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End homology and duality

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Abstract. This paper gives a simplicial chain complex description of the compactly supported cohomology and Borel-Moore homology for polyhedra. These are compared to ordinary cohomology and homology theories, called end cohomology and end homology. The end homology of an open manifold M of dimension n satisfies Poincaré duality in dimension $n-1$.

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The purpose of this paper is to give a simplicial chain complex description of the compactly supported cohomology and Borel–Moore homology for polyhedra. These are compared to ordinary cohomology and homology and the difference is measured by suitable cohomology and homology theories, called end cohomology and end homology. The end homology and end cohomology of an open manifold M of dimension n satisfy Poincaré duality in dimension $n-1$. If M is the interior of a compact manifold with boundary ∂M then the end homology (resp. cohomology) of M is realized as the ordinary homology (resp. cohomology) of the space ∂M. Many results are well-known from sheaf theory or Alexander–Spanier theory; our aim is to present an elementary simplicial construction. The algebraic core of the paper is a study of Borel–Moore duals and limits of chain complexes which might also have some independent interest.

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1. Duality and limits of chain complexes

Let R be a principal ideal domain. Denote by $C(R)$ the category of chain complexes of R-modules and chain maps. We allow chain groups also in negative degrees but require the complexes to be bounded below. Let $C^*(R)$ denote the corresponding category of cochain complexes. If K is the fraction field of R we consider the canonical injective resolution

$$K \to K/R$$

of R both as a chain complex R_* and as a cohain complex R^* with K in degree 0.

Definition 1.1. The *dual of a chain complex* C_* is the cochain complex $DC_* = \operatorname{Hom}_R(C_*, R_*)$ and the *dual of the cochain complex* C^* is the chain complex $DC^* = \operatorname{Hom}_R(C^*, R^*)$.

This agrees with the notion of the dual of a differential graded module by Borel and Moore [BM, p. 142]. More precisely, we have

$$DC_*^n = \operatorname{Hom}_R(C_n, K) \oplus \operatorname{Hom}_R(C_{n-1}, K/R)$$

with the differentials induced from C_* and R_*, and likewise

$$DC_n^* = \operatorname{Hom}_R(C^n, K) \oplus \operatorname{Hom}_R(C^{n+1}, K/R).$$

(Note that if C^* starts in dimension 0 then DC^* starts in dimension -1. This is the reason why we consider complexes bounded below instead of non-negative ones.)

Chain maps induce dual chain maps and we get functors

$$D: C(R) \to C^*(R), \quad D: C^*(R) \to C(R).$$

Lemma 1.2. i) *If a chain map* $f: C_1 \to C_2$ *induces isomorphism on homology, so does its dual* $Df: DC_2 \to DC_1$.

ii) *There are natural short exact sequences*

$$0 \to \operatorname{Ext}_R(H_{n-1}(C_*), R) \to H^n(DC_*) \to \operatorname{Hom}_R(H_n(C_*), R) \to 0$$
$$0 \to \operatorname{Ext}_R(H^{n+1}(C^*), R) \to H_n(DC^*) \to \operatorname{Hom}_R(H^n(C^*), R) \to 0$$

which are split.

Proof. ii) is clear since R_* is an injective resolution of R: the spectral sequence of the double complex DC breaks into exact sequences.

i) follows from ii) and the five lemma. □

The "naive" duality functor Hom(, R) satisfies the universal coefficient formulas of Lemma 1.2 only on the subcategory of free complexes. If the free complexes are moreover finitely generated, one has natural double duality

$$\eta : C \cong \mathrm{Hom}(\mathrm{Hom}(C, R), R)$$

given by evaluation. This is generalized as follows. We say that a complex C has *locally finite homology*, if the groups $H_n(C)$ (resp. $H^n(C)$) are finitely generated R-modules for each n.

Proposition 1.3. *If C has locally finite homology then the canonical map $\eta : C \to DDC$ is a homology isomorphism.*

Proof. Suppose first that C is a chain complex. Since it has locally finite homology and R is Noetherian, we may find a homology isomorphism $f : P_* \to C_*$ where P_n is finitely generated and free for each n. By Lemma 1.2 the map DDf is a homology isomorphism, so it is enough to prove the claim for P_*.

$$\begin{array}{ccc} P_* & \xrightarrow{\eta_P} & DDP_* \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle DDf} \\ C_* & \xrightarrow{\eta_C} & DDC_* \end{array}$$

Since P_* is free, the inclusion $\mathrm{Hom}(P_*, R) \xrightarrow{j} DP_*$ is a homology isomorphism. Now P_* is finitely generated, so that $\mathrm{Hom}(P_*, R)$ is again free and the inclusion

$$k : \mathrm{Hom}(\mathrm{Hom}(P_*, R), R) \to D\mathrm{Hom}(P_*, R)$$

is a homology isomorphism. But $P_* \cong \mathrm{Hom}(\mathrm{Hom}(P_*, R), R)$ and the result follows from the commuting triangle

$$\begin{array}{ccc} & & D\mathrm{Hom}(P_*, R) \\ & \overset{k}{\nearrow} & \uparrow{\scriptstyle Dj} \\ P_* & & \\ & \underset{\eta_P}{\searrow} & \\ & & DDP_* \end{array}$$

where k and Dj are homology isomorphisms. For cochain complexes one may decompose the cohomology groups into free and torsion parts, $H^n(C) = F^n(C) \oplus T^n(C)$, and prove the claim by considering the universal coefficient sequence of Lemma 1.2.. We omit the details. □

Remark. The derived category is defined as the localization of the category of complexes and homotopy classes of chain maps with respect to homology isomorphisms. In this language Proposition 1.3 may be expressed by saying that the duality functors establish a natural equivalence between the derived categories of chain and cochain complexes with locally finite homology, cf. [Ha, V. § 1].

We next turn to questions of exactness and limits. A sequence of complexes and chain maps is called *exact*, if it is exact at each degree. As R_* and R^* consists of injective modules, it is clear that the duality functor D is exact. The *direct limit* $C = \varinjlim C(i)$ of a direct sequence of complexes and chain maps

$$C(\cdot) : C(0) \to C(1) \to C(2) \to \cdots$$

is defined by taking direct limit in each degree separately. The direct limit is an exact functor so that the homology $H_n(C) = \varinjlim H_n(C(i))$ only depends on the homology of the system $C(\cdot)$. The (degreewise) *inverse limit* $\varprojlim C(i)$ of an inverse sequence of complexes is not an exact functor, and an inverse limit of acyclic complexes need not be acyclic. However, duals of direct sequences behave properly:

Proposition 1.4. *Let* $C(0) \xrightarrow{f_0} C(1) \xrightarrow{f_1} \ldots$ *be a direct sequence of complexes.*

i) *The complexes* $D \varinjlim C(i)$ *and* $\varprojlim DC(i)$ *are canonically isomorphic.*

ii) *There are natural short exact sequences*

$$0 \to \varprojlim{}^1 H^{n-1}(DC(i)) \to H^n(\varprojlim DC(i)) \to \varprojlim H^n(DC(i)) \to 0$$

$$0 \to \varprojlim{}^1 H_{n+1}(DC(i)) \to H_n(\varprojlim DC(i)) \to \varprojlim H_n(DC(i)) \to 0$$

for chain and cochain complexes, respectively.

Proof. i) The direct limit $\varinjlim C(i)$ can be defined as the cokernel in the short exact sequence

$$(*) \qquad 0 \to \bigoplus_{i=0}^{\infty} C(i) \xrightarrow{f} \bigoplus_{i=0}^{\infty} C(i) \to \varinjlim C(i) \to 0$$

where

$$f(c_0, c_1, c_2, \ldots) = (c_0, c_1 - f_0(c_0), c_2 - f_1(c_1), \ldots).$$

The exact functor D turns $(*)$ to the short exact sequence

$$0 \to D \varinjlim C(i) \to \prod_{i=0}^{\infty} DC(i) \xrightarrow{Df} \prod_{i=0}^{\infty} DC(i) \to 0$$

where

$$Df(d_0, d_1, d_2, \dots) = (d_0 - Df_0(d_1), d_1 - Df_1(d_2), \dots).$$

But the kernel of Df is by definition $\varprojlim DC(i)$. This proves i).

The claim ii) follows from the homology sequence of (∗∗) using the definition of $\varprojlim^1$ [M1, p. 338] as the derived functor of the inverse limit. □

Remark. The argument shows more generally that any inverse system $C(\cdot)$ of complexes for which the complex $\varprojlim^1 C$ is acyclic satisfies the Milnor sequences of Proposition 1.4ii). This holds in particular when $\varprojlim^1 C = 0$, e.g. each map $C(n) \to C(n-1)$ is surjective. A homotopy invariant inverse limit of complexes is obtained by using $\varprojlim^1$-acyclic resolutions. In particular, if each $C(i)$ has locally finite homology, the inverse system $DDC(i)$ is $\varprojlim^1$-acyclic and the homotopy inverse limit

$$\operatorname{ho}\varprojlim C(i) = \varprojlim DDC(i) = D \varinjlim DC(i)$$

has good formal properties. In fact it is the derived functor of $\varprojlim$ on the category of complexes bounded below with locally finite homology in the sense of [Ha, I § 5].

2. Homology of polyhedra

We shall work in the category of polyhedra of [RS]. A subset $X \subset \mathbb{R}^m$ is a *polyhedron* if each point $a \in X$ has a cone neighborhood $N = aL$ in X where L is compact. A polyhedron can be triangulated as a locally finite simplicial complex [RS, p. 12]. Conversely, an abstract simplicial complex can be geometrically realized as a polyhedron if and only if it is countable, locally finite and finite-dimensional [Sp, p. 120].

Let X be a polyhedron. An *exhaustion* (K_i) of X is a sequence of compact subpolyhedra

$$K_0 \subset K_1 \subset K_2 \subset \dots$$

such that $X = \bigcup K_i$ and $K_i \subset \operatorname{Int} K_{i+1}$ for each i. Exhaustions always exist and X may be triangulated so that the sets K_i are finite subcomplexes [RS, proof of Th. 2.2 p. 12]. We fix one exhaustion and a corresponding triangulation.

The closed subset $X \setminus \operatorname{Int} K_i$ is a subcomplex of X. Consider the exact sequence of complexes of simplicial cochains

$$0 \to C^*(X, X \setminus \operatorname{Int} K_i) \to C^*(X) \to C^*(X \setminus \operatorname{Int} K_i) \to 0$$

with coefficients in R. As the cochain complex of a relatively finite simplicial complex $C^*(X, X \setminus \operatorname{Int} K_i)$ is finitely generated and free. The exact sequences of complexes form a direct system and the limit

$$(2.1) \qquad 0 \to C_c^*(X) \to C^*(X) \to C_e^*(X) \to 0$$

is still exact, where $C_c^*(X) = \varinjlim C^*(X, X \setminus \mathrm{Int}\, K_i)$ and $C_e^*(X) = \varinjlim C^*(X \setminus \mathrm{Int}\, K_i)$. More precisely, if S_n is the set of n-simplices of X, then

$$C_c^n(X) = \bigoplus_{S_n} R, \quad C^n(X) = \prod_{S_n} R, \quad C_e^n(X) = \bigoplus_{S_n} R / \prod_{S_n} R.$$

Definition 2.2. Let X be a polyhedron with exhaustion (K_i). The *cohomology* of X with *compact supports* is

$$H_c^n(X) = H^n(C_c^*(X)) = \varinjlim H^n(X, X \setminus \mathrm{Int}\, K_i)$$

and the *end cohomology* is

$$H_e^n(X) = H^n(C_e^*(X)) = \varinjlim H^n(X \setminus \mathrm{Int}\, K_i).$$

Remark. End cohomology of polyhedra in dimensional 0 has been used in the classical theory of Freudenthal, Hopf and Specker in connection with surfaces, 3-manifolds and discrete groups. For a modern exposition see e.g. [AS], [E], [M2] or [R]. The following results are collected from an unpublished Master's Thesis of Jussi Talsi [T]. The *space of ends* $e(X)$ of a polyhedron X (or more generally any locally path connected, locally compact space) in the sense of [F] and [Ho] can be interpreted as the inverse limit

$$e(X) = \varprojlim \pi_0(X \setminus K)$$

where K runs through the compact subsets of X, or a suitable cofinal collection like an exhaustion. It is always a Hausdorff totally disconnected space and $X^e = X \cup e(X)$ has a natural locally compact topology where X is an open and dense set. If X has only finitely many components, the space X^e is compact, called the *Freudenthal* or *end compactification* of X. For a polyhedron with finitely many components, the end space $e(X)$ is homeomorphic to a closed subset of the Cantor set, and the connection with end cohomology is that $H_e^0(X)$ is isomorphic to the Čech cohomology $\check{H}^0(e(X))$ of the space of ends $e(X)$.

End cohomology has been introduced under the name cohomology of the ideal boundary by Raymond [R, p. 951] in Alexander-Spanier theory and Bredon [B, p. 79] in sheaf theory and under the name cohomology at infinity by Godbillon [G] in Čech theory. Massey [Ma] gives a chain complex description based on finitely valued Alexander-Spanier cochains.

As homology commutes with direct limits, the long exact sequence of (2.1) takes the form

$$(2.3) \qquad 0 \to H_c^0(X) \to \cdots \to H_c^n(X) \to H^n(X) \to H_e^n(X) \to H_c^{n+1}(X) \to \cdots$$

Let Y be another polyhedron with exhaustion (L_i). If $f: X \to Y$ is a continuous proper map, i.e. $f^{-1}(C)$ is compact whenever $C \subset Y$ is compact, then $f(X \setminus \operatorname{Int} K_i)$ is contained in some $Y \setminus \operatorname{Int} L_j$ for each i. Hence f induces a mapping of direct systems $(H^n(Y, Y \setminus \operatorname{Int} L_j)) \to (H^n(X, X \setminus \operatorname{Int} K_i))$ and in the limit a map $f^*: H_c^n(Y) \to H_c^n(X)$. Properly homotopic maps give the same induced map (by the ordinary homotopy axiom). In particular, $H_c^n(X)$ and similarly $H_e^n(X)$ depend only on the proper homotopy type of X, but not on the chosen triangulation and exhaustion. The sequence (2.3) is functorial with respect to proper maps.

The groups $H^n(X, X \setminus \operatorname{Int} K_i)$ are always finitely generated since $(X, X \setminus \operatorname{Int} K_i)$ is a relatively finite complex. Consequently the groups $H_c^n(X)$ are countably generated, hence countable if R is countable. On the contrary $H^n(X)$ and $H_e^n(X)$ may well be uncountably generated.

Example 2.4. If X is a discrete countable set then

$$H_c^0(X) = \bigoplus_X R, \quad H^0(X) = \prod_X R, \quad H_e^0(X) = \prod_X R \Big/ \bigoplus_X R$$

and the other groups vanish. When the coefficient ring R is a field, the vector space $H_c^0(X)$ has countable dimension but the dimension of both $H^0(X)$ and $H_e^0(X)$ is the cardinality of the continuum. Hence H_c^* is "small" cohomology whereas H^* and H_e^* are "big" cohomologies.

Each compact set $K \subset X$ is contained in some K_i. Hence the sets $X \setminus \operatorname{Int} K_i$ form a cofinal system system in the set of all cobounded sets. Since Čech cohomology and sheaf cohomology agree with ordinary cohomology on polyhedral pairs, it follows from [Sp, Th. 6.6.15, p. 322] that $H_c^*(X)$ is isomorphic to the cohomology of X with compact supports in the sense of Čech cohomology or sheaf cohomology.

The simplicial chain complexes of the pair $(X, X \setminus \operatorname{Int} K_i)$

$$0 \to C_*(X \setminus \operatorname{Int} K_i) \to C_*(X) \to C_*(X, X \setminus \operatorname{Int} K_i) \to 0$$

form an inverse system of complexes. Taking ordinary inverse limits is now clearly inappropriate, since $C_0(X \setminus \operatorname{Int} K_i)$ form a decreasing sequence with

$$\varprojlim C_*(X \setminus \operatorname{Int} K_i) = \bigcap_{i=0}^{\infty} C_*(X \setminus \operatorname{Int} K_i) = 0\,.$$

Instead we replace the sequence with

$$C_*(X) \xrightarrow{p_i} C_*(X, X \setminus \operatorname{Int} K_i) \to C_*(p_i)$$

where $C_*(p_i)$ is the algebraic mapping cone of p_i. Taking inverse limits leads to a cofibration sequence

$$(2.5) \qquad C_*(X) \xrightarrow{p} C_*^{lf}(X) \to C_*^e(X)[1]$$

where $C_*^{lf}(X) = \varprojlim C_*(X, X \setminus \operatorname{Int} K_i)$ and $C_*^e(X) = \varprojlim C_*(p_i)[-1]$. More precisely if S_n is the set of n-simplices of X, then

$$C_n(X) = \bigoplus_{S_n} R, \quad C_n^{lf}(X) = \prod_{S_n} R, \quad C_n^e(X) = \bigoplus_{S_n} R \oplus \prod_{S_{n+1}} R.$$

Definition – Proposition 2.6. Let X be a polyhedron with exhaustion (K_i). The *locally finite homology* of X is $H_n^{lf}(X) = H_n(C_*^{lf}(X))$ and the *end homology* of X is $H_n^e(X) = H_n(C_*^e(X))$. There are short exact sequences

$$0 \to \varprojlim{}^1 H_{n+1}(X, X \setminus \operatorname{Int} K_i) \to H_n^{lf}(X) \to \varprojlim H_n(X, X \setminus \operatorname{Int} K_i) \to 0$$
$$0 \to \varprojlim{}^1 H_{n+1}(X \setminus \operatorname{Int} K_i) \to H_n^e(X) \to \varprojlim H_n(X \setminus \operatorname{Int} K_i) \to 0\,.$$

Proof. The inverse systems $C_*(X, X \setminus \operatorname{Int} K_i)$ and $C_*(p_i)$ consist of surjective chain mappings so their limits satisfy the Milnor exact sequence (Remark after Prop. 1.4). Since p_i is surjective, the homology of $C_*(p_i)$ is the homology of $\operatorname{Ker} p_i = C_*(X \setminus \operatorname{Int} K_i)$ translated by one. □

Remark. The locally finite homology seems to be folklore. End homology in the Borel-Moore homology corresponding to Alexander-Spanier theory is treated in Raymond [R, Section 2] and Massey [Ma, Section 10.2]. We could also have defined $C_*^e(X)$ as the quotient $C_*^{lf}(X)/C_*(X)$ translated by one as in the case of end cohomology. The above definition was chosen in order to represent the chain complex $C_*^e(X)$ as a suitable inverse limit.

The chain complexes $C_*(X)$ and $C_*^{lf}(X)$ start at dimension 0 but because of the translation $C_*^e(X)$ starts at dimension -1. Hence the exact sequence in homology of (2.5) is now

$$(2.7) \qquad \cdots \to H_n^e(X) \to H_n(X) \to H_n^{lf}(X) \to H_{n-1}^e(X) \to \cdots \to H_{-1}^e(X) \to 0\,.$$

The possibility $H_{-1}^e(X) \neq 0$ is not excluded:

Example 2.4, continuation. If X is a countable discrete set then

$$H_0(X) = \bigoplus_X R, \quad H_0^{lf}(X) = \prod_X R, \quad H_{-1}^e(X) = \prod_X R \Big/ \bigoplus_X R$$

and the other groups vanish. Hence H_* is a "small" homology and H_*^{lf} and H_*^e are "big" ones.

As in the case of cohomology one sees that $H_*^{lf}(X)$ and $H_*^e(X)$ only depend on the proper homotopy type of X and that 2.7 is functorial with respect to proper maps.

The ordinary cohomology $H^*(X)$ is dual to the ordinary homology $H_*(X)$ in the sense that $H^n(X)$ is determined by $H_{n-1}(X)$ and $H_n(X)$ via the universal coefficient theorem. It turns out that locally finite homology $H_*^{lf}(X)$ is dual to compactly supported cohomology $H_c^*(X)$.

Proposition 2.8. *For any polyhedron X there is an exact sequence*

$$0 \to \mathrm{Ext}_R(H_c^{n+1}(X), R) \to H_n^{lf}(X) \to \mathrm{Hom}(H_c^n(X), R) \to 0 .$$

Proof. $H_c^*(X)$ is the homology of $C_c^*(X) = \varinjlim C^*(X, X \setminus \mathrm{Int}\, K_i)$. The dual of $C_c^*(X)$ is

$$DC_c^*(X) = D \varinjlim C^*(X, X \setminus \mathrm{Int}\, K_i) = \varprojlim DC^*(X, X \setminus \mathrm{Int}\, K_i) .$$

Since $C_*(X, X \setminus \mathrm{Int}\, K_i)$ is a free chain complex, $C^*(X, X \setminus \mathrm{Int}\, K_i)$ is equivalent to $DC_*(X, X \setminus \mathrm{Int}\, K_i)$. By local finiteness Proposition 1.3 implies that the canonical map $C_*(X, X \setminus \mathrm{Int}\, K_i) \to DC^*(X, X \setminus \mathrm{Int}\, K_i)$ induces an isomorphism on homology. The homology of both inverse limits is determined by Milnor exact sequences (Prop. 1.4.ii) so that

$$C_*^{lf}(X) = \varprojlim C_*(X, X \setminus \mathrm{Int}\, K_i) \to \varprojlim DC^*(X, X \setminus \mathrm{Int}\, K_i)$$

is a homology isomorphism. The claim now follows from Lemma 1.2.ii). □

As a corollary it follows that the locally finite homology $H_*^{lf}(X)$ agrees with the sheaf-theoretic Borel-Moore homology with closed supports, and with Massey's version of Steenrod homology, since both homologies satisfy similar split exact sequences [B, p. 184], [Ma, p. 115]. This does not establish functoriality; it can be obtained on the chain level in the derived category by composing the equivalence of simplicial and singular homology and the equivalence of singular cohomology and sheaf alias Alexander-Spanier cohomology.

Incidentally, a more concrete proof of Proposition 2.8 can be obtained by noting that the matrices of the boundary maps $\partial_n : C_n(X) = \oplus_{S_n} R \to \oplus_{S_{n-1}} R = C_{n-1}(X)$ have only finitely many non-zero elements in each degree. The compactly supported cochain complex has the same chain groups but transposed differentials $d^n = \partial_n^T$, and $C_*^{lf}(X) = (\prod_{S_n} R, \partial_n)$ is the naive dual of the free complex $C_c^*(X)$. Hence 2.8 follows from the ordinary universal coefficient theorem.

Let us symbolize the dualities obtained so far by two arrows $H_n \xrightarrow{D} H^n$, and $H_c^n \xrightarrow{D} H_n^{lf}$, and insert them between the sequence 2.3 und 2.7:

$$\begin{array}{ccccccc} \cdots \to & H_c^n(X) & \to & H^n(X) & \to & H_e^n(X) & \to \cdots \\ & D\downarrow & & \uparrow D & & & \\ \cdots \leftarrow & H_n^{lf}(X) & \leftarrow & H_n(X) & \leftarrow & H_n^e(X) & \leftarrow \cdots \end{array}$$

To be precise, the arrows should be interpreted as duality functors on the chain level. Since they point to opposite directions, there is little hope for any duality between $H_e^*(X)$ and $H_*^e(X)$ without further assumptions. If X is assumed to have finitely generated homology, then we show in the next Proposition that H_*^e is the universal coefficient dual of H_e^*. This can be explained by noting that the right hand duality D can then be inverted, at least if we are using field coefficients (and in the derived category in general). On the other hand if X is a manifold, we shall see in the next section that Poincaré duality holds for $H_e(X)$ with a shift of dimensions.

Proposition 2.9. *If X is a polyhedron with finitely generated homology $H_*(X)$, there is an exact sequence*

$$0 \to \mathrm{Ext}_R(H_e^{n+1}(X), R) \to H_n^e(X) \to \mathrm{Hom}_R(H_e^n(X), R) \to 0\,.$$

Proof. By the universal coefficient theorem $C^*(X)$ has finitely generated homology. Since $C^*(X, X \setminus \mathrm{Int}\, K_i)$ has always finite homology, so does $C^*(X \setminus \mathrm{Int}\, K_i)$. The claim follows now by dualizing the complex $C_e^*(X) = \varinjlim C^*(X \setminus \mathrm{Int}\, K_i)$ as in the proof of Proposition 2.8 and using the Milnor exact sequences. □

The assumption that $H_*(X)$ is finitely generated does not imply that any of the groups $H_*^{lf}(X)$, $H_*^e(X)$, $H_c^*(X)$ or $H_e^*(X)$ is finitely generated.

Example 2.10. A tree with infinitely many branches. Let X be the union of the positive real axis and countably many non-intersecting half-lines starting at the integer points. Then X is contractible,

$$H_c^1(X) = \bigoplus_{i=0}^{\infty} R, \quad H^0(X) = R, \quad H_e^0(X) = \bigoplus_{i=0}^{\infty} R$$

and the other cohomology groups vanish. The dualities 2.8 and 2.9 then show that

$$H_1^{lf}(X) = \prod_{i=0}^{\infty} R, \quad H_0(X) = R, \quad H_0^e(X) = \prod_{i=0}^{\infty} R$$

are the only non-trivial homology groups.

3. Poincaré duality

Let M be an unbounded *PL-manifold of dimension n*, i.e. a polyhedron such that each point has a neighborhood PL-homeomorphic to an open set in $\mathbb{R}^n$. Then we may choose an exhaustion

$$M_0 \subset M_1 \subset M_2 \subset \cdots$$

such that each M_i is a compact PL-manifold with boundary by using regular neighborhoods [RS, Ch. 3]. Notice that if M is the interior of a compact manifold N with boundary ∂N we can use a collar on ∂N to choose an exhaustion such that all inclusions $M \setminus \operatorname{Int} M_{i+1} \subset M \setminus \operatorname{Int} M_i$ are homotopy equivalences, each space being homeomorphic to $\partial N \times [0, 1)$. Then the end cohomology and the end homology of M equal the ordinary cohomology and homology of ∂N,

$$H_e^*(M) \cong H^*(\partial N), \quad H_*^e(M) \cong H_*(\partial N)$$

and moreover

$$H_c^*(M) \cong H^*(N, \partial N), \quad H_*^{lf}(M) \cong H_*(N, \partial N).$$

As the inclusion $M \subset N$ is a homotopy equivalence, it is clear in this case that we get Poincaré dualities between $H_c^*(M)$ and $H_*(M)$, between $H^*(M)$ and $H_*^{lf}(M)$, and between the end theories $H_e^*(M)$ and $H_*^e(M)$ in dimension $n-1$.

We shall now indicate how Lefschetz duality for compact manifolds implies Poincaré duality for M and the end theories of M in general. This is well-known for ordinary homology and it is due to Raymond in the case of end theory [R, Th. 3.1]. We assume either that M is oriented or that char $R = 2$. Hence every M_i gets an R-orientation, compatible with each other.

Theorem 3.1. *Let M be an unbounded PL-manifold of dimension n. Then there is a commutative diagram with exact rows*

$$\begin{array}{ccccccccc}
\cdots \to & H_c^k(M) & \to & H^k(M) & \to & H_e^k(M) & \to & H_c^{k+1}(M) & \to \cdots \\
& \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
\cdots \to & H_{n-k}(M) & \to & H_{n-k}^{lf}(M) & \to & H_{n-k-1}^e(M) & \to & H_{n-k-1}(M) & \to \cdots
\end{array}$$

where the vertical maps are isomorphisms induced by capping with a fundamental class in $H_n^{lf}(M)$.

Proof. We have fundamental classes $\mu_i \in C_n(M, M \setminus \operatorname{Int} M_i) \cong C_n(M_i, \partial M_i)$ which are compatible to give in the limit a fundamental class $\mu \in C_n^{lf}(M)$. It is the sum of all n-simplices of M suitably oriented. Capping with μ_i gives rise to homomorphisms

$$\cap \mu_i : C^*(M) \to C_{n-*}(M, M \setminus \operatorname{Int} M_i)$$

which are again compatible to yield a cap homomorphism

$$\cap\mu : C^*(M) \to C^{lf}_{n-*}(M)$$

in the limit. It can be considered as a chain map with suitable signs. As $\cap\mu$ maps the subcomplex $C^*_c(M)$ to $C_{n-*}(M)$ we get a commutative diagram

$$\begin{array}{ccccccccc} 0 \to & C^*_c(M) & \to & C^*(M) & \to & C^*_e(M) & \to 0 \\ & \cap\mu\downarrow & & \cap\mu\downarrow & & \vdots\downarrow & \\ 0 \to & C_{n-*}(M) & \to & C^{lf}_{n-*}(M) & \to & C^{lf}_{n-*}(M)/C_{n-*}(M) & \to 0. \end{array}$$

There is a unique chain map making the right hand square commute. As we have a canonical homology equivalence $C^e_*(M)[1] \cong C^{lf}_*(M)/C_*(M)$ (Remark after 2.6) we get a commutative ladder with exact rows as claimed.

We are left with showing that the vertical arrows are isomorphisms. The maps $\cap\mu : H^*_c(M) \to H_{n-*}(M)$ are direct limits of Lefschetz isomorphisms [RS, Th. 6.11, 6.12, p. 84]

$$\cap\mu_i : H^*(M, M\setminus \operatorname{Int} M_i) \cong H^*(M_i, \partial M_i) \to H_{n-*}(M_i)$$

and thus isomorphisms. Similarly the cap products

$$\cap\mu_i : H^*(M_i) \to H_{n-*}(M_i, \partial M_i) \cong H_{n-*}(M, M\setminus \operatorname{Int} M_i)$$

form isomorphisms between the inverse systems and by the Milnor exact sequence the maps $\cap\mu : H^*(M) \to H^{lf}_{n-*}(M)$ are isomorphisms as well. It follows from the five lemma that $H^*_e(M)$ and $H^e_*(M)$ satisfy Poincaré duality in dimension $n-1$. □

Remark 3.2. The simplicial techniques we use are suitable for *PL*-manifolds, in particular for smooth manifolds. The duality theorem holds for topological manifolds, too, but in that case more refined tools such as sheaf cohomology and Borel-Moore cohomology are needed, see [R].

The (-1)-dimensional homology in Example 2.4 gets now a natural explanation: since $M = X$ is a zero-dimensional manifold, $H^0_e(M) \cong H^e_{-1}(M)$. For general manifolds $H_0(M)$ is the free module on all components of M and $H^{lf}_0(M)$ is the direct product of R over the compact components. Since M always has at most countably many components, we get

$$H^e_{-1}(M) \cong H^n_e(M) = \begin{cases} 0, & M \text{ has finitely many compact components,} \\ \prod_{i=0}^{\infty} R/\bigoplus_{i=0}^{\infty} R, & M \text{ has infinitely many compact components.} \end{cases}$$

Hence in order to realize $H_*^e(M)$ as the homology of a space it is reasonable to assume that M has only finitely many compact components. The problem of fitting a boundary to M is much deeper and leads to an obstruction in the projective class group, cf. [Si]

We finally point out that if M is a manifold with finitely generated homology, then $H_c^*(M)$ and $H_*^{lf}(M)$ are finitely generated by Poincaré duality. The long exact sequences (3.1) show that $H_e^*(M)$ and $H_*^e(M)$ are also finitely generated.

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