a (finite index) subgroup that acts on γ via a translation, it is easy to see that there are no other V-invariant geodesics in T, yielding its uniqueness. This finishes the argument for our Claim 2. $\qquad\square$

Having established some basic properties of the Γ-action on T, we now return to the main argument. Recall that, by combining our Claim 1 with the Fact, we have reduced the proof of the Main Theorem to understanding the cokernel of the relative assembly map

$$\rho : H_n^\Gamma(E_\mathcal{F}\,\Gamma; \mathbb{K}R^{-\infty}) \longrightarrow H_n^\Gamma(E_{\mathcal{VC}}\,\Gamma; \mathbb{K}R^{-\infty}),$$

where $\mathcal{F}$ is the family of subgroups consisting of all virtually cyclic subgroups of Γ that can be conjugated into either G_i.

Now recall that in [**23**] the authors introduced the notion of a collection of subgroups to be *adapted* to a nested family of subgroups, and, in the presence of an adapted family, showed how a model for the classifying space with isotropy in the smaller family could be 'promoted' to a model for the classifying space with isotropy in the larger family. In a subsequent paper [**24**], the authors used some recent work of Lück and Weiermann [**27**] to give an alternative model for this classifying space, which had the additional advantage of providing *explicit splittings* for the cokernel of the relative assembly maps. Let us briefly recall the relevant definitions.

Given a nested pair of families $\mathcal{F} \subset \tilde{\mathcal{F}}$ of subgroups of Γ, we say that a collection $\{H_\alpha\}_{\alpha \in I}$ of subgroups of Γ is *adapted* to the pair $(\mathcal{F}, \tilde{\mathcal{F}})$ provided that the following hold.

(1) For all $H_1, H_2 \in \{H_\alpha\}_{\alpha \in I}$, either $H_1 = H_2$, or $H_1 \cap H_2 \in \mathcal{F}$.

(2) The collection $\{H_\alpha\}_{\alpha \in I}$ is *conjugacy closed*, that is, if $H \in \{H_\alpha\}_{\alpha \in I}$ then $gHg^{-1} \in \{H_\alpha\}_{\alpha \in I}$ for all $g \in \Gamma$.

(3) Every $H \in \{H_\alpha\}_{\alpha \in I}$ is *self-normalizing*, that is, $N_\Gamma(H) = H$.

(4) For all $G \in \tilde{\mathcal{F}} \setminus \mathcal{F}$, there exists $H \in \{H_\alpha\}_{\alpha \in I}$ such that $G \leqslant H$.

In [**24**], we applied this result to the nested family $\mathcal{FIN} \subset \mathcal{VC}$ for relatively hyperbolic groups (for which an adapted collection of subgroups is easy to find).

In our present context, we would like to find a collection of subgroups adapted to the nested pair of families $\mathcal{F} \subset \mathcal{VC}$. Recall that $\mathcal{F}$ consists of all virtually cyclic subgroups that can be conjugated into either G_i, and $\mathcal{VC}$ consists of all virtually cyclic subgroups of Γ.

CLAIM 3. The collection $\{H_\alpha\}$ of subgroups of Γ consisting of all *maximal* virtually cyclic subgroups in $\mathcal{VC} - \mathcal{F}$ is adapted to the pair $(\mathcal{F}, \mathcal{VC})$.

Proof. To verify this, we first note that properties (2) and (4) in the definition of an adapted family are immediate. Property (3) follows easily from Claim 2. We let $V \in \{H_\alpha\}$ be given, and consider $N_\Gamma(V)$. We know that V leaves invariant a unique geodesic $\gamma \subset T$. Furthermore, for every $g \in \Gamma$, we see that gVg^{-1} leaves $g \cdot \gamma$ invariant. The uniqueness of the V-invariant geodesic γ now implies that γ is actually $N_\Gamma(V)$-invariant. In particular, we have containments $V \leqslant N_\Gamma(V) \leqslant \text{Stab}_\Gamma(\gamma)$. But from Claim 2 we know that $\text{Stab}_\Gamma(\gamma)$ is virtually cyclic, and the maximality of V now forces all of the containments to be equalities, and, in particular, $V = N_\Gamma(V)$, as required by property (3).

For property (1), let $V_1, V_2 \in \{H_\alpha\}$. We want to establish that either $V_1 = V_2$, or that $V_1 \cap V_2 \in \mathcal{F}$. So let us assume that $V_1 \neq V_2$. We know from Claim 2 that each V_i stabilizes a unique geodesic γ_i, and, from the maximality of the groups V_i, we actually have $V_i = \text{Stab}_\Gamma(\gamma_i)$. Since $V_1 \neq V_2$, we have that $\gamma_1 \neq \gamma_2$. There are now two possibilities: (i) either $\gamma_1 \cap \gamma_2 = \emptyset$, or (ii) $\gamma_1 \cap \gamma_2$ is a path in T. We claim that in both cases the intersection $H = V_1 \cap V_2 \leqslant V_i$ has the property that the H-action on the corresponding γ_i *fixes a point*.

To see this, let us first consider possibility (i). Since $\gamma_1 \cap \gamma_2 = \emptyset$, one can consider the (unique) minimal length geodesic segment η joining γ_1 to γ_2. We observe that, since H stabilizes both γ_i, it must leave the segment η invariant. In particular, H must fix the vertex $v_i = \eta \cap \gamma_i \in \gamma_i$, as desired. Next consider possibility (ii). If $\gamma_1 \cap \gamma_2 \neq \emptyset$, then the intersection will be a subpath $\eta \subset \gamma_i$. Note that η is either a geodesic *segment*, or is a geodesic *ray*, and in both cases will be invariant under the group H. If η is a geodesic ray (that is, homeomorphic to $[0, \infty)$), then there is a (topologically) distinguished point inside η that will have to be fixed by H. If η is a geodesic segment, then each element in H either fixes η, or reverses η (note that the latter can only occur if η has *even* length, as Γ acts on T without inversions). In particular, we see that, if η has odd length, then every point in η is fixed by H, while, if η has even length, then the (combinatorial) midpoint is fixed.

Finally, we observe that $H \leqslant V_i$ acts on γ_i, and fixes a point. This immediately implies that H contains a subgroup of index at most two that *acts trivially* on γ_i, that is, $H' \leqslant \text{Fix}_\Gamma(\gamma_i)$.

But recall that the latter group is finite (see the proof of Claim 2), completing the proof of property (1). We conclude that the collection $\{H_\alpha\}$ is an adapted collection of subgroups for the nested families $(\mathcal{F}, \mathcal{VC})$, as desired. $\qquad\square$
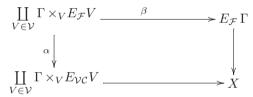
Finally, we exploit the adapted family that we have just constructed to establish the following.

CLAIM 4.   We have an identification:
$$\operatorname{coker}(\rho) \cong \bigoplus_{V \in \mathcal{V}} H_*^V(E_{\mathcal{FIN}}V \longrightarrow *; \mathbb{K}R^{-\infty}),$$
where the groups $H_*^V(E_{\mathcal{FIN}}V \to *; \mathbb{K}R^{-\infty})$ are the cokernels of the relative assembly maps associated with the virtually cyclic groups $V \in \mathcal{V}$.

Proof.   The argument for this is virtually identical to the one given in [**24**, Corollary 3.2]; we reproduce the argument here for the convenience of the reader. From Claim 3, we have an adapted family for the pair $(\mathcal{F}, \mathcal{VC})$, consisting of all maximal subgroups in $\mathcal{VC} - \mathcal{F}$. From this adapted collection of subgroups, [**24**, Proposition 3.1] establishes (using [**27**, Theorem 2.3]) a method for constructing an $E_{\mathcal{VC}}\Gamma$; namely, if $\mathcal{V}$ is a complete set of representatives of the conjugacy classes within the adapted collection of subgroups $\{H_\alpha\}$, then we form the following cellular Γ-pushout.

$$
\begin{array}{ccc}
\coprod\limits_{V \in \mathcal{V}} \Gamma \times_V E_{\mathcal{F}}V & \xrightarrow{\quad\beta\quad} & E_{\mathcal{F}}\,\Gamma \\
{\scriptstyle\alpha}\Big\downarrow & & \Big\downarrow \\
\coprod\limits_{V \in \mathcal{V}} \Gamma \times_V E_{\mathcal{VC}}V & \xrightarrow{\qquad\qquad} & X
\end{array}
$$

Then the resulting space X is a model for $E_{\mathcal{VC}}\Gamma$ (we refer the reader to [**24**, Proposition 3.1] for a more precise discussion of this result, including a description of the maps α and β in the above cellular Γ-pushout). Note that the map ρ whose cokernel we are trying to understand is precisely the map on (equivariant) homology induced by the second vertical arrow in the above cellular Γ-pushout.

Since X is the double-mapping cylinder of the maps α and β in the above diagram, one has a natural Γ-equivariant decomposition of X by taking A and B to be the $[0, 2/3)$ and $(1/3, 1]$, respectively, portions of the double-mapping cylinder. Applying the homology functor $H_*^\Gamma(-; \mathbb{K}R^{-\infty})$ (and omitting the coefficients to shorten notation), we have the Mayer–Vietoris sequence
$$\cdots \longrightarrow H_*^\Gamma(A \cap B) \longrightarrow H_*^\Gamma(A) \oplus H_*^\Gamma(B) \longrightarrow H_*^\Gamma(X) \longrightarrow H_{*-1}^\Gamma(A \cap B) \longrightarrow \cdots.$$

But now observe that we have obvious Γ-equivariant homotopy equivalences between:
 (i) $A \simeq_\Gamma \coprod_{V \in \mathcal{V}} \Gamma \times_V E_{\mathcal{VC}}V$;
 (ii) $B \simeq_\Gamma E_{\mathcal{F}}\,\Gamma$;
 (iii) $A \cap B \simeq_\Gamma \coprod_{V \in \mathcal{V}} \Gamma \times_V E_{\mathcal{F}}V$.
Furthermore, the homology theory that we have takes disjoint unions into direct sums. Combining this with the induction structure, we obtain the following isomorphisms:
$$H_*^\Gamma(A) \cong \bigoplus_{V \in \mathcal{V}} H_*^\Gamma(\Gamma \times_V E_{\mathcal{VC}}V) \cong \bigoplus_{V \in \mathcal{V}} H_*^V(E_{\mathcal{VC}}V),$$
$$H_*^\Gamma(A \cap B) \cong \bigoplus_{V \in \mathcal{V}} H_*^\Gamma(\Gamma \times_V E_{\mathcal{F}}V) \cong \bigoplus_{V \in \mathcal{V}} H_*^V(E_{\mathcal{F}}V).$$

Now observing that the groups $V \in \mathcal{V}$ are all virtually cyclic, we have that each $E_{\mathcal{VC}}V$ can be taken to be a point. Furthermore, for the groups $V \in \mathcal{V}$, we have that the restriction of the family $\mathcal{F}$ to V coincides with the family of finite subgroups of V, that is, $E_{\mathcal{F}}V = E_{\mathcal{FIN}}V$. Substituting all of this in the above Mayer–Vietoris sequence, we get the long exact sequence

$$\cdots \longrightarrow \bigoplus_{V \in \mathcal{V}} H_*^V(E_{\mathcal{FIN}}V) \longrightarrow H_*^{\Gamma}(E_{\mathcal{F}}\,\Gamma) \oplus \bigoplus_{V \in \mathcal{V}} H_*^V(*) \longrightarrow H_*^{\Gamma}(E_{\mathcal{VC}}\,\Gamma) \longrightarrow \cdots .$$

Now observe that the each of the maps $H_*^V(E_{\mathcal{FIN}}V) \to H_*^V(*)$ are split injective (from Bartels' result [4]). Since the map $\rho : H_*^{\Gamma}(E_{\mathcal{F}}\,\Gamma) \to H_*^{\Gamma}(E_{\mathcal{VC}}\,\Gamma)$ is also split injective (from Claim 1), we now have an identification:

$$\operatorname{coker}(\rho) \cong \bigoplus_{V \in \mathcal{V}} \operatorname{coker}\left(H_*^V(E_{\mathcal{FIN}}V) \longrightarrow H_*^V(*)\right)$$

completing the proof of Claim 4. $\qquad \square$

Finally, combining the Fact with Claim 1 and Claim 4, we see that we have the desired splitting:

$$\operatorname{Nil}_*^W(RH; R[G_1 - H], R[G_2 - H]) \cong \bigoplus_{V \in \mathcal{V}} H_*^V(E_{\mathcal{FIN}}V \longrightarrow *; \mathbb{K}R^{-\infty}),$$

where the groups $H_*^V(E_{\mathcal{FIN}}V \to *; \mathbb{K}R^{-\infty})$ denote the cokernels appearing in Claim 4. This completes the proof of the Main Theorem.

REMARK 5. One of the key ingredients in our proof was the Fact, established by Davis in [11, Lemma 7]. Prior to learning of Davis's preprint, the authors had an alternative argument for the Fact. For the sake of the interested reader, we briefly outline our alternative approach.

Anderson and Munkholm [3, Section 7] defined a functor K_*^{cc}, continuously controlled K-theory, from the category of diagrams of holink type to the category of spectra. Munkholm and Prassidis [28, Theorem 2.1] showed that the Waldhausen Nil-group that we are interested in can be identified with the cokernel of a natural split injective map $\tilde{K}_{*+1}^{cc}(\xi^+) \to K_*(\mathbb{Z}\Gamma)$, where ξ^+ is a suitably defined diagram of holink type associated to the splitting $\Gamma = G_1 *_H G_2$ (see [3, Section 9]). Furthermore, Anderson and Munkholm [3, Theorem 9.1] have shown that there is a natural isomorphism $\tilde{K}_{*+1}^{cc}(\xi^+) \cong \tilde{K}_{*+1}^{bc}(\xi^+)$, where the latter is the boundedly controlled K-theory defined by Anderson and Munkholm [2]. Finally, there are Atiyah–Hirzebruch spectral sequences computing both the groups $\tilde{K}_{*+1}^{bc}(\xi^+)$ (see [2, Theorem 4.1]) and $H_*^{\Gamma}(T; \mathbb{K}\mathbb{Z}^{-\infty})$ (see [30, Section 8]). It is easy to verify that the two spectral sequences are *canonically identical*: they have the same E^2-terms and the same differentials. Combining these results, and keeping track of the various maps appearing in the sequence of isomorphisms, one can give an alternative proof of the Fact.

REMARK 6. We also point out that, from the Γ-action on the Bass–Serre tree T, it is easy to obtain constraints on the isomorphism type of groups inside the collection $\mathcal{V}$. Indeed, any such group must be the stabilizer of a bi-infinite geodesic $\gamma \subset T$ (see Claim 2), and must act cocompactly on γ. Recall that infinite virtually cyclic subgroups are of two types: those that surject onto the infinite dihedral group D_∞, and those that do not.

The groups that surject onto D_∞ always split as an amalgamation $A *_C B$, with all three groups A, B, and C being finite, and C of index two in both A and B. Observe that, if $V \in \mathcal{V}$ is of this type, then, under the action of V on $\gamma \subset T$, the groups A and B can be identified with the stabilizers of a pair of vertices v and w, and C can be identified with the stabilizer of the segment joining v to w. In particular, C *must be a subgroup of an edge stabilizer*, and hence

is conjugate (within Γ) to a finite subgroup of H. Furthermore, since A and B both stabilize a pair of vertices, they must be conjugate (within Γ) to a finite subgroup of either G_1 or G_2.

The groups that do not surject onto D_∞ are of the form $F \rtimes_\alpha \mathbb{Z}$, where F is a finite group, and $\alpha \in \mathrm{Aut}(F)$ is an automorphism. If $V \in \mathcal{V}$ is of this form, then, for the action of V on $\gamma \subset T$, one has that F can be identified with the subset of V that pointwise fixes γ, while the $\mathbb{Z}$ component acts on γ via a translation. In particular, F is again conjugate (within Γ) to a finite subgroup of H.

In particular, if we are given an explicit amalgamation $\Gamma = G_1 *_H G_2$, and we have knowledge of the finite subgroups inside the groups H, G_1 and G_2, then we can readily identify up to isomorphism the possible groups arising in the collection $\mathcal{V}$. If one has knowledge of the Nil-groups associated with these various groups, our Main Theorem can be used to get corresponding information about the Waldhausen Nil-group associated to Γ.

REMARK 7.   The attentive reader will notice that the hypothesis of 'acylindricity' was used only in the proof of Claim 2, where it was used to show that, for every geodesic $\gamma \subset T$, the subgroup $\mathrm{Fix}_\Gamma(\gamma) \leqslant \Gamma$ consisting of elements that fix γ pointwise is, in fact, finite. In particular, the conclusion of the Main Theorem holds in a slightly more general setting; namely, rather than requiring amalgamations $\Gamma = G_1 *_H G_2$ to be acylindrical, it is sufficient to require them to have the property that, for the action of Γ on the associated Bass–Serre tree, every geodesic $\gamma \subset T$ satisfies $|\mathrm{Fix}_\Gamma(\gamma)| < \infty$. The authors do not know of any way to ensure this a priori more general condition, other than by showing that the amalgamation is acylindrical.

REMARK 8.   We remark that there is also a version of the Waldhausen Nil-groups associated to HNN extensions. The proof of our Main Theorem should extend to provide a similar splitting for HNN extensions, under an identical hypothesis on the action of Γ on the Bass–Serre tree associated to the HNN extension (that is, acylindricity, or, more generally, finiteness of the subgroups fixing geodesics in the Bass–Serre tree). The key missing step in the HNN case is the analog of the Fact, which the authors have not verified, but believe to be true. Modulo this result, the rest of our argument extends verbatim to the HNN setting.

It is also perhaps worth mentioning that, unlike the amalgamation case, in the HNN case there are easy examples of extensions satisfying the more general condition discussed in Remark 7. For example, if we consider the classical Baumslag–Solitar groups $\mathrm{BS}(1, n) := \{a, b \mid aba^{-1} = b^n\}$, then these come with a natural realization as an HNN extension, with the vertex group generated by the subgroup $\langle b \rangle$, the edge group isomorphic to $\mathbb{Z} = \langle c \rangle$, and the two edge to vertex inclusions given by $c \mapsto b$ and $c \mapsto b^n$. In the associated Bass–Serre tree, we note that there is a single distinguished end that is fixed by the entire $\mathrm{BS}(1, n)$, while all of the other ends of the Bass–Serre tree are only left invariant by the trivial element. In particular, since every geodesic $\eta \subset T$ has two distinct ends, one of the two has a trivial stabilizer, and hence the fixed group of η must likewise be trivial. This implies that the HNN extension satisfies the property discussed in our Remark 7. On the other hand, every finite subtree of T has an infinite stabilizer (isomorphic to $\mathbb{Z}$), and hence the amalgamation is *not* acylindrical. We thank M. Forester for bringing this simple example to our attention.

3. *Applications*

Having completed the proof of our Main Theorem, we now proceed to isolate a few interesting corollaries. As mentioned earlier, from the viewpoint of topological applications, the most interesting situation is the case where $R = \mathbb{Z}$, that is, integral group rings.

COROLLARY 1. *Let $\Gamma = G_1 *_H G_2$ be an amalgamation, and assume that the Farrell–Jones isomorphism conjecture holds for the groups $\Gamma, G_1,$ and $G_2,$ and that H is finite. Then the associated Waldhausen Nil-group $\mathrm{Nil}_*^W(\mathbb{Z}H; \mathbb{Z}[G_1 - H], \mathbb{Z}[G_2 - H])$ is either trivial, or an infinitely generated torsion group.*

Proof. Note that, since H is finite, the amalgamation is acylindrical, and our Main Theorem applies. So the Waldhausen group we are interested in splits as a direct sum of Nil-groups associated to a particular collection $\mathcal{V}$ of virtually cyclic subgroups. It is well known that the Nil-groups associated to virtually cyclic groups are either trivial or infinitely generated (see [**14**, **17**, **31**]). Furthermore, these groups are known to be purely torsion (see [**10**, **18**, **22**, **37**]), giving us the second statement. $\square$

A special case of the above corollary is worth mentioning.

COROLLARY 2. *Let $\Gamma = G_1 *_H G_2$ be an amalgamation, and assume that $G_1, G_2,$ and H are all finite. Then the associated Waldhausen Nil-group $\mathrm{Nil}_*^W(\mathbb{Z}H; \mathbb{Z}[G_1 - H], \mathbb{Z}[G_2 - H])$ is either trivial, or an infinitely generated torsion group.*

Proof. The two groups G_1 and G_2 trivially satisfy the Farrell–Jones isomorphism conjecture, as they are finite. Furthermore, the group Γ is δ-hyperbolic, and hence, by a recent result of Bartels, Lück, and Reich [**5**], also satisfies the isomorphism conjecture. Hence the hypotheses of our previous corollary are satisfied. $\square$

As was kindly pointed out to the authors by J. Grunewald, the 'torsion group' conclusion in our Corollary 2 has also been independently obtained, via different methods, by Bartels, Lück, and Reich [**6**, Theorem 0.15].

Finally, we observe that the Bartels, Lück, and Reich [**5**] result establishes the isomorphism conjecture when Γ is a δ-hyperbolic group, in all dimensions, and for arbitrary coefficient rings R. In particular, the hypotheses of our Main Theorem hold for arbitrary amalgamations of finite groups, giving the following corollary.

COROLLARY 3. *Let $\Gamma = G_1 *_H G_2$ be an amalgamation, where $G_1, G_2,$ and H are all finite. Then, for arbitrary rings with unity R, we have the following isomorphisms:*

$$\mathrm{Nil}_*^W(RH; R[G_1 - H], R[G_2 - H]) \cong \bigoplus_{V \in \mathcal{V}} H_*^V(E_{\mathcal{FIN}}V \longrightarrow *; \mathbb{K}R^{-\infty}),$$

where $H_*^V(E_{\mathcal{FIN}}V \to *; \mathbb{K}R^{-\infty})$ are the cokernels of the relative assembly maps associated to the virtually cyclic subgroups $V \in \mathcal{V},$ and the collection $\mathcal{V}$ is as in the statement of our Main Theorem.

We point out that the special case of the modular group $\Gamma = \mathrm{PSL}_2(\mathbb{Z}) = \mathbb{Z}_2 * \mathbb{Z}_3$ has also been independently studied by Davis, Khan, and Ranicki [**12**, Section 3.3].

REMARK 9. As was mentioned in Remark 1, another convenient class of acylindrical amalgamations are those of the form $\Gamma = G_1 *_H G_2$, where H has the property that $|gHg^{-1} \cap H| < \infty$ for every $g \in G_1 - H$ (we call this property *conjugacy $\mathcal{FIN}$-separability*). In the case where both G_i are δ-hyperbolic, and H is also assumed to be quasi-convex in both G_i, then the resulting amalgamation Γ is known to be δ-hyperbolic (see [**7**, **8**; **21**, Theorem 2]). Combining

these two comments, we see that, in the situation where $H \leqslant G$ is a quasi-convex, conjugacy $\mathcal{FIN}$-separable subgroup of a δ-hyperbolic group G, the resulting amalgamation $\Gamma = G *_H G$ is an acylindrical amalgamation, with both vertex groups G and the amalgamated group Γ satisfying the Farrell–Jones isomorphism conjecture (by the recent work of Bartels, Lück, and Reich [5]). In particular, this would give rise to examples of amalgamations satisfying all of the hypotheses of our Main Theorem.

In the case where G is torsion-free, the property of conjugacy $\mathcal{FIN}$-separability reduces to the more familiar notion of *malnormality*. Examples of quasi-convex malnormal subgroups in torsion-free δ-hyperbolic groups are plentiful. For example, in the situation where G is non-elementary, any maximal virtually cyclic subgroup $H < G$ is automatically quasi-convex and malnormal. As another example of how common such subgroups are, we mention a result of Kapovich [20]: every non-elementary subgroup K of a torsion-free δ-hyperbolic group G contains a further subgroup $F_2 \cong H < K$, isomorphic to a free group on two generators, that is malnormal and quasi-convex in the ambient G. Of course, these results do not yield interesting examples, since the lack of torsion elements implies that all of the groups involved will have vanishing K-theory (by [5]), and hence both sides of the isomorphism in our Main Theorem would trivially vanish.

It would perhaps be of some interest to see whether, in a non-elementary δ-hyperbolic group G with non-trivial torsion, one can find examples of quasi-convex subgroups $H < G$ that are conjugacy $\mathcal{FIN}$-separable. By the discussion above, such a pair would give rise, by doubling along the subgroup H, to a group $\Gamma = G *_H G$ satisfying the hypotheses of our Main Theorem, and for which the algebraic K-theory could potentially be non-trivial. The authors suspect that such subgroups should exist, and, in view of the results mentioned in the previous paragraph for the torsion-free case, we expect such examples to be plentiful.

To give a concrete example that fits into the general context discussed in our Remark 9 above, we establish our final corollary.

COROLLARY 4. *Let $G_i \leqslant O^+(3,1) = \mathrm{Isom}(\mathbb{H}^3)$ be a pair of uniform lattices in the isometry group of hyperbolic 3-space. Assume that $H_i \leqslant G_i$ are a pair of maximal infinite virtually cyclic subgroups that are abstractly isomorphic to a group H. Consider the amalgamated group $\Gamma = G_1 *_H G_2$, where H is identified with the subgroups H_i. Then, for $* \leqslant 1$, the associated Waldhausen Nil-group satisfies*

$$\mathrm{Nil}_*^W(\mathbb{Z}H; \mathbb{Z}[G_1 - H_1], \mathbb{Z}[G_2 - H_2]) = 0.$$

Proof. Since the groups G_i are uniform lattices in the semi-simple Lie group $O^+(3,1)$, they both satisfy the Farrell–Jones isomorphism conjecture (see [15]). Next we observe that, since the H_i are maximal in the respective G_i, they are automatically malnormal, and hence the amalgamation Γ is acylindrical. Finally, since we are amalgamating δ-hyperbolic groups along malnormal infinite virtually cyclic subgroups, the resulting group Γ is also δ-hyperbolic (see [7, 8, 21]), and so also satisfies the Farrell–Jones isomorphism conjecture ([5]). This completes the verification of the hypotheses of our Main Theorem, which allows us to conclude that there is a splitting:

$$\mathrm{Nil}_*^W(\mathbb{Z}H; \mathbb{Z}[G_1 - H_1], \mathbb{Z}[G_2 - H_2]) \cong \bigoplus_{V \in \mathcal{V}} H_*^V(E_{\mathcal{FIN}}V \longrightarrow *; \mathbb{K}\mathbb{Z}^{-\infty}).$$

We are now left with two goals: to identify the infinite virtually cyclic groups in the collection $\mathcal{V}$, and to show that the cokernels appearing on the left-hand side of the above expression all vanish.

Next we recall that, in the proof of our Main Theorem, we were able to identify groups in $\mathcal{V}$ with infinite stabilizers of geodesics γ in the Bass–Serre tree associated to the splitting $\Gamma = G_1 *_H G_2$ (see our Claim 2). Returning to the notation of our Claim 2, we now proceed to exploit the short exact sequence

$$0 \longrightarrow \mathrm{Fix}_\Gamma(\gamma) \longrightarrow \mathrm{Stab}_\Gamma(\gamma) \longrightarrow \mathrm{Simp}_{\Gamma,\gamma}(\mathbb{R}) \longrightarrow 0.$$

Recall that $\mathrm{Fix}_\Gamma(\gamma)$ is the subgroup fixing γ pointwise, and $\mathrm{Simp}_{\Gamma,\gamma}(\mathbb{R})$ is the induced simplicial action on $\mathbb{R}$ (and, in the present situation, it is isomorphic to either $\mathbb{Z}$, or D_∞). We now claim that, under our hypotheses, we also have control of the group $\mathrm{Fix}_\Gamma(\gamma)$.

Indeed, taking any two consecutive edges e_1 and e_2 along the geodesic g, and letting v be the common vertex, we note that $\mathrm{Fix}_\Gamma(\gamma)$ can be identified with a subgroup F of $\mathrm{Stab}_\Gamma(v)$. The latter is conjugate to one of the two groups G_i, a subgroup of the isometry group of $\mathbb{H}^3$. Now recall that the two subgroups $\mathrm{Stab}_\Gamma(e_i) \leqslant \mathrm{Stab}_\Gamma(v) \cong G_i$ can be identified geometrically as follows: if we think of $\mathrm{Stab}_\Gamma(v)$ acting isometrically on $\mathbb{H}^3$ (via the identification with the appropriate G_i), then each of the two groups $\mathrm{Stab}_\Gamma(e_1) \cong H \cong \mathrm{Stab}_\Gamma(e_2)$ correspond to the stabilizers of two distinct geodesics $\gamma_1, \gamma_2 \subset \mathbb{H}^3$. Now, since $\mathrm{Fix}_\Gamma(\gamma)$ also fixes the two incident edges e_1 and e_2, the corresponding subgroup $F \leqslant \mathrm{Stab}_\Gamma(v)$ acting on $\mathbb{H}^3$ is also a subgroup in both $\mathrm{Stab}_\Gamma(e_i)$, that is, it *must leave both of the geodesics γ_1 and γ_2 invariant*. But, given two distinct geodesics in $\mathbb{H}^3$, the subgroup of $\mathrm{Isom}(\mathbb{H}^3)$ leaving both geodesics invariant is either trivial or isomorphic to $\mathbb{Z}_2$. This implies that the group $\mathrm{Fix}_\Gamma(\gamma)$ is either trivial or $\mathbb{Z}_2$. It is not too hard from the geometry (a slight modification of the argument in [**23**, Proposition 3.6]) to see that the short exact sequence must split as a direct product, forcing the groups that we are interested in to each be abstractly isomorphic to one of the four groups $\mathbb{Z}, D_\infty, \mathbb{Z}_2 \times \mathbb{Z}$, or $\mathbb{Z}_2 \times D_\infty$. Finally, when V is any of the four virtually cyclic groups listed above, the cokernels of the relative assembly map $H_*^V(E_{\mathcal{F}\mathcal{I}\mathcal{N}}V \to *; \mathbb{K}\mathbb{Z}^{-\infty})$ are known to vanish for $* \leqslant 1$ (see [**16**] for $\mathbb{Z}$ and D_∞, and [**29**] for $\mathbb{Z}_2 \times \mathbb{Z}$ and $\mathbb{Z}_2 \times D_\infty$). We conclude that every term in the direct sum splitting for the Waldhausen Nil-group vanishes, completing the proof of our corollary. $\square$

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