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T.Y. Lam

Exercises in Modules and Rings

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To Chee King

A constant source of strength and inspiration

Preface

The idea of writing this book came roughly at the time of publication of my graduate text *Lectures on Modules and Rings*, Springer GTM Vol. 189, 1999. Since that time, teaching obligations and intermittent intervention of other projects caused prolonged delays in the work on this volume. Only a lucky break in my schedule in 2006 enabled me to put the finishing touches on the completion of this long overdue book.

This book is intended to serve a dual purpose. First, it is designed as a "problem book" for *Lectures.* As such, it contains the statements and full solutions of the many exercises that appeared in *Lectures.* Second, this book is also offered as a reference and repository for general information in the theory of modules and rings that may be hard to find in the standard textbooks in the field.

As a companion volume to *Lectures*, this work covers the same mathematical material as its parent work; namely, the part of ring theory that makes substantial use of the notion of modules. The two books thus share the same table of contents, with the first half treating projective, injective, and flat modules, homological and uniform dimensions, and the second half dealing with noncommutative localizations and Goldie's theorems, maximal rings of quotients, Frobenius and quasi-Frobenius rings, concluding with Morita's theory of category equivalences and dualities. Together with the coverage of my earlier text *First Course in Noncommutative Rings*, Springer GTM Vol. 131, these topics comprise a large part of the foundational material in the classical theory of rings.

An integral part of *Lectures* is the large collection of six hundred exercises distributed over the seven chapters of the book. With the exception of two or three (which are now deemed too difficult for inclusion), all exercises are solved in full in this problem book. Moreover, some 40 new exercises have been added to the present collection to further broaden its coverage. To facilitate the cross-referencing, I have by and large used the same numbering scheme for the exercises in the two books. Some exceptions to this rule are explained in the *Notes to the Reader* section on page xiii.

Problem solving is something truly special in mathematics. Every student trying to learn a mathematical subject with any degree of seriousness finds it helpful or even necessary to do a suitable number of exercises along the way, to help consolidate his or her understanding of the subject matter, and to internalize the myriad of information being offered. This exercise book is intended not to supplant this process, but rather, to facilitate it. There are certainly more exercises in *Lectures* than the author can realistically expect his readers to do; for instance, §3 alone contains as many as 61 exercises. If my teaching experience is any guide, most students appreciate doing some exercises in detail, and learning about others by reading. And, even in cases where they solved exercises on their own, they find it helpful to compare their solutions with more "official" versions of the solutions, say prepared by the author. This is largely the *raison d'être* of a problem book, such as the present one.

What this book offers, however, is more than exercise solutions. Among the exercises in *Lectures*, only a rather small number are of a routine nature. The others range from nontrivial to medium-difficult, difficult, challenging, to very challenging, although they are not explicitly identified as such. In quite a few cases, the "exercises" are based on original results of other authors published in the research literature, for which no convenient expositions are available. This being the situation, a problem book like this one where all exercise solutions are independently written and collected in one place should be of value to students and researchers alike. For some problems that can be approached from several different angles, sometimes more than one solution is given. Many of the problem solutions are accompanied by a *Comment* section giving relevant bibliographical, historical or anecdotal information, pointing out latent connections to other exercises, or offering ideas on further improvements and generalizations. These *Comment* sections rounding out the solutions of the exercises are intended to be a particularly useful feature of this problem book.

This book is an outgrowth of my lecture courses and seminars over the years at the University of California at Berkeley, where many of the problem solutions were presented and worked over. As a result, many of my students and seminar participants have offered corrections and contributed useful ideas to this work; I thank them all. As usual, the warm support of my family (Chee King; Juwen, Fumei, Juleen, and Tsai Yu) was instrumental to the completion of this project, for which no words of acknowledgement could be adequate enough.

T.Y.L.

Berkeley, California
September 25, 2006

Contents

Notes to the Reader

Since this Problem Book is based on the author's "Lectures on Modules and Rings", the two books share a common organization. Thus, just as in *Lectures,* the main text of this book contains seven chapters, which are divided into nineteen sections. For ease of reference, the sections are numbered consecutively, independently of the chapters, from §1 to §19. The running heads offer the quickest and most convenient way to tell what chapter and what section a particular page belongs to. This should make it very easy to find an exercise with a given number.

Each section begins with its own introduction, in which the material in the corresponding section in *Lectures* is briefly recalled and summarized. Such short introductions thus serve as recapitulations of the theoretical underpinnings of the exercises that follow. The exercises in §8, for instance, are numbered 8.1, 8.2, etc., with occasional aberrations such as 8.5A and 8.5B. The exercise numbers are almost always identical to those in *Lectures,* though in a few cases, some numbers may have shifted by one. Exercises with numbers such as 8.5A, 8.5B are usually added exercises that did not appear before in *Lectures.*

For the exercise solutions, the material in *Lectures* is used rather freely throughout. A code such as *LMR*-(3.7) refers to the result (3.7) in *Lectures on Modules and Rings.* Occasional references are also made to my earlier Springer books *FC* (*First Course in Noncommutative Rings,* 2nd ed., 2001) and *ECRT* (*Exercises in Classical Ring Theory,* 2nd ed., 2003). These are usually less essential references, included only for the sake of making further connections.

The ring theory conventions used in this book are the same as those introduced in *LMR.* Thus, a ring R means a ring with identity (unless otherwise stated). A subring of R means a subring containing the identity of R (unless otherwise stated). The word "ideal" always means a two-sided ideal; an adjective such as "noetherian" likewise means both right and left noetherian. A ring homomorphism from R to S is supposed to take the identity of R to that of S. Left and right R-modules are always assumed to be unital; homomorphisms between modules are (usually) written on the opposite side of the scalars. "Semisimple rings" are in the sense of Wedderburn, Noether and Artin: these are rings R that are semisimple as (left or right) modules over themselves. Rings with Jacobson radical zero are called Jacobson semisimple (or semiprimitive) rings.

Throughout the text, we use the standard notations of modern mathematics. For the reader's convenience, a partial list of the ring-theoretic notations used in this book is given on the following pages.

Partial List of Notations

$\mathbb{Z}$	the ring of integers
$\mathbb{N}$	the set of natural numbers
$\mathbb{Q}$	the field of rational numbers
$\mathbb{R}$	the field of real numbers
$\mathbb{C}$	the field of complex numbers
$\mathbb{Z}_n$	the ring (or the cyclic group) $\mathbb{Z}/n\mathbb{Z}$
C_{p^∞}	the Prüfer p-group
$\emptyset$	the empty set
$\subset$, $\subseteq$	used interchangeably for inclusion
$\subsetneqq$	strict inclusion
$\lvert A\rvert$, $\mathrm{Card}\, A$	used interchangeably for cardinality of A
$A \setminus B$	set-theoretic difference
$A \rightarrowtail B$	injective mapping from A into B
$A \twoheadrightarrow B$	surjective mapping from A onto B
δ_{ij}	Kronecker deltas
E_{ij}	standard matrix units
M^t, M^T	transpose of the matrix M
$\mathbb{M}_n(S)$	set of $n \times n$ matrices with entries from S
$\mathrm{GL}_n(S)$	group of invertible $n \times n$ matrices over S
$\mathrm{GL}(V)$	group of linear automorphisms of a vector space V
$Z(G)$	center of the group (or the ring) G
$C_G(A)$	centralizer of A in G
$[G : H]$	index of subgroup H in a group G
$[K : F]$	field extension degree
$\mathfrak{M}_R$, ${}_R\mathfrak{M}$	category of right (left) R-modules
$\mathfrak{M}_R^{fg}$, ${}_R^{fg}\mathfrak{M}$	category of f.g. right (left) R-modules

M_R, ${}_RN$	right R-module M, left R-module N
${}_RM_S$	(R,S)-bimodule M
$M \otimes_R N$	tensor product of M_R and ${}_RN$
$\mathrm{Hom}_R(M,N)$	group of R-homomorphisms from M to N
$\mathrm{End}_R(M)$	ring of R-endomorphisms of M
nM	$M \oplus \cdots \oplus M$ (n times)
$M^{(I)}$	$\bigoplus_{i\in I} M$ (direct sum of I copies of M)
M^I	$\prod_{i\in I} M$ (direct product of I copies of M)
$\Lambda^n(M)$	n-th exterior power of M
$\mathrm{soc}(M)$	socle of M
$\mathrm{rad}(M)$	radical of M
$\mathrm{Ass}(M)$	set of associated primes of M
$E(M)$	injective hull (or envelope) of M
$\tilde{E}(M)$	rational hull (or completion) of M
$\mathcal{Z}(M)$	singular submodule of M
length M	(composition) length of M
u.dim M	uniform dimension of M
rank M	torsionfree rank or (Goldie) reduced rank of M
$\rho(M)$, $\rho_R(M)$	ρ-rank of M_R
rk M	rank (function) of a projective module M
M^*	R-dual of an R-module M
M', M^0	character module $\mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Q}/\mathbb{Z})$ of M_R
$\hat{M}$, $M\hat{}$	k-dual of a k-vector space (or k-algebra) M
N^{**}, $\mathrm{cl}(N)$	Goldie closure of a submodule $N \subseteq M$
$N \subseteq_e M$	N is an essential submodule of M
$N \subseteq_d M$	N is a dense submodule of M
$N \subseteq_c M$	N is a closed submodule of M
R^{op}	the opposite ring of R
$\mathrm{U}(R)$, R^*	group of units of the ring R
$\mathrm{U}(D)$, D^*, $\dot{D}$	multiplicative group of the division ring D
$\mathcal{C}_R$	set of regular elements of a ring R
$\mathcal{C}(N)$	set of elements which are regular modulo the ideal N
rad R	Jacobson radical of R
Nil^*R	upper nilradical of R
Nil_*R	lower nilradical (a.k.a. prime radical) of R
$\mathrm{Nil}\,(R)$	nilradical of a commutative ring R
$A^\ell(R)$, $A^r(R)$	left, right artinian radical of R
$\mathrm{Max}(R)$	set of maximal ideals of a ring R
$\mathrm{Spec}(R)$	set of prime ideals of a ring R
$\mathcal{I}(R)$	set of isomorphism classes of indecomposable injective modules over R
$\mathrm{soc}(R_R)$, $\mathrm{soc}({}_RR)$	right (left) socle of R
$\mathcal{Z}(R_R)$, $\mathcal{Z}({}_RR)$	right (left) singular ideal of R
$\mathrm{Pic}(R)$	Picard group of a commutative ring R

R_S	universal S-inverting ring for R
RS^{-1}, $S^{-1}R$	right (left) Ore localization of R at S
$R_{\mathfrak{p}}$	localization of (commutative) R at prime ideal $\mathfrak{p}$
$Q^r_{\max}(R)$, $Q^\ell_{\max}(R)$	maximal right (left) ring of quotients for R
$Q^r_{cl}(R)$, $Q^\ell_{cl}(R)$	classical right (left) ring of quotients for R
$Q_{cl}(R)$, $Q(R)$	the above when R is commutative
$Q^r(R)$, $Q^l(R)$	Martindale right (left) ring of quotients
$Q^s(R)$	symmetric Martindale ring of quotients
$\mathrm{ann}_r(S)$, $\mathrm{ann}_\ell(S)$	right, left annihilators of the set S
$\mathrm{ann}^M(S)$	annihilator of S taken in M
kG, $k[G]$	(semi)group ring of the (semi)group G over the ring k
$k[x_i : i \in I]$	polynomial ring over k with commuting variables $\{x_i : i \in I\}$
$k\langle x_i : i \in I\rangle$	free ring over k generated by $\{x_i : i \in I\}$
$k[[x_1, \ldots, x_n]]$	power series ring in the x_i's over k

Partial List of Abbreviations

RHS, LHS	right-hand side, left-hand side
ACC	ascending chain condition
DCC	descending chain condition
IBN	"Invariant Basis Number" property
PRIR, PRID	principal right ideal ring (domain)
PLIR, PLID	principal left ideal ring (domain)
FFR	finite free resolution
QF	quasi-Frobenius
PF	pseudo-Frobenius
PP	"principal implies projective"
PI	"polynomial identity" (ring, algebra)
CS	"closed submodules are summands"
QI	quasi-injective (module)
Obj	objects (of a category)
iff	if and only if
resp.	respectively
ker	kernel
coker	cokernel
im	image
rk	rank
f.cog.	finitely cogenerated
f.g.	finitely generated
f.p.	finitely presented
f.r.	finitely related
l.c.	linearly compact

pd	projective dimension
id	injective dimension
fd	flat dimension
wd	weak dimension (of a ring)
r.gl.dim	right global dimension (of a ring)
l.gl.dim	left global dimension (of a ring)

Chapter 1

Free Modules, Projective, and Injective Modules

§1. Free Modules

Free modules are basic because they have bases. *A* right free module F over a ring R comes with a *basis* $\{e_i : i \in I\}$ (for some indexing set I) so that every element in F can be uniquely written in the form $\sum_{i \in I} e_i r_i$, where all but a finite number of the elements $r_i \in R$ are zero. Free modules can also be described by a *universal property*, but the definition given above is more convenient for working inside the free module in question. We can also work with F by identifying it with $R^{(I)}$, the direct sum of I copies of R (or more precisely R_R). The direct sum $R^{(I)}$ is contained (as *a* submodule) in the direct product R^I, which is usually "much bigger": we have the equality $R^{(I)} = R^I$ iff I is finite or R is the zero ring.

If $F = R^{(I)}$, the cardinality $|I|$ of I may be (and usually is) called "the rank" of F, with the understanding that it may not be uniquely defined by F as *a* free module. If I is infinite (and $R \neq 0$), then the rank of the free module F happens to be uniquely determined. However, f.g. (finitely generated) free modules need not have a unique rank. This leads to a definition: R is said to have (right) IBN (invariant basis number) if, for any natural numbers n, m, $R^n \cong R^m$ (as right modules) implies that $n = m$. These are the rings for which *any* free right module has a unique rank. For instance, local rings and nonzero commutative rings have this property.

A ring R is *Dedekind-finite* if $xy = 1$ implies $yx = 1$ in$R$, and R is *stably finite* if any matrix ring $\mathbb{M}_n(R)$ $(n < \infty)$ is Dedekind-finite. The latter notion is related to IBN, in that *any nonzero stably finite ring has* IBN.

Two more conditions arise naturally in connection with the study of free modules. We say that R satisfies the *rank condition* if the existence of an epimorphism $R^k \to R^n$ implies that $k \geq n$; and we say that R satisfies the *strong rank condition* if the existence of a monomorphism $R^m \to R^n$ implies that $m \leq n$. (Here, k, m, n are natural numbers.) This terminology is justified by the fact that the strong rank condition implies the rank condition (though not conversely). For any nonzero ring R, we have

$$\text{Stable finiteness} \implies \text{rank condition} \implies \text{IBN},$$

and all three are left–right symmetric conditions. The "strong rank condition" defined above, however, is *not* left–right symmetric, and should more properly be called the *right* strong rank condition.

Much more about the relationships between the various notions above is given in *LMR*-§1. Some of it is highly nontrivial, e.g., the fact that any nonzero commutative ring satisfies the strong rank condition. There is also an interesting relationship between the rank condition and stable finiteness: *a ring R satisfies the rank condition iff it* admits a homomorphism into a nonzero *stably finite ring.* This is a result of P. Malcolmson (see *LMR*-(1.26)).

The exercises in this section cover additional material on the various conditions above, as well as the hopfian and cohopfian conditions on modules. For instance, the behavior of many of these conditions under the formation of matrix rings, polynomial rings and power series rings is studied. Some of these exercises require rather serious work, e.g. the construction of a Dedekind-finite ring R for which $\mathbb{M}_2(R)$ is not Dedekind-finite (Exercise 18). The example of a "free group ring" gives a very good illustration of the possible failure of the strong rank condition (Exercise 29).

Exercises for §1

Ex. 1.1. Give a matrix-theoretic proof for "stable finiteness ⇒ rank condition" (for nonzero rings).

Solution. Assume that the rank condition fails for a ring $R \neq 0$. Then, for some integers $n > k \geq 1$, there exist an $n \times k$ matrix A and an $k \times n$ matrix B over R such that $AB = I_n$. (See *LMR*-(1.24). The matrices A, B are obtained from an epimorphism $R_R^k \to R_R^n$ and a splitting thereof, with $n > k \geq 1$.) Let $A' = (A,\ 0)$ and $B' = \begin{pmatrix} B \\ 0 \end{pmatrix}$ be $n \times n$ completions of the rectangular matrices A, B. Then

$$A'B' = (A,\ 0)\begin{pmatrix} B \\ 0 \end{pmatrix} = AB = I_n, \text{but}$$
$$B'A' = \begin{pmatrix} B \\ 0 \end{pmatrix}(A,\ 0) = \begin{pmatrix} BA & 0 \\ 0 & 0 \end{pmatrix} \neq I_n,$$

so R is not stably finite.

Comment. Instead of using the completions A' and B' above, we can also write $A = \begin{pmatrix} C \\ C' \end{pmatrix}$, $B = (D,\ D')$ where $C, D \in \mathbb{M}_k(R)$. From $AB = I_n$, we get $CD = I_k$, $C'D = 0$, $CD' = 0$ and $C'D' = I_{n-k}$. From these, we see easily that $DC \neq I_k$, so R is not stably finite.

Ex. 1.2. A student gave the following argument to show that any algebra A over a field k has IBN. "Suppose A is generated over k by $\{x_i : i \in I\}$ with certain relations. Let be the quotient of A obtained by introducing the further relations $x_i\, x_j - x_j x_i = 0\, (\forall i,\ j)$. Then A has a natural surjection onto $\bar{A}$. Since the commutative ring $\bar{A}$ has IBN, it follows from *LMR*-(1.5) that A has IBN." Is this argument valid?

Solution. No! The flaw of this argument lies in the fact that the commutative ring $\bar{A}$ may be the zero ring. In that case, $\bar{A}$ *does not* satisfy IBN, so we cannot apply *LMR*-(1.5).

Ex. 1.3. Let V be a free right module of infinite rank over a nonzero ring k, and let $R = \text{End}\ (V_k)$. Show that, for any positive integers n, m, $\mathbb{M}_n(R)$ and $\mathbb{M}_m(R)$ are isomorphic as rings.

Solution. The endomorphism ring R was introduced and studied in *LMR*-(1.4). It is shown there that $R_R^n \cong R_R^m$ for any $n,\ m \geq 1$. Taking endomorphism rings of these free right R-modules, we then obtain a ring isomorphism $\mathbb{M}_n(R) \cong \mathbb{M}_m(R)$. (This means, of course, that R itself is isomorphic to all the matrix rings $\mathbb{M}_n(R)$!)

Ex. 1.4. Does every simple ring have IBN?

Solution. The answer is "no." Take any nonzero ring R which does not have IBN. Let I be a maximal ideal of R. Then $S = R/I$ is a simple ring, and we have a surjection $R \to S$. Since R does not have IBN, it follows from *LMR*-(1.5) that S also does not have IBN.

Comment. A much harder question is: *Does every domain have* IBN? The answer also turns out to be "no": see the discussion in the paragraphs following *LMR*-(9.16).

Ex. 1.5. Suppose the ring R admits an additive group homomorphism T into an abelian group $(A,\ +)$ such that $T(cd) = T(dc)$ for all $c,\ d \in R$. (Such a T is called a *trace map.*) If $T(1)$ has infinite additive order in A, show that R must have IBN.

Solution. Suppose $R_R^m \cong R_R^n$. Then there exist an $m \times n$ matrix A and an $n \times m$ matrix B over R such that $AB = I_m$ and $BA = I_n$. Taking (ordinary) traces, we have

$$m = \mathrm{tr}(AB) = \sum_i \sum_k A_{ik} B_{ki}, \quad \text{and}$$
$$n = \mathrm{tr}(BA) = \sum_k \sum_i B_{ki} A_{ik}.$$

Applying T, we get

$$m\,T(1) = \sum_i \sum_k T(A_{ik}\, B_{ki}) = \sum_i \sum_k T(B_{ki}\, A_{ik}) = n\,T(1).$$

Since $T(1) \in A$ has *infinite* additive order, we must have $m = n$, as desired.

Ex. 1.6. A module M_R is said to be *cohopfian* if every R-monomorphism $\varphi : M \to M$ is an isomorphism. Show that, if M_R is an artinian module, then M is cohopfian.

Solution. Assume, for the moment, that $\varphi(M) \subsetneq M$. Since φ is an injection, $\varphi^2(M) \subsetneq \varphi(M)$. Repeating this argument, we see that $\varphi^{n+1}(M) \subsetneq \varphi^n(M)$ for all n. Therefore, we have a strictly descending chain

$$M \supsetneq \varphi(M) \supsetneq \varphi^2(M) \supsetneq \cdots,$$

contradicting the fact that M is an artinian module.

Comment. A module M_R is said to be *hopfian* if every R-epimorphism $\varphi : M \to M$ is an isomorphism. The "dual" version of this exercise is also true: *if M_R is a noetherian module, then M is hopfian.* This is proved in *LMR*-(1.14) by a dual argument.

Ex. 1.7. A ring that is Dedekind-finite is also known as von Neumann finite. Is every von Neumann regular ring von Neumann finite?

Solution. The answer is "no." Let $V = e_1 k \oplus e_2 k \oplus \cdots$ where k is any division ring. Since V_k is a semisimple module, $R := \mathrm{End}(V_k)$ is a von Neumann regular ring by *FC*-(4.27). Let $x,\ y \in R$ be defined by

$$y(e_i) = e_{i+1} \quad \text{and} \quad x(e_i) = e_{i-1} \ \ (i \geq 1),$$

where e_0 is taken to be 0. Then $xy = 1 \in R$, but $yx \neq 1$ (since $yx(e_1) = 0$). Therefore, R is not von Neumann finite.

Comment. Although a general von Neumann regular ring need not be von Neumann finite, various classes of von Neumann regular rings are known to be von Neumann finite. For instance, let R be a *unit-regular ring*, in the sense that any $a \in R$ can be written in the form aua for some *unit* $u \in R$. Then, whenever $ab = 1$, we have $1 = ab = (aua)b = au$. This shows that $a = u^{-1} \in \mathrm{U}(R)$, so R is von Neumann finite.

Ex. 1.8. A module M_R is said to be *Dedekind-finite* if $M \cong M \oplus N$ (for some R-module N) implies that $N = 0$. Consider the following statements:

(A) M is Dedekind-finite.
(B) The ring $E := \text{End}(M_R)$ is Dedekind-finite.
(C) M is hopfian (any surjective $a \in E$ is bijective).

Show that (C) $\Rightarrow$ (A) $\Leftrightarrow$ (B), and that (C) $\Leftrightarrow$ (A) if every R-epimorphism $M \to M$ splits (e.g., if M is a projective module). Give an example to show that, in general, $(A) \not\Rightarrow (C)$.

Solution. (A)$\Rightarrow$ (B). Let $a,\ b \in E$ be such that $ab = 1$. Then $b : M \to bM$ is an isomorphism (since $bm = 0 \Rightarrow 0 = a(bm) = m$). Also, we check easily that $M = bM \oplus \ker(a)$, so (A) forces $\ker(a) = 0$. Since a is clearly surjective, we have $a \in \mathrm{U}(E)$, and hence $ba = 1 \in E$.

(B)$\Rightarrow$ (A). Suppose M_R is not Dedekind-finite. Then we have a decomposition $M = M_0 \oplus N$ where $N \neq 0$ and there exists an isomorphism $\pi : M \to M_0$. Let $b \in E$ be the composition of π with the inclusion map $M_0 \subseteq M$, and let $a \in E$ be defined by $a|N \equiv 0$ and $a|M_0 = \pi^{-1}$. Then clearly $ab = 1 \in E$; but $ba \neq 1 \in E$ since $ba(N) = 0$. Thus, E is not Dedekind-finite.

(C)$\Rightarrow$ (A). Suppose again that M is not Dedekind-finite. Using the notations in the last paragraph, we have a surjection $a :\ M \to M$ that is not bijective (since $a(N) = 0$).

Next, assume that every R-epimorphism $M \to M$ splits. To prove $(A) \Rightarrow (C)$, suppose M is Dedekind-finite, and consider every surjection $f :\ M \to M$. Let g be any splitting for f. Then $M = \ker(f) \oplus im(g)$ and $im(g) \cong M$, so (A) implies that $\ker(f) = 0$, as desired.

In general, (A) does not imply (C). For instance, over $\mathbb{Z}$, let M be the Prüfer p-group $\bigcup_{n=1}^{\infty} C_{p^n}$ where C_{p^n} denotes a cyclic group of order $p^n (p = prime)$. It is well-known that M is an indecomposable group, so $M_{\mathbb{Z}}$ is Dedekind-finite. However, $M/C_p \cong M$ yields an epimorphism $M \to M$ with kernel C_p, so M is not hopfian. (The endomorphism ring E here is the ring of p-adic integers; this is a commutative domain, which is, of course, Dedekind-finite.)

Comment. The following consequences of the exercise are noteworthy. In the case where $M = R_R$, we have $E = \text{End}(R_R) \cong R$. Therefore, the exercise implies that *R_R is Dedekind-finite (as a module) iff R is Dedekind-finite (as a ring)*. Applying the exercise to R_R^n (for all $n < \infty$), we see also that *R is stably finite iff all free modules R_R^n are Dedekind-finite, iff all R_R^n are hopfian.* This was stated without proof in *LMR*-(1.7).

Ex. 1.8*. (1) If M_R is not Dedekind-finite, show that M contains a copy of $N \oplus N \oplus \cdots$ for some nonzero module N. Is the converse true? (2) Is a submodule of a Dedekind-finite module also Dedekind-finite?

Solution. Say $M = M_1 \oplus N$, where $M_1 \cong M$, and $N \neq 0$. Then $M_1 \cong M_2 \oplus N$, where $M_2 \cong M$. Continuing this construction, we arrive at a submodule of M that is isomorphic to $N \oplus N \oplus \cdots$.

The converse of the first part of (1) is not true. For instance, let R be the $\mathbb{Z}$-algebra generated by x, y, with the relations $y^2 = yx = 0$. By *FC*-(1.26),

$$(*) \qquad R = \mathbb{Z}[x] \oplus \mathbb{Z}[x]y = \mathbb{Z}[x] \oplus \bigoplus\nolimits_{i=0}^{\infty} \mathbb{Z} \cdot x^i y$$

is a left noetherian ring, and hence Dedekind-finite (say by the first part of (1)). This in turn implies that $M = R_R$ is Dedekind-finite. However, $(*)$ shows that M contains a copy of $N \oplus N \oplus \cdots$ where N is the right R-module given by $\mathbb{Z}$ with x and y acting trivially. (Each $\mathbb{Z} \cdot x^i y$ is a right ideal isomorphic to N_R.)

The construction above also gives a negative answer to (2), since M_R is Dedekind-finite, while $N \oplus N \oplus \cdots$ is not.

Comment. For the question in (1), we have used a noncommutative example. We can equally well give a *commutative* one. Take an (additive) abelian group $N \neq 0$, and let $I = N_1 \oplus N_2 \oplus \cdots$ where $N_i \cong N$. Now make $R = I \oplus \mathbb{Z}$ into a (commutative) ring by introducing the (unique) multiplication with respect to which $I^2 = 0$. The module $M = R_R$ is Dedekind-finite (since R is commutative). But, for each i, $N_i \subseteq R$ is an ideal isomorphic to N_R with I acting trivially. Thus, M contains a copy of $N_R \oplus N_R \oplus \cdots$.

Ex. 1.9. Show that a ring R is not Dedekind-finite iff there exists an idempotent $e \neq 1$ in R such that $eR \cong R$ as right R-modules.

Solution. By the *Comment* following Exercise 8, R is not Dedekind-finite iff there exists an R-isomorphism $R_R \cong R_R \oplus X$ for some R-module $X \neq 0$. This means that $R = eR \oplus (1-e)R$ for some idempotent $e \neq 1$ such that $(eR)_R \cong R_R$.

Ex. 1.10. Let M, N be modules over a ring R with a surjection $f : N \to M$ and an injection $g : N \to M$. Show that f is an isomorphism under either one of the following assumptions:

(1) N is a noetherian module, or
(2) R is commutative and M is finitely generated.

Solution. (1) We shall think of g as an inclusion map, so that N "becomes" a submodule of M. Consider the following submodules of N:

$$X_n = \{x \in N : \ f(x) \in N, \ \ldots, \ f^{n-1}(x) \in N, \ \text{and} \ f^n(x) = 0\},$$

which form an ascending chain. Thus, under (1), we have $X_{n-1} = X_n$ for some n. Let $x_1 \in \ker(f)$. Since f is surjective, there exist $x_2, \ldots, x_n \in N$ such that

$$x_1 = f(x_2), \ \ x_2 = f(x_3), \ \ \ldots, \ \ \text{and} \ x_{n-1} = f(x_n).$$

Letting $x = x_n$, we have then $f(x) = x_{n-1}, \dots, f^{n-1}(x) = x_1$, and $f^n(x) = f(x_1) = 0$. Thus, $x \in X_n$. But then $x \in X_{n-1}$, which yields $0 = f^{n-1}(x) = x_1$. This proves that $\ker(f) = 0$, so f is an isomorphism.

(2) Under the hypotheses in (2), assume instead that there exists $n \in \ker(f) \setminus \{0\}$. Let $m_1, \dots, m_k$ generate M, and pick $n_i \in N$ such that $f(n_i) = m_i$. Say $n_i = \sum_j r_{ij} m_j$, and $n = \sum_j r_j m_j$, where $r_{ij}, r_j \in R$. Let R_0 be the subring of R generated (over $\mathbb{Z}\cdot 1$) by the finite set $\{r_{ij}, r_j\}$. By the Hilbert Basis Theorem, this is a (commutative) noetherian ring. Now let M_0 (resp. N_0) be the R_0-submodule generated by $\{m_i\}$ (resp. $\{n_i, n\}$). Then $N_0 \subseteq N \cap M_0$, and f restricts to a surjection $f_0 : N_0 \to M_0$, with $n \in \ker(f_0) \setminus \{0\}$. But N_0 is f.g. over R_0, so it is a *noetherian* R_0-module. This contradicts what we have proved in (1).

Comment. This very interesting exercise is taken from p. 61 of the book of Balcerzyk and Jósefiak, "Commutative Noetherian Rings and Krull Rings", Halsted Press/Polish Sci. Publishers, 1989. Several special cases of this exercise are noteworthy.

Let M be a noetherian module over any ring R. Case (1) above implies that M is hopfian: this case was done in *LMR*-(1.14). It also implies that $M \oplus X$ cannot be embedded into M unless $X = 0$: this case was done in *LMR*-(1.36). The present conclusion is thus much stronger. Next, assume R is commutative and M is f.g. The conclusion in Case (2) generalizes a well known result of Vasconcelos and Strooker, which says that M is hopfian. (For more details and relevant citations on this case, see *ECRT*-(20.9).) The conclusion in the present exercise, covering the case $N \subseteq M$, is again stronger. For instance, if $N \cong R^k$, the exercise yields the following:

> *If M is generated by $m_1, \dots, m_k$, and contains linearly independent elements $n_1, \dots, n_k$, then $m_1, \dots, m_k$ form a free R-basis for M.*

This case is covered (with a matrix-theoretical proof) in Thm. 5.10 in W.C. Brown's book, "Matrices over Commutative Rings", M. Dekker, New York, 1993.

Ex. 1.11. (Jacobson, Klein) Let R be a ring for which there exists a positive integer n such that $c^n = 0$ for any nilpotent element $c \in R$. Show that R is Dedekind-finite.

Solution. Let $ab = 1 \in R$. Following a construction of Jacobson, let $e_{ij} = b^i(1 - ba)a^j$ $(i, j \geq 0)$. In *FC*-(21.26), it is shown that these elements in R satisfy the matrix unit relations $e_{ij}e_{k\ell} = \delta_{jk}e_{i\ell}$ where δ_{jk} are the Kronecker deltas. Imitating the construction of a nilpotent Jordan matrix, let $c = e_{01} + e_{12} + \cdots + e_{n-1,n}$, where n is as given in the exercise. By a direct calculation using the above matrix unit relations, we see that

$$c^2 = e_{02} + e_{13} + \cdots + e_{n-2,n},$$
$$c^3 = e_{03} + e_{14} + \cdots + e_{n-3,n},$$

etc. In particular, $c^{n+1} = 0$. By the given hypothesis, we have $0 = c^n = e_{0,n} = (1-ba)a^n$. Now right multiplication by b^n gives $ba = 1$.

A somewhat different approach to the problem, due to A. A. Klein, is as follows. Given $ab = 1$, note that

$$(1) \qquad (1-b^i a^i)(1-b^j a^j) = 1 - b^j a^j \quad \text{for} \quad i \geq j, \quad \text{and}$$

$$(2) \qquad a(1-b^i a^i) = (1-b^{i-1}a^{i-1})a \quad \text{for} \quad i \geq 1.$$

Using these, we can check that, for $d = (1-b^n a^n)a$, we have

$$\begin{aligned} d^2 &= (1-b^n a^n)a(1-b^n a^n)a \\ &= (1-b^n a^n)(1-b^{n-1}a^{n-1})a^2 \\ &= (1-b^{n-1}a^{n-1})a^2, \end{aligned}$$

and inductively,

$$d^i = (1-b^{n-i+1}a^{n-i+1})a^i \quad \text{for} \quad i \leq n+1.$$

In particular, $d^{n+1} = 0$. By the given hypothesis, we have $0 = d^n = (1-ba)a^n$, and so $ba = 1$ as before.

Comment. In the two solutions given above, the elements c and d in R look different. However, they have the same n-th power, $(1-ba)a^n$, so one may wonder if they are the same to begin with. Indeed, it can be shown that $c = d$ (so the two solutions offered for the exercise are mathematically equivalent!). To see this, we induct on n, the case $n = 1$ being clear. Assuming the conclusion is true for $n-1$, we have

$$\begin{aligned} c &= (e_{01} + \cdots + e_{n-2,\,n-1}) + e_{n-1,\,n} \\ &= (1-b^{n-1}a^{n-1})a + b^{n-1}(1-ba)a^n \\ &= \left[(1-b^{n-1}a^{n-1}) + b^{n-1}(1-ba)a^{n-1}\right] a \\ &= (1-b^n a^n)a \\ &= d. \end{aligned}$$

Jacobson's construction of the e_{ij}'s appeared in his classical paper "Some remarks on one-sided inverses," Proc. Amer. Math. Soc. **1**(1950), 352–355. A predecessor of Jacobson's paper is that of R. Baer, "Inverses and zero-divisors," Bull. Amer. Math. Soc. **48**(1942), 630–638. Klein's construction of the element d comes from his paper "Necessary conditions for embedding rings into fields," Trans. Amer. Math. Soc. **137**(1969), 141–151. A ring S is said to satisfy *Klein's Nilpotency Condition* if, for any n, any nilpotent matrix $N \in \mathbb{M}_n(S)$ satisfies $N^n = 0$. The results presented in this Exercise (applied to matrix rings) imply that *any ring S satisfying Klein's Nilpotency Condition is always stably finite.*

Klein has also shown, however, that Dedekind-finite rings need not have the bounded nilpotency property in the statement of this exercise.

Ex. 1.12. For any ring R, we can embed R into $S = \mathbb{M}_n(R)$ by sending $r \in R$ to diag $(r, \ldots, r)$. Therefore, S may be viewed as an (R, R)-bimodule. Show that $S_R \cong R_R^{n^2}$ and ${}_RS \cong ({}_RR)^{n^2}$, with the matrix units $\{E_{ij} : 1 \leq i,\ j \leq n\}$ as basis.

Solution. With respect to the given right R-action on S, the scalar multiplication $E_{ij} \cdot r$ $(r \in R)$ is easily seen to be a matrix with (i, j)-th entry r and other entries zero. Therefore, any matrix $s = (a_{ij}) \in S$ can be expressed in the form $s = \sum E_{ij}\, a_{ij}$. Clearly, this implies that S_R is free on the basis $\{E_{ij} : 1 \leq i,\ j \leq n\}$ so $S_R \cong R_R^{n^2}$. The same result for ${}_RS$ follows similarly.

Ex. 1.13. (Montgomery) Let I be an ideal of a ring R contained in rad R (the Jacobson radical of R). Show that R is stably finite iff R/I is.

Solution. We shall use the following crucial fact about the Jacobson radical of a ring S (see *FC*-(4.8)):

(1) *If J is an ideal in* rad S *and $\bar{S} = S/J$, then an element $a \in S$ has an inverse* (*resp. left inverse*) *iff $\bar{a} \in \bar{S}$ does.*

Using this, we see immediately that

(2) *S is Dedekind-finite iff $\bar{S}$ is.*

For the given ring R in the problem and any $n \geq 1$, let $S = \mathbb{M}_n(R)$. Then, by *FC*-p.61,

$$\begin{aligned} J :&= \mathbb{M}_n(I) \subseteq \mathbb{M}_n(\text{rad } R) = \text{rad } S, \quad \text{and} \\ S/J &= \mathbb{M}_n(R)/\mathbb{M}_n(I) \cong \mathbb{M}_n(R/I). \end{aligned}$$

By (2), $S = \mathbb{M}_n(R)$ is Dedekind-finite iff $S/J \cong \mathbb{M}_n(R/I)$ is. Since this holds for all n, it follows that R is stably finite iff R/I is.

Comment. The result in this exercise is an observation in M.S. Montgomery's paper "von Neumann finiteness of tensor products of algebras," Comm. Algebra **11**(1983), 595–610.

For the following exercises ((14) *to* (17)), *let "P" denote any one of the properties*: IBN, *the rank condition, or stable finiteness.*

Ex. 1.14. Let $S = \mathbb{M}_n(R)$, where $n \geq 1$. Show that R satisfies the property "P" iff S does.

Solution. (1) Suppose S has IBN. Since we have a ring homomorphism $\varepsilon :\ R \to S$ sending $r \in R$ to diag $(r, \ldots, r)$, *LMR*-(1.5) implies that R also has IBN. Now suppose S does not have IBN. Then there exist positive integers $p \neq q$ and matrices $A,\ B$ of sizes $p \times q$ and $q \times p$ over S such that $AB = I_p$ and $BA = I_q$. Viewing $A,\ B$ as matrices over R, of sizes $np \times nq$ and $nq \times np$ (where $np \neq nq$), we see that R does not have IBN. (We could also have used Exercise 12 for this conclusion.)

(2) Suppose S satisfies the rank condition. By *LMR*-(1.23), the existence of the ring homomorphism $\varepsilon: R \to S$ above implies that R also satisfies the rank condition. Now suppose S does not satisfy the rank condition. By *LMR*-(1.24), there exist integers $p > q \geq 1$ and matrices A, B of sizes $p \times q$ and $q \times p$ over S such that $AB = I_p$. Viewing A, B as matrices over R, of sizes $np \times nq$ and $nq \times np$ (where $np > nq \geq 1$), we see that R does not satisfy the rank condition. (Alternatively, use Exercise 12.)

(3) Suppose S is stably finite. Since $\varepsilon: R \to S$ is an embedding of rings, *LMR*-(1.9) implies that R is stably finite. Now suppose S is not stably finite. Then for some p, there exist matrices $A, B \in M_p(S)$ such that $AB = I_p \neq BA$. Since $\mathbb{M}_p(S) = \mathbb{M}_{pn}(R)$, it follows that R is not stably finite.

Ex. 1.15. (Small) Let $S = R[[x]]$ (power series ring in one variable x over R). Show that R satisfies the property "P" iff S does.

Solution. (1) Note that there is a homomorphism from R to S (the inclusion map) and also a homomorphism from S to R (sending $\sum_{i=0}^{\infty} a_i x^i$ to a_0). From *LMR*-(1.5) and *LMR*-(1.23), it follows that, for "P"= IBN or the rank condition, R satisfies "P" iff S does.

(2) We are now left with the case "P"= stable finiteness. Suppose S is stably finite. Clearly, the subring $R \subseteq S$ is also stably finite. Finally, suppose R is stably finite. Consider the ideal $(x) \subseteq S$ generated by x. Since $1 + (x)$ consists of units of S, $(x) \subseteq \text{rad}\, S$. We have $S/(x) \cong R$, so by Exercise 13, the fact that R is stably finite implies that S is stably finite.

Ex. 1.16. (Small) Let $S = R[x]$. Show that R satisfies the property "P" iff S does.

Solution. If "P" = IBN or the rank condition, the same proof given in (1) of the last exercise works here. Thus, we need only handle the case where "P" is stable finiteness. If $R[x]$ is stably finite, certainly so is the subring R. Finally, assume R is stably finite. By the last exercise, $R[[x]]$ is stably finite. Therefore, the subring $R[x] \subseteq R[[x]]$ is also stably finite.

Ex. 1.17. (Cohn) Let $R = \varinjlim R_i$ (direct limit of a directed system of rings $\{R_i : i \in I\}$). Show that, if each R_i satisfies the property "P," so does R.

Solution. (1) Suppose R fails to have IBN. Then there exist natural numbers $n \neq m$ and matrices A, B over R of sizes $m \times n$ and $n \times m$ respectively, such that $AB = I_m$ and $BA = I_n$. For a suitably chosen index $i \in I$, there exist matrices A_1, B_1 over R_i of sizes $m \times n$ and $n \times m$ which map to the matrices A, B under the natural map $R_i \to R$. The fact that $AB = I_m$ and $BA = I_n$ implies that, for some index $j \geq i$, both of the matrices $A_1B_1 - I_m$ and $B_1A_1 - I_n$ over R_i map to the zero matrix over R_j. This implies that R_j does not have IBN.

(2) If R fails to satisfy the rank condition, then (by *LMR*-(1.24)) there exist integers $m > n \geq 1$ and matrices A, B over R of sizes $m \times n$ and $n \times m$ such that $AB = I_m$. The same argument used in (1) shows that such matrices can already be found over some R_j, so R_j fails to satisfy the rank condition.

(3) Suppose $AB = I_n$ where $A, B \in \mathbb{M}_n(R)$. For a suitable index j, we can find $A', B' \in \mathbb{M}_n(R_j)$ with $A'B' = I_n$ such that A', B' map to A, B respectively under the map $\mathbb{M}_n(R_j) \to \mathbb{M}_n(R)$. If R_j is stably finite, then $B'A' = I_n$ over R_j, and so $B'A' = I_n$ over R. This shows that R is also stably finite.

Ex. 1.18. Construct a ring R such that R is Dedekind-finite but $\mathbb{M}_2(R)$ is not Dedekind-finite. (In particular, R is not stably finite.)

Solution. (Sketch) Following a construction of Shepherdson (in the 2×2 case), let R be the k-algebra generated over a field k by $\{s, t, u, v; w, x, y, z\}$ with relations dictated by the matrix equation $AB = I_2$ where $A = \begin{pmatrix} s & u \\ t & v \end{pmatrix}$ and $B = \begin{pmatrix} x & y \\ z & w \end{pmatrix}$. Thus there are four relations:

$$(*) \qquad sx + uz = 1, \;\; sy + uw = 0, \;\; tx + vz = 0, \quad \text{and} \quad ty + vw = 1.$$

Our goal is to show that R is a *domain* (in particular R is Dedekind-finite), but that $\mathbb{M}_2(R)$ is not Dedekind-finite.

The idea here is that we can bring the elements of R to a "normal form" by using the relations $(*)$. Notice that these relations enable us to make the following "replacements":

$$(**) \qquad sx = 1 - uz, \;\; sy = -uw, \;\; tx = -vz, \;\; ty = 1 - vw.$$

Let us say that a monomial M in $\{s, t, u, v; w, x, y, z\}$ is in *normal form* if there is no occurrence of the strings sx, sy, tx, ty in M. More generally, we say that a sum of monomials $\sum_i M_i$ is in normal form if each M_i is in normal form. Upon making the replacements $(**)$ repeatedly, we can write each element in normal form. Note that the symbols s, t, x, y do not appear on the RHS's of the equations in $(**)$, so each replacement does get rid of one "bad" string (sx, sy, tx or ty) without creating another one. This guarantees the "convergence" of our algorithm for reducing elements of R to normal form.

The same observation, basically, also guarantees the *uniqueness* of a normal form of an element of R. (The point is that there is no "interference" among the four relations written out in the form $(**)$, so that our algorithm can proceed essentially uniquely.)

We can now define a degree function $\delta : \; R \setminus \{0\} \to \{0, 1, 2, \ldots\}$ by taking the degree of the nonzero elements of R, expressed in their unique normal forms. The fact that R is a domain will follow if we can show that for nonzero $\alpha, \beta \in R$ (in normal form), $\alpha\beta$ also has a degree, namely $\delta(\alpha) + \delta(\beta)$. For this, we may assume that α, β are homogeneous, say of degree m and n. Consider a typical monomial α_0 (resp. β_0) appearing in α (resp. β). The product $\alpha_0\beta_0$ is already in normal form unless, say α_0 ends in s and β_0 begins with x. In this case we would replace the string sx in $\alpha_0\beta_0$ by $1 - uz$. The term 1 will only contribute to terms of lower degree (than $m + n$), so if $\alpha_0 = \sigma s$ and $\beta_0 = x\tau$, we have

$$\alpha_0\beta_0 = \sigma(sx)\tau = \sigma(1 - uz)\tau \equiv -\sigma uz\tau$$

modulo terms of lower degree. Here, $\sigma uz\tau$ is in normal form, and can be "cancelled out" only if there is another monomial $\alpha_1 = \sigma u$ in α, and another monomial $\beta_1 = z\tau$ in β. But then $\alpha\beta$ contains $\alpha_1\beta_0 = \sigma ux\tau$ which is in normal form and *cannot be* cancelled out in the full expansion of $\alpha\beta$. This shows that $\alpha\beta \neq 0$ and in fact $\delta(\alpha\beta) = \delta(\alpha) + \delta(\beta)$.

It is now easy to see that $\mathbb{M}_2(R)$ is *not* Dedekind-finite. Indeed, the matrices A, $\beta \in \mathbb{M}_2(R)$ satisfy, by definition, $AB = I_2$; but $BA \neq I_2$ since its (1, 2)-entry has the nonzero normal form $xu + yv$.

Comment. J.C. Shepherdson's construction appeared in his paper "Inverses and zero-divisors in matrix rings," Proc. London Math, Soc. **1**(1951), 71–85. Using similar methods, P.M. Cohn has constructed (for any $n \geq 1$) a ring R for which $\mathbb{M}_r(R)$ is Dedekind-finite for all $r \leq n$, but $\mathbb{M}_{n+1}(R)$ is not Dedekind-finite; see his paper "Some remarks on invariant basis property," Topology **5**(1966), 215–228. Noting that $\mathbb{M}_n(R) \cong R \otimes_k \mathbb{M}_n(k)$, one can consider more generally finiteness questions for tensor products of k-algebras; see M.S. Montgomery, "von Neumann finiteness of tensor products of algebras," Comm. Alg. **11**(1983), 595–610. Among other results, Montgomery showed that the tensor product of a stably finite algebra with a *PI* algebra remains stably finite, and that, if K/k is an algebraic field extension of degree > 1, there always exists a k-domain A such that $A \otimes_k K$ is *not* Dedekind-finite.

There is a result due to P.M. Cohn which asserts that any domain can be embedded into a simple domain. Let R denote again Shepherdson's domain constructed in the solution to this exercise, and let S be any simple domain containing R. Then S is Dedekind-finite, but $\mathbb{M}_2(S)$ is not since $AB = I_2 \neq BA$ in $\mathbb{M}_2(R) \subseteq \mathbb{M}_2(S)$. This enables us to produce a "simple" example for the present exercise. (I thank P.M. Cohn and K. Goodearl for relaying to me this interesting remark.)

Ex. 1.18′. *True or False*: if an upper triangular matrix $\begin{pmatrix} x & z \\ 0 & y \end{pmatrix}$ over a ring k is invertible in $\mathbb{M}_2(k)$, then x, y are units in k?

Solution. Taking an inverse $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ for $\begin{pmatrix} x & z \\ 0 & y \end{pmatrix}$ in $\mathbb{M}_2(k)$, we have the equations $ax = 1 = yd$. Thus, x is left-invertible, and y is right-invertible. If k is Dedekind-finite, we can conclude that $x, y \in \mathrm{U}(k)$. But if k is *not*

Dedekind-finite, this conclusion need not hold, as the following construction shows!

Let $x, y \in k$ be such that $yx = 1 \neq xy$. Then xy is an idempotent, and $z := 1 - xy$ is its "complementary" idempotent. Consider the triangular matrices

$$A = \begin{pmatrix} x & z \\ 0 & y \end{pmatrix}, \quad \text{and} \quad B = \begin{pmatrix} y & 0 \\ z & x \end{pmatrix}$$

in $R = \mathbb{M}_2(k)$. Since $zx = x - xyx = 0$ and $yz = y - yxy = 0$, we have

$$AB = \begin{pmatrix} xy + z^2 & zx \\ yz & yx \end{pmatrix} = I_2, \quad \text{and} \quad BA = \begin{pmatrix} yx & yz \\ zx & z^2 + xy \end{pmatrix} = I_2.$$

Thus, A, B are invertible in R, but $x, y \notin \mathrm{U}(k)$.

In a similar spirit, we can also take

$$C = \begin{pmatrix} x & 1 \\ 0 & y \end{pmatrix}, \quad \text{and} \quad D = \begin{pmatrix} y & -1 \\ z & x \end{pmatrix},$$

with $CD = DC = I_2$.

Comment. To complete the discussion, we might add the comment that, if an upper triangular matrix is invertible in the ring of upper triangular matrices, then its diagonal entries are indeed units in the coefficient ring, and conversely. The first counterexample offered in the solution above is from the author's paper: "Corner ring theory: a generalization of Peirce decompositions", in Algebra, Rings, and Their Representations, Proc. Int'l Conf. in Algebras, Modules, and Rings (Lisbon, 2003), pp. 153–182, World Scientific, Singapore, 2006.

Ex. 1.19. Show that a ring $R \neq 0$ satisfies the strong rank condition iff, for any right R-module M generated by n elements, any $n + 1$ elements in M are linearly dependent.

Solution. Suppose first that, in any right R-module M generated by n elements, any $n + 1$ elements in M are linearly dependent. Then, for any n, R_R^n cannot contain a copy of R_R^{n+1}. This implies that R satisfies the strong rank condition. Conversely, assume that R satisfies the strong rank condition. Let M be any right R-module generated by $x_1, \ldots, x_n$, and let $y_1, \ldots, y_{n+1} \in M$. Let $\pi : R^n \to M$ by the R-epimorphism defined by $\pi(e_i) = x_i$ (where $\{e_i\}$ is the standard basis in R^n), and let $f_i \in R^n (1 \leq i \leq n + 1)$ be such that $\pi(f_i) = y_i$. By hypothesis, $f_1, \ldots, f_{n+1}$ must be linearly dependent. Applying π, we see that $y_1, \ldots, y_{n+1}$ are likewise linearly dependent.

Ex. 1.20. Let $f : R \to S$ be a ring homomorphism such that S becomes a flat left R-module under f (i.e., the functor $- \otimes_R S$ is exact on $\mathfrak{M}_R$). Show that if S satisfies the right strong rank condition, so does R. Using this, show that if S satisfies the strong rank condition, then so does the direct product $S \times T$ for any ring T.

Solution. Consider any R-monomorphism $R_R^m \to R_R^n$. Tensoring this with the flat left R-module S, we get an S-monomorphism $S_S^m \to S_S^n$. If S satisfies the strong rank condition, then this implies that $m \leq n$, so we have shown that R satisfies the strong rank condition.

Now suppose S satisfies the strong rank condition, and let $R = S \times T$, where T is any ring. Here we have the projection map $f : R \to S$. The module ${}_RS$ (defined via f) is a direct summand of ${}_RR$. Thus, ${}_RS$ is a projective, and hence flat, module. By the first part of this exercise, we conclude that $R = S \times T$ also satisfies the strong rank condition.

Ex. 1.21. If a product ring $R = S \times T$ satisfies the strong rank condition, show that either S or T must satisfy the strong rank condition.

Solution. Suppose neither S nor T satisfies the strong rank condition. Then there exist an embedding $S_S^{m+1} \to S_S^m$ for some m, and an embedding $T_T^{n+1} \to T_T^n$ for some n. We may assume that $m = n$. (If, say $m > n$, add T_T^{m-n} to both sides of $T_T^{n+1} \to T_T^n$.) By taking products, we obtain easily an embedding $R_R^{m+1} \to R_R^m$, which contradicts the assumption that R satisfies the strong rank condition.

Ex. 1.22. Let "P" be the strong rank condition. Redo Exercises (14), (16), and prove the "if" part of Exercise 15 for this "P."

Solution. (1) Suppose $S = \mathbb{M}_n(R)$ satisfies "P." Consider the usual embedding $\varepsilon : R \to S$, defined by $\varepsilon(r) = \text{diag}\,(r, \ldots, r)$. Viewing S as a left R-module via ε, we have ${}_RS \cong {}_RR^{n^2}$ by Exercise 12. In particular, ${}_RS$ is a *flat* module. Applying Exercise 20, we see that R satisfies "P". Now suppose S does not satisfy "P". Then, for some m, there exists an embedding $S_S^{m+1} \to S_S^m$. Since we also have $S_R \cong R_R^{n^2}$, this leads to an embedding $R_R^{n^2(m+1)} \to R_R^{n^2 m}$. Hence R does not satisfy "P".

(2) Suppose $A = R[[x]]$ satisfies "P". Viewing A as a left R-module via the embedding $R \to A$, we have ${}_RA \cong R \times R \times \cdots$, which is a flat module. Therefore, it follows from Exercise 20 again that R satisfies "P".

(3) If $T = R[x]$ satisfies "P", then so does R since here ${}_RT \cong R \oplus R \oplus \cdots$, which is free and hence flat. Finally, suppose R satisfies "P". Consider a homogeneous system of linear equations

$$(*) \qquad \sum_{j=1}^{m} a_{ij}(x) y_j = 0 \quad (1 \leq i \leq n)$$

over T, with $m > n$. Write

$$a_{ij}(x) = \sum_{k=0}^{d} a_{ijk} x^k \quad \text{and} \quad y_j = \sum_{\ell=0}^{d'} y_{\ell j} x^\ell.$$

The system of equations amounts to

$$0 = \sum_j \left(\sum_k a_{ijk} x^k\right)\left(\sum_\ell y_{\ell j} x^\ell\right) = \sum_{k,\ell} \left(\sum_j a_{ijk} y_{\ell j}\right) x^{k+\ell}$$

for all i. Since i ranges here from 1 to n, and $k+\ell$ ranges from 0 to $d+d'$, this leads to a homogeneous linear system over R with $n(d+d'+1)$ equations, in the $m(d'+1)$ unknowns $\{y_{\ell j}\}$. Since R satisfies "P", we can solve this system nontrivially over R if d' is chosen so large that

$$d'(m-n) > n(d+1) - m.$$

Therefore, the given system of equations $(*)$ has a nontrivial solution over T. This shows that T has the property "P".

Ex. 1.23. If R satisfies the (right) strong rank condition, does the same hold for $S = R[[x]]$?

Solution. The answer, in general, is "no". However, it is not easy to find a counterexample. To present as short a solution as possible, let us appeal to a later exercise in §10 and also make full use of the results in *LMR*. Let R be a domain. According to Exercise (10.21), R satisfies the strong rank condition iff R is a right Ore domain; that is, iff $aR \cap bR \neq 0$ for any $a, b \neq 0$ in R. Now, by *LMR*-(10.31A), there exist (right and left) Ore domains R for which $S = R[[x]]$ is not right Ore. By Exercise (10.21), such domains R satisfy the strong rank condition, but the power series rings $R[[x]]$ over them do not.

I do not know if it is any easier to find a counterexample where R is allowed to have 0-divisors.

Ex. 1.24. Let R be a ring that satisfies the strong rank condition, and let $\beta : R^{(I)} \to R^{(J)}$ be a monomorphism from the free (right) module $R^{(I)}$ to the free module $R^{(J)}$, where I, J are (possibly infinite) sets. Show that $|I| \leq |J|$.

Solution. We may clearly assume that $|I|, |J|$ are both infinite cardinals. Let $\{e_i : i \in I\}$ and $\{e'_j : j \in J\}$ be bases for $R^{(I)}$, $R^{(J)}$ respectively. For each $i \in I$, fix a finite set $S_i \subseteq$ J such that $\beta(e_i)$ is in the span of $\{e'_j : j \in S_i\}$. Since $|J|$ is infinite, the collection of sets $\{S_i : i \in I\}$ is a set of cardinality $\leq |J|$. If $|I| > |J|$, there would exist infinitely many indices, say $i_1, i_2, \ldots \in I$ such that $S_{i_1} = S_{i_2} = \cdots = S$ (say). But then β would induce an injection from $e_{i_1}R \oplus e_{i_2}R \oplus \cdots$ to $R(S)$, which contradicts the strong rank condition.

Ex. 1.25. Let $R \neq 0$ be a commutative ring such that any ideal in R is free as an R-module. Show that R is a PID.

Solution. We first check that R has no 0-divisors. Let $a \neq 0$ in R. If $ar = 0 (r \in R)$, then $(aR) \cdot r = 0$. Since aR is a nonzero free right R-module, we must have $r = 0$. Therefore, R is a (commutative) domain. Let A be any nonzero ideal in R. By the given hypothesis, A has a free R-basis $\{e_i : i \in I\}$. If I contains two elements $i \neq j$, the equation $e_i e_j - e_j e_i = 0 \in I$ would give a contradiction. Therefore, we must have $|I| = 1$, and so A is a principal ideal, as desired.

Comment. For a noncommutative version of this exercise, see Exercise 10.26.

Ex. 1.26. Let R be any ring such that any right ideal in R is free as an R-module. Show that any submodule of a free right R-module is free.

Solution. Since free modules are projective, the hypothesis implies that R is a right hereditary ring. A quick way to solve this problem is to use a basic result on right hereditary rings due to I. Kaplansky. According to Kaplansky's Theorem (*LMR*-(2.24)), any submodule M of a free right R-module is isomorphic to a direct sum $\bigoplus_i I_i$ where the I_i's are suitable right ideals in R. Since each I_i is free (by assumption), it follows that M itself is free.

Ex. 1.27. Let R be a ring and $\mathfrak{B} \subseteq R$ be an ideal that is free as a left R-module with a basis $\{b_j : j \in J\}$. For any free left R-module A with a basis $\{a_i : i \in I\}$, show that $\mathfrak{B}A$ is a free left R-module with a basis $\{b_j a_i : j \in J, i \in I\}$.

Solution. Since $A = \bigoplus_i Ra_i$ and $\mathfrak{B}$ is a right ideal,

$$\mathfrak{B}A = \bigoplus_i \mathfrak{B}R \cdot a_i = \bigoplus_i \mathfrak{B}a_i.$$

Writing $\mathfrak{B} = \bigoplus_j Rb_j$, we have

$$\mathfrak{B}A = \bigoplus_i \left(\bigoplus_j Rb_j\right) a_i = \bigoplus_{i,j} R \cdot b_j a_i.$$

If $r(b_j a_i) = 0$ where $r \in R$, then $rb_j = 0$ (since Ra_i is free on a_i) and hence $r = 0$ (since Rb_j is free on b_j). Therefore, $R \cdot b_j \cdot a_i$ is free on $b_j a_i$, and hence $\mathfrak{B}A$ is free on $\{b_j a_i : j \in J, \ i \in I\}$.

Ex. 1.28. Let R and $\mathfrak{B}$ be as in Exercise 27, and let $\mathfrak{A} \supseteq \mathfrak{B}$ be a left ideal in R that is free as a left R-module. Show that (1) for each $i \geq 0$, $\mathfrak{B}^i\mathfrak{A}/\mathfrak{B}^{i+1}\mathfrak{A}$ and $\mathfrak{B}^i/\mathfrak{B}^{i+1}$ are both free left $R/\mathfrak{B}$-modules; (2) there is a long exact sequence of left $R/\mathfrak{B}$-modules:

$$(*) \qquad \cdots \to \frac{\mathfrak{B}^2}{\mathfrak{B}^3} \to \frac{\mathfrak{B}\mathfrak{A}}{\mathfrak{B}^2\mathfrak{A}} \to \frac{\mathfrak{B}}{\mathfrak{B}^2} \to \frac{\mathfrak{A}}{\mathfrak{B}\mathfrak{A}} \to \frac{R}{\mathfrak{B}} \to \frac{R}{\mathfrak{A}} \to 0,$$

where all modules except $R/\mathfrak{A}$ are free over $R/\mathfrak{B}$. (Such a sequence is called a *free resolution* for the $R/\mathfrak{B}$-module $R/\mathfrak{A}$.)

Solution. Applying the last exercise to $A = \mathfrak{A}$, we see that $\mathfrak{B}\mathfrak{A}$ is R-free. By induction on i, it follows that each $\mathfrak{B}^i\mathfrak{A}$ is R-free, and therefore $\mathfrak{B}^i\mathfrak{A}/\mathfrak{B}^{i+1}\mathfrak{A}$ is $R/\mathfrak{B}$-free. In the special case when $\mathfrak{A} = R$, this implies that $\mathfrak{B}^i/\mathfrak{B}^{i+1}$ is also $R/\mathfrak{B}$-free. Since $\mathfrak{B} \subseteq \mathfrak{A}$ (and $\mathfrak{B}$ is an ideal), we have a filtration of left ideals:

$$R \supseteq \mathfrak{A} \supseteq \mathfrak{B} \supseteq \mathfrak{B}\mathfrak{A} \supseteq \mathfrak{B}^2 \supseteq \mathfrak{B}^2\mathfrak{A} \supseteq \mathfrak{B}^3 \supseteq \cdots .$$

From this, we get the long exact sequence $(*)$, where, as we have just observed, all modules except $R/\mathfrak{A}$ are free over $R/\mathfrak{B}$.

Ex. 1.29. Let G be a free group on a set of generators $\{x_i : i \in I\}$ and let R be the group ring kG, where k is a commutative ring. Show that, as a left R-module, the augmentation ideal $\mathfrak{A}$ (the kernel of the augmentation map $\varepsilon : R \to k$ defined by $\varepsilon\left(\sum_{z \in G} a_z z\right) = \sum_z a_z$) is R-free with basis $\{x_i - 1 : i \in I\}$.

Solution. Our strategy will be to show that, for any left kG-module M and any given set of elements $\{m_i : i \in I\} \subseteq M$, there exists a unique R-homorphism $f : \mathfrak{A} \to M$ such that $f(x_i - 1) = m_i$ for all $i \in I$. It is easy to see that $\mathfrak{A} = \sum_i R \cdot (x_i - 1)$, so the uniqueness of f is clear. In the following, we must show the *existence* of f.

To construct f, we use the notion of a cross-homomorphism from group cohomology. A map $c : G \to {}_{kG}M$ is called a *cross-homomorphism* if it satisfies

$$c(yz) = yc(z) + c(y) \quad (\forall\, y, z \in G).$$

In the case of a free group G as above, for any set of elements $\{m_i : i \in I\} \subseteq M$, there exists a cross-homomorphism $c : G \to M$ such that $c(x_i) = m_i$ for all $i \in I$. To define c on G, note that if c does exist, we must have $c(1) = 0$ and therefore $c(x_i^{-1}) = -x_i^{-1}c(x_i) = -x_i^{-1}m_i$. Using this information as a guide, we can define c on a reduced word $w \in G$ by induction on the length of w, as follows. We start with $c(1) = 0$, $c(x_i) = m_i$, and $c(x_i^{-1}) = -x_i^{-1}m_i$ for words of length ≤ 1. For a reduced word $w = yw_0$ where y has length 1, we take

$$c(w) := \left\{\begin{matrix} x_i c(w_0) + m_i & \text{if } y = x_i, \\ x_i^{-1}c(w_0) - x_i^{-1}m_i & \text{if } y = x_i^{-1} \end{matrix}\right\}.$$

With these definitions, it is straightforward to verify (again by induction on word length) that $c : G \to M$ is a cross-homomorphism. With this map c in place, we can then define $f : \mathfrak{A} \to M$ by

$$f\left(\sum_{z \in G} a_z z\right) = \sum_{z \in G} a_z c(z) \quad \left(a_z \in k,\ \sum_z a_z = 0\right).$$

(Of course, all sums involved are finite sums.) Clearly,

$$f(x_i - 1) = c(x_i) - c(1) = c(x_i) = m_i \quad (\forall\, i \in I),$$

so we need only verify that f is a kG-homomorphism. Since f is clearly k-linear, we are done by carrying out the following computation, where $y \in G$ and $\sum_z a_z z \in \mathfrak{A}$:

$$\begin{aligned} f\left(y \cdot \sum_z a_z z\right) &= \sum_z a_z\, c(yz) \\ &= \sum_z a_z(yc(z) + c(y)) \\ &= y \sum_z a_z c(z) + \left(\sum_z a_z\right) c(y) \\ &= y \cdot f\left(\sum_z a_z z\right). \end{aligned}$$

Comment. Certainly, the cross-homomorphism $c: G \to M$ can be constructed in a more conceptual way. Using the action of G on M, we can construct a semidirect product $A = M \rtimes G$ which contains M as a normal subgroup with G as a complement. Given the elements $\{m_i : i \in I\} \subseteq M$, we can define a splitting φ of the surjection $A \to G$ by taking

$$\varphi(x_i) = (m_i,\ x_i) \in A.$$

For any $z \in G$, $\varphi(z)$ has then the form $(c(z),\ z) \in A$, and $z \mapsto c(z)$ is the desired cross-homomorphism.

The fact that $\{x_i - 1 : i \in I\}$ forms a free kG-basis for $\mathfrak{A}$ means that, for any element $\alpha \in \mathfrak{A}$, there exist unique elements $\partial_i\alpha \in kG$ such that

$$(*) \qquad \alpha = \sum\nolimits_{i\in I} (\partial_i\alpha)\,(x_i - 1).$$

We can extend each ∂_i to a k-linear map $kG \to kG$ by specifying that $\partial_i(k) = 0$. The maps ∂_i $(i \in I)$ are called the "Fox derivations with respect to x_i"; they are k-linear endomorphisms of kG characterized by the derivation properties

$$\partial_i(\alpha + \beta) = \partial_i(\alpha) + \partial_i(\beta), \quad \partial_i(\alpha\beta) = \alpha\partial_i(\beta) + \partial_i(\alpha)\varepsilon(\beta) \quad (\alpha,\ \beta \in kG),$$

and the property that $\partial_i(x_j) = \delta_{ij}$ (the Kronecker deltas). More generally, it can be shown that each ideal power $\mathfrak{A}^n$ $(n \geq 1)$ is kG-free with basis

$$\{(x_{i_1} - 1)\ \cdots\ (x_{i_n} - 1)\},$$

and there is a formula for $\alpha \in \mathfrak{A}^n$ analogous to $(*)$ using higher Fox derivations.

It follows from this exercise that, if $|I| \geq 2$ and $k \neq 0$, then $R = kG$ *does not* satisfy the left (or right) strong rank condition, although R does satisfy the rank condition since k does and we have the augmentation homomorphism ε from R to k. In the case where $k = \mathbb{Z}$, the freeness of $\mathfrak{A}$ as a (left) kG-module implies that the n-th homology (and cohomology) of G in any $\mathbb{Z}G$-module $\mathbb{M}$ is trivial for $n > 1$.

Ex. 1.30. Let G and k be as in the preceding exercise, and let H be a subgroup of G. It is known that H is also a free group, say, on a set of generators $\{y_j : j \in J\}$. Let G/H be the coset space $\{gH : g \in G\}$ viewed as a left G-set, and let $k[G/H]$ be the permutation kG-module with k-basis G/H. Let $\alpha : kG \to k[G/H]$ be the kG-module homomorphism induced by the natural G-map $G \to G/H$. Show that, as a left kG-module, $\ker(\alpha)$ is free with basis $\{y_j - 1 : j \in J\}$. (This generalizes the last exercise, which corresponds to the case $H = G$.)

Solution. Let $R = kG$, $\mathfrak{B} = \ker(\alpha)$, and

$$\mathfrak{B}_0 = \sum\nolimits_j R \cdot (y_j - 1) \subseteq \mathfrak{B}.$$

It is easy to see that all $y \in H$ map to 1 in the quotient module $R/\mathfrak{B}_0$. From this, we see that $\mathfrak{B}_0 = \mathfrak{B}$. Thus, ${}_R\mathfrak{B}$ is generated by $\{y_j - 1 \ : \ j \in J\}$, so it only remains to prove that the elements $y_j - 1$ $(j \in J)$ are left linearly independent over R. Suppose $\sum_j r_j(y_j - 1) = 0$, where $r_j \in R$. Fix a coset decomposition $G = \bigcup_\ell g_\ell H$ with respect to H. Then $R = kG = \bigoplus_\ell g_\ell \cdot kH$, so we can write $r_j = \sum_\ell g_\ell \, s_{j\ell}$ where $s_{j\ell} \in kH$. From

$$0 = \sum_j r_j(y_j - 1) = \sum_j \sum_\ell g_\ell s_{j\ell}(y_j - 1) = \sum_\ell g_\ell \sum_j s_{j\ell}(y_j - 1),$$

we see that $\sum_j s_{j\ell}(y_j - 1) = 0 \in kH$ for each ℓ. Since the augmentation ideal of kH is free on $\{y_j - 1 \ : \ j \in J\}$ as a left kH-module (by Exercise 29), it follows that $s_{j\ell} = 0$ for each j and each ℓ. Therefore, $r_j = \sum_\ell g_\ell s_{j\ell} = 0$ for each j, as desired.

Ex. 1.31. Let k be any commutative ring, and E be any (multiplicative) group. Fix a presentation of E by generators and relations, say,

$$1 \to H \to G \to E \to 1,$$

where G (and hence H) is free. (Here H is the normal subgroup of G generated by the "relations.") Let $\mathfrak{A} = \ker(\varepsilon)$ and $\mathfrak{B} = \ker(\alpha)$ be as in the last two exercises. Show that k, viewed as a left kE-module with the trivial E-action, has the following free resolution.

$$(*) \qquad \cdots \to \frac{\mathfrak{B}^2}{\mathfrak{B}^3} \to \frac{\mathfrak{B}\mathfrak{A}}{\mathfrak{B}^2\mathfrak{A}} \to \frac{\mathfrak{B}}{\mathfrak{B}^2} \to \frac{\mathfrak{A}}{\mathfrak{B}\mathfrak{A}} \to \frac{R}{\mathfrak{B}} \to k \to 0$$

in the category of left kE-modules.

Solution. As shown in the last two exercises, $\mathfrak{A}, \mathfrak{B} \subseteq kG$ are both free as left kG-modules, and we have clearly $\mathfrak{B} \subseteq \mathfrak{A}$. Since H is normal in G with $G/H \cong E$, we can identify $k\,[G/H]$ with the group ring kE. The map α in Exercise 30 is just the k-algebra homomorphism induced by the projection $G \to E$. Thus $\mathfrak{B} = \ker(\alpha)$ is an ideal in $R = kG$, with $R/\mathfrak{B} \cong kE$. The quotient $R/\mathfrak{A} \cong k$ is a left module over kG (and kE) with the trivial group action. Therefore, Exercise 28 yields the desired resolution $(*)$ of k by free left modules over $R/\mathfrak{B} \cong kE$.

Comment. For $k = \mathbb{Z}$, $(*)$ is known as *Gruenberg resolution* of the trivial $\mathbb{Z}E$-module $\mathbb{Z}$. This free resolution is of basic importance in the cohomology theory of groups, and is especially useful in computing the low dimensional homology and cohomology groups of the group E.

Ex. 1.32. Show that any nonzero submodule M of a free module F_R contains a copy of a nonzero principal right ideal aR.

Solution. We may assume M is a cyclic module xR. In this case, we may further assume that F is a free module of finite rank, say $F = R^n$. We write $x = (x_1, \ldots, x_n)$ $(x_i \in R)$, and induct on the number m of nonzero coordinates x_i. If $m = 1$, say

$$x_1 \neq 0 = x_2 = \cdots = x_n,$$

then $xR \cong x_1R$ and we are done. If $m \geq 2$, assume, for convenience, that $x_1, \ldots, x_m$ are the nonzero coordinates, and consider $\text{ann}_r(x_i)$ $(1 \leq i \leq m)$. In case these annihilators are all equal, we will have $xR \cong x_1R$ again (by the isomorphism $x_1y \mapsto xy$, $y \in R$). Therefore, we may assume, say $\text{ann}_r(x_1) \not\subseteq \text{ann}_r(x_2)$. Fix an element $y \in R$ such that $x_1y = 0$ but $x_2y \neq 0$, and let

$$x' = xy = (0, x_2y, \ldots, x_my, 0, \ldots, 0) \neq 0,$$

which has fewer than m nonzero coordinates. Since $x'R = xyR \subseteq xR$, we are done by applying the inductive hypothesis to $x'R$.

Comment. The result and its proof in this exercise come from the paper of H. Bass, "Finitistic dimension and a homological generalization of semi-primary rings," Trans. Amer. Math. Soc. **95**(1960), 466–488.

Of course, the exercise remains true if the free module F_R is replaced by a projective module P_R, since P can be embedded into some free F_R.

Ex. 1.33. Let F_R be a free R-module on the basis $\{e_1, \ldots, e_n\}$, $\alpha = e_1a_1 + \cdots + e_na_n \in F$ $(a_i \in F)$, and $A = \sum_i Ra_i$. Let f be an idempotent in R. Show that the following are equivalent: (1) $A = Rf$; (2) $\alpha \cdot R$ is a direct summand of F isomorphic to fR with $\alpha \leftrightarrow f$.

Solution. (1) $\Rightarrow$ (2). Write $f = \sum_i r_ia_i$ where $r_i \in R$. Since $a_i \in Rf$, we have $a_if = a_i$. Let $\varphi \in \text{Hom}_R(F, fR)$ be defined by $\varphi(e_i) = fr_i$, and $\psi \in \text{Hom}_R(fR, F)$ be defined by $\psi(fr) = \alpha r (\forall r \in R)$. Note that ψ is well-defined, for, if $fr = 0$, then $a_ir = a_ifr = 0$ for all i, so $\alpha r = 0$. Since

$$\varphi\psi(fr) = \varphi(\alpha r) = \sum_i fr_ia_ir = f^2r = fr$$

for all $r \in R$, φ is a surjection split by ψ. In particular, $\varphi(\alpha) = f$ defines an isomorphism $\alpha R \cong fR$, and $F = \alpha R \oplus \ker(\varphi)$.

(2) $\Rightarrow$ (1). Write $F = \alpha \cdot R \oplus K$ and let $\varphi : \alpha R \to fR$ be an isomorphism with $\varphi(\alpha) = f$. We may think of φ as in $\text{Hom}_R(F,\ fR)$ with $\varphi(K) = 0$. Let $\varphi(e_i) = fr_i$ where $r_i \in R$. Since

$$f = \varphi(\alpha) = \sum_i \varphi(e_i)a_i \in \sum_i Ra_i = A,$$

we have $Rf \subseteq A$. On the other hand,

$$\varphi(\alpha f) = \varphi(\alpha)f = f^2 = f = \varphi(\alpha)$$

implies that $\alpha f = \alpha$. Therefore $a_i = a_if \in Rf$ for all i. This shows that $Rf = A$, as desired.

Comment. The result in this exercise comes from the same paper of H. Bass referenced in the *Comment* on the last exercise.

Ex. 1.34. ("Unimodular Column Lemma") Let $F = \bigoplus_{i=1}^{n} e_i R$ and $\alpha = \sum_i e_i\, a_i \in F$ be as in Exercise 33.

(1) Show that $\sum_{i=1}^{n} Ra_i = R$ iff $\alpha \cdot R$ is a direct summand of F free on $\{\alpha\}$.
(2) In case $\sum_{i=1}^{n} Ra_i = R$, show that a direct complement of $\alpha \cdot R$ in F is free of rank $n-1$ iff there exists a matrix $(a_{ij}) \in \mathrm{GL}_n(R)$ with $a_{i1} = a_i$ for all i.

Solution. (1) follows by applying Exercise 33 to $A = \sum_i Ra_i$ with the idempotent $f = 1 \in R$. For (2), assume that $\sum_i Ra_i = R$. Note that the isomorphism type of a direct complement of $\alpha \cdot R$ is uniquely determined (being isomorphic to $F/\alpha \cdot R$). If such a complement is $\cong R^{n-1}$, there exists a basis $\{\alpha_1, \ldots, \alpha_n\}$ on F with $\alpha_1 = \alpha$. Then the matrix expressing the $\{\alpha_j\}$ in terms of the $\{e_i\}$ is in $\mathrm{GL}_n(R)$ with first column $(a_1, \ldots, a_n)^t$. Conversely, if there exists $(a_{ij}) \in \mathrm{GL}_n(R)$ with $a_{i1} = a_i$ for all i, we can define $\alpha_j = \sum_i e_i a_{ij} \in F\,(1 \le j \le n)$. These form a basis for F with $\alpha_1 = \sum_i e_i a_i = \alpha$, so $\alpha \cdot R$ has a direct complement $\bigoplus_{i=2}^{n} \alpha_i R \cong R^{n-1}$.

Ex. 1.35. Let R be a ring with IBN such that any direct summand of R_R^n is free (for a fixed n). Show that $\sum_{i=1}^{n} Ra_i = R$ iff the column vector $(a_1, \ldots, a_n)^t$ can be completed to a matrix in $\mathrm{GL}_n(R)$.

Solution. The "if" part is clear by considering an inverse of a matrix in $\mathrm{GL}_n(R)$ completing the column vector $(a_1, \ \ldots \ , a_n)^t$. Conversely, assume that $\sum_{i=1}^{n} Ra_i = R$. Then, for $\alpha = \sum e_i a_i$ in $R^n = \bigoplus_{i=1}^{n} e_i R$, we have $R^n = \alpha R \oplus K$ for some K. By assumption, K must be free (and f.g.), and IBN implies that $K \cong R^{n-1}$. By the last exercise, it follows that $(a_1, \ldots, a_n)^t$ can be completed to a matrix in $\mathrm{GL}_n(R)$.

Comment. The best known case in which the conclusion of the exercise applies is when R is a commutative PID. In the case $R = \mathbb{Z}$, the exercise goes all the way back to Charles Hermite. In general, a unimodular column $(a_1, \ldots, a_n)^t$ need not be completable to an invertible matrix. The standard counterexample is over the coordinate ring of the 2-sphere, $R = \mathbb{R}[x, y, z]$, with the relation $x^2 + y^2 + z^2 = 1$. Here, the column $(x, y, z)^t$ is unimodular, but the existence of a matrix in $\mathrm{GL}_3(R)$ with first column $(x, y, z)^t$ would contradict the "Hairy Ball Theorem" in topology; see p. 34 of my book *Serre's Problem on Projective Modules*, Springer-Verlag, 2006. A direct complement to the free module generated by $\alpha = xe_1 + ye_2 + ze_3$ in R^3 is stably free, but not free, over R; this module corresponds to the tangent bundle of the 2-sphere.

§2. Projective Modules

Projective modules are important because they arise as direct summands of free modules. In general, they need not be free. A good example of a (f.g.)

nonfree projective module is given by a nonprincipal ideal in the ring of algebraic integers in a given number field. For a noncommutative example, take R to be the triangular ring $\begin{pmatrix} k & k \\ 0 & k \end{pmatrix}$ over a field k; both $P = \begin{pmatrix} k & k \\ 0 & 0 \end{pmatrix}$ and $Q = \begin{pmatrix} 0 & 0 \\ 0 & k \end{pmatrix}$ are projective right R-modules, and they are not free since any f.g. free R-module must have k-dimension divisible by 3.

It is convenient to think of projective modules as direct summands of free modules, but the "textbook definition" for P_R to be projective is that the functor $\mathrm{Hom}_R(P, -)$ be *exact* (instead of just left exact). The required right exactness for $\mathrm{Hom}_R(P, -)$ amounts to the usual "lifting property" for homomorphisms from P (to an epimorphic image of any module). A very important characterization for a projective module is the *Dual Basis Lemma* in *LMR*-(2.9). This lemma implies that any projective module P *injects* into its double dual by the canonical map $\varepsilon: P \to P^{**}$. In case P is f.g. (and projective), P^* is also f.g. projective, and ε is an isomorphism (Exercise 7). In general, however, neither statement is true (Exercise 8).

In the case of a *commutative* ring R, the (f.g.) "rank 1 projectives" over R present an important class of modules for study. They can be put together into an abelian group, with binary operation given by the tensor product. Of course, the elements of this group are not quite the projective modules themselves, but rather *isomorphism classes* of them, denoted by $[P]$. The abelian group formed this way is $\mathrm{Pic}(R)$, called the *Picard group* of R; its identity element is $[R]$ since

$$[R] \cdot [P] = [R \otimes_R P] = [P].$$

Various examples of Picard groups are given in *LMR*-§2.

The "rank" of a projective module P over a (nonzero) commutative ring R is defined via localizations. In general, the "rank" of a f.g. locally free module P over R is only a function

$$\mathrm{rk}\, P: \ \mathrm{Spec}\, R \to \mathbb{Z},$$

where $\mathrm{Spec}\, R$ is the Zariski spectrum of R. If this happens to be a constant function n, then we say P has rank n. Exercise 21 summarizes a number of standard facts about the rank from Bourbaki. In particular, we learn from this exercise that, among the f.g. modules, the projective ones are precisely those P for which $\mathrm{rk}\, P$ is a *continuous* (or equivalently, *locally constant*) function. Much more information about f.g. projectives in the commutative case is given in Exercises 22–31.

Two important classes of rings defined via the notion of projective modules are the right hereditary (resp. semihereditary) rings. A ring R is *right hereditary* (resp. *semihereditary*) if every right ideal (resp. f.g. right ideal) in R is projective (as a right R-module). Over a right hereditary ring R, any projective right module is isomorphic to a direct sum of right ideals (Theorem of Kaplansky), and over a right semihereditary ring R, any f.g.

projective right R-module is isomorphic to a direct sum of f.g. right ideals (Theorem of Albrecht).

In studying the notions of hereditary and semihereditary rings, the following examples should be kept in mind: von Neumann regular rings are always semihereditary and, among the commutative domains, the hereditary ones are the Dedekind domains, and the semihereditary ones are the Prüfer domains. Dedekind domains are just the *noetherian* Prüfer domains. In the noncommutative case, we have to be more careful as usual, since right (semi)-hereditary rings need not be left (semi)-hereditary, as shown by examples in *LMR*-(2.33) and *LMR*-(2.34).

One of the nicest hereditary rings is the ring of $n \times n$ upper triangular matrices over a division ring k. Some interesting characterizations of right artinian right hereditary rings are given in *LMR*-(2.35).

The last section of *LMR*-§2 introduces the notion of the *trace ideal* of a right R-module P_R: this is defined to be $\sum \operatorname{im}(f)$ where f ranges over the dual $P^* = \operatorname{Hom}_R(P, R)$. Denoted by $\operatorname{tr}(P)$, this is always an ideal of R, and in fact an idempotent ideal if P_R happens to be projective. An important result in this connection is that *a f.g. projective module P over a commutative ring R is faithful iff* $\operatorname{tr}(P) = R$ (or, iff P is a *generator*, in the terminology to be introduced in a later section). Over any ring R, the f.g. projective generators are important since they can be used to produce other rings that are "Morita equivalent" to R.

The most challenging exercise in the following may be $(8')$, which is assigned as an "Extra Credit" problem! Though not essential to the main development in the text, this exercise provides a good test for the reader's problem-solving skill.

Exercises for §2

Ex. 2.1. Let S, R be rings and let ${}_SP_R$ be an (S, R)-bimodule such that P_R is a projective right R-module. Show that, for any projective S-module M_S, the tensor product $M \otimes_S P$ is a projective right R-module. In particular, if there is a given ring homomorphism $S \to R$, whereby we can view R as an (S, R)-bimodule, then for any projective S-module M_S, $M \otimes_S R$ is a projective right R-module . Deduce that, for any ideal $\mathfrak{A} \subseteq S$, if M_S is any projective S-module, then $M/M\mathfrak{A}$ is a projective right $S/\mathfrak{A}$-module.

Solution. Fix an S-module N_S such that $M \oplus N \cong S^{(I)}$ (free right S-module with a basis indexed by I). Then

$$(M \otimes_S P) \oplus (N \otimes_S P) \cong (M \oplus N) \otimes_S P \cong S^{(I)} \otimes_S P.$$

The RHS is just a direct sum of $|I|$ copies of P, so it is a projective right R-module. It follows that $M \otimes_S P$ is also a projective right R-module.

The statement concerning the ring homomorphism $S \to R$ follows by taking $P = {}_S R_R$. For the last statement in the exercise, consider the projection map $S \to S/\mathfrak{A}$. For any projective right S-module M_S, it follows from the above that $M \otimes_S (S/\mathfrak{A}) \cong M/M\mathfrak{A}$ is a projective right $S/\mathfrak{A}$-module.

Ex. 2.2. Show that a principal right ideal aR in a ring R is projective as a right R-module iff $\text{ann}_r(a)$ (the right annihilator of a) is of the form eR where e is an idempotent of R.

Solution. Consider the exact sequence of right R-modules

$$0 \longrightarrow \text{ann}_r(a) \longrightarrow R \xrightarrow{f} aR \longrightarrow 0,$$

where f is defined by left multiplication by a. If aR is projective, this sequence splits. Then $\text{ann}_r(a)$ is a direct summand of R_R, so $\text{ann}_r(a) = eR$ for some $e = e^2 \in R$. Conversely, if $\text{ann}_r(a) = eR$ where $e = e^2 \in R$, then the direct sum decomposition $R = eR \oplus (1-e)R$ implies that

$$aR \cong R/eR \cong (1-e)R,$$

which is a projective right R-module.

Comment. The following important special cases of this exercise should be kept in mind.

(1) If $\text{ann}_r(a) = 0$, then aR is a free R-module of rank 1, with a singleton basis $\{a\}$.

(2) If a is itself an idempotent, then indeed aR is projective. In this case, $\text{ann}_r(a) = eR$ for the complementary idempotent $e := 1 - a$.

Rings R for which *all* principal right ideals are projective ("principal $\Rightarrow$ projective") are called *right PP-rings,* or *right Rickart rings.* For more information on these rings and examples of them, see *LMR*-§7D.

Ex. 2.3. (Ojanguren-Sridharan) Let a, b be two noncommuting elements in a division ring D, and let $R = D[x, y]$. Define a right R-homomorphism $\varphi : R^2 \to R$ by $\varphi(1, 0) = x + a$, $\varphi(0, 1) = y + b$, and let $P = \ker(\varphi)$. Show that (1) P is a f.g. projective R-module with $P \oplus R \cong R^2$, and (2) P is isomorphic to a right ideal of R.

Solution. The image of φ contains

$$(x + a)(y + b) - (y + b)(x + a) = ab - ab \in \mathrm{U}(D) = \mathrm{U}(R).$$

Therefore, φ is *onto.* Since R_R is projective, φ splits. It follows that

$$R^2 \cong \ker(\varphi) \oplus \text{im}(\varphi) = P \oplus R,$$

so P is a f.g. projective (in fact stably free) R-module .

It is quite easy to see that P is isomorphic to a right ideal of R. In fact, for the right ideal $A, = (x + a)R \cap (y + b)R$, we have an R-monomorphism

$\psi: A \to P$ defined by $\psi(f) = (g, -h)$ if

$$f = (x+a)g = (y+b)h \in A.$$

This map is also surjective since, for *any* $(g, -h) \in P$,

$$0 = \varphi(g, -h) = (x+a)g - (y+b)h$$

implies that $(g, -h) = \psi(f)$ for $f = (x+a)g = (y+b)h \in A$. Thus, $P \cong A$.

Comment. This example is drawn from the paper of M. Ojanguren and R. Sridharan, "Cancellation of Azumaya algebras," J. Algebra **18**(1971), 501–505. In this paper, it is also shown that the stably free R-module P constructed above is not free, but $3 \cdot P (= P \oplus P \oplus P)$ is free.

The general study of f.g. projective right modules over $R_n = D[x_1, \ldots, x_n]$ was started by A. Suslin (see Trudy Mat. Inst. Steklov. **148**(1978), 233–252), who proved that if D is finite-dimensional over its center $Z(D)$, then any f.g. projective module P of rank(*) > 1 over R_n is free. Later, in "Projective modules over polynomial extensions of polynomial rings," Invent. Math. **59**(1980), 105–117, J. T. Stafford proved that, as long as $Z(D)$ is infinite, *any f.g. projective right R_n-module is either free or else isomorphic to a right ideal of R_n*. It follows that (assuming $Z(D)$ is infinite), for any projective right ideal $P \subseteq R_n$, $r \cdot P$ is in fact free for any $r \geq 2$.

(*Note.* The finite direct sum $r \cdot P$ of r copies of P is often written also as P^r.)

Ex. 2.4. Let P be a projective right R-module that has R as a direct summand. If $P \oplus R^m \cong R^n$ where $n > m$, show that P^{m+1} is free.

Solution. Say $P \cong R \oplus Q$, where Q is a suitable right R-module. Then

(1) $$P^{m+1} \cong P \oplus R^m \oplus Q^m \cong R^n \oplus Q^m.$$

On the other hand,

(2) $$R^{m+1} \oplus Q \cong R^m \oplus (R \oplus Q) \cong R^m \oplus P \cong R^n$$

is free. Adding copies of Q to the two sides, and using the fact that $n \geq m+1$, we see that $R^{m+1} \oplus Q^m$ is also free. Going back to (1), we conclude that P^{m+1} is free.

Comment. This exercise is taken from the author's paper, "Series summation of stably free modules," Quart. J. Math. **27**(1976), 37–46.

Ex. 2.5. Suppose R has IBN and f.g. projective right R-modules are free. Show that R is stably finite, and hence R satisfies the rank condition.

(*) Here, the rank of P may be defined as $\dim_D P/P(x_1, \ldots, x_n)$.

Solution. To check that R is stably finite, suppose $R^n \cong R^n \oplus L$, where L is some R-module. By assumption, $L \cong R^s$ for some s, so IBN yields $n = n + s$. Thus, $L = 0$, and we have checked that R^n is Dedekind-finite for any n; that is, R is stably finite. The fact that R satisfies the rank condition now follows from *LMR*-(1.22).

Ex. 2.6. Use a cardinality argument to show that the $\mathbb{Z}$-module $M = \mathbb{Z} \times \mathbb{Z} \times \cdots$ is not a projective $\mathbb{Z}$-module. (Do not assume that subgroups of free abelian groups are free.)

Solution. We modify the proof given in *LMR*-(2.8) as follows. Assume that M is projective. Then $M \subseteq F$ for a suitable free abelian group F with basis $\{e_i : i \in I\}$. Since $P := \mathbb{Z} \oplus \mathbb{Z} \oplus \cdots \subseteq M$ is countable, we can decompose I into a disjoint union $I_1 \cup I_2$ such that I_1 is countable and P is contained in the span F_1 of $\{e_i : i \in I_1\}$. Let F_2 be the span of $\{e_i : i \in I_2\}$. The group $M/M \cap F_1$ has an embedding into the free abelian group $F/F_1 \cong F_2$. We will get a contradiction if we can show that $M/M \cap F_1$ contains a nonzero element α that is divisible by 2^n for any $n \geq 1$. Consider the set

$$S = \{(2\varepsilon_1, 4\varepsilon_2, 8\varepsilon_3, \ldots) : \varepsilon_i = \pm 1\} \subseteq M.$$

Since F_1 is countable but S is not, S contains an element

$$a = (2\varepsilon_1, 4\varepsilon_2, 8\varepsilon_3, \ldots) \notin F_1.$$

Then $\alpha := a + M \cap F_1 \in M/(M \cap F_1)\backslash\{0\}$ is divisible by 2^n for any n since

$$\begin{aligned}\alpha &= (2\varepsilon_1, \ldots, 2^{n-1}\varepsilon_{n-1}, 0, 0, \ldots) + (0, \ldots, 0, 2^n\varepsilon_n, 2^{n+1}\varepsilon_{n+1}, \ldots) + M \cap F_1 \\ &= 2^n(0, \ldots, 0, \varepsilon_n, 2\varepsilon_{n+1}, \ldots) + M \cap F_1.\end{aligned}$$

Comment. The result in this Exercise is due to Reinhold Baer. In connection to this classical result, one can ask a number of questions, the answers to which may or may not be easy. For instance, *are the following subgroups of M free*?

(1) $Q_1 = \{(a_1, a_2, \ldots) : a_n \in 2\,\mathbb{Z}$ for sufficiently large $n\}$.
(2) $Q_2 = \{(2b_1, 4b_2, 8b_3, \ldots) : b_i \in \mathbb{Z}\}$.
(3) $Q_3 = \{(a_1, a_2, \ldots)$: the a_i's are eventually constant$\}$.
(4) $Q_4 = \{(a_1, a_2, \ldots)$: the a_i's are bounded$\}$.

For Q_1 and Q_2, the answers are "no". In fact, $Q_1 \supseteq 2M$ and $2M \cong M$ is not free, so Q_1 cannot be free. The map

$$(b_1, b_2, \ldots) \mapsto (2b_1, 4b_2, \ldots)$$

defines an isomorphism from M to Q_2, so Q_2 is also not free. However, Q_3 and Q_4 turn out to be free. In fact, by a general theorem of Specker and Nöbeling, if X is *any* set, then any additive group of bounded functions from X to $\mathbb{Z}$ (where functions are added pointwise) is free. For the relevant literature, see E. Specker: "Additive Gruppen von Folgen ganzen Zahlen," Portugaliae Math. **9**(1950), 131–140, and G. Nöbeling: "Verallgemeinerung eines Satzes von Herrn E. Specker," Invent. Math. **6**(1968), 41–55. For a ring-theoretic treatment of the Specker–Nöbeling Theorem and its generalizations, see G. Bergman: "Boolean rings of projection maps," J. London Math. Soc. **4**(1972), 593–598.

There is, of course, also the question of generalizing the results from $\mathbb{Z}$-modules to modules over other rings. Again, the situation is far from easy. I. Emmanouil has pointed out to me that the techniques used here can be generalized to show that, over any commutative noetherian domain R which is not a field, the R-module $R \times R \times \cdots$ is not free. Much earlier, S. Chase has shown that for any domain R with a nonzero element p such that $\bigcap p^n R = 0$, the *left* R-module $R \times R \times \cdots$ is not projective: see his paper "On direct sums and products of modules," Pac. J. Math. **12**(1962), 847–854.

In general, one can consider *arbitrary* direct products, instead of just countable ones. (This was done in Chase's article, where a lot of useful information can be found.) For any infinite set I, the direct product $\mathbb{Z}^I$ is clearly still nonfree, since it contains a copy of $\mathbb{Z}^{\mathbb{N}}$. However, another theorem of Specker guarantees that any *countable* subgroup of $\mathbb{Z}^I$ is free. (For a proof, see p. 115 of D. J. S. Robinson's book "A Course in the Theory of Groups", Springer-Verlag, 1982.) For general products R^I over arbitrary rings, the freeness questions become more murky as it involves subtle consideration of cardinal numbers.

Ex. 2.7. Let P be a f.g. projective right R-module, with a pair of dual bases $\{a_i, f_i : 1 \le i \le n\}$. For $a \in P$, let $\hat{a} \in P^{**}$ be defined by $f\,\hat{a} = f(a)$, for every $f \in P^*$. (Note that, since P^* is a left R-module, we write linear functionals on P^* on the right.) Show that

(1) $\{f_i, \hat{a}_i\}$ is a pair of dual bases for P^*;
(2) P^* is a f.g. projective left R-module; and
(3) the natural map $\varepsilon : P \to P^{**}$ defined by $\varepsilon(a) = \hat{a}$ (for every $a \in P$) is an isomorphism of right R-modules.

Solution. (1) We must verify that $f = \sum (f\,\hat{a}_i)\, f_i$ for every $f \in P^*$. Indeed, computing the RHS on any $a \in P$, we have

$$\left(\sum (f\,\hat{a}_i) f_i\right)(a) = \sum f(a_i) f_i(a) = f\left(\sum a_i f_i(a)\right) = f(a).$$

(2) By the Dual Basis Lemma, (1) implies (2).

(3) In *LMR*-(2.10), it is already shown that ε is *injective*. By the equation $f = \sum (f\ \hat{a}_i) f_i$, we know that $\{f_i\}$ is a generating set for P^*. Applying this conclusion to the pair of dual bases $\{f_i,\ \hat{a}_i\}$ for P^*, we see that $\{\hat{a}_i\}$ is a generating set for P^{**}. Since $\hat{a}_i = \varepsilon(a_i)$, it follows that $\varepsilon : P \to P^{**}$ is *surjective*, so ε is an isomorphism.

Comment. The fact that $\varepsilon : P \to P^{**}$ is an isomorphism (in case P is f.g. projective) can also be proved by making a reduction to the case where P is f.g. free. In the case $P = R_R$, we have $P^* = {}_RR$, and $P^{**} = R_R$, (using suitable identifications). The map ε is now the identity map from R to R. Taking finite direct sums, we see that ε is also an isomorphism for $P = R^n$. For a general f.g. projective module P, fix a module Q such that $P \oplus Q \cong R^n$. Since $\varepsilon_{P\oplus Q} = \varepsilon_P \oplus \varepsilon_Q$ is an isomorphism, it follows that ε_P is also an isomorphism.

What about the f.g. assumption? Ostensibly, the formal calculation in (1) above works for any pair of dual bases $\{a_i, f_i\}$. However, this is not so. If the set $\{a_i, f_i\}$ is *infinite*, we have no way of knowing that, for any $f \in P^*, f\ \hat{a}_i = f(a_i)$ is zero for all but finitely many i. Without this, the sum $\sum_i (f\ \hat{a}_i)f_i$ becomes meaningless. More information on the relation between P and P^{**} in the non-f.g. case can be found in the next two exercises.

Ex. 2.8. Give examples of (necessarily non-f.g.) projective right R-modules P, P_1 such that (1) the first dual P^* of P is *not* a projective left R-module, and (2) the natural embedding of P_1 into P_1^{**} is *not* an isomorphism.

Solution. (1) Let $R = \mathbb{Z}$, and let P be the non-f.g. free R-module $\mathbb{Z} \oplus \mathbb{Z} \oplus \cdots$. Then

$$P^* = \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z} \oplus \mathbb{Z} \oplus \cdots,\ \mathbb{Z}) \cong \mathbb{Z} \times \mathbb{Z} \times \cdots .$$

By *LMR*-(2.8), this is *not* a projective $\mathbb{Z}$-module.

(2) Let $R = \mathbb{Q}$, and $P_1 = \mathbb{Q} \oplus \mathbb{Q} \oplus \cdots$, a (free) $\mathbb{Q}$-vector space of countable dimension. As in (1), we have $P_1^* \cong \mathbb{Q} \times \mathbb{Q} \times \cdots$, which is a $\mathbb{Q}$-vector space of uncountable dimension. Clearly, its dual P_1^{**} is also a $\mathbb{Q}$-vector space of uncountable dimension. Therefore, the natural embedding $\varepsilon : P_1 \to P_1^{**}$ cannot be an isomorphism.

Comment. In the solution to (2), we worked with the field $\mathbb{Q}$ since the fact that $\mathbb{Q}$ is countable makes it easy to see that $\mathbb{Q} \times \mathbb{Q} \times \cdots$ is of uncountable $\mathbb{Q}$-dimension. In general, it is true that $k \times k \times \cdots$ is of uncountable k-dimension over any field k, so we could have chosen $P_1 = k \oplus k \oplus \cdots$ over any field k in the solution of (2).

Ex. 2.8′. (Extra Credit) Let M be the $\mathbb{Z}$-module $\mathbb{Z} \times \mathbb{Z} \times \cdots$ and let $e_1, e_2, \cdots \in M$ be the standard unit vectors.

(1) For any $f \in \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Z})$, show that $f(e_i) = 0$ for almost all i.

(2) Using (1), show that, for the free $\mathbb{Z}$-module $P = \mathbb{Z} \oplus \mathbb{Z} \oplus \cdots$, the natural map $\varepsilon : P \to P^{**}$ is an isomorphism.

Solution. We first prove (2), assuming (1). Since P is $\mathbb{Z}$-free, ε is an injection by *LMR*-(2.10). To show that ε is a surjection, consider any $f \in \mathrm{Hom}_{\mathbb{Z}}(P^*, \mathbb{Z})$. As in Exercise 8, we may identify P^* with M. By (1), there exists n such that $f(e_i) = 0$ for $i > n$. Let $N = \mathbb{Z}e_{n+1} \times \mathbb{Z}e_{n+2} \times \cdots$, and let $g = f|N$. Since g is zero on $\mathbb{Z}e_{n+1} \oplus \mathbb{Z}e_{n+2} \oplus \ldots$, *LMR*-(2.8)′ implies that $g \equiv 0$. Therefore, for $a_i = f(e_i)(i \geq 1)$ and $a = (a_1, a_2, \ldots) \in P$, we have

$$\begin{aligned} f(x_1, x_2, \ldots) &= f(x_1, \ldots, x_n, 0, \ldots) + f(0, \ldots, 0, x_{n+1}, x_{n+2}, \ldots) \\ &= x_1 f(e_1) + \cdots + x_n f(e_n) \\ &= \sum\nolimits_{i=1}^{\infty} x_i a_i \\ &= \varepsilon(a)(x_1, x_2, \ldots). \end{aligned}$$

Thus, $f = \varepsilon(a)$, as desired.

To prove (1), let us assume, instead, that $f(e_i) \neq 0$ for infinitely many i's. By keeping only these i's, we may assume that $f(e_i) \neq 0$ for all $i \geq 1$. Also, replacing e_i by $-e_i$ if necessary, we may assume that each $a_i := f(e_i) > 0$. Fix any prime $p \nmid a_1$, and define two sequences $\{y_n, x_n : n \geq 1\} \subseteq \mathbb{N}$ inductively on n as follows: $y_1 = x_1 = 1$, and for $n > 1$, $y_n = x_1 a_1 + \cdots + x_{n-1} a_{n-1}$, $x_n = p y_n$. Note that

$$y_n = y_{n-1} + x_{n-1} a_{n-1} = y_{n-1}(1 + p a_{n-1}) \quad (\forall n \geq 2),$$

so $x_n = p y_n$ is divisible by y_i for all $i \leq n$. Now, for any n:

$$\begin{aligned} f(x_1, x_2, \ldots) &= f(x_1, \ldots, x_{n-1}, 0, 0, \ldots) + f(0, \ldots, 0, x_n, x_{n+1}, \ldots) \\ &= y_n + f(0, \ldots, 0, x_n, x_{n+1}, \ldots) \end{aligned}$$

is divisible by y_n (since each of $x_n, x_{n+1}, \ldots$ is). As the sequence $y_n \to \infty$, this implies that $f(x_1, x_2, \ldots) = 0$. From

$$\begin{aligned} f(x_1, x_2, \ldots) &= f(1, 0, 0, \ldots) + f(0, p y_2, p y_3, \ldots) \\ &= a_1 + p f(0, y_2, y_2, \ldots), \end{aligned}$$

We have therefore $p | a_1$, a contradiction!

Comment. The analysis of the homomorphisms from $M = \mathbb{Z} \times \mathbb{Z} \times \cdots$ to $\mathbb{Z}$ comes from E. Specker's study of "growth types"; see his paper referenced in the *Comment* on Exercise 6. In particular, Specker showed that the $\mathbb{Z}$-dual of M is $P = \mathbb{Z} \oplus \mathbb{Z} \oplus \cdots$. The generalization of this to an arbitrary direct product $M = \mathbb{Z}^I$ was taken up by E. C. Zeeman, in what appears to be his maiden paper, "On direct sums of free cycles," J. London Math. Soc. **30**(1955), 195–212. Zeeman showed that the $\mathbb{Z}$-dual of the direct product $\mathbb{Z}^I$ is the direct sum $\mathbb{Z}^{(I)}$, assuming, however, the "axiom of accessibility of ordinals." Zeeman's work was motivated by algebraic topology. He remarked that $\mathbb{Z}^{(I)}$ and $\mathbb{Z}^I$ are the groups of finite integral chains and infinite integral cochains, so the statement that

$$\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}^I, \mathbb{Z}) \cong \mathbb{Z}^{(I)} \quad \text{(along with} \quad \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}^{(I)}, \mathbb{Z}) \cong \mathbb{Z}^I)$$

expresses a duality between them. Our *ad hoc* treatment in the case $I = \mathbb{N}$ follows the paper of G. A. Reid, "Almost free abelian groups," Tulane University Lecture Notes, 1966/67. I thank Rad Dimitrić for pointing out the references above, as well as for providing the remarks below.

In general, over any ring R, a right R-module E is said to be *slender* if, for any R-homomorphism

$$f: R \times R \times \cdots \to E,$$

we have $f(e_i) = 0$ for almost all i (where, as before, $e_1, e_2, \ldots$ are the standard unit vectors in $R \times R \times \cdots$). Part (1) of the Exercise says precisely that the $\mathbb{Z}$-module $\mathbb{Z}$ is slender. Let us say that a ring R is (right) slender if the right regular module R_R is slender; it is known that this is the case iff the free right R-module $P = R \oplus R \oplus \cdots$ is "reflexive", i.e. isomorphic to its double dual by the map ε. (This, however, need not imply that any free right R-module is reflexive.) There do exist rings which are not slender. For instance, any field k is certainly non-slender; see the *Comment* on Exercise 8. It might appear to a casual reader that the proof for $\mathbb{Z}$ being slender in this exercise would generalize to show that a PID with a nonzero prime element p is slender. This is not the case since the ring $R = \hat{\mathbb{Z}}_p$ of p-adic integers turns out to be *not* slender. An R-homomorphism $f: R \times R \times \cdots \to R$ with $f(e_i) \neq 0$ for *all* i is given by

$$f(a_1, a_2, \ldots) = \sum_{n=1}^{\infty} a_n p^n,$$

noting that the series $\sum_{n=1}^{\infty} a_n p^n$ is convergent in $\hat{\mathbb{Z}}_p$ since the n-th term $a_n p^n \to 0$ in the ultrametric topology. More generally, any module that is complete in a nondiscrete metrizable linear topology is *not* slender. For this and much more information on slender modules/rings and their applications, see Dimitric's forthcoming book "Products, Sums and Chains of Modules," Cambridge University Press.

Ex. 2.9. Let $R = \mathbb{Z}[\theta]$ $(\theta^2 = -5)$ be the full ring of algebraic integers in the number field $\mathbb{Q}(\theta)$.

(A) Show that the ideal $\mathfrak{B} = (3,\ 1+\theta)$ is invertible, and compute $\mathfrak{B}^{-1}$ explicitly.
(B) Show that $\mathfrak{B}$ and the ideal $\mathfrak{A} = (2,\ 1+\theta)$ (considered in *LMR*-(2.19D)) represent the same element in $\text{Pic}(R)$.
(C) Show that $\mathfrak{A} \oplus \mathfrak{A}$ is free of rank 2, and construct a basis for it explicitly.

Solution. We first compute $\mathfrak{B}^{-1} = \{x \in \mathbb{Q}(\theta) :\ x\mathfrak{B} \subseteq R\}$. Since $3 \in \mathfrak{B}$, any element in $\mathfrak{B}^{-1}$ has the form $x = (a + b\theta)/3$ where $a,\ b \in \mathbb{Z}$. The condition for $x \in \mathfrak{B}^{-1}$ is that $(a + b\theta)\,(1 + \theta)\ \in 3R$. Now

$$(a + b\theta)(1 + \theta) = (a - 5b) + (a + b)\theta,$$

so the condition is $3|(a + b)$. Writing $a = 3n - b$ (where $n \in \mathbb{Z}$), we have

$$x = \frac{(3n - b) + b\theta}{3} = n - b \cdot \frac{1 - \theta}{3}.$$

Therefore, $\mathfrak{B}^{-1} = \mathbb{Z} \cdot 1 + \mathbb{Z} \cdot \sigma$ where $\sigma = (1 - \theta)/3$. Since $\mathfrak{B}\mathfrak{B}^{-1}$ contains

$$3 \cdot 1 - (1 + \theta) \cdot \sigma = 3 - (1 - \theta^2)/3 = 3 - 2 = 1,$$

we see that $\mathfrak{B}\mathfrak{B}^{-1} = R$, so $\mathfrak{B}$ is an invertible ideal. (Of course, R is a Dedekind ring, so we expect every nonzero ideal to be invertible.) Moreover,

$$(1) \qquad \sigma \cdot \mathfrak{B} = \sigma(3,\ 1+\theta) = (1-\theta,\ 2) = (2, 1+\theta) = \mathfrak{A},$$

so *multiplication by* σ *gives an* R-isomorphism *from* $\mathfrak{B}$ *to* $\mathfrak{A}$. In particular, $[\mathfrak{B}] = [\mathfrak{A}] \in \text{Pic}(R)$.

Clearly, $\mathfrak{A}, \mathfrak{B}$ are comaximal ideals, so $\mathfrak{A}\mathfrak{B} = \mathfrak{A} \cap \mathfrak{B}$. We have therefore an exact sequence:

$$(2) \qquad 0 \longrightarrow \mathfrak{A}\mathfrak{B} \stackrel{h}{\longrightarrow} \mathfrak{A} \oplus \mathfrak{B} \stackrel{k}{\longrightarrow} R \longrightarrow 0,$$

where $h(c) = [c,\ -c]$ for $c \in \mathfrak{A}\mathfrak{B}$, and $k[a,\ b] = a + b$ for $a \in \mathfrak{A}$ and $b \in \mathfrak{B}$. (Here, we use square brackets to express the elements of the external direct sum.) Now

$$\mathfrak{A}\mathfrak{B} = (6,\ 2(1+\theta),\ 3(1+\theta),\ 2(\theta-2)) = (6,\ 1+\theta,\ 2(\theta-2))$$

is just $(1+\theta)R$, since $2(\theta-2) = 2(1+\theta) - 6$ and $6 = (1+\theta)\ (1-\theta)$. Therefore, from the (split) exact sequence (2), we have

$$(3) \qquad \mathfrak{A} \oplus \mathfrak{B} \cong R \oplus \mathfrak{A}\mathfrak{B} \cong R \oplus R.$$

A free R-basis for $\mathfrak{A} \oplus \mathfrak{B}$ is thus given by

$$(4) \qquad e_1 = [-2,\ 3] \quad \text{and} \quad e_2 = h(1+\theta) = [1+\theta,\ -(1+\theta)].$$

In view of (1), a free R-basis for $\mathfrak{A} \oplus \mathfrak{A}$ is given by

$$(5) \quad f_1 = [-2,\ 3\sigma] = [-2,\ 1-\theta] \quad \text{and} \quad f_2 = [1+\theta,\ -\sigma(1+\theta)] = [1+\theta,\ -2].$$

For a little computational fun, we might also check *directly* that f_1, f_2 are linearly independent in $\mathfrak{A} \oplus \mathfrak{A}$ and that they do span this R-module.

The former follows quickly from the fact that

$$\det \begin{pmatrix} -2 & 1+\theta \\ 1-\theta & -2 \end{pmatrix} = 4 - (1-\theta^2) = -2 \neq 0,$$

and the latter follows from the equations

$$
\begin{aligned}
&(6) & 2f_1 + (1-\theta)f_2 &= [2,\ 0], \\
&(7) & (1+\theta)f_1 + 3f_2 &= [1+\theta,\ 0], \\
&(8) & (1+\theta)f_1 + 2f_2 &= [0,\ 2], \\
&(9) & (-2+\theta)f_1 + (1+\theta)f_2 &= [0,\ 1+\theta],
\end{aligned}
$$

and the fact that the four elements on the right generate $\mathfrak{A} \oplus \mathfrak{A}$.

Comment. It is well-known that $\mathbb{Q}(\theta)$ has class number 2, so $\text{Pic}(R) \cong \mathbb{Z}/2\mathbb{Z}$. Since $\mathfrak{A}$ is not a principal ideal (See *LMR*-(2.19D)), we have in fact $\text{Pic}(R) = \{1,\ [\mathfrak{A}]\} = \{1,\ [\mathfrak{B}]\}$. In general, it follows from the Steinitz Isomorphism

$$A \oplus B \cong R \oplus AB \cong R \oplus (A \otimes B)$$

(for nonzero ideals A, B in a Dedekind ring R) that, if $[A] \in \mathrm{Pic}(R)$ has order $m < \infty$, then $A \oplus \cdots \oplus A$ (n copies) is R-free iff $m|n$.

Ex. 2.10. (1) Show that a Dedekind ring R has trivial Picard group iff R is a PID, iff R is a unique factorization domain. (2) Deduce from (1) that a semilocal Dedekind ring is a PID.

Solution. (1) If R is a PID, it is well-known that R is a UFD. If R is a UFD, then *LMR*-(2.22) (F) gives $\mathrm{Pic}(R) = \{1\}$. Finally, if $\mathrm{Pic}(R) = \{1\}$, then every invertible ideal in R is principal. Since R is a Dedekind ring, every nonzero ideal is invertible, and hence principal. This shows that R is a PID.

(2) If R is (commutative) semilocal, we know from *LMR*-(2.22)(D) that $\mathrm{Pic}(R) = \{1\}$. If R is also Dedekind, then (1) shows that R is a PID.

Comment. For a more powerful statement than (2), see Ex. 2.11B(2) below.

Ex. 2.11A. (Quartararo-Butts) (1) Let I be a f.g. ideal in a commutative ring R such that $\mathrm{ann}_R(I) = 0$. If $A_1, \ldots, A_n$ are proper ideals of R, show that $\bigcup_{i=1}^n (A_i\, I) \subsetneq I$. (2) Show that, if an invertible ideal of R is contained in the union of a finite number of ideals in R, then it is contained in one of them.

Solution. (1) We induct on n. To start the induction, we must show first $A_1 I \subsetneq I$. If $A_1 I = I$, then, since I is f.g., a familiar determinant argument shows that $(a-1)I = 0$ for some $a \in A_1$. But $\mathrm{ann}_R(I) = 0$, so $a = 1$, which contradicts $A_1 \neq R$. For $n \geq 2$, we may assume that the desired conclusion is true for $n-1$ ideals.

Case 1. $A_1 + A_2 \neq R$. By the inductive hypothesis, there exists $x \in I$ such that $x \notin (A_1 + A_2)\, I$ and $x \notin \bigcup_{i=3}^n (A_i\, I)$. Clearly, $x \notin A_j\, I$ for all j.

Case 2. In view of Case 1, we may assume here that the ideals $A_1, \ldots, A_n$ are pairwise comaximal. Then each A_i is also comaximal to $B_i = \prod_{j \neq i} A_j$. If $B_i I \subseteq A_i I$ for some i, then

$$I = (A_i + B_i)\, I = A_i I + B_i I = A_i I,$$

which is impossible by the case $n = 1$. Thus, for each i, there exists $x_i \in B_i I \backslash A_i I$. For $x = x_1 + \cdots + x_n \in I$, we have then $x \notin \bigcup_{i=1}^n (A_i\, I)$, as desired.

(2) Suppose an invertible ideal I is *not* contained in any one of the ideals $C_1, \ldots, C_n$. Then $C_1' := C_i \cap I \subsetneq I$. Let I^{-1} denote the usual "inverse" of I in the total ring of quotients of R. Then each $C_i' I^{-1}$ is a *proper* ideal of R (for $C_i' I^{-1} = R$ would imply $C_i' = I$ by multiplication by I). Since I is f.g. (by *LMR*-(2.17)), and

$$\mathrm{ann}_R(I) \subseteq \mathrm{ann}_R\left(I\, I^{-1}\right) = \mathrm{ann}_R(R) = 0,$$

we can apply (1) to get

$$I \supsetneqq \bigcup_{i=1}^{n} \left((C_i' I^{-1}) I \right) = \bigcup_{i=1}^{n} C_i',$$

which clearly implies $I \not\subseteq \bigcup_{i=1}^{n} C_i$.

Comment. The above results appeared in the paper of P. Quartararo and H.S. Butts: "Finite unions of ideals and modules", Proc. Amer. Math. Soc. **52**(1975), 91–96. The special case of (1) for invertible ideals I has appeared earlier in the paper of R. Gilmer and W. Heinzer: "On the number of generators of an invertible ideal", J. Algebra **14**(1970), 139–151.

Quartararo and Butts defined an R-module M to be a *u-module* if $M = M_1 + \cdots + M_n$ for submodules $\{M_1, \ldots, M_n\}$ implies that $M = M_i$ for some i. Part (2) of this exercise is tantamount to the assertion that *any invertible ideal in R is a u-module over R.* A (commutative) ring R is called a *u-ring* if every ideal in R is a u-module, and a *um*-ring if (more strongly) every R-module is a u-module. Such rings are characterized in the paper of Quartararo and Butts cited above.

Ex. 2.11B. (Gilmer–Heinzer) Let I be an invertible ideal in a commutative ring R. If an element $a \in I$ is contained in only finitely many maximal ideals $\mathfrak{m}_1, \ldots, \mathfrak{m}_n$ of R, show that $I = Ra + Rb$ for some $b \in I$.

Solution. By Ex. 2.11A(1) (applied to our invertible ideal I), there exists an element $b \in I \backslash \bigcup_{i=1}^{n} (\mathfrak{m}_i I)$. Let B be the ideal $I^{-1}b$. Then $IB = Rb$, and $b \notin \mathfrak{m}_i I$ implies that $B \not\subseteq \mathfrak{m}_i$ for each i. It follows that $Ra + B$ is contained in no maximal ideals of R, and so $Ra + B = R$. Multiplying this equation by I, we get $aI + Rb = I$, and a fortiori, $I = Ra + Rb$.

Comment. This exercise may be thought of as a generalization of the fact that, in a (commutative) semilocal ring, any invertible ideal is principal. In fact, this statement corresponds precisely to the case $a = 0$.

The result in this exercise appeared in the paper of Gilmer and Heinzer cited in the *Comment* on Ex. 2.11A. The proof above shows that the result can be improved a bit. The conclusion of the exercise would have held if we had imposed a weaker hypothesis, namely, that the ideal $I^{-1}a$ is contained in only a finite number of maximal ideals $\mathfrak{m}_1, \ldots, \mathfrak{m}_n$. Under this hypothesis (and by the same work), we would have $I^{-1}a + B = R$, and multiplication by I would have given $Ra + Rb = I$ for a tighter argument.

An interesting corollary of the Gilmer-Heinzer result is the following: *if every non* 0*-divisor in a commutative ring R is contained in only finitely many maximal ideals, then every invertible ideal of R is generated by two elements a, b, where a can be any prescribed non* 0*-divisor in I* (which always exists by *LMR*-(2.17)). For a comparable result, see Ex. 2.11D below.

Ex. 2.11C. For any Dedekind domain R, deduce the following conclusions from the last exercise:

(1) For any ideal $I \subseteq R$ and any nonzero element $a \in R$, there exists $b \in I$ such that $I = Ra + Rb$.
(2) R is either J-semisimple or a PID.

Solution. (1) follows from the last paragraph of the *Comment* on Ex. 2.11B, since any nonzero ideal of a Dedekind ring R is invertible, and any nonzero element in R is divisible by only finitely many prime ideals of R.
(2) Say $J := \text{rad}(R) \neq 0$. For any nonzero ideal I, we have $J \cdot I \neq 0$, so there exists a nonzero element $a \in J \cdot I \subseteq I$. By (1), we have $I = Ra + Rb$ for some $b \in I$. This yields $I = J \cdot I + Rb$, so Nakayama's Lemma implies that $I = Rb$.

Comment. (1) above is a well-known classical result, but (2) seems to have been observed in few textbooks. The latter is a rather striking statement: for instance, it "subsumes" the fact (Ex. 2.10(2)) that a semilocal Dedekind ring is a PID. (If a Dedekind ring R has only finitely many maximal ideals $\mathfrak{m}_1, \ldots, \mathfrak{m}_n$, then

$$0 \neq \mathfrak{m}_1 \cdots \mathfrak{m}_n \subseteq \bigcap\nolimits_{i=1}^{n} \mathfrak{m}_i = \text{rad}\,(R)\,,$$

and this implies that R is a PID by (2).)

Ex. 2.11D. (Gilmer, Matlis) If an invertible ideal I in a commutative ring R is contained in only finitely many maximal ideals $\mathfrak{m}_1, \ldots, \mathfrak{m}_n$ of R, show that $I = Ra + Rb$ for some $a,\ b \in I$.

Solution. As in the solution to Ex. 2.11B, there exists an element $b \in I \setminus \bigcup_{i=1}^{n} \mathfrak{m}_i I$. The ideal $I^{-1}b$ cannot be contained in any $\mathfrak{m}_i$ (since $I^{-1}b \subseteq \mathfrak{m}_i$ would imply that $b \in \mathfrak{m}_i I$, by the invertibility of I). Therefore, $I + I^{-1}b$ is contained in no maximal ideals of R, and so $I + I^{-1}b = R$. Multiplying this equation by I, we get $I^2 + Rb = I$. Now the ideal I/Rb in the ring R/Rb is f.g. and idempotent. By *LMR*-(2.43), this implies that I/Rb is generated by a suitable (idempotent) element $a + Rb$ in R/Rb. From this, we have clearly $I = Ra + Rb$.

Comment. The result in this exercise appeared in E. Matlis's paper: "Decomposable modules", Trans. Amer. Math. Soc. **125**(1966), 147–179, and somewhat later in the Gilmer-Heinzer paper cited in the *Comment* on Ex. 2.11A. In a footnote to the latter paper commenting on this result, Gilmer and Heinzer stated that "We may have been aware of its validity before Matlis was", citing another related result given without proof in Gilmer's paper "Overrings of Prüfer rings" in J. Algebra 4(1966), 331–340.

Since Prüfer domains are natural generalizations of Dedekind domains, the results in Ex. 2.11B–D put into focus the question whether any f.g. ideal in a Prüfer domain is 2-generated (i.e. generated by two elements). This was proved to be the case for Prüfer domains of (Krull) dimension 1 by J. Sally and W. Vasconcelos in "Stable rings", J. Pure & Applied Algebra

4(1974), 319–336. But for dimension 2, the question was later answered in the negative by H.-W. Schülting in his paper: "Über die Erzeugendenanzahl invertierbarer Ideale in Prüferringen", Comm. Algebra **7**(1979), 1331–1349. Schülting considered the "real holomorphy ring" of the rational function field $\mathbb{R}(x,\ y)$. This is a Prüfer domain of dimension 2, and Schülting showed that the fractional ideal generated by $\{1,\ x,\ y\}$ is *not* 2-generated. Prior to Schülting's result, R.C. Heitmann has shown that, in any (commutative) domain of dimension n, any invertible ideal is $(n+1)$-generated; see his paper: "Generating ideals in Prüfer domains", Pac. J. Math. **62**(1976), 117–126. Later, for any n, R. Swan constructed an n-dimensional Prüfer domain with a f.g. ideal that is (necessarily) $(n+1)$-generated, but not n-generated; see his paper "n-generator ideals in Prüfer domains", Pac. J. Math. **111**(1984), 433–446.

Ex. 2.12. Let R be a commutative local ring. Since $\text{Pic}(R) = \{1\}$ by *LMR*-(2.22)(C), we know that any invertible ideal I of R is principal. Give a direct proof for this fact without using the notion of projective modules.

Solution. Since I is invertible, there exists an equation $\sum_i b_i a_i = 1$, where $a_i \in I$ and $b_i \in I^{-1} \subseteq Q$ (the total ring of quotients of R). Since R is a local ring, we may assume that $u := b_1 a_1 \in \bigcup(R)$ (see *FC*-(19.1)(5″)). Then, for any $a \in I$, we have $a = a_1(b_1 a)u^{-1} \in a_1 R$, so $I = a_1 R$, as desired.

Ex. 2.13. In *LMR*-§2D, it is stated that "Pic" is a covariant functor from the category of commutative rings to the category of abelian groups. Supply the details for a full verification of this fact.

Solution. Let P be any f.g. projective (right) R-module of rank 1, and let $f:\ R \to S$ be any homomorphism from R to a commutative ring S. We know already that $P \otimes_R S$ is a f.g. projective S-module. Let us now verify that it has rank 1. For any prime ideal $\mathfrak{P}$ of S, $\mathfrak{p} = f^{-1}(\mathfrak{P})$ is a prime ideal in R, and we have a commutative diagram

$$\begin{array}{ccc} R & \xrightarrow{\ f\ } & S \\ \downarrow & & \downarrow \\ R_{\mathfrak{p}} & \xrightarrow{\ f'\ } & S_{\mathfrak{P}} \end{array}$$

where f' is a "localization" of f. Since the localization $P_{\mathfrak{p}}$ has rank 1, we have

$$(P \otimes_R S) \otimes_S S_{\mathfrak{P}} \cong (P \otimes_R R_{\mathfrak{p}}) \otimes_{R_{\mathfrak{p}}} S_{\mathfrak{P}} \cong R_{\mathfrak{p}} \otimes_{R_{\mathfrak{p}}} S_{\mathfrak{P}} \cong S_{\mathfrak{P}}.$$

This checks that $P \otimes_R S$ is a rank 1 (f.g. projective) S-module. Therefore $[P] \mapsto [P \otimes_R S]$ defines a map

$$f_*:\ \text{Pic}\,(R) \to \text{Pic}\,(S).$$

For $[P], [Q] \in \text{Pic}(R)$, we have a map

$$\varphi:\ (P \otimes_R S) \otimes_S (Q \otimes_R S) \longrightarrow (P \otimes_R Q) \otimes_R S$$

defined by $\varphi((p \otimes s) \otimes (q \otimes s')) = (p \otimes q) \otimes ss'$ with an inverse defined by $(p \otimes q) \otimes s \mapsto (p \otimes s) \otimes (q \otimes 1)$. It is easy to check that φ is an S-module homomorphism, so it is an isomorphism. We have therefore

$$\begin{aligned} f_* [P] \cdot f_*[Q] &= [P \otimes_R S] \cdot [Q \otimes_R S] \\ &= [(P \otimes_R S) \otimes_S (Q \otimes_R S)] \\ &= [(P \otimes_R Q) \otimes_R S] \\ &= f_* [P \otimes_R Q] \\ &= f_* ([P] \cdot [Q]), \end{aligned}$$

which shows that f_* is a group homomorphism. If $f : R \to S$ and $g : S \to T$ are homomorphisms of commutative rings, it is routine to verify that $(g \circ f)_* = g_* \circ f_*$ as group homomorphisms from $\text{Pic}(R)$ to $\text{Pic}(T)$. Therefore, "Pic" is a covariant functor from the category of commutative rings to the category of abelian groups.

Ex. 2.14. Let R be the (commutative) ring of real-valued continuous functions on $[0, 1]$, with pointwise addition and multiplication for functions. Let

$$P = \{f \in R : f \text{ vanishes on } [0, \varepsilon] \text{ for some } \varepsilon = \varepsilon(f) \in (0, 1)\}.$$

Show that P is the union of a strictly ascending chain of principal ideals $A_1 \subsetneq A_2 \subsetneq \cdots$ in R (In particular, R does not satisfy ACC on principal ideals.)

Solution. For $n \geq 2$, let $a_n \in P$ be the piecewise linear function $[0,\ 1] \to \mathbb{R}$ which is zero on $[0,\ 1/n]$ and whose graph on $[1/n,\ 1]$ is the line segment joining the two points $(1/n,\ 0)$ and $(1,\ 1)$. Note that $a_{n+1} \notin a_n R$ since any function in $a_n R$ must vanish on $[1/(n+1), 1/n]$ but a_{n+1} doesn't. On the other hand, $a_n \in a_{n+1} R$. To see this, define a function g_n on $[0,\ 1]$ by $g_n(x) = 0$ if $x \in [0,\ 1/n]$, and $g_n(x) = a_n(x)/a_{n+1}(x)$ if $x \in (1/n,\ 1]$. This g_n is *continuous* on $(1/(n+1),\ 1)$ and is zero on $[0,\ 1/n]$, so $g_n \in R$. It follows that $a_n = a_{n+1} \cdot g_n \in a_{n+1} R$, and we have an ascending chain of principal ideals

$$a_2 R \subsetneq a_3 R \subsetneq \cdots \subsetneq a_n R \subsetneq \cdots$$

in P. For any $f \in P$, take $n \in \mathbb{N}$ such that f vanishes on $[0,\ 1/n]$. Using the above argument for f (instead of a_n), we see that $f \in a_{n+1} R$. Therefore, $P = \bigcup_{n \geq 2} a_n R$, as desired.

Comment. The reason the ideal P is of interest is that it is a countably (but not finitely) generated projective R-module. This is shown in *LMR*-(2.12D).

Ex. 2.15. (1) Let R be a commutative ring, with $[P]\,[Q] = 1$ in $\text{Pic}(R)$. If P can be generated by two elements, show that $P \oplus Q \cong R^2$. (2) For the

Schanuel modules

$$P_r = (1 + rg,\, g^2), \quad P_{-r} = (1 - rg,\, g^2) \quad (r \in R)$$

introduced in *LMR*-(2.15), construct an explicit isomorphism $P_r \oplus P_{-r} \cong R^2$.

Solution. (1) Since P can be generated by two elements, there exists a surjection $\alpha : R^2 \twoheadrightarrow P$. Therefore, $P \oplus P' \cong R^2$ where $P' = \ker(\alpha)$. Forming the exterior algebras of the two sides, we get

$$\begin{aligned} \Lambda\left(R^2\right) &\cong \Lambda\left(P \oplus P'\right) \\ &\cong \Lambda\left(P\right) \otimes \Lambda\left(P'\right) \\ &\cong \left(R \oplus P\right) \otimes \left(R \oplus P'\right) \\ &\cong R \oplus \left(P \oplus P'\right) \oplus \left(P \otimes P'\right). \end{aligned}$$

Therefore, we must have

$$P \otimes P' \cong \Lambda^2(R^2) \cong R, \quad \text{and} \quad P \oplus P' \cong \Lambda^1(R^2) \cong R^2.$$

The former shows that $P' \cong P^* \cong Q$, so the latter yields $P \oplus Q \cong R^2$.

(2) Let us first recall the construction of Schanuel modules. Let S be a commutative ring containing R, and let $g \in S$ be an element such that $g^2,\ g^3 \in R$ (and therefore $g^n \in R$ for all $n \geq 2$). For any $r \in R$, the R-submodule $P_r = (1+rg,\ g^2) \subseteq S$ generated by $1+rg$ and g^2 is projective of rank 1, called a Schanuel module. A direct calculation (see *LMR*-(2.15)) shows that the inverse of $[P_r] \in \text{Pic}(R)$ is given by the Schanuel module P_{-r}. Thus, by the general fact proved in (1), we should have $P_r \oplus P_{-r} \cong R^2$. The point of (2) is to construct an *explicit* isomorphism.

The procedure for finding such an isomorphism is to set up first the surjection $\alpha : R^2 \twoheadrightarrow P_r$ defined by

$$\alpha(e_1) = 1 + rg, \quad \alpha(e_2) = g^2,$$

where $\{e_1,\ e_2\}$ is the standard basis for R^2. The rest of the calculation consists of showing that $\ker(\alpha) \cong P_{-r}$. In order to shorten the calculation, we shall only present the net results below.

Along with $\alpha : R^2 \twoheadrightarrow P_r$ constructed above, let us set up another homomorphism $\beta : R^2 \to P_{-r} = (1 - rg,\ g^2)$, defined by

$$\beta(e_1) = -r^4 g^2, \quad \beta(e_2) = \left(1 + r^2 g^2\right)\left(1 - rg\right).$$

We shall verify that the map $(\alpha,\ \beta) : R^2 \to P_r \oplus P_{-r}$ *is onto.* Since both R^2 and $P_r \oplus P_{-r}$ are projective of rank 2, the map $(\alpha,\ \beta)$ must then be an isomorphism. Consider the following elements in R^2:

$$\begin{aligned} f_1 &= \left(1 - r^4 g^4\right) e_1 + r^4 g^2 \left(1 + rg\right) e_2, \\ f_2 &= g^2 \left(1 + r^2 g^2\right) \left(1 - rg\right) e_1 + r^4 g^4 e_2, \\ f_3 &= -g^4 e_1 + g^2 \left(1 + rg\right) e_2, \\ f_4 &= -g^2 \left(1 + r^2 g^2\right) \left(1 - rg\right) e_1 + \left(1 - r^4 g^4\right) e_2. \end{aligned}$$

By direct computations:

$$\begin{cases} \alpha\,(f_1) = \left(1-r^4g^4\right)(1+rg) + r^4g^2\,(1+rg)\,g^2 = 1+rg, \\ \beta\,(f_1) = \left(1-r^4g^4\right)\left(-r^4g^2\right) + r^4g^2\,(1+rg)\,(1-rg)\left(1+r^2g^2\right) = 0, \end{cases}$$

$$\begin{cases} \alpha\,(f_2) = g^2\left(1+r^2g^2\right)(1-rg)\,(1+rg) + r^4g^4g^2 = g^2, \\ \beta\,(f_2) = g^2\left(1+r^2g^2\right)(1-rg)\left(-r^4g^2\right) + r^4g^4\left(1+r^2g^2\right)(1-rg) = 0. \end{cases}$$

This shows that the image of $(\alpha,\ \beta):\ R^2 \to P_r \oplus P_{-r}$ contains P_r. We finish by showing that the image of $(\alpha,\ \beta)$ also contains P_{-r}. For this, we compute

$$\begin{cases} \alpha\,(f_3) = -g^4\,(1+rg) + g^2\,(1+rg)\,g^2 = 0, \\ \beta\,(f_3) = -g^4\left(-r^4g^2\right) + g^2\,(1+rg)\,(1-rg)\left(1+r^2g^2\right) = g^2, \end{cases}$$

$$\begin{cases} \alpha\,(f_4) = -g^2\left(1+r^2g^2\right)(1-rg)\,(1+rg) + \left(1-r^4g^4\right)g^2 = 0, \\ \beta\,(f_4) = -g^2\left(1+r^2g^2\right)(1-rg)\left(-r^4g^2\right) + \left(1-r^4g^4\right)\left(1+r^2g^2\right)(1-rg) \\ \qquad\quad = \left(1+r^2g^2\right)(1-rg)\,. \end{cases}$$

This shows that the image of $\alpha \oplus \beta$ contains $0 \oplus Q$ where Q is the R-submodule $(g^2, (1+r^2g^2)(1-rg)) \subseteq P_{-r}$. Noting that $g^2(1-rg) = g^2 - rg^3 \in R$, we see that Q also contains

$$r^4g^2\,(1-rg)\cdot g^2 + \left(1-r^2g^2\right)\cdot\left(1+r^2g^2\right)(1-rg) = 1-rg.$$

Therefore, $Q = P_{-r}$, and we have shown that im $(\alpha,\ \beta) \supseteq P_{-r}$, as desired.

We stress again that the idea of the above calculation is that we decompose R^2 into $(Rf_1 + Rf_2) \oplus \ker(\alpha)$ with $\alpha : Rf_1 + Rf_2 \cong P_r$, and then construct an explicit isomorphism $\beta:\ \ker\ \alpha \cong P_{-r}$ by realizing $\ker(\alpha)$ as $Rf_3 + Rf_4$. The upshot of the whole calculation is that the following two elements

$$[\alpha\,(e_1)\,,\,\beta\,(e_1)] = \left[1+rg,\ -r^4g^2\right],$$
$$[\alpha\,(e_2)\,,\,\beta\,(e_2)] = \left[g^2, \left(1+r^2g^2\right)(1-rg)\right]$$

form a free basis for the direct sum $P_r \oplus P_{-r}$!

Ex. 2.16. (Modified Projectivity Test) Let $\mathfrak{B}$ be a class of objects in $\mathfrak{M}_R$ such that any module in $\mathfrak{M}_R$ can be embedded in some module in $\mathfrak{B}$. Show that, in testing whether a right module P is projective, it is sufficient to check that, for any R-epimorphism $g:\ B \to C$ where $B \in \mathfrak{B}$ and $C \in \mathfrak{M}_R$, any R-homomorphism $h:\ P \to C$ can be "lifted" to a homomorphism $f:\ P \to B$ (such that $h = g \circ f$).

Solution. Suppose the lifting property above holds for P. To check that P is projective, consider any R-epimorphism $g' : A \to D$, and any R-homomorphism $h' : P \to D$, where A and D are *arbitrary* right R-modules. We would like to "lift" h' to an R-homomorphism $f':\ P \to A$.

Fix an R-module $B \in \mathfrak{B}$ for which we have an embedding $A \subseteq B$. Construct the module

$$C = (B \oplus D)/\{(a,\ -g'(a)) :\ a \in A\},$$

for which we have the "pushout" diagram

$$\begin{array}{ccc} A & \xrightarrow{\ \ g'\ \ } & D \\ \Big\downarrow & & \Big\downarrow i \\ B & \xrightarrow{\ \ g\ \ } & C \end{array}$$

where $g(b) = \overline{(b,\ 0)}$ and $i(d) = \overline{(0,\ d)}$. It is easy to check that i is injective and g is surjective. Let

$$h = i \circ h' : P \to C.$$

Since $B \in \mathfrak{B}$, the given hypothesis implies that there is a homomorphism $f :\ P \to B$ such that $h = g \circ f$. Now, for any $p \in P$, we have

$$h(p) = \overline{(0,\ h'(p))} \quad \text{and} \quad (g \circ f)(p) = \overline{(f(p),\ 0)}.$$

Therefore, $(f(p), 0) - (0,\ h'(p)) \in \{(a, -g'(a) :\ a \in A\}$, which implies that $f(p) \in A$. This means that $f(P) \subseteq A$. Letting $f' :\ P \to A$ be the map defined by f (with its codomain B replaced by A), we have then $h' = g' \circ f$ (in view of the injectivity of i).

Comment. Assuming the material in *LMR*-§3, we'll see that $\mathfrak{B}$ may be taken, for instance, to be the class of all *injective* right R-modules. Thus, we get a criterion for projective modules formulated in terms of the liftability of homomorphisms to injective modules.

Ex. 2.17. Let P be a projective right module over a von Neumann regular ring R. Show that any f.g. submodule $M \subseteq P$ is a direct summand of P (and hence also a projective module).

Solution. Since P can be embedded in a free module, we may as well assume that P is free. Since M is f.g., we may also assume that $P \cong R^n$ for some $n < \infty$. Suppose M can be generated by m elements. By adding copies of R to P, we may assume that $n \geq m$. Then there exists $f \in \text{End}_R(P)$ with $f(P) = M$. Now

$$\text{End}_R(P) \cong \text{End}_R(R^n) \cong \mathbb{M}_n(R)$$

is a von Neumann regular ring (see *ECRT*-(21.10B)). Therefore, there exists $g \in \text{End}_R(P)$ such that $f = fgf$. Then $e := fg$ is an idempotent endomorphism of P. We have

$$f(P) = fgf(P) \subseteq e(P) \subseteq f(P),$$

so $M = f(P) = e(P)$ is a direct summand of P, with direct complement $(1 - e)(P)$.

Comment. In the case $P = R_R$, this exercise says that any f.g. right ideal of R is a direct summand. This is a well-known property of a von Neumann regular ring: see *FC*-(4.23). It gives a good motivation for the present exercise. Note that the solution above actually provides another proof of this property, independently of *FC*-(4.23), but using the fact that any finite matrix ring over a von Neumann regular ring is von Neumann regular.

The fact that a f.g. submodule of a projective module P_R is projective actually holds over any right semihereditary ring R, according to *LMR*-(2.29).

Ex. 2.18. Show that any f.g. projective right R-module P can be represented as $e(R^n)$, where $e : R^n \to R^n$ is left multiplication by some idempotent matrix $(a_{ij}) \in \mathbb{M}_n(R)$. With respect to this representation, show that $\mathrm{tr}(P) = \sum Ra_{ij}R$, and deduce that $\mathbb{M}_n(\mathrm{tr}(P)) = \mathbb{M}_n(R)\, e\, \mathbb{M}_n(R)$.

Solution. Choose a suitable module Q such that $P \oplus Q = R^n$, and let

$$e = (a_{ij}) \in \mathrm{End}_R(R^n) = \mathbb{M}_n(R)$$

be the projection onto P with respect to this decomposition. Then clearly $e^2 = e$, and $eR^n = P$. Let a_j be the jth column of e (so $a_j \in P$), and let f_j be the linear functional on P mapping any vector in P to its jth coordinate. We claim that $\{a_j,\ f_j :\ 1 \le j \le n\}$ are a pair of dual bases for P (as in the Dual Basis Lemma). In fact, for any vector $x = (x_1, \ldots,\ x_n)^t \in P$, we have

$$x = ex = \begin{pmatrix} b_{11}x_1 + b_{12}x_2 + \cdots \\ \vdots \\ b_{n1}x_1 + b_{n2}x_2 + \cdots \end{pmatrix} = \sum\nolimits_j \begin{pmatrix} b_{1j} \\ \vdots \\ b_{nj} \end{pmatrix} x_j = \sum\nolimits_j a_j f_j(x).$$

(This calculation is hardly surprising, since the $\{a_j,\ f_j\}$ on P are constructed exactly as in the general proof of the Dual Basis Lemma.)

Now we can calculate the trace ideal $\mathrm{tr}(P)$ of P. According to *LMR*-(2.41), $\mathrm{tr}(P)$ is generated as an ideal by the elements $\{f_i(a_j)\}$ (with respect to any pair of dual bases). Since $f_i(a_j)$ is just a_{ij}, we have $\mathrm{tr}(P) = \sum Ra_{ij}R$.

To calculate the ideal $\mathbb{M}_n(R)\, e\, \mathbb{M}_n(R)$ in $\mathbb{M}_n(R)$, we consider the matrix product $(rE_{ij})\, e\, (E_{kl}r')$, where $r,\ r' \in R$, and $E_{ij},\ E_{kl}$ are matrix units. By explicit matrix calculation, we see that

$$rE_{ij}\ eE_{kl}\, r' = ra_{jk}\, r'\, E_{il}.$$

Forming sums of matrices of this type, we see that $\mathbb{M}_n(R)\, e\, \mathbb{M}_n(R)$ is just $\mathbb{M}_n(\mathrm{tr}(P))$.

Ex. 2.19. If, for any n, any idempotent in $\mathbb{M}_n(R)$ is conjugate to some diag $(1, \ldots,\ 1,\ 0, \ldots,\ 0)$, show that any f.g. projective right R-module is free. Show that the converse is also true if R has IBN.

Solution. In general, if e, e' are conjugate idempotents in $\mathbb{M}_n(R)$, say $g^{-1}e'g = e$ where $g \in \mathrm{GL}_n(R)$, then g restricts to an R-module isomorphism $eR^n \to e'R^n$. Now consider any f.g. projective right R-module P. By Exercise 18, we can represent P in the form eR^n where $e = e^2 \in \mathbb{M}_n(R)$ (for some n). By assumption, e is conjugate to some $e' = \mathrm{diag}(1, \ldots, 1, 0, \ldots, 0)$ with, say, m ones ($m \leq n$). Then $eR^n \cong e'R^n \cong R^m$ is free. Conversely, assume R has IBN, and that any f.g. projective right R-module is free. Consider any $e = e^2 \in \mathbb{M}_n(R)$. Then $R^n = eR^n \oplus (\ker\, e)$, so by the given assumption, there exist R-isomorphisms

$$\varphi :\ eR^n \to R^m, \quad \psi :\ \ker(e) \to R^t$$

for suitable integers m, t. By IBN, we must have $n = m + t$. Now let $g = \varphi \oplus \psi$, which is an isomorphism from R^n to $R^m \oplus R^t = R^n$, so $g \in \mathrm{GL}_n(R)$. Let e' be the projection from R^n to R^m, so in matrix form $e' = \mathrm{diag}(1, \ldots, 1, 0, \ldots, 0)$ with m ones. By the above construction, we have clearly $ge = e'g$ so e is conjugate to e', as desired.

Comment. An *explicit* example of an idempotent matrix not conjugate to any $\mathrm{diag}(1, \ldots, 1, 0, \ldots, 0)$ should help. We know exactly how to construct one: using a non-free f.g. projective module. For the Dedekind domain $\mathbb{Z}[\theta]$ with $\theta^2 = -5$, take the familiar rank 1 projective ideal $\mathfrak{A} = (2,\ 1+\theta)$ in Exercise 9. In that exercise, we worked out a basis

$$f_1 = [-2,\ 1-\theta], \qquad f_2 = [1+\theta,\ -2]$$

on $\mathfrak{A} \oplus \mathfrak{A}$. Let $\pi \in \mathrm{End}_R(\mathfrak{A}\oplus\mathfrak{A})$ be the projection map onto the first coordinate. Using the basis $\{f_1, f_2\}$ on $\mathfrak{A}\oplus\mathfrak{A}$, and referring to the equations (6), (7) in the solution to Exercise 9, we have

$$\begin{aligned} \pi(f_1) &= [-2, 0] &&= -2f_1 + (\theta-1)f_2, \\ \pi(f_2) &= [1+\theta, 0] &&= (\theta+1)f_1 + 3f_2. \end{aligned}$$

Therefore π "corresponds" to the idempotent matrix $e = \begin{pmatrix} -2 & \theta+1 \\ \theta-1 & 3 \end{pmatrix}$, whose kernel and image are both isomorphic to $\mathfrak{A}$. Since $\mathfrak{A}$ is not free by *LMR*-(2.19) (*D*), e is not conjugate to $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. (This fact can be verified directly by a matrix computation, using the fact that $\mathrm{U}(R) = \{\pm 1\}$.) Of course, e cannot be conjugate to 0 or I_2 either.

Ex. 2.20. For right modules A, B over a ring R, define

$$\sigma = \sigma_{A,B} :\ B \otimes_R A^* \longrightarrow \mathrm{Hom}_R(A, B)$$

by $\sigma(b \otimes f)(a) = bf(a)$, where $b \in B$, $a \in A$, and $f \in A^* = \mathrm{Hom}_R\,(A, R)$. (Recall that $A^* \in {}_R\mathfrak{M}$.) Show that, for any given $A \in \mathfrak{M}_R$, the following are equivalent:

(1) A is a f.g. projective module.
(2) $\sigma_{A,\,B}:\ B \otimes_R A^* \to \mathrm{Hom}_R(A,\ B)$ is an isomorphism for all $B \in \mathfrak{M}_R$.
(3) $\sigma_{B,\,A}:\ A \otimes_R B^* \to \mathrm{Hom}_R(B,\ A)$ is an isomorphism for all $B \in \mathfrak{M}_R$.
(4) $\sigma_{A,\,A}:\ A \otimes_R A^* \to \mathrm{End}_R(A)$ is an epimorphism (resp. isomorphism).

Solution. We shall show (1) $\Rightarrow$ (2) $\Rightarrow$ (4) $\Rightarrow$ (1), and (1) $\Rightarrow$ (3) $\Rightarrow$ (4).

(1) $\Rightarrow$ (2) and (3). We fix a pair of dual bases $\{a_i,\ f_i : l \le i \le n\}$ for A. Construct

$$\tau : \mathrm{Hom}_R\,(A,\ B)\ \to B \otimes_R A^* \quad \text{by } \tau(h) = \sum h(a_i) \otimes f_i,$$
$$\rho : \mathrm{Hom}_R\,(B,\ A)\ \to A \otimes_R B^* \quad \text{by } \rho(g) = \sum a_i \otimes (f_i g),$$

where $h \in \mathrm{Hom}_R(A,\ B)$ and $g \in \mathrm{Hom}_R(B,\ A)$. We claim that $\tau,\ \rho$ are respectively the inverses for $\sigma_{A,\,B}$ and $\sigma_{B,\,A}$. Indeed

$$(\sigma_{A,\,B}\,\tau(h))\,(a) = \sum h(a_i)\, f_i(a) = h\left(\sum a_i f_i(a)\right) = h(a)$$

for any $a \in A$, so $\sigma_{A,B}\tau = \mathrm{Id}$. Also,

$$\begin{aligned} \tau\,\sigma_{A,\,B}\,(b \otimes f) &= \sum \sigma_{A,\,B}\,(b \otimes f)\ (a_i) \otimes f_i \\ &= \sum bf(a_i) \otimes f_i \\ &= b \otimes \sum f(a_i) f_i \\ &= b \otimes f \end{aligned}$$

in view of Exercise 2.7(1). Therefore, $\tau\sigma_{A,\,B} = Id$. The proof for ρ being the inverse for $\sigma_{B,A}$ is similar (and easier since it doesn't use Exercise 2.7(1)).

(3) $\Rightarrow$ (4) and (2) $\Rightarrow$ (4) are trivial.

(4) $\Rightarrow$ (1). Let $\sum a_i \otimes f_i \in A \otimes_R A^*$ be a preimage of $\mathrm{Id}_A \in \mathrm{End}_R(A)$ under the surjection $\sigma_{A,\,A}$. Then for any $a \in A$:

$$a = \sigma_{A,\,A}\left(\sum a_i \otimes f_i\right)\ (a) = \sum a_i f_i(a).$$

By the Dual Basis Lemma *LMR*-(2.9), A is a f.g. projective R-module.

In Exercises 21–31 *below, R denotes a commutative ring.*

Ex. 2.21. (Bourbaki) Let P_R be a f.g. R-module. We say that P is *locally free* if the localization $P_{\mathfrak{p}}$ of P at any maximal (or prime) ideal $\mathfrak{p}$ is free over the local ring $R_{\mathfrak{p}}$. (It turns out that these P's are exactly the f.g. flat modules; see Exercise 4.15.) For such a locally free (f.g.) module P, define $\mathrm{rk}\ P : \mathrm{Spec}\ R \to \mathbb{Z}$ by $(\mathrm{rk}\ P)(\mathfrak{p})$ = the (uniquely defined) rank of the free module $P_{\mathfrak{p}}$ over $R_{\mathfrak{p}}$. Here, $\mathbb{Z}$ is given the discrete topology, and the prime spectrum Spec R is given the Zariski topology. (The Zariski closed sets are of the form $V(\mathfrak{A}) = \{\mathfrak{p} : \mathfrak{p} \supseteq \mathfrak{A}\}$, where $\mathfrak{A}$ ranges over the ideals of R.) Show that the following are equivalent:

(1) P is a projective R-module.
(2) P is finitely presented; that is, there exists an exact sequence $R^m \to R^n \to P \to 0$ for some integers m, n.
(3) rk P is a continuous function from Spec R to $\mathbb{Z}$.
(4) rk P is a "locally constant" function; i.e., for any $\mathfrak{p} \in$ Spec R, rk P is constant on a suitable neighborhood of $\mathfrak{p}$.

Solution. Since $\mathbb{Z}$ here is given the discrete topology, the continuity of rk P : Spec $R \to \mathbb{Z}$ amounts to the fact that the preimage of any singleton set is open. This in turn amounts to the fact that the function rk P is locally constant. Thus, we have (3) $\Leftrightarrow$ (4). Since clearly (1) $\Rightarrow$ (2), it will suffice to show that (4) $\Rightarrow$ (2) and (2) $\Rightarrow$ (1) + (4).

(2) $\Rightarrow$ (1)+(4). Assume P is finitely presented. Then, for any R-module N and any multiplicative set $S \subseteq R$, the natural map

$$(*) \qquad \theta : S^{-1}\left(\mathrm{Hom}_R(P, N)\right) \longrightarrow \mathrm{Hom}_{S^{-1}R}\left(S^{-1}P,\, S^{-1}N\right)$$

is an *isomorphism.* This is easily proved by first checking the case where $P \cong R^k$ (for any finite k), and then going to the case of a finitely presented P by using

$$R^m \longrightarrow R^n \longrightarrow P \longrightarrow 0$$

and the left-exactness of "Hom" (cf. Exercise 4.12). Suppose (rk P)($\mathfrak{p}$) $= k$ at a given prime $\mathfrak{p}$. Then there is an isomorphism $R_{\mathfrak{p}}^k \cong P_{\mathfrak{p}}$. Using the isomorphism $(*)$ for $S = R \setminus \mathfrak{p}$, it is easy to construct an element $f \in S$ for which there is an isomorphism $R_f^k \cong P_f$. (Here and in the following, the subscript f means localization at $\{f^i : i \geq 0\}$.) Therefore, rk P takes the constant value k on the open neighborhood

$$D(f) := \{\mathfrak{p} \in \mathrm{Spec}\, R : f \notin \mathfrak{p}\}$$

of $\mathfrak{p}$. This checks (4). To check (1), we must show that, for any surjection $A \twoheadrightarrow B$ of R-modules, the induced map

$$\rho : \mathrm{Hom}_R(P, A) \longrightarrow \mathrm{Hom}_R(P, B)$$

is also a surjection. It suffices to check that $\rho_{\mathfrak{p}}$ is surjective for any prime $\mathfrak{p}$. Using $(*)$, we can "identify" $\rho_{\mathfrak{p}}$ with the map

$$\mathrm{Hom}_{R_{\mathfrak{p}}}\left(P_{\mathfrak{p}}, A_{\mathfrak{p}}\right) \longrightarrow \mathrm{Hom}_{R_{\mathfrak{p}}}\left(P_{\mathfrak{p}}, B_{\mathfrak{p}}\right),$$

which is certainly surjective since $P_{\mathfrak{p}}$ is free.

(4) $\Rightarrow$ (2). (Sketch) Assuming (4), we first prove the following:

($\dagger$) *For any prime $\mathfrak{p}$, there exists $f \notin \mathfrak{p}$ such that P_f is R_f-free.*

To see this, let $k = (\operatorname{rk} P)(\mathfrak{p})$. Since P is f.g., we can find an R-homomorphism $\tau : R^k \to P$ which localizes to an $R_{\mathfrak{p}}$-isomorphism $\tau_{\mathfrak{p}} : R^k_{\mathfrak{p}} \to P_{\mathfrak{p}}$. Then, for a suitable $g \notin \mathfrak{p}$, τ localizes to an R_g-epimorphism $R^k_g \to P_g$. Now, by our assumption (4), there exists an $h \notin \mathfrak{p}$ such that $P_{\mathfrak{p}'}$ is free of rank k over $R_{\mathfrak{p}'}$ for any $\mathfrak{p}' \in D(h)$. By rank consideration, we see that $\tau_{\mathfrak{p}'} : R^k_{\mathfrak{p}'} \to P_{\mathfrak{p}'}$ is an isomorphism for every $\mathfrak{p}' \in D(f)$, where $f := gh$. This means that $\tau_f : R^k_f \to P_f$ localizes to an isomorphism at every prime of R_f, so τ_f is itself an isomorphism. This proves (†). From this, it follows that the set of $f \in R$ for which P_f is R_f-free generates the unit ideal in R. Therefore, there exists a finite set $\{f_1, \ldots, f_r\} \subseteq R$ with $\sum Rf_i = R$ such that P_{f_i} is R_{f_i}-free for all i. Now let $T = \prod_{i=1}^r R_{f_i}$, and consider the natural (injective) map $R \to T$. Since $\sum Rf_i = R$, the functor $- \otimes_R T$ from R-modules to T-modules is exact. Fix any exact sequence of R-modules

$$0 \to X \to R^n \to P \to 0.$$

If we can show that X is f.g., then P is finitely presented. Tensoring the above sequence with T, we get

$$0 \to X \otimes_R T \to T^n \to P \otimes_R T \to 0.$$

Since each $P \otimes_R R_{f_i} \cong P_{f_i}$ is R_{f_i}-free, the T-module $P \otimes_R T$ is projective. Thus the last sequence splits, so $X \otimes_R T$ is f.g. as a T-module. Pick elements $\{x_i \otimes_{t_i} : 1 \le i \le N\}$ which generate $X \otimes_R T$, and let $Y = \sum Rx_i \subseteq X$. Then $X \otimes_R T = Y \otimes_R T$ leads to $(X/Y) \otimes_R T = 0$, which in turn leads to $X/Y = 0$. Therefore $X = Y = \sum Rx_i$ is f.g., as desired!

Comment. The proof of the last implication (4) ⇒ (2) above is a bit sketchy. We did not include all the details in order to keep the length of the proof within reasonable bounds. Readers who desire to see more details should consult Bourbaki's "Commutative Algebra," pp. 109–111, Hermann/Addison-Wesley, 1972, or the author's "Serre's Problem on Projective Modules", Chapter I, Monographs in Math., Springer-Verlag, 2006.

The argument used in the last part of the proof of (4) ⇒ (2) is known as the method of "faithfully flat descent." We have couched the argument in such a way that we can reach the desired conclusions without another axiomatic detour. A more detailed treatment of faithfully flat descent can be found in *LMR*-§4H.

In general, *a f.g. locally free (= f.g. flat) module P over a* commutative *ring R need not be projective.* For instance, Let R be any non-noetherian commutative von Neumann regular ring, and let I be a non-principal ideal. Let P be the cyclic R-module R/I. Then P is locally free since every localization $\mathrm{R}_{\mathfrak{p}} (\mathfrak{p} \in \operatorname{Spec} R)$ is a field by *LMR*-(3.71). However, P is *not* projective, for otherwise $0 \to I \to R \to P \to 0$ would split, and I would be a principal ideal. For certain classes of rings (e.g. noetherian rings), f.g. flat modules will be projective. For results of this nature, see *LMR*-(4.38) and Exercises (4.16), (4.21).

Ex. 2.22. Keeping the notations in Exercise 21, show that a subset $S \subseteq \text{Spec } R$ is clopen (closed and open) iff $S = V(eR)$ for some idempotent $e \in R$. Using this, show that the following statements are equivalent for any nonzero ring R:

(a) R has no idempotents other than 0, 1.
(b) Spec R is connected.
(c) Every f.g. projective R-module has constant rank.

Solution. We first prove the equivalence of (a), (b) and (c), assuming the first part of the exercise.

$(a) \Rightarrow (b)$. Suppose Spec $R = S \cup T$, where S, T are disjoint open (and therefore clopen) sets. By the first part of the exercise, $S = V(eR)$ for some idempotent $e \in R$. By (a), we have either $e = 0$ or $e = 1$, so S is either Spec R or $\emptyset$.

$(b) \Rightarrow (c)$. $\text{rk}P:\ \text{Spec } R \to \mathbb{Z}$ is a continuous function according to Exercise 21. Since $\mathbb{Z}$ here has the discrete topology and Spec R is connected, the image of the map $\text{rk}\,P$ must be a singleton.

$(c) \Rightarrow (a)$. Suppose R has an idempotent $e \neq 0,\ 1$. Then the cyclic projective R-module $P = eR$ has rank 1 at any prime containing $1-e$, and rank 0 at any prime containing e. (In other words, $\text{rk}\,P$ is the characteristic function for the nonempty subset $V(R(1-e)) \subsetneq \text{Spec } R$.)

To prove the first statement in the exercise, consider any clopen set $S \subseteq \text{Spec } R$. The complement T of S is also clopen, so we have $S = V(\mathfrak{A})$, $T = V(\mathfrak{B})$ for suitable ideals $\mathfrak{A}$, $\mathfrak{B} \subseteq R$. Since no prime ideal can contain both $\mathfrak{A}$ and $\mathfrak{B}$, $\mathfrak{A} + \mathfrak{B} = R$. On the other hand, any prime contains $\mathfrak{A} \cap \mathfrak{B}$ (since it contains either $\mathfrak{A}$ or $\mathfrak{B}$). Therefore

$$\mathfrak{A} \cap \mathfrak{B} \subseteq \bigcap\nolimits_{\mathfrak{p} \in \text{Spec } R} \mathfrak{p} = \text{Nil}(R)$$

Fix an equation $1 = a + b$ where $a \in \mathfrak{A}$ and $b \in \mathfrak{B}$. Then $(ab)^n = 0$ for some $n \geq 1$. Let $\mathfrak{A}' = a^n R$ and $\mathfrak{B}' = b^n R$. Since

$$V(\mathfrak{A}) \subseteq V(\mathfrak{A}'),\ \ V(\mathfrak{B}) \subseteq V(\mathfrak{B}'),\ \text{ and }\ V(\mathfrak{A}') \cap V(\mathfrak{B}') = \emptyset,$$

we must have $V(\mathfrak{A}) = V(\mathfrak{A}')$ and $V(\mathfrak{B}) = V(\mathfrak{B}')$. As before, $\mathfrak{A}' + \mathfrak{B}' = R$, so there is an equation $1 = a^n s + b^n t$ for suitable $s,\ t \in R$. Letting $e = a^n s$ and $f = b^n t$, we have $e^f = a^n b^n st = 0$ and $e + f = 1$. Multiplying the latter equation by e shows that $e^2 = e$, and the relations

$$V(\mathfrak{A}') \subseteq V(eR),\ \ V(\mathfrak{B}') \subseteq V(fR),\ \ V(eR) \cap V(fR) = \emptyset$$

show that $V(eR) = V(\mathfrak{A}') = V(\mathfrak{A}) = S$.

Comment. For a follow-up to this exercise, see Exercise 7.31 in Chapter 3.

Ex. 2.23. The *support* of an R-module P is defined to be

$$\operatorname{supp} P = \{\mathfrak{p} \in \text{Spec } R : \ P_{\mathfrak{p}} \neq 0\}.$$

For any f.g. $P \in \mathfrak{M}_R$, show that supp $P = V(\text{ann } P)$. Deduce that $P_{\mathfrak{p}} \neq 0$ for all primes $\mathfrak{p}$ iff ann $P \subseteq \text{Nil}(R)$.

Solution. In general (without any assumption on P), we have supp $P \subseteq V(\text{ann } P)$. For, if a prime $\mathfrak{p} \notin V(\text{ann } P)$, then there exists $r \in (\text{ann } P)\backslash\mathfrak{p}$. Since $Pr = 0$ and r localizes to a unit in $R_{\mathfrak{p}}$, we have $P_{\mathfrak{p}} = 0$, so $\mathfrak{p} \notin$ supp P.

Conversely, consider any prime $\mathfrak{p} \notin$ supp P, so that $P_{\mathfrak{p}} = 0$. If P is f.g., we will have $Pr = 0$ for some $r \notin \mathfrak{p}$. Now $r \in (\text{ann } P)\backslash\mathfrak{p}$, so $\mathfrak{p} \notin V(\text{ann } P)$. This proves the equality supp $P = V(\text{ann } P)$.

For the second conclusion in the exercise, note that $P_{\mathfrak{p}} \neq 0$ for all primes iff Spec $R =$ supp $P = V(\text{ann } P)$, iff ann $P \subseteq \bigcap \{\text{all primes}\} = \text{Nil}(R)$.

Ex. 2.24. For any f.g. projective R-module P, show that P is faithful iff the function $rk\ P$: Spec $R \to \mathbb{Z}$ is everywhere positive. (In particular, a f.g. projective module P_R of rank $n > 0$ is always faithful.)

Solution. By *LMR*-(2.44), ann P always has the form fR for some idempotent $f \in R$. By the exercise above, rk P is everywhere positive iff $fR =$ ann $P \subseteq \text{Nil}(R)$, iff ann $P = 0$, iff P is faithful.

Comment. In *LMR*-(2.44), it is also shown that P is faithful iff the "trace ideal"

$$\text{tr}(P) = \sum \{\text{im}(\varphi) : \ \varphi \in P^*\}$$

of P is the whole ring R.

Ex. 2.25. Suppose P, $Q \in \mathfrak{M}_R$ are such that $P \otimes_R Q \cong R^n$ where $n > 0$. Show that P and Q must be faithful f.g. projective R-modules.

Solution. Fix a generating set $\{p_i \otimes q_i : 1 \leq i \leq m\}$ for the R-module $P \otimes_R Q$, and consider the R-homomorphism $f : R^m \to P$ defined by sending the standard basis vectors of R^m to the p_i's. Then

$$f \otimes_R Q : \ R^m \otimes_R Q \longrightarrow P \otimes_R Q \cong R^n$$

is a *split* epimorphism. Now tensor the above with P to get a *split* epimorphism:

$$f \otimes_R (Q \otimes P) : \ R^m \otimes_R (Q \otimes_R P) \longrightarrow P \otimes_R (Q \otimes_R P).$$

Identifying $Q \otimes_R P$ with R^n, we see that $f \otimes_R (Q \otimes P)$ is essentially "n copies" of f. An easy argument shows that f itself must already be a *split* epimorphism. Therefore, P is a f.g. projective R-module.

Localizing the isomorphism $P \otimes_R Q \cong R^n$ at a prime $\mathfrak{p}$, we get

$$R_{\mathfrak{p}}^n \cong (P \otimes_R Q)_{\mathfrak{p}} \cong P_{\mathfrak{p}} \otimes_{R_{\mathfrak{p}}} Q_{\mathfrak{p}}.$$

Since $R_\mathfrak{p} \neq 0$, we have $P_\mathfrak{p} \neq 0$. Therefore, by Exercise 24, P is a faithful R-module.

Comment. The last conclusion can also be deduced by noting that ann $(P) \subseteq$ ann $(P \otimes_R Q) = \text{ann}(R^n) = 0$.

Ex. 2.26. Deduce from Exercise 25 that $P \in \mathfrak{M}_R$ is f.g. projective of rank 1 iff there exists $Q \in \mathfrak{M}_R$ such that $P \otimes_R Q \cong R$. In this case, show that necessarily $Q \cong P^*$.

Solution. If P is f.g. projective of rank 1, it is already shown in *LMR*-§2D that $P \otimes_R P^* \cong R$. Conversely, let $P \in \mathfrak{M}_R$ be such that $P \otimes_R Q \cong R$ for some $Q \in \mathfrak{M}_R$. By Exercise 25, P, Q are both f.g. projective. For any prime $\mathfrak{p}$, we have

$$R_\mathfrak{p} \cong (P \otimes_R Q)_\mathfrak{p} \cong P_\mathfrak{p} \otimes_{R_\mathfrak{p}} Q_\mathfrak{p}$$

Since $P_\mathfrak{p}$, $Q_\mathfrak{p}$ are f.g. free modules over $R_\mathfrak{p}$, we must have $P_\mathfrak{p} \cong R_\mathfrak{p}$ (for any $\mathfrak{p}$), so rk $P = 1$. Finally, tensoring $R \cong P \otimes_R Q$ with P^*, we get

$$P^* \cong P^* \otimes_R P \otimes_R Q \cong R \otimes_R Q \cong Q,$$

as desired

Ex. 2.26′. (1) Show that a f.g. projective module P_R of rank 1 is a direct sum of cyclic modules iff $P \cong R$. (2) Deduce from (1) that the Picard group of a (commutative) von Neumann regular ring is trivial.

Solution. (1) ("Only if" part) Say $P = Rx_1 \oplus \cdots \oplus Rx_n$, where $x_i \in P$. Let $x = x_1 + \cdots + x_n$, and $Q = Rx \subseteq P$. Consider any prime ideal $\mathfrak{p} \subset R$. Since $P_\mathfrak{p} \cong R_\mathfrak{p}$ and $R_\mathfrak{p}$ is indecomposable, we must have $(Rx_i)_\mathfrak{p} = 0$ for all but one i, and hence $P_\mathfrak{p} = Q_\mathfrak{p}$. Since this holds for all primes $\mathfrak{p}$, we have $P = Q$. Therefore, the map $R \to P$ sending 1 to x is a split surjection, which must then be an isomorphism since rk $P = 1$.

(2) Let R be von Neumann regular, and P_R be f.g. projective of rank 1. Since R is a semihereditary ring. Albrecht's Theorem (*LMR*-(2.29)) implies that $P = P_1 \oplus \cdots \oplus P_n$ where each P_i is isomorphic to a f.g. ideal of R. But then $P_i \cong e_i R$ for suitable idempotents $e_i \in R$, so P is a direct sum of cyclic modules. By (1), it follows that $P \cong R$, so we have $\text{Pic}(R) = \{1\}$.

Comment. Results such as (1), (2) are typical folklore in commutative algebra. For an explicit reference (where these results are used to prove a nice fact on invertible ideals in commutative domains of Krull dimension 1), see the paper of R. C. Heitmann and L. S. Levy: "$1\frac{1}{2}$ and 2 generator ideals in Prüfer rings," Rocky Mountain J. Math. **5**(1975), 361–373.

Ex. 2.27. Show that a f.g. projective module P_R has rank 1 iff the natural map $\lambda: R \to \text{End}_R(P)$ (defined by $\lambda(r)(p) = pr$) is an isomorphism of rings.

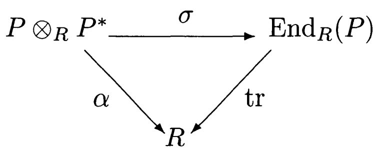
Solution. By familiar localization facts, λ is an isomorphism iff, for any prime $\mathfrak{p} \subset R$, the localization

$$\lambda_{\mathfrak{p}} : R_{\mathfrak{p}} \to (\mathrm{End}_R(P))_{\mathfrak{p}} \overset{\tau}{\cong} \mathrm{End}_{R_{\mathfrak{p}}}(P_{\mathfrak{p}})$$

is an isomorphism. (For the isomorphism τ used above, see Exercise 4.12.) Since $P_{\mathfrak{p}}$ is free, $\lambda_{\mathfrak{p}}$ is an isomorphism iff $P_{\mathfrak{p}} \cong R_{\mathfrak{p}}$. Therefore, λ is an isomorphism iff $\mathrm{rk}\ P = 1$.

Ex. 2.28. Let P be a f.g. projective R-module. Show that there is a natural way to define the trace of an R-endomorphism of P so that we get an R-module homomorphism $\mathrm{tr} : \mathrm{End}_R(P) \to R$. The definition should be such that, in case $P = R^n$, we get back the usual trace map on $n \times n$ matrices, upon identifying $\mathrm{End}_R(R^n)$ with $\mathbb{M}_n(R)$.

Solution. We shall use the isomorphism $\sigma = \sigma_{P,P}$ from $P \otimes_R P^*$ to $\mathrm{End}_R(P)$ defined in Exercise 20. We also have a map $\alpha : P \otimes_R P^* \to R$ which is well-defined by the equation $\alpha(b \otimes f) = f(b)$ (for $b \in P$ and $f \in P^*$). It is easy to check that both σ and α are R-module homomorphisms. We can therefore define $\mathrm{tr} : \mathrm{End}_R(P) \to R$ by the equation $\mathrm{tr} = \alpha \circ \sigma^{-1}$, so that we have a commutative diagram



Clearly, "tr" is also an R-module homomorphism.

In the case where $P = R^n$, we identify $\mathrm{End}_R(P)$ with $\mathbb{M}_n(R)$ by using the standard basis $\{e_i : 1 \le i \le n\}$ on R^n for writing matrices. Let $\{e_j^* : 1 \le j \le n\}$ be the dual basis for P^*. Then $\sigma(e_i \otimes e_j^*)$ is the matrix unit E_{ij} in $\mathbb{M}_n(R)$. Therefore, the trace map defined above will assign to E_{ij} the value

$$\alpha\left(e_i \otimes e_j^*\right) = e_j^*(e_i) = \delta_{ij} \quad \text{(the Kronecker deltas).}$$

This happens to be the usual trace of the matrix E_{ij}. Thus, our trace map on $\mathbb{M}_n(R)$ agrees with the usual trace map on the matrix units. Since $\mathbb{M}_n(R)$ is generated over R by the matrix units, it follows that our trace map agrees with the usual trace on *all* matrices in $\mathbb{M}_n(R)$.

Comment. The reader should try to work out a few properties of the trace function defined above on $\mathrm{End}_R(P)$. For instance, if $\varphi \in \mathrm{End}_R(P)$ and $\psi \in \mathrm{End}_R(Q)$ (where Q is also f.g. projective), we can define an endomorphism $\varphi \oplus \psi$ on $P \oplus Q$. It can be shown that $\mathrm{tr}(\varphi \oplus \psi)$ is exactly $\mathrm{tr}(P) + \mathrm{tr}(Q)$ in R, etc.

In the *noncommutative* case, the "tr" construction above does not work since there is no natural map from $P \otimes_R P^*$ to R. (The rule $b \otimes f \mapsto f(b)$ no longer gives a well-defined group homomorphism.)

Ex. 2.29. Let P be a f.g. projective R-module, and let $\{a_i, f_i\}$ $(1 \le i \le n)$ be a pair of dual bases as in LMR-(2.9) (so that $a_i \in P$, $f_i \in P^*$, and $a = \sum_i a_i f_i(a)$ for every $a \in P$). Show that $\tau(P) := \sum_i f_i(a_i) \in R$ is an invariant of P (not depending on the choice of $\{a_i, f_i\}$). Is the same conclusion true if R is not commutative?

Solution. Again we shall use the isomorphism $\sigma : P \otimes_R P^* \to \mathrm{End}_R(P)$. We have, for any $a \in P$:

$$\sigma\left(\sum_i a_i \otimes f_i\right)(a) = \sum_i a_i f_i(a) = a,$$

so $\sigma(\sum_i a_i \otimes f_i) = \mathrm{Id}_P$. Therefore, by definition,

$$\mathrm{tr}(\mathrm{Id}_P) = \alpha\left(\sum_i a_i \otimes f_i\right) = \sum_i f_i(a_i),$$

where α is the map from $P \otimes_R P^*$ to R introduced in the last exercise. This shows that $\tau(P) = \mathrm{tr}(\mathrm{Id}_P)$ is determined by P independently of the choice of $\{a_i, f_i\}$.

In the *noncommutative* case, the element $\sum_i a_i \otimes f_i^* \in P \otimes_R P^*$ is still uniquely determined by P (since it corresponds to Id_P under the isomorphism σ. However, the element $\sum_i f_i(a_i)$ is no longer uniquely determined by P. For instance, consider the case $P = eR$ where $e \in R$ is an idempotent. A pair of dual bases is given by $\{a_1, f_1\}$ where $a_1 = e$ and $f_1 \in P^*$ is the inclusion map $e_1 R \hookrightarrow R$. We have here $f_1(a_1) = a_1 = e$. However, we may have another idempotent $e' \in R$ with $P = eR \cong e'R$, so the element $e \in R$ is *not* determined by the isomorphism type of P.

Ex. 2.30. Recall that any idempotent (square) matrix defines a f.g. projective module (as in Exercise 18). Show that if $e, e' \in \mathbb{M}_n(R)$ are idempotent matrices that define isomorphic projective modules, then trace $(e) =$ trace(e'). Is the same conclusion true if R is not commutative?

Solution. Let $P = eR^n = \sum a_j R$ where a_j denotes the jth column of the (idempotent) matrix $e = (a_{ij})$. Let f_j be the linear functional on P mapping any vector in P to its jth coordinate. We have seen in the solution to Exercise 18 that $\{a_j, f_j : 1 \le j \le n\}$ are a pair of dual bases for P. Thus, the invariant $\tau(P)$ defined in the last exercise can be calculated as follows:

$$\tau(P) = \sum_j f_j(a_j) = \sum_j a_{jj} = \mathrm{trace}(e).$$

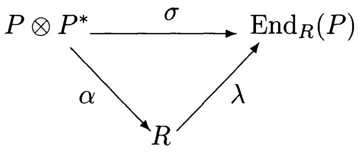
Thus, if e' is another idempotent of $\mathbb{M}_n(R)$ defining a projective module $P' \cong P$, we'll have

$$\mathrm{trace}(e') = \tau(P') = \tau(P) = \mathrm{trace}(e).$$

In the noncommutative case, the conclusion fails again, as we can show by exactly the same example used in the last exercise. The 1×1 idempotent matrix $e \in R$ defines the projective right R-module eR. We may have $eR \cong e'R$ for another idempotent $e' \in R$ without having $e = e'$.

Ex. 2.31. Let P_R be a f.g. projective R-module of rank 1.

(a) For $a, b \in P$ and $f \in P^*$, show that $af(b) = bf(a) \in P$.
(b) Show that the following diagram is commutative:



Here, $\sigma = \sigma_{P,P}$, λ, and α are defined, respectively, in Exercises 20, 27–28.
(c) Show that the trace map $\mathrm{tr} : \mathrm{End}_R(P) \to R$ defined in Exercise 28 is the same as λ^{-1}.
(d) Show that $\tau(P) = 1$ (in the notation of Exercise 29).

Solution. (a) It suffices to check the desired equation in every localization $P_\mathfrak{p}$ ($\mathfrak{p} \in \mathrm{Spec}\, R$). Let $f_\mathfrak{p} : P_\mathfrak{p} \to R_\mathfrak{p}$ be the localization of the functional f, and fix a basis vector e for $P_\mathfrak{p} \cong R_\mathfrak{p}$. Then $a = ex$ and $b = ey$ in $P_\mathfrak{p}$ for suitable x, $y \in R_\mathfrak{p}$. Now

$$\begin{aligned} af(b) &= ex \cdot f_\mathfrak{p}(ey) = e \cdot x f_\mathfrak{p}(e) y, \quad \text{and} \\ bf(a) &= ey \cdot f_\mathfrak{p}(ex) = e \cdot y f_\mathfrak{p}(e) x \end{aligned}$$

in $P_\mathfrak{p}$, so $a\, f(b) = bf(a) \in P_\mathfrak{p}$, as desired.

(b) Take any generator $b \otimes f \in P \otimes_R P^*$, where $b \in P$ and $f \in P^*$. For any $a \in P$, we have

$$(\sigma\,(b \otimes f))\,(a) = bf(a).$$

On the other hand,

$$(\lambda\alpha(b \otimes f))(a) = \lambda(f(b))(a) = f(b)a.$$

Since $bf(a) = f(b)\, a$ by (a), we see that $\sigma(b \otimes f)$ and $\lambda\alpha(b \otimes f)$ are equal as endomorphisms of P. Hence, $\sigma = \lambda\alpha$.

(c) Note that all maps in the commutative diagram are isomorphisms. By definition, tr: $\mathrm{End}_R(P) \to R$ is the map $\alpha\sigma^{-1}$, which is just $\alpha(\lambda\alpha)^{-1} = \alpha\alpha^{-1}\lambda^{-1} = \lambda^{-1}$.

(d) By Exercise 29 and (c) above,

$$\tau(P) = \mathrm{tr}(\mathrm{Id}_P) = \lambda^{-1}(\mathrm{Id}_P) = 1 \in R.$$

Comment. Here is an explicit example. Over the ring $R = \mathbb{Z}[\theta]$ $(\theta^2 = -5)$, we have the familiar rank 1 projective R-module $\mathfrak{A} = (2, 1+\theta) \subseteq R$ from Exercise 9. As we saw in the *Comment* on Exercise 19, $\mathfrak{A}$ is isomorphic to the image P of the idempotent operator on R^2 given by $e = \begin{pmatrix} -2 & 1+\theta \\ \theta - 1 & 3 \end{pmatrix}$. By the solution to Exercise 30, we have, indeed, $\tau(P) = \text{trace}(2) = -2 + 3 = 1$.

Ex. 2.32. Let P_R be a projective module that is not f.g. Show that there is a split monomorphism $f : P \to \bigoplus_{i \in I} P$ for a suitable infinite indexing set I such that $f(P)$ is not contained in $\bigoplus_{i \in J} P$ for any finite subset $J \subseteq I$.

Solution. By the Dual Basis Lemma (*LMR*-(2.9)), we have elements $p_i \in P$, $f_i \in P^*$ for i ranging over a suitable (necessarily infinite) indexing set I such that, for any $p \in P$, $f_i(p) = 0$ for almost all i, and $p = \sum_i p_i f_i(p)$. The map $f : P \to \bigoplus_{i \in I} P$ defined by $f(p) = (p_i f_i(p))_{i \in I}$ is then an R-module homomorphism, split by the homomorphism $g : \bigoplus_{i \in I} P \to P$ given by $g((a_i)_{i \in I}) = \sum_i a_i$. Assume, for the moment, that $f(P) \subseteq \bigoplus_{i \in J} P$ for some finite subset $J \subseteq I$. Then for any $p \in P$, $p_i f_i(p) = 0$ for any $i \notin J$. But then

$$p = \sum_{i \in I} p_i f_i(p) = \sum_{i \in J} p_i f_i(p)$$

implies that P is generated by the finite set $\{p_i : i \in J\}$, a contradiction.

Ex. 2.33. In a ring theory monograph, the following statement appeared: "If G is a finite group, every projective module over (the integral group ring) $\mathbb{Z}G$ is free." Give a counterexample.

Solution. For $G = \langle g \rangle$, a cyclic group of prime order p, let $R = \mathbb{Z}G$, $S = \mathbb{F}_p$, $R_1 = \mathbb{Z}$, and $R_2 = \mathbb{Z}\,[\zeta]$ where ζ is a primitive pth root of unity. We have a pullback diagram

(1)
$$\begin{array}{ccc} R & \xrightarrow{\;i_1\;} & R_1 \\ {\scriptstyle i_2}\downarrow & & \downarrow{\scriptstyle j_1} \\ R_2 & \xrightarrow{\;j_2\;} & S \end{array}$$

where i_1 is the augmentation map, j_1 is the natural projection, and i_2, j_2 are ring homomorphisms defined by $i_2(g) = \zeta$ and $j_2(\zeta) = 1$. (To say that the above diagram is a "pullback" means that, for $r_1 \in R_1$ and $r_2 \in R_2$ such that $j_1(r_1) = j_2(r_2)$, there exists a unique $r \in R$ such that $i_1(r) = r_1$ and $i_2(r) = r_2$.)

We shall construct (for suitable primes p) a f.g. projective R-module that is not free. To simplify the construction, we shall use a standard result in Milnor's "Introduction to Algebraic K-theory," Princeton University Press, 1971. By §2 of this reference, we have the following existence theorem on f.g. projective R-modules for a ring R in a pullback diagram as in (1) with at least one of j_1, j_2 surjective. (In our situation, of course, both j_1, j_2 are surjective.) For f.g. projective modules P_i over R_i and any S-isomorphism $\alpha : j_1^* P_1 \to j_2^* P_2$, let

$$\begin{array}{ccc} P & \longrightarrow & P_1 \\ \big\downarrow & & \big\downarrow \alpha j_1^* \\ P_2 & \xrightarrow{\ j_2^*\ } & j_2^* P_2 \end{array}$$

be a pullback diagram, that is

(2) $$P = \{(p_1, p_2) \in P_1 \oplus P_2 \ : \ \alpha j_1^*(p_1) = j_2^*(p_2)\},$$

with the R-action

(3) $$r \cdot (p_1, p_2) = (i_1(r)p_1,\ i_2(r)p_2), \quad (p_1, p_2) \in P.$$

(In the above, $j_\alpha^* P_\alpha$ denotes the S-module $S \otimes_{R_\alpha} P_\alpha$.) Then P is a f.g. projective R-module such that $i_\alpha^* P \cong P_\alpha$ for $\alpha = 1, 2$.

For our specific pullback diagram, it is then easy to produce an example of a nonfree f.g. projective module over $R = \mathbb{Z}G$. Let p be any prime such that $R_2 = \mathbb{Z}[\zeta]$ is not a PID; say $P_2 \subseteq R_2$ is a nonprincipal ideal. Since R_2 is a Dedekind ring, P_2 is f.g. projective (but not free) over R_2. For the prime ideal $p = (\zeta - 1)R_2$, the unique factorization theorem for ideals in R_2 implies that there is no ideal strictly between $\mathfrak{p}P_2$ and P_2. Thus, $P_2/\mathfrak{p}P_2 \cong R_2/\mathfrak{p} \cong \mathbb{F}_p$. Let $P_1 = \mathbb{Z}$ and fix an isomorphism α from $j_1^* P_1 = \mathbb{F}_p$ to $j_2^* P_2 \cong P_2/\mathfrak{p}P_2$. Then the f.g. projective R-module P defined in (2) is *not* free since the R_2-module $i_2^* P \cong P_2$ is not free.

We finish by giving an alternative description of the projective R-module P. Fix an element $x_0 \in P_2 \backslash \mathfrak{p}P_2$, and make the abelian group $Q = \mathbb{Z} \oplus P_2$ into an R-module by defining

(4) $$g \cdot (n,\, x) = (n,\, \zeta x + n x_0) \quad \text{for} \quad n \in \mathbb{Z} \quad \text{and} \quad x \in P_2.$$

This gives a well-defined action, since under this definition:

$$g^p \cdot (n,\ x) = (n,\ \zeta^p x + (\zeta^{p-1} + \cdots + \zeta + 1) n x_0) = (n,\ x).$$

If the isomorphism $\alpha : j_1^* P_1 \to j_2^* P_2$ is chosen in such a way that

$$\alpha(\bar{1}) = \bar{x}_0 \in P_2/\mathfrak{p}P_2 = j_2^* P_2,$$

then the R-module P *defined in* (2) *is isomorphic to* Q. Indeed, if we define $\theta : Q \to P_1 \oplus P_2$ by

(5) $$\theta(n,\ x) = (n,\ (\zeta - 1)x + n x_0) \quad \text{for} \quad n \in \mathbb{Z} \quad \text{and} \quad x \in P_2,$$

then $\theta(Q) \subseteq P$ since

$$\overline{(\zeta - 1)x + n x_0} = \bar{n}\ \bar{x}_0 = \bar{n}\ \alpha(\bar{1}) = \alpha(\bar{n}).$$

In fact, we have $\theta(Q) = P$. For, if $(n,\ y)\ \in P$, then $\bar{y} = \alpha(\bar{n}) = \overline{nx_0}$ implies that $y - nx_0 = (\zeta - 1)x$ for some $x\ \in P_2$, and hence $(n,\ y) = \theta(n, x)$ from (5). Clearly, θ is injective, so it defines a group isomorphism $Q \to P$. The following verifies that this is an R-isomorphism:

$$\begin{aligned} \theta(g \cdot (n,\ x)) &= \theta(n,\ \zeta x + nx_0) \\ &= (n,\ (\zeta - 1)(\zeta x + nx_0) + nx_0) \\ &= (n,\ \zeta[(\zeta - 1)x + nx_0]) \\ &= g \cdot (n,\ (\zeta - 1)x + nx_0) \\ &= g \cdot \theta(n,\ x). \end{aligned}$$

Comment. Note that the R-module Q above contains the R_2-module $P_2 \oplus (0)$ (viewed as an R-module via i_2), and it is a (nonsplit) extension of this submodule by the trivial $\mathbb{Z}G$-module $\mathbb{Z}$. A direct proof for the fact that Q is a nonfree projective over $\mathbb{Z}G$ (not using Milnor's results on pullback diagrams (1)) can be found in W.H. Gustafson's paper, "The representation ring of a group of prime order", Comm. Algebra **25**(1997), 2681–2686. In fact, Gustafson showed that, if Q' is constructed from another ideal $P_2' \subseteq R_2$ such that $P_2 + P_2' = R_2$ and P_2P_2' is principal, then $Q \oplus Q' \cong R \oplus R$. This shows explicitly that Q, Q' are projective over R.

The question whether there exist f.g. nonfree projective modules over integral group rings of finite groups was first raised in Cartan-Eilenberg's "Homological Algebra" (p. 241). The first such example, Q above over $\mathbb{Z}\langle g\rangle$, appeared in D. S. Rim's paper, "Modules over finite groups," Annals of Math. **69**(1959), 700–712. Rim also showed that the Grothendieck group of f.g. projectives over $\mathbb{Z}\langle g\rangle$ is isomorphic to $\mathbb{Z} \oplus \text{Pic}(\mathbb{Z}[\zeta])$ in the notation above.

A class of very easily constructed examples of f.g. projective modules over group rings appeared later in R. G. Swan's paper, "Periodic resolutions for finite groups," Annals of Math. **72**(1960), 267–291. For any finite group G and any integer r prime to $|G|$, let P_r be the left ideal of $R = \mathbb{Z}G$ generated by r and $\sum_{g \in G} g$. Swan showed that if s is the inverse of r modulo $|G|$, then $P_r \oplus P_s \cong R^2$. Thus, *P_r is always a projective* R-module. While P_r is always free for G cyclic, Swan showed, for instance, that P_3 is not free for the quaternion group of order 8.

Later, for the generalized quaternion group G of order 32, Swan constructed a nonfree $\mathbb{Z}G$-module P such that $P \oplus \mathbb{Z}G \cong \mathbb{Z}G \oplus \mathbb{Z}G$. Therefore, *even stably free $\mathbb{Z}G$-modules need not be free*, and f.g. projective modules over $\mathbb{Z}G$ fail to satisfy the cancellation law. For the details, see Swan's paper, "Projective modules over group rings and maximal orders," Annals of Math. **76**(1962), 55–61. On the other hand, if G is the quaternion group of order 8, then f.g. projective $\mathbb{Z}G$-modules satisfy the cancellation law; in particular, f.g. stably free $\mathbb{Z}G$-modules are free. This is a result of

J. Martinet, in "Modules sur l'algèbre du groupe quaternionien," Ann. Sci. École Norm. Sup. **4**(1971), 399–408.

For non-f.g. modules, the situation is different. For any finite abelian group G, for instance, it follows from the results of H. Bass that *any non-f.g. projective module over* $\mathbb{Z}G$ *is free*; see his paper "Big projective modules are free," Ill. J. Math. **7**(1963), 24–31.

The next two problems are suggested by R. Swan. Recall that, for any commutative ring R, $Q(R)$ denotes its *total ring of quotients*, i.e. the localization of R at the multiplicative set of its non 0-divisors.

Ex. 2.34. For any commutative ring S, show that there exists another commutative ring $R \supseteq S$ with the following properties:

(1) $Q(R) = R$;
(2) S is a "retract" of R (i.e. the inclusion map $S \to R$ is split by a ring homomorphism $R \to S$);
(3) $\text{Nil}(S) = \text{Nil}(R)$; and
(4) $\text{U}(S) = \text{U}(R)$.

Solution. Let $\{\mathfrak{m}_i\}$ be the family of maximal ideals of S. Construct the commutative ring $S_1 = S[\{x_i\}]$ with the relations $x_i\mathfrak{m}_i = 0$ for all i.

We note the following properties of S_1.

(a) *S is a retract of S_1*. This is clear in view of the map $S_1 \to S$ defined by $x_i \mapsto 0$ for all i.

(b) *If $i \neq j$, then $x_i x_j = 0 \in S_1$*, since $x_i x_j$ is annihilated by $\mathfrak{m}_i$, $\mathfrak{m}_j$, and therefore by $\mathfrak{m}_i + \mathfrak{m}_j = R$.

(c) By (a) *any element $f \in S_1$ has the form $s + \sum_i \sum_{k \geq 1} a_{ik}\, x_i^k$ where $s \in S$, $a_{ik} \in S$. Furthermore, $s \in S$ and $\bar{a}_{ik} \in S/\mathfrak{m}_i$ are uniquely determined by f.* This uniqueness is seen by using the homomorphism $\theta_i : S_1 \to (S/\mathfrak{m}_i)[x_i]$ defined by the natural map $S\,[x_i] \to (S/\mathfrak{m}_i)\,[x_i]$ and by sending all x_j to 0 for $j \neq i$.

(d) From (c), we see, in particular, that each $x_i \neq 0$ in S_1. Thus, *any nonunit of S becomes a 0-divisor in S_1*. In fact, if $t \in S$ is a nonunit, then $t \in \mathfrak{m}_i$ for some i, so $x_i t = 0$.

(e) $\text{Nil}(S) = \text{Nil}(S_1)$. In fact, let

$$f = s + \sum_i \sum_{k \geq 1} a_{ik}\, x_i^k \in \text{Nil}\,(S_1).$$

Applying the homomorphism θ_i above, we have

$$\theta_i(f) = \bar{s} + \sum_{k \geq 1} \bar{a}_{ik}\, x_i^k \in \text{Nil}\,((S/\mathfrak{m}_i)[x_i]) = 0.$$

Therefore, $a_{ik} \in \mathfrak{m}_i$ for all i, k, and so

$$f = s \in S \cap \text{Nil}(S_1) = \text{Nil}(S).$$

(f) $\mathrm{U}(S) = \mathrm{U}(S_1)$. To see this, suppose $fg = 1$ in S_1, where f is expressed as in (c). For each i, $\theta_i(f)\theta_i(g) = 1$ in $(S/\mathfrak{m}_i)[x_i]$ shows again that $a_{ik} \in \mathfrak{m}_i$ for all k. Thus, $f = s \in S$, and similarly $g = t \in S$. Now $1 = fg = st$ implies that $f \in \mathrm{U}(S)$.

Repeating the above construction, we get a chain of rings $S \subseteq S_1 \subseteq S_2 \subseteq \cdots$. Now let $R = \bigcup_{i \geq 1} S_i \supseteq S$. Clearly, S is still a retract of R. Any element $r \in \mathrm{Nil}(R)$ belongs to $\mathrm{Nil}(S_i)$ for some i, and therefore belongs to $\mathrm{Nil}(S)$ by (e). Similarly, any $r \in \mathrm{U}(R)$ belongs to $\mathrm{U}(S)$ by (f). Finally, let r be a non 0-divisor of R. Then r is a non 0-divisor in some S_i. By (d) (applied to S_i), we see that $r \in \mathrm{U}(S_i) \subseteq \mathrm{U}(R)$. Therefore, $Q(R) = R$, as desired.

Ex. 2.35. In *LMR*-(2.22)(A), an example of a commutative ring R is given such that $Q(R) = R$ and $\mathrm{Pic}(R) \neq \{1\}$. However, this ring R has nonzero nilpotent elements. Now, use Exercise 2.34 to construct a *reduced* ring R with the same properties.

Solution. Start with any commutative reduced ring S with $\mathrm{Pic}(S) \neq \{1\}$. With respect to this S, let $R \supseteq S$ be the commutative ring constructed in Exercise 34. By the functorial property of "Pic", the fact that S is a retract of R implies that $\mathrm{Pic}(S)$ is (isomorphic to) a direct summand of $\mathrm{Pic}(R)$. In particular, $\mathrm{Pic}(R) \neq \{1\}$. By construction, $Q(R) = R$, and $\mathrm{Nil}(R) = \mathrm{Nil}(S) = \{0\}$ so R is reduced.

Comment. A (commutative) ring R with the property $Q(R) = R$ is called a *classical ring* (or a *ring of quotients*) in *LMR*-(11.4). It may be somewhat surprising that, even when a commutative ring is reduced and classical, it may *not* have a trivial Picard group.

Ex. 2.35′. (Ishikawa) Show that a commutative semihereditary ring R is integrally closed in its total ring of quotients $K = Q(R)$ (that is, every element $x \in K$ integral over R is in R).

Solution. Fix an integrality equation $x^{n+1} + a_n x^n + \cdots + a_0 = 0$, where $a_i \in R$, and pick a non 0-divisor $r \in R$ such that $rx^i \in R$ for $1 \leq i \leq n$. Consider the f.g. ideal $I = \sum_{i=1}^n Rrx^i$ in R. By hypothesis, ${}_RI$ is projective, so by the Dual Basis Lemma (*LMR*-(2.9)), there exist linear functionals $f_i : I \to R$ such that $b = \sum_{i=1}^n f_i(b)\, r\, x^i$ for all $b \in I$. Now from the integrality equation, we have $rx^{n+1} \in I$, so we can write

$$rx = \sum_{i=1}^n f_i(rx)\, rx^i = \sum_{i=1}^n f_i(r^2 x^{i+1}) = r \sum_{i=1}^n f_i(rx^{i+1}).$$

Cancelling r, we conclude that $x = \sum_{i=1}^n f_i(rx^{i+1}) \in R$.

Comment. This exercise comes from Lemma 1 of T. Ishikawa's paper "A theorem on flat couples," Proc. Japan Academy **36**(1960), 389–391. The same property of semihereditary rings was also proved in Corollary 2 of S. Endo's paper "On semihereditary rings," J. Math. Soc. Japan **13**(1961), 109–119.

The next two problems are from a paper of H. Bass.

Ex. 2.36. For any right R-module P and $x \in P$, let $o_P(x) = \{f(x) : f \in P^*\}$ where, as usual, P^* denotes the left R-module $\mathrm{Hom}_R(P, R)$.

(1) If P' is any R-module and $F = P \oplus P'$, show that $o_P(x) = o_F(x)$ for any $x \in P \subseteq F$.
(2) If P is a projective right R-module and $x \in P$, show that $o_P(x)$ is a f.g. left ideal of R, with $o_P(x) \neq 0$ if $x \neq 0$. Deduce that the natural map $P \to P^{**}$ is a monomorphism.

Solution. (1) Any functional $h : F \to R$ can be represented as a pair (f, g) where $f \in P^*$ and $g \in P'^*$. With respect to this representation, we have, for any $x \in P$:

$$h(x, 0) = f(x) + g(0) = f(x).$$

From this, it is clear that $o_P(x) = o_F(x)$.

(2) Clearly, $o_P(x)$ is a left ideal in R, without any assumption on P. Now assume P_R is a *projective* module. Pick a module P'_R such that $F = P \oplus P'$ is free, say with a basis $\{e_i : i \in I\}$. The given element $x \in P$ then lies in a f.g. free direct summand F_1 of F. Applying (1) twice, we have $o_P(x) = o_F(x) = o_{F_1}(x)$. Replacing P by F_1, we may, therefore, assume that P is a free module with a finite basis, say, $e_1, \ldots, e_n$. Write $x = e_1a_1 + \cdots + e_na_n$. A typical functional $f \in P^*$ is determined by the scalars $b_i := f(e_i)$, and we have

$$f(x) = f(e_1)a_1 + \cdots + f(e_n)a_n = b_1a_1 + \cdots + b_na_n.$$

Therefore, $o_p(x) = \sum_{i=1}^n Ra_i$, as desired. If $x \neq 0$, then the a_i's are not all zero, so $o_P(x) \neq 0$. This means that, for any projective P_R and $x \neq 0$ in P, there exists $f \in P^*$ such that $f(x) \neq 0$. Therefore, $P \to P^{**}$ is a monomorphism.

Comment. In *LMR*-(4.65)(a), the (right) R-modules P with the property that $P \to P^{**}$ is injective are seen to be the *torsionless* modules, i.e. those modules that can be embedded into some direct product $(R^I)_R$. In particular, any submodule P of a free module (e.g. any projective module) has the property that $P \to P^{**}$ is injective.

Ex. 2.37. For any right R-module P and $x \in P$, let

$$o'_P(x) = \{y \in P : \ \forall f \in P^*, \ f(x) = 0 \Rightarrow f(y) = 0\}.$$

(1) If P' is any projective right R-module and $F = P \oplus P'$, show that $o'_P(x) = o'_F(x)$ for any $x \in P \subseteq F$.
(2) If P is a projective right R-module and $x \in P$, show that $o'_P(x) \cong o_P(x)^*$, where $o_P(x)$ is the left ideal associated with $x \in P$ in Exercise 36.
(3) Under the same hypothesis as in (2), show that $o'_P(x)$ is a direct summand of P iff $o_P(x)$ is a projective left R-module.

Solution. (1) Clearly, $o'_P(x) \subseteq o'_F(x)$, since any functional on F restricts to a functional on P. Conversely, let $y_1 = (y, y') \in P \oplus P'$ be in $o'_F(x)$. If $y' \neq 0$, there exists a functional $f' : P' \to R$ with $f'(y') \neq 0$, according to (2) of the last exercise. Let $f_1 : F \to R$ be the 0-extension of f' to F (with $f_1(P) = 0$). Then $f_1(x) = 0$, but $f_1(y_1) = f'(y') \neq 0$, a contradiction. Therefore, we must have $y' = 0$. Now $y_1 = y \in P$ and it is clear that y must belong to $o'_P(x)$.

(2) For the rest, we assume P_R is projective. Exploiting (1) and arguing as in (2) of the last exercise, we may assume that P is a f.g. free R-module. In particular, the natural map $P \to P^{**}$ is an isomorphism. We shall identify P^{**} with P in the following. For $M = P/xR$, let $\pi : P \to M$ be the projection map, and let $\alpha : R \to P$ be defined by $\alpha(r) = xr$. Note that the kernel of $P = P^{**} \to M^{**}$ is exactly $o'_P(x)$. Applying the left exact functor $\mathrm{Hom}_R(-, R)$ to $R \xrightarrow{\alpha} P \xrightarrow{\pi} M \to 0$, we get an exact sequence $0 \to M^* \xrightarrow{\pi^*} P^* \xrightarrow{\alpha^*} R^*$. Upon identifying R^* with ${}_RR$ in the usual way, we see that im (α^*) is exactly $o_P(x)$. Therefore, we have an exact sequence

$$\text{(A)} \qquad 0 \longrightarrow M^* \xrightarrow{\pi^*} P^* \xrightarrow{h} o_P(x) \longrightarrow 0.$$

Applying $\mathrm{Hom}_R(-, R)$ again, we get an exact sequence

$$\text{(B)} \qquad 0 \longrightarrow o_P(x)^* \xrightarrow{h^*} P^{**} \xrightarrow{\pi^{**}} M^{**}.$$

By our earlier observation, $o'_P(x) \cong \ker(\pi^{**}) \cong o_P(x)^*$.

(3) We go back to $P \oplus P' = F$ where P' is a projective R-module and F is a free module. As in (2), we have $Q := o'_P(x) = o'_F(x)$. Clearly, Q is a direct summand of P iff Q is a direct summand of F. Applying this observation twice, we are again reduced to the case where P *is a f.g. free module* in verifying (3). We shall use the notations in (2). First assume $o_P(x)$ is projective. Then the surjection h in (A) splits, and so does the injection h^* in (B). This means that $Q = o'_P(x) = \mathrm{im}(h^*)$ splits in $P^{**} = P$. Conversely, assume that Q splits in $P = P^{**}$. Then h^* in (B) is a split injection, so h^{**} in the diagram below is a split surjection:

$$\text{(C)} \qquad \begin{array}{ccc} P^{***} & \xrightarrow{\quad h^{**} \quad} & o_P(x)^{**} \\ {\scriptstyle i}\uparrow & & \uparrow{\scriptstyle j} \\ P^* & \xrightarrow{\quad h \quad} & o_P(x) \end{array}$$

Since P^* is f.g. projective, i is an isomorphism, and hence j is onto. But j is also one-one, since the inclusion map $o_P(x) \hookrightarrow R$ is a functional on $o_P(x)$ that takes any nonzero element of $o_P(x)$ to a nonzero element of R. Thus, j is an isomorphism, and it follows from (C) that h is a split surjection. Since P^* is a projective left R-module, so is $o_P(x)$, as desired.

Comment. The last two exercises are taken from Bass's paper: "Projective modules over free groups are free," J. Algebra **1**(1964), 367–373. In the case where R is a left semihereditary ring, it follows from (2) above that any element x in a projective right R-module P is contained in a f.g. (projective) direct summand of P, namely, $o'_P(x) \cong o_P(x)^*$. Coupled with some arguments due to I. Kaplansky, this fact can be used to show that P is a direct sum of f.g. right R-modules, each isomorphic to the dual of a f.g. left ideal. (In particular, *if each f.g. left ideal of R is free, then so is each projective right* R-module.)

In the case of a *commutative* semihereditary ring R, the above result on the structure of a projective module P_R was first proved by Kaplansky; see his paper: "Projective modules," Annals of Math. **68**(1958), 372–377. It was generalized to the case of a *right* semihereditary ring R in F. Albrecht's paper: "On projective modules over semihereditary rings," Proc. Amer. Math. Soc. **12**(1961), 638–639. Bass's result, on the other hand, covers the case where R is a *left* semihereditary ring. Later, G. Bergman obtained a similar result for what he called *weakly semihereditary* rings, which include all 1-sided semihereditary rings; see his paper: "Hereditarily and cohereditarily projective modules," Proc. Conf. on Ring Theory at Park City, Utah, 1971 (R. Gordon, ed.), 29–62, Academic Press, N. Y., 1972.

Ex. 2.38. Let $P \subseteq M$ be right R-modules where P is f.g. projective. Show that P is a direct summand of M iff the induced map $M^* \to P^*$ is onto (that is, iff every linear functional on P extends to one on M).

Solution. The "only if" part is clear. Conversely, let $\varepsilon : P \to M$ be the inclusion map and assume $\varepsilon^* : M^* \to P^*$ is onto. Since P^* is also projective, ε^* splits, and thus it induces a split injection $\varepsilon^{**} : P^{**} \to M^{**}$. Consider the following commutative diagram

$$\begin{array}{ccc} P & \xrightarrow{\ \varepsilon\ } & M \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\beta} \\ P^{**} & \xrightarrow{\ \varepsilon^{**}\ } & M^{**} \end{array}$$

where α, β are the natural maps from P and M to their double duals. Pick any $\gamma : M^{**} \to P^{**}$ such that $\gamma\varepsilon^{**} = 1_{P^{**}}$. Noting that α is an isomorphism (Ex. 7), we can define $\delta = \alpha^{-1}\gamma\beta : M \to P$. Then

$$\delta\varepsilon = \alpha^{-1}\gamma\beta\varepsilon = \alpha^{-1}\gamma\varepsilon^{**}\alpha = \alpha^{-1}1_{P^{**}}\alpha = 1_P,$$

so δ provides a splitting for the injection ε, as desired.

Ex. 2.39. (Hinohara) Let $p \in P$, where P is a projective right module over a ring R. If $p \notin P\mathfrak{m}$ for every maximal left ideal $\mathfrak{m}$ of R, show that pR is a free direct summand of P with basis $\{p\}$.

Solution. Say $P \oplus Q = F$, where F is free with basis $\{e_i : i \in I\}$, and write $p = e_1a_1 + \cdots + e_na_n$ $(a_i \in R)$. (For convenience, we assume $\{1, 2, \ldots, n\} \subseteq I$.) Decompose e_i into $p_i + q_i$, where $p_i \in P$ and $q_i \in Q$. Then $p = p_1a_1 + \cdots + p_na_n$ implies that $\sum_{i=1}^{r} Ra_i$ cannot be contained in a maximal left ideal $\mathfrak{m}$ (for otherwise $p \in P\mathfrak{m}$). Thus, $\sum_{i=1}^{r} Ra_i = R$, so by the Unimodular Column Lemma (Ex. 1.34), pR is free on p, and $pR \oplus X = F$ for a suitable submodule $X \subseteq F$. But then $pR \oplus (X \cap P) = P$, as desired.

Comment. The result in this exercise appeared in Y. Hinohara's paper: "Projective modules over semilocal rings", Tôhuku Math. J. **14**(1962), 205–211. Of course, the converse of the exercise is also true.

Ex. 2.40. (Hinohara) Let R be a commutative semilocal ring, and P be a f.g. projective R-module.

(1) If P has constant rank n, show that $P \cong R^n$.
(2) In general, show that P is isomorphic to $\bigoplus_i Re_i$ for suitable idempotents $e_i \in R$.

Solution. Let $\mathfrak{m}_1, \ldots, \mathfrak{m}_k$ be the maximal ideals of R.

(1) Each localization $P_{\mathfrak{m}_i}$ is free of rank n. Pick $x_{i1}, \ldots, x_{in} \in P$ such that their images in $P_{\mathfrak{m}_i}$ form a free basis for $P_{\mathfrak{m}_i}$. By the Chinese Remainder Theorem, we can find x_j (for $1 \le j \le n$) in P such that

$$x_j \equiv x_{ij} \pmod{\mathfrak{m}_i P}$$

for all i. It is then easy to check that $x_1, \ldots, x_n$ form a free basis for each $P_{\mathfrak{m}_i}$ upon localization. If we define $f : R^n \to P$ by $f(e_j) = x_j$ for all unit vectors e_j, then $f_{\mathfrak{m}_i}$ is an isomorphism for each i. From this, it follows that f is an isomorphism itself.

(2) Since $R/\text{rad}\ R \cong \prod_{i=1}^{k} R/\mathfrak{m}_i$, R cannot be the direct product of more than k nonzero rings. Therefore, we can write $R = R_1 \times \cdots \times R_r$, where each R_j is an indecomposable ring (that is, a ring with only trivial idempotents). Clearly, R_j is still a semilocal ring. We have $P = P_1 \oplus \cdots \oplus P_r$, where each P_j is a f.g. projective R_j-module on which all other components R_i $(i \neq j)$ act trivially. By (1), $P_j \cong R_j^{n_j}$, and so $P \cong R_1^{n_1} \oplus \cdots \oplus R_r^{n_r}$. Here, each summand R_j is isomorphic to Re for some idempotent e, as desired.

Comment. The result in this exercise first appeared in Y. Hinohara's paper "Note on finitely generated projective modules", Proc. Japan Acad. **37**(1961), 478–481 (and was independently proved by S. Endo). If R is a commutative semilocal ring with only trivial idempotents, Hinohara has also proved that *any* projective R-module P is free, and if R is allowed to have idempotents, then P is still a direct sum $\bigoplus_i Re_i$, where the e_i's are idempotents of R. See his paper cited in the *Comment* on the last exercise.

For any module P over a commutative semilocal ring R, it is true that P being f.g. projective is a *local condition*, that is, P is f.g. projective iff $P_{\mathfrak{m}}$ is f.g. projective over $R_{\mathfrak{m}}$ for each maximal ideal $\mathfrak{m} \subset R$. This was another result proved in Hinohara's first paper cited above. Using the techniques of flat modules, this result can be deduced easily, for instance, from the *Comment* on Ex. 4.46B.

§3. Injective Modules

A module I_R is called *injective* if the (contravariant) functor $\text{Hom}_R(-, I_R)$ is exact. Comparing this to the definition of a projective module, we see that "injectivity" and "projectivity" are in some sense dual notions. Some explicit analogies may be made between projective and injective modules by "turning arrows around". However, we should not assume that any true statement about projective modules always has a true analogue for injective ones, or vice versa. Over any ring R, R_R is free and hence always projective, but R_R is seldom injective. Rings for which R_R happens to be injective are called *right self-injective* rings. An important source of such rings is the class of *Frobenius algebras* over a field k, treated in §3B. These are finite-dimensional k-algebras R for which R and $\hat{R} = \text{Hom}_k(R,\ k)$ are isomorphic as right (or equivalently, left) R-modules. (In general, $\hat{R}$ is an (R, R)-bimodule, since R itself has a natural (R, R)-bimodule structure.) A classical family of Frobenius algebras is given by the group algebras kG for finite groups G. These are, in fact, *symmetric algebras* in the later terminology of §16. Exercises 12–17 are devoted to the theme of self-injectivity and Frobenius algebras.

An injective module I_R is characterized by the property that it is a direct summand of any "containing" module $M \supseteq I$. (Admittedly, this characterization sounds better in German.) A very powerful way to check the injectivity of I_R is to apply "Baer's Test", i.e. to check that any homomorphism f from a right ideal $\mathfrak{A}$ to I can be extended to $f' : R \to I$. This test is special to injective modules, and has apparently no analogue for projective modules.

Another fact distinguishing injective modules from projective ones is the existence of an *injective hull* for any module M_R. In general, M gives rise to a module $E(M)$ (its injective hull), which is "minimal injective" over M, or equivalently, "maximal essential" over M. Here we utilize the notion of an *essential* extension that is fundamental to module theory: a module $N \supseteq M$ is said to be essential over M (written $M \subseteq_e N$) if every nonzero submodule of N intersects M nontrivially; or equivalently, for any nonzero $x \in N$, $xr \in M \backslash \{0\}$ for a suitable $r \in R$. Some of the properties of essential extensions are collected in Exercises 6–8 below; they will be used freely later. Many examples of injective hulls are presented in *LMR*-(3.43).

In general, direct products of injectives are injective, but direct sums of injectives need not be injective. The Bass-Papp Theorem (*LMR*-(3.46)) states that *direct sums of injective right R-modules are injective iff R is a right noetherian ring*, and a theorem of Matlis adds a third equivalent condition: *iff any injective right* R-module *is a direct sum of indecomposable submodules* ((1) $\Leftrightarrow$ (2) in *LMR*-(3.48)). Some basic material on indecomposable injective modules is presented in §3F: these are the modules that are injective hulls of *uniform* modules (nonzero modules in which the intersection of any two nonzero submodules is nonzero).

The notion of *divisibility* is closely related to that of injectivity. The reader should note, however, that the definition of a divisible module in *LMR*-§3C is a little different from that in some other texts. In fact, the definition of I_R being divisible in *LMR*-(3.16) amounts to the module being *principally injective* in the sense of Nicholson and Yousif; that is, one requires that, for any $a \in R$, any R-homomorphism $f : aR \to I$ extends to some $f' : R \to I$. Exercises 45–47 are related to this theme.

Returning to indecomposable injectives, the case of commutative noetherian rings R is particularly worth our study. Here, the set of indecomposable injectives is given by $\{E(R/\mathfrak{p}) : \mathfrak{p} \in \text{Spec } R\}$. Matlis' Theory in this case describes the injective hulls $E(R/\mathfrak{p})$, and computes their endomorphism rings. The analysis is carried out by first passing to the localization $R_\mathfrak{p}$; under this localization, $E(R/\mathfrak{p})$ is basically unchanged. It thus suffices to consider the case of a commutative noetherian local ring $(R, \mathfrak{m})$, and to compute the so-called "standard module" $E(R/\mathfrak{m})$. This module is described via a certain filtration (see *LMR*-(3.82)), and its endomorphism ring turns out to be the $\mathfrak{m}$-adic completion of $(R, \mathfrak{m})$ (see *LMR*-(3.84)). Some relevant exercises are Exercises 38–42.

An easy corollary of the Matlis Theory is that, over a commutative *artinian* ring, $E(M)$ is f.g. for any f.g. module M. (This was also proved without Matlis' Theory in *LMR*-(3.64).) However, over a *noncommutative* artinian ring, this need not be the case even for a cyclic module M. Exercise 34 offers the relevant counterexample.

In the special case where $R = k[x_1, \ldots, x_r]$ is a polynomial ring over a field k and $\mathfrak{p} = (x_1, \ldots, x_r)$, there is a special construction for the injective hull of $R/\mathfrak{p} = k$ (with trivial x_i-actions). In fact, in *LMR*-§3J, it is shown that $E_R(k)$ is given by the "module of inverted polynomials" $T = k[x_1^{-1}, \ldots, x_r^{-1}]$ (on which R has a natural action). The last two exercises ((51) and (52)) give a bit more information on this theme.

Exercises for §3

Ex. 3.1. Let R be a domain that is not a division ring. If a module M_R is both projective and injective, show that $M = 0$.

Solution. We may assume M is a submodule of a free R-module F, say with basis $\{e_i : i \in I\}$. Suppose M has a nonzero element $m = \sum_i e_i r_i$, with $r_{i_0} \neq 0$. We claim that any nonzero element $r \in R$ has a left inverse (so that R is a division ring). Indeed, since M_R is divisible by *LMR*-(3.17)′, $rr_{i_0} \neq 0$ implies that $m \in M\mathrm{rr}_{i_0} \subseteq F\, rr_{i_0}$. Comparing the coefficients of e_{i_0}, we get $r_{i_0} = s\,rr_{i_0}$ for some $s \in R$. Therefore, $sr = 1$, as desired.

Comment. In *LMR*, this exercise was stated for commutative domains. The present improvement was suggested to me by Dennis Keeler. Note that the hypothesis cannot be further weakened to R being a prime ring. In fact, for the (simple) prime ring $R = \mathbb{M}_n(k)$ where k is a division ring, every module is both projective and injective.

Ex. 3.2. Let R be a right self-injective ring.

(1) Show that an element of R has a left inverse iff it is not a left 0-divisor in R.
(2) If R has no nontrivial idempotents, show that R is a local ring, and that the unique maximal (left, right) ideal $\mathfrak{m}$ of R consists of all left 0-divisiors of R.
(3) If R is a domain, show that it must be a division ring.

Solution. (1) The "only if" part is true without any assumption on R : If $r \in R$ has a left inverse, say $xr = 1$ in R, then

$$ry = 0 \implies xry = 0 \implies y = 0,$$

so r is not a left 0-divisor. The converse is a consequence of the fact that the injective module R_R is a divisible module. We repeat the argument from *LMR*-(3.18) here. Suppose $r \in R$ is not a left 0-divisor. Then $f : rR \to R_R$ given by $f(ry) = y$ is a well-defined right R-module homomorphism. Since R_R is injective, f is induced by left multiplication by some $x \in R$. In particular, for $y = 1$, we get $xr = f(r) = 1$.

(2) The assumption on R in (2) means that R_R is an *indecomposable* injective module. By *LRM*-(3.52), $R \cong \mathrm{End}(R_R)$ is a local ring. Let $\mathfrak{m}$ be the unique maximal (left, right) ideal of R. Consider any $r \in \mathfrak{m}$. If r is not a left 0-divisor in R, then, by (1), $xr = 1$ for some $x \in R$. This gives $1 \in \mathfrak{m}$, a contradiction. Therefore, each $r \in \mathfrak{m}$ is a left 0-divisor, and of course each left 0-divisor must lie in $\mathfrak{m}$.

(3) This follows immediately from (2).

Comment. Let R be a right self-injective ring. *If $r \in R$ is neither a left nor a right 0-divisor, then in fact $r \in \mathrm{U}(R)$.* (A ring R satisfying this property is known as a classical ring.) See Exercise 11.8.

For (1) in this exercise, the hypothesis that R be right self-injective can be somewhat weakened: see Exercise 13.18.

Ex. 3.3. In a ring theory text, the following exercise appeared: "Every simple projective module is injective." Find a counterexample.

Solution. A counterexample can be found in *LMR*-(3.74B); namely, if ${}_kV$ is an infinite-dimensional vector space over a division ring k and $E = \text{End}({}_kV)$ (defined as a ring of right operators on (V), then V_E is a simple, projective module that is not injective. This is a counterexample in which the ring is von Neumann regular. In the following, we offer another counterexample in which the ring is artinian.

Let $E = \begin{pmatrix} k & k \\ 0 & k \end{pmatrix}$ where k is a division ring. For the idempotent $e = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \in E$, $P : = eE = \begin{pmatrix} 0 & 0 \\ 0 & k \end{pmatrix}$ is a projective right E-module. Since $\dim_k P = 1$, P is clearly simple. We finish by showing that P_E is not divisible (which implies that it is not injective). Consider the ring element $a = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$. For any $r = \begin{pmatrix} x & y \\ 0 & z \end{pmatrix} \in E$, note that

$$ar = 0 \Rightarrow \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x & y \\ 0 & z \end{pmatrix} = 0 \Rightarrow z = 0 \Rightarrow er = \begin{pmatrix} 0 & 0 \\ 0 & z \end{pmatrix} = 0.$$

However, $Pa = \begin{pmatrix} 0 & 0 \\ 0 & k \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = 0$, so in particular, $e \notin Pa$. This shows that P_E is not a divisible module.

Ex. 3.4. True or False: If I_S is injective and $f : S \to R$ is a ring homomorphism, then $I \otimes {}_SR$ is injective as a right R-module? (*Note.* It goes without saying that R is viewed here as an (S, R)-bimodule, with the left S-structure coming from f.)

Solution. The statement is easily seen to be *false.* For instance, if S is a right self-injective ring, we can take the injective module $I = S_S$. Then $I \otimes {}_SR \cong R_R$, so if R is not right self-injective, we have a counterexample.

To be more explicit, take S to be a field, $R = S[t]$ (polynomial ring in t), and f to be the inclusion map of S into R. Then S_S is injective, but R_R is not injective by Exercise 1.

Comment. The reason for asking the question in this Exercise is that, if P_S is a *projective* S-module, then $P \otimes {}_SR$ is a projective R-module. The proper "dual statement" of this for injective modules is the following: *If I is an injective right S-module, then* $\text{Hom}_S(R_S, I_S)$ *is an injective right R-module.* Here R is viewed as an (R, S)-biomodule, with the right S-structure coming from f, and the right R-module structure on $\text{Hom}_S(R_S, I_S)$ comes from the left R-module structure on R. The statement in italics above on injective modules is a special case of the "Injective Producing Lemma" proved in *LMR*-(3.5).

Ex. 3.5. Let a, b be elements in a ring R such that $ab = 1$ and $bR \subseteq_e R_R$. Show that $ba = 1$.

Solution. First note that $bR \cap \text{ann}_r(a) = 0$. In fact, if $bx \in \text{ann}_r(a)$, then $0 = a(bx) = x$, so $bx = 0$. Since $bR \subseteq_e R_R$, it follows that $\text{ann}_r(a) = 0$. Therefore, $a(ba - 1) = (ab)a - a = 0$ implies that $ba = 1$.

Ex. 3.6. (A) If $M_i \subseteq_e E$ for $1 \leq i \leq n$, show that $\bigcap_{i=1}^n M_i \subseteq_e E$. Does the same statement hold for an *infinite* family of essential submodules?
(B) If $M_i \subseteq_e E_i$ for $i = 1, 2, \ldots$, does it follow that $\prod_i M_i \subseteq_e \prod_i E_i$?

Solution. (A) Consider any $e \in E \backslash \{0\}$. Since $M_1 \subseteq_e E$, there exists $r_1 \in R$ such that $er_1 \in M_1 \backslash \{0\}$. Since $M_2 \subseteq_e E$, there exists $r_2 \in R$ such that $er_1 r_2 \in M_2 \backslash \{0\}$. Repeating this argument a finite number of times, we get $er_1 r_2 \cdots r_n \in M_n \backslash \{0\}$ for suitable $r_i \in R$. Clearly $er_1 \cdots r_n \in \bigcap_{i=1}^n M_i$, so we have shown $\bigcap_{i=1}^n M_i \subseteq_e E$. For an infinite family $\{M_1, M_2, \ldots\}$, the conclusion no longer holds in general. For instance, in the free module $E = \mathbb{Z}$ over $R = \mathbb{Z}$, the submodules $M_i = i\,\mathbb{Z}$ $(i = 1, 2, \ldots)$ are each essential in E, but $\bigcap_{i=1}^\infty M_i = 0$ is not.

(B) In general, the conclusion $\prod_i M_i \subseteq_e \prod_i E_i$ does not follow. For an example, take $E_i = \mathbb{Z}$ and $M_i = i\,\mathbb{Z} \subseteq_e E_i$ $(i = 1, 2, \ldots)$ over the ring of integers again. For the element $e = (1, 1, \ldots) \in \prod_i E_i$, there clearly exists no nonzero $r \in \mathbb{Z}$ such that $er = (r, r, \ldots) \in \prod_i M_i$.

Comment. The reason for asking the question in (B) is, of course, that for any family $M_i \subseteq_e E_i$ (finite or infinite), one does have $\bigoplus_i M_i \subseteq_e \bigoplus_i E_i$, as was shown in *LMR*-(3.38).

Ex. 3.7. Let $f : E' \to E$ be a homomorphism of right R-modules. If $M \subseteq_e E$, show that $f^{-1}(M) \subseteq_e E'$. (In particular, if $E' \subseteq E$, then $M \subseteq_e E$ implies that $M \cap E' \subseteq_e E'$.) Use this to give a proof for the first part of Exercise 6.

Solution. Consider any $e' \in E' \backslash f^{-1}(M)$. Then $f(e') \neq 0$, so there exists $r \in R$ such that $f(e')r \in M \backslash \{0\}$. Then clearly $e'r \in f^{-1}(M) \backslash \{0\}$. This shows that $f^{-1}(M) \subseteq_e E'$.

The fact above applies well to the first part of Exercise 6. By induction, it suffices to show $M_i \subseteq_e E$ for $i = 1, 2$ implies that $M_1 \cap M_2 \subseteq_e E$. By the above, $M_1 \subseteq_e E$ implies $M_1 \cap M_2 \subseteq_e M_2$. Since $M_2 \subseteq_e E$, the transitivity property of essential extensions (*LMR*-(3.27)(2)) implies that $M_1 \cap M_2 \subseteq_e E$.

Comment. Given $M \subseteq_e E$ and any element $y \in E$, let $f : R_R \to E$ be defined by $f(r) = yr$. Then the Exercise implies that

$$f^{-1}(M) = y^{-1}M := \{r \in R : \ yr \in M\} \subseteq_e R_R.$$

This is a very useful conclusion, even in the special case where M is an essential right ideal in R.

Ex. 3.8. Let U be an R-module that contains a direct sum $\bigoplus_{i \in I} V_i$, and let $V_i \subseteq_e E_i \subseteq U$ for every $i \in I$. Show that the sum $\sum_i E_i$ must also be a *direct* sum.

Solution. It is sufficient to prove the desired conclusion in the case $|I| < \infty$. For $n = |I| < \infty$, an easy induction on n reduces the proof to the case $n = 2$. So, let us assume that $I = \{1, 2\}$. Since $(V_1 \cap E_2) \cap V_2 = 0$, we must have $V_1 \cap E_2 = 0$ since $V_2 \subseteq_e E_2$. Now $(E_2 \cap E_1) \cap V_1 = 0$, so we must have $E_2 \cap E_1 = 0$ since $V_1 \subseteq_e E_1$.

Ex. 3.9. Show that a module M_R is semisimple iff no submodule $N \neq M$ is essential in M.

Solution. First suppose M is semisimple. Then, for any submodule $N \neq M$, there exists a submodule $T \neq 0$ such that $M = N \oplus T$. Since $N \cap T = 0$, N cannot be essential in M. Conversely, suppose no submodule $N \neq M$ is essential in M. We will show that M is semisimple by checking that any submodule $S \subseteq M$ is a direct summand. By Zorn's Lemma, there exists a submodule C maximal with respect to the property $C \cap S = 0$. Then $S \oplus C$ is essential in M (for otherwise there exists a nonzero submodule T such that $T \cap (S \oplus C) = 0$, and $(C \oplus T) \cap S = 0$ contradicts the maximality of C). By the given hypothesis on M, $S \oplus C = M$ so S is a direct summand of M, as desired.

Ex. 3.10. (Matlis) Show that a ring R is right hereditary iff the sum of two injective submodules of any right R-module is injective.

Solution. The key fact used for this exercise is that *R is right hereditary iff quotients of right injective R-modules are injective*: see *LMR*-(3.22).

First assume R is right hereditary, and let I_1, I_2 be injective submodules of a right R-module M. Then $I_1 \oplus I_2$ is still injective and $I_1 + I_2$ is a quotient thereof. By *LMR*-(3.22) quoted above, $I_1 + I_2$ is also injective.

For the converse, assume the property on sums of injective submodules in the exercise, and consider a quotient M/N of an injective module M_R. By *LMR*-(3.22) again, we are done if we can show that M/N is injective. Let $\pi : M \to M/N$ be the projection map and consider the R-monomorphism $g : M \to M \oplus (M/N)$ given by $g(m) = (m, \pi(m))$ for $m \in M$. Clearly, $M \oplus (M/N)$ is the sum of the two submodules $M \oplus (0)$ and $g(M)$, both of which are injective. By assumption, $M \oplus (M/N)$ is injective, so M/N is also injective, as desired.

Comment. The result of this Exercise comes from E. Matlis' classical paper "Injective modules over noetherian rings," Pacific J. Math. **8**(1958), 511–528. In connection with this Exercise, it is of interest to point out that, if M_R is any *nonsingular* module over any ring R (i.e. no nonzero element of M is killed by an essential right ideal), then the sum of two injective submodules of M is *always* injective: see Exercise 7.15 below.

Ex. 3.11. (Osofsky) Show that a ring R is semisimple iff the intersection of two injective submodules of any right R-module is injective.

Solution. The "only if" part is clear since every right module is injective over a semisimple ring. For the "if" part, make use of the following observation over any ring R:

(∗) *For any right R-module N, there exists a right R-module containing two isomorphic injective submodules A, B such that $N \cong A \cap B$.*

If R is such that the intersection of two injective submodules of any right R-module is injective, then (∗) implies that any N_R is injective, and hence R is semisimple by FC-(2.9).

To prove (∗), let $M = E(N)$ and $M' = M_1 \oplus M_2$ where $M_1 = M_2 = M$. Let $N' = \{(n, n) : n \in N\} \subseteq M'$, and let π be the projection map from M' onto M'/N'. Clearly, $\pi(M_i) \cong M_i = M$, so $\pi(M_1)$, $\pi(M_2)$ are isomorphic injective submodules of M'/N'. Now, the map

$$n \mapsto \pi(n,\ 0) = \pi(0,\ -n)$$

defines an isomorphism from N to $\pi(M_1) \cap \pi(M_2)$, as desired.

Ex. 3.12. If R, S are (finite-dimensional) Frobenius k-algebras (over a field k), show that $R \times S$ and $R \otimes_k S$ are also Frobenius k-algebras. Using this and the Wedderburn–Artin Theorem, show that any finite-dimensional semisimple k-algebra is a Frobenius algebra.

Solution. Let $B : R \times R \to k$ and $C : S \times S \to k$ be the nonsingular k-bilinear maps with the associativity property imposed in the definition of Frobenius k-algebras. We first deal with the direct product $T = R \times S$, which is viewed as a k-algebra by the diagonal action of k. We can define a pairing $D : T \times T \to k$ by

$$D\left((r,\ s),\ (r',\ s')\right) = B(r,\ r') + C(s,\ s') \quad (r,\ r' \in R;\ \ s,\ s' \in S).$$

This is easily checked to be k-bilinear, and we have

$$\begin{aligned} D\left((r,\ s),\ (r',\ s')(r'',\ s'')\right) &= D\left((r,\ s),\ (r'r'',\ s's'')\right) \\ &= B(r,\ r'r'') + C(s,\ s's'') \\ &= B(rr',\ r'') + C(ss',\ s'') \\ &= D\left((r,\ s)(r',\ s'),\ (r'',\ s'')\right). \end{aligned}$$

To check the nonsingularity of D, suppose $D((r, s), (r',\ s')) = 0$ for all r', s', where $r \in R$, $s \in S$ are fixed. Choosing $s' = 0$, we see that $B(r, r') = 0$ for all $r' \in R$, so $r = 0$. Similarly, we can check that $s = 0$. This shows that T is a Frobenius k-algebra.

Next we deal with the tensor product $W = R \otimes_k S$, which is viewed as a k-algebra in the usual way. This time, we define a k-bilinear pairing $E : W \times W \to k$ by

$$E\left(\sum\nolimits_i r_i \otimes s_i,\ \sum\nolimits_j r'_j \otimes s'_j\right) = \sum\nolimits_{i,j} B\left(r_i, r'_j\right) C(s_i, s'_j).$$

Note that the matrix of E is the Kronecker product of those of B and C with respect to suitable bases. From this observation, it follows easily that the nonsingularity of the pairings B and C implies that of E. A calculation similar to the one given in the case $B \times C$ shows that E has the associativity property (if B and C do). This checks that $W = B \otimes_k C$ is a Frobenius k-algebra.

For the last part of the problem, let A be a finite-dimensional semisimple k-algebra. By the Wedderburn–Artin Theorem, $A = A_1 \times \cdots \times A_r$ where the $A_i = \mathbb{M}_{n_i}(D_i)$ for suitable (finite-dimensional) k-division algebras D_i. Using the direct product reduction, we need only show that each $\mathbb{M}_{n_i}(D_i)$ is a Frobenius k-algebra. Now $\mathbb{M}_{n_i}(D_i) \cong D_i \otimes_k \mathbb{M}_{n_i}(k)$. By *LMR*-(3.15C), D_i is a Frobenius k-algebra, so it suffices to show that each matrix algebra $R = \mathbb{M}_n(k)$ is Frobenius. To this end, we define a k-bilinear pairing $B : R \times R \to k$ by $B(\alpha, \beta) = \mathrm{tr}(\alpha\beta)$ where "tr" denotes the trace function on $n \times n$ matrices. Clearly, B has the associativity property, so we need only check that B is *nonsingular*. Say $\alpha \in R$ is such that $\mathrm{tr}(\alpha\beta) = 0$ for all $\beta \in R$. For the matrix units E_{pq}, we have

$$0 = \mathrm{tr}\,(\alpha \mathrm{E}_{pq}) = \mathrm{tr}\left(\sum\nolimits_{i,j} \alpha_{ij} E_{ij} E_{pq}\right) = \mathrm{tr}\left(\sum\nolimits_{i} \alpha_{ip} E_{iq}\right) = \alpha_{qp},$$

so $\alpha = (\alpha_{ij}) = 0$ in $\mathbb{M}_n(k)$, as desired.

Ex. 3.13. In *LMR*-(3.12), it is shown that, if S is a principal right ideal domain, then for any nonzero element $b \in S$ such that $bS = Sb$, the factor ring S/bS is a right self-injective ring. In the commutative case, generalize this by showing that, for any Dedekind domain S and any nonzero ideal $\mathfrak{B} \subseteq S$, the factor ring $R = S/\mathfrak{B}$ is a self-injective ring.

Solution. Let $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$ be the prime ideals containing $\mathfrak{B}$, and let $T = S \backslash (\mathfrak{p}_1 \cup \cdots \cup \mathfrak{p}_n)$. Consider S_T, the localization of S at the multiplicative set T. This is a semilocal Dedekind domain (whose maximal ideals are $(\mathfrak{p}_1)_T, \ldots, (\mathfrak{p}_n)_T$), so by Ex. 2.11, it is a principal ideal domain. We claim that the natural ring homomorphism

$$\varepsilon : \; S/\mathfrak{B} \longrightarrow S_T/\mathfrak{B}_T$$

is an *isomorphism*. Once this is proved, then $S/\mathfrak{B} \cong S_T/\mathfrak{B}_T$ is a self-injective ring by the result *LMR*-(3.12) quoted above.

To prove that ε is one-one, consider any $\bar{s} \in \ker(\varepsilon)$. Then $s = bt^{-1}$ for some $b \in \mathfrak{B}$ and $t \in T$. Since no prime ideal contains t and $\mathfrak{B}$, we have an equation $at + b' = 1$ where $a \in S$ and $b' \in \mathfrak{B}$. Then

$$s = s(at + b') = a(st) + sb = ab + sb' \in \mathfrak{B},$$

so $\bar{s} = 0 \in S/\mathfrak{B}$. To show that ε is onto, consider any element $st^{-1} + \mathfrak{B}_T$ in $S_T/\mathfrak{B}_T$, where $s \in S$ and $t \in T$. As above, there exists an equation $at + b' = 1$ where $a \in S$ and $b' \in \mathfrak{B}$. Then

$$st^{-1} - sa = st^{-1}(1 - ta) = (b's)t^{-1} \in \mathfrak{B}_T,$$

and so $st^{-1} + \mathfrak{B}_T = sa + \mathfrak{B}_T = \varepsilon(sa + \mathfrak{B})$, as desired.

Comment. Another way to show that $S/\mathfrak{B}$ is self-injective is to apply Baer's Criterion (*LMR*-(3.7)). For any ideal $\mathfrak{A}/\mathfrak{B}$ in $R = S/\mathfrak{B}$, we need to show that the map

$$\mathrm{Hom}_R(R,\, R) \to \mathrm{Hom}_R(\mathfrak{A}/\mathfrak{B},\, R)$$

is onto. Again, this is something which can be checked "locally". Yet another way of proving the self-injectivity of $S/\mathfrak{B}$ is mentioned below in the *Comment* on Exercise 15.

Ex. 3.14. For any finite-dimensional commutative local algebra R over a field k, show that the following are equivalent:

(1) R is a Frobenius k-algebra;
(2) R is self-injective;
(3) R has a unique minimal ideal.

Solution. (1) $\Rightarrow$ (2) is just *LMR*-(3.14).

(2) $\Rightarrow$ (3). Let $\mathfrak{A}, \mathfrak{B}$ be minimal ideals of R. Then there exists an R-isomorphism $f : \mathfrak{A} \to \mathfrak{B}$ (since $\mathfrak{A}, \mathfrak{B}$ are both isomorphic to $R/\mathfrak{m}$ where $\mathfrak{m}$ is the maximal ideal of R). Since R_R is injective, the isomorphism f is induced by multiplication by a suitable element $r \in R$. In particular, we have $r\,\mathfrak{A} = \mathfrak{B}$. Since $r\,\mathfrak{A} \subseteq \mathfrak{A}$ and $\mathfrak{A}$ is minimal, we must have $\mathfrak{A} = r\,\mathfrak{A} = \mathfrak{B}$.

(3) $\Rightarrow$ (1). Let $\mathfrak{A}$ be the unique minimal ideal of R. Fix a nonzero element $a \in \mathfrak{A}$ and take a hyperplane H in R that does not contain a. Clearly, H does not contain any nonzero ideal, since any such contains $\mathfrak{A}$. Therefore, by *LMR*-(3.15), R is a Frobenius k-algebra.

Comment. There is certainly room for improvement in this result: note that R being local is not needed in the argument for (3) $\Rightarrow$ (1), and the argument for (2) $\Rightarrow$ (3) shows more generally that any two isomorphic minimal ideals in the socle of a commutative self-injective ring must be equal. The local algebras in this exercise are known to commutative algebraists and algebraic geometers as *zero-dimensional* local Gorenstein algebras. Examples of such algebras can be found in *LMR*-(3.15B) and *LMR*-(3.15C).

For a generalization of this exercise to *artinian* rings, see *LMR*-(15.27); and for a noncommutative version of the same, see Exer. (16.1).

Ex. 3.15. Let k be a field, and R be a finite-dimensional k-algebra that is a proper factor ring of a Dedekind k-domain S. Show that R is a Frobenius k-algebra.

Solution. Say $R = S/\mathfrak{B}$ where $\mathfrak{B} \subsetneq S$ is a nonzero ideal. Let $\mathfrak{B} = \mathfrak{p}_1^{n_1} \cdots \mathfrak{p}_r^{n_r}$ be the unique factorization of $\mathfrak{B}$ into prime ideals. By the Chinese Remainder Theorem (applicable here since the ideals $\{\mathfrak{p}_i^{n_i}\}$ are pairwise comaximal), we have $R \cong \prod_i R/\mathfrak{p}_i^{n_i}$. According to Exercise 12, it suffices to show that each $R/\mathfrak{p}_i^{n_i}$ is a Frobenius k-algebra. Now $R/\mathfrak{p}_i^{n_i}$

is a finite-dimensional local k-algebra (with unique maximal ideal $\mathfrak{p}_i/\mathfrak{p}_i^{n_i}$), and it has a unique minimal ideal $\mathfrak{p}_i^{n_i-1}/\mathfrak{p}_i^{n_i}$. Therefore, the last exercise applies to show that $R/\mathfrak{p}_i^{n_i}$ is a Frobenius algebra for every i.

Comment. In the case where no field k is present, the same method can be used to give another solution to Exercise 13.

Ex. 3.16. Let K/k be a field extension, and let R be a finite-dimensional k-algebra. Show that R is a Frobenius algebra over k iff $R^K = R \otimes_k K$ is a Frobenius algebra over K.

Solution. The "only if" part is easy. For, if R is a Frobenius algebra, then by definition, there exists a nonsingular bilinear pairing $B: R \times R \to k$ with the "associativity" property $B(xy, z) = B(x, yz)$ for all $x, y, z \in R$. For any field extension K/k, we can extend B to $B^K : R^K \times R^K \to K$ in the usual way, and it is easy to check that B^K is nonsingular, and inherits the associativity property of B. Therefore, R^K is also a Frobenius K-algebra.

Conversely, assume that, for some field extension K/k, R^K is a Frobenius K-algebra. We try to compare the following two right R-modules: $M = R_R$, and $N = \mathrm{Hom}_k\,({}_RR, k)$. Upon scalar extension to K, we have $M^K = R_R \otimes_k K \cong (R^K)_{R^K}$, and

$$N^K = \mathrm{Hom}_k\,(R^R, k) \otimes_k K \cong \mathrm{Hom}_K\left({}_{R^K}(R^K), K\right)$$

(see *FC*-(7.4)). Since R^K is a Frobenius algebra, we have $M^K \cong N^K$ as right R^K-modules. By the Noether–Deuring Theorem (*FC*-(19.25)), we have $M \cong N$ as right R-modules, so R is a Frobenius k-algebra.

Ex. 3.17. Let R be a finite-dimensional algebra over a field k. For any f.g. left R-module M, let $\hat{M} = \mathrm{Hom}_k(M, k)$, which has a natural structure of a right R-module. It is known that the isomorphism type of this right R-module is independent of the choice of k: see *LMR*-(19.32). Prove the following special case of this fact: Let K be a field extension of k within the center of R (so R is also a finite-dimensional K-algebra). Then for any f.g. left R-module M, $\mathrm{Hom}_k(M, k)$ and $\mathrm{Hom}_K(M, K)$ are isomorphic as right R-modules.

Solution. Fix a nonzero k-linear functional $t: K \to k$. We define a k-linear map

$$f : \mathrm{Hom}_K(M, K) \longrightarrow \mathrm{Hom}_k(M, k)$$

by $f(\lambda) = t \circ \lambda$ for any $\lambda \in \mathrm{Hom}_K(M, K)$. This is a right R-homomorphism between the two dual spaces. In fact, for λ as above, and $r \in R$, $m \in M$, we have

$$\begin{aligned} f(\lambda r)(m) &= (t \circ \lambda r)(m) = t((\lambda r)(m)) \\ &= t(\lambda(rm)) = (t \circ \lambda)(rm) \\ &= f(\lambda)(rm) = (f(\lambda)r)(m), \end{aligned}$$

so $f(\lambda r) = f(\lambda)r$. It is easy to check that f is an injection. For, suppose $\lambda \neq 0$. Then $\lambda(M) = K$ and so

$$f(\lambda)(M) = t(\lambda(M)) = t(K) = k$$

shows that $f(\lambda) \neq 0$. Computing k-dimensions, we have

$$\begin{aligned}\dim_k \operatorname{Hom}_K(M,K) &= [K:k] \dim_K \operatorname{Hom}_K(M,K)\\ &= [K:k] \dim_K M\\ &= \dim_k M\\ &= \dim_k \operatorname{Hom}_k(M,k).\end{aligned}$$

Therefore, the k-linear injection f must be an isomorphism (of right R-modules).

Comment. The work above comes close to, but is not sufficient for proving the independence of the right module $\operatorname{Hom}_k(M,k)$ on k. The difficulty stems from the fact that if k_1, k_2 are fields in $Z(R)$ (the center of R) over both of which R is a finite-dimensional algebra, then R may not be finite-dimensional over $k_1 \cap k_2$; on the other hand, while R is a f.g. module over $k_1 \cdot k_2$, the latter may not be a field. Therefore, the present exercise may not yield a direct comparison between the k_1-dual and the k_2-dual. Of course, in the case where $Z(R)$ is a field, or where $Z(R)$ contains a smallest field over which R is finite-dimensional, this exercise will give the independence of $\operatorname{Hom}_k(M, k)$ over k (independently of (19.32)).

Ex. 3.18. Use the fact that $\mathbb{Z}/n\mathbb{Z}$ is self-injective (*LMR*-(3.13)(1)) to prove Prüfer's Theorem, which states that any abelian group G of finite exponent n is isomorphic to a direct sum of cyclic groups, necessarily of exponents dividing n. (Your proof should show, in particular, that any element of order n generates a direct summand in G.)

Solution. Since G is the direct sum of its primary components, we are immediately reduced to the case when $n = p^r$ where p is a prime. Let R be the ring $\mathbb{Z}/p^r\mathbb{Z}$ and view G as a (left) R-module. Then G contains at least one copy of ${}_RR$. Consider the family $\mathfrak{F}$ of direct sums $\bigoplus_i A_i$ of submodules $A_i \subseteq M$ where each $A_i \cong R$, and define $\bigoplus_i A_i \leq \bigoplus_j B_j$ in $\mathfrak{F}$ to mean that each A_i is equal to some B_j. It is easy to show that any linearly ordered family in $\mathfrak{F}$ has an upper bound. Therefore, by Zorn's Lemma, $\mathfrak{F}$ has a maximal member, say $H = \bigoplus_k C_k$. Since R is a noetherian self-injective ring, ${}_RH \cong \bigoplus_k ({}_RR)$ is injective by *LMR*-(3.46). Therefore, $G = H \bigoplus H'$ for a suitable R-module H'. Since $\bigoplus_k C_k$ is maximal in $\mathfrak{F}$, H' cannot contain a copy of R. Therefore, we must have $p^{r-1}H' = 0$. Invoking an inductive hypothesis at this point, we may assume that $H' = \bigoplus_\ell D_\ell$ where the D_ℓ's are cyclic groups of exponent dividing p^{r-1}. Now $G = \bigoplus_k C_k \oplus \bigoplus_\ell D_\ell$ gives what we want.

Comment. The induction above can be started at $r = 0$, for which the desired result is trivial. The inductive step from $r = 0$ to $r = 1$ is essentially the argument for the existence of a basis for a vector space over the field $\mathbb{F}_p$. The self-injectivity of the ring $\mathbb{Z}/n\mathbb{Z}$ enables us to extend this argument from vector spaces to abelian groups.

Prüfer's Theorem should more appropriately be called the Prüfer–Baer Theorem. It was proved by H. Prüfer for countable abelian groups in 1923, and by R. Baer for arbitrary abelian groups (of finite exponent) in 1934.

Ex. 3.19. For any field k and any nonzero polynomial $f(t)$ in $k[t]$, it is proved in *LMR*-(3.13)(2) that $k[t]/(f(t))$ is a self-injective ring. Explain how this would impact upon the proof of the Jordan Canonical Form Theorem.

Solution. Let T be a linear operator on a finite-dimensional k-vector space V such that the minimal polynomial $f(t)$ of T splits completely over k. By passing to the generalized eigenspaces of V ("primary decomposition"), we may assume that $f(t)$ has the form $(t-\lambda)^m$. By replacing T by $T - \lambda I$, we may further assume that $\lambda = 0$. Then V is a left module over the k-algebra $R = k[t]/(t^m)$ where t acts on V via T. We shall make use of the fact that R is a self-injective ring. The idea is the same as that used in the solution of Exercise 18, except that things are easier here due to $\dim_k V < \infty$.

Since $T^{m-1} \neq 0$ on V, there exists an R-submodule of V isomorphic to ${}_RR$. Let U be a free R-module of the largest k-dimension in V. Since ${}_RU$ is an injective module, $V = U \oplus W$ for a suitable R-module W. Clearly W cannot contain a copy of ${}_RR$, so T^{m-1} acts as zero on W and we can handle $T|W$ by induction. For each copy of ${}_RR$, say $R \cdot u$ in U, we can use the k-basis $\{u,\ Tu, \dots, T^{m-1}u\}$, with respect to which the action of T is expressed in the form of a nilpotent $m \times m$ Jordan block. This leads to the Jordan Canonical Form of T on V.

Ex. 3.20. Let S be a submodule of a right module M over a ring R. Show that there exists a submodule $C \subseteq M$ such that $E(M) \cong E(S) \oplus E(C)$.

Solution. By Zorn's Lemma, there exists a submodule $C \subseteq M$ maximal with respect to the property $S \cap C = 0$. (Any such submodule C is called a *complement* to S in M.) As we have seen in the solution to Exercise 9, $S \oplus C \subseteq_e M$. By *LMR*-(3.33)(2) and (3.39), it follows that

$$E(M) \cong E(S \oplus C) \cong E(S) \oplus E(C),$$

as desired.

Ex. 3.21. For any noetherian right module M over any ring R, show that $E(M)$ is a finite direct sum of indecomposable injective R-modules.

Solution. We may assume that $M \neq 0$. We claim that each submodule $M' \subsetneq M$ can be expressed as $M_1 \cap \cdots \cap M_n$ in such a way that each M/M_i is uniform. For, if otherwise, there would exist a maximal M' which *cannot* be so expressed (since M_R is noetherian). In particular $M/M' \neq 0$ cannot be uniform, so there exist submodules $X,\ Y \supsetneq M'$ such that $X \cap Y = M'$. But then $X = X_1 \cap \cdots \cap X_r$ and $Y = Y_1 \cap \cdots \cap Y_s$, where M/X_i and M/Y_j are all uniform. Since

$$M' = X \cap Y = X_1 \cap \cdots \cap X_r \cap Y_1 \cap \cdots \cap Y_s,$$

we have arrived at a contradiction.

Applying our claim above to $M' = 0$, we have an equation $0 = M_1 \cap \cdots \cap M_n$ where each M/M_i is uniform. We may assume that this expression of the zero submodule is *irredundant*, in the sense that no M_i can contain the intersection of the other M_j's. Then, for the embedding map

$$f : M \longrightarrow N = \bigoplus\nolimits_i M/M_i,$$

we have $N_i := f(M) \cap (M/M_i) \neq 0$ for each i. Since M/M_i is uniform, $N_i \subseteq_e M/M_i$ and hence $\bigoplus_i N_i \subseteq_e N$ by *LMR*-(3.38). From $\bigoplus_i N_i \subseteq f(M)$, we have then $f(M) \subseteq_e N$ and it follows from *LMR*-(3.39) that

$$E(M) \cong E\left(f(M)\right) = E\left(\bigoplus\nolimits_{i=1}^n N_i\right) \cong \bigoplus\nolimits_{i=1}^n E\left(M/M_i\right).$$

By *LMR*-(3.52), each $E(M/M_i)$ is indecomposable, as desired.

Comment. The method of proving the existence of the expression $M' = M_1 \cap \cdots \cap M_n$ in the first part of the Exercise is called "noetherian induction"; it goes back essentially to Emmy Noether. In fact, the whole analysis above is very close to Noether's proof of the Primary Decomposition Theorem for ideals in a (commutative) noetherian ring.

With the later material on uniform dimensions, this Exercise can be further improved. In *LMR*-(6.12), it is shown that, for any module M_R, *$E(M)$ is a finite direct sum of indecomposable injective R-modules iff M has finite uniform dimension* (that is, iff M contains no infinite direct sum of nonzero submodules). The present exercise is a special case of this, since any noetherian module has finite uniform dimension. For more information on the indecomposable summands of $E(M)$ and the "associated primes" of M (in the case of finite uniform dimension), see Exercise 6.4.

Ex. 3.22. Show that any injective module I_R is the injective hull of a direct sum of cyclic modules.

Solution. We may assume that $I \neq 0$. Consider the family of subsets X of nonzero elements in I with the property that the sum $\sum_{x \in X} xR$ is direct. By applying Zorn's Lemma to this family, we come out with a *maximal* X in this family (with respect to inclusion). Since I is injective, we can write $I = E\left(\bigoplus_{x \in X} xR\right) \oplus M$ for some submodule $M \subseteq I$. If $M \neq 0$, we can pick $y \in M \backslash \{0\}$ and get a direct sum $\bigoplus_{x \in X} xR \oplus yR$, in contradiction to the maximality of X. Therefore $M = 0$, and we have $I = E\left(\bigoplus_{x \in X} xR\right)$ as desired.

Ex. 3.23. Let $H = \text{End}\,(I_R)$ where I is an injective right R-module. For $f,\, h \in H$, show that $f \in H \cdot h$ iff $\ker(h) \subseteq \ker(f)$.

Solution. If $f = gh$ for some $g \in H$, clearly $h(x) = 0$ implies $f(x) = g(h(x)) = 0$ for $x \in I$, so we have $\ker(h) \subseteq \ker(f)$ without any condition on I. Conversely, assume I_R is injective and $\ker(h) \subseteq \ker(f)$. For $N = h(I)$, we can define $g_0 \in \text{Hom}_R(N,\, I)$ by $g_0(h(x)) = f(x)$ (for $x \in I$). Note that g_0 is well-defined since

$$h(x) = h(x') \implies x - x' \in \ker(h) \subseteq \ker(f) \implies f(x) = f(x').$$

Since I is injective, g_0 extends to some $g \in \text{Hom}_R(I,\, I) = H$, which then satisfies $f = gh$.

Comment. The property needed on I for the above proof to work is precisely that, for any submodule $N \subseteq I$, any R-homomorphism $g_0 : N \to I$ extends to an endomorphism $g \in \text{End}(I_R)$. This property defines the notion of I being a *quasi-injective* (or *weakly injective*) R-module. Most properties provable for $\text{End}(I_R)$ for injective modules I are provable by the same token for $\text{End}(I_R)$ for quasi-injective modules. For a select list of such properties, see *LMR*-(13.1) and (13.2).

The special case of the Exercise for the right regular module R_R is as follows: *If R is a right self-injective ring, then $f \in Rh$ iff* $\text{ann}_r(h) \subseteq \text{ann}_r(f)$. (It is true that assuming R_R quasi-injective would be sufficient. However, by Baer's Criterion, this is already equivalent to the right self-injectivity of R.)

Ex. 3.24. Let I_R be an injective module. If every surjective endomorphism of I is an automorphism, show that every injective endomorphism of I is an automorphism. How about the converse?

Solution. Let $f : I \to I$ be injective, with $J = f(I)$. Since $J \cong I$ is injective, $I = J \oplus K$ for a suitable submodule K. Now let $g : J \to I$ be the inverse of $f : I \to J$, and consider $g \oplus 0 : J \oplus K \to I$. This is a surjective endomorphism of I, so by assumption $K = \ker(g \oplus 0) = 0$, which shows that f is onto.

The converse statement is not true in general. For instance, let I be the Prüfer p-group over $\mathbb{Z}$ where p is any prime. Then $I = \bigcup_{n=1}^{\infty} C_{p^n}$ where C_{p^n} is cyclic of order p^n. We claim that *any injective $h : I \to I$ is onto.* Indeed, $h\,(C_{p^n}) \cong C_{p^n}$ implies that $h\,(C_{p^n}) = C_{p^n}$ (since C_{p^n} is the only subgroup of order p^n in I), so h is clearly onto. However, $I/C_p \cong I$, so we have a surjective $I \to I$ with kernel $C_p \neq 0$.

Comment. The result in this exercise appeared in G. Birkenmeier's paper "On the cancellation of quasi-injective modules," Comm. Algebra **4**(1976), 101–109.

Ex. 3.25. Let M_R be any module, and let $f \in \text{End}_R(E(M))$. If $f|_M$ is an automorphism of M, show that f is an automorphism of $E(M)$.

Solution. Since $f|_M$ is injective, we have $M \cap \ker(f) = 0$. Therefore, $M \subseteq_e E(M)$ implies that $\ker(f) = 0$, so f is a monomorphism. Now consider $f(E(M)) \supseteq f(M) = M$. Since $f(E(M)) \cong E(M)$ is injective, $f(E(M))$ must be *equal* to the injective hull $E(M)$ of M. Therefore, f is also an epimorphism.

Comment. In general, the converse of this exercise is not true. However, if M is a *quasi-injective module* in the sense of *LMR*-§6 G (any homomorphism from a submodule of M to M extends to an endomorphism of M), the converse becomes true, since in this case M is a fully invariant submodule of $E(M)$ (by *LMR*-(6.74)).

Ex. 3.26. Show that Baer's Criterion for Injectivity *LMR*-(3.7) can be further modified as follows: To check that a module I_R is injective, it is sufficient to show that, for any right ideal $\mathfrak{A} \subseteq_e R_R$, any $f \in \mathrm{Hom}_R(\mathfrak{A}, I)$ can be extended to R.

Solution. Suppose the given condition on I is satisfied, and let $g \in \mathrm{Hom}_R(\mathfrak{B}, I)$, where $\mathfrak{B} \subseteq R$ is any right ideal. Consider the family $\mathfrak{F}$ of pairs $(\mathfrak{B}', g')$ where $g' \in \mathrm{Hom}_R(\mathfrak{B}', I)$, $\mathfrak{B}' \supseteq \mathfrak{B}$ and $g'|\mathfrak{B} = g$. We partially order $\mathfrak{F}$ by defining $(\mathfrak{B}', g') \leq (\mathfrak{B}'', g'')$ if $\mathfrak{B}' \subseteq \mathfrak{B}''$ and $g''|\mathfrak{B}' = g'$. Then a routine application of Zorn's Lemma shows the existence of a maximal member $(\mathfrak{A}, f)$ in $\mathfrak{F}$. We must have $\mathfrak{A} \subseteq_e R_R$, for otherwise $\mathfrak{A} \cap \mathfrak{A}' = 0$ for some nonzero right ideal $\mathfrak{A}'$, and we could have extended f to $f \oplus 0 : \mathfrak{A} \oplus \mathfrak{A}' \to I$. Now by the given hypothesis f can be extended to $R \to I$. Therefore, by Baer's Criterion *LMR*-(3.7), I_R is an injective module.

Ex. 3.27. (P. Freyd) Give a direct proof for the validity of the modified Baer's Criterion in the last exercise by using the fact that a module I_R is injective iff it has no proper essential extensions.

Solution. Suppose any R-homomorphism from an essential right ideal of R to I_R can be extended to R_R. Our job is to check that I has no proper essential extensions. Consider any essential extension $I \subseteq_e E$ and any element $y \in E$. By the *Comment* following Exercise 7,

$$\mathfrak{A} := \{a \in R : ya \in I\} \subseteq_e R_R.$$

Consider $f \in \mathrm{Hom}_R(\mathfrak{A}, I)$ defined by $f(a) = ya$ for all $a \in \mathfrak{A}$. By assumption, f can be extended to some $g \in \mathrm{Hom}_R(R, I)$. For $x := g(1) \in I$, we have therefore $f(a) = ya = xa$ for all $a \in \mathfrak{A}$. *We claim that* $I \cap (y-x)R = 0$. In fact, if $z = (y-x)r \in I$ where $r \in R$, then $yr = z + xr \in I$ implies that $r \in \mathfrak{A}$, and hence $z = yr - xr = 0$. Since $I \subseteq_e E$, it follows that $(y-x)R = 0$; that is, $y = x \in I$. Hence $I = E$, as desired.

Ex. 3.28. (1) For an R-module M_R and an ideal $J \subseteq R$, let $P = \{m \in M : mJ = 0\}$. If M is an injective R-module, show that P is an injective R/J-module.

(2) Use the above to give a proof for the fact that any proper factor ring of a commutative PID is a self-injective ring.

Solution. (1) View P as an R/J-module, and let $E = E(P_{R/J})$. Then $P \subseteq_e E$ as R/J-modules, and hence as R-modules. Since M_R is injective, we may assume that E is embedded in N, that is, $P \subseteq E \subseteq N$. But then $EJ = 0$ implies that $E \subseteq P$, so $P = E$ is injective as an R/J-module.

(2) Consider R/J, where R is a communtative PID, and J is a nonzero ideal in R. Let K be the quotient field of R. Then K and hence $M := K/J$ are divisible R-modules. In particular, M_R is injective (by *LMR*-(3.17)$'$). Using the fact that J is principal, it is easy to check that

$$P := \{m \in M : mJ = 0\} = R/J.$$

Therefore, by (1) above, $P = R/J$ is an injective R/J-module, as desired.

Ex. 3.29. Let M_R be an R-module, and $J \subseteq R$ be an ideal such that $MJ = 0$. By Exercise 28, if M_R is injective, then $M_{R/J}$ is injective. Is the converse also true?

Solution. The converse is clearly false. For $R = \mathbb{Z}$ and $J = 2\mathbb{Z}$, consider the R-module $M = \mathbb{Z}/2\mathbb{Z}$, which satisfies $MJ = 0$. Here M is an injective R/J-module (since R/J is a field), but M is clearly not an injective R-module.

Comment. Curiously enough, the converse mentioned in the Exercise is "sometimes" true. For instance, let $R = S \times J$ where S, J are two rings. Let M_S be an S-module, viewed as an R-module via the projection $R \to S$. Then $R/J \cong S$, $MJ = 0$, and it is proved in *LMR*-(3.11A) that M_R is injective iff $M_A = M_{R/J}$ is injective. For some generalizations of this, see Ex. 4.29$'$. For the "quasi-injective" analogues of this and the last exercise, see Ex. 6.27A (and the ensuing *Comment*).

Ex. 3.30. Let $S = R[X]$, where X is any commuting set of indeterminates over R. For any essential right ideal $\mathfrak{A} \subseteq_e R_R$, show that $\mathfrak{A}[X] \subseteq_e S_S$. What if $S = R\langle Y\rangle$ where Y is a noncommuting set of indeterminates?

Solution. For $f(X) \neq 0$ in S, it suffices to show that $f(X)r \in \mathfrak{A}[X]\backslash\{0\}$ for some $r \in R$. Let $f(X) = b_1\beta_1 + \cdots + b_n\beta_n$ where $b_i \in R\backslash\{0\}$ and the β_i's are distinct monomials in the variables in X. Pick $r_1 \in R$ such that $b_1r_1 \in \mathfrak{A}\backslash\{0\}$ Then $f(X)\,r_1 = b_1r_1\beta_1 + g(X)$ where

$$g(X) = b_2r_1\beta_2 + \cdots + b_nr_1\beta_n.$$

If $g(X) = 0$, then we already have $f(X)r_1 = b_1r_1\beta_1 \in \mathfrak{A}[X] \setminus \{0\}$. Otherwise, we may assume, by invoking an inductive hypothesis, that there exists $r' \in R$ such that $g(X)\,r' \in \mathfrak{A}[X] \setminus \{0\}$. For $r = r_1r' \in R$, we have now

$$f(X)\,r = (b_1r_1\beta_1 + g(X))\,r' = b_1r_1r'\beta_1 + g(X)\,r' \in \mathfrak{A}[X] \setminus \{0\},$$

as desired.

The argument above clearly did not depend on the fact that the variables in X commute with one another. Therefore, in the case $S = R\langle Y\rangle$, the same argument yields $\mathfrak{A}\langle Y\rangle \subseteq_e S_S$.

Ex. 3.31. ("Schröder-Bernstein for Injectives") Let A, B be injective R-modules that can be embedded in each other. Show that $A \cong B$.

Solution. Since B is injective, we may assume that $A = B \oplus X$ and that there exists a monomorphism $f : A \to B$. Note first that if

$$x_0 + f(x_1) + \cdots + f^n(x_n) = 0$$

where $x_i \in X$, then all $x_i = 0$. In fact, $x_0 \in \text{im}(f) \subseteq B$ implies $x_0 = 0$, and so

$$x_1 + f(x_2) + \cdots + f^{n-1}(x_n) = 0$$

(since f is a monomorphism). By induction, we see that all $x_i = 0$. Therefore, we have

$$C := X \oplus f(X) \oplus f^2(X) \oplus \cdots \subseteq A.$$

Let $E = E(f(C)) \subseteq B$, and write $B = E \oplus Y$. Since $C = X \oplus f(C)$,

$$E(C) = E(X \oplus f(C)) \cong E(X) \oplus E(f(C)) = X \oplus E.$$

On the other hand, $E(C) \cong E(f(C)) = E$, so $X \oplus E \cong E$. From this, we deduce that

$$A = X \oplus B = X \oplus E \oplus Y \cong E \oplus Y = B.$$

Comment. This exercise was proposed as a problem in Carl Faith's ring theory seminar at Rutgers University around 1964. R. Bumby's solution to the problem, using ideas similar to those in one of the proofs of the set-theoretic Schröder-Bernstein Theorem, appeared in Archiv der Math. **16**(1965), 184–185. This paper also contained the refinement in the following exercise.

Ex. 3.32. Suppose A, B are R-modules that can be embedded in each other. Show that $E(A) \cong E(B)$, but that we may have $A \ncong B$.

Solution. Fix an embedding $f : A \to B$. Then $f(A) \subseteq B \subseteq E(B)$, so $E(B)$ contains a copy of $E(f(A)) \cong E(A)$. By symmetry, $E(A)$ also contains a copy of $E(B)$. Since $E(A)$, $E(B)$ are *injective*, it follows from Exercise 31 that $E(A) \cong E(B)$.

In general, we may not have $A \cong B$. For instance, let $B = \mathbb{Z}_4 \oplus \mathbb{Z}_4 \oplus \cdots$ and $A = \mathbb{Z}_2 \oplus B$ over the ring $\mathbb{Z}$. Then B is a submodule of A, and A can be embedded into B by

$$a + (b_1, b_2, \dots) \mapsto (i(a), b_1, b_2, \dots) \quad (a \in \mathbb{Z}_2,\ b_i \in \mathbb{Z}_4),$$

where i is the embedding of $\mathbb{Z}_2$ into $\mathbb{Z}_4$. Every element of order 2 in B is divisible by 2, but the same is not true in A. Therefore, we have $A \ncong B$.

In this example, neither A nor B is injective (as a $\mathbb{Z}$-module). It is just as easy to produce an example where one of A, B is injective. Just take

$$B = \mathbb{Q}/\mathbb{Z} \oplus \mathbb{Q}/\mathbb{Z} \oplus \cdots \quad \text{and} \quad A = \mathbb{Z}_2 \oplus B,$$

and the same argument will work (since $\mathbb{Z}_2$ embeds into $\mathbb{Q}/\mathbb{Z}$). Here B is injective and A is not, so in particular, $A \not\cong B$. (In both counterexamples, it is easy to see directly that $E(A) \cong E(B)$.) Finitely generated examples are also easy to find : for instance, over the commutative domain $R = \mathbb{Q}[x, y]$, we can take $A = R$ and $B = xR + yR \subseteq A$. Clearly, A can be embedded in B, but $A \not\cong B$ as R-modules.

Ex. 3.33. Let $J = \operatorname{rad} R$, where R is a semilocal ring (i.e. R/J is semisimple). Let V_R be a semisimple module, and $E = E(V)$. Show that there is an R-isomorphism $E/V \cong \operatorname{Hom}_R(J, E)$. (Here, the right R-action on $\operatorname{Hom}_R(J, E)$ comes from the left R-action on J.) If, moreover, $J^2 = 0$, show that $E/V \cong \operatorname{Hom}_R(J, V)$.

Solution. Consider the map $\varphi : E \to \operatorname{Hom}_R(J, E)$ defined by $\varphi(e)(j) = ej$ for $e \in E$ and $j \in J$. It is straightforward to check that φ is a right R-module homomorphism (with the right R-module structure on $\operatorname{Hom}_R(J, E)$ as described above). Since E_R is an injective R-module, the map φ is *onto*. We claim that $\ker(\varphi) = V$. In fact, since V is semisimple, $VJ = 0$ so $\varphi(V) = 0$. On the other hand, if $e \in \ker(\varphi)$, then $0 = eJ = (eR)J$, so eR may be viewed as an R/J-module. Since R/J is a semisimple ring, eR is a semisimple R-module, whence $eR \subseteq V$ (as $V \subseteq_e E$). This shows that $e \in V$, so we have shown that $\ker(\varphi) = V$. Therefore, φ induces an isomorphism $E/V \cong \operatorname{Hom}_R(J, E)$.

Now assume also $J^2 = 0$. Then, as above, we see that J_R is a semisimple module. For any $f \in \operatorname{Hom}_R(J, E)$, $f(J)$ is also semisimple, and so $f(J) \subseteq V$(as $V \subseteq_e E$). This shows that

$$\operatorname{Hom}_R(J, V) = \operatorname{Hom}_R(J, E) \cong E/V.$$

Ex. 3.34. (Big Injective Hulls over Artinian Rings) Let $A \subseteq B$ be division rings such that $\dim({}_AB) < \infty$ but $\dim(B_A) = \infty$. (Such pairs of division rings were first constructed by P.M. Cohn, in answer to a question of E. Artin.) Let $R = \begin{pmatrix} A & B \\ 0 & B \end{pmatrix}$. By *FC*-(1.22), R is an artinian ring. Since $J = \operatorname{rad} R = \begin{pmatrix} 0 & B \\ 0 & 0 \end{pmatrix}$ with $R/J \cong A \times B$, the two simple right R-modules V', V may be thought of as A and B respectively, with R acting via the projection $R \to A \times B$.

(1) Show that V' is injective.
(2) Determine the quotient module $E(V)/V$ and show that $E(V)_R$ is not finitely generated.

Solution. Since $J^2 = 0$, the last conclusion of Exercise 32 applies.

(1) Since $E(V')/V' \cong \mathrm{Hom}_R(J, V')$, it suffices to show that any $f \in \mathrm{Hom}_R(J,\ V')$ is zero. This follows since, for any $x \in B$:

$$f\begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix} = f\left(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & x \end{pmatrix}\right) \in V' \cdot \begin{pmatrix} 0 & 0 \\ 0 & x \end{pmatrix} = 0.$$

(2) Since $E(V)/V \cong \mathrm{Hom}_R(J,\ V)$, it suffices to compute the latter. Define a map $\theta :\ B \to \mathrm{Hom}_R(J,\ V)$ by $\theta(b) = g$ where

$$g\begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix} = bx \in B\ =\ V.$$

Clearly, $g = 0$ only if $b = 0$, so θ is injective. For any $f \in \mathrm{Hom}_R(J,\ V)$, let $b := f\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. Then

$$f\begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix} = f\left(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & x \end{pmatrix}\right) = bx \quad (\forall x \in B),$$

so $f = \theta(b)$. This shows that θ is a group isomorphism. For any $r = \begin{pmatrix} a & y \\ 0 & z \end{pmatrix} \in R$, note that

$$(fr)\begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix} = f\left(\begin{pmatrix} a & y \\ 0 & z \end{pmatrix}\begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix}\right) = f\begin{pmatrix} 0 & ax \\ 0 & 0 \end{pmatrix} = bax.$$

Thus, θ is an R-isomorphism $\mathrm{Hom}_R(J,\ V) \cong B_A$, where B_A is viewed as an R-module via the projection $R \to A$. Since $\dim(B_A) = \infty$, $E(V)/V$ is not finitely generated as an R-module, and so $E(V)$ is not finitely generated as an R-module.

Comment. The result of this Exercise and the approach used herein come from the paper of A. Rosenberg and D. Zelinsky, "On the finiteness of injective hulls," Math. Zeit. **70**(1959), 372–380. Right artinian rings R with the property that $E(V)$ is f.g. for every simple right R-module V have a special significance. In fact, in *LMR*-(19.74), it is shown that these are precisely the right artinian rings which admit a Morita duality (with some other ring S).

Ex. 3.35. Over a right noetherian, right self-injective ring R, show that any projective module P_R is injective.

Solution. Take a suitable module Q_R such that $F := P \oplus Q$ is free. Since R is right noetherian and R_R is injective, F (being a direct sum of copies of R_R) is also injective, by *LMR*-(3.46), and hence P is injective.

Comment. A ring R satisfying the hypotheses of this exercise is called a *quasi-Frobenius* (QF) *ring.* Much more information about such a ring and its modules can be found in *LMR*-§§15-16. For instance, injective right modules over R are also projective, and R turns out to be an artinian self-injective ring.

Ex. 3.36. *True or False*: Let $R \subseteq S$ be rings such that $R \subseteq_e S_R$, and S_S is injective. Then $S = E(R_R)$.

Solution. The conclusion $S = E(R_R)$ is not true in general since S_R may not be an injective R-module. To construct a counterexample, let k be a field, and $S = k[t]$ with the relation $t^4 = 0$. Since S is a factor ring of the principal ideal domain $k[x]$, S_S is injective by *LMR*-(3.12). Let R be the subring $k \oplus kt^2 \oplus kt^3$. Then $R \subseteq_e S_R$. To see this, consider any

$$f = a + bt + ct^2 + dt^3 \in S \backslash R,$$

where $a, b, c, d \in k$. Then $b \neq 0$, and so $ft^2 = at^2 + bt^3 \in R \backslash \{0\}$. We finish by checking that S_R *is not injective.* Consider the ideal $\mathfrak{A} = kt^3$ and the map $\varphi : \mathfrak{A} \to S$ given by $\varphi(dt^3) = dt^2$ for every $d \in k$. This is clearly an R-module homomorphism. If S_R was injective, there would exist $g \in R$ such that $\varphi(t^3) = gt^3$. But $gt^3 \in kt^3$ (since kt^3 is an ideal in S), so now $t^2 \in kt^3$, a contradiction.

Comment. The k-algebra R here is actually isomorphic to $A = k[u, v]$ with the relations $u^2 = v^2 = uv = 0$. (An explicit isomorphism φ from A to R is given by

$$\varphi(a + cu + dv) = a + ct^2 + dt^3,$$

for any $a, c, d \in k$.) Now the injective hull of A_A was computed in *LMR*-(3.69). Using this computation, and identifying R with A, we see that $\dim_k E(R_R) = 6$. Thus, although the ring S is self-injective, it is "too small" to be the injective hull of R_R.

For the genesis of the example $R \subseteq S$ used above, see the *Comment* following the solution to Ex. 13.24.

In some sense, the statement given in this exercise is "not too far" from being true. Most notably, when the ring R is assumed to be "right nonsingular," then this statement becomes true, as a consequence of a theorem of R. E. Johnson: see *LMR*-(13.39).

Ex. 3.37. (Douglas–Farahat) Let M be an additive abelian group, and let $R = \text{End}_{\mathbb{Z}}(M)$ (operating on the left of M). Show that ${}_RM$ is a projective module in case (1) M is a f.g. abelian group, or (2) $nM = 0$ for some positive integer n.

Solution. Case (1) is already handled in part in *LMR*-(2.12B). In order not to repeat the explanations there, let us try to give a somewhat different formulation here.

First note that, for any abelian group X, $\text{Hom}_{\mathbb{Z}}(X, M)$ is a left R-module since $M = {}_RM$. Of course, for $X = M$, this gives back the left regular module ${}_RR$. Let us make the following observation.

Lemma. *Suppose $M = X \oplus Y$ where X, Y are subgroups of M. Then the left R-module* $\mathrm{Hom}_{\mathbb{Z}}(X, M)$ *is isomorphic to a direct summand of* ${}_RR$.

Proof. Let $i: X \to M$ and $j: Y \to M$ be the inclusion maps. The natural group isomorphism

$$\varphi: \mathrm{Hom}_{\mathbb{Z}}(M, M) \longrightarrow \mathrm{Hom}_{\mathbb{Z}}(X, M) \oplus \mathrm{Hom}_{\mathbb{Z}}(Y, M)$$

is given by $\varphi(f) = (fi, fj)$. The map φ is a left R-module homomorphism since, for any $g \in R$,

$$\begin{aligned}\varphi(gt) &= ((gf)i,\ (gf)j) = (g(fi),\ g(fj))\\ &= g(fi,\ fj) = g\varphi(f).\end{aligned}$$

Therefore, $\mathrm{Hom}_{\mathbb{Z}}(X, M)$ is isomorphic, as a left R-module, to a direct summand of $\mathrm{Hom}_{\mathbb{Z}}(M, M) = {}_RR$. □

To solve the exercise (independently of *LMR*-(2.12B)), we go into the following two cases.

Case 1. M is a f.g. abelian group. We work first in the case where M has rank ≥ 1. Here we can write $M = \mathbb{Z} \oplus Y$. By the Lemma, ${}_RM \cong \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}, M)$ is isomorphic to a direct summand of ${}_RR$, so ${}_RM$ is a projective R-module. If rank $M = 0$, then M is a finite torsion group. This case is clearly subsumed in the case below.

Case 2. $nM = 0$. We may assume that n is chosen least. Let $n = p_1^{r_1} \cdots p_k^{r_k}$ be the prime factorization of n, and let M_i be the p_i-primary component of M. Then M_i has exponent $p_i^{r_i}$ so by Exercise 18, $M_i = X_i \oplus Y_i$ where $X_i \cong \mathbb{Z}/p_i^{r_i}\mathbb{Z}$, and Y_i is a suitable subgroup of M_i. Then

$$M = M_1 \oplus \cdots \oplus M_k = X \oplus Y,$$

where $Y = Y_1 \oplus \cdots \oplus Y_k$, and $X = X_1 \oplus \cdots \oplus X_k \cong \mathbb{Z}/n\mathbb{Z}$. *We claim again that* $\mathrm{Hom}_{\mathbb{Z}}(X, M) \cong M$ *as left R-modules.* To see this, let x be a generator of the cyclic group X, and define $\psi: \mathrm{Hom}_{\mathbb{Z}}(X, M) \to M$ by $\psi(g) = g(x)$ for any $g \in \mathrm{Hom}_{\mathbb{Z}}(X, M)$. It is easy to verify that ψ is an R-homomorphism. If $\psi(g) = 0$, then $g(x) = 0$ and so $g = 0$. This shows that ψ is *one-one.* For any $z \in M$, we have $nz = 0$, so there is a homomorphism $g: X \to M$ defined by $g(x) = z$. Then $\psi(g) = g(x) = z$ shows that ψ is *onto.* This proves the claim, and the Lemma shows that ${}_RM$ is isomorphic to a direct summand of ${}_RR$. Hence ${}_RM$ is again a projective R-module.

Comment. This exercise is a result of A. J. Douglas and H. K. Farahat from their paper "The homological dimension of an abelian group as a module over its ring of endomorphisms," Monatshefte Math. **69**(1965), 294–305. The case where M is a p-primary abelian group was independently noted in F. Richman and E. A. Walker, "Primary abelian groups as modules over their endomorphism rings," Math. Zeit. **89**(1965), 77–81. This paper also offered several criteria for a p-primary abelian group M to be projective as a left module over $R = \text{End}_{\mathbb{Z}}(M)$. This is certainly not always the case. For instance, if M is the Prüfer p-group $\varinjlim \mathbb{Z}/p^n\mathbb{Z}$, then $R = \varprojlim \mathbb{Z}/p^n\mathbb{Z}$ is the ring of p-adic integers. This is a discrete valuation ring, so $\text{pd}_R(M) \leq 1$ by *LMR*-(5.14), where "pd" denotes "projective dimension". Here ${}_RM$ is *not* R-projective since it cannot be embedded in a free R-module (the latter being torsion-free). Therefore, we have $\text{pd}_R(M) = 1$.

In their 1965 paper referred to above, Douglas and Farahat asked for the possible values of $\text{pd}_R(M)$ where M is any abelian group. They showed that for any *torsion* group M, $\text{pd}_R(M) \leq 1$. In a later paper (Monatshefte Math. **76**(1972), 109–111), they showed that, for any *divisible* M, one also has $\text{pd}_R(M) \leq 1$, with strict inequality iff M is torsion-free. In the review of this paper in *Math. Reviews*, E. A. Walker observed that the direct sum M of the Prüfer p-group with the additive group of the p-adic integers satisfies $\text{pd}_R M = 2$. In another sequel to their paper (Monatshefte Math. **80**(1975), 37–44), Douglas and Farahat constructed abelian groups M with $\text{pd}_R M = \infty$.

Ex. 3.38. Use the fact that injective modules are divisible to prove the following: Let $E = E(R/\mathfrak{A})$ where $\mathfrak{A} \subsetneq R$ is a left ideal, and let $s \in R$. If $sE = 0$, then $t\,s = 0$ for some $t \in R \backslash \mathfrak{A}$. Deduce that, if $R \backslash \mathfrak{A}$ consists of non 0-divisors. (e.g. R is a domain, or R is a local ring with maximal ideal $\mathfrak{A}$), then E is a faithful R-module.

Solution. Assume, on the contrary, that $\text{ann}_l(s) \subseteq \mathfrak{A}$. Consider the nonzero element $a = 1 + \mathfrak{A} \in R/\mathfrak{A} \subseteq E$. For any $x \in R$, we have

$$xs = 0 \implies x \in \mathfrak{A} \implies xa = x \cdot (1 + \mathfrak{A}) = 0.$$

Since E is injective and hence divisible, the above implies that $a = se$ for some $e \in E$. But then $a \in sE = 0$, a contradiction.

The last conclusion of the exercise is now clear. Note that this conclusion generalizes the faithfulness of the "standard module" (injective hull of the residue class field) of a local ring.

Ex. 3.39. Let $(R, \mathfrak{m})$ be a commutative noetherian local ring. Use the faithfulness of $E(R/\mathfrak{m})$ to show that $\bigcap_{n=0}^{\infty} \mathfrak{m}^n = 0$.

Solution. Let $E = E(R/\mathfrak{m})$ and $E_n = \text{ann}^E(\mathfrak{m}^n)$. By *LMR*-(3.76), we have $\text{ann}^R(E_n) = \mathfrak{m}^n$, and by *LMR*-(3.78), $\bigcup_{n=0}^{\infty} E_n = E$. Therefore,

$$\bigcap_{n=0}^{\infty} \mathfrak{m}^n = \bigcap_{n=0}^{\infty} \text{ann}^R(E_n) = \text{ann}^R(E) = 0$$

by the faithfulness of E.

Comment. The fact that $\bigcap_{n=0}^{\infty} \mathfrak{m}^n = 0$ in a commutative noetherian local ring $(R, \mathfrak{m})$ is a special case of Krull's Intersection Theorem: see Theorem 74 in Kaplansky's "Commutative Rings," University of Chicago Press, 1974.

The following exercises, 40A *through* 40G, *collect a few basic facts about associated primes and primary decompositions, mostly in a commutative setting. Some of these facts will prove to be useful in future chapters when we consider the case of commutative rings.*

Ex. 3.40A. Let R be a commutative noetherian ring, and let $\mathfrak{q}$ be a meet-irreducible ideal in R. By a classical theorem of Emmy Noether, $\mathfrak{q}$ must be a primary ideal,(*) say with radical $\mathfrak{p}$. Show that $E(R/\mathfrak{q}) \cong E(R/\mathfrak{p})$. Does this isomorphism still hold if $\mathfrak{q}$ is only assumed to be a primary ideal?

Solution. Since $\mathfrak{p}$ is a finitely generated ideal, $\mathfrak{p}^n \subseteq \mathfrak{q}$ for some n, which we may choose to be smallest. If $n = 1$, we have $\mathfrak{p} = \mathfrak{q}$, and there is no problem. So assume $n > 1$. Take $b \in \mathfrak{p}^{n-1} \backslash \mathfrak{q}$. Viewing $\bar{b}$ as a (nonzero) element in $R/\mathfrak{q} \subseteq E(R/\mathfrak{q})$, we have $\mathfrak{p} \cdot \bar{b} = 0$ (since $\mathfrak{p}^n \subseteq \mathfrak{q}$). Conversely, if $r \cdot \bar{b} = 0 \in R/\mathfrak{q}$, then $rb \in \mathfrak{q}$, and so $b \notin \mathfrak{q}$ forces $r \in \mathfrak{p}$. This shows that $\text{ann}^R(\bar{b}) = \mathfrak{p}$. In particular, $R \cdot \bar{b} \cong R/\mathfrak{p}$ as R-modules. Since $\mathfrak{q}$ is meet-irreducible, $E(R/\mathfrak{q})$ is indecomposable by *LMR*-(3.52). From this, it follows that $E(R/\mathfrak{q}) = E(R \cdot \bar{b}) \cong E(R/\mathfrak{p})$.

If $\mathfrak{q}$ is only given to be a primary ideal of R, with radical $\mathfrak{p}$, the isomorphism $E(R/\mathfrak{q}) \cong E(R/\mathfrak{p})$ need not follow. In fact, if we let $R = \mathbb{Q}[u, v]$ with relations $u^2 = v^2 = uv = 0$ and take the standard example $\mathfrak{q} = (0)$ which is primary but not meet-irreducible, then $\mathfrak{p} := \text{rad}(\mathfrak{q}) = (u, v)$ and $E(R/\mathfrak{p})$ is indecomposable, but $E(R/\mathfrak{q}) = E(R)$ is decomposable since R_R is not uniform. Thus, $E(R/\mathfrak{q}) \ncong E(R/\mathfrak{p})$. In fact, we have here

$$E(R/\mathfrak{q}) \cong E(R/\mathfrak{p}) \oplus E(R/\mathfrak{p}),$$

by the computations in *LMR*-(3.69).

Ex. 3.40B. Let I be an ideal in a commutative noetherian ring R. Show that:

(1) I is primary iff $|\text{Ass}(R/I)| = 1$.
(2) By the Lasker–Noether Theorem (which we assume), there exists an (irredundant) primary decomposition $I = \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_n$, where the $\mathfrak{q}_i$'s are primary ideals with distinct radicals $\mathfrak{p}_i$. ("Irredundant" here means I is not the intersection of a smaller subset of the $\mathfrak{q}_i$'s.) Show that

$$\text{Ass}(R/I) = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_n\}.$$

(3) Show that any prime minimal over I lies in $\text{Ass}(R/I)$. Conclude from this that the minimal members in $\text{Ass}(R/I)$ (with respect to inclusion) are exactly the primes in R that are minimal over I.
(4) Show that I is a radical ideal (i.e., $I = \sqrt{I}$) iff $\mathfrak{q}_i = \mathfrak{p}_i$ for all i. In this case, show that *each* $\mathfrak{p}_i$ is a minimal prime over I.

(*) An ideal $\mathfrak{q} \subsetneq R$ is said to be *primary* if, whenever $xy \in \mathfrak{q}$, we have either $x \in \mathfrak{q}$ or else $y^n \in \mathfrak{q}$ for some n. The radical of a primary ideal is always prime. Noether's Theorem referred to here is usually proved by the ascending chain argument in the proof of *LMR*-(3.78).

(5) Part (2) above shows that the primes $\mathfrak{p}_i$ arising as radicals of the $\mathfrak{q}_i$'s in a primary decomposition depend only on I, and not on the decomposition chosen. Give an example to show, however, that the $\mathfrak{q}_i$ themselves may not be uniquely determined.

Solution. We first prove the "only if" part in (1). Let I be primary, say with radical $\mathfrak{p}$. Then, for any nonzero element $\bar{a} \in R/I$, we have ann $(\bar{a}) \subseteq \mathfrak{p}$. In fact, any $r \in \text{ann}(\bar{a})$ satisfies $ra \in I$, and $a \notin I$ implies that $r \in \mathfrak{p}$. Now consider any $P \in \text{Ass}(R/I)$, say $P = \text{ann}(\bar{a})$ for some $\bar{a}$ as above (see *LMR*-(3.56)). Then $P \subseteq \mathfrak{p}$, and conversely, for $b \in \mathfrak{p}$, we have $b^n \in I \subseteq P$ (for some n), which implies that $b \in P$. Therefore $P = \mathfrak{p}$. Since $\text{Ass}(R/I) \neq \emptyset$ (by *LMR*-(3.58)), we have $\text{Ass}(R/I) = \{\mathfrak{p}\}$.

The "if" part of (1) will clearly follow from (2), so it suffices to prove (2). Using the given primary decomposition for I, we have an inclusion $R/I \hookrightarrow \bigoplus_{i=1}^n R/\mathfrak{q}_i$. Therefore, by *LMR*-(3.57)(4) (and what we have proved above):

$$\begin{aligned}\text{Ass}(R/I) &\subseteq \text{Ass}\left(\bigoplus\nolimits_{i=1}^n R/\mathfrak{q}_i\right) \\ &= \bigcup\nolimits_{i=1}^n \text{Ass}(R/\mathfrak{q}_i) \\ &= \{\mathfrak{p}_1, \ldots, \mathfrak{p}_n\}.\end{aligned}$$

We finish by proving, say, $\mathfrak{p}_1 \in \text{Ass}(R/I)$. Let $J = \mathfrak{q}_2 \cap \cdots \cap \mathfrak{q}_n \supsetneq I$. Since R is noetherian, $\mathfrak{p}_1^k \subseteq \mathfrak{q}_1$ for some k, so

$$\mathfrak{p}_1^k J \subseteq \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_n = I.$$

We may assume k is chosen smallest for such an inclusion to hold, so that there exists an element $a \in \mathfrak{p}_1^{k-1} J \setminus I$. Since $a \in J$, we must have $a \notin \mathfrak{q}_1$. Now for the nonzero element $\bar{a} \in R/I$, we can argue as before that $\text{ann}(\bar{a}) \subseteq \mathfrak{p}_1$. (If $ra \in I$, then $ra \in \mathfrak{q}_1$, and $a \notin \mathfrak{q}_1$ implies $r \in \mathfrak{p}_1$.) On the other hand, $\mathfrak{p}_1 a \subseteq \mathfrak{p}_1^k J \subseteq I$ implies that $\mathfrak{p}_1 \subseteq \text{ann}(\bar{a})$. Therefore, $\mathfrak{p}_1 = \text{ann}(\bar{a}) \in \text{Ass}\,(R/I)$, as desired.

To prove (3), let $\mathfrak{p}$ be a prime that is minimal over I. Since

$$\mathfrak{q}_1 \cdots \mathfrak{q}_n \subseteq \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_n = I \subseteq \mathfrak{p},$$

we have $\mathfrak{q}_i \subseteq \mathfrak{p}$ for some i. But then $\mathfrak{p}$ also contains the radical $\mathfrak{p}_i$ of $\mathfrak{q}_i$. Since $\mathfrak{p}$ is minimal over I, we must have $\mathfrak{p} = \mathfrak{p}_i \in \text{Ass}(R/I)$. Clearly, $\mathfrak{p}$ must be a minimal member in $\text{Ass}(R/I)$. Conversely, if $\mathfrak{p}'$ is any minimal member in Ass (R/I), then $\mathfrak{p}'$ contains a prime $\mathfrak{p}$ minimal over I. (This holds for any prime $\mathfrak{p}'$ containing I, by a standard Zorn's Lemma argument.) By the first part of (3), $\mathfrak{p} \in \text{Ass}(R/I)$, so our assumption on $\mathfrak{p}'$ implies that $\mathfrak{p}' = \mathfrak{p}$.

To prove (4), first assume $\mathfrak{q}_i = \mathfrak{p}_i$ for all i. Then $I = \bigcap \mathfrak{q}_i = \bigcap \mathfrak{p}_i$. Since each $\mathfrak{p}_i$ is a radical ideal, so is I. Conversely, assume that I is a radical ideal. Let $\mathfrak{p}_1, \ldots, \mathfrak{p}_t$ $(t \leq n)$ be the minimal primes over I. It is well-known that $\mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_t = I$ (see *ECRT*, Ex. 10.14). Since $I \subseteq \mathfrak{q}_i \subseteq \mathfrak{p}_i$, this implies that $\mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_t = I$. The irredundancy of the primary decomposition

$I = \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_n$ now implies that $t = n$. Finally, to prove that $\mathfrak{q}_i = \mathfrak{p}_i$, let $a \in \mathfrak{p}_i$. Since $\mathfrak{p}_i$ is minimal over I, we have $\bigcap_{j \neq i} \mathfrak{p}_j \not\subseteq \mathfrak{p}_i$, so there exists an element $b \in \left(\bigcap_{j \neq i} \mathfrak{p}_j\right) \setminus \mathfrak{p}_i$. Then

$$ab \in \mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_n = I \subseteq \mathfrak{q}_i,$$

so $b \notin \mathfrak{p}_i$ implies that $a \in \mathfrak{q}_i$. This shows that $\mathfrak{q}_i = \mathfrak{p}_i$, as desired.

Finally, for (5), let $I = (y^2, xy)$ in the polynomial ring $R = k[x, y]$ over a field k. Fix any $a \in k$, and let

$$\mathfrak{q}_1 = (y), \quad \mathfrak{q}_2 = (y^2, \ x + ay).$$

We claim that $I = \mathfrak{q}_1 \cap \mathfrak{q}_2$. The inclusion "$\subseteq$" is clear. For "$\supseteq$", note that if

$$g = f_1 y^2 + f_2(x + ay) \in \mathfrak{q}_1 \cap \mathfrak{q}_2 \quad (f_1, f_2 \in R),$$

then $f_2 x \in \mathfrak{q}_1 \Rightarrow f_2 = y f_3$ for some $f_3 \in R$, and hence $g \in I$. Here, $\mathfrak{q}_1$ is prime with radical $\mathfrak{p}_1 = \mathfrak{q}_1 = (y)$, and it is easy to show that $\mathfrak{q}_2$ has radical $\mathfrak{p}_2 = (x, y)$. Since this latter radical is a maximal ideal, it is well-known that $\mathfrak{q}_2$ is primary.(*) Therefore, for different choices of a, we have different (irredundant) primary decompositions $I = \mathfrak{q}_1 \cap \mathfrak{q}_2$. This example was noted in Footnote 12 of Emmy Noether's famous paper "Idealtheorie in Ringbereichen," Math. Ann. **83** (1921), 24–66.

Comment. Note that, in (5), $\mathfrak{p}_1 = (y)$ and $\mathfrak{p}_2 = (x, y)$ are associated primes of $M = R/I$ also on account of $\mathfrak{p}_1 = \text{ann}(\bar{x})$ and $\mathfrak{p}_2 = \text{ann}(\bar{y})$ for $\bar{x}$, $\bar{y} \in M$. In the language of commutative algebra, $\mathfrak{p}_1$ is an "isolated prime" of I (a prime minimal over I), while $\mathfrak{p}_2$ ($\supsetneq \mathfrak{p}_1$) is an "embedded prime" of I. In the general primary decomposition theory of ideals, it is known that the primary component $\mathfrak{q}_1$ corresponding to the isolated prime $\mathfrak{p}_1$ is uniquely determined by I, but "the" primary component $\mathfrak{q}_2$ corresponding to the embedded prime $\mathfrak{p}_2$ is not. See, e.g. Zariski-Samuel's "Commutative Algebra," vol. **1**. For more information about the ring R/I, see *LMR*-(12.23)(a).

Ex. 3.40C. Let M be a module over a commutative noetherian ring R with $\text{Ass}(M) = \{\mathfrak{p}\}$ for some prime $\mathfrak{p}$. Show that any $c \in M$ is killed by some power of $\mathfrak{p}$.

Solution. We may assume that $c \neq 0$. Let $I = \text{ann}(c)$. Then $R/I \cong R \cdot c \subseteq M$ implies that Ass $(R/I) = \{\mathfrak{p}\}$. By the last exercise, we see that I is primary with radical $\mathfrak{p}$. Taking a large n, we have $\mathfrak{p}^n \subseteq I$ (R being noetherian), so $\mathfrak{p}^n c = 0$.

Comment. The result in this exercise was proved for a *uniform* R-module M in *LMR*-(3.78). The present exercise is an improvement of this.

(*) In fact, $\mathfrak{q}_2$ is even meet-irreducible, as noted by Emmy Noether.

Ex. 3.40D. Let M_R be a f.g. module over a commutative noetherian ring R, and let $I = \text{ann}(M)$. In a commutative algebra monograph, it was claimed that $\text{Ass}(M)$ is the same as $\text{Ass}(R/I)$. Find a counterexample to this statement; then state (and prove) a corrected version thereof.

Solution. The statement is definitely false. For instance, if R is such that there exist prime ideals $\mathfrak{p}_1 \subsetneq \mathfrak{p}_2$ in R, then $M = (R/\mathfrak{p}_1) \oplus (R/\mathfrak{p}_2)$ has annihilator $I = \mathfrak{p}_1$ so $\text{Ass}(R/I) = \{\mathfrak{p}_1\}$. But by *LMR*-(3.57) (4), $\text{Ass}(M) = \{\mathfrak{p}_1, \mathfrak{p}_2\}$.

We claim, however, that $\text{Ass}(R/I) \subseteq \text{Ass}(M)$ in general (as in the example above). Take $\mathfrak{p} \in \text{Ass}(R/I)$. Then for some $a \in R \backslash I$, we have

$$\mathfrak{p} = (I : a) = \{r \in R : \ ar \in I\}.$$

Then $N := Ma \neq 0$ is a submodule with $\text{ann}(N) = \mathfrak{p}$ since, for $r \in R$, we have

$$Nr = 0 \iff Mar = 0 \iff ar \in I \iff r \in \mathfrak{p}.$$

Now M is a noetherian module, so $N = \sum_{i=1}^{n} x_i R$ for suitable $x_1, \ldots, x_n \in N$. Let $\mathfrak{A}_i = \text{ann}(x_i) \supseteq \mathfrak{p}$. Then

$$\mathfrak{A}_1 \cdots \mathfrak{A}_n \subseteq \mathfrak{A}_1 \cap \cdots \cap \mathfrak{A}_n = \text{ann}(N) = \mathfrak{p}$$

implies that $\mathfrak{A}_i \subseteq \mathfrak{p}$ for some i, and so

$$\mathfrak{p} = \mathfrak{A}_i = \text{ann}(x_i) \in \text{Ass}(M),$$

as desired.

We also note that, although $\text{Ass}(M)$ may not be equal to $\text{Ass}(R/I)$, the minimal members in these two sets (with respect to inclusion) are the same. First, any minimal member in $\text{Ass}(R/I)$ is a prime minimal over I by Exercise 3.40B(3). Since this prime belongs to $\text{Ass}(M)$, it is clearly a minimal member in $\text{Ass}(M)$. Conversely, consider any minimal member $\mathfrak{p}$ in $\text{Ass}(M)$. Since $\mathfrak{p} \supseteq I$, $\mathfrak{p}$ must contain a prime $\mathfrak{p}'$ minimal over I. But then $\mathfrak{p}' \in \text{Ass}(M)$ by the above, so $\mathfrak{p} = \mathfrak{p}'$, as desired.

Ex. 3.40E. Let M be a module over a commutative ring R.

(1) Show that any maximal member of the family $\{\text{ann}(m) : \ 0 \neq m \in M\}$ is in $\text{Ass}(M)$.
(2) If R is noetherian, show that $\bigcup\{\mathfrak{p} : \ \mathfrak{p} \in \text{Ass}(M)\}$ is precisely the set of elements of R that act as 0-divisors on M.
(3) Does (2) still hold if R is not noetherian?

Solution. (1) Let $\mathfrak{p} = \text{ann}(m)$ be a maximal member of the family given in (1), where $0 \neq m \in M$. Clearly, $\mathfrak{p} \neq R$. Let $a, b \in R$ be such that $ab \in \mathfrak{p}$ but $b \neq \mathfrak{p}$. Then $m' := bm \neq 0$ in M, and $\text{ann}(m') \supseteq \text{ann}(m)$ implies that $\text{ann}(m') = \text{ann}(m) = \mathfrak{p}$. Thus, $am' = abm = 0$ yields $a \in \mathfrak{p}$. This checks that $\mathfrak{p}$ is a prime ideal in R.

(2) Let zd $(M) \subseteq R$ be the set (including 0) of 0-divisors on M. Since every $\mathfrak{p} \in \mathrm{Ass}(M)$ is of the form $\mathrm{ann}(m)$ for some $m \neq 0$ in M (*LMR*-(3.56)), we have $\mathfrak{p} \subseteq \mathrm{zd}(M)$. Conversely, if r is an element in $\mathrm{zd}(M)\backslash\{0\}$,then $r \in \mathrm{ann}(m_0)$ for some $m_0 \in M\backslash\{0\}$. Since R is noetherian, the family of $\mathrm{ann}(m) \supseteq \mathrm{ann}(m_0)$ (where $0 \neq m \in M$) has a maximal member. By (1), this maximal member must be some $\mathfrak{p} \in \mathrm{Ass}(M)$, so we have $r \in \mathrm{ann}(m_0) \subseteq \mathfrak{p}$, as desired.

(3) The following example shows that the noetherian assumption on R in (2) is essential. Let R be the commutative local ring $\mathbb{Q}[x_1, x_2, \ldots]$ with the relations $x_1^2 = x_2^2 = \ldots = 0$. Then $\mathrm{Ass}(R) = \emptyset$ (by *LMR*-(3.57) (5)), so $\bigcup\{\mathfrak{p} : \mathfrak{p} \in \mathrm{Ass}(R)\} = \emptyset$. But zd (R) is the ideal generated by the x_i's.

Comment. It is worth pointing out that the maximal members of $\{\mathrm{ann}(m) : 0 \neq m \in M\}$ are also the same as the maximal members of $\{\mathrm{ann}(N) : 0 \neq N_R \subseteq M\}$, provided that R is commutative.

In the case where R is a commutative noetherian ring and M_R is a f.g. R-module, the next exercise implies that $|\mathrm{Ass}(M)| < \infty$. Therefore, zd (M) is a *finite* union of primes: this is one of the most basic results in the module theory over commutative noetherian rings.

Ex. 3.40F. Let M_R be a noetherian module over an *arbitrary* ring R.

(1) Show that $|\mathrm{Ass}(M)| < \infty$.
(2) If R has ACC on ideals and $M \neq (0)$, show that there exists a filtration $(0) = M_0 \subsetneq \cdots \subsetneq M_n = M$ such that each filtration factor M_i/M_{i-1} is a prime module.
(3) If R is commutative and noetherian, show that the filtration for M above may be chosen such that each $M_i/M_{i-1} \cong R/\mathfrak{p}_i$ for a suitable prime ideal $\mathfrak{p}_i \subset R$.

Solution. (1) Let $\mathcal{F}$ be the family of submodules $N \subseteq M$ for which $|\mathrm{Ass}(N)| < \infty$. This family is nonempty since $\{0\} \in \mathcal{F}$ (with $|\mathrm{Ass}(\{0\})| = 0$). Therefore, there is a maximal member $N_0 \in \mathcal{F}$ (with respect to inclusion), and *LMR*-(3.57)(3) gives

$$(*) \qquad \mathrm{Ass}(M) \subseteq \mathrm{Ass}(N_0) \cup \mathrm{Ass}(M/N_0).$$

If $\mathrm{Ass}(M/N_0) = \emptyset$ we are done, so we may assume that there exists a prime $\mathfrak{p} \in \mathrm{Ass}(M/N_0)$, say $\mathfrak{p} = \mathrm{ann}(N_1/N_0)$, where N_1/N_0 is a suitable prime submodule of M/N_0. Now $\mathrm{Ass}(N_1/N_0) = \{\mathfrak{p}\}$, and another application of *LMR*-(3.57)(3) shows that

$$\begin{aligned}\mathrm{Ass}(N_1) &\subseteq \mathrm{Ass}(N_0) \cup \mathrm{Ass}(N_1/N_0)\\ &= \mathrm{Ass}(N_0) \cup \{\mathfrak{p}\}.\end{aligned}$$

Therefore, $|\mathrm{Ass}(N_1)| < \infty$, and we have $N_1 \in \mathcal{F}$. This contradicts the maximality of N_0, as $N_0 \subsetneq N_1$.

(2) Here, let $\mathcal{S}$ be the family of nonzero submodules of M that have filtrations of the kind described in (2). Again, $\mathcal{S}$ is nonempty. For, if $\text{ann}(N')$ is a maximal member among all

$$\{\text{ann}(N) : 0 \neq N \subseteq M\},$$

then N' is clearly a prime module and $N' \in \mathcal{S}$. (The existence of the maximal annihilator $\text{ann}(N')$ follows from the ACC on ideals in R.) As in (1), let N_0 be a maximal member of $\mathcal{S}$. If $N_0 \neq M$, we can repeat the argument with M/N_0 to get a prime submodule $N_1/N_0 \subseteq M/N_0$. Then $N_1 \in \mathcal{S}$ and we would have a contradiction. Thus, we must have $M = N_0 \in \mathcal{S}$, as desired.

(3) This follows by repeating the argument in (2) for the family $\mathcal{S}'$ consisting of nonzero submodules (of M) that have filtrations of the kind described in (3). (That $\mathcal{S}'$ is nonempty follows from the fact that any associated prime $\mathfrak{p} \in \text{Ass}(M)$ has the form $\text{ann}(m)$ for some $m \in M$, for then $mR \cong R/\mathfrak{p}$ implies $mR \in \mathcal{S}'$.)

Ex. 3.40G. Let R be the Boolean ring $\prod_{i \in I} \mathbb{Z}_2$.

(1) Show that $\text{Ass}(R)$ consists of all prime ideals of the form $\mathfrak{p}_J = \prod_{i \in J} \mathbb{Z}_2$, where J is any subset of I such that $|I \backslash J| = 1$.

(2) If I is infinite, show that there exist primes of R that are *not* in $\text{Ass}(R)$.

Solution. (1) A typical $\mathfrak{p} \in \text{Ass}(R)$ is a prime ideal of the form $\text{ann}(r)$, where $r = (r_i)_{i \in I}$, $r_i \in \{0, 1\}$. Let $J = \{i \in I : r_i = 0\}$. Then clearly $\text{ann}(r) = \prod_{i \in J} \mathbb{Z}_2$. Since $R/\text{ann}(r) = \prod_{i \in I \backslash J} \mathbb{Z}_2$, $\text{ann}(r)$ is a prime ideal iff $|I \backslash J| = 1$. Therefore, the associated primes of R are precisely the $\mathfrak{p}_J$'s listed in the statement of the exercise.

(2) Assume now I is infinite. Consider the ideal $A = \bigoplus_{i \in I} \mathbb{Z}_2$ in R. Since $A \neq R$, there exists a maximal ideal $\mathfrak{m} \supseteq A$. Clearly, $\mathfrak{m}$ is *not* one of the associated primes $\mathfrak{p}_J$ of R listed in (1). (The existence of $\mathfrak{m}$, however, depends on Zorn's Lemma.)

Comment. In general, the prime (= maximal) ideals in the Boolean ring R are "classified" by the *ultrafilters* on the indexing set I. For any ultrafilter $\mathcal{F}$ on I, the corresponding prime ideal in R is

$$\mathfrak{p} = \{a = (a_i)_{i \in I} : \text{supp}(a) \notin \mathcal{F}\}.$$

If we take a principal ultrafilter, say all subsets of I containing a *fixed* index i_0, the corresponding prime in R is

$$\mathfrak{p} = \{a = (a_i)_{i \in I} : a_{i_0} = 0\} = \prod_{i \in J} \mathbb{Z}_2,$$

where $J = I \backslash \{i_0\}$. Therefore, the associated primes of R are precisely those arising from the principal ultrafilters on I, and the others arise from the nonprincipal ultrafilters.

If I is infinite, (1) shows that there are infinitely many associated primes. But these do not exhaust the prime ideals since in this case there always exist *nonprincipal* ultrafilters. For instance, consider the Fréchet filter $\mathcal{F}_0 = \{S \subseteq I : |I \backslash S| < \infty\}$. Clearly, any ultrafilter $\mathcal{F}$ refining $\mathcal{F}_0$ is not a principal ultrafilter. Thus, the prime ideal $\mathfrak{p}$ corresponding to $\mathcal{F}$ is *not* one of the primes in (1). Such $\mathfrak{p}$ will, indeed, be a maximal ideal of R containing $A = \bigoplus_{i \in I} \mathbb{Z}_2$. However, the existence of the ultrafilter $\mathcal{F}$ depends again on Zorn's Lemma.

For an application of this *Comment*, see the solution to Exercise 8.18.

Ex. 3.41. Let R be a commutative noetherian ring, and $E = E(R/\mathfrak{p})$ where $\mathfrak{p}$ is a prime ideal of R. For any ideal $J \subseteq \mathfrak{p}$, let $\bar{R} = R/J$, $\bar{\mathfrak{p}} = \mathfrak{p}/J$, and let $E' = \text{ann}^E(J) \subseteq E$. Show that E' is isomorphic to the injective hull of the $\bar{R}$-module $\bar{R}/\bar{\mathfrak{p}}$.

Solution. The key step here is to apply Exercise 3.28. Since E is injective over R, this exercise implies that E' is injective over $\bar{R} = R/J$. Now the copy of $R/\mathfrak{p}$ in E is certainly killed by $J \subseteq \mathfrak{p}$, so we may view $\bar{R}/\bar{\mathfrak{p}} \cong R/\mathfrak{p}$ as contained in E'. The fact that $R/\mathfrak{p} \subseteq_e E$ implies that $\bar{R}/\bar{\mathfrak{p}} \subseteq_e E'$, so it follows that $E' = E_{\bar{R}}(\bar{R}/\bar{\mathfrak{p}})$.

Ex. 3.42. (Vámos, Faith-Walker) Show that a ring R is right artinian iff every injective right R-module is a direct sum of injective hulls of simple R-modules.

Solution. First assume R is right artinian. Then by the Hopkins–Levitzki Theorem *FC*-(4.15) R is also right noetherian. By *LMR*-(3.48), any injective M_R can be expressed as $\bigoplus_i M_i$ where each M_i is injective and indecomposable. Finally, by *LMR*-(3.63), each M_i is the injective hull of some simple R-module.

Conversely, assume that every injective right R-module is a direct sum of injective hulls of simple right R-modules. If R is not right artinian, consider a descending $\mathfrak{A}_1 \supsetneq \mathfrak{A}_2 \supsetneq \cdots$ of right ideals $\mathfrak{A}_i$ and let $\mathfrak{A} = \bigcap \mathfrak{A}_i$. By assumption $E(R/\mathfrak{A}) = \bigoplus_{i=1}^n E(V_i)$ where each V_i is a simple R-module. (Note that the direct sum here is necessarily finite since $R/\mathfrak{A}$ is a cyclic module.) Let $V = V_1 \oplus \cdots \oplus V_n$. This is a module of finite length, so the descending chain

$$V \cap (\mathfrak{A}_1/\mathfrak{A}) \supseteq V \cap (\mathfrak{A}_2/\mathfrak{A}) \supseteq \cdots$$

must stabilize, say $V \cap (\mathfrak{A}_k/\mathfrak{A}) = V \cap (\mathfrak{A}_{k+1}/\mathfrak{A}) = \cdots$ for some k. Now, by *LMR*-(3.38), $V \subseteq_e E(R/\mathfrak{A})$, so $V \cap (\mathfrak{A}_k/\mathfrak{A}) \neq 0$. The last equation then shows $\bigcap_{i=1}^{\infty} (\mathfrak{A}_i/\mathfrak{A}) \neq 0$, a contradiction.

Comment. The second part of the solution above actually proved more: *A ring R is right artinian if (and only if) the injective hull of every cyclic module M_R is a direct sum of injective hulls of artinian modules.*

It is of interest to compare the characterization of a right artinian ring in this exercise with the following characterization of a right noetherian ring, proved in *LMR*-(3.48): *R is right noetherian iff every injective right module is a direct sum of indecomposables.* From these characterization statements, we can retrieve the well-known fact that a right antinian ring is always right noetherian.

The result in this exercise appeared as Theorem 1 in *P*. Vámos' paper, "The dual of the notion of 'finitely generated'", J. London Math. Soc. **43**(1968), 643–646. Vámos' proof is an application of his new notion of a "finitely cogenerated" module. He showed that a module M is artinian iff all quotients of M are finitely cogenerated. The result in this exercise follows by applying this characterization to the module R_R. The same result was obtained independently by C. Faith and E. A. Walker in "Direct sum representations of injective modules," J. Algebra **5**(1967), 203–221.

Ex. 3.43. Define a module I_R to be *fully divisible* if the following condition is satisfied: For any families $\{u_\alpha\} \subseteq I$ and $\{a_\alpha\} \subseteq R$ such that

$$(*) \qquad \sum a_\alpha x_\alpha = 0 \text{ (finite sum, } x_\alpha \in R) \implies \sum u_\alpha x_\alpha = 0,$$

there exists $v \in I$ such that $u_\alpha = v a_\alpha$ for all α. Show that I_R is fully divisible iff it is injective.

Solution. First suppose I_R is injective, and let $\{u_\alpha\}$, $\{a_\alpha\}$ be families in I and in R satisfying $(*)$. Then, for the right ideal $\mathfrak{A} = \sum a_\alpha R$, the rule $f(\sum a_\alpha x_\alpha) = \sum u_\alpha x_\alpha$ gives a well-defined R-homomorphism $f : \mathfrak{A} \to I$. Since I_R is injective, there exists $v \in I$ such that $f(a) = va$ for all $a \in \mathfrak{A}$. In particular, for each α, $u_\alpha = f(a_\alpha) = v a_\alpha$, so we have shown that I_R is fully divisible.

Conversely, assume that I is fully divisible. We check the injectivity of I by applying Baer's Test *LMR*-(3.7). Let $f \in \mathrm{Hom}_R(\mathfrak{A}, I)$ where $\mathfrak{A}$ is any right ideal. Fix a family $\{a_\alpha\}$ such that $\mathfrak{A} = \sum a_\alpha R$, and let $u_\alpha = f(a_\alpha) \in I$. Whenever $\sum a_\alpha x_\alpha = 0$ holds with $\{x_\alpha\} \subseteq R$, we have

$$0 = f\left(\sum a_\alpha x_\alpha\right) = \sum f(a_\alpha) x_\alpha = \sum u_\alpha x_\alpha \in I,$$

so $(*)$ holds. Since I_R is fully divisible, there exists $v \in I$ such that $u_\alpha = v a_\alpha$ for all α. Now f extends to a homomorphism $g : R_R \to I$ defined by $g(r) = vr$ (for all $r \in R$), as desired.

Ex. 3.44. *A* ring R is defined to be *right principally injective* if R_R is a principally injective (or equivalently, divisible) module.(*) For instance, a right self-injective ring is right principal injective.

(1) Show that a von Neumann regular ring is right (and left) principally injective.
(2) Give an example of a right principally injective ring R that is neither von Neumann regular nor right self-injective.

(*) A good alternative name for such R would be a *right divisible ring.*

Solution. (1) If R is von Neumann regular, we know that *any* module M_R is principally injective (see *LMR*-(3.18)). This applies, in particular, to R_R.

(2) According to *LMR*-(3.17), R is right principally injective iff $\text{ann}_l(\text{ann}_r(a)) = Ra$ for every $a \in R$. Using this criterion, it is easy to check that a direct product $R = \prod_{i\in I} A_i$ is right principally injective iff each factor A_i is. Now take A to be a von Neumann regular ring that is not right self-injective, and take B to be a right self-injective ring that is not von Neumann regular (e.g. $\mathbb{Q}[x]/(x^2)$). Then $R := A \times B$ is right principally injective since A, B are. But R is not von Neumann regular since B is not, and R is not right self-injective since A is not (by *LMR*-(3.11B)).

Comment. The trick used to construct the counterexample in (2) was characterized as "dirty" by one of my students. Those who agreed are encouraged to come up with a counterexample that is indecomposable as a ring.

Ex. 3.45. Prove the following for any right principally injective ring R:

(1) $a \in R$ has a left inverse iff a is not a left 0-divisor.
(2) R is Dedekind-finite iff any non left 0-divisor in R is a unit.

Solution. (1) is a generalization of Exercise 2. In fact, the solution given for that exercise (for the "if" part) carries over verbatim, since in that solution, we need only extend a certain R-homomorphism $aR \to R$ to an endomorphism of R_R.

(2) The "if" part is true for any R. In fact, if $ba = 1$, then $ax = 0 \Rightarrow x = xba = 0$. Hence a is not a left 0-divisor and by assumption $a \in \text{U}(R)$. For the "only if" part, assume R is Dedekind-finite and consider $a \in R$ that is not a left 0-divisor. By (1), a has a left inverse, so Dedekind-finitenes implies that $a \in \text{U}(R)$.

Comment. For any ring R, R being Dedekind-finite means that R_R is "hopfian" (any surjective endomorphism is an isomorphism), and "any non left 0-divisor is a unit" means that R_R is "cohopfian" (any injective endomorphism is an isomorphism). (See *ECRT*-(1.12), *ECRT*-(4.16).) In general, R_R cohopfian $\Rightarrow$ R_R hopfian. This exercise means essentially that *the converse is true for any right principally injective ring R.*

This and the following two exercises are taken from W. K. Nicholson and M. F. Yousif, "Principally injective rings," J. Algebra **174**(1995), 77–93.

Ex. 3.46. Let R be a right principally injective ring, and $f = f^2 \in R$. If $I \subseteq R$ is a right ideal isomorphic to the right ideal fR, show that $I = eR$ for some $e = e^2 \in R$.

Solution. Since $I \cong fR$, I is a principal right ideal of R. Fix an isomorphism $\varphi : I \to fR \subseteq R$, and say $\varphi(a) = f$, where $a \in I$. As R_R is a principally injective module, φ extends to an endomorphism of R_R, so we have $f = ra$ for some $r \in R$. From $\varphi(af) = ff = f$, we see that $a = af = ara$. Therefore, $e := ar \in I$ is an idempotent, and $a = ea \in eR$ implies that $I = eR$, as desired.

Comment. The conclusion in this exercise (that any right ideal isomorphic to fR is itself generated by an idempotent) is sometimes called (C_2): it is a property that holds only for certain rings (including the right principally injective rings). For more information on this, see Exercises 35–38 in §6.

Ex. 3.47. For any right principally injective ring R, prove the following:

(1) If we have a direct sum of principal left ideals $\bigoplus_{i=1}^n Ra_i$ in R, then any R-homomorphism $g: \sum_i a_iR \to R$ extends to an endomorphism of R_R.
(2) If $\bigoplus_{i=1}^n A_i$ is a direct sum of ideals in R, then for any left ideal B, $B \cap (\bigoplus_i A_i) = \bigoplus_i (B \cap A_i)$.

Solution. (1) By assumption, $g|a_iR$ is left multiplication by some b_i, and $g|(a_1 + \cdots + a_n)R$ is left multiplication by some b. Therefore,

$$\begin{aligned} b(a_1 + \cdots + a_n) &= g(a_1 + \cdots + a_n) \\ &= g(a_1) + \cdots + g(a_n) \\ &= b_1a_1 + \cdots + b_na_n. \end{aligned}$$

Since the sum $\sum Ra_i$ is supposed to be direct, we have $b_ia_i = ba_i$ for every i. It follows that g on $\sum_i a_iR$ is just left multiplication by $b \in R$, as desired.

(2) We need only prove the inclusion "$\subseteq$". Let $b \in B \cap (\bigoplus_i A_i)$, and write $b = a_1 + \cdots + a_n$ where $a_i \in A_i$. Clearly, the sums $\sum_i Ra_i$ and $\sum_i a_iR$ are both direct (since the A_i's are ideals). Let

$$\pi_j : \bigoplus_i a_iR \longrightarrow a_jR \subseteq R$$

be the j^{th} projection map (followed by the inclusion). By (1), π_j is left multiplication by some $r_j \in R$. Therefore, $a_j = \pi_j(b) = r_jb \in B \cap A_j$ (since B is a left ideal), so now $b \in \bigoplus_j (B \cap A_j)$, as desired.

Ex. 3.48. Let R be a commutative noetherian complete local ring, and E be its standard module. For any R-submodules A, $B \subseteq E$, show that any R-homomorphism φ from A to B is given by a multiplication by an element of R.

Solution. Viewing φ as a homomorphism from A into E, we can extend φ to some $\psi \in \text{End}_R(E)$. Now by Matlis' Theorem *LMR*-(3.84), $\text{End}_R(E) \cong R$ (since R is complete). Therefore, ψ is given by a multiplication by some element of R on E.

Ex. 3.49. Let $(R, \mathfrak{m})$ be a commutative noetherian local ring with $\mathfrak{m}$-adic completion $\tilde{R}$ and standard module $E = E(R/\mathfrak{m})$. (You may assume that $\tilde{R}$ is also a noetherian local ring.) Upon identifying $\text{End}_R(E)$ with $\tilde{R}$ by Matlis' Theorem *LMR*-(3.84), show that

(1) the $\tilde{R}$-module E can be identified with the standard module $\tilde{E}$ of $\tilde{R}$;
(2) the R-submodules of E are the same as its $\tilde{R}$-submodules.

Solution. Let $\varepsilon : R \to \varprojlim R/\mathfrak{m}^n = \tilde{R}$ be the natural map from R to $\tilde{R}$. Since $\bigcap_{n=0}^{\infty} \mathfrak{m}^n = 0$ by Exercise 39, ε is an injection. Identifying R with $\varepsilon(R)$, we may then view R as a subring of $\tilde{R}$. Let $\tilde{E}$ be the standard module of $\tilde{R}$. The crucial step is to show that *every R-submodule $X \subseteq \tilde{E}$ is an $\tilde{R}$-submodule.* To see this, let $\tilde{\mathfrak{m}}$ be the maximal ideal of $\tilde{R}$, and consider any $x \in X$ and any $\alpha \in \tilde{R}$. We may represent α in the form $(a_0, a_1, a_2, \ldots)$ where the $a_i \in R$ are such that

$$a_{i+1} \equiv a_i \pmod{\mathfrak{m}^{i+1}}$$

for every i. Then $\alpha \equiv a_i \pmod{\tilde{\mathfrak{m}}^{i+1}}$ for every i. Now by *LMR*-(3.78), $\tilde{\mathfrak{m}}^{n+1}x = 0$ for some n. Therefore, $(\alpha - a_n)x = 0$ and so $\alpha x = a_n x \in X$ since X is an R-module.

Let us identify $\tilde{R}/\tilde{\mathfrak{m}}$ with $R/\mathfrak{m}$. Since $R/\mathfrak{m} \subseteq_e E$ as R-modules, we also have $R/\mathfrak{m} \subseteq_e E$ as $\tilde{R}$-modules. Therefore, we may assume that E is embedded in the standard module $\tilde{E}$ of $\tilde{R}$. By the injectivity of E as an R-module, $\tilde{E} = E \oplus X$ for some R-submodule X of $\tilde{E}$. But then the last paragraph implies that X is an $\tilde{R}$-submodule of $\tilde{E}$. Since $\tilde{E}$ is an indecomposable $\tilde{R}$-module, we must have $X = 0$ and so $E = \tilde{E}$. This proves (1), and (2) follows from what we have proved about $\tilde{E}$.

Comment. It is interesting to note that the above method shows that $E = \tilde{E}$ without going through an argument for the injectivity of E as an $\tilde{R}$-module. The conclusion in (2) is also quite useful. In a later result in *LMR*-(19.56), it is shown that $\tilde{E}$ is an artinian $\tilde{R}$-module. Therefore, it follows that, even without a completeness assumption on R, *E is an artinian R-module.* A more general fact holding over any commutative noetherian ring R can be found in Ex. 19.8.

Ex. 3.50. In the notation of the last exercise, let T be an R-submodule of E (so that it is also an $\tilde{R}$-submodule). Show that $T = E$ iff T is a faithful $\tilde{R}$-module. (You may assume that $_RE$ is artinian.)

Solution. The "only if" part follows from the fact that $\tilde{R} \cong \text{End}_R(E)$, which is moved in *LMR*-(3.84). For the converse, assume that $T \subsetneq E$. Since $_RE$ is artinian, so is $_R(E/T)$. In particular, E/T contains a simple submodule P/T. Fix an isomorphism $P/T \to R/\mathfrak{m} \subseteq E$ and extend it to an R-homomorphism $\varphi : E/T \to E$. The composition $E \to E/T \xrightarrow{\varphi} E$ is then a nonzero element $\alpha \in \tilde{E}$ with $\alpha(T) = 0$. This shows that T is not a faithful $\tilde{R}$-module, as desired.

Comment. The above exercise was suggested to me by Craig Huneke, who also pointed out that $T = E$ will *not* follow in general if T is only assumed to be a faithful R-module.

Ex. 3.51. Let $A = k[x_1, \ldots, x_r]$ where k is a field. In *LMR*-§3J, it is shown that the A-module k (with trivial x_i-action for all i) has injective hull $T = k[x_1^{-1}, \ldots, x_r^{-1}]$, where the A-action on T is defined by usual multiplication, with the convention that after multiplication, any term $x_1^{a_1} \ldots x_r^{a_r}$ ($a_i \in \mathbb{Z}$) not lying in T is discarded. Show that T is isomorphic to a quotient of the A-module of Laurent polynomials $k[x_1^{\pm 1}, \ldots, x_r^{\pm 1}]$. In the case $r = 1$ (where we write $A = k[x]$), show that T is isomorphic to the x-primary component of the torsion A-module $k(x)/k[x]$.

Solution. Let L be the set of all Laurent monomials $\alpha = x_1^{a_1} \cdots x_r^{a_r}$ ($a_i \in \mathbb{Z}$), and

$$D = \{\alpha \in L: \text{ all } a_i \geq 0\}, \quad D^{-1} = \{\alpha \in L: \text{ all } a_i \leq 0\},$$
$$C = \{\alpha \in L: \text{ some } a_i > 0\}.$$

We denote the k-spans of these sets by kL, kD, kD^{-1} and kC. Thus,

$$kD = A, \quad kD^{-1} = T, \quad \text{and} \quad kL = k[x_1^{\pm 1}, \ldots, x_r^{\pm 1}].$$

Since $D \cdot C \subseteq C$, kC is an A-submodule of kL. From $L = D^{-1} \cup C$ (disjoint union), we have the k-span decomposition $kL = T \oplus kC$. The definition of T as an A-module given in the statement of the exercise shows that ${}_AT \cong kL/kC$.

In the case $r = 1$, write $x = x_1$. Here $kL = k[x, x^{-1}]$ and $kC = x\,k[x]$. Hence ${}_AT \cong k[x, x^{-1}]/xk[x]$. Multiplying numerator and denominator by x^{-1}, we see that

$${}_AT \cong k[x, x^{-1}]/k[x].$$

Since $k[x, x^{-1}] = \bigcup_{n \geq 0} x^{-n}k[x]$, the RHS above is just the x-primary component of the A-torsion module $k(x)/k[x]$. (This primary component, is, of course, well-known to be the injective hull $E(A/xA) = E(k)$ to begin with: see *LMR*-(3.63).)

Ex. 3.52. Let $A = k[x_1, \ldots, x_r]$, $P = k[[x_1, \ldots, x_r]]$, and $T = k[x_1^{-1}, \ldots, x_r^{-1}]$, where k is a field. As in *LMR*-§3J, there is a natural P-module structure on T extending the A-module structure described in the last exercise. Show that T is a faithful P-module, and use Exercises (49) and (50) above to give an alternative proof for the fact that $T = E_A(k) = E_P(k)$, where k denotes the A-module (resp. P-module) with trivial x_i-action for all i.

Solution. To show the faithfulness of ${}_PT$, let $0 \neq f \in P$. Of all the monomials appearing in f, let $\alpha = x_1^{a_1} \cdots x_r^{a_r}$ be such that $(a_1, \ldots, a_r)$ is smallest in the lexicographical ordering on $\mathbb{Z} \times \cdots \times \mathbb{Z}$. We are done if we can show that, for the inverted monomial $\alpha^{-1} \in T$, $f * \alpha^{-1} \neq 0 \in T$. To see this, note that, for any monomial $\beta = x_1^{b_1} \cdots x_r^{b_r} \neq \alpha$ appearing in f, we have, say $b_1 = a_1, \ldots, b_i = a_i$, and $b_{i+1} > a_{i+1}$ for some $i < r$, and therefore, under the P-action on T, $\beta * \alpha^{-1} = 0$ (since x_{i+1} appears with a positive exponent $b_{i+1} - a_{i+1}$ in $\beta\alpha^{-1}$). It follows that $f * \alpha^{-1} = c\alpha^* \alpha^{-1} = c$ for some $c \in k^*$ (the coefficient of α in f).

To prove the other statements in the exercise, let R be the localization of A at $(x_1, \ldots, x_r)$. We have the natural inclusions $A \subseteq R \subseteq P$, and

P is the completion of R. It is easy to see that $k \subseteq_e T$ as A-modules. Therefore, we may assume $k \subseteq T \subseteq E$ where $E := E_A(k)$. Then E is an R-module by *LMR*-(3.77), with endomorphism ring P by Matlis' Theorem *LMR*-(3.84). Since T is a faithful P-module, it follows from Exercise 50 that $T = E = E_A(k)$. By Exercise 49, E is also the standard module for P, so $T = E = E_P(k)$, as desired.

Comment. The above argument was shown to me by Craig Huneke. It provides a way to prove the injectivity of T as an A-module (resp. P-module) without using the Artin–Rees Lemma as in *LMR*-§3J.

Ex. 3.53. For any r-tuple $a = (a_1, \ldots, a_r)$ over a field k, let $\mathfrak{m}_a$ be the maximal ideal $(x_1 - a_1, \ldots, x_r - a_r)$ in the polynomial ring $A = k[x_1, \ldots, x_r]$. Let $k_a = A/\mathfrak{m}_a$, so that k_a is the A-module k on which each x_i acts as multiplication by a_i. Construct the injective hull $E_A(k_a)$.

Solution. Let us first review what happens when $a = 0$. Here, $\mathfrak{m}_0$ is the ideal $(x_1, \ldots, x_r) \subset A$, and we have a decomposition $A = A_0 \oplus A_1 \oplus \cdots$ where A_n is the space of homogeneous polynomials (in $x_1, \ldots, x_r$) of degree n. Let $M = \hat{A} = \text{Hom}_k(A,\ k)$ be the k-dual of A, which can be identified with $\prod_{n=0}^{\infty} \hat{A}_n$, where $\hat{A}_n$ means

$$\{f \in \hat{A}\ :\ f(A_i) = 0 \quad \text{for every} \quad i \neq n\}.$$

Then $\hat{A}_0 \cong k_0$ (the A-module k with trivial x_i-actions), and by the analysis in *LMR*-§ 3J,

$$\begin{aligned} E_A(k_0) \cong N :&= \{f \in \hat{A} :\ f(A_i) = 0 \text{ for large } i\} \\ &= \hat{A}_0 \oplus \hat{A}_1 \oplus \hat{A}_2 \oplus \cdots. \end{aligned}$$

To treat the case $a = (a_1, \ldots, a_r)$, let $y_i = x_i - a_i$ and note that

$$\mathfrak{m}_a = (y_i, \ldots, y_r) \subseteq A = k[y_1, \ldots, y_r].$$

The k-dual $\hat{A} = \text{Hom}_k(A,\ k)$ is "unchanged", but with respect to the new variables y_i, we have a similar decomposition $A = B_0 \oplus B_1 \oplus B_2 \oplus \cdots$ into spaces of homogeneous polynomials in the y_i's. The A-module k_a is now found as $\hat{B}_0$, which is *not* the same as $\hat{A}_0$ inside $\hat{A}$. Nevertheless, by what we said above (applied to $A = k[y_1, \ldots, y_r]$), we have

$$\begin{aligned} E_A(k_a) \cong N_a :&= \{f \in \hat{A} :\ f(B_i) = 0 \quad \text{for large } i\} \\ &= \hat{B}_0 \oplus \hat{B}_1 \oplus \hat{B}_2 \oplus \cdots. \end{aligned}$$

This describes the A-module $E_A(k_a)$ explicitly. Note that for different $a \in k^n$, the $E_A(k_a)$'s are mutually nonisomorphic injective A-submodules of the injective A-module $\hat{A}$.

Of course, by the theory developed in *LMR*-§3J, $E_A(k_a)$ can also be described as $k[y_1^{-1}, \ldots, y_r^{-1}]$ on which $A = k[y_1, \ldots, y_r]$ acts in a natural way. Note, however, that although $k[y_1^{-1}, \ldots, y_r^{-1}]$ is isomorphic to $k[x_1^{-1}, \ldots, x_r^{-1}]$ as k-algebras, the obvious k-algebra isomorphism given by

$y_i^{-1} \mapsto x_i^{-1}$ is *not* an A-module isomorphism. In fact, as we have remarked above, with varying $a \in k^n$, $k[y_1^{-1}, \ldots, y_r^{-1}]$ will give mutually different (injective) A-modules.

Ex. 3.54. Let $(A, \mathfrak{m}_A)$ be a commutative noetherian local ring, with (right) standard module ω_A. Let $(B, \mathfrak{m}_B)$ be a right artinian local ring that is a module-finite algebra over A such that $1_B \cdot \mathfrak{m}_A \subseteq \mathfrak{m}_B$. Show that the right standard module $\omega_B = E_B((B/\mathfrak{m}_B)_B)$ of B is given by $\mathrm{Hom}_A(B, \omega_A)$. [Here, B is viewed as a (B, A)-bimodule, and the right B-module structure on $\mathrm{Hom}_A(B, \omega_A)$ comes from the left B-structure on B.]

Solution. Since ω_A is an injective right A-module, the Injective Producing Lemma (*LMR*-(3.6B)) implies that $H := \mathrm{Hom}_A(B, \omega_A)$ is an injective right B-module. We are done if we can show that H is an essential extension of a copy of $(B/\mathfrak{m}_B)_B$. Let us consider the following B-submodule of H,

$$(1) \qquad S := \{\psi \in H : \ \psi(\mathfrak{m}_B) = 0\},$$

and write $k_A = A/m_A$, $k_B = B/m_B$. Then

$$(2) \qquad S \cong \mathrm{Hom}_A(k_B, \omega_A) = \mathrm{Hom}_A(k_B, k_A),$$

since the annihilator of $\mathfrak{m}_A$ in ω_A is the copy of k_A sitting in ω_A. Now the assumptions on B imply that k_B is a finite-dimensional vector space over k_A. Therefore, (2) above shows that $S \cong k_B$ as B-modules. We finish by checking that $S \subseteq_e H$. Let $0 \neq \varphi \in H$. Since B is right artinian, $\mathfrak{m}_B$ is nilpotent, so there exists a minimal $r \geq 1$ with $\varphi(\mathfrak{m}_B^r) = 0$. Then $\varphi(b) \neq 0$ for some $b \in \mathfrak{m}_B^{r-1}$. Therefore, $(\varphi \cdot b)(1) = \varphi(b) \neq 0$ implies that $\varphi \cdot b \neq 0$. On the other hand,

$$(\varphi \cdot b)(\mathfrak{m}_B) = \varphi(b\ \mathfrak{m}_B) \subseteq \varphi(\mathfrak{m}_B^r) = 0$$

yields $\varphi \cdot b \in S$, so $S \subseteq_e H$, as desired.

Ex. 3.55. Let R be a right noetherian ring, and $\mathfrak{p}$ be a prime ideal of R. If $R/\mathfrak{p}$ is a domain, show that $\mathfrak{p}$ is right meet-irreducible (that is, if A, B are right ideals such that $A \cap B = \mathfrak{p}$, then $A = \mathfrak{p}$ or $B = \mathfrak{p}$).

Solution. By passing to $R/\mathfrak{p}$ (which remains right noetherian), we may assume that $\mathfrak{p} = 0$, i.e. R is a domain. Assume for the time, that there exist *nonzero* right ideals A, $B \subseteq R$ with $A \cap B = 0$. Take nonzero elements $a \in A$ and $b \in B$. We derive a contradiction (to R being right noetherian) by showing that the following infinite sum

$$aR + baR + b^2aR + b^3aR + \cdots$$

is direct. Indeed, suppose $ar_0 + bar_1 + b^2ar_2 + \cdots + b^nar_n = 0$. Then $-ar_0 = b(ar_1 + bar_2 + \cdots)$ shows that $r_0 = 0$ and $ar_1 + bar_2 + \cdots = 0$. Repeating this argument, we see that all $r_i = 0$, as desired.

Comment. The converse of the exercise is true too, but is much deeper: its proof requires Goldie's Theorem in *LMR*-§11 (see *LMR*-(11.25)).

Chapter 2

Flat Modules and Homological Dimensions

§4. Flat Modules

A right R-module P_R is called *flat* if $P \otimes_R -$ is an exact functor on ${}_R\mathfrak{M}$, the category of left R-modules. Specifically, this requires that, if $A \to B$ is injective in ${}_R\mathfrak{M}$, then $P \otimes_R A \to P \otimes_R B$ is injective also. Projective modules are flat, but flat modules enjoy an important property not shared by projective modules: they are *closed w.r.t. direct limits.* In particular, a module is flat if all f.g. submodules are flat. Over $\mathbb{Z}$, the flat modules are just the torsion-free abelian groups.

While flat modules are generalizations of projective modules, they are also related to injective modules. If, for $P \in \mathfrak{M}_R$, we define $P' = \operatorname{Hom}_{\mathbb{Z}}(P, \mathbb{Q}/\mathbb{Z}) \in {}_R\mathfrak{M}$ (called the *character module* of P), then Lambek's Theorem (*LMR*-(4.9)) says that *P is flat in $\mathfrak{M}_R$ iff P' is injective in ${}_R\mathfrak{M}$.* It may happen that *all* modules in $\mathfrak{M}_R$ are flat; in fact, this is the case iff R is a von Neumann regular ring. In particular, over such a ring, *any* character module P' above is injective.

There are various tests for flatness, some of which are presented in *LMR*-§4. We shall recall a couple of them here. Let

$$0 \longrightarrow K \longrightarrow F \longrightarrow P \longrightarrow 0$$

be a fixed exact sequence in $\mathfrak{M}_R$, where F is flat. Then *P is flat iff $K \cap F\mathfrak{A} = K\mathfrak{A}$ for every left ideal $\mathfrak{A} \subseteq R$* (*LMR*-(4.14)). If F is assumed to be free, then *P is flat iff, for any $c \in K$, there exists $\theta \in \operatorname{Hom}_R(F, K)$ such that $\theta(c) = c$* (*LMR*-(4.23)). There are also two important "Equational Criteria" for flatness. Since their statements are more technical, we refer the reader to *LMR*-(4.24) here.

A module P_R is called *finitely related* (f.r.) if there is an exact sequence

$$0 \longrightarrow K \longrightarrow F \longrightarrow P \longrightarrow 0$$

in $\mathfrak{M}_R$ where F is free (of any rank) and K is f.g. If, in addition, the free module F is *also* f.g., then we say P is *finitely presented* (f.p.). In general, *a module P is f.p. iff it is both f.g. and f.r.*

The equational criteria for flatness can be reformulated in terms of f.p. modules: P_R is flat iff, for any f.p. module M_R, any R-homomorphism $\lambda: M \to P$ can be factored through a f.g. free module (*LMR*-(4.32)). This result can be used to prove the important theorem of Lazard and Govorov (*LMR*-(4.34)), which states that *a module P_R is flat iff it is a direct limit of f.g. free modules.*

Over certain rings, f.g. flat right modules turn out to be projective. Examples include: right noetherian rings, local rings, and domains satisfying the strong rank condition in §1. Exercise 21 adds semiperfect rings to this list.

In studying the flatness property of a direct product of flat modules, the coherent property of a ring comes to the fore. A ring R is said to be *left coherent* if every f.g. left ideal of R is f.p. (as a left R-module). The theorem of Chase (*LMR*-(4.47)) states that, over a ring R, *any direct product of flat right R-modules is flat iff R is left coherent.* Coherent left R-modules can be defined as well: these are f.g. modules in which every f.g. submodule is f.p. *A ring R is left coherent iff every f.p. left R-module is coherent* (*LMR*-(4.52)).

The notion of a torsionless module, due to Bass, is related to flatness. By definition, a module B_R is *torsionless* if, for any $b \neq 0$ in B, there exists a functional $f : B \to R_R$ such that $f(b) \neq 0$. An equivalent statement is that B can be embedded into some direct product $(R^I)_R$. We have the following theorem of Chase (*LMR*-(4.67)) characterizing the left semihereditary rings: *R is left semihereditary iff all torsionless right R-modules are flat, iff R is left coherent and all left* (*resp. right*) *ideals are flat.* This theorem leads to many different characterizations of Prüfer domains (*LMR*-(4.69)).

Faithfully flat modules are useful in ring theory and algebraic geometry for making certain "descent" arguments. By definition, P_R is *faithfully flat* if P is flat and, for any left R-module M, $P \otimes_R M = 0$ implies that $M = 0$. Given the flatness of P, the second condition may also be replaced by: $P\mathfrak{m} \neq P$ or any maximal left ideal $\mathfrak{m}$ of R. A faithfully flat module is both faithful and flat, though not conversely.

A ring homomorphism φ: $R \to S$ is called *right faithfully flat* if the module S_R is faithfully flat. In this case, φ must be injective, and we may refer to S as a (*right*) *faithfully flat extension* of R. For instance, if R, S are commutative rings, this amounts to S_R being flat and φ^* : Spec $S \to$ Spec R being onto. Some typical ways in which faithfully flat extensions can be used are given at the end of *LMR*-§4H.

The last major topic in §4 is pure exact sequences. A short exact sequence

$$\mathcal{E}: 0 \longrightarrow A \xrightarrow{\varphi} B \longrightarrow C \longrightarrow 0$$

in $\mathfrak{M}_R$ is called *pure* if $\mathcal{E} \otimes_R M$ is exact for every left module M. (In this case, we also say that $\varphi(A)$ is a *pure submodule* of B.) It turns out that $\mathcal{E}$ is pure iff it is the direct limit of a direct system of *split* short exact sequences

$$0 \longrightarrow A \longrightarrow B_i \longrightarrow C_i \longrightarrow 0$$

(*LMR*-(4.89)). The notion of pure exact sequences is related to flatness as follows: *a module C_R is flat iff every short exact sequence $0 \to A \to B \to C \to 0$ is pure* (*LMR*-(4.85)).

A pure submodule $A \subseteq B$ has the property that $A \cap B\mathfrak{A} = A\mathfrak{A}$ for any left ideal $\mathfrak{A} \subseteq R$, but the converse is not true in general. Over a commutative PID, however, the purity of $A \subseteq B$ does amount to its usual definition: $A \cap Br = Ar$ for any $r \in R$ (*LMR*-(4.93)). Finally, for any f.g. module B_R over a right noetherian ring, the pure submodules of B are simply its direct summands (*LMR*-(4.91)).

The exercises in this section cover many aspects of flatness, faithful flatness, purity, as well as facts about finitely presented modules. There are also various exercises on the notion of flatness in the commutative case (e.g. Exercises 13–18 and Exercises 34–37). We also introduce the notions of *locally split* monomorphism and epimorphisms in Ex. 38 and Ex. 39, and study these notions in some detail in the subsequent exercises due to Azumaya, Fieldhouse, Zimmermann-Huisgen and others.

Exercises for §4

Ex. 4.1. For any ring R, show that (a) every f.g. projective right R-module is f.p., and that (b) every f.p. right R-module is projective iff R is von Neumann regular.

Solution. (a) Let P_R be f.g. projective, say $P \oplus Q = R^n$ where $n < \infty$. Let π be the projection from R^n onto $Q \subseteq R^n$ with respect to this direct sum decomposition. Then the presentation $R^n \xrightarrow{\pi} R^n \to P \to 0$ shows that P is f.p.

(b) First assume R is von Neumann regular, and consider any right R-module P. Then P is flat by *LMR*-(4.21). If moreover P is f.p., then by *LMR*-(4.30), P is projective. Conversely, assume that every f.p. right R-module is projective. Consider any principal right ideal $aR \subseteq R$. Then R/aR is f.p., and therefore projective. This implies that the exact sequence

$$0 \longrightarrow aR \longrightarrow R \longrightarrow R/aR \longrightarrow 0$$

splits, so aR is a direct summand of R_R. Therefore, R is von Neumann regular.

Ex. 4.2. Prove the following slight generalization of *LMR*-(4.5): If every f.g. submodule P_0 of a module P_R is contained in a flat submodule P_1 of P, then P itself is flat.

Solution. By the Modified Flatness Test *LMR*-(4.12), it suffices to show that, for any left ideal $\mathfrak{A} \subseteq R$, any $x \in \ker(P \otimes_R \mathfrak{A} \to P)$ is zero. Now $x \in P \otimes_R \mathfrak{A}$ is the image of some $x_0 \in P_0 \otimes_R \mathfrak{A}$ for a suitable f.g. submodule $P_0 \subseteq P$. By assumption, $P_0 \subseteq P_1$ for some *flat* submodule $P_1 \subseteq P$. Therefore x_0 maps to an element in $\ker(P_1 \otimes_R \mathfrak{A} \to P_1) = 0$. It follows that x_0 maps to zero in $P \otimes_R \mathfrak{A}$; that is, $x = 0$.

Ex. 4.3. In a ring theory text, the following statement appeared: "A module is flat iff every f.g. submodule is flat." Give a counterexample to the "only if" part of this statement. (The "if" part is true by *LMR*-(4.4).)

Solution. Consider the case where R is a commutative ring. The module R_R is free and therefore flat. If the quoted statement was true, every f.g. ideal $I \subseteq R$ would be flat as an R-module. Take $R = k[x, y]$, where k is a field. We know from *LMR*-(4.19) that the ideal $I = xR + yR$ is not flat.

Ex. 4.4. In a ring theory text, the following statement appeared: "If $0 \to C \to Q \to P \to 0$ is exact with C and Q f.g., then P is f.p." Give a counter-example.

Solution. Let P be a module (over a suitable ring) that is f.g. but not f.p. Then $0 \to C \to Q \xrightarrow{f} P \to 0$ is exact with $C = 0$ and $Q = P$ both f.g. (and $f = \mathrm{Id}_P$), but P is not f.p. by choice.

Ex. 4.5. In a ring theory text, the following statement appeared: "For right R-modules $N \subseteq M$, if $N \cap Mr = Nr$ for every $r \in R$, then $N \cap M\mathfrak{A} = N\mathfrak{A}$ for every left ideal $\mathfrak{A} \subseteq R$." Give a counterexample.

Solution. The following commutative counterexample is due to G. Bergman. Let $R = k[x, y]$, where k is a field. Let $M = R^2$, and $N = (x, y) \cdot R \subseteq M$. Then $N \cap Mr = Nr$ for all $r \in R$. For, if $(f, g)r = (x, y)s$ where $s \in R$, then, assuming (as we may) $r \neq 0$, we can show by unique factorization that $f = xf_0$, $g = yg_0$ for suitable $f_0, g_0 \in R$. Now $f_0 r = s = g_0 r$ implies that $f_0 = g_0$, so

$$(f, g) = (x, y)(f_0 r) \in Nr.$$

On the other hand, for the ideal $\mathfrak{A} = Rx + Ry$, we have

$$M\mathfrak{A} = (R \oplus R)\mathfrak{A} = \mathfrak{A} \oplus \mathfrak{A} \supseteq N,$$

so $N \cap M\mathfrak{A} = N \neq N\mathfrak{A}$, as desired.

Ex. 4.6. (a) Let M, N be submodules of a module E such that $M+N$ is flat. Show that $M \cap N$ is flat iff M and N are both flat. (b) Give an example of a flat module with two submodules M, N such that M, N, and $M \cap N$ are all flat, but $M+N$ is not flat.

Solution. (a) The key fact to be used for the proof is the following result from *LMR*-(4.86): If $0 \to A \to B \to C \to 0$ is exact and C is flat, then B is flat iff A is flat. For the given modules M, $N \subseteq E$, consider the exact sequence

$$0 \longrightarrow M \cap N \xrightarrow{\varphi} M \oplus N \xrightarrow{\psi} M + N \longrightarrow 0,$$

where $\varphi(x) = (x, -x)$ and $\psi(x, y) = x + y$. Since $M+N$ is flat, it follows from the aforementioned result that $M \cap N$ is flat iff $M \oplus N$ is flat, iff M and N are both flat.

(b) We use again the example $R = k[x, y]$, where k is a field. Let $M = xR$ and $N = yR$. Both of these R-modules are flat, as is $M \cap N = xyR$. However, it is shown in *LMR*-(4.19) that $M + N = xR + yR$ is *not* flat.

Ex. 4.7. Show that $\mathbb{Q}$ is isomorphic to a direct summand of $G = \mathbb{Q}/\mathbb{Z} \times \mathbb{Q}/\mathbb{Z} \times \cdots$.

Solution. If suffices to get an embedding of $\mathbb{Q}$ into G, for then the injectivity of $\mathbb{Q}$ (as a $\mathbb{Z}$-module) implies that $\mathbb{Q}$ is isomorphic to a direct summand of G.

For any $a \in \mathbb{Q}\backslash\{0\}$, *LMR*-(4.7) guarantees that there exists a homomorphism $\varphi_a : \mathbb{Q} \to \mathbb{Q}/\mathbb{Z}$ such that $\varphi_a(a) \neq 0$. Now define a homomorphism

$$\Phi : \mathbb{Q} \longrightarrow \prod\nolimits_{a \in \mathbb{Q}\backslash\{0\}} \mathbb{Q}/\mathbb{Z}$$

by $\Phi(q) = (\varphi_a(q))_{a \in \mathbb{Q}\backslash\{0\}}$. Since $\varphi_a(a) \neq 0$ for every nonzero a, it follows that Φ is an embedding. This gives what we want, since the countability of $\mathbb{Q}$ implies that $\prod_{a \in \mathbb{Q}\backslash\{0\}} \mathbb{Q}/\mathbb{Z}$ is isomorphic to $\mathbb{Q}/\mathbb{Z} \times \mathbb{Q}/\mathbb{Z} \times \cdots = G$.

The point of the solution above was to illustrate the use of *LMR*-(4.7). Without using this result, an embedding Ψ of $\mathbb{Q}$ into G can be defined *directly* by taking

$$\Psi(q) = (\psi\,(q/2)\,,\ \psi\,(q/4)\,,\ \psi\,(q/8)\,,\ \ldots)\,,$$

where ψ is the natural projection from $\mathbb{Q}$ to $\mathbb{Q}/\mathbb{Z}$. The injectivity of the homomorphism Ψ is seen from the fact that, for any $q \in \mathbb{Q}$, $q \in \bigcap_{n \geq 1} 2^n\mathbb{Z}$ implies that $q = 0$.

Ex. 4.8. For any exact sequence $0 \to M' \to M \to M'' \to 0$ in ${}_R\mathfrak{M}$, prove the following statements (in *LMR*-(4.54)):

(1) If M is f.p. and M' is f.g., then M'' is f.p.
(2) If M' and M'' are f.p., then M'' is f.p.
(3) A direct sum $M_1 \oplus \cdots \oplus M_n$ is f.p. iff each M_i is f.p.

Solution. (1) Express M in the form R^n/X, where $n < \infty$ and X is f.g. Then we can express M' (up to isomorphism) in the form Y/X for some submodule $Y \subseteq R^n$. Since $M'' \cong M/M' \cong R^n/Y$, we have a presentation $0 \to Y \to R^n \to M'' \to 0$. Now the finite generation of X and $Y/X \cong M'$ implies that of Y. Therefore, M'' is f.p.

(2) Clearly, M is f.g. Express M as before in the form R^n/X. Here, we have to prove that X is f.g. Using the same notations as in the above proof, the fact that M'' is f.p. implies that Y is f.g. Since $0 \to X \to Y \to M' \to 0$ is exact, the fact that M' is f.p. now implies that X is f.g., by *LMR*-(4.26)(b).

(3) By induction, we may assume $n = 2$. In this case, the "if" part follows from (2), and the "only if" part follows from (1).

Ex. 4.9. Let $0 \to M' \to M \to M'' \to 0$ be exact in ${}_R\mathfrak{M}$ as above. If M, M'' are f.p. and M' is f.g., is M' necessarily f.p.?

Solution. The answer is "no". For a counterexample, take a ring R that is not left coherent. Let $M' \subseteq R$ be a left ideal that is f.g. but not f.p., and take $M = {}_RR$, $M'' = R/M'$. Then M, M'' are clearly f.p. However, M' is only f.g., and is not f.p. by choice.

For an explicit commutative example of R and M', we can take $R = \mathbb{Q}[y, x_1, x_2, \ldots]$ with the relations $yx_i = 0$ for all i (see *LMR*-(4.46)(d)). The principal ideal $M' = (y)$ is not f.p. in view of the exact sequence

$$0 \longrightarrow (x_1, x_2, \ldots) \longrightarrow R \xrightarrow{\varphi} (y) \longrightarrow 0,$$

where φ is defined by multiplication by y.

Comment. The solution to this exercise essentially implies that, over a ring R that is not left coherent, the category of f.p. left R-modules do not form an abelian category.

Ex. 4.10. Let F_1, F_2 be left exact contravariant additive functors from ${}_R\mathfrak{M}$ to abelian groups, and let $\theta: F_1 \to F_2$ be a natural transformation. If $\theta(R): F_1(R) \to F_2(R)$ is a monomorphism (resp. isomorphism), show that $\theta(M): F_1(M) \to F_2(M)$ is also a monomorphism (resp. isomorphism) for every f.g. (resp. f.p.) module ${}_RM$. State and prove the analogue of this for right exact covariant additive functors.

Solution. First assume M is f.p. Apply $F_1 \xrightarrow{\theta} F_2$ to a finite presentation $R^m \to R^n \to M \to 0$. We get a commutative diagram with exact rows:

$$\begin{array}{ccccccc} 0 & \longrightarrow & F_1(M) & \longrightarrow & F_1(R)^n & \longrightarrow & F_1(R)^m \\ & & \downarrow{\scriptstyle \theta(M)} & & \downarrow{\scriptstyle \theta(R)^n} & & \downarrow{\scriptstyle \theta(R)^m} \\ 0 & \longrightarrow & F_2(M) & \longrightarrow & F_2(R)^n & \longrightarrow & F_2(R)^m. \end{array}$$

Here, $\theta(R)^n$ and $\theta(R)^m$ are isomorphisms. An easy diagram chase ("4-Lemma") shows that $\theta(M)$ is also an isomorphism. If M is only f.g.

instead of f.p., we take $R^n \to M \to 0$ and argue just with the first square above.

For the last part of the exercise, let F_1, F_2 be right exact covariant additive functors, and let $\theta: F_1 \to F_2$ be a natural transformation. If $\theta(R): F_1(R) \to F_2(R)$ is an epimorphism (resp. isomorphism), then $\theta(M): F_1(M) \to F_2(M)$ is also an epimorphism (resp. isomorphism) for every f.g. (resp. f.p.) module M. The proof is analogous to the one we have given: just change the directions of all the horizontal arrows in the commutative diagram above.

Ex. 4.11. Recall that, for arbitrary right R-modules P and M, there exists a natural map

$$\sigma_{M,\,P}: P \otimes_R M^* \longrightarrow \mathrm{Hom}_R(M, P),$$

where $M^* := \mathrm{Hom}_R(M, R)$ is viewed, as usual, as a left module (see Exercise 2.20).

(1) Assume that P is flat and M is f.p. Show that $\sigma_{M,P}$ is an isomorphism. Using this, show that, for any R-homomorphism $\lambda: M \to P$ there exist R-homomorphisms $M \xrightarrow{\nu} R^n \xrightarrow{\mu} P$ (for some $n < \infty$) such that $\lambda = \mu \circ \nu$.
(2) Show that $\sigma_{M,P}$ is also an isomorphism if we assume, instead, that M is projective and P is f.p.

Solution. (1) Fixing the flat module P, let us define

$$F_1(M) = P \otimes_R M^*, \quad F_2(M) = \mathrm{Hom}_R(M, P)$$

for any right R-module M. Defining F_1 and F_2 on morphisms in the obvious way, we have here two contravariant additive functors, and σ defines a natural transformation from F_1 to F_2. Both functors F_1, F_2 are left exact since the "Hom" functor is left exact, and the tensor functor $P \otimes_R -$ is exact (by the flatness of P). By Exercise 10, the first part of (1) will follow if we can show that

$$\sigma_{R,\,P}: P \otimes_R R^* \longrightarrow \mathrm{Hom}_R(R, P)$$

is an isomorphism. After making the usual identifications $R^* = {}_RR$, $P \otimes_R R^* = P \otimes_R R = P$ and $\mathrm{Hom}_R(R, P) = P$, we check easily that $\sigma_{R,P}$ is just the identity map Id_P, so we are done.

For the second part of (1), write $\lambda \in \mathrm{Hom}_R(M, P)$ in the form $\sigma_{M,P}(\sum_{i=1}^n p_i \otimes f_i)$, where $p_i \in P$ and $f_i \in M^*$. Defining $\nu: M \to R^m$ by $\nu(m) = (f_1(m), \ldots, f_n(m))$ and $\mu: R^n \to P$ by $\mu(r_1, \ldots, r_n) = \sum_{i=1}^n p_i r_i$, we have

$$\mu v(m) = \mu(f_1(m), \ldots, f_n(m)) = \sum p_i f_i(m) = \lambda(m)$$

for any $m \in M$, as desired.

(2) This time, we fix a projective module M and define

$$F_1(P) = P \otimes_R M^*, \quad F_2(P) = \mathrm{Hom}_R(M, P)$$

for any right R-module P. Here, we have two covariant additive functors F_1, F_2, and σ defines a natural transformation from F_1 to F_2. Both functors are right exact since the tensor functor $-\otimes_R M^*$ is right exact, and $\mathrm{Hom}_R(M, -)$ is exact (by the projectivity of M). By Exercise 10 again, (2) will follow if we can show that

$$\sigma_{M,\,R} : \; R \otimes_R M^* \longrightarrow \mathrm{Hom}_R(M, R)$$

is an isomorphism. After identifying $R \otimes_R M^*$ with M^*, we check easily that $\sigma_{M,R}$ is just the identity map Id_{M^*}, so we are done as before.

Comment. In *LMR*-(4.32), it is shown that *P_R is flat iff any R-homomorphism from a f.p. module M_R into P factors through a homomorphism $M \to R^n$ (for some n).* The second part of (1) above corresponds to the "only if" part of this statement. The proof we gave for this second part of (1) is quite different from that of *LMR*-(4.32).

Ex. 4.12. Let $\varphi : \; R \to R'$ be a ring homomorphism. Assume that R is commutative, $\varphi(R)$ is in the center of R', and that R' is a flat R-module via φ. Let M be a f.g. (resp. f.p.) left R-module. Show that, for any left R-module N, the natural map

$$\theta_{M,\,N} : \; R' \otimes_R \mathrm{Hom}_R(M, N) \longrightarrow \mathrm{Hom}_{R'}(R' \otimes_R M, \; R' \otimes_R N)$$

is a monomorphism (resp. isomorphism).

Solution. Viewing N as fixed, let $F_1(M)$ and $F_2(M)$ be the two abelian groups above. As in the last exercise, we get two functors F_1, F_2, and the maps above lead to a natural transformation from F_1 to F_2. Both functors are left exact since the "Hom" functor is left exact, and $R' \otimes_R -$ is exact (by the flatness of R'_R). Therefore, by Exercise 10, the conclusions will follow if they hold for

$$\theta_{R,\,N} : \; R' \otimes_R \mathrm{Hom}_R(R, N) \longrightarrow \mathrm{Hom}_{R'}(R' \otimes_R R, \; R' \otimes_R N).$$

After identifying $R' \otimes_R \mathrm{Hom}_R(R, N)$ with $R' \otimes_R N$ and $\mathrm{Hom}_{R'}(R' \otimes_R R, R' \otimes_R N)$ with $\mathrm{Hom}_{R'}(R', \; R' \otimes_R N) = R' \otimes_R N$, we can check that $\theta_{R,N}$ becomes the identity map on $R' \otimes_R N$, so we are done.

Comment. The most concrete case of this exercise is where R is a commutative ring and R' is the localization $S^{-1}R$ of R at a multiplicative set $S \subseteq R$. The exactness of localization implies that $S^{-1}R$ is flat as an R-module. In this case the θ-map above has the form

$$S^{-1}(\mathrm{Hom}_R(M, N)) \to \mathrm{Hom}_{S^{-1}R}(S^{-1}M, \; S^{-1}N).$$

If, moreover, $S = R \setminus \mathfrak{p}$ where $\mathfrak{p}$ is a prime ideal of R, then the θ-map has the form

$$(\mathrm{Hom}_R(M, N))_{\mathfrak{p}} \to \mathrm{Hom}_{R\mathfrak{p}}(M_{\mathfrak{p}}, N_{\mathfrak{p}})$$

in the usual notation for localizations at prime ideals. The fact that the map above is an isomorphism when M is a f.p. R-module has been used before in some exercises in §2.

Ex. 4.13. Let $\mathcal{E}: \ 0 \to A \to B \to C \to 0$ be an exact sequence in $\mathfrak{M}_R$. Assume that R is commutative and C is f.p. Show that $\mathcal{E}$ is split iff the localization of $\mathcal{E}$ at every maximal ideal is split. Does this remain true if C is not f.p.?

Solution. We need only prove the "if" part, so assume $\mathcal{E}$ splits upon localization at every maximal ideal. We claim that the map

$$f_* : \ \mathrm{Hom}_R(C, B) \longrightarrow \mathrm{Hom}_R(C, C)$$

induced by $f: \ B \to C$ is surjective. If so, then a preimage of Id_C under f_* will be an R-homomorphism from C to B splitting f. To check that f_* is onto, it suffices to check that, for every maximal ideal $\mathfrak{p} \subset R$,

$$(f_*)_{\mathfrak{p}} : \ (\mathrm{Hom}_R(C, B))_{\mathfrak{p}} \longrightarrow (\mathrm{Hom}_R(C, C))_{\mathfrak{p}}$$

is onto. Since C is f.p., the *Comment* following the last exercise enables us to "identify" $(f_*)_{\mathfrak{p}}$ with the map

$$\mathrm{Hom}_{R_{\mathfrak{p}}}(C_{\mathfrak{p}}, \ B_{\mathfrak{p}}) \longrightarrow \mathrm{Hom}_{R_{\mathfrak{p}}}(C_{\mathfrak{p}}, \ C_{\mathfrak{p}}).$$

This map is certainly surjective, since by assumption the localized exact sequence $\mathcal{E}_{\mathfrak{p}}$ splits.

The "if" part of this exercise may not hold if C is not assumed to be f.p. This can be shown by the same example used in the *Comment* on Exercise 2.21. Let R be a commutative non-noetherian von Neumann regular ring, and let $C = R/I$ where I is a non-f.g. ideal in R. The exact sequence

$$\mathcal{E}: \ 0 \longrightarrow I \longrightarrow R \longrightarrow C \longrightarrow 0$$

is split at every localization $R_{\mathfrak{p}}$ since $R_{\mathfrak{p}}$ is in fact a field by *LMR*-(3.71). However, $\mathcal{E}$ itself is not split since I is not f.g. According to this Exercise, C must be not f.p.—a fact that is also clear from the presentation of C given by the exact sequence $\mathcal{E}$.

Ex. 4.14. Over a commutative ring R, show that a module P is flat iff, for every maximal ideal $\mathfrak{m} \subset R$, the localization $P_{\mathfrak{m}}$ is flat over $R_{\mathfrak{m}}$.

Solution. The "only if" part follows from the functoriality of flatness: see *LMR*-(4.1). For the converse, assume that $P_{\mathfrak{m}}$ is flat over $R_{\mathfrak{m}}$ for every maximal ideal $\mathfrak{m}$. Consider any exact sequence of R-modules:

$$\mathcal{E}: \ 0 \to A_1 \to A_2 \to A_3 \to 0.$$

Since $\mathcal{E}_\mathfrak{m}$ remains exact, the flatness of $P_\mathfrak{m}$ over $R_\mathfrak{m}$ implies that $P_\mathfrak{m} \otimes_{R_\mathfrak{m}} \mathcal{E}_\mathfrak{m}$ is exact. Using the isomorphism

$$P_\mathfrak{m} \otimes_{R_\mathfrak{m}} (A_i)_\mathfrak{m} \cong (P \otimes_R A_i)_\mathfrak{m},$$

we see that $(P \otimes_R \mathcal{E})_\mathfrak{m}$ is exact. Since this holds for every maximal ideal $\mathfrak{m}$, it follows that $P \otimes_R \mathcal{E}$ is exact, so we have proved that P is flat.

Ex. 4.15. Show that the following are equivalent for any f.g. module P over a commutative ring R:

(1) P is flat;
(2) For any maximal ideal $\mathfrak{m} \subset R$, $P_\mathfrak{m}$ is free over $R_\mathfrak{m}$.

If P is f.p., show that (1), (2) are further equivalent to P being projective.

Solution. First, (2) $\Rightarrow$ (1) follows from the last exercise (even without the f.g. assumption on P). Conversely, if P is f.g. flat over R, then for any maximal ideal $\mathfrak{m} \subset R$, $P_\mathfrak{m}$ is f.g. flat over $R_\mathfrak{m}$. By *LMR*-(4.38)(2), this implies that $P_\mathfrak{m}$ is f.g. free over the local ring $R_\mathfrak{m}$. If P is in fact f.p., Exercise 2.21 (or *LMR*-(4.30)) further implies that P is projective over R.

Ex. 4.15A. Let R be a commutative ring whose total ring of quotients $K := Q(R)$ is von Neumann regular. Show that, for any multiplicative set $S \subseteq R$, the total ring of quotients (see Exer. 2.34) of the localization R_S is given by K_S; that is, $Q(R_S) \cong Q(R)_S$.

Solution. Every element $x \in K_S$ has the form $x = k/s$ where $k \in K$ and $s \in S$. Pick a non 0-divisor r in R such that $rk \in R$. Then $rx = rk/s \in R_S$. Since $r/1$ is (easily seen to be) a non 0-divisor in R_S, the above work shows that $K_S \subseteq Q(R_S)$. Next, we claim that K_S is a von Neumann regular ring. Once this claim is proved, we can conclude that $Q(R_S) = Q(K_S) = K_S$. To see that K_S is von Neumann regular, consider again an arbitrary element $x = k/s$ as above. Since K is von Neumann regular, $k = kk'k$ for some $k' \in K$. But then

$$x = \frac{k}{s} = \frac{kk'k}{s} = \frac{k}{s} \cdot k's \cdot \frac{k}{s} = x(k's)x,$$

as desired.

Comment. Without some assumptions on K or on R, the ring of quotients $Q(R_S)$ need not be given by $Q(R)_S$, even in the case where $K = R$. For an explicit example, let R be the local ring $\mathbb{Q}[[x, y, z]]/(x^2, xy, xz)$. Since $\bar{x}$ kills the maximal ideal of R, we have $K = Q(R) = R$. But if we localize R at the multiplicative set S generated by $\bar{y}$, the element $\bar{x}$ goes to 0, and the element $\bar{z}$ goes to a non 0-divisor of R_S that is not invertible. Hence $Q(R)_S = R_S$ is *not* its own total ring of quotients.

Ex. 4.15B. (S. Endo) For $R \subseteq K$ as in the last exercise, show that a f.g. ideal $P \subseteq R$ is projective iff it is flat.

Solution. The "only if" part is true for any R-module. For the "if" part, assume P is flat and let $A = \text{ann}^R(P)$. Our first major step is to prove the following:

Claim. $PP^{-1}+A = R$ (*where* $P^{-1} = \{x \in K : xP \subseteq R\}$)

Assume this is not the case. Then there exists a maximal ideal $\mathfrak{m} \supseteq PP^{-1}+A$. By Ex. 15, $P_\mathfrak{m} \cong R_\mathfrak{m}^n$ for some n. If $n = 0$, there would exist $r \in R\backslash\mathfrak{m}$ such that $rP = 0$, which contradicts $A \subseteq \mathfrak{m}$. Thus, we must have $n = 1$. (A quick way to rule out $n \geq 2$ is to recall that any nonzero commutative ring, $R_\mathfrak{m}$ in this case, satisfies the "Strong Rank Condition" in *LMR*-§1D.) Therefore, there exists $b \in R$ which is a non 0-divisor in $R_\mathfrak{m}$ such that $P_\mathfrak{m} = bR_\mathfrak{m}$. Let $S = R\backslash\mathfrak{m}$ and write $P = p_1R + \cdots + p_kR$. Then

$$p_i = bc_i/s \in R_\mathfrak{m} \quad \text{and} \quad b = (p_1q_1 + \cdots + p_kq_k)/s_1 \in R_\mathfrak{m}$$

for some c_i, $q_i \in R$ and s, $s_1 \in S$. By the last exercise, we can take $Q(R_\mathfrak{m})$ to be $K_\mathfrak{m} = S^{-1}K$. Thus, there exist $d \in K$ and $s_2 \in S$ such that $b^{-1} \in Q(R_\mathfrak{m})$ can be written as $d/s_2 \in S^{-1}K$. For a suitable $s_3 \in S$, we have thus

$$s_3(sp_i - bc_i) = s_3(s_1b - p_1q_1 - \cdots - p_kq_k) = 0 \in R, \text{ and}$$

$$s_3(s_2 - db) = 0 \in K.$$

From these, $ds_3sp_i = ds_3bc_i = s_3s_2c_i \in R$ for all i, and hence $ds_3s \in P^{-1}$. But then

$$ss_1s_2s_3 = ss_1s_3bd = \sum_i ds_3s(p_iq_i) \in PP^{-1}$$

contradicts $PP^{-1} \subseteq \mathfrak{m}$ (since $ss_1s_2s_3 \in S = R\backslash\mathfrak{m}$).

We have thus proved the *Claim*, from which we see that $PP^{-1} \cap A = 0$. (For, if $x \in PP^{-1} \cap A$, then $xR \subseteq x(PP^{-1}+A) \subseteq A \cdot PP^{-1} = 0$.) Therefore, the *Claim* actually gives $R = PP^{-1} \oplus A$, and we can write $PP^{-1} = eR$ for some indempotent $e \in R$. Let $B = P \oplus (1-e)R \subseteq R$. Clearly, $P^{-1}e \subseteq B^{-1}$, so

$$BB^{-1} \supseteq B + PP^{-1}e \supseteq (1-e)R + eR = R.$$

Thus, B is an *invertible* ideal of R. It follows that B is a projective R-module, and so is its direct summand P, as desired.

Comment. For an application of this exercise to the characterization of commutative semihereditary rings, see the *Comment* on Ex. 7.37 below. A general criterion for a f.g. flat ideal in a commutative ring to be projective is given in (6) of the next exercise, although it does not seem to be of much help here.

Ex. 4.16. (Vasconcelos) Let P be a f.g. flat module over a *commutative* ring R. Define the n^{th} *invariant factor* of P to be $I_n(P) = \text{ann}(\Lambda^n(P))$

where $\Lambda^n(P)$ denotes the n^{th} exterior power of P. Let $\operatorname{rk} P : \operatorname{Spec} R \to \mathbb{Z}$ be the *rank function* of P, as defined in Exercise 2.21. Show that:

(1) For any $\mathfrak{p} \in \operatorname{Spec} R$, $(\operatorname{rk} P)(\mathfrak{p}) \geq n$ iff $I_n(P) \subseteq \mathfrak{p}$.
(2) $\{\mathfrak{p} \in \operatorname{Spec} R : (\operatorname{rk} P)(\mathfrak{p}) = n\} = V(I_n(P)) \backslash V(I_{n+1}(P))$, where $V(\mathfrak{A})$ denotes the Zariski closed set $\{\mathfrak{p} : \mathfrak{p} \supseteq \mathfrak{A}\}$ in Spec R.
(3) For any n and any prime $\mathfrak{p}$, $I_n(P)_{\mathfrak{p}}$ is either (0) or $R_{\mathfrak{p}}$. Using this, show that $I_n(P)^2 = I_n(P)$.
(4) Show that P is projective iff $I_n(P)$ is f.g. for all n.
(5) Show that, if R has no nontrivial idempotent ideals, any f.g. flat module P_R is projective.
(6) Deduce from (4) that a f.g. ideal $P \subseteq R$ is projective iff P is flat and $\operatorname{ann}(P)$ is f.g.

Solution. (1) Consider any prime ideal $\mathfrak{p}$ such that $(\operatorname{rk} P)(\mathfrak{p}) \geq n$ Then $P_{\mathfrak{p}}$ is free of rank $\geq n$, so $\Lambda^n(P_{\mathfrak{p}})$ is $R_{\mathfrak{p}}$-free of positive rank. For any $a \in I_n(P)$, we have $a\Lambda^n(P) = 0$ and so $a\Lambda^n(P_{\mathfrak{p}}) = 0$. Therefore, $a/1 = 0 \in R_{\mathfrak{p}}$, which is possible only if $a \in \mathfrak{p}$. This shows that $I_n(P) \subseteq \mathfrak{p}$. For the converse, assume that $\mathfrak{p}$ is such that $(\operatorname{rk} P)(\mathfrak{p}) < n$. Then $0 = \Lambda^n(P_{\mathfrak{p}}) \cong (\Lambda^n(P))_{\mathfrak{p}}$. Since $\Lambda^n(P)$ is f.g., there exists $a \in R \backslash \mathfrak{p}$ such that $a\Lambda^n(P) = 0$. This shows that $I_n(P) \nsubseteq \mathfrak{p}$.

(2) Follows by applying (1) to n and to $n+1$.

(3) It is routine to check that $I^n(P)_{\mathfrak{p}} = I^n(P_{\mathfrak{p}})$ for any prime ideal $\mathfrak{p}$. Therefore, it suffices to verify that $I^n(P_{\mathfrak{p}})$ is either (0) or $R_{\mathfrak{p}}$. This is clear since $P_{\mathfrak{p}}$ is a free $R_{\mathfrak{p}}$-module of finite rank and hence so is $\Lambda^n(P_{\mathfrak{p}})$. Now let $I = I_n(P)$ (for a fixed n). Since $I_{\mathfrak{p}}$ is either (0) or $R_{\mathfrak{p}}$, we have $I_{\mathfrak{p}} = I_{\mathfrak{p}}^2$. Thus, the inclusion $I^2 \subseteq I$ localizes to an equality at every $\mathfrak{p}$, so we must have $I^2 = I$.

(4) First assume that P is (f.g.) projective. Then so is $\Lambda^n(P)$; hence, its annihilator $I_n(P)$ is generated by an idempotent, by *LMR*-(2.44). Conversely, assume that $I_n(P)$ is f.g. for all n. Since $I_n(P)$ is an idempotent ideal, *LMR*-(2.43) implies that $I_n(P) = e_n R$ for some idempotent $e_n \in R$. Writing S^c for the complement of a set S in Spec R, we have by (2):

$$\begin{aligned}(\operatorname{rk} P)^{-1}(n) &= V(e_n) \setminus V(e_{n+1}) \\ &= V(1-e_n)^c \cap V(e_{n+1})^c \\ &= (V(1-e_n) \cup V(e_{n+1}))^c\end{aligned}$$

for any n. Since this is an open set in Spec R, $\operatorname{rk} P$ is locally constant. It follows then from Exercise 2.21 that the f.g. R-module P is *projective.*

(5) For any n, $I_n(P)$ is an idempotent ideal by (3), so by assumption, $I_n(P) = (0)$ or R. In particular, $I_n(R)$ is f.g. By (4), P is projective.

(6) First assume the f.g. ideal $P \subseteq R$ is projective. Then of course P is flat, and $\operatorname{ann}(P) = I_1(P)$ is f.g. by (4) (or by *LMR*-(2.44)). Conversely, assume

that the f.g. ideal P is flat with $\operatorname{ann}(P)$ f.g. Then $I_1(P) = \operatorname{ann}(P)$ is f.g., and for $n \geq 2$, $\Lambda^n(P) = 0$ shows that $I_n(P) = R$ is also f.g. It follows from (4) that P is a projective ideal.

Comment. The result in this exercise comes from the paper of W. Vasconcelos, "On finitely generated flat modules," Trans. Amer. Math. Soc. **138**(1969), 505–512.

Ex. 4.17. (Vasconcelos) Construct a principal ideal $P = aR$ in a commutative ring R such that P is flat but not projective, as follows. Let $R_0 = (\mathbb{Z}/2\mathbb{Z}) \oplus (\mathbb{Z}/2\mathbb{Z}) \oplus \cdots$, viewed as a (commutative) ring without 1, with addition and multiplication defined componentwise. Let $R = \mathbb{Z} \oplus R_0$ be the ring obtained by adjoining an identity $1 \in \mathbb{Z}$ to R_0. For $a = (2, 0) \in \mathbb{Z} \oplus R_0 = R$, show that:

(1) the principal ideal $P = aR$ is not f.p. (so R is not coherent), and
(2) P is flat but not projective.

Solution. (1) Consider the exact sequence

$$0 \longrightarrow \operatorname{ann}(a) \longrightarrow R \xrightarrow{\varphi} aR \longrightarrow 0,$$

where φ is multiplication by a. Since $\operatorname{ann}(a)$ is not a f.g. ideal of R, it follows from *LMR*-(4.26)(b) that $P = aR$ is not f.p.

(2) Since $P = aR$ is not f.p., it cannot be projective. It remains to show that P is flat. We do this by checking that each localization $P_{\mathfrak{p}}$ is free over $R_{\mathfrak{p}}$ ($\mathfrak{p} \in \operatorname{Spec} R$). If $R_0 \not\subseteq \mathfrak{p}$, say b is an element in $R_0 \setminus \mathfrak{p}$, then

$$ab = 0 \Longrightarrow bP = 0 \Longrightarrow P_{\mathfrak{p}} = (0).$$

Now assume $R_0 \subseteq \mathfrak{p}$. Then for any $e \in R_0$, $(1-e)e = 0 \Rightarrow e = 0 \in R_{\mathfrak{p}}$. This implies that a is not a 0-divisor in $R_{\mathfrak{p}}$. Therefore, $P_{\mathfrak{p}} = aR_{\mathfrak{p}}$ is also free, as desired.

Ex. 4.18. Show that, over a commutative ring R, the tensor product of any two flat (resp. faithfully flat) modules is flat (resp. faithfully flat).

Solution. Say A, B are flat (right) R-modules. For any short exact sequence $\mathcal{E}$ (in $\mathfrak{M}_R$), $B \otimes_R \mathcal{E}$ is short exact (since B is flat), and $A \otimes_R (B \otimes_R \mathcal{E})$ is short exact (since A is flat). Identifying $A \otimes_R (B \otimes_R \mathcal{E})$ with $(A \otimes_R B) \otimes_R \mathcal{E}$, we see that $A \otimes_R B$ is flat.

Now assume A, B are faithfully flat. For any R-module M:

$$0 = (A \otimes_R B) \otimes_R M \cong A \otimes_R (B \otimes_R M) \Rightarrow B \otimes_R M = 0 \Rightarrow M = 0.$$

Since $A \otimes_R B$ is flat, this implies that $A \otimes_R B$ is in fact faithfully flat.

Ex. 4.19. Let P_R be a flat right module and ${}_RM$ be a left module with submodules M_1, M_2. Show that

$$P \otimes_R (M_1 \cap M_2) = (P \otimes_R M_1) \cap (P \otimes_R M_2) \text{ in } P \otimes_R M.$$

Solution. We may assume, without loss of generality, that $M = M_1 + M_2$. Let $N = M_1 \cap M_2$ and $X = (P \otimes_R M_1) \cap (P \otimes_R M_2)$, noting that $P \otimes_R M_1$ and $P \otimes_R M_2$ may be thought of as subgroups of $P \otimes_R M$. Since

$$P \otimes_R M = (P \otimes_R M_1) + (P \otimes_R M_2),$$

the map φ below is an isomorphism (where all tensor products are over R):

$$\begin{array}{ccc}
\dfrac{P \otimes M_1}{P \otimes N} & \xrightarrow{\quad\cong\quad} & P \otimes \dfrac{M_1}{N} \\
{\scriptstyle\psi}\downarrow & & \downarrow{\scriptstyle\cong} \\
\dfrac{P \otimes M_1}{X} & \xrightarrow{\ \varphi\ } \dfrac{P \otimes M}{P \otimes M_2} \cong & P \otimes \dfrac{M}{M_2}
\end{array}$$

It follows that the map ψ is also an isomorphism. Therefore, we must have $X = P \otimes_R N$.

Ex. 4.20. Let P_R be a projective module, and K be a submodule of rad P (the intersection of maximal submodules of P: see *FC*-(24.3)). If P/K is flat, show that $K = 0$.

Solution. Pick a module Q such that $F = P \oplus Q$ is free. It is easy to see that $\text{rad}\,F \supseteq \text{rad}\ P \supseteq K$. Also, if P/K is flat, so is $F/K \cong (P/K) \oplus Q$. Therefore, after replacing P by F, we may assume that P is free, say with a basis $\{e_i : i \in I\}$. For $x = \sum e_i a_i \in K$, consider $\mathfrak{A} = \sum R a_i$, a f.g. left ideal in R. Since P/K is flat, *LMR*-(4.14) implies that $K \cap P\mathfrak{A} = K\mathfrak{A}$ so we have $x = \sum x_i a_i$ for suitable $x_i \in K$. Let $J = \text{rad}\ R$. By *FC*-(24.6)(2), $\text{rad}\ P = \sum e_i J$. From $x_i \in K \subseteq \text{rad}\ P$, we see that

$$x = \sum x_i a_i \in \sum_i e_i J \mathfrak{A}.$$

Comparison with $x = \sum e_i a_i$ shows that $\mathfrak{A} = J\mathfrak{A}$, so Nakayama's Lemma (applied to the f.g. left R-module $\mathfrak{A}$) yields $\mathfrak{A} = 0$. Hence $x = 0$ and $K = 0$.

Ex. 4.21. (This problem, due to H. Bass, assumes familiarity with the class of semiperfect rings introduced in *FC*-§23.) Let R be a semiperfect ring. Show that any f.g. flat module M_R is projective.

Solution. Since R is semiperfect, the f.g. module M has a projective cover by *FC*-(24.12). This means that there exists a projective module P and an epimorphism $\pi : P \to M$ such that $K := \ker(\pi)$ is a small submodule of P (in the sense that $K + L = P \Rightarrow L = P$ for any submodule $L \subseteq P$). By *FC*-(24.4)(1), the small submodule K must lie in rad P. Since $P/K \cong M$ is flat, Exercise 20 implies that $K = 0$. Therefore, $M \cong P$ is projective.

Comment. It is of interest to recall the result (*FC*-(24.25)) that a ring R is right perfect iff every flat right R-module is projective. If R is only semiperfect, this exercise shows that every *f.g.* flat right R-module is projective. The converse is not true. For instance, every f.g. flat $\mathbb{Z}$-module is certainly projective (even free); however, $\mathbb{Z}$ is not semiperfect.

Ex. 4.22. (This problem, due to S. Chase, assumes familiarity with the class of right perfect rings introduced in *FC*-§23.) Let R be a commutative ring. Show that R is coherent and perfect iff R is artinian.

Solution. If R is artinian, then by the Hopkins-Levitzki Theorem *FC*-(4.15), R is also noetherian, and therefore coherent. On the other hand, an artinian ring is semiprimary, and hence perfect. Conversely, assume that R is coherent and perfect. By *FC*-(23.11), $R \cong R_1 \times \cdots \times R_n$ where the R_i's are (commutative) local rings. Each R_i is also coherent and perfect. Therefore, we may as well assume that R itself is local, say with maximal ideal $\mathfrak{m}$. By *FC*-(23.20), R satisfies DCC on principal ideals. In particular, R has a minimal ideal $\mathfrak{A}$. Since R is coherent, $R/\mathfrak{m} \cong \mathfrak{A}$ is f.p. From the exact sequence

$$0 \to \mathfrak{m} \to R \to R/\mathfrak{m} \to 0,$$

it follows (see *LMR*-(4.26)(b)) that $\mathfrak{m}$ is f.g. Since $\mathfrak{m}$ is nil (R being perfect), it must be nilpotent. Now each $\mathfrak{m}^i/\mathfrak{m}^{i+1}$ is f.g. and semisimple, and hence has finite length. It follows that R_R also has finite length, so R is an artinian ring.

Ex. 4.23. Let J be a f.g. left ideal in a left coherent ring R. For any finite set $A \subseteq R$, show that $K = \{r \in R : \; rA \subseteq J\}$ is a f.g. left ideal. From this, conclude that, for any two f.g. ideals I, J in a commutative coherent ring,

$$(J : \; I) = \{r \in R : \; rI \subseteq J\}$$

is also a f.g. ideal.

Solution. For any $a \in A$, $Ja^{-1} = \{r \in R : ra \in J\}$ is a f.g. left ideal in R, by *LMR*-(4.60). By the same result, the intersection of any two f.g. left ideals in R is f.g. By induction, it follows that

$$K = \{r \in R : \; rA \subseteq J\} = \bigcap\nolimits_{a \in A} Ja^{-1}$$

is also f.g. Now assume that R is commutative (and coherent), and let I, $J \subseteq R$ be f.g. ideals. Let A be a finite set of generators for I. Then

$$(J : I) = \{r \in R : \; rA \subseteq J\},$$

which is f.g. by the first part of this exercise.

Ex. 4.24. In an algebra text, the following statement appeared: "A direct sum $\bigoplus_{i \in I} M_i$ of R-modules is faithfully flat iff each M_i is flat and at least one of the M_i's is faithfully flat." Give a counterexample to the "only if" part of this statement.

Solution. Let R be a commutative ring, and consider the direct sum $S = \bigoplus_{\mathfrak{p}} R_{\mathfrak{p}}$, where $\mathfrak{p}$ ranges over Spec R. Each $R_{\mathfrak{p}}$ is R-flat, and by *LMR*-(4.72) (1), S is faithfully flat. However, none of the $R_{\mathfrak{p}}$'s may be faithfully flat. For instance, for $R = \mathbb{Z}$, if $\mathfrak{p} = (p)$ where p is a prime, then for any prime $\ell \neq p$, the nonzero $\mathbb{Z}$-module $\mathbb{Z}/\ell\,\mathbb{Z}$ localizes to zero with respect to $\mathfrak{p}$. If $\mathfrak{p} = (0)$, any torsion $\mathbb{Z}$-module localizes to zero with respect to $\mathfrak{p}$. This shows that no $\mathbb{Z}_\mathfrak{p}$ is faithfully flat over $\mathbb{Z}$.

Ex. 4.25. For any ring extension $R \subseteq S$, show that the following are equivalent:

(1) $R \subseteq S$ is a (right) faithfully flat extension;
(2) S is a pure, flat extension of R in $\mathfrak{M}_R$;
(3) For any system of linear equations $\sum_{i=1}^m x_i b_{ij} = a_j$ $(a_j, b_{ij} \in R,$ $1 \leq j \leq n)$, any solution $(s_1, \dots, s_m) \in S^m$ can be expressed in the form $s_i = r_i + \sum_k t_k c_{ki}$, where $(r_1, \dots, r_m)$ is a solution of the system in R^m, and for each k, $t_k \in S$ and $(c_{k1}, \dots, c_{km})$ is a solution of the associated homogeneous system $\sum_{i=1}^m x_i b_{ij} = 0$ in R^m.

Solution. (1) $\Rightarrow$ (2). Certainly S_R is flat since it is faithfully flat. The faithful flatness of S_R also implies that $(S/R)_R$ is flat, by *LMR*-(4.74) (4). This, in turn, implies that $0 \to R \to S \to (S/R)_R \to 0$ is pure, by *LMR*-(4.85).

(2) $\Rightarrow$ (1). Assume (2). Then, from the pure sequence

$$0 \longrightarrow R \longrightarrow S \longrightarrow (S/R)_R \longrightarrow 0,$$

the flatness of S_R implies that of $(S/R)_R$, by *LMR*-(4.86) (1). This, coupled with the flatness of S_R, implies that $R \subseteq S$ is a (right) faithfully flat extension, again by *LMR*-(4.85).

(2) $\Rightarrow$ (3). Consider any solution $(s_1, \dots, s_m) \in S^m$ for the given linear system. Since $R \subseteq S_R$ is pure, *LMR*-(4.89) implies that there is also a solution $(r_1, \dots, r_m) \in R^m$. Thus, $(s_1 - r_1, \dots, s_m - r_m)$ is a solution in S^m for the homogeneous system $\sum_{i=1}^m x_i b_{ij} = 0$. Since S_R is flat, *LMR*-(4.24) implies that $s_i - r_i = \sum_k t_k c_{ki}$ where, for each k, $t_k \in S$ and $(c_{k1}, \dots, c_{km})$ is a solution of the same homogeneous system in R^m. By transposition, we obtain the desired expression $s_i = r_i + \sum_k t_k c_{ki}$ for the solution $(s_1, \dots, s_m) \in S^m$.

(3) $\Rightarrow$ (2). Note that the two results used above, namely *LMR*-(4.89) and *LMR*-(4.24), are both of an "iff" nature, characterizing, respectively, purity and flatness. Therefore, by reversing the argument given for (2) $\Rightarrow$ (3), we obtain (3) $\Rightarrow$ (2).

Ex. 4.26. Let $\mathcal{E}: 0 \to A \to B \to C \to 0$ be exact in $\mathfrak{M}_R$, where A, C are flat and one of them is faithfully flat. Show that B must be faithfully flat.

Solution. First, by *LMR*-(4.13), B is flat. Second, by *LMR*-(4.85), C being flat implies that $\mathcal{E}$ is a pure exact sequence. Therefore, for any left R-module M, $\mathcal{E} \otimes_R M$ remains short-exact. Suppose $B \otimes_R M = 0$. Then the exactness of $\mathcal{E} \otimes_R M$ implies that

$$A \otimes_R M = 0 = C \otimes_R M.$$

Assuming that one of A, C is faithfully flat, we see that $M = 0$. Therefore, we have shown that B is faithfully flat.

Ex. 4.27. Let $R \subseteq S$ be a (right) faithfully flat extension. If S is a left noetherian (resp. artinian), show that R is also left noetherian (resp. artinian).

Solution. It is known (*LMR*-(4.74)(2)) that any left ideal $\mathfrak{A} \subseteq R$ has the property that $\mathfrak{A} = R \cap S\,\mathfrak{A}$. Assume that S is left noetherian (resp. artinian). For any ascending (resp. descending) sequence of left ideals $\mathfrak{A}_i$ $(i = 1, 2, \ldots)$ in R, we must have $S\,\mathfrak{A}_n = S\,\mathfrak{A}_{n+1} = \cdots$ for some n. Contracting to R then yields $\mathfrak{A}_n = \mathfrak{A}_{n+1} = \cdots$, so we have proved that R is left noetherian (resp. artinian).

Ex. 4.28. In *LMR*-(4.33), it is proved that a module P_R is flat iff, for any R-epimorphism $\varphi : Q \to P$ and any f.p. R-module M, any homomorphism $\lambda : M \to P$ can be "lifted" to some $\psi : M \to Q$. Give another proof for this result by using properties of pure exact sequences.

Solution. First assume P is flat and consider any epimorphism $\varphi : Q \to P$. Then

$$\mathcal{E} :\ 0 \longrightarrow \ker(\varphi) \longrightarrow Q \longrightarrow P \longrightarrow 0$$

is pure by *LMR*-(4.85). For any f.p. module M, *LMR*-(4.89) (5) guarantees that $\mathrm{Hom}_R(M,\ \mathcal{E})$ remains exact. This means precisely that any $\lambda : M \to P$ can be "lifted" to some $\psi : M \to Q$.

Conversely, let P be a right R-module with the stated lifting property. Fix an exact sequence

$$\mathcal{E} :\ 0 \longrightarrow K \longrightarrow Q \longrightarrow P \longrightarrow 0$$

with Q free. The given lifting property means that $\mathrm{Hom}_R(M,\ \mathcal{E})$ is exact for all f.p. modules M_R. Therefore, by *LMR*-(4.89) again, $\mathcal{E}$ is pure. Since Q_R is free and therefore flat, *LMR*-(4.86) guarantees that P_R is also flat.

Ex. 4.29. Show that a ring R is von Neumann regular iff all short exact sequences in $\mathfrak{M}_R$ are pure, iff all right ideals are pure in R_R.

Solution. First suppose R is von Neumann regular. By *LMR*-(4.21), every left module ${}_RN$ is flat. Therefore, for any exact sequence $\mathcal{E}$ in $\mathfrak{M}_R$, $\mathcal{E} \otimes_R N$ remains exact. Since this holds for all ${}_RN$, $\mathcal{E}$ is pure.

Conversely, assume that every right ideal $\mathfrak{A}$ is pure in R_R. For any cyclic module M_R, there exists an exact sequence

$$0 \longrightarrow \mathfrak{A} \longrightarrow R \longrightarrow M \longrightarrow 0.$$

Since this sequence is pure and R_R is flat, *LMR*-(4.86) implies that M_R is flat. Now that we know every cylic right R-module is flat, *LMR*-(4.21) implies that R is von Neumann regular.

Ex. 4.29′. Let J be an ideal in a ring R, and $\bar{R} = R/J$.

(1) If $\bar{R}$ is flat as a left R-module, show that any injective right $\bar{R}$-module is injective over R.
(2) If R is a von Neumann regular ring, show that any injective right $\bar{R}$-module is injective over R.

Solution. Clearly (2) follows from (1), since any left module over a von Neumann regular ring is flat, by *LMR*-(4.21).

To prove (1), assume ${}_R\bar{R}$ is flat and let M be any injective right $\bar{R}$-module. We'll show that M_R remains injective by checking that any monomorphism $\varphi\colon M \to A$ in $\mathfrak{M}_R$ splits. Since ${}_R\bar{R}$ is flat, we have an induced monomorphism

$$\varphi' : M \otimes_R \bar{R} \longrightarrow A \otimes_R \bar{R}.$$

We can identify $A \otimes_R \bar{R}$ with A/AJ, and $M \otimes_R \bar{R}$ with $M/MJ = M$, and view φ' as a monomorphism $\varphi'' : M \to A/AJ$ in $\mathfrak{M}_{\bar{R}}$. Since $M_{\bar{R}}$ is injective, φ'' splits. From this, it is clear that φ splits.

Comment. Besides (2), a good special case of (1) is where R has the form $S \times J$. It is easy to verify that $R/J \cong S$ is flat as a left module over R, so any right injective S-module remains injective over R. This property was already pointed out in *LMR*-(3.11A).

For related exercises, see Exs. (3.27), (3.28), and (6.27A).

Ex. 4.30. Let $K \subseteq A \subseteq B$ be right R-modules. Show that

(1) if A is pure in B, then A/K is pure in B/K, and
(2) if we assume K is pure in B, the converse of (1) also holds.

Solution. (1) Let ${}_RN$ be any left R-module. The purity of $A \subseteq B$ implies that $A \otimes_R N \subseteq B \otimes_R N$. Quotienting out the submodule $\text{im}(K \otimes_R N) \subseteq A \otimes_R N$, we have an injection

$$\text{(I)} \qquad (A \otimes_R N)/\text{im}(K \otimes_R N) \longrightarrow (B \otimes_R N)/\text{im}(K \otimes_R N).$$

Using the right exactness of the tensor-functor, we can identify the above map with

$$\text{(II)} \qquad (A/K) \otimes_R N \longrightarrow (B/K) \otimes_R N.$$

Therefore, this map is an injection for all ${}_RN$, which implies that $A/K \subseteq B/K$ is pure.

(2) Again, let N be any left R-module. Here, $K \otimes_R N$ embeds into both $B \otimes_R N$ and $A \otimes_R N$ (since $K \subseteq B$ is pure). If $A/K \subseteq B/K$ is pure,

then (II) is an injection, and so is (I). From the latter, it follows that $A \otimes_R N \to B \otimes_R N$ is an injection, and hence A is pure in B.

Ex. 4.31. Let A, A' be submodules of a module B_R.

(1) If $A + A'$ and $A \cap A'$ are pure in B, show that A and A' are pure in B.
(2) If A, A' are pure in B and $A + A'$ is flat, show that $A \cap A'$ is pure in B.
(3) Construct an example of A, $A' \subseteq B$ such that A, A', $A + A'$ are all pure in B, but $A \cap A'$ is not.
(4) Construct an example of A, $A' \subseteq B$ such that A, A', $A \cap A'$ are all pure in B, but $A + A'$ is not.

Solution. Consider the exact commutative diagram

$$
\text{(I)} \qquad
\begin{array}{ccccccccc}
0 & \longrightarrow & A \cap A' & \xrightarrow{\alpha} & A \oplus A' & \xrightarrow{\beta} & A + A' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & B & \xrightarrow{\gamma} & B \oplus B & \xrightarrow{\delta} & B & \longrightarrow & 0,
\end{array}
$$

where $\alpha(a) = (a, -a)$, $\beta(a, a') = a + a'$, $\gamma(b) = (b, -b')$, and $\delta(b, b') = b + b'$. Tensoring this with any left R-module X, we get the following exact commutative diagram

$$
\text{(II)} \qquad
\begin{array}{ccccccccc}
 & & (A \cap A') \otimes X & \longrightarrow & (A \oplus A') \otimes X & \longrightarrow & (A + A') \otimes X & \longrightarrow & 0 \\
 & & \downarrow f & & \downarrow g & & \downarrow h & & \\
0 & \longrightarrow & B \otimes X & \longrightarrow & (B \oplus B) \otimes X & \longrightarrow & B \otimes X & \longrightarrow & 0.
\end{array}
$$

(1) Here f is injective since $A \cap A' \subseteq B$ is pure, and h is injective since $A + A' \subseteq B$ is pure. By a simple diagram chase, we see that g is also injective. This implies that $A \otimes X \to B \otimes X$ and $A' \otimes X \to B \otimes X$ are both injective, so $A \subseteq B$ and $A \subseteq B$ are both pure.

(2) Since $A + A'$ is flat, the top sequence in (I) is pure by *LMR*-(4.85). Therefore, the top sequence in (II) remains exact if we add a 0-term to the left. Next, the purity of A, A' in B implies that the map g in (II) is injective. It follows easily that f is also injective, and so $A \cap A' \subseteq B$ is pure.

For (3) and (4), we shall produce examples over $R = \mathbb{Z}$,

(3) Let $B = \mathbb{Z} \oplus \mathbb{Z}_4$. Here, $A = (1, \bar{1}) \cdot \mathbb{Z}$ and $A' = (1, \bar{2}) \cdot \mathbb{Z}$ are both direct summands in $B = A + A'$. But $A \cap A' = (4, 0) \cdot \mathbb{Z}$ is not pure in B, since it is not even pure in $\mathbb{Z} \oplus (0)$.

(4) Here, let $B = \mathbb{Z} \oplus \mathbb{Z}$ instead, and let $A = (1, 0) \cdot \mathbb{Z}$, $A' = (1, 2) \cdot \mathbb{Z}$. Again, A, A' are direct summands in B, with $A \cap A' = (0)$. But $A + A' = \mathbb{Z} \oplus 2\mathbb{Z}$ is not pure in $B = \mathbb{Z} \oplus \mathbb{Z}$ since $2\mathbb{Z}$ is not pure in $\mathbb{Z}$.

Ex. 4.32. Let A be a submodule of a f.p. module B_R. Show that A is a direct summand of B iff A is f.g. and pure in B.

Solution. The "only if" part is clear, since a direct summand of any f.g. module is always f.g. For the "if" part, assume that A is f.g. and is a pure submodule of the f.p. module B. By Exercise 8 applied to the short exact sequence

$$0 \longrightarrow A \longrightarrow B \longrightarrow B/A \longrightarrow 0,$$

B/A is f.p. Now by *LMR*-(4.91), this fact together with the purity of the short exact sequence implies that the sequence splits, so A must be a direct summand of B.

Ex. 4.33. Show that, over any domain R, a right ideal $\mathfrak{B}$ is pure in R_R iff $\mathfrak{B}$ is (0) or R.

Solution. ("Only if") Assume $\mathfrak{B}$ is pure in R_R. Then

$$0 \longrightarrow \mathfrak{B} \longrightarrow R \longrightarrow R/\mathfrak{B} \longrightarrow 0$$

is pure exact. Since R_R is flat, the purity of this sequence implies that $R/\mathfrak{B}$ is flat, by *LMR*-(4.86) (1). Now over a domain, the flat module $R/\mathfrak{B}$ must be torsion-free, by *LMR*-(4.18). Since the element $\overline{1}$ is killed by anything in $\mathfrak{B}$, this is possible only when $\mathfrak{B} = (0)$ or R.

Ex. 4.34. Over a commutative ring R, show that $A \subseteq B$ is pure (in $\mathfrak{M}_R$) iff $A_\mathfrak{m} \subseteq B_\mathfrak{m}$ is pure (in $\mathfrak{M}_{R_\mathfrak{m}}$) for every maximal ideal $\mathfrak{m} \subset R$.

Solution. First assume $A \subseteq B$ is pure. For any R-module N, $A \otimes_R N \to B \otimes_R N$ remains injective. Localizing this at a maximal ideal $\mathfrak{m}$, we see that $(A \otimes_R N)_\mathfrak{m} \to (B \otimes_R N)_\mathfrak{m}$ is injective, or equivalently,

$$A_\mathfrak{m} \otimes_{R_\mathfrak{m}} N_\mathfrak{m} \longrightarrow B_\mathfrak{m} \otimes_{R_\mathfrak{m}} N_\mathfrak{m}$$

is injective. Since any $R_\mathfrak{m}$-module is isomorphic to some $N_\mathfrak{m}$, it follows from the above that $A_\mathfrak{m} \subseteq B_\mathfrak{m}$ is pure for every maximal ideal $\mathfrak{m}$. The converse is proved by reversing this argument.

Ex. 4.35. (Prüfer) Show that a subgroup A of an abelian group B is pure iff any coset β with respect to the subgroup A contains an element b whose order ($\leq \infty$) equals the order of β in B/A.

Solution. Recall from *LMR*-(4.93) that A is pure in B iff $A \cap nB = nA$ for any natural number n. First assume A is pure in B and consider any coset $\beta = b_0 + A$. If $\beta \in B/A$ has infinite order, clearly $b_0 \in B$ also has infinite order, so we are done. If β has finite order n in B/A, then $nb_0 \in A$, so $nb_0 = na$ for some $a \in A$. Therefore, for $b := b_0 - a$, we have $nb = 0$ and $\beta = b_0 + A = b + A$. Since β has order n in B/A, the order of b must be n also. For the converse, assume that the condition on cosets in the statement of the exercise holds. Let $a \in A \cap nB$ where n is any natural number; say $a = nb_0$ where $b_0 \in B$. Then $\beta := b_0 + A$ has order $m|n$ in B/A. By assumption, there exists $b \in B$ of order m such that $\beta = b + A$. Then $a_0 := b_0 - b \in A$, and we have $na_0 = nb_0 - nb = nb_0 = a$, so $a \in nA$, as desired.

Ex. 4.36. For a module B_R over a commutative domain R, let B_t denote its torsion submodule

$$\{b \in B : \ br = 0 \quad \text{for some } r \in R \setminus \{0\}\}.$$

(1) If R is a Prüfer domain, show that (a) B_t is a pure submodule of B, and that (b) B_t is a direct summand of B in case B_R is f.g.
(2) Show that in general B_t need not be pure in B.

Solution. First assume R is a Prüfer domain. The quotient module B/B_t is torsionfree, and therefore flat by *LMR*-(4.69). It follows from *LMR*-(4.85) that B_t is pure in B. If B is f.g., then so is B/B_t, and hence by *LMR*-(4.40), the flat module B/B_t is projective. This implies that B_t is a direct summand of B, as desired.

For an example of a nonpure torsion submodule of a f.g. module, take $R = k[x, y]$ where k is a field, and take $B = R^2/(xy, y^2) \cdot R$. Suppose $\overline{(f,\ g)} \in B_t$. Then $(f, g)r = (xy, y^2)s$ for some r, $s \in R$ with $r \neq 0$. As in the solution to Exercise 5, we can use unique factorization to show that $f = xf_0$, $g = yg_0$ for suitable f_0, $g_0 \in R$, which must then be equal. Therefore, $(f, g) = (x, y)f_0$. This shows that $B_t = \overline{(x,\ y)} \cdot R$. *We claim that* $B_t \subseteq B$ *is not pure.* Indeed, let $\mathfrak{A}$ be the ideal $xR + yR \subseteq R$. If B_t is pure in B, we would have (by *LMR*-(4.92)) $B_t \cap B\mathfrak{A} = B_t\mathfrak{A}$. Since

$$\overline{(x,\ y)} = \overline{(1,\ 0)} \cdot x + \overline{(0,\ 1)} \cdot y \in B\mathfrak{A},$$

it would follow that

$$\overline{(x,\ y)} = \overline{(x,\ y)} \cdot (xf_1 + yg_1)$$

for some f_1, $g_1 \in R$. But then

$$(x,\ y) \cdot (xf_1 + yg_1 - 1) = (xy,\ y^2) \cdot h = (x,\ y) \cdot yh$$

for some $h \in R$, and hence $xf_1 + yg_1 - 1 = yh \in R$, which is clearly impossible. Therefore, B_t is not pure in B.

Comment. (1) and (2) above are parts of the following more general result. For any commutative domain R, the following properties are equivalent:

(A) *For any R-module* B, B_t *is pure in* B.
(B) *For any f.g. R-module* B, B_t *is pure in* B.
(C) *For any f.g. R-module* B, B_t *is a direct summand of* B.
(D) R *is a Prüfer domain.*

Here, the equivalence of (C) and (D) is due to I. Kaplansky; see his paper "A characterization of Prüfer rings," J. Indian Math. Soc. **24**(1960), 279–281. Part (1a) of the exercise (also due to Kaplansky) gives (D) $\Rightarrow$ (A), and (A) $\Rightarrow$ (B) is a tautology. The implication (B) $\Rightarrow$ (D) was proved in an unpublished manuscript of S. H. Man and P. F. Smith (ca. 1999).

Ex. 4.37. Let B be an additive abelian group, viewed as a $\mathbb{Z}$-module. *True or False*: the (pure) torsion subgroup B_t is always a direct summand of B?

Solution. The statement is false! We shall provide two (counter) examples, both of which are standard.

For the first counterexample, let $P = \{2,\ 3,\ 5, \ldots\}$ be the set of all primes, and let $B = \prod_{p \in P} \mathbb{Z}_p$. It is easy to see that $B_t = \bigoplus_{p \in P} \mathbb{Z}_p$ (using the fact that every nonzero element of $\mathbb{Z}_p$ has order p). We claim that *the nonzero element*

$$\alpha = (1,\ 1,\ 1,\ \ldots) + B_t \in B/B_t$$

is divisible by every prime. If so, then B_t cannot be a direct summand, for otherwise $B \cong B_t \oplus (B/B_t)$ but B itself clearly does not have a nonzero element divisible by every prime. To check the claim, let q be any prime. Note that

$$\alpha = (1,\ 1, \ldots,\ 1,\ 0,\ 1,\ 1,\ \ldots) + B_t,$$

where the 0 occurs in the q^{th} coordinate. Now every coordinate of the new coset representative for α above is divisible by q, so α itself is divisible by q in B/B_t.

The second counterexample is based on similar ideas, and is in some sense a "local version" of the above. Fix a prime p and let $B = \mathbb{Z}_p \times \mathbb{Z}_{p^2} \times \mathbb{Z}_{p^3} \times \cdots$. Here B_t consists of sequences $(a_1, a_2, \ldots)$ where the a_i's have a bounded exponent. (Notice that now B_t is bigger than the direct sum $\mathbb{Z}_p \oplus \mathbb{Z}_{p^2} \oplus \cdots$.) Clearly, B has no nonzero element that is divisible by p^n for all $n \geq 1$. Thus, we are done as before by showing that B/B_t has a nonzero element that is divisible by p^n for any $n \geq 1$. Now the element

$$\alpha = (0, p, 0, p^2, 0, p^3,\ \ldots) + B_t \in B/B_t$$

is clearly nonzero, since the $2n^{\text{th}}$ coordinate $p^n \in \mathbb{Z}_{p^{2n}}$ has order p^n (so the orders of the coordinates are unbounded). On the other hand, for any $n \geq 1$,

$$\begin{aligned} \alpha &= (0, p, \ldots, 0, p^{n-1}, 0, p^n, 0, p^{n+1}, 0, \ldots) + B_t \\ &= (0, \ldots, 0, p^n, 0, p^{n+1}, 0, \ldots) + B_t \\ &= p^n(0, \ldots, 0, 1, 0, p, 0, \ldots) + B_t, \end{aligned}$$

so $\alpha \in B/B_t$ is indeed divisible by p^n for any $n \geq 1$.

Comment. In spite of the examples of the above nature, in the theory of abelian groups there are theorems which guarantee that, under suitable hypotheses, B_t is a direct summand of B. For instance, this is the case if either (1) B/B_t is f.g., or (2) B_t is divisible. In fact, in case (1), B/B_t is free and hence projective as a $\mathbb{Z}$-module; in case (2), B_t is injective as a $\mathbb{Z}$-module. Therefore, in either case, the short exact sequence $0 \to B_t \to B \to B/B_t \to 0$ splits. A result of Prüfer guarantees that if B_t (or for that matter any pure subgroup of B) has a finite exponent, then it is a direct summand of B: see §8 of Kaplansky's book "Infinite Abelian Groups," (revised edition), Univ. of Michigan Press, 1968. Both of our examples offered above are taken from this source. These examples B are good choices since in either case B_t is not divisible and does not have finite exponent. Once we showed that B_t is not a direct summand of B, it also follows that the torsion-free abelian group B/B_t is not f.g. and not free.

Ex. 4.38. A monomorphism $\varphi: A \to B$ in $\mathfrak{M}_R$ is said to be *locally split* if, for any $a \in A$, there exists $\sigma \in \mathrm{Hom}_R(B, A)$ such that $\sigma(\varphi(a)) = a$. In this case, the argument used in the last part of the proof of *LMR*-(4.23) shows that, for any $a_1, \ldots, a_n \in A$, there exists $\sigma \in \mathrm{Hom}_R(B, A)$ such that $\sigma(\varphi(a_i)) = a_i$ for all i. Using this, show that if φ is locally split, then $0 \to A \xrightarrow{\varphi} B \to B/A \to 0$ is pure.

Solution. To check the purity of the sequence, we shall apply the criterion (3) in *LMR*-(4.89). Let $a_j \in A$ $(1 \le j \le n)$, $b_i \in B$ $(1 \le i \le m)$ and $s_{ij} \in R\,(1 \le i \le m,\ 1 \le j \le n)$ be given such that $a_j = \sum_i b_i\, s_{ij}$ for all j. Fix $\sigma \in \mathrm{Hom}_R(B, A)$ such that $\sigma(a_j) = a_j$ for all j. (Here, we think of φ as an inclusion map, and suppress it in order to simplify the notations.) Applying σ to the equations for a_j, we get

$$a_j = \sigma(a_j) = \sum\nolimits_i \sigma(b_i) s_{ij}$$

where $\sigma(b_i) \in A$ for $1 \le i \le m$. This verifies the criterion (3) in *LMR*-(4.89), so A is pure in B.

Comment. It is equally easy to apply the alternate criterion (4) in *LMR*-(4.89) for checking purity. This requires that, for any commutative diagram

$$\begin{array}{ccc} R^n & \xrightarrow{\ \gamma\ } & R^m \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\beta} \\ A & \xrightarrow{\ \varphi\ } & B \end{array}$$

there exists $\delta \in \mathrm{Hom}_R(R^m, A)$ such that $\delta\gamma = \alpha$ (i.e. making the upper triangle commutative). All we need to do is to take basis elements $\{e_i : 1 \le i \le n\}$ in R^n, and choose $\sigma \in \mathrm{Hom}_R(B, A)$ such that $\sigma\varphi\ (\alpha(e_i)) = \alpha(e_i)$ (for all i). Upon defining $\delta := \sigma\beta$, we'll have

$$\delta\,\gamma(e_i) = \sigma\,\beta\,\gamma(e_i) = \sigma\,\varphi\,\alpha(e_i) = \alpha(e_i)$$

for all i, and therefore $\delta\gamma = \alpha$, as desired.

It follows immediately from this exercise that *if the R-module A is* f.g., *then any locally split monomorphism $\varphi: A \to B$ is in fact split.*

Finally, a note on terminology is in order. For people working in commutative algebra, the phrase "$\varphi: A \hookrightarrow B$ being locally split" might suggest the condition that the localization of φ is split at every prime ideal of the (commutative) ground ring R. The notion studied in this exercise is certainly not in this spirit; after all, the ground ring R need not be commutative. In dealing with the "locally split" notions in this and the next few exercises, we simply have to ignore localizations for the time being.

Ex. 4.39. (Azumaya) An epimorphism $\psi : B \to C$ in $\mathfrak{M}_R$ is said to be *locally split* if, for any $c \in C$, there exists $\tau \in \mathrm{Hom}_R(C, B)$ such that $\psi\,\tau\,(c) = c$. In this case, prove the following statements.

(1) For any $c_1, \ldots, c_n \in C$, there exists $\tau_n \in \mathrm{Hom}_R(C, B)$ such that $\psi\,\tau_n(c_i) = c_i$ for $1 \le i \le n$.
(2) For any countably generated submodule $D \subseteq C$, the epimorphism $\psi^{-1}(D) \to D$ induced by ψ is split. (In particular, if C itself is countably generated, then ψ is already split.)
(3) The short exact sequence $\mathcal{E} : 0 \to \ker\psi \to B \to C \to 0$ is pure.

Solution. (1) We construct τ_n by induction on n, the case $n = 1$ being covered by the hypothesis. Suppose we have already constructed τ_{n-1}. To construct τ_n, consider the element $c_n - \psi\tau_{n-1}(c_n) \in C$. By the hypothesis again, there exists $\alpha \in \mathrm{Hom}_R(C, B)$ such that

$$(*) \qquad \psi\,\alpha(c_n - \psi\,\tau_{n-1}(c_n)) = c_n - \psi\,\tau_{n-1}(c_n).$$

Now take $\tau_n := \tau_{n-1} + \alpha - \alpha\psi\tau_{n-1} \in \mathrm{Hom}_R(C, B)$. For $i \le n-1$, we have

$$(\dagger) \qquad \begin{aligned} \tau_n(c_i) &= \tau_{n-1}(c_i) + \alpha(c_i) - \alpha\psi\tau_{n-1}(c_i) \\ &= \tau_{n-1}(c_i) + \alpha(c_i) - \alpha(c_i) = \tau_{n-1}(c_i), \end{aligned}$$

and therefore $\psi\,\tau_n(c_i) = \psi\,\tau_{n-1}(c_i) = c_i$. For the last element c_n, we have

$$\begin{aligned} \psi\,\tau_n(c_n) &= \psi[\tau_{n-1}(c_n) + \alpha(c_n) - \alpha\,\psi\,\tau_{n-1}(c_n)] \\ &= \psi\,\tau_{n-1}(c_n) + \psi\,\alpha(c_n - \psi\,\tau_{n-1}(c_n)) \\ &= c_n \end{aligned}$$

by $(*)$. Thus, τ_n is the homomorphism we want.

(2) Express D in the form $\sum_{n=1}^{\infty} c_n R \subseteq C$. Applying the inductive construction in (1) to the elements $c_1, c_2, \ldots$, we obtain $\tau_n \in \mathrm{Hom}_R(C, B)$ for $n = 1, 2, \ldots$, with the property that, for each n, $\psi\tau_n(c_i) = c_i$ for $1 \le i \le n$. Note further from $(\dagger)$ that, on the module $\sum_{i=1}^{n-1} c_i R$, the homomorphism τ_n is no different from τ_{n-1}. Therefore, by taking direct limit, we obtain a homomorphism $\tau \in \mathrm{Hom}_R(D, B)$ whose restriction to $\sum_{i=1}^{n} c_i R$ is given by τ_n, for any n. Since $\psi\tau_n(c_i) = c_i$ for $1 \le i \le n$, it follows that $\psi\tau(c_i) = c_i$ for all $i \ge 1$. Thus $\psi\,\tau = \mathrm{Id}_D$, so the surjection $\psi^{-1}(D) \to D$ is split by τ, as desired.

(3) To show that the sequence $\mathcal{E}$ is pure, it suffices to check that, for any f.p. module M, any homomorphism $\gamma : M \to C$ can be "lifted" to a homomorphism $\lambda : M \to B$. (see *LMR*-(4.89)(5)). In fact, we can verify this lifting property for any countably generated module M. Let $D = \gamma(M) \subseteq C$, which is also countably generated. By (2), there exists

$\tau \in \text{Hom}_R(D,\ B)$ such that $\psi\tau = \text{Id}_D$. Taking $\lambda := \tau\gamma \in \text{Hom}_R(M, B)$, we have $\psi\lambda = \psi\tau\gamma = \gamma$, as desired.

Comment. The result in this exercise comes from the paper of G. Azumaya: "Locally pure-projective modules," Contemporary Math. **124**(1992), 17–22. See also his earlier paper, "Finite splitness and finite projectivity," J. Algebra **106**(1987), 114–134. The property (2) is implicit in Azumaya's 1992 paper, and is independently observed by G. Bergman. It seems to be a rather remarkable fact that if a surjection $\psi : B \to C$ is locally split and if C is countably generated, then ψ must be split. For this special case of (2), see Cor. 2.2.2 in the paper of L. Gruson and M. Raynaud, "Critères de platitude et de projectivité," Invent. Math. **13**(1971), 1–89, as well as Prop. 4.1 in the paper of K. M. Rangaswamy: "The theory of separable mixed abelian groups," Comm. Algebra **12**(1984), 1813–1834. On the other hand, if an injection $\varphi :\ A \to B$ is locally split and A is countably generated, φ need not be split: see the first example constructed in the solution to the next exercise!

Ex. 4.40. Let $\mathcal{E} :\ 0 \to A \xrightarrow{\varphi} B \xrightarrow{\psi} C \to 0$ be exact in $\mathfrak{M}_R$.

(1) If φ is locally split, does it follow that ψ is locally split?
(2) If $\mathcal{E}$ is pure, does it follow that one of φ, ψ is locally split?

Solution. *Both answers are "no"!* For (1), work over the ring $R = \mathbb{Z}$ and take

$$A = \mathbb{Z} \oplus \mathbb{Z} \oplus \cdots, \qquad B = \mathbb{Z} \times \mathbb{Z} \times \cdots,$$

with φ given by the inclusion map $A \hookrightarrow B$. Any element $a \in A$ lies in a finite direct sum $A_0 = \mathbb{Z} \oplus \cdots \oplus \mathbb{Z}$ (say n copies). If we define $\sigma :\ B \to A_0$ by projection onto the first n coordinates, we have clearly $\varphi\sigma(a) = a$. Therefore, φ is locally split. We claim that the surjection $\psi :\ B \to C = B/A$ is *not* locally split. In fact, as in *LMR*-(4.88), for the element

$$c = (2,\ 2^2,\ 2^3,\ \ldots) + A \in C,$$

we have $c \in 2^n d_n + A$ for suitable $d_n \in B$ ($\forall\, n \geq 1$). Therefore, c is divisible by 2^n in C for every n, and hence $\tau(c) = 0$ for any $\tau \in \text{Hom}_{\mathbb{Z}}(C,\ B)$. In particular, $\psi\tau(c) \neq c$ no matter how we choose τ. [Note that the exact sequence in question is indeed pure (as is predicted by Exercise 38), since, upon tensoring φ with any $\mathbb{Z}$-module M, the resulting map is essentially the inclusion

$$M \oplus M \oplus \cdots \subseteq M \times M \times \cdots.$$

Of course, it is also easy to see that $A \cap rB = rA$ for any $r \in \mathbb{Z}$.]

A somewhat easier solution to (1), due to K. M. Rangaswamy, is as follows. Againlet the ground ring be $\mathbb{Z}$, $B = \mathbb{Z} \oplus \mathbb{Z} \oplus \cdots$, and $C = \mathbb{Q}$.

Take any surjection $\psi:\ B \to C$ and let $A = \ker(\psi)$. Since $\mathbb{Q}$ is $\mathbb{Z}$-flat, the inclusion map $\varphi:\ A \to B$ is locally split by *LMR*-(4.23). However, $\mathrm{Hom}_{\mathbb{Z}}(C,\ B) = 0$, so clearly ψ cannot be locally split.

To construct a counterexample to (2), recall first that all exact sequences over a von Neumann regular ring R are pure (Exercise 29). If we produce a *nonsplit* short exact sequence

$$\mathcal{E}:\ 0 \longrightarrow A \xrightarrow{\varphi} B \xrightarrow{\psi} C \longrightarrow 0$$

over such a ring R with A and C f.g., then $\mathcal{E}$ will be a counterexample to (2) since $\mathcal{E}$ is automatically pure, and the finite generation of A and C implies that neither φ nor ψ is locally split. For an explicit construction, take $T = k \times k \times \cdots$ where k is any field, and let R be the subring of T consisting of all sequences that are eventually constant. It is easy to verify that R is a (commutative) von Neumann regular ring. Also, note that R is essential in T_R (since any $(a_1, a_2, \ldots) \in T$ with $a_n \neq 0$ can be multiplied by the n^{th} unit vector $e_n \in R$ to get a nonzero element $(0, \ldots, a_n, 0, \ldots)$ of R). Therefore, if we take any $t \in T \setminus R$, then

$$\mathcal{E}:\ 0 \to R \to R + tR \to (R + tR)/R \to 0$$

is nonsplit since R is essential in (but not equal to) $(R + tR)_R$. Here, $A = R$ and $C = (R + tR)/R$ are both *cyclic* R-modules.

Comment. In parallel to (1), it is of course also natural to ask, for any short exact sequence $\mathcal{E}$, whether ψ being locally split would imply φ being locally split. According to part (2) of Ex. 39, the answer to this question was "yes" if the module C is *countably generated.* It will be shown in part (2) of Ex. 42 below that the answer is again "yes" if the module B is *projective.* However, the answer is "no" in general, according to G. Azumaya; see p. 133 of the second paper of his listed in the *Comment* on Ex. 4.39.

Ex. 4.41. (Villamayor, Fieldhouse) For any submodule A of a flat module B_R, show that the following are equivalent:

(1) $A \cap B\mathfrak{A} = A\mathfrak{A}$ for any left ideal $\mathfrak{A} \subseteq R$;
(2) B/A is flat;
(3) A is pure in B.

If B is in fact a *projective* module, show that these statements are also equivalent to:

(4) The inclusion map $A \hookrightarrow B$ is locally split.

Solution. The equivalence of (1), (2) and (3) is all in *LMR*-§4. Indeed, (1) $\Leftrightarrow$ (2) is *LMR*-(4.14), (3) $\Rightarrow$ (1) is *LMR*-(4.92), and (2) $\Rightarrow$ (3) follows from *LMR*-(4.85). Also, Exercise 38 above gives (4) $\Rightarrow$ (3). Note that the last three implications *do not* require the flatness of B.

Finally, assume that B is projective. We shall finish by proving (1) $\Rightarrow$ (4). This was done in *LMR*-(4.23) with the assumption that B is free. The case where B is projective is a routine extension, which we supply below for the sake of completeness. Take an R-module B' such that $F := B \oplus B'$ is free with a basis $\{e_i\}$, and let $\pi: F \to B$ be the projection on B with respect to this decomposition. To check that the inclusion map $A \subseteq B$ is locally split, take any $a \in A$. Writing $a = e_{i_1}r_1 + \cdots + e_{i_n}r_n$ (with distinct i_j's), we have $a = \pi(a) = \sum_j \pi(e_{i_j})r_j$. Therefore, by (1) (applied to $\mathfrak{A} = \sum_j Rr_j$), we can write $a = \sum_j a_{i_j}r_j$ for suitable $a_{i_j} \in A$. Now take any $\sigma \in \mathrm{Hom}_R\,(F, A)$ such that $\sigma(e_{i_j}) = a_{i_j}$ for all j. Such σ exists since F is free and the i_j's are distinct. Then, clearly,

$$\sigma(a) = \sum_j \sigma(a_{i_j})\,r_j = \sum_j a_{i_j}r_j = a,$$

which proves (4).

Comment. We pointed out above that (2) $\Rightarrow$ (3) $\Rightarrow$ (1) holds without any assumptions on B. In general, the converses of these implications do not hold. For a counterexample to (3) $\Rightarrow$ (2), take $B = A \oplus C$ with C any nonflat module. A counterexample to (1) $\Rightarrow$ (3), due to I. Emmanouil, was given at the end of *LMR*-§4.

If B is only assumed to be flat (instead of projective), then the equivalent conditions (1), (2), (3) need not imply (4). To show this, consider any von Neumann regular ring R that is not right self-injective. Then $A = R_R$ is nonsplit in $B = E(A)$ (the injective hull of A). Since A is cyclic, this means that $A \hookrightarrow B$ is *not* locally split. But A is pure in B by Exercise 29. (Here, B_R is flat, as is every module over R, but B is not projective.)

The key reference for this exercise is D. J. Fieldhouse's paper, "Pure theories," Math. Ann. **184** (1969), 1–18, where a lot of related information can be found.

Ex. 4.42. Let $\mathcal{E}:\ 0 \to A \xrightarrow{\varphi} B \xrightarrow{\psi} C \to 0$ be a short exact sequence in $\mathfrak{M}_R$ where B is projective.

(1) If A is f.g. and pure in B, show that A is a direct summand of B (and hence also a projective R-module).
(2) If ψ is locally split, show that φ is also locally split.

Solution. (1) By the implication (3) $\Rightarrow$ (4) in Exercise 41, A being pure in B implies that φ is locally split. Since A is also f.g., the last paragraph in the *Comment* on Exercise 38 shows that the exact sequence $\mathcal{E}$ splits.

(2) Assume that ψ is locally split. By Exercise 39, $\mathcal{E}$ is pure. Again by (3) $\Rightarrow$ (4) in Exercise 41, φ is locally split.

Comment. Let us recall Exercise 2.17, which states that, over a von Neumann regular ring R, any f.g. submodule A of a projective module B_R is a direct summand. This now follows from (1) since over a von Neumann regular ring R, any submodule of a module is pure, by Exercise 29. Thus, it is useful for us to think of (1) above as a generalization of the well-known fact about von Neumann regular rings from Exercise 2.17.

It is also worth pointing out that part (2) above remains true as long as B is a direct sum of countably generated modules. This is a result obtained independently by K. M. Rangaswamy and B. Zimmermann-Huisgen: see Prop. 12 in Zimmermann-Huisgen's paper, "On the abundance of $\aleph_1$-separable modules," in Abelian Groups and Noncommutative Rings, Contemp. Math. **130** (1992), 167–180. Note that this result of Rangaswamy and Zimmermann-Huisgen is more general than part (2) of this Exercise since, by a well-known theorem of Kaplansky (Ann. Math. **68** (1958), 372–377), every projective module (over any ring) is a direct sum of countably generated (projective) modules.

Ex. 4.43. (Zimmermann-Huisgen) Let R be a left noetherian ring, and C be an arbitrary direct product R^I, viewed as a *right* R-module. For any f.g. submodule $D \subseteq C$, show that there exists $\rho \in \text{Aut}(C_R)$ such that $\rho(D) \subseteq R^J$ for some *finite* subset $J \subseteq I$, where, by R^J, we mean the direct summand of R^I consisting of $(x_i)_{i \in I}$ with $x_i = 0$ for all $i \notin J$.

Solution. We first deal with the case where D is a cyclic module, with a generator $c = (c_i)_{i \in I}$. Since R is left noetherian, there exists a finite subset $J \subseteq I$ such that $\{c_j :\ j \in J\}$ generates $\sum_{i \in I} Rc_i$ as a left ideal. Writing $K = I \backslash J$, we have equations $c_k = \sum_{j \in J} r_{kj} c_j$ for $k \in K$ and $r_{kj} \in R$. Decompose $C = R^I$ into the direct sum $R^J \oplus R^K$, and define $\rho \in \text{End}(C_R)$ as follows: $\rho \,|\, R^K = \text{Id}$, and for the standard basis $\{e_j :\ j \in J\}$ on R^J,

$$\rho(e_j) = e_j - (r_{kj})_{k \in K} \quad (j \in J).$$

Here $(r_{kj})_{k \in K}$ means the I-tuple whose k^{th} coordinate is r_{kj} for $k \in K$, and whose J-coordinates are all zero. Note that $\rho \in \text{Aut}(C_R)$ since ρ is the identity on both R^K and R^I/R^K. Now

$$\begin{aligned}
\rho(c) &= \rho\left(\sum\nolimits_{j \in J} e_j c_j\right) + \rho\left((c_k)_{k \in K}\right) \\
&= \sum\nolimits_{j \in J} \left(e_j - (r_{kj})_{k \in K}\right) c_j + (c_k)_{k \in K} \\
&= \sum\nolimits_{j \in J} e_j c_j - \left(\sum\nolimits_{j \in J} r_{kj} c_j\right)_{k \in K} + (c_k)_{k \in K} \\
&= (c_j)_{j \in J} \in R^J,
\end{aligned}$$

as desired.

For the general case of a f.g. $D \subseteq C$, we can induct on the number of generators for D. Having taken care of the cyclic case, we may assume that $D = D' + cR$ where the result is true for D'. After applying an automorphism to R^I, we may therefore assume that $D' \subseteq R^{J'}$ for some finite $J' \subseteq I$. Let $K' = I \backslash J'$ and write $c = c' + c_1$ where $c' \in R^{J'}$ and $c_1 \in R^{K'}$. By the cyclic case, we know that there exists $\rho_1 \in \text{Aut}(R^{K'})$

such that $\rho_1(c_1) \in R^L$ where L is a finite subset of K'. For

$$\rho := \mathrm{Id} \oplus \rho_1 \in \mathrm{Aut}(R^{J'} \oplus R^{K'}) = \mathrm{Aut}(C_R),$$

we have $\rho(c) = c' + \rho_1(c_1) \in R^{J'} \oplus R^L = R^J$ for the finite set $J = J' \cup L$. Since ρ is the identity on $D' \subseteq R^{J'}$ we have now $\rho(D) \subseteq D' + \rho(c)R \subseteq R^J$, as desired.

Comment. The result in this exercise is taken from the paper of B. Zimmermann-Huisgen: "Pure submodules of direct products of free modules," Math. Ann. **224** (1976), 233–245. Our formulation here follows the suggestions of G. Bergman.

Ex. 4.44. (Zimmermann-Huisgen) Let $\mathcal{E}:\ 0 \to A \xrightarrow{\varphi} B \xrightarrow{\psi} C \to 0$ be a short exact sequence of right R-modules over a left noetherian ring R. If C is a direct product $(R^I)_R$ where I is any set, show that ψ is locally split.

Solution. Assume that $C = R^I$, and consider any $c = (c_i)_{i\in I}$ in C. By Exercise 43, there exists $\rho \in \mathrm{Aut}(C_R)$ such that $\rho(c) = d = (d_i)_{i\in I}$, where $d_i = 0$ outside of a *finite* set $J \subseteq I$. Therefore, $\rho(c) \in R^J$ where $R^J \subseteq R^I$ has the same meaning as in the last exercise. Since R^J is free (and hence projective), $(\rho \circ \psi)^{-1}\ (R^J) \twoheadrightarrow R^J$ is split by a suitable R-homomorphism θ_0. We can extend θ_0 to

$$\theta :\ R^I \longrightarrow (\rho \circ \psi)^{-1}(R^J) \subseteq B$$

by making $\theta = 0$ on $R^{I\setminus J}$. Now let $\tau := \theta \circ \rho \in \mathrm{Hom}_R(C,\ B)$. Since $(\rho \circ \psi)(\theta_0(d)) = d$, we have

$$\psi\,\tau(c) = (\psi\,\theta)(\rho(c)) = \psi\,\theta_0(d) = \rho^{-1}(d) = c,$$

so we have checked that ψ is locally split.

Comment. The result in this exercise is taken from the paper of B. Zimmermann-Huisgen referenced in the *Comment* on Exercise 43. Zimmermann-Huisgen defined a right R-module C to be *locally projective* if every surjection $\psi :\ B \to C$ is locally split. In the paper cited above, Zimmermann-Huisgen proved more generally that, over a left noetherian ring R, any pure submodule of a direct product $(R^I)_R$ is locally projective. On the other hand, she proved that, over any ring R, any locally projective right module is a pure submodule of some direct product $(R^I)_R$.

Ex. 4.45. Use Exercises (42) and (44) to construct a short exact sequence $\mathcal{E}:\ 0 \to A \xrightarrow{\varphi} B \xrightarrow{\psi} C \to 0$ over some ring R in which φ and ψ are *both* locally split, but $\mathcal{E}$ itself is not split.

Solution. (Bergman) Consider any left noetherian ring R for which some direct product $C = (R^I)_R$ is *not* projective. (For instance, take $R = \mathbb{Z}$ and I to be any infinite set. It follows from Exercise 2.6 that $C = \mathbb{Z}^I$ is not

$\mathbb{Z}$-projective.) Now take a sufficiently large free module B_R that admits a surjection ψ onto C, and let $\mathcal{E}$ be the associated short exact sequence

$$0 \longrightarrow A \xrightarrow{\varphi} B \xrightarrow{\psi} C \longrightarrow 0.$$

By Exercise 44, ψ is locally split. Since B is projective, Exercise 42 implies that φ is also locally split. However, $\mathcal{E}$ itself is not split, since otherwise C would be isomorphic to a direct summand of B and hence projective as an R-module, a contradiction.

Ex. 4.46. (Jøndrup) Let $R \subseteq S$ be rings, and P_R be a f.g. flat R-module. Show that P is a projective R-module iff $P \otimes_R S$ is a projective S-module.

Solution. It suffices to prove the "if" part, so assume $P \otimes_R S$ is a projective S-module. Fix an exact sequence $0 \to K \hookrightarrow F \to P \to 0$ where $F = R^m$ for some integer m. By *LMR*-(4.85), this exact sequence is pure, so the induced sequence

$$0 \longrightarrow K \otimes_R S \longrightarrow F \otimes_R S \longrightarrow P \otimes_R S \longrightarrow 0$$

is also exact. Since $P \otimes_R S$ is projective, this sequence splits, and so $K \otimes_R S$ is f.g. over S. Noting that $k \otimes s = (k \otimes 1)s$ (for $k \in K$ and $s \in S$), we can fix a finite generating set of the form $\{k_1 \otimes 1, \ldots, k_n \otimes 1\}$ for $(K \otimes_R S)_S$. Now by *LMR*-(4.23), there exists $\theta \in \mathrm{Hom}_R(F, K)$ such that $\theta(k_i) = k_i$ for every i. We finish by showing that $\theta(k) = k$ for every $k \in K$, for this will imply that $0 \to K \hookrightarrow F \to P \to 0$ splits, which will in turn imply that P_R is projective.

Given $k \in K$, write $k \otimes 1 = \sum_i (k_i \otimes 1)\, s_i$ where $s_i \in S$. Calculating in $K \otimes_R S$, we get

$$\begin{aligned}\theta(k) \otimes 1 &= (\theta \otimes 1)(k \otimes 1) = \sum\nolimits_i (\theta \otimes 1)(k_i \otimes s_i)\\ &= \sum\nolimits_i \theta(k_i) \otimes s_i = \sum\nolimits_i k_i \otimes s_i = k \otimes 1.\end{aligned}$$

Now by *LMR*-(4.86)(2), K_R is also flat, so the injection of left R-modules $R \to S$ induces an injection

$$K = K \otimes_R R \longrightarrow K \otimes_R S$$

of abelian groups. Thus, from $\theta(k) \otimes 1 = k \otimes 1 \in K \otimes_R S$, we may conclude that $\theta(k) = k$, as desired.

Comment. The result in this exercise comes from S. Jøndrup's paper: "On finitely generated flat modules II", Math. Scand. **27** (1970), 105–112. Applying this result to a commutative domain R and its quotient field S, we obtain, for instance, Cartier's theorem that any f.g. flat module over R is projective. For even more powerful applications of Jøndrup's result, see the next exercise.

Instead of making the change of rings $R \hookrightarrow S$, one can also consider the change of rings $R \twoheadrightarrow R/J$, where J is an ideal of R. Of course, one cannot expect the same result to hold. But what about the special case where $J = \text{rad}\, R$ (the Jacobson radical of R)? This question was raised by Jøndrup in his later paper: "Flat and projective modules," Math. Scand. **43** (1978), 336–342. Jøndrup reduced the consideration of this question to the case where $R/\text{rad}\, R \cong A \times A$, where A is either $\mathbb{Z}$ or a finite field. However, the answer seemed to have remained unknown in these two crucial cases.

Ex. 4.46A. Let R be a ring that is embeddable in a right noetherian ring, or a local ring, or a semiperfect ring. Show that any f.g. flat right R-module is projective.

Solution. Say $R \subseteq S$, where S is either right noetherian, or local, or semiperfect. By *LMR*-(4.38) and Ex. 21 above, any f.g. flat right S-module is projective. For any f.g. flat right R-module M, $M \otimes_R S$ remains f.g. and flat as a right S-module (by *LMR*-(4.1)), and hence it is projective by the above remark. It then follows from Ex. 4.46 that M_R is already projective.

Comment. Of course, what underlies this exercise is the fact that the class $\mathcal{C}$ of rings whose f.g. flat right modules are projective is "closed" with respect to taking subrings. For instance, if R is a semiprime right Goldie ring, then by Goldie's Theorem, the classical ring of right quotients S of R is a semisimple ring (see *LMR*-§11). Since $S \in \mathcal{C}$, it follows immediately that $R \in \mathcal{C}$. For another example, consider any commutative semilocal ring R, with maximal ideals $\mathfrak{m}_1, \ldots, \mathfrak{m}_n$. Then R embeds in the ring

$$S := R_{\mathfrak{m}_1} \times \cdots \times R_{\mathfrak{m}_n}.$$

Each $R_{\mathfrak{m}_i} \in \mathcal{C}$, and so $S \in \mathcal{C}$. From this, we deduce that $R \in \mathcal{C}$. More generally, if R_0 is a commutative ring whose set of 0-divisors is a finite union of prime ideals, then its classical ring of quotients R is a (commutative) semilocal ring (see Ex. 12.10). From the above, $R \in \mathcal{C}$, and hence $R_0 \in \mathcal{C}$ also! It follows, for instance, that *any commutative ring with* ACC *on annihilator ideals belongs to* $\mathcal{C}$. (That such a ring has the property of R_0 is proved in *LMR*-(8.31)(2).)

S. Jøndrup has obtained many other interesting properties of the class $\mathcal{C}$. For instance, for any ring R, $R \in \mathcal{C}$ iff $R[[x]] \in \mathcal{C}$, and if R is commutative, then $R \in \mathcal{C}$ iff $R[x] \in \mathcal{C}$. See his paper: "On finitely generated flat modules", Math. Scand. **26** (1970), 233–240.

Ex. 4.47. Let C be a right R-module and M, N be left R-modules. Let $\mathcal{F}: \ 0 \to C' \to M \to N \to 0$ be an exact sequence in ${}_R\mathfrak{M}$, where $C' = \text{Hom}_{\mathbb{Z}}\,(C, \mathbb{Q}/\mathbb{Z})$ is the character module of C. Show that if $\mathcal{F}$ is pure, then it splits.

Solution. Consider any right R-module X. Since $\mathcal{F}$ is pure, $X \otimes_R \mathcal{F}$ remains exact, so by *LMR*-(4.8), $(X \otimes_R \mathcal{F})'$ is also exact. Now, by the Hom-$\otimes$ adjointness, $(X \otimes_R \mathcal{F})'$ is just $\text{Hom}_R(\mathcal{F}, X')$, so the latter is exact.

Applying this to $X = C$, we see that

$$\mathrm{Hom}_R(M, C') \to \mathrm{Hom}_R(C', C')$$

is surjective. This means precisely that the injection $C' \to M$ splits.

Ex. 4.48. For any exact sequence $\mathcal{E}:\ 0 \to A \to B \to C \to 0$ in $\mathfrak{M}_R$, show that the following are equivalent:

(1) $\mathcal{E}$ is pure;
(2) $\mathcal{E}'$ is pure;
(3) $\mathcal{E}'$ is split.

Solution. Since $\mathcal{E}'$ has the form $0 \to C' \to B' \to A' \to 0$, the last exercise gives the equivalence of (2) and (3). In the following, we shall prove (1) $\Leftrightarrow$ (3).

(1) $\Rightarrow$ (3) Assume that $\mathcal{E}$ is pure. Consider any left R-module Y. Then $\mathcal{E} \otimes_R Y$ is exact, and so is $(\mathcal{E} \otimes_R Y)' \cong \mathrm{Hom}_R(Y,\ \mathcal{E}')$. For $Y = A'$, the surjection

$$\mathrm{Hom}_R(A',\ B') \longrightarrow \mathrm{Hom}_R(A',\ A')$$

implies that $\mathcal{E}'$ splits.

(3) $\Rightarrow$ (1) Assume that $\mathcal{E}'$ is split. To show that $\mathcal{E}$ is pure, we must show that $\mathcal{E} \otimes_R Y$ is exact, where Y is any left R-module. Now

$$(\mathcal{E} \otimes_R Y)' \cong \mathrm{Hom}_R(Y,\ \mathcal{E}')$$

is exact, since $\mathcal{E}'$ is split. It follows from *LMR*-(4.8) that $\mathcal{E} \otimes_R Y$ is exact, as desired.

Ex. 4.49. Show that any right R-module A is naturally embedded in A'' as a pure submodule.

Solution. The map $\varepsilon:\ A \to A''$ is defined in the usual way: for $a \in A$, $\varepsilon(a):\ A' \to \mathbb{Q}/\mathbb{Z}$ is defined by $\varepsilon(a)(f) = f(a)$ for $f \in A'$. It is easy to check that ε is a homomorphism of right R-modules. If $\varepsilon(a) = 0$, then $f(a) = 0$ for every $f \in A' = \mathrm{Hom}_{\mathbb{Z}}(A,\ \mathbb{Q}/\mathbb{Z})$. By *LMR*-(4.7), this implies that $a = 0$. Therefore, ε embeds A into A''. Consider the exact sequence:

$$\mathcal{E}:\ 0 \longrightarrow A \xrightarrow{\ \varepsilon\ } A'' \longrightarrow A''/A \longrightarrow 0.$$

To show that $\mathcal{E}$ is pure, it suffices to verify that

$$\mathcal{E}':\ 0 \longrightarrow (A''/A)' \longrightarrow A''' \xrightarrow{\ \varepsilon'\ } A' \longrightarrow 0$$

is split, according to Exercise 48. Here ε' is the left R-module homomorphism induced by the right R-module homomorphism ε. Now, for the left R-module A', we have also an embedding $\delta:\ A' \to A'''$. Thus, we are done if we can check that $\varepsilon'\delta = \mathrm{Id}_{A'}$, for then δ splits the surjection ε'. Now, for any $f \in A'$ and any $a \in A$:

$$(\varepsilon' \delta (f)) (a) = (\delta (f) \circ \varepsilon) (a) = \delta (f) (\varepsilon (a))$$
$$= \varepsilon (a) (f) = f (a).$$

Therefore, $\varepsilon'\delta (f) = f$ for all $f \in A'$, as desired.

Comment. A similar idea is also used in *LMR*-(19.34).

Ex. 4.50. Let R be a (commutative) UFD, and let x, y be two nonzero elements of R with $\gcd(x, y) = 1$. If the ideal $\mathfrak{A} = xR + yR$ is flat, show that $\mathfrak{A} = R$.

Solution. Let F be the free module $xR \oplus yR$ (external direct sum), and let K be the free submodule $cR \subseteq F$ generated by $c := (xy, xy) \in F$. Then the sequence

$$(*) \quad 0 \longrightarrow K \longrightarrow F \xrightarrow{\varphi} \mathfrak{A} \longrightarrow 0 \quad \text{defined by } \varphi (xr, ys) = xr - ys$$

is exact. To see this, first note that $\varphi(c) = 0$ so $K \subseteq \ker(\varphi)$. If $\varphi(xr, ys) = 0$, then since R is a UFD and $\gcd(x, y) = 1$, we can write $s = xs_0$ for some $s_0 \in R$. Then $xr = ys = yxs_0$ implies that $r = ys_0$, so

$$(xr, ys) = (xys_0, xys_0) = cs_0 \in K.$$

Now use the assumption that $\mathfrak{A}$ *is flat*. Since F is free, the exactness of $(*)$ implies that $K\mathfrak{A} = K \cap F\mathfrak{A}$ by *LMR*-(4.14). However, $F\mathfrak{A} = x\mathfrak{A} \oplus y\mathfrak{A}$ contains c, so $K \cap F\mathfrak{A}$ is just K. Thus, we have $K\mathfrak{A} = K$. Since K is a nonzero free module, this implies that $\mathfrak{A} = R$.

Ex. 4.51. Let R be a (commutative) UFD. Show that R is a PID iff all ideals of R are flat, iff all torsion-free R-modules are flat.

Solution. If R is a PID, then by *LMR*-(4.20), all torsion-free R-modules (and hence all ideals) are flat. Conversely, assume that all ideals of R are flat. By Exercise 50, if $x, y \neq 0$ in R have $\gcd(x, y) = 1$, then $xR + yR = R$. Clearly, this implies that, if $x, y \neq 0$ in R have $\gcd(x, y) = z$, then $xR + yR = zR$. It follows that any f.g. ideal is principal, i.e. R is a Bézout domain. We finish now by proving the following result, which is independent of the notion of flatness.

Lemma. *A Bézout domain R is a* PID *iff it is a* UFD.

Proof. It is well-known that any PID is a UFD. Conversely, assume that R is a UFD. Then principal ideals in R satisfy ACC. Since R is Bézout, this implies that f.g. ideals satisfy ACC. It follows that R is noetherian, and hence R is a PID.

Comment. Another solution to the Exercise, using more results from *LMR*-§4, can be given as follows. Assume R is a UFD and all ideals in R are flat. By *LMR*-(4.69), R is a Prüfer domain; that is, any nonzero f.g. ideal $\mathfrak{A}$ is invertible. Since R is a UFD, *LMR*-(2.22F) implies that $\mathfrak{A}$ is principal, so R is a Bézout domain. Now apply the Lemma.

Let us call a commutative ring R a "flat ideal ring" if all ideals of R are flat (see *LMR*-(4.66)). The next exercise offers a local characterization of this property.

Ex. 4.52. For any commutative ring R, show that the following are equivalent:

(1) R is a flat ideal ring.
(2) $R_{\mathfrak{p}}$ is a valuation domain for every prime ideal $\mathfrak{p} \subset R$.
(3) $R_{\mathfrak{m}}$ is a valuation domain for every maximal ideal $\mathfrak{m} \subset R$.

Solution. (2) $\Rightarrow$ (3) is a tautology.

(3) $\Rightarrow$ (1). By *LMR*-(4.4), it suffices to show that any f.g. ideal $I \subseteq R$ is flat. Now, for any maximal ideal $\mathfrak{m} \subset R$, the localization $IR_{\mathfrak{m}}$ is a principal ideal (since $R_{\mathfrak{m}}$ is a valuation domain). From Ex. 4.14, it follows that I is flat.

(1) $\Rightarrow$ (2). Assume R is a flat ideal ring, and consider any prime ideal $\mathfrak{p} \subset R$. Since any ideal of $R_{\mathfrak{p}}$ "comes from" an ideal of R (by localization), Ex. 4.14 referred to above implies that $R_{\mathfrak{p}}$ is also a flat ideal ring. After changing notations, we may thus assume that R is local with maximal ideal $\mathfrak{p}$. Any f.g. ideal of R is then flat (by assumption), and hence free (by *LMR*-(4.38)(2) and *FC*-(19.29)). To see that R is a domain, let $0 \neq a \in R$. Since aR is free (and nonzero), we have clearly

$$ra = 0 \implies r(aR) = 0 \implies r = 0,$$

as desired. To complete the proof, it suffices to show that, for any nonzero $x, y \in R$, we have either $yR \subseteq xR$ or $xR \subseteq yR$. Now $xR + yR$ is free, so it must be free of rank 1 (since R is a domain). Say $xR + yR = aR$. Then $R = \frac{x}{a}R + \frac{y}{a}R$ implies that either $x/a \in \mathrm{U}(R)$ or $y/a \in \mathrm{U}(R)$, and we have $yR \subseteq xR$ or $xR \subseteq yR$ accordingly.

Comment. This exercise comes from Proposition 11 in S. Endo's paper "On semihereditary rings," J. Math. Soc. Japan **13** (1961), 109–119. It is to be compared with Kaplansky's result (*LMR*-(3.71)) that a commutative ring R is von Neumann regular iff $R_{\mathfrak{m}}$ is a field for any maximal ideal $\mathfrak{m} \subset R$.

For Endo's characterizations of (commutative) semihereditary rings, see Ex. 7.37 below.

Ex. 4.53. Let R be any commutative ring with total ring of quotients $K = Q(R)$. For any K-module M, show that M is flat as a K-module iff it is flat as an R-module.

Solution. First suppose M_k is flat. Consider any injection of R-modules $A \to B$. This induces an injection $K \otimes_R A \to K \otimes_R B$, which in turn induces an injection

$$M \otimes_K (K \otimes_R A) \longrightarrow M \otimes_K (K \otimes_R B)$$

by the flatness of M_K. Now for any R-module C, $M \otimes_K (K \otimes_R C)$ may be identified with $M \otimes_R C$ (where M is viewed as an R-module). Thus, the work above shows that $M \otimes_R A \to M \otimes_R B$ is injective. This checks the flatness of M_R.

Conversely, assume M_R is flat, and consider any K-module monomorphism $\varphi: P \to Q$. Since P may be viewed as $K \otimes_R P$, we have

$$M \otimes_K P \cong M \otimes_K (K \otimes_R P) \cong M \otimes_R P.$$

Therefore, $M \otimes_K \varphi$ may be identified with $M \otimes_R P \to M \otimes_R Q$. This is injective since M_R is flat. Hence we have checked that M_K is flat.

Comment. The result in this exercise works well in the setting of Ore localizations too, although we will not repeat it in the noncommutative setting in §10 below. One reason we put the above exercise in this section rather than in §10 is that it will be useful for some exercises in §7 on total rings of quotients of commutative rings; see, specifically, the solution to Ex. 7.37 and the ensuing *Comment.*

The analogue of this exercise holds for *injective* modules as well; see Ex. 10.29.

§5. Homological Dimensions

In this section, we present the theory of homological dimensions of rings following Kaplansky's idea of using suitable "shift operators".

The starting point is Schanuel's Lemma *LMR*-(5.1), which says that if M_R is a right R-module and

$$0 \to K \to P \to M \to 0, \qquad 0 \to L \to Q \to M \to 0$$

are exact sequences in $\mathfrak{M}_R$ with P and Q projective, then $K \oplus Q \cong L \oplus P$. This leads us to define that two R-modules A, B are *projectively equivalent* if $A \oplus Q \cong B \oplus P$ for suitable projective R-modules P, Q. The projective equivalence classes of R-modules $\{[A] : A \in \mathfrak{M}_R\}$ form a commutative semigroup with identity $[0]$, where addition is defined by the direct sum. The *projective shift* $\mathcal{P}$ on this semigroup is defined by $\mathcal{P}[M] = [K]$ where M and K are as above. Using this shift operator, one can define the projective dimension of M to be

$$\text{pd}(M) = \text{pd}_R(M) = \min\{n : \mathcal{P}^n[M] = 0\}.$$

If no such n exists, we define $\text{pd}(M) = \infty$. In some sense, $\text{pd}(M)$ is a measure of how far M is from being a projective module.

The definition of $\mathrm{pd}(M)$ leads us to a global homological invariant of the base ring R: we define the *right global dimension* of R to be

$$\text{r.gl.dim}\, R = \sup\{\mathrm{pd}_R(M) : \ M \in \mathfrak{M}_R\} \leq \infty,$$

and define l.gl. dim R similarly. Rings with r.gl. dim $R = 0$ are the semisimple rings, and those with r.gl. dim $R \leq 1$ are the right hereditary rings. The fact that there exist right hereditary rings which are not left hereditary shows that in general, r.gl. dim $R \neq$ l.gl. dim R.

There is also an injective version of Schanuel's Lemma, which is obtained by replacing the projective modules by injective ones, and reversing arrows. This enables us to define an *injective shift* $\mathcal{I}$ on a semigroup of injective equivalence classes of right R-modules, and to define as before the notion of an *injective dimension* of M, denoted by $\mathrm{id}(M)$. This leads to another global invariant, r.inj.gl. dim R (by taken supremum of $\mathrm{id}(M)$ for $M \in \mathfrak{M}_R$). Fortuitously, it turns out that this is just r.gl. dim R, by *LMR*-(5.45).

For modules, there is also a flat dimension too. Since there is no "flat version" of Schanuel's Lemma, the flat equivalence classes have to be defined via the character modules: K_1, K_2 in $\mathfrak{M}_R$ are said to be *flat equivalent* if there exist flat modules F_1, F_2 such that

$$(K_1 \oplus F_1)^0 \cong (K_2 \oplus F_2)^0.$$

Here, for any right module M_R, M^0 denotes the left R-module $\mathrm{Hom}_{\mathbb{Z}}$ $(M, \mathbb{Q}/\mathbb{Z})$ (called the *character module* of M). Flat equivalence enables us to talk about flat equivalence classes of modules, define a flat shift $\mathcal{F}$, and hence define $\mathrm{fd}(M)$, the *flat dimension* of any M_R. The *right weak dimension*, $\mathrm{r.wd}(R)$, is taken to be $\sup\{\mathrm{fd}(M)\}$, but this time, we get the surprising equation $\mathrm{r.wd}(R) = \mathrm{l.wd}(R)$ for any ring R (by *LMR*-(5.63)).

For any right noetherian ring R, we have

$$\text{r.gl.dim}\, R = \mathrm{r.wd}(R) \leq \text{l.gl.dim}\, R$$

by *LMR*-(5.59). In particular, r.gl.dim$R$ = l.gl.dim$R$ for any noetherian ring R, by *LMR*-(5.60). The same equation also holds for any semiprimary ring, by *LMR*-(5.71).

The above approach to homological dimensions enables us to prove most facts about projective, injective and flat dimensions without defining the Ext and Tor functors in homological algebra. Basically, in our simplified approach, we just need to work with the conditions $\mathrm{Ext}(A, B) = 0$ and $\mathrm{Tor}(C, D) = 0$ without having to define the full abelian groups $\mathrm{Ext}(A, B)$ and $\mathrm{Tor}(C, D)$.

After a short excursion on the homological properties of semiprimary rings, *LMR*-§5 closes with some material on commutative rings. For any commutative noetherian local ring $(R, \mathfrak{m})$ with residue field $k = R/\mathfrak{m}$, it is shown that gl. dim $R = \mathrm{id}(k) = \mathrm{pd}(k)$.This leads to the famous result of

Serre (*LMR*-(5.84)), which states that, for $(R, \mathfrak{m})$ above, gl. dim $R < \infty$ *iff R is a regular local ring* (and that, in this case, gl. dim R equals the Krull dimension of R). This result, in particular, implies that the localizations of R at prime ideals are also regular local rings (*LMR*-(5.88)), which is a conjecture of Krull dating from the 1930s.

For a general commutative *noetherian* ring R, gl. dim (R) can be computed as sup $\{\text{pd}(S)\}$, where S ranges over a complete set of simple R-modules. One can define R to be a *regular ring* (not to be confused with a von Neumann regular ring) if all localizations of R at prime ideals are regular local rings. There are a few equivalent conditions for this, one of which, for instance, is that $\text{pd}_R(M) < \infty$ for any f.g. R-module M. For such regular rings R, we have gl. dim R = dim R, although this common dimension is not necessarily finite in general, by an example of Nagata given in *LMR*-(5.96).

The exercises in this section cover various properties of projective, injective and flat dimensions of modules, and offer examples on change of modules and rings. There are also a few exercises on *finite free resolutions*, which is a topic of increasing importance. Some of the exercises assume a knowledge of UFD's (unique factorization domains), and the Auslander-Buchsbaum Theorem which states that (commutative) regular local rings are UFD's. Although this theorem is not proved in *LMR*-§5, such exercises are deemed interesting and instructive from the viewpoint of this section.

Exercises for §5

Ex. 5.0. Let $0 \to A \to B \to C \to 0$ be an exact sequence in $\mathfrak{M}_R$. The basic results relating the projective dimensions pd(A), pd(B) and pd(C) are stated in *LMR*-(5.20) and *LMR*-(5.23), respectively, as follows:

(a) (1) If $\text{pd}(A) < \text{pd}(B)$ then $\text{pd}(C) = \text{pd}(B)$.
(2) If $\text{pd}(A) > \text{pd}(B)$, then $\text{pd}(C) = \text{pd}(A) + 1$.
(3) If $\text{pd}(A) = \text{pd}(B)$, then $\text{pd}(C) \leq \text{pd}(A) + 1$.
(b) $\text{pd}(B) \leq \max\{\text{pd}(A),\ \text{pd}(C)\}$, with equality unless $\text{pd}(C) = \text{pd}(A)+1$.

Show that these two formulations are equivalent.

Solution. In *LMR*-(5.23), it is already shown that the statement (a) implies the statement (b). Let us now assume statement (b), and try to prove statement (a).

(1) *Assume that* $\text{pd}(A) < \text{pd}(B)$. In the inequality in (b), the RHS must be pd(C), and equality is supposed to hold (i.e. $\text{pd}(B) = \text{pd}(C)$) except possibly when $\text{pd}(C) = \text{pd}(A) + 1$. But in this case, $\text{pd}(C) \leq \text{pd}(B)$, so we'll again have $\text{pd}(C) = \text{pd}(B)$.

(2) *Assume that* $\text{pd}(A) > \text{pd}(B)$. If the desired conclusion $\text{pd}(C) = \text{pd}(A) + 1$ did not hold, we would have, by (b),

$$\text{pd}(B) = \max\{\text{pd}(A),\ \text{pd}(C)\} \geq \text{pd}(A),$$

a contradiction. Thus, we must have $\text{pd}(C) = \text{pd}(A) + 1$.

(3) Assume, finally, that $\text{pd}(A) = \text{pd}(B)$. If the desired conclusion $\text{pd}(C) \leq \text{pd}(A) + 1$ fails, then $\text{pd}(C) > \text{pd}(A) + 1$. According to (b), we are supposed to have

$$\text{pd}(B) = \max\{\text{pd}(A),\ \text{pd}(C)\}.$$

But the RHS is $\text{pd}(C)$, which is bigger than $\text{pd}(A) = \text{pd}(B)$, a contradiction. Thus, we must have $\text{pd}(C) \leq \text{pd}(A) + 1$.

Ex. 5.1. Let R be a ring with $\text{r.gl.dim}\, R = n \geq 1$, and let B be a right R-module with $\text{pd}(B) = n - 1$. Show that $\text{pd}(A) \leq n - 1$ for any submodule $A \subseteq B$.

Solution. If $\text{pd}(A) \leq n - 1$ was not true, then $\text{pd}(A) \geq n > \text{pd}(B)$, so by (a)(2) in the above exercise, we would have

$$\text{pd}(B/A) = \text{pd}(A) + 1 \geq n + 1,$$

which would contradict $\text{r.gl.dim}\, R = n$. Therefore, we must have $\text{pd}(A) \leq n - 1$.

Ex. 5.2. Show that $\text{r.gl.dim}\, R = \infty$ iff there exists a right R-module M such that $\text{pd}(M) = \infty$

Solution. The "if" part follows from the definition of $\text{r.gl.dim}\, R$. For the "only if" part, assume that $\text{r.gl.dim}\, R = \infty$. Then there exists a sequence of modules $M_1, M_2, \ldots$ in $\mathfrak{M}_R$ such that $\text{pd}(M_n) \to \infty$ as $n \to \infty$. Here, some $\text{pd}(M_n)$ may already be ∞, in which case we are done. In any case, for the module $M = \bigoplus_{n=1}^{\infty} M_n$, *LMR*-(5.25) implies that $\text{pd}(M) = \sup\{\text{pd}(M_n)\} = \infty$.

Comment. There is a good reason for us to point out that, in the above situation, some (or all) of the M_n may have infinite projective dimension. In fact, $\text{r.gl.dim}\, R = \infty$ may not imply the existence of $M_1, M_2, \ldots.$ with $\text{pd}(M_n) < \infty$ for all n and with $\text{pd}(M_n) \to \infty$. A rather extreme case is provided by the ring $R = \mathbb{Z}/4\mathbb{Z}$, which has global dimension ∞, but any R-module M is either free or has $\text{pd}(M) = \infty$. (This can be deduced from *LMR*-(5.18)(4) and Prüfer's Theorem on abelian groups of finite exponent.) In general, Bass defined the *right finitistic dimension* of a ring R to be $\sup\{\text{pd}(M)\}$, where M ranges over all right R-modules with $\text{pd}(M) < \infty$. There do exist rings R with $\text{r.gl.dim}\ R = \infty$ and with *arbitrarily prescribed* right finitistic dimension.

Ex. 5.3. Let $0 \to K_i \to F_i \to M \to 0 \quad (i = 1, 2)$ be exact in $\mathfrak{M}_R$, where F_1, F_2 are flat. Show that $K_1 \oplus F_2$ may not be isomorphic to $K_2 \oplus F_1$.

Solution. Over $R = \mathbb{Z}$, let $M = \mathbb{Q}/\mathbb{Z}$, $F_1 = \mathbb{Q}$ and $K_1 = \mathbb{Z}$ (with the obvious maps), and let $0 \to K_2 \to F_2 \to M \to 0$ be any projective resolution of M. Indeed, F_1 and F_2 are flat. Here $K_1 \oplus F_2$ is projective, and it cannot be isomorphic to $K_2 \oplus F_1$ since otherwise $F_1 = \mathbb{Q}$ would also be projective (over $\mathbb{Z}$), which it is not.

Ex. 5.4. Recall that for any right R-module, A' denotes the character module $\text{Hom}_{\mathbb{Z}}(A, \mathbb{Q}/\mathbb{Z})$, which is a left R-module. Show that $A' \cong B'$ need not imply $A \cong B$.

Solution. Let $R = \mathbb{Z}$ and consider the two exact sequences constructed in Exercise 3. Taking character modules, we have new exact sequences

$$0 \longrightarrow M' \longrightarrow F_1' \longrightarrow K_1' \longrightarrow 0 \quad \text{and} \quad 0 \longrightarrow M' \longrightarrow F_2' \longrightarrow K_1' \longrightarrow 0.$$

By the Injective Schanuel's Lemma, $K_1' \oplus F_2' \cong K_2' \oplus F_1'$. Therefore, for $A = K_1 \oplus F_2$ and $B = K_2 \oplus F_1$, we have $A' \cong B'$. However, $A \ncong B$ by what we said in the last exercise.

Ex. 5.5. Let x, y, z be central elements in a ring R such that $x\,R \cap y\,R = zR$ and x, y are not 0-divisors. For $I = xR + yR$, show that there exists a free resolution $0 \to R \to R^2 \to I_R \to 0$. (In particular, $\text{pd}_R(I) \leq 1$.)

Solution. We shall use the fact that x, y, z are central only in the last step, so work without this assumption for now. Write $z = xu = yv$ where u, $v \in R$, and define a sequence of right R-modules

$$R \xrightarrow{\alpha} R^2 \xrightarrow{\beta} I \longrightarrow 0$$

by $\alpha(1) = (u, -v) \in R^2$ and $\beta(a, b) = xa + yb \in I$ for a, $b \in R$. We claim that the sequence is exact. First

$$\beta\,\alpha(1) = \beta(u, -v) = xu - yv = 0,$$

so we need only show that $\ker(\beta) \subseteq \text{im}(\alpha)$. Say $\beta(a, b) = 0$. Then

$$xa = -yb \in xR \cap yR = zR$$

implies that, for some $r \in R$, $xa = zr = xur$ and $-yb = zr = yvr$. Since x, y are non (left) 0-divisors, we have $a = ur$ and $b = -vr$. Therefore, $(a, b) = (u, -v)\,r \in \text{im}(\alpha)$.

Assuming now x, y, z are central, we claim that α is injective. Say $0 = \alpha(s) = (u, -v)s$. Then $us = 0$. Since

$$xy = yx \in xR \cap yR = zR = Rz,$$

$xy = tz$ for some $t \in R$. Then

$$xys = tzs = txus = 0$$

implies that $s = 0$, since x, y are not 0-divisors.

Comment. The free resolution obtained in this exercise is not to be confused with the Koszul resolution for I over a commutative ring R. Although the two resolutions look alike (with the same β), they are derived from different assumptions on x and y, and do not seem to be directly related to each other.

Ex. 5.6. Prove the exactness of the Koszul resolution in *LMR*-§5B for $n = 3$. (over a commutative ring).

Solution. Let x, y, z be a regular sequence in a commutative ring R generating an ideal I. The Koszul resolution in *LMR*-(5.35) looks like

$$(*) \qquad 0 \longrightarrow \Lambda^3(R^3) \xrightarrow{\alpha} \Lambda^2(R^3) \xrightarrow{\beta} \Lambda^1(R^3) \xrightarrow{\gamma} R \longrightarrow R/I \longrightarrow 0,$$

with the appropriate maps α, β and γ. As in *LMR*-(5.35), we make the identifications

$$\Lambda^3(R^3) \cong R, \quad \Lambda^2(R^3) \cong R^3, \quad \Lambda^1(R^3) \cong R^3,$$

using the basis $e_1 \wedge e_2 \wedge e_3$ on Λ^3 (R^3), the basis

$$f_1 = e_2 \wedge e_3, \quad f_2 = -e_1 \wedge e_3, \quad f_3 = e_1 \wedge e_2$$

on $\Lambda^2(R^3)$, and the natural basis e_1, e_2, e_3 on $\Lambda^1(R^3) = R^3$. As observed in the text after *LMR*-(5.35), α, β and γ are given respectively by the matrices

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix}, \quad \begin{pmatrix} 0 & z & -y \\ -z & 0 & x \\ y & -x & 0 \end{pmatrix} \quad \text{and} \quad (x,\ y,\ z).$$

By matrix multiplications, we see readily that $\beta\alpha = 0$ and $\gamma\beta = 0$.

To prove that $\ker(\gamma) \subseteq \text{im}\ (\beta)$, suppose $(x,\ y,\ z)\begin{pmatrix} r \\ s \\ t \end{pmatrix} = 0$. Then $xr + ys = -zt$ implies that $t = -xv + yu$ for some u, $v \in R$ (since z is not a 0-divisor modulo $xR + yR$). Gathering terms, we have

$$x(r - zv) + y\,(s + zu) = 0,$$

Therefore, $s + zu = wx$ for some $w \in R$ (since y is not a 0-divisor modulo xR). Cancelling the non 0-divisor x, we get $r - zv + yw = 0$. Therefore

$$\begin{pmatrix} r \\ s \\ t \end{pmatrix} = \begin{pmatrix} 0 & z & -y \\ -z & 0 & x \\ y & -x & 0 \end{pmatrix} \begin{pmatrix} u \\ v \\ w \end{pmatrix} = \beta \begin{pmatrix} u \\ v \\ w \end{pmatrix},$$

as desired.

To prove that $\ker(\beta) \subseteq \text{im}(\alpha)$, suppose $\beta\begin{pmatrix} u \\ v \\ w \end{pmatrix} = 0$, so that

$$zv - yw = 0, \quad -zu + xw = 0 \quad \text{and} \quad yu - xv = 0.$$

Here, $yu = xv$ implies that $u = xd$ for some $d \in R$. Plugging this into $yu = xv$ and cancelling x, we get $v = yd$. Now $xw = zu = zxd$ yields $w = zd$, so $\begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} d = \alpha(d)$. This completes the (elementary) proof of the exactness of the Koszul resolution $(*)$.

Ex. 5.7. Show that, if $I = \sum x_i R$, where $x_1, \ldots, x_n$ is a regular sequence in R (in the sense of *LMR*-(5.31)), then I/I^2 is a free right R/I-module with basis $x_1 + I^2, \ldots, x_n + I^n$.

Solution. We proceed by induction on $n \geq 0$, the case $n = 0$ being trivial. Our job is to show that $\sum_{i=1}^n x_i a_i \in I^2$ implies that all $a_i \in I$. Since $I^2 = \sum_{i=1}^n x_i I$, we can write $\sum_{i=1}^n x_i a_i = \sum_{i=1}^n x_i y_i$ for suitable $y_i \in I$. From

$$(*) \qquad \sum_{i=1}^{n-1} x_i(a_i - y_i) + x_n(a_n - y_n) = 0$$

and the fact that x_n is not a 0-divisor in $R/\sum_{i=1}^{n-1} x_i R$, we have

$$a_n - y_n = \sum\nolimits_{i=1}^{n-1} x_i z_i$$

for suitable $z_i \in R$. Since x_n is central, $(*)$ can be rewritten as

$$\sum\nolimits_{i=1}^{n-1} x_i(a_i - y_i - x_n z_i) = 0.$$

By the inductive hypothesis (applied to the regular sequence $x_1, \ldots, x_{n-1}$), we have

$$a_i - y_i - x_n z_i \in \sum\nolimits_{j=1}^{n-1} x_j R \quad \text{for} \quad i \leq n-1,$$

so $a_1, \ldots, a_{n-1} \in I$. Finally, $a_n = y_n + \sum_{i=1}^{n-1} x_i z_i \in I$, as desired.

Ex. 5.8. Let $(R, \mathfrak{m})$ be a commutative noetherian local ring. Using Exercise 7, show that R is regular iff $\mathfrak{m}$ can be generated (as an ideal) by a regular sequence in R.

Solution. Let $d = \dim R$ (Krull dimension of R) and $t = \dim_k \mathfrak{m}/\mathfrak{m}^2$ where $k = R/\mathfrak{m}$. In general, t is the smallest number of ideal generators of $\mathfrak{m}$ and $t \geq d$. By definition, R is a *regular local ring* if $t = d$.

If R is a regular local ring (of dimension d), then $\mathfrak{m} = \sum_{i=1}^d x_i R$ for suitable $x_1, \ldots, x_d \in \mathfrak{m}$. In the paragraphs following *LMR*-(5.84), we have already explained that $x_1, \ldots, x_d$ form a regular sequence in R. Conversely, suppose there exists a regular sequence, say $x_1, \ldots, x_n$, generating the ideal $\mathfrak{m}$. By the last exercise, $x_1 + \mathfrak{m}, \ldots, x_n + \mathfrak{m}$ form a k-basis of $\mathfrak{m}/\mathfrak{m}^2$, so we must have $n = t$. Let us induct on this integer, the case $n = t = 0$ being trivial. Consider the noetherian local ring $(\bar{R}, \bar{\mathfrak{m}})$, where "bar" means quotienting out $x_1 R$. Since $\bar{\mathfrak{m}}$ is generated by the regular sequence

$\bar{x}_2, \ldots, \bar{x}_t$, the inductive hypothesis implies that $\bar{R}$ is a regular local ring of dimension $t-1$. Thus, there exists a prime chain

$$\mathfrak{p}_1 \subsetneq \mathfrak{p}_2 \subsetneq \cdots \subsetneq \mathfrak{p}_t \; (= \mathfrak{m})$$

in R with $x_1 R \subseteq \mathfrak{p}_1$. Now $\mathfrak{p}_1$ *cannot* be a minimal prime of R, since a minimal prime must consist of 0-divisors (see Kaplansky's "Commutative Algebra", Thm. 84), but x_1 is not a 0-divisor. Therefore, there exists a prime ideal $\mathfrak{p}_0 \subsetneq \mathfrak{p}_1$, so we have a prime chain

$$\mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \cdots \subsetneq \mathfrak{p}_t$$

of length t in R. This shows that $d \geq t$, so equality must hold, and R is a regular local ring.

Comment. Recalling the fact that a regular local ring must be an integral domain, we see that, in the above notation, we must have $\mathfrak{p}_1 = x_1 R$, and $\mathfrak{p}_0 = (0)$. However, we did not have to know these facts in the course of the proof above.

Ex. 5.9. Let $(R, \mathfrak{m})$ be a right noetherian local ring with $\text{ann}_r(\mathfrak{m}) \neq 0$. Show that any f.g. right R-module P with $\text{pd}(P) < \infty$ is free. Deduce that $\text{r.gl.dim}\, R = \infty$ or else R is a division ring.

Solution. Suppose P is not free. Then P is also not projective, so $n := \text{pd}(P) > 0$. Fix a f.g. representative P' for the projective shift $\mathcal{P}^{n-1}(P)$, so we have $\text{pd}(P') = 1$. Choose $a_1, \ldots, a_n \in P'$ such that $\bar{a}_1, \ldots, \bar{a}_n$ form a basis of $P'/P'\mathfrak{m}$ as a right vector space over the division ring $R/\mathfrak{m}$. Then $P' = P'\mathfrak{m} + \sum a_i R$, and Nakayama's Lemma (*FC*-(4.22)) implies that $P' = \sum a_i R$. Let F be a free right R-module with basis $e_1, \ldots, e_n$ and let $\varphi : F \to R$ be the epimorphism defined by $\varphi(e_i) = a_i$. If $\sum e_i r_i \in K := \ker(\varphi)$, then $\sum \bar{a}_i r_i = 0 \in P'/P'\mathfrak{m}$ and hence all $r_i \in \mathfrak{m}$. This shows that $K \subseteq F\mathfrak{m}$. Fix a nonzero element $a \in \text{ann}_r(\mathfrak{m})$. Then $Ka \subseteq F\mathfrak{m}a = 0$. On the other hand, the exact sequence $0 \to K \to F \to P' \to 0$ shows that

$$[K] = \mathcal{P}[P'] = \mathcal{P}^{n-1}[P] = 0;$$

hence K is projective. Since K_R is also f.g., it is free, so $Ka = 0$ implies that $K = 0$. But then $P' \cong F$ is free, contradicting $\text{pd}(P') = 1$.

Assume now $\text{r.gl.dim}(R) < \infty$. Then $\text{pd}(\mathfrak{m}_R) < \infty$. Since $\mathfrak{m}_R$ is also f.g., the first part of the exercise implies that $\mathfrak{m}_R$ is free. But $\mathfrak{m}a = 0$ for some $a \neq 0$, so we must have $\mathfrak{m} = 0$, which means that R is division ring.

The next three exercises are based on Kaplansky's book, "Fields and Rings," pp. 176–181, University of Chicago Press, 1972.

Ex. 5.10. Let $\overline{R} = R/Rx$, where x is a central element in the ring R. Let M be a right R-module, and let $\overline{M} = M/Mx$. If x is not a 0-divisor on R_R and on M_R, show that $\text{pd}_{\overline{R}}(\overline{M}) \leq \text{pd}_R(M)$.

Solution. If $\text{pd}_R(M) = \infty$, there is nothing to prove. Thus, we may assume that $n := \text{pd}_R(M) < \infty$. We proceed by induction on n. If $n = 0$, M is a projective R-module, so $M \oplus N \cong F$ for some free R-module F. Reducing modulo (x), we have $\overline{M} \oplus \overline{N} \cong \overline{F}$, so $\overline{M}$ is $\overline{R}$-projective, and $\text{pd}_{\overline{R}}(\overline{M}) = 0$ as well. If $n > 0$, fix an exact sequence

$$0 \longrightarrow K \xrightarrow{\varphi} F \xrightarrow{\psi} M \longrightarrow 0$$

in $\mathfrak{M}_R$ with F free. *We claim that the induced sequence* $0 \to \overline{K} \to \overline{F} \to \overline{M} \to 0$ *remains exact in* $\mathfrak{M}_{\overline{R}}$. If this is the case, then by the inductive hypothesis, we have

$$\text{pd}_R(M) = 1 + \text{pd}_R(K) \geq 1 + \text{pd}_{\overline{R}}(\overline{K}) = \text{pd}_{\overline{R}}(\overline{M}),$$

as desired. (Note that the inductive hypothesis applies to K since x is a non 0-divisor on F and hence also on K.)

To prove the exactness of the sequence in $\mathfrak{M}_{\overline{R}}$, let $k \in K$ be such that $\varphi(k) \in Fx$, say $\varphi(k) = fx$ where $f \in F$. Then $0 = \psi\varphi(k) = \psi(f)x$ implies that $\psi(f) = 0$, since x is a non 0-divisor on M. Thus $f = \varphi(k')$ for some $k' \in K$, and $\varphi(k) = fx = \varphi(k'x)$ gives $k = k'x \in Kx$. Finally, let $f \in F$ be such that $\psi(f) \in Mx$, so $\psi(f) = mx$ for some $m \in M$. Writing $m = \psi(f')$ for some $f' \in F$), we have $\psi(f) = \psi(f')x = \psi(f'x)$ so $f - f'x = \varphi(k)$ for some $k \in K$. Therefore, $f \in Fx + \text{im}(\varphi)$, as desired. (Of course, the last part of this argument could have been omitted if we assume the right exactness of the functor $- \otimes_R R/Rx$.)

Ex. 5.11. Keep the hypotheses in Exercise 10, and assume, in addition, that R is right noetherian, $x \in \text{rad}\, R$, and M is f.g. Show that $\text{pd}_{\overline{R}}(\overline{M}) = \text{pd}_R(M)$.

Solution. This time, let $n := \text{pd}_{\overline{R}}(\overline{M})$, which we may assume to be finite, by the last exercise. We shall prove that $\text{pd}_R(M) = n$, by induction on n. To start the induction, assume $n = 0$, i.e. $\overline{M}$ is projective over $\overline{R}$. We must prove that M is projective over R. Suppose this is true when "projective" is replaced by "free". Fix an exact sequence

$$0 \longrightarrow K \longrightarrow F \longrightarrow M \longrightarrow 0$$

in $\mathfrak{M}_R$ with F f.g. free, and let $A = K \oplus M$. As in the solution to the last exercise, $0 \to \overline{K} \to \overline{F} \to \overline{M} \to 0$ remains exact in $\mathfrak{M}_{\overline{R}}$. If $\overline{M}$ is $\overline{R}$-projective, this sequence splits, and so $\overline{A} \cong \overline{K} \oplus \overline{M} \cong \overline{F}$ is $\overline{R}$-free. Supposedly, this implies that A is R-free, and so M is R-projective.

To handle the free case, assume now $\overline{M}$ is $\overline{R}$-free, and let $m_1, \ldots, m_k \in M$ be such that $\overline{m}_1, \ldots, \overline{m}_k$ form an $\overline{R}$-basis for $\overline{M}$. Then $\sum m_i R + Mx = M$, and Nakayama's Lemma implies that $\sum m_i R = M$. We claim that the m_i's are right linearly independent. Indeed, if $\sum m_i r_i = 0$, reducing modulo (x) shows that $r_i = s_i x$ for suitable $s_i \in R$. Then

$$\sum m_i s_i x = 0 \Longrightarrow \sum m_i s_i = 0$$

and we have $s_i = t_i x$, etc. If $r_i \neq 0$, the fact that $x \in \operatorname{rad} R$ would imply that $r_i R \subsetneq s_i R \subsetneq t_i R \subsetneq \cdots$ is a *strictly* ascending chain of right ideals, contradicting the assumption that R is right noetherian. Thus, all $r_i = 0$, so M is R-free on $\{m_1, \dots, m_k\}$.

For the inductive step, assume now $n = \operatorname{pd}_{\overline{R}}(\overline{M}) > 0$, and fix the sequence $0 \to K \to F \to M \to 0$ as before. Then $\overline{M}$ is not $\overline{R}$-projective implies that M is not R-projective, so we'll have

$$(*) \qquad \operatorname{pd}_R M = \operatorname{pd}_R K + 1 \quad \text{and} \quad \operatorname{pd}_{\overline{R}}(\overline{M}) = \operatorname{pd}_{\overline{R}} \overline{K} + 1.$$

Now K remains f.g. since R is right noetherian, and x is a non 0-divisor on F and hence also on K. Applying the inductive hypothesis to K, we have $\operatorname{pd}_R K = \operatorname{pd}_{\overline{R}} \overline{K}$, and hence the equations $(*)$ imply that $\operatorname{pd}_R M = \operatorname{pd}_{\overline{R}} \overline{M}$.

Ex. 5.12. Keep the hypotheses in Exercise 11 and assume that $R \neq 0$ and $n = \text{r.gl.}\dim \overline{R} < \infty$. Show that $\text{r.gl.}\dim R = n + 1$.

Solution. For any f.g. M_R, fix a short exact sequence

$$0 \longrightarrow K \longrightarrow F \longrightarrow M \longrightarrow 0$$

in $\mathfrak{M}_R$ with F f.g. free. Then K is also f.g. with x acting as a non 0-divisor. Applying Exercise 11 to K, we have $\operatorname{pd}_R(K) = \operatorname{pd}_{\overline{R}}(\overline{K}) \leq n$. From the first exact sequence, it follows that $\operatorname{pd}_R(M) \leq n + 1$. Since this holds for all f.g. M in $\mathfrak{M}_R$, *LMR*-(5.51) implies that $\text{r.gl.}\dim R \leq n + 1$. On the other hand, even without assuming $x \in \operatorname{rad} R$, *LMR*-(5.30) gives

$$\text{r.gl.}\dim R \geq \text{r.gl.}\dim \overline{R} + 1 = n + 1.$$

Therefore, equality holds.

Ex. 5.13. (Auslander-Buchsbaum, Small, Strooker) For any right noetherian ring $A \neq (0)$, show that

$$\text{r.gl.}\dim A[[x]] = 1 + \text{r.gl.}\dim A,$$

where $A[[x]]$ denotes the power series ring in one variable over A.

Solution. Let $R = A[[x]]$. Since any power series in $1 + xR$ is invertible, we have $x \in \operatorname{rad} R$, and of course x is not a 0-divisor in R. By the "power series version" of the Hilbert Basis Theorem, A being right noetherian implies that R is right noetherian. Also, note that $\overline{R} := R/xR \cong A$.

First assume that $\text{r.gl.}\dim A < \infty$. By the last exercise, we have

$$\text{r.gl.}\dim R = 1 + \text{r.gl.}\dim \overline{R} = 1 + \text{r.gl.}\dim A.$$

Finally, assume that $\text{r.gl.}\dim A = \infty$. By Exercises 3, there exists a module N_A such that $\operatorname{pd}_A(N) = \infty$. Let M be the right R-module $N \otimes_A R$. Then $\overline{M} := M/Mx \cong N$ as A-modules, and Exercise 10 gives

$$\operatorname{pd}_R(M) \geq \operatorname{pd}_{\overline{R}}(\overline{M}) = \operatorname{pd}_A(N) = \infty.$$

This implies that $\text{r.gl.}\dim R = \infty$ as well.

Ex. 5.14. A right R-module P is said to be stably free if, for some integer $n \geq 0$, $P \oplus R^n$ is free. If R is commutative, and P is f.g. stably free R-module of rank 1, show that $P \cong R$.

Solution. Let $\mathfrak{p}$ be any prime ideal of R. Then $P_{\mathfrak{p}} \cong R_{\mathfrak{p}}$ as $R_{\mathfrak{p}}$-modules, so for any $k \geq 2, (\Lambda^k(P))_{\mathfrak{p}} \cong \Lambda^k(P_{\mathfrak{p}}) = 0$. This implies that $\Lambda^k(P) = 0$. Take an n such that $P \oplus R^n$ is free. Then $P \oplus R^n \cong R^{n+1}$. Forming the $(n+1)$-st exterior power, we get

$$\begin{aligned} R \cong \Lambda^{n+1}(R^{n+1}) &\cong \bigoplus_{i+j=n+1} \Lambda^i(P) \otimes_R \Lambda^j(R^n) \\ &\cong \Lambda^1(P) \otimes_R \Lambda^n(R^n) \cong P \otimes_R R \cong P, \end{aligned}$$

as desired.

Comment. A more elementary solution, not using the exterior algebra, can be given as follows. Using an isomorphism $P \oplus R^n \cong R^{n+1}$, we can think of P as the "solution space" of a right-invertible $(n+1) \times n$ matrix A over the commutative ring R. It is easy to see that *this solution space is* $\cong R$ *iff* A can be completed into a matrix in $\mathrm{GL}_{n+1}(R)$. Thus, it suffices to show that the maximal minors $a_1, \ldots, a_{n+1}$ of A satisfy $\sum a_i R = R$ (for, if $\sum a_i b_i = 1$, we can "complete" A by adding a last row $b_1, \ldots, b_{n+1}$ with appropriate signs). Assume instead, $\sum a_i R \neq R$. Then $\sum a_i R \subseteq \mathfrak{m}$ for some maximal ideal $\mathfrak{m} \subset R$. Working over $\overline{R} = R/\mathfrak{m}$, we do have $\overline{P} \cong \overline{R}$, so $\overline{A}$ can be completed to a matrix $X \in \mathrm{GL}_{n+1}(\overline{R})$. However, $\det(X) \in \sum \overline{a}_i \overline{R} = 0$, a contradiction.

The following standard example shows that this exercise is, indeed, special to rank 1. Let R be the coordinate ring of the real 2-sphere; that is, $R = \mathbb{R}[x, y, z]$ with the relation $x^2 + y^2 + z^2 = 1$. Let $\varphi : R^3 \to R$ be the R-homomorphism given by mapping the unit vectors e_1, e_2 $e_3 \in R^3$ to x, y, $z \in R$, and let $P = \ker(\varphi)$. As is observed in *LMR*-(17.36), $P \oplus R \cong R^3$ so P is stably free *of rank* 2, but P itself is not free (in fact not even decomposable). A proof for $P \not\cong R^2$ is usually based on the fact that the matrix $A = (x,\ y,\ z)$ of φ *cannot* be completed into a matrix in $\mathrm{GL}_3(R)$, which can be checked easily by the *nonexistence* of a continuous vector field of nowhere vanishing tangent vectors on the sphere S^2.

Remarkably enough, the assumption of *commutativity* on R is also essential for the present exercise. Many examples of noncommutative rings R are known that have *nonfree* left ideals P with the property that $P \oplus R \cong R^2$. An explicit example of such a ring is the polynomial ring $R = K[x, y]$, where K is any noncommutative division ring. For an exposition of this result of Ojanguren and Sridharan, see Chapter II of the author's monograph "Serre's Problem on Projective Modules," Springer-Verlag, 2006.

Ex. 5.15. A right R-module M is said to have a finite free resolution (FFR) if, for some integer $n \geq 0$, there exists a long exact sequence

$$(*) \qquad 0 \longrightarrow F_n \longrightarrow \cdots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow M \longrightarrow 0 \quad \text{in} \quad \mathfrak{M}_R,$$

where the F_i's are f.g. free right R-modules. If such a module M is projective, show by induction on n that M is stably free.

Solution. We induct on n, the case $n = 0$ being trivial. For $n \geq 1$, let $X = \text{im}(\varphi)$ where φ is the homomorphism from F_1 to F_0. The short exact sequence $0 \to X \to F_0 \to M \to 0$ is split since M is assumed to be projective. Therefore, we have $M \oplus X \cong F_0$. Applying the inductive hypothesis to the FFR:

$$0 \longrightarrow F_n \longrightarrow \cdots \longrightarrow F_2 \longrightarrow F_1 \to X \longrightarrow 0,$$

we see that X is stably free, say $X \oplus R^s \cong R^t$. But then

$$M \oplus R^t \cong M \oplus X \oplus R^s \cong F_0 \oplus R^s,$$

so M itself is stably free.

Comment. Of course, the inductive argument above shows directly that

$$M \oplus F_1 \oplus F_3 \oplus \cdots \cong F_0 \oplus F_2 \oplus F_4 \oplus \cdots,$$

so $M \oplus R^t$ is free for $R^t = F_1 \oplus F_3 \oplus \cdots$.

Let $R = k[x_1, \ldots, x_n]$, where k is a field. A classical result of Hilbert, known as his Syzygy Theorem, states that any f.g. R-module has a FFR. In view of this, the present exercise implies that any f.g. projective R-module M is stably free. Many years after Hilbert's work, Quillen and Suslin proved in 1976 that M is, in fact, *free*, solving a famous problem raised by Serre in his paper "Faisceaux algébriques cohérents" (FAC, ca. 1955). For an exposition on Serre's Problem, see the author's monograph referenced in the *Comment* on the previous exercise.

Ex. 5.16. Let P be a f.g. projective module of rank 1 over a commutative ring R. If P has a FFR, use Exercises (14) and (15) to show that $P \cong R$.

Solution. Since P is f.g. projective, Exercise 15 implies that P is stably free. The fact that P has rank 1 now implies that $P \cong R$, according to Exercise 14.

Ex. 5.17. Let R be a right coherent ring over which every f.g. projective right module is stably free. Show that every f.g. module M_R with $d = \text{pd}(M) < \infty$ has a FFR as in Exercise 15 with $n = 1 + d$.

Solution. We induct on d. First assume $d = 0$. In this case M is f.g. projective, so by assumption M is stably free. From an isomorphism $M \oplus R^s \cong R^t$, we get a FFR: $0 \to R^s \to R^t \to M \to 0$. Now assume $d > 0$. Since R is right coherent, M is f.p. (finitely presented) by *LMR*-(4.52). Fix a short exact sequence $0 \to M' \to F_0 \to M \to 0$ where F_0 is f.g. free and M' is f.g. Then $\mathrm{pd}(M') = d - 1$ by *LMR*-(5.12), so by the inductive hypothesis there exists a FFR:

$$0 \longrightarrow F_{d+1} \longrightarrow \cdots \longrightarrow F_2 \to F_1 \longrightarrow M' \longrightarrow 0.$$

Splicing this long exact sequence with $0 \to M' \to F_0 \to M \to 0$ then yields the desired FFR for M.

The following three exercises assume some familiarity with UFD's (Unique Factorization Domains) and the Auslander-Buchsbaum Theorem (that commutative regular local rings are UFD's).

Ex. 5.18. Let R be a commutative regular domain. Show that R is a UFD iff $\mathrm{Pic}(R) = \{1\}$ (i.e. invertible ideals of R are principal).

Solution. The "only if" part is true for any domain R (without the assumption of regularity), by *LMR*-(2.22F). For the "if" part, assume that $\mathrm{Pic}(R) = \{1\}$. To show that R is a UFD, it suffices to check that any height one prime ideal $M \subset R$ is principal. Consider any prime ideal $\mathfrak{p} \subset R$. If $\mathfrak{p} \not\supseteq M$, then $M_\mathfrak{p} = R_\mathfrak{p}$. If $\mathfrak{p} \supseteq M$, then $M_\mathfrak{p}$ is a height 1 prime in $R_\mathfrak{p}$. In the latter case, $R_\mathfrak{p}$ is a regular local ring by *LMR*-(5.94), so by the Auslander-Buchsbaum Theorem, $R_\mathfrak{p}$ is a UFD, and so the height 1 prime $M_\mathfrak{p} \subseteq R_\mathfrak{p}$ is principal. Thus, *in all cases*, $M_\mathfrak{p} \cong R_\mathfrak{p}$. Since R is noetherian, M is a finitely presented R-module, so $M_\mathfrak{p} \cong R_\mathfrak{p}$ for all primes $\mathfrak{p}$ implies that M is projective as an R-module (see Exercise (2.21)). By *LMR*-(2.17), M is an invertible ideal, which must then be principal since $\mathrm{Pic}(R) = \{1\}$.

Comment. The result in this Exercise may be thought of as a "globalization" of the Auslander-Buchsbaum Theorem.

Ex. 5.19. Let R be a commutative noetherian domain over which any f.g. module has a FFR. Show that R is a UFD.

Solution. The hypothesis implies that $\mathrm{pd}_R(M) < \infty$ for any f.g. R-module M. By *LMR*-(5.94), this guarantees that R is a regular domain. Now consider any invertible ideal $I \subseteq R$. By assumption, I has a FFR, so Exercise 16 implies that $I \cong R$. This means that I is principal, so we have shown that $\mathrm{Pic}(R) = \{1\}$. By the last exercise, R must be a UFD.

Ex. 5.20. Let R be a commutative regular domain over which all f.g. projectives are stably free. Show that R is a UFD.

Solution. Since R is assumed to be noetherian, it is also coherent. Let M be any f.g. R-module. By *LMR*-(5.94), $\mathrm{pd}(M) < \infty$. Thus, by Exercise 17, M has a FFR. This checks the hypothesis of Exercise 19, so by the conclusion of that exercise, R must be a UFD.

Ex. 5.21A. Let R be a ring with IBN, and let M be a right R-module with a FFR as in Exercise 15.

(1) Using *RM*-(5.5), show that the integer

$$\chi(M) := \sum_{i=0}^{n} (-1)^i \text{ rank } (F_i)$$

depends only on M; i.e. it is independent of the particular FFR chosen. ($\chi(M)$ is called the *Euler characteristic* of M.)

(2) If R is commutative and $S \subseteq R$ is any multiplicative set, show that $\chi(M) = \chi(M_S)$, where M_S denotes the RS^{-1}-module obtained by localizing M at S.

(3) If R is a commutative domain with quotient field K, show that $\chi(M) = \dim_K(M \otimes_R K)$.

Solution. (1) Suppose $0 \to G_n \to \cdots \to G_1 \to G_0 \to 0$ is another FFR. (It is harmless to use the same n since we can always "add" zero modules to either sequence.) By the generalized version of Schanuel's Lemma *LMR*-(5.5), we have

$$F_0 \oplus G_1 \oplus F_2 \oplus \cdots \cong G_0 \oplus F_1 \oplus G_2 \oplus \cdots .$$

Taking ranks, we set

$$\text{rk } F_0 + \text{rk } G_1 + \text{rk } F_2 + \cdots = \text{rk } G_0 + \text{rk } F_1 + \text{rk } G_2 + \cdots ,$$

and transposition yields

$$\text{rk } F_0 - \text{rk } G_1 + \text{rk } F_2 - \cdots = \text{rk } G_0 - \text{rk } F_1 + \text{rk } G_2 - \cdots .$$

This shows that $\chi(M) \in \mathbb{Z}$ is well-defined.

(2) Follows easily from the fact that localization is exact.

(3) By (2), $\chi_R(M) = \chi_K(M \otimes_R K)$. As K is a field, $M \otimes_R K$ is K-free, so the RHS is just $\dim_K(M \otimes_R K)$.

Ex. 5.21B. Let M be an R-module with a FFR of length n as in $(*)$ in Ex. 5.15. If M can be generated by m elements, show that there exists a FFR

$$0 \longrightarrow G_s \longrightarrow \cdots G_1 \longrightarrow G_0 \longrightarrow M \to 0$$

with $G_0 = R^m$ and $s = \max\{2, n\}$.

Solution. Given $(*)$ as in Ex. 5.15, let $X_i = \text{im}(\alpha_i)$, Fix an exact sequence $0 \to Y_1 \to G_0 \to M \to 0$ where $G_0 = R^m$. By Schanuel's Lemma, $Y_1 \oplus F_0 \cong X_1 \oplus G_0$, so Y_1 is f.g. Now take an exact sequence

$$0 \longrightarrow Y_2 \longrightarrow G_1 \longrightarrow Y_1 \longrightarrow 0$$

where G_1 is f.g. free. Applying the generalized version of Schanuel's Lemma (*LMR*-(5.5)) to the sequences

$$0 \longrightarrow X_2 \longrightarrow F_1 \longrightarrow F_0 \longrightarrow M \longrightarrow 0,$$

$$0 \longrightarrow Y_2 \longrightarrow G_1 \longrightarrow G_0 \longrightarrow M \longrightarrow 0,$$

we get $Y_2 \oplus F_1 \oplus G_0 \cong X_2 \oplus G_1 \oplus F_0$, so Y_2 is f.g. Continuing this construction, we arrive at

$$0 \longrightarrow Y_n \longrightarrow G_{n-1} \longrightarrow Y_{n-1} \longrightarrow 0$$

with all G_i f.g. free and

$$Y_n \oplus F_{n-1} \oplus G_{n-2} \oplus \cdots \cong X_n \oplus G_{n-1} \oplus F_{n-2} \oplus \cdots,$$

Since $X_n \cong F_n$ is f.g. free, we have $Y_n \oplus R^\ell \cong R^k$ for some $k,\ \ell \geq 0$. If $n \geq 2$, we can "add" R^ℓ to Y_n and G_{n-1} in the sequence

$$0 \longrightarrow Y_n \longrightarrow G_{n-1} \longrightarrow G_{n-2} \longrightarrow \cdots \longrightarrow G_0 \longrightarrow M \longrightarrow 0$$

to get a FFR for M (of the same length n) with $G_0 = R^m$. If $n = 1$ instead, we can add a zero term $F_2 = 0$ to the original resolution in the construction above, ending with a FFR

$$0 \longrightarrow G_2 \longrightarrow G_1 \longrightarrow R^m \longrightarrow M \longrightarrow 0.$$

Comment. For a nice application of this exercise to $\mu(M)$, the least number of generators for the module M_R, see Ex. 5.22F and Ex. 12.12.

Ex. 5.21C. Let R be a ring, and let $0 \to L \xrightarrow{f} M \xrightarrow{g} N \to 0$ be exact in $\mathfrak{M}_R$. If each of L, N has a FFR, show that M also has a FFR. Assuming that R has IBN, show that $\chi(M) = \chi(L) + \chi(N)$ in $\mathbb{Z}$.

Solution. Let $0 \to F_n \xrightarrow{\alpha_n} \cdots \to F_0 \xrightarrow{\alpha_0} L \to 0$ be a FFR for L, and $0 \to H_m \xrightarrow{\gamma_m} \cdots \to H_0 \xrightarrow{\gamma_0} N \to 0$ be a FFR for N. Since H_0 is free, there exists $\tau : H_0 \to M$ such that $g \circ \tau = \gamma_0$. Letting $G_0 = F_0 \oplus H_0$, we can define $\beta_0 : G_0 \to M$ by $\beta_0(x, y) = f\alpha_0(x) + \tau(y)$ for $x \in F_0$ and $y \in H_0$. It is routine to check that the following diagram is commutative:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & F_0 & \xrightarrow{f_0} & G_0 & \xrightarrow{g_0} & H_0 & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \alpha_0} & & \downarrow{\scriptstyle \beta_0} & & \downarrow{\scriptstyle \gamma_0} & & \\
0 & \longrightarrow & L & \xrightarrow{f} & M & \xrightarrow{g} & N & \longrightarrow & 0.
\end{array}$$

Here the top row is the natural exact sequence associated with the direct sum $G_0 = F_0 \oplus H_0$. Now let L_0, M_0, and N_0 be the kernels of α_0, β_0, and γ_0 respectively. The Snake Lemma yields an exact sequence $0 \to L_0 \to M_0 \to N_0 \to 0$, so we can now repeat the construction (since F_1 maps onto L_0 by α_1, and H_1 maps onto N_0 by γ_1). In a finite number of steps, we will arrive at a FFR for M.

Now assume that R has IBN. The above construction yields a FFR for M with the free modules $G_i = F_i \oplus H_i$. Therefore, we have

$$\begin{aligned}\chi(M) &= \sum (-1)^i \operatorname{rank}(G_i) \\ &= \sum (-1)^i \left[\operatorname{rank}(F_0) + \operatorname{rank}(H_0)\right] \\ &= \chi(L) + \chi(N).\end{aligned}$$

Comment. It can be shown, more generally, that if any two of L, M, N have FFRs then so does the third. For the remaining two cases not covered by this Exercise, we refer the reader to Lang's "Algebra", pp. 843–844.

Ex. 5.22A. Let $(R, \mathfrak{m})$ be a commutative local ring with the following property:

$(*)$ $\qquad \operatorname{ann}(I) \neq 0$ for any f.g. ideal $I \subseteq \mathfrak{m}$.

Show that any R-module M with FFR is free (and therefore $\chi(M) = \operatorname{rk}(M)$.

Solution. With an obvious induction on the length of a FFR, we may work in the case of a short exact sequence

$$0 \longrightarrow R^s \longrightarrow R^t \longrightarrow M \longrightarrow 0.$$

Fix an epimorphism $\varphi : R^r \to M$ where r is the minimal number of generators for M, and let $K = \ker(\varphi)$. Then $\subseteq (R^r)\mathfrak{m}$. By Schanuel's Lemma (*LMR*-(5.1)), we have $K \oplus R^t \cong R^r \oplus R^s$. Hence K is f.g. projective, and therefore free. Expressing a (finite) basis of K by the canonical basis of R^r involves a finite number of coefficients from $\mathfrak{m}$, which, by $(*)$, can be killed by a nonzero element $b \in R$. But then $Kb = 0$, which implies that $K = 0$ since K is free. Therefore, $M \cong R^r$ is free, as claimed.

Ex. 5.22B. For any nonzero element a in a commutative ring R, let $\mathfrak{p}$ be a minimal prime over the ideal $J = \operatorname{ann}(a)$. Show that the local ring $R_\mathfrak{p}$ has the property $(*)$ in Ex. 5.22A. Deduce from this that $\chi(M) \geq 0$ for any module M with FFR over a nonzero commutative ring R.

Solution. Since a cannot be killed by any element in $R \setminus \mathfrak{p}$, we have $a/1 \neq 0 \in R_\mathfrak{p}$. To check the alleged property $(*)$ for $R_\mathfrak{p}$, consider any f.g. ideal $I \subseteq \mathfrak{p}R_\mathfrak{p}$. As $\mathfrak{p}R_\mathfrak{p}$ is a minimal prime over $J_\mathfrak{p}$, $I^n \subseteq J_\mathfrak{p}$ for some n. Thus, $aI^n = 0$ in $R_\mathfrak{p}$. Rechoosing n to be minimal for $aI^n = 0$, we have then $n \geq 1$ and $0 \neq aI^{n-1} \subseteq \operatorname{ann}(I)$, as desired.

For the second part of the exercise, apply the above to $a = 1$. Here, $\mathfrak{p}$ is any minimal prime for R, and $R_\mathfrak{p}$ has the property $(*)$. For any R-module M with FFR, $M_\mathfrak{p}$ is *free* by Ex. 5.22A. Thus, $\chi(M) = \chi(M_\mathfrak{p}) = \operatorname{rk}(M_\mathfrak{p}) \geq 0$.

Comment. In *LMR*-(1.38), two proofs were given for the fact that any nonzero commutative ring R satisfies the "Strong Rank Condition"; that is, if $F_1 \subseteq F_0$ are free R-modules of finite rank, then rk $(F_1) \leq$ rk (F_0). It is worth noting that the last part of this exercise provides yet another proof of this, since the short free resolution $0 \to F_1 \to F_0 \to F_0/F_1 \to 0$ leads to rk $(F_0) -$ rk $(F_1) = \chi(F_1/F_0) \geq 0$. For more information on the case where rk $(F_0) =$ rk (F_1), see Ex. 5.23B.

The result that $\chi(M) \geq 0$ can also be proved for modules M_R with FFR over a right noetherian ring R. The proof for this, however, requires some new noncommutative techniques: see the solution to Ex. 12.11 below.

For the next exercise, it is convenient to use a concept and a notation introduced in *LMR*-§8. For an ideal A in a commutative ring R, we say that A is *dense* in R (written $A \subseteq_d R$) if $\text{ann}(A) = 0$. The following exercise gives a nice criterion for $\chi(M)$ to be positive over a nonzero commutative ring.

Ex. 5.22C. (Vasconcelos) Let M be any module with FFR over a commutative ring $R \neq 0$. Use the two previous exercises to show that:

(1) $\chi(M) > 0 \implies \text{ann}(M) = 0$.
(2) $\chi(M) = 0 \implies \text{ann}(M) \subseteq_d R$.

In particular, $\chi(M) > 0$ iff M is faithful as an R-module.

Solution. It suffices to prove (1) and (2), since the zero ideal cannot be dense in $R \neq 0$.

(1) Assume $\chi(M) > 0$ but there is a *nonzero* $a \in \text{ann}(M)$. Let $\mathfrak{p}$ be a minimal prime over $\text{ann}_R(a)$ (so that $a/1 \neq 0$ in $R_\mathfrak{p}$ as we've noted earlier). By Exercises 5.22A and 5.22B, $M_\mathfrak{p} \cong R_\mathfrak{p}^n$ for some n. But then $n = \chi(M_\mathfrak{p}) = \chi(M) > 0$, and $M_\mathfrak{p}\, a = 0$ provides a contradiction.

(2) Assume $\chi(M) = 0$, but there exists $a \in R \setminus \{0\}$ annihilating $\text{ann}(M)$. Let $\mathfrak{p}$ be as above and argue as in (1). Here, $\chi(M) = 0$ yields $M_\mathfrak{p} = 0$. Since M is f.g. as an R-module, $\text{ann}(M)_\mathfrak{p} = \text{ann}(M_\mathfrak{p}) = R_\mathfrak{p}$. (For a detailed proof of the first inequality, see the solution to Ex. 8.5A.) But this is killed by $a/1 \in R_\mathfrak{p} \setminus \{0\}$, a contradiction.

Comment. The results in this exercise appeared in W. V. Vasconcelos's paper "Annihilators of modules with a finite free resolution", Proc. A.M.S. **29** (1971), 440–442. In this paper, as well as in other expositions on the subject of FFR (e.g. Northcott's book "Finite Free Resolutions", Cambridge Univ. Press, 1976), the proofs of the results in this exercise were couched in the language of the *Fitting invariant* $\mathfrak{F}(M)$ of the module M. Our formulation of the solution above shows that the use of $\mathfrak{F}(M)$ is not necessary. In fact, since the Fitting invariant $\mathfrak{F}(M)$ is an ideal between $\text{ann}(M)$ and $\text{ann}(M)^n$ (where n is the number of generators of M), and in general $\text{ann}(M) \subseteq_d R$ iff $\text{ann}(M)^n \subseteq_d R$, it follows from (2) above that $\chi(M) = 0 \Rightarrow \mathfrak{F}(M) \subseteq_d R$, however $\mathfrak{F}(M)$ is defined!

Ex. 5.22D. Let M, R be as in Ex. 5.22C, and assume that R satisfies the ACC on annihilator ideals (e.g. R is a nonzero commutative noetherian ring). Show that $\chi(M) = 0$ iff $\text{ann}(M)$ contains a non 0-divisor of R.

Solution. This follows from Ex. 5.22C, assuming Lance Small's result that, under the current assumptions on R, an ideal of R is dense iff it contains a non 0-divisor. Since this result is proved in full in *LMR*-(8.31)(2), we will not repeat the work here.

Comment. If no chain condition is imposed on the commutative ring R, it may happen that $\chi(M) = 0$ without $\text{ann}(M)$ containing any non 0-divisor of R. In Vasconcelos's paper referred to in the *Comment* on Ex. 5.22C, a commutative (non-noetherian) local ring $(R, \mathfrak{m})$ is constructed for which $M := R/\mathfrak{m}$ has a free resolution

$$(*) \qquad 0 \longrightarrow R \longrightarrow R^2 \longrightarrow R \xrightarrow{\pi} M \longrightarrow 0$$

(so $\chi(M) = 0$), but $\text{ann}(M) = \mathfrak{m}$ is composed of 0-divisors.

In this example, the module M has a free resolution $(*)$ of length 2. It turns out that this is the shortest length of an FFR for which such a counterexample is possible, as Ex. 5.23B below shows.

Ex. 5.22E. If a nonzero ideal I of a commutative ring R has a FFR (as an R-module), show that $I \subseteq_d R$ and $\chi(I) = 1$.

Solution. Let $0 \to F_n \to \cdots \to F_0 \xrightarrow{\varphi} I \to 0$ be a FFR. Then

$$0 \longrightarrow F_n \longrightarrow \cdots \longrightarrow F_0 \xrightarrow{\varphi} R \longrightarrow R/I \longrightarrow 0$$

is a FFR for R/I, so

$$\chi(R/I) = 1 - \text{rk } F_0 + \text{rk } F_1 - \cdots = 1 - \chi(I).$$

Since $\text{ann}(R/I) = I \neq 0$, Ex. 5.22C gives $\chi(R/I) = 0$, and so $\chi(I) = 1$. Also by Ex. 5.22C, $\chi(R/I) = 0$ implies that $I = \text{ann}(R/I) \subseteq_d R$.

Comment. The conclusion $I \subseteq_d R$ here cannot be strengthened into "I contains a non 0-divisor". In fact, in Vasconcelos's example mentioned in the *Comment* on the last exercise, the maximal ideal $\mathfrak{m} \subset R$ has $\chi(\mathfrak{m}) = 1$, but $\mathfrak{m}$ (is dense and) consists of 0-divisors.

A nice illustration for this exercise is given by the case where I is generated by a regular sequence of length $n \geq 1$. Here, an FFR for I is given by the Koszul resolution

$$0 \longrightarrow \Lambda^n(R^n) \longrightarrow \cdots \longrightarrow \Lambda^2(R^n) \longrightarrow \Lambda^1(R^n) \to I \longrightarrow 0.$$

(See *LMR*-(5.33).) From this explicit FFR, we get

$$\chi(I) = \binom{n}{1} - \binom{n}{2} + \cdots + (-1)^{n-1}\binom{n}{n} = \binom{n}{0} = 1.$$

In this example, we have, of course, $I \subseteq_d R$, since the first element of the regular sequence is a non 0-divisor in R.

Ex. 5.22F. Let M be a module with FFR over a nonzero commutative ring R. Show that $\chi(M) \leq \mu(M)$ (the least number of generators for M), with equality iff M is free.

Solution. Let $m = \mu(M)$. By Ex. 5.21B, there exists a FFR

$$0 \longrightarrow G_s \longrightarrow \cdots \longrightarrow G_1 \xrightarrow{\alpha} G_0 \longrightarrow M \longrightarrow 0$$

with $G_0 = R^m$. Let $K = \operatorname{im}(\alpha) \subseteq G_0$. Then

$$\chi(M) = \operatorname{rk} G_0 - \operatorname{rk} G_1 + \cdots = m - \chi(K) \leq m$$

by Ex. 5.22B. If M is free, of course $\chi(M) = \operatorname{rk}(M) = m$. Conversely, if $\chi(M) = m$, the above work show that $\chi(K) = 0$. Consider any vector $\sum e_i r_i \in K$ where $\{e_1, \ldots, e_m\}$ is a basis for $G_0 = R^m$. For any $r \in \operatorname{ann}(K)$, we have $\sum e_i r_i r = 0$. Thus, $r_i r = 0$ for all i; that is, $r_i \in \operatorname{ann}_R(\operatorname{ann}(K))$. Since $\operatorname{ann}(K) \subseteq_d R$ by Exer. 5.22C(2), each $r_i = 0$. This shows that $K = 0$, and hence $M \cong R^m$ is free.

Ex. 5.23A. For an $n \times m$ matrix $A = (a_{ij})$ over a commutative ring, the ith determinantal ideal $D_i(A)$ is defined to be the ideal in R generated by all $i \times i$ minors of A (if any). (For convenience, we define $D_0(A)$ to be R.) When $R \neq 0$, the (McCoy) rank of A is, by definition,

$$\begin{aligned}\operatorname{rank}(A) &= \max\{i : D_i(A) \subseteq_d R\}\\ &= \max\{i : \operatorname{ann}(D_i(A)) = 0\}.\end{aligned}$$

(1) Show that $D_0(A) \supseteq D_1(A) \supseteq \cdots$, and that $\operatorname{rank}(A) \leq \min\{m, n\}$.

(2) Show that, in the case where R is a commutative domain, rank(A) coincides with the usual rank of A as a matrix over the quotient field of R.

(3) (McCoy) show that the linear system $A(x_1, \ldots, x_m\}^T = 0$ has a nontrivial solution (over R) iff $\operatorname{rank}(A) < m$.

Solution. (1) $D_i(A) \supseteq D_{i+1}(A)$ follows by computing $(i+1) \times (i+1)$ minors via a row expansion. If $R \neq 0$ and $i > \min\{m, n\}$, then $D_i(A) = 0$ cannot be dense, so $\operatorname{rank}(A)$ cannot exceed $\min\{m, n\}$.

(2) is clear since a field F has only two ideals, namely, F (which is dense), and (0) (which is not dense). In view of this, $D_i(A)$ being dense over a field simply means that *some* $i \times i$ minor of A is nonzero.

(3) For the "only if" part, let $(x_1, \ldots, x_m)^T \neq 0$ be a solution of the linear system. If B is any $m \times m$ submatrix of A, then $B \cdot (x_1, \ldots, x_m)^T = 0$. Multiplying this from the left by $\operatorname{adj}(B)$, we get $(\det B)\,(x_1, \ldots, x_m)^T = 0$. This shows that $D_m(A)$ is killed by $\sum x_i R \neq 0$, so $\operatorname{rank}(A) < m$.

Conversely, assume that $r := \operatorname{rank}(A) < m$. By adding zero rows to A if necessary, we may assume that $r < n$ as well. Fix a nonzero $a \in R$ that kills $D_{r+1}(A)$. If $r = 0$, then a kills all a_{ij}, so $(a, \ldots, a)^T \neq 0$ solves our linear system. We may thus assume $r > 0$. There exists an $r \times r$ submatrix B of A with $a \cdot \det(B) \neq 0$. For convenience, let us assume that B is in the upper left corner of A. Let C be the $(r+1) \times (r+1)$ matrix in the upper left corner of A, and left $d_1, \dots, d_{r+1}$ be the cofactors of C associated with the entries on the last row of C. *We finish by checking that the vector*

$$(*) \qquad (ad_1, \dots, ad_{r+1}, \ 0, \dots, 0)^T \in R^m$$

is a solution to our linear system (and noting that, by choice, $ad_{r+1} = a \cdot \det(B) \neq 0$). Indeed, taking the "dot product" of $(*)$ with the ith row of A, we get $a \sum_{j=1}^{r+1} a_{ij} d_j$. If $i \leq r$, this is zero, since the summation is the determinant of the matrix C with its last row replaced by $(a_{i1}, \dots, a_{i,\,r+1})$. On the other hand, if $i \geq r+1$, the summation is an $(r+1) \times (r+1)$ minor of A, which is killed by a. This completes the proof that $(*)$ is a (nontrivial) solution of the linear system in question.

Comment. It is perhaps not surprising to note that (3) above also implies the Strong Rank Condition for a nonzero commutative ring R. Indeed, if $n < m$, then

$$\operatorname{rank}(A) \leq \min\{m, n\} = n < m,$$

So the n equations $\sum_{j=1}^{m} a_{ij} x_j = 0$ have a nontrivial solution by (3).

The determinntal ideals $D_i(A)$ are important invariants of the $n \times m$ matrix A over the commutative ring R. If M is the cokernel of the R-homomorphism $R^m \to R^n$ define by the matrix A, the $D_i(A)$'s are the *Fitting invariants* of the finitely presented R-module M. Of special interest is the *initial Fitting invariant* $\mathfrak{F}(M) := D_n(A)$, which bears the following relationship to the annihilator ideal of M:

$$\operatorname{ann}(M)^n \subseteq \mathfrak{F}(M) \subseteq \operatorname{ann}(M).$$

From this, it is easy to see that $\operatorname{ann}(M) \subseteq_d R$ iff $\mathfrak{F}(M) \subseteq_d R$, as was already pointed out in the *Comment* on Ex. 5.22C.

The McCoy rank of a rectangular matrix over a commutative ring was defined by Neal McCoy in his paper, "Remarks on divisors of zero," MAA Monthly **49** (1942), 286–295. The result (3) in the present exercise is taken from this paper. For an excellent exposition on the McCoy rank, see McCoy's Carus Monograph "Rings and Ideals," Math. Assoc. America, 1948. For some applications of part (3) above, see the next exercise, as well as Ex. 11.9 in Chapter 4.

Ex. 5.23B. Let R be a nonzero commutative ring.

(1) Deduce from (3) of Ex. 5.23A that the columns of a matrix $A \in \mathbb{M}_n(R)$ are linearly independent iff $\delta := \det(A)$ is a non 0-divisor of R.

(2) Use (1) to show that, if an R-module M has a FFR of length 1 and $\chi(M) = 0$, then $\operatorname{ann}(M)$ contains a non 0-divisor of R.

Solution. (1) The linear independence of the columns amounts to $A \cdot (x_1, \ldots, x_n)^T = 0$ having only the trivial solution. By Ex. 5.23A, this amounts to $r = n$, where $r = \text{rank}(A)$. Since $D_n(A) = \delta R$, $r = n$ simply means that δ is a non 0-divisor.

(2) Given a free resolution $0 \to F_1 \to F_0 \to M \to 0$ where $\chi(M) = 0$, we may assume that $F_0 = R^n$ and that F_1 is freely generated by the linearly independent columns $c_1, \ldots, c_n$ of some matrix $A \in \mathbb{M}_n(R)$. For $\delta = \det(A)$, we have an equation

$$\text{diag}(\delta_1, \ldots, \delta_n) = A \cdot \text{adj}(A) = (c_1, \ldots, c_n)(b_{ij})$$

where $(b_{ij}) = \text{adj}(A)$. Comparing the ith columns shows that $e_i\delta \in$ span $\{c_1, \ldots, c_n\} = F_1$ (where e_i's are the unit vectors in R^n). Thus, $\delta \in \text{ann}(R^n/F_1) = \text{ann}(M)$, and we know that δ is a non 0-divisor from (1).

Comment. As was pointed out by McCoy, the two conditions in (1) above are also equivalent to each of the following:

(a) A is not a left 0-divisor in $\mathbb{M}_n(R)$;
(b) A is not a right 0-divisor in $\mathbb{M}_n(R)$.

The equivalence again follows easily from part (3) of Ex. 5.23A.

Ex. 5.24. Let $R = \left\{ \begin{pmatrix} x & u & v \\ 0 & x & w \\ 0 & 0 & y \end{pmatrix} \right\}$, where x, y, u, v, w are arbitrary elements in a division ring k. Show that the artinian ring R has exactly two simple right modules M_1, M_2, each 1-dimensional over k, with $\text{pd}_R(M_1) = \infty$ and $\text{pd}(M_2) = 0$. What are the projective shifts of M_1 and the Jacobson radical of R?

Solution. First, the Jacobson radical of R is given by $J = \left\{ \begin{pmatrix} 0 & u & v \\ 0 & 0 & w \\ 0 & 0 & 0 \end{pmatrix} \right\}$, since J is nilpotent, and $R/J \cong k \times k$.

This enables us to construct the two simple (right) modules of R. They are just the two simple (right) modules M_1, M_2 of $k \times k$, with right R-action given via the projection $R \longrightarrow k \times k$ sending $\begin{pmatrix} x & u & v \\ 0 & x & w \\ 0 & 0 & y \end{pmatrix}$ to (x, y). Thus, we can take M_1, M_2 to be each a copy of k, where $\begin{pmatrix} x & u & v \\ 0 & x & w \\ 0 & 0 & y \end{pmatrix}$ acts on M_1 (resp. M_2) by right multiplication by x (resp. y).

Let $e_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ and $e_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. These are idempotents in R with sum 1, giving rise to the projective right ideals:

$$P_1 = e_1R = \left\{ \begin{pmatrix} x & u & v \\ 0 & x & w \\ 0 & 0 & 0 \end{pmatrix} \right\} \quad \text{and} \quad P_2 = e_2R = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & y \end{pmatrix} \right\}.$$

Clearly $P_2 = P_2/P_2J \cong M_2$, and we see easily that $P_1J = J$ and $P_1/P_1J \cong M_1$. The latter shows that P_1 is indecomposable, so $\{P_1, P_2\}$ is the full set of the principal indecomposables. Let $\mathcal{P}$ denote the projective shift operator. Of course, $\text{pd}(M_2) = 0$ and $\mathcal{P}^n[M_2] = 0$. It remains to determine $\mathcal{P}^n(J)$ and $\mathcal{P}^n[M_1]$.

The first projective shift $\mathcal{P}[M_1]$ is easy: the exact sequence $0 \longrightarrow J \longrightarrow P_1 \longrightarrow M_1 \longrightarrow 0$ shows that $\mathcal{P}[M_1] = [J]$, where, of course, J is viewed as a right R-module. Now look at the two elements

$$a = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad b = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

An easy computation shows that

$$\text{ann}_r\, a = \left\{ \begin{pmatrix} 0 & u & v \\ 0 & 0 & 0 \\ 0 & 0 & y \end{pmatrix} \right\} = bR \quad \text{and} \quad \text{ann}_r(b) = \left\{ \begin{pmatrix} 0 & u & v \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right\} = aR.$$

Thus, we are in the situation of *LMR*-(5.16). In particular, we have the exact sequences

$$0 \longrightarrow aR \hookrightarrow R \xrightarrow{\varphi} bR \to 0 \quad \text{and} \quad 0 \longrightarrow bR \hookrightarrow R \xrightarrow{\psi} aR \longrightarrow 0,$$

with $\varphi(s) = bs$ and $\psi(s) = as$. Notice that bR is not projective, for otherwise the first sequence would split and aR would contain a nonzero idempotent. This is impossible since $(aR)^2 = 0$. Thus, as in *LMR*-(5.16), $\mathcal{P}[aR] = [bR]$, $\mathcal{P}[bR] = [aR]$, and $\text{pd}(aR) = \text{pd}(bR) = \infty$. Finally, observe that there is an R-isomorphism $\sigma : J_R \longrightarrow bR$ given by

$$\sigma \begin{pmatrix} 0 & u & v \\ 0 & 0 & w \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & u & v \\ 0 & 0 & 0 \\ 0 & 0 & w \end{pmatrix},$$

and a similar construction shows that $J \cong aR \oplus M_2$. Thus, aR, bR, *and* J *are all projectively equivalent.* Since $\mathcal{P}(M_1) = [J]$, we have

$$\mathcal{P}^n(M_1) = \mathcal{P}^n(J) = [J] = [aR] = [bR]$$

for all $n > 0$. The module J_R acts somewhat like a "black hole" for the projective shift. In particular, $\text{pd}(J) = \text{pd}(M_1) = \infty$.

Ex. 5.25. Let $\varphi : R \to S$ be a ring homomorphism, and let $n = \text{pd}(S_R)$ where S is viewed as a right R-module via φ. Show that, for any right S-module M, $\text{pd}(M_R) \leq n + \text{pd}(M_S)$.

Solution. We may assume that $n < \infty$ and $m := \text{pd}(M_S) < \infty$. The proof proceeds by induction on m. First assume $m = 0$. Then M_S is projective, so $M \oplus X \cong F$ for some S-modules X, F, where F is S-free. Then by *LMR*-(5.25),

$$\text{pd}(M_R) \leq \text{pd}(F_R) = \text{pd}(S_R) = n,$$

so the desired conclusion holds in this case. Now assume $m > 0$. Fix an exact sequence $0 \to K \to F \to M \to 0$ in $\mathfrak{M}_S$ where F is S-free. Then $\text{pd}(K_S) = m - 1$, so by the inductive hypothesis, $\text{pd}(K_R) \leq n + (m-1)$. Now by (a) of Exercise 1 (applied to $K_R \subseteq F_R$),

$$\begin{aligned} \text{pd}(M_R) &\leq \max\{\text{pd}(K_R),\ \text{pd}(F_R)\} + 1 \\ &\leq \max\{n + m - 1,\ n\} + 1 \\ &= (n + m - 1) + 1 = n + m, \end{aligned}$$

as desired.

Ex. 5.26. (Bass) Let R be a right noetherian ring and let $\{M_i : i \in I\}$ be a direct system of right R-modules, with direct limit M. If $\text{id}(M_i) \leq n$ for all $i \in I$, show that $\text{id}(M) \leq n$.

Solution. (Sketch) The idea here is to construct a "canonical injective resolution" of any module N_R. Let U be the injective hull of $\bigoplus R/\mathfrak{A}$, where $\mathfrak{A}$ ranges over all right ideals of R. For any N_R, let $\tilde{N} = \text{Hom}_R(N, U)$ and let $F(N)$ be the direct product $\prod_{\tilde{N}} U$, which is an injective R-module. We have a natural R-homomorphism $\varepsilon_N : N \longrightarrow F(N)$ defined by $\varepsilon_N(a) = \{f(a)\}_{f \in \tilde{N}}$ for any $a \in N$. It is easy to check that F is a (covariant) functor, and ε defines a natural transformation from the identity functor to the functor F. Now, for any $0 \neq a \in N$, $aR \cong R/\mathfrak{A}$ for some right ideal $\mathfrak{A}$. Since U contains a copy of $R/\mathfrak{A}$ and is injective, there exists $f \in \tilde{N}$ such that $f(a) \neq 0$, and so $\varepsilon_N(a) \neq 0 \in F(N)$. This shows that ε_N is an *embedding*. We can now form $F(N)/\varepsilon_N(N)$ and repeat the construction. In this way, we get a canonical ("functorial") injective resolution for N.

Suppose now $\text{id}(M_i) \leq n$ for all $i \in I$ as in the exercise. The above constructions lead to a direct system of injective resolutions of length n for the system $\{M_i\}_{i \in I}$. Taking the direct limit of these resolutions (see *Comment*), we get an *injective* resolution of length n for $\varinjlim M_i = M$, since, over a right noetherian ring, the direct limit of injective right modules remains injective (see *LMR*-(3.46)). This proves that $\text{id}(M) \leq n$, as desired.

Comment. In the above proof, we have used the exactness of the direct limit over direct systems of R-modules. For a proof of this, see p. 271 of Matsumura's book, "Commutative Ring Theory," Cambridge University Press, 1986.

For any R-module N, let $E(N)$ denote its injective hull. It is well-known that this leads to a "minimal injective resolution" of N: we form $E(N)/N$ and embed it into its injective hull $E(E(N)/N)$, etc. Unfortunately, "E" is not a functor, since a map $N \longrightarrow N'$ does not induce a natural map from $E(N)$ to $E(N')$. Therefore, we could not have used the minimal injective resolution in the argument above. The idea is to replace this minimal resolution with the one we used in the solution. In the terminology of *LMR*-§19, the module U is an "injective cogenerator" for the category of right R-modules. The construction of this "canonical" injective resolution was attributed by H. Bass to C. Watts in Bass' paper, "Injective dimensions in noetherian rings," Trans. Amer. Math. Soc. **102** (1962), 18–29.

Ex. 5.27. (Osofsky) For any right noetherian ring R, prove that $\text{r.gl.}\dim R = \sup\{\text{id}(C)\}$, where C ranges over all cyclic right R-modules. [This is the injective dimension analogue of the formula $\text{r.gl.}\dim R = \sup\{\text{pd}(C)\}$ (due to M. Auslander) proved in *LMR*-(5.51).]

Solution. We need only prove that $\text{r.gl.}\dim\ R \leq \sup\{\text{id}(C)\}$. For this, we may assume that $n = \sup\{\text{id}(C)\} < \infty$. We shall prove that, for any f.g. right R-module N, $\text{id}(N) \leq n$. Since any right R-module M is a direct limit of f.g. modules, the last exercise will imply that $\text{id}(M) \leq n$, and therefore $\text{r.gl.}\dim R \leq n$ by *LMR*-(5.45).

To prove the inequality $\text{id}(N) \leq n$, we induct on the number of generators needed to generate N as an R-module. If N can be generated by one element, then N is cyclic and $\text{id}(N) \leq n$ by the definition of n. If N is generated by $m \geq 2$ elements, then there exists a cyclic submodule $C \subseteq N$ such that N/C can be generated by $m - 1$ elements. Assuming the injective dimension analogue of *LMR*-(5.23) (which can be proved in the same manner as the latter), we have

$$\text{id}(N) \leq \max\{\text{id}(C),\ \text{id}(N/C)\}.$$

Since $\text{id}(N/C) \leq n$ by the inductive hypothesis and $\text{id}(C) \leq n$ by the definition of n, we have $\text{id}(N) \leq n$, as desired.

Comment. The result in this exercise appeared in Osofsky's paper "Global dimensions of valuation rings," Trans. Amer. Math. Soc. **126** (1967), 136–149. For a right noetherian ring R, Osofsky also proved that $\text{r.gl.}\dim R = \sup\{\text{id}(\mathfrak{A})\}$, where $\mathfrak{A}$ ranges over all the right ideals of R.

Chapter 3
More Theory of Modules

§6. Uniform Dimensions, Complements, and CS Modules

A module M_R is said to have *uniform dimension* n (written $\text{u.dim}\, M = n$) if there is an essential submodule $V \subseteq_e M$ that is a direct sum of n uniform submodules. Here, n is a nonnegative integer. (The fact that $\text{u.dim}\, M$ is well-defined is a consequence of the Steinitz Replacement Theorem *LMR*-(6.1).) If no such integer n exists, we define $\text{u.dim}\, M$ to be the symbol ∞.

We have $\text{u.dim}\, M = 0$ *iff* $M = (0)$, and $\text{u.dim}\, M = 1$ *iff* M *is uniform.* Modules with $\text{u.dim}\, M = \infty$ also have an interesting characterization; that is, that M contains an infinite direct sum of nonzero submodules. This requires a proof, which is given in *LMR*-(6.4). Given this information, one can further check that, for any module M, $\text{u.dim}\, M$ can be computed as the supremum of

$$\{k : M \text{ contains a direct sum of } k \text{ nonzero submodules}\}.$$

In some books, this is taken as the definition of $\text{u.dim}\, M$. Another important interpretation of finite uniform dimension is the following: $\text{u.dim}\, M = n$ *iff the injective hull* $E(M)$ *is a direct sum of* n *indecomposable injective modules.*

In general, the uniform dimension is additive over finite direct sums, but *not* over short exact sequences! Also, the usual dimension formula relating the dimensions of sums and intersections of vector subspaces does not work for the uniform dimension. For instance, it is possible for M to contain two submodules A, B such that u.dim A and u.dim B are finite, but u.dim $(A + B) = \infty$: see Exercises 7–8.

The notion of complements is quite useful in module theory. Given a submodule $S \subseteq M_R$, a submodule $C \subseteq M$ is a *complement* to S in M if C is maximal with respect to the property that $C \cap S = 0$. A submodule $C \subseteq M$ is said to be a complement (written $C \subseteq_c M$) if it is a complement to *some* submodule S. An important fact here is that $C \subseteq_c M$ *iff* C *is closed in* M (in the sense that M contains no submodule that is a proper essential extension of C). The many useful facts proved for closed submodules in *LMR*-§6B,C are worth reviewing before proceeding to the exercises in this section.

Two other topics covered in *LMR*-§6D,G are CS modules and QI modules. *A module* M_R *is called* CS *if every* $C \subseteq_c M$ *is a direct summand.* The class of CS modules includes all uniform modules, semisimple modules, and injective modules. A noteworthy property of a CS module M is that u.dim $M < \infty$ iff M is a finite direct sum of uniform modules. An important recent theorem of Osofsky and Smith (*LMR*-(6.44)) offers a way to check that certain modules are finite direct sums of uniform modules. A classical result of Utumi (*LMR*-(6.48)) asserts that if a ring R is such that ${}_RR$ and R_R are CS, then R must be Dedekind-finite. Both of these are fairly deep results.

For exercises, we confine ourselves to a classification of f.g. abelian groups which are CS over $\mathbb{Z}$: see Exercise 19 (in four parts). Exercise 36 explores two useful subclasses of CS modules, called *continuous* and *quasi-continuous* modules. In the context of von Neumann regular rings R, the condition that R_R be CS (or continuous, or quasi-continuous), turns out to be equivalent to a certain continuity axiom studied already many years ago by J. von Neumann: see Exercise 38.

A module M_R is called QI (*quasi-injective*) if, for any submodule $L \subseteq M$, any $f \in \text{Hom}_R(L, M)$ can be extended to an endomorphism of M. These are the modules M that are "fully invariant" in $E(M)$, by *LMR*-(6.74). In general, we have

$$\text{injective} \Longrightarrow \text{QI} \Longrightarrow \text{CS},$$

and any semisimple module is QI. However, the direct sum of two QI modules need not be QI. Exercises 27–39 offer many useful results on QI modules.

Sections §6E and §6F in *LMR* contain a survey of the many finiteness conditions that can be imposed on a ring R. This includes the ACC and DCC on right ideals, principal right ideals, right complements in R, right annihilators in R, and right direct summands in R (and their left analogues). There are various relations among these conditions, for instance, right artinian implies right noetherian, ACC on right annihilators is equivalent to DCC on left annihilators, and ACC on right summands in R

is equivalent to DCC on left summands, etc. The study of the behavior of the various finiteness conditions under the change of rings (subrings, polynomial rings and matrix rings) provides further interesting results in *LMR*-§6F. An easy, but nevertheless surprising, result is that ACC (or DCC) on right annihilators in any ring is inherited by any of its subrings, according to *LMR*-(6.61). A deeper result is that, if $S = R[x]$, then u.dim S_S = u. dim R_R; this is a theorem of R. Shock(*LMR*-(6.65)). In particular, if ACC holds for right summands in R, then ACC also holds for right summands in S. In Exercise 26, these facts are generalized to a polynomial ring over R in any number of commuting variables.

Some of the exercises in this section make use of the set Ass(M) of associated primes of a module M_R, so *LMR*-§3F should be quickly reviewed before the reader proceeds to these exercises. In the case where R is commutative, a prime ideal $\mathfrak{p}$ is an associated prime of M iff $\mathfrak{p}$ has the form ann(m) for some element $m \in M$, or equivalently, iff M contains a copy of the cyclic R-module $R/\mathfrak{p}$.

Exercises for §6

Ex. 6.1A. Recall that a module M_R is called Dedekind-finite (cf. Exercise 1.8) if $M \oplus N \cong M$ (for some module N) implies that $N = 0$. Show that any module M with u.dim $M < \infty$ is Dedekind-finite.

Solution. Suppose $M \oplus N \cong M$. Then

$$\text{u.}\dim M = \text{u.}\dim(M \oplus N) = \text{u.}\dim M + \text{u.}\dim N.$$

Cancelling the finite number u.dim M, we get u.dim $N = 0$, and hence $N = 0$.

Comment. D. Khurana pointed out the following strengthening of this exercise (which is also easily proved): *If* u.dim $M < \infty$, *then* M *is weakly cohopfian*, in the sense that any injective endomorphism on M has an essential image. For an investigation on weakly cohopfian modules in general, see the paper by A. Haghany and M. R. Vedadi in J. Algebra **243** (2001), 765–779.

For readers familiar with the notion of co-uniform dimensions, we might mention the following dual version of Khurana's observation: *if M has finite co-uniform dimension, then M is weakly hopfian*, in the sense that any surjective endomorphism of M has a small kernel K ($K + B = M \Rightarrow B = M$ for any submodule M).

In *LMR*-(6.11), the example of $\mathbb{Q} \to \mathbb{Q}/\mathbb{Z}$ was given to show that a module of finite uniform dimension may have a factor module that has an infinite uniform dimension. In this direction, Khurana has also suggested the following strengthening.

Ex. 6.1B. Give an example of a module M_R with finite uniform dimension such that some factor module of M is not directly finite.

Solution. The following example was constructed by G. Bergman. Let U, V be right vector spaces over a field k such that U has infinite dimension and V has finite dimension > 0. Let $H = \mathrm{Hom}_k(U, V)$ and let R be the triangular ring $\begin{pmatrix} k & H \\ 0 & k \end{pmatrix}$ (with formal matrix multiplication)(*). Then $M := U \oplus V$ is a right module over R with the formal action

$$(u, v)\begin{pmatrix} a & h \\ 0 & b \end{pmatrix} = (ua,\ h(u) + vb).$$

The R-submodule V is essential in M_R since, for any nonzero $u \in U$, there exists $h \in H$ with $h(u) \neq 0$ and hence $(u, v)\begin{pmatrix} 0 & h \\ 0 & 0 \end{pmatrix}$ is a nonzero element in V. Therefore, we have

$$\mathrm{u.dim}\, M = \mathrm{u.dim}\, V = \dim_k V < \infty.$$

But the factor module $M/V \cong U$ has infinite dimension over k and is consequently not Dedekind-finite as a module over R.

Ex. 6.2. Show that $|\mathrm{Ass}(M)| \leq \mathrm{u.dim}\, M$ for any module M.

Solution. Here, we are working under the convention that $|\mathrm{Ass}(M)|$, u.dim M are either finite integers or the symbol ∞. We may thus assume that u.dim $M = n < \infty$. By *LMR*-(6.12),

$$E(M) = N_1 \oplus \cdots \oplus N_n,$$

where the N_i's are indecomposable injective modules. By *LMR*-(3.52), each N_i is uniform, so by *LMR*-(3.59), $|\mathrm{Ass}(N_i)| \leq 1$. Using *LMR*-(3.57) (2) and *LMR*-(3.57) (4), we see that

$$|\mathrm{Ass}(M)| = |\mathrm{Ass}(E(M))| \leq n.$$

Ex. 6.3. Give an example of a f.g. module M (over any given ring $R \neq 0$) such that $|\mathrm{Ass}(M)| = 1$ and u.dim $M = n$ (a prescribed integer).

Solution. Let S be any simple (right) R-module. Then S is a prime module, and $\mathfrak{p} = \mathrm{ann}(S)$ is the unique prime ideal associated to S. For $M = S \oplus \cdots \oplus S$ (n copies), we have clearly u.dim $M = n$, and by *LMR*-(3.57)(3), $\mathrm{Ass}(M) = \{\mathfrak{p}\}$.

Ex. 6.4. Let M_R be any module with u.dim $M = n < \infty$. Show that there exist closed submodules $M_i \subseteq_c M$ $(1 \leq i \leq n)$ with the following properties:

(1) Each M/M_i is uniform.
(2) $M_1 \cap \cdots \cap M_n = 0$.

(*) Here, $H = \mathrm{Hom}_k(U, V)$ is viewed as a (k, k)-bimodule in the natural way: $(ahb)(u) = h(u)\,ab$ for $h \in H$, $a, b \in k$, and $u \in U$.

(3) $E(M) \cong \bigoplus_{i=1}^n E(M/M_i)$.
(4) $\operatorname{Ass}(M) = \bigcup_{i=1}^n \operatorname{Ass}(M/M_i)$.

Solution. Suppose we have found $M_i \subseteq_c M$ $(1 \le i \le n)$ satisfying (1) and (2). Then we have an embedding $f: M \to \bigoplus_{i=1}^n M/M_i$. Since both sides have uniform dimension n, $f(M) \subseteq_e \bigoplus_{i=1}^n M/M_i$. From this, it follows that

$$E(M) \cong E(f(M)) = E\left(\bigoplus_{i=1}^n M/M_i\right) \cong \bigoplus_{i=1}^n E(M/M_i).$$

Also, from *LMR*-(3.57)(2) and *LMR*-(3.57)(4), we have

$$\operatorname{Ass}(M) = \operatorname{Ass}(f(M)) = \operatorname{Ass}\left(\bigoplus_{i=1}^n M/M_i\right) = \bigcup_{i=1}^n \operatorname{Ass}(M/M_i).$$

(Note that this implies $|\operatorname{Ass}(M)| \le n$, since each $|\operatorname{Ass}(M/M_i)| \le 1$ by *LMR*-(3.59). This gives a somewhat more sophisticated view of the result in Exercise 2.)

It remains for us to prove the existence of the M_i's satisfying (1), (2). We do this by induction on n, the case $n = 1$ being clear (upon choosing $M_1 = 0$). For $n \ge 2$, fix a uniform submodule $U \subseteq M$. After replacing U by an essential closure, we may assume that $U \subseteq_c M$. By *LMR*-(6.35), $\text{u.dim}\, M/U = n - 1$. Invoking the inductive hypothesis, we can find $M_1, \cdots, M_{n-1} \supseteq U$ such that $M_i/U \subseteq_c M/U$ and $\text{u.dim}\, M/M_i = 1$ for $i \le n-1$, and with $M_1 \cap \cdots \cap M_{n-1} = U$. Since $U \subseteq_c M$, $M_i/U \subseteq_c M/U$ implies that $M_i \subseteq_c M$ by *LMR*-(6.28). It only remains to construct M_n. Let M_n be a complement to U in M. Then the image of U is essential in M/M_n so $\text{u.dim}\, M/M_n = 1$. Finally, we have

$$M_i \cap \cdots \cap M_{n-1} \cap M_n = U \cap M_n = 0,$$

as desired.

Ex. 6.5. Let R be the factor ring $k[x,y]/(x,y)^n$, where k is a field. Show that $\text{u.dim}\, R_R = n$.

Solution. Note that R is a commutative local ring with unique maximal ideal $\mathfrak{m} = (\bar{x}, \bar{y})$ having index of nilpotency equal to n. By expressing the elements of R in the form $\beta_0 + \beta_1 + \cdots + \beta_{n-1}$ where $\beta_d =$ homogeneous polynomial in $\bar{x}, \bar{y}$ of degree d, it is easy to show that

$$\operatorname{ann}(\mathfrak{m}) = \mathfrak{m}^{n-1} = \bigoplus_{i=0}^{n-1} \bar{x}^i\, \bar{y}^{n-1-i} k.$$

Here each $\bar{x}^i\, \bar{y}^{n-1-i} k = \bar{x}^i\, \bar{y}^{n-1-i} R$ is a uniform ideal (since it has k-dimension 1). On the other hand, we see easily that $\operatorname{ann}(\mathfrak{m}) \subseteq_e R$. It follows immediately that $\text{u.dim}\, R_R = n$.

Ex. 6.6. Give an example of a module of finite uniform dimension that is neither noetherian nor artinian.

Solution. The $\mathbb{Z}$-module $\mathbb{Q}$ is uniform, so $\text{u.dim}_{\mathbb{Z}}\,\mathbb{Q} = 1$. However, $\mathbb{Q}$ has an infinite ascending chain $\mathbb{Z} \subsetneq \frac{1}{2}\mathbb{Z} \subsetneq \frac{1}{4}\mathbb{Z} \subsetneq \cdots$ and an infinite descending chain $\mathbb{Z} \supsetneq 2\mathbb{Z} \supsetneq 4\mathbb{Z} \supsetneq \cdots$, so as a $\mathbb{Z}$-module $\mathbb{Q}$ is neither noetherian nor artinian.

Ex. 6.7. For any submodules A, B of a module M, show that

$$(*) \qquad \text{u.}\dim A + \text{u.}\dim B \leq \text{u.}\dim (A \cap B) + \text{u.}\dim (A + B).$$

Solution. Without loss of generality, we may assume that $M = A + B$. For $C = A \cap B$, we have an exact sequence

$$0 \to C \xrightarrow{f} A \oplus B \xrightarrow{g} M \to 0,$$

where $f(c) = (c, -c)$ for $c \in C$, and $g((a, b)) = a + b$. By *LMR*-(6.35), we have

$$\begin{aligned}\text{u.}\dim A + \text{u.}\dim B &= \text{u.}\dim (A \oplus B) \leq \text{u.}\dim C + \text{u.}\dim M \\ &= \text{u.}\dim (A \cap B) + \text{u.}\dim (A + B).\end{aligned}$$

Comment. In an exercise in a ring theory textbook, it was asserted that the uniform dimension inequality $(*)$ above is an equality. The fact that this is not true in general was pointed out by Camillo and Zelmanowitz. The next exercise shows in a general way how equality may break down in $(*)$.

Ex. 6.8. Let $C \subseteq D$ be modules such that $\text{u.dim}\,D < \infty$ and $\text{u.dim}\,D/C = \infty$, and let

$$C' = \{(c, -c) : c \in C\} \subseteq D \oplus D, \quad M = (D \oplus D)/C'.$$

(a) Show that $\text{u.dim}\,M = \infty$.

(b) Let $A = \overline{D \oplus 0} \subseteq M$ and $B = \overline{0 \oplus D} \subseteq M$. Show that $A \cong B \cong D$ and $A + B = M$ (so it is possible for $\text{u.dim}\,A$, $\text{u.dim}\,B$ to be finite and $\text{u.dim}(A + B) = \infty$ in Exercise 7).

Solution. (a) The homomorphism $\varphi : D \to M$ defined by

$$\varphi(d) = \left(\overline{d, -d}\right) \in M$$

has obviously kernel C. Therefore, M contains a submodule $\varphi(D) \cong D/C$, so $\text{u.dim}\,M = \infty$.

(b) The projection map $D \oplus D \to M$ is injective on $D \oplus 0$ and on $0 \oplus D$, so $A = \overline{D \oplus 0} \cong D$ and $B = \overline{0 \oplus D} \cong D$. Clearly $A + B = \overline{D \oplus D} = M$.

Comment. This exercise is adapted from the paper of V. Camillo and J. Zelmanowitz, "On the dimension of a sum of modules", Comm. Algebra **6** (1978), 345–352. In this paper, the authors studied conditions under which the inequality $(*)$ in Exercise 7 becomes an equality.

Ex. 6.9. Show that an abelian group $M \neq 0$ is a uniform $\mathbb{Z}$-module iff $M \subseteq \mathbb{Q}$, or $M \cong \mathbb{Z}/p^n\mathbb{Z}$, or $M \cong \varinjlim \mathbb{Z}/p^n\mathbb{Z}$, where p is a prime. Generalize this to a (commutative) PID.

Solution. First assume $M \subseteq \mathbb{Q}$. Any two nonzero cyclic subgroups $\mathbb{Z}\cdot a/b$, $\mathbb{Z}\cdot a'/b' \subseteq M$ contain a common nonzero element aa', so M is uniform. If $M \subseteq \varinjlim \mathbb{Z}/p^n\mathbb{Z}$, then the subgroups of M form a chain, so M is also uniform. Conversely, let M be any uniform $\mathbb{Z}$-module. Then the injective hull $E(M)$ is an indecomposable injective $\mathbb{Z}$-module, so it is either $\mathbb{Q}$, or the Prüfer group $C_{p^\infty} = \varinjlim \mathbb{Z}/p^n\mathbb{Z}$ for some prime p. If $E(M) \cong \mathbb{Q}$, then M embeds in $\mathbb{Q}$. If $E(M) \cong C_{p^\infty}$, then M embeds in C_{p^∞}, so C is (isomorphic to) either C_{p^∞}, or $\mathbb{Z}/p^n\mathbb{Z}$ (for some n), since these are the only nonzero subgroups of C_{p^∞}.

If we work over a commutative PID, say R, the argument is exactly the same, upon replacing $\mathbb{Q}$ by the quotient field K of R, and replacing the Prüfer group C_{p^∞} by $\varinjlim R/p^nR$, where p ranges over a complete set (up to associates) of nonzero prime elements of R.

Ex. 6.10. Let R be a commutative domain with quotient field K. For any R-module M with torsion submodule $t(M)$, show that $\dim_K(M \otimes_R K) = \text{u.}\dim M/t(M)$; this number is called the "torsion-free rank" of M. If u.dim $t(M) < \infty$, show that the torsion-free rank of M is given by u.dim M − u.dim $t(M)$.

Solution. Write N for the torsion-free module $M/t(M)$. Then $t(M) \otimes_R K = 0$ implies that $M \otimes_R K \cong N \otimes_R K$. By *LMR*-(6.14), we have therefore

$$\text{u.}\dim_R N = \dim_K(N \otimes_R K) = \dim_K(M \otimes_R K).$$

For the last part of the exercise, note that $t(M)$ is essentially closed in M. For, if $t(M) \subseteq_e A \subseteq M$, then for any $0 \neq a \in A$, there exists $r \in R$ such that $ar \in t(M) \setminus \{0\}$, so $(ar)r' = 0$ for some nonzero $r' \in R$, which shows that $a \in t(M)$. Therefore, by *LMR*-(6.35),

$$\text{u.}\dim_R M = \text{u.}\dim_R t(M) + \text{u.}\dim_R N.$$

If u.dim$_R\, t(M) < \infty$, it is meaningful to write this equation as

$$\text{u.}\dim_R N = \text{u.}\dim_R M - \text{u.}\dim_R t(M).$$

By the first part of the exercise, the LHS is the torsion-free rank of M.

Ex. 6.11. Show that a module M_R is noetherian iff every essential submodule of M is f.g.

Solution. If M_R is noetherian, then every submodule of M is f.g. Conversely, assume every *essential* submodule of M is f.g. Consider any submodule $S \subseteq M$. Let C be a complement to S (that is, $C \subseteq M$ is a submodule maximal with respect to $C \cap S = 0$). Clearly $C \oplus S \subseteq_e M$.

Hence $C \oplus S$ is f.g., and so is S. Now every submodule of M is f.g., so M is noetherian.

Ex. 6.12. For any module M_R, let soc(M) (the socle of M) be the sum of all simple submodules of M (with $\mathrm{soc}(M) = 0$ if there are no simple submodules). Show that

(1) $M \cdot \mathrm{soc}(R_R) \subseteq \mathrm{soc}(M)$;
(2) $\mathrm{soc}(M) = \bigcap \{N : N \subseteq_e M\}$;
(3) For any submodule $N \subseteq M$, $\mathrm{soc}(N) = N \cap \mathrm{soc}(M)$;
(4) If $N \subseteq_e M$, then $\mathrm{soc}(N) = \mathrm{soc}(M)$;
(5) A maximal submodule $N \subseteq M$ is essential in M iff $N \supseteq \mathrm{soc}(M)$;
(6) $\mathrm{soc}\left(\bigoplus_{i \in I} M_i\right) = \bigoplus_i \mathrm{soc}(M_i)$;
(7) For any idempotent $f \in R$, $\mathrm{soc}(fR) = f \cdot \mathrm{soc}(R_R)$.

Solution. (1) For any $m \in M$, $m \cdot \mathrm{soc}(R_R)$ is an epimorphic image of $\mathrm{soc}(R_R)$, and hence a semisimple submodule of M. Therefore, $m \cdot \mathrm{soc}(R_R) \subseteq \mathrm{soc}(M)$.

(2) Let $E = \bigcap \{N : N \subseteq_e M\}$. Any simple submodule of M is contained in every $N \subseteq_e M$, so $\mathrm{soc}(M) \subseteq E$. To see the equality, it suffices to show that E itself is a semisimple module. Consider any submodule $S \subseteq E$. For any complement C to S in M, we have, as in the previous exercise, $C \oplus S \subseteq_e M$. Hence $E \subseteq C \oplus S$, and it follows that $E = S \oplus (C \cap E)$. This checks the semisimplicity of E.

(3) Since $\mathrm{soc}(N)$ is semisimple, it is contained in $N \cap \mathrm{soc}(M)$. Conversely, the semisimplicity of $N \cap \mathrm{soc}(M)$ implies that it is contained in $\mathrm{soc}(N)$.

(4) As in (3), we have $\mathrm{soc}(N) \subseteq \mathrm{soc}(M)$. If $N \subseteq_e M$, (2) implies that $\mathrm{soc}(M) \subseteq N$, so we also have $\mathrm{soc}(M) \subseteq \mathrm{soc}(N)$.

(5) The "only if" part follows from (4) even without N being maximal. Now assume N is maximal, and that $N \supseteq \mathrm{soc}(M)$. If N is not essential in M, we would have $N \cap X = 0$ for some nonzero submodule $X \subseteq M$. Clearly $N \oplus X = M$, so $X \cong M/N$ is a simple module. Hence $X \subseteq \mathrm{soc}(M) \subseteq N$, a contradiction. This shows that $N \subseteq_e M$.

(6) Since $\bigoplus_i \mathrm{soc}(M_i)$ is semisimple, it lies in $\mathrm{soc}\left(\bigoplus_i M_i\right)$. To prove the equality of the two, it suffices to show that any simple submodule $S \subseteq \bigoplus_i M_i$ is contained in $\bigoplus_i \mathrm{soc}(M_i)$. Consider any nonzero element $s \in S$, say $s = s_{i_1} + \cdots + s_{i_n}$ where $s_{i_j} \in M_{i_j}$. Then $S = sR$ and $sr \mapsto s_{i_j} r$ gives a well-defined homomorphism φ_j from S to M_{i_j}. The image $S_j = \varphi_j(S)$ is either (0) or a simple R-module, so $S_j \subseteq \mathrm{soc}\left(M_{i_j}\right)$. Therefore,

$$S \subseteq \bigoplus_j S_j \subseteq \bigoplus_i \mathrm{soc}(M_i).$$

(7) Let $S = \mathrm{soc}(R_R)$. Then $(fS)_R$ is an epimorphic image of S_R, so it is a semisimple submodule of fR. Therefore, $fS \subseteq \mathrm{soc}(fR)$. Conversely, $\mathrm{soc}(fR) \subseteq \mathrm{soc}(R_R) = S$ gives $\mathrm{soc}(fR) \subseteq f \cdot \mathrm{soc}(fR) \subseteq fS$.

Comment. The description of soc(M) in (2) above is due to Kasch and Sandomierski. It may be thought of as "dual" to the description of the radical of M as the sum of all superfluous (or small) submodules of M: see FC-(24.4). (The definition of a small submodule is dual to that of an essential submodule: a submodule $S \subseteq M$ is small if, for any submodule $N \subseteq M$, $N + S = M \Rightarrow N = M$.)

Ex. 6.13. If R is a semisimple ring with Wedderburn decomposition $\mathbb{M}_{n_1}(D_1) \times \cdots \times \mathbb{M}_{n_r}(D_r)$ where $D_1, \ldots, D_r$ are division rings, show that $\text{u.dim}\, R_R = n_1 + \cdots + n_r$.

Solution. If S_i denotes the unique simple module of $\mathbb{M}_{n_i}(D_i)$, with zero action by $\mathbb{M}_{n_j}(D_j)$ for $j \neq i$, we have

$$R_R \cong n_1 S_1 \oplus \cdots \oplus n_r\, S_r.$$

Since this is a semisimple module, u.dim (R_R) is just the number of composition factors of R_R, which is $n_1 + \cdots + n_r$.

Ex. 6.14. Let $S \subseteq R$ be fields such that $\dim_S R = \infty$. Let T be the triangular ring $\begin{pmatrix} R & R \\ 0 & S \end{pmatrix}$. By FC-(1.22), T is left artinian but not right noetherian. Show that u.dim $({}_T T) = 2$ and u.dim $(T_T) = \infty$.

Solution. Let $\bigoplus_{i=1}^{\infty} V_i$ be an infinite direct sum of nonzero S-subspaces of R_S. Then T contains $\bigoplus_{i=1}^{\infty} \begin{pmatrix} 0 & V_i \\ 0 & 0 \end{pmatrix}$ where each $\begin{pmatrix} 0 & V_i \\ 0 & 0 \end{pmatrix}$ is a (nonzero) right ideal. This shows that u.dim $(T_T) = \infty$. To compute u.dim $({}_T T)$, note that rad T (the Jacobson radical of T) is $\begin{pmatrix} 0 & R \\ 0 & 0 \end{pmatrix}$, with $T/\text{rad}\, T \cong R \times S$. Therefore, T has exactly two simple right modules, R_T and S_T, where T acts on the right of R (resp. S) via the projection map $T \to R \times S$ followed by the projection onto R (resp. S). The projection $T_T \to S_T$ given by $\begin{pmatrix} r & r' \\ 0 & s \end{pmatrix} \mapsto s$ is a T-epimorphism with kernel

$$\mathfrak{m} = \begin{pmatrix} R & R \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} R & 0 \\ 0 & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & R \\ 0 & 0 \end{pmatrix},$$

where the summands on the right are easily checked to be left ideals of T both isomorphic to R_T. Therefore, if we can show that $\mathfrak{m} \subseteq_e {}_T T$, it will follow that u.dim $({}_T T) = 2$. Now for any $\begin{pmatrix} r & r' \\ 0 & s \end{pmatrix}$ with $s \neq 0$, we have

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} r & r' \\ 0 & s \end{pmatrix} = \begin{pmatrix} 0 & s \\ 0 & 0 \end{pmatrix} \in \mathfrak{m} \setminus \{0\}.$$

So indeed $\mathfrak{m} \subseteq_e {}_T T$, as desired.

Ex. 6.15. Let R be a commutative PID, and $C \subseteq M$ be right R-modules. Show that $C \subseteq_c M$ iff, for every nonzero prime element $p \in R$, $C \cap Mp = Cp$. Using this, show that any pure submodule of M is a complement.

Solution. For the "only if" part, assume C is a complement to some S in M, and let $m \in M$ be such that $mp \in C$. We want to show that $mp \in Cp$. Note that $(C \oplus S)/C \subseteq_e M/C$. Since $(C + mR)/C \subseteq M/C$ is either zero or isomorphic to the simple R-module R/pR, we have

$$(C + mR)/C \subseteq (C \oplus S)/C.$$

Therefore, $m = c + s$ for some $c \in C$ and $s \in S$. Now

$$sp = mp - cp \in C \cap S = 0,$$

so $mp = cp \in Cp$.

For the "if" part, suppose C is *not* a complement in M. Then C is not closed by *LMR*-(6.32), so $C \subseteq_e C' \subseteq M$ for some $C' \neq C$. Fix an element $m \in C' \backslash C$. Then $mp \in C \backslash \{0\}$ for some $p \in R$. After replacing m by a suitable R-multiple, we may assume that p is a (nonzero) prime element of R. If $mp = cp$ for some $c \in C$, then $(m - c) \cdot R \cong R/pR$ is a simple submodule of C' not contained in C, in contradiction to $C \subseteq_e C'$. Therefore, we have $C \cap Mp \neq Cp$, as desired.

Finally, let $C \subseteq M$ be a pure submodule. By *LMR*-(4.93), we have $C \cap Mr = Cr$ for *any* $r \in R$. In particular, by the "if" part above, C is a complement in M.

Ex. 6.16. (1) Give an example of a complement $C \subseteq_c M$ (over a commutative PID if possible) such that C is not a pure submodule of M.
(2) Give an example of a pure submodule $C \subseteq M$ (over some ring) such that C is not a complement in M.

Solution. (1) An example can already be found over the ring $R = \mathbb{Z}$. Indeed, consider the abelian group $M = \langle a \rangle \oplus \langle s \rangle$ where a has order 8 and s has order 2. The subgroup $C = \langle 2a + s \rangle \cong \mathbb{Z}_4$ is a complement in M, as is shown in *LMR*-(6.17) (5). (Or more directly, note that

$$C \cap 2M = C \cap \langle 2a \rangle = \langle 4a \rangle = 2C$$

and apply the last exercise.) However, $C \cap 4M = C \cap \langle 4a \rangle = \langle 4a \rangle$ is not equal to $4C = 0$, so C is not a pure submodule of M.

Alternatively, consider a commutative domain R and a module M_R. The torsion submodule C of M is always a complement in M by *LMR*-(6.34), but C may not be pure in M by Exercise 4.35.

(2) Let R be any von Neumann regular ring that is not semisimple. By *FC*-(2.9), there must exist a non-injective right R-module C. Let $M = E(C)$. Since all short exact sequences of right R-modules are pure (by Exercise (4.29)), C is a pure submodule of M. However, C is not closed in M, since $M \supsetneq C$ is a proper essential extension of C. Therefore, by *LMR*-(6.32), C is not a complement in M.

Ex. 6.17. Decide which of the following statements is true:

(1) If T is a direct summand of a module M_R, then any submodule $N \subseteq M$ with $N \cap T = 0$ can be enlarged to a direct complement of T in M.
(2) Let $f \in \text{Hom}_R(M, M')$. Then $L \subseteq_c M$ implies that $f(L) \subseteq_c f(M)$.

Solution. (1) is a false statement in general, even over $R = \mathbb{Z}$. Let $M = \mathbb{Z}_2 \oplus \mathbb{Z}_4$ and T be the direct summand $\mathbb{Z}_2 \oplus (0)$. Then the subgroup N generated by $(\bar{1}, \bar{2})$ satisfies $N \cap T = 0$. If $N \subseteq T'$ with $T' \oplus T = M$, then $T' \cong \mathbb{Z}_4$, and so $N = 2T' \subseteq 2M$, which is not the case.

(2) is also false: the conclusion $f(L) \subseteq_c f(M)$ need not follow even if L is a direct summand of M. For a counterexample, again take $M = \mathbb{Z}_2 \oplus \mathbb{Z}_4$ and $N = \mathbb{Z} \cdot (\bar{1}, \bar{2})$ as above. Let f be the quotient map from M to $M' = M/N$. Here, $L = \mathbb{Z}_2$ is a direct summand of M that maps onto the subgroup of order 2 in M', which is not closed in M'.

Comment. Note that (2) becomes a true statement if we assume that $L \supseteq \ker(f)$: this follows from *LMR*-(6.28)(1).

Ex. 6.18. Show that a subgroup C of a divisible abelian group M is a direct summand iff $C \cap Mp = Cp$ for every prime p.

Solution. The "only if" part is trivial. For the "if" part, assume that $C \cap Mp = Cp$ for every prime p. We are done if we can show that C itself is a divisible group, for then C is an injective $\mathbb{Z}$-module, and hence a direct summand of M. To show the divisibility of C, it suffices to prove the divisibility of every element $c \in C$ by every prime p. Since M is divisible, $c = mp$ for some $m \in M$. From this, it follows that $c \in C \cap Mp = Cp$, so $c = c_0 p$ for some $c_0 \in C$.

Alternatively, we can also deduce the "if" part above from Exercise 15. If $C \cap Mp = Cp$ for every prime p, that exercise implies that C is an (essentially) closed submodule. Since M is an injective $\mathbb{Z}$-module, it follows from *LMR*-(6.32) that C is a direct summand of M.

The following four exercises are intended to give a complete determination of all f.g. abelian groups that are CS modules over the ring of integers $\mathbb{Z}$.

Ex. 6.19A. Show that a free abelian group F is CS as a $\mathbb{Z}$-module iff F has finite rank.

Solution. If rank $F < \infty$, we already know from *LMR*-(6.42)(4) that M is CS. *Now assume* rank F *is infinite*, and let $\{e_1, e_2, \ldots\} \dot{\cup} B$ be a basis of F. Define a homomorphism $f: F \to \mathbb{Q}$ by taking $f(B) = 0$ and $f(e_n) = 1/n$ for $n \geq 1$. Then $K := \ker(f) \subseteq_c F$. For, if $K \subseteq_e L \subseteq F$, then for any $\ell \in L$, we have $m\ell \in K$ for some $m \geq 1$ and so $0 = f(m\ell) = mf(\ell)$ implies that $f(\ell) = 0$. This shows that $L = K$. The closed submodule K cannot be a direct summand of F, for otherwise

$$F \cong K \oplus (F/K) \cong K \oplus \mathbb{Q},$$

which is impossible. Therefore, F is not a CS module.

Ex. 6.19B. Let M be a f.g. abelian group of rank $n \geq 1$. Show that M is a CS module over $\mathbb{Z}$ iff $M \cong \mathbb{Z}^n$.

Solution. We need only prove the "only if" part. Write $M = T \oplus \mathbb{Z}^n$ where $|T| < \infty$, and suppose M is CS. We want to show that $T = 0$. Since the direct summand $T \oplus \mathbb{Z}$ is CS, we may as well assume that $n = 1$. Assume for now $T \neq 0$. Then $pT \neq T$ for some prime p. Say $t_0 \in T \backslash pT$. For $C := \mathbb{Z} \cdot (t_0, p)$, we have $T \cap C = 0$. *We claim that C is a complement to T.* Indeed, if there exists a subgroup $D \supsetneqq C$ with $T \cap D = 0$, then, writing "bars" for images modulo T, we have $\overline{D} \supsetneqq \overline{C}$. Since $\left[\overline{M} : \overline{C}\right] = p$, we must have $\overline{D} = \overline{M} \cong \mathbb{Z}$. Therefore,

$$[D : C] = \left[\overline{D} : \overline{C}\right] = \left[\overline{M} : \overline{C}\right] = p.$$

It follows that $C = pD \subseteq pM = pT \oplus p\mathbb{Z}$, and so $t_0 \in pT$, a contradiction. This shows that $C \subseteq_c M$. Since M is CS, we have $M = C \oplus X$ for some subgroup $X \subseteq M$. From the fact that rank $M = 1$, we see that X must be torsion. But then $X \subseteq T$, and so we have $M = C \oplus T$, contradicting $\left[\overline{M} : \overline{C}\right] = p$.

Ex. 6.19C. Let p be any prime number, and $r \geq 1$.

(1) Show that $\mathbb{Z}_{p^r} \oplus \mathbb{Z}_{p^{r+i}}$ (for $i \geq 2$) is not CS as a $\mathbb{Z}$-module.
(2) Show that $(\mathbb{Z}_{p^r})^k \oplus \left(\mathbb{Z}_{p^{r+1}}\right)^\ell$ is CS as a $\mathbb{Z}$-module.

Solution. (1) Let $M = \mathbb{Z}_{p^r} \oplus \mathbb{Z}_{p^{r+i}}$ where $i \geq 2$, and let $A = \mathbb{Z} \cdot \left(1, p^{i-1}\right) \cong \mathbb{Z}_{p^{r+1}}$. *We claim that $A \subseteq_c M$.* Indeed, if otherwise, we have $A \subsetneqq_e B$ for some subgroup B, which we may assume to have cardinality p^{r+2}. If B is decomposable, A would be a direct summand of B, which contradicts $A \subseteq_e B$. If B is indecomposable, then $B \cong \mathbb{Z}_{p^{r+2}}$ and hence

$$A = pB \subseteq pM = p\mathbb{Z}_{p^r} \oplus p\mathbb{Z}_{p^{r+i}},$$

a contradiction. This proves our claim that $A \subseteq_c M$. But clearly A is not a direct summand of M, so M is not CS.

(2) Let $N = K \oplus L$, where $K = (\mathbb{Z}_{p^r})^k$ and $L = \left(\mathbb{Z}_{p^{r+1}}\right)^\ell$. If either $k = 0$ or $\ell = 0$, then N is a quasi-injective module by *LMR*-(6.72)(3) and *LMR*-(6.77), so N is a CS module by *LMR*-(6.80). It is not strictly necessary to use these facts, but they do help to somewhat simplify the arguments, so we shall use them. Let us first prove the following special case of our desired conclusion.

Lemma. *Any indecomposable closed submodule $C \subseteq_c N$ is a direct summand.*

Proof. Let $|C| = p^s$. If $s = r+1$, then C is a direct summand of N since N has exponent p^{r+1} (cf. the solution to Exercise (3.18)). Now assume $s \leq r$. Then

$$C \subseteq_c N_0 := \{x \in N : p^r x = 0\} = K \oplus pL \cong (\mathbb{Z}_{p^r})^{k+\ell}.$$

Since $(\mathbb{Z}_{p^r})^{k+\ell}$ is a CS module (as observed above), C is a direct summand of N_0. Therefore, $C \cong \mathbb{Z}_{p^r}$. Let

$$c = (c_1, \ldots, c_k,\ pd_{k+1}, \ldots, pd_{k+\ell}) \in K \oplus L$$

be a generator of C. Then some $c_i \notin p \cdot \mathbb{Z}_{p^r}$, for otherwise $c = pd$ for some $d \in N$, and $C \subsetneq_e \langle d \rangle$ contradicts $C \subseteq_c N$. Say $c_1 \notin p \cdot \mathbb{Z}_{p^r}$. Let $e_1, \ldots, e_k, e_{k+1}, \ldots, e_{k+\ell}$ be fixed generators for the $k+\ell$ direct summands in the given decomposition of N. Since $\mathbb{Z} \cdot c_1 = \mathbb{Z} \cdot e_1$, it follows that $N = C \oplus \bigoplus_{i=2}^{k+\ell} \mathbb{Z} \cdot e_i$, which proves the Lemma.

Now consider *any* nonzero $D \subseteq_c N$. Take any indecomposable direct summand C of D. Then $C \subseteq_c D \subseteq_c N$ implies that $C \subseteq_c N$ (see *LMR*-(6.24)(1)), so by the Lemma, C is a direct summand of N. Writing "bars" for taking images modulo C, we have $\overline{D} \subseteq_c \overline{N}$ (by *LMR*-(6.28)), where $\overline{N}$ is a group of the same type as N (namely, a direct sum of $\mathbb{Z}_{p^r}$'s and $\mathbb{Z}_{p^{r+1}}$'s). Invoking an induction hypothesis at this point, we may assume that $\overline{D}$ is a direct summand of $\overline{N}$. Thus, there exists a subgroup $X \subseteq N$ with $X \cap D = C$ such that $\overline{N} = \overline{D} \oplus \overline{X}$. Since C is a direct summand of N, it is also a direct summand of X and of D, say $X = C \oplus Y$. Thus, $N = D + X = D \oplus Y$, as desired.

Ex. 6.19D. Show that a f.g. abelian group M is a CS module over $\mathbb{Z}$ iff either $M \cong \mathbb{Z}^n$ for some n, or M is finite and for any prime p, the p-primary part M_p of M is of the form $(\mathbb{Z}_{p^r})^k \oplus (\mathbb{Z}_{p^{r+1}})^\ell$ for some r, k and ℓ (depending on p).

Solution. First, $\mathbb{Z}^r$ is CS. Next, consider $M = \bigoplus_{i=1}^{n} M_{p_i}$ where each M_{p_i} is as described. By the last exercise, M_{p_i} is CS. If $C \subseteq_c M$, we can decompose C into $\bigoplus_{i=1}^{n} C_{p_i}$ where clearly $C_{p_i} \subseteq_c M_{p_i}$. Then each C_{p_i} is a direct summand of M_{p_i}, so C is also a direct summand of M. This checks that M is a CS module.

Conversely, let M be any f.g. CS module over $\mathbb{Z}$. If rank $M = n \geq 1$, Exercise (6.19B) implies that $M \cong \mathbb{Z}^n$. Therefore, let us assume rank $M = 0$, i.e. $|M| < \infty$. Since direct summands of M are also CS, we may assume that M is a (nontrivial) p-group (for some prime p). Consider the Krull-Schmidt decomposition $M \cong \bigoplus_{i=r}^{m} (\mathbb{Z}_{p^i})^{k_i}$, with $k_r > 0$. If any of k_{r+2}, $k_{r+3}, \ldots$ is nonzero, M would contain a direct summand $\mathbb{Z}_{p^r} \bigoplus \mathbb{Z}_{p^{r+i}}$ with $i \geq 2$, which is not CS according to Exercise (6.19C). This contradicts the fact that M (and hence any direct summand thereof) is CS. It follows that $M \cong (\mathbb{Z}_{p^r})^k \bigoplus (\mathbb{Z}_{p^{r+1}})^\ell$ with $k = k_r$ and $\ell = k_{r+1}$.

Comments. Similar ideas (and more) can be used to determine *all* abelian groups that are CS as $\mathbb{Z}$-modules. For more complete information on CS modules over commutative domains and noetherian rings, see the papers of M. A. Kamal and B. J. Müller in Osaka J. Math. **25** (1988), 531–538.

Ex. 6.20. Show that, if R is a von Neumann regular ring, then the 20 finiteness conditions formulated in *LMR*-§6D are each equivalent to R being semisimple.

Solution. If R is semisimple, then of course R satisfies all the ACC and DCC conditions formulated in *LMR*-§6D. Conversely, suppose R satisfies the *weakest* of these conditions, namely, R has no infinite family of orthogonal nonzero idempotents, or equivalently, ACC holds for the family of right (resp. left) direct summands of R. If R is not semisimple, it cannot be right noetherian by *FC*-(4.25). Therefore, there exists a chain of f.g. right ideals $\mathfrak{A}_1 \subsetneq \mathfrak{A}_2 \subsetneq \cdots$. Since R is von Neumann regular, each $\mathfrak{A}_i$ is a direct summand in R_R, so ACC fails to hold for right direct summands of R.

Comment. In contrast, a Dedekind-finite (or even stably finite) von Neumann regular ring is pretty far from being semisimple.

Ex. 6.21. Show that R satisfies DCC on right annihilators iff, for any set $S \subseteq R$, there exists a finite subset $S_0 \subseteq S$ such that $\text{ann}_r(S) = \text{ann}_r(S_0)$.

Solution. For the "only if" part, consider any right annihilator $\text{ann}_r(S)$. Pick a minimal member $\mathfrak{A}$ from the family $\{\text{ann}_r(T)\}$, where T ranges over all finite subsets of S, say

$$\mathfrak{A} = \text{ann}_r(S_0) \quad (S_0 \subseteq S,\ |S| < \infty).$$

For any $s \in S$, $\text{ann}_r(S_0 \cup \{s\}) \subseteq \text{ann}_r(S_0) = \mathfrak{A}$ implies that $\text{ann}_r(S_0 \cup \{s\}) = \mathfrak{A}$. From this, it follows that $\text{ann}_r(S) = \mathfrak{A}$.

Conversely, consider any descending chain of right annihilators

$$(*) \qquad \text{ann}_r(S_1) \supseteq \text{ann}_r(S_2) \supseteq \cdots,$$

where we may assume the sets S_i are such that $S_1 \subseteq S_2 \subseteq \cdots$. (Simply replace S_i by $\text{ann}_\ell(\text{ann}_r S_i)$.) Let $S = \bigcup_i S_i$ and pick a finite subset $S_0 \subseteq S$ such that $\text{ann}_r(S) = \text{ann}_r(S_0)$. Then $\text{ann}_r(S_i) = \text{ann}_r(S_0)$ whenever $S_i \supseteq S_0$. Since $S_i \supseteq S_0$ for sufficiently large i, the descending chain $(*)$ stabilizes, as desired.

Ex. 6.22. Show that R *does not* satisfy ACC on right (resp. left) annihilators iff there exist elements $s_i, t_i \in R \ (i = 1, 2, \ldots)$ such that $s_i t_i \neq 0$ for all i and $s_i t_j = 0$ for all $i > j$ (resp. for all $i < j$).

Solution. By left-right symmetry, it suffices to give the proof in the case of right annihilators. Suppose the elements s_i, t_i exist as above. For $S_i := \{s_i, s_{i+1}, \ldots\}$, we have

$$(1) \qquad t_i \in \text{ann}_r(S_{i+1}) \setminus \text{ann}_r(S_i),$$

so we get a strictly ascending chain

$$(2) \qquad \text{ann}_r(S_1) \subsetneq \text{ann}_r(S_2) \subsetneq \cdots.$$

Conversely, if we have a chain (2) for some subsets $S_i \subseteq R$, we may assume (as in the last exercise) that $S_1 \supseteq S_2 \supseteq \cdots$. Pick t_i as in (1) and $s_i \in S_i$ such that $s_i t_i \neq 0$, for $i = 1, 2, \ldots$. Then clearly $s_i t_j = 0$ for $i > j$, as desired.

Ex. 6.23. Show that "ACC on right annihilators" and "DCC on right annihilators" are independent conditions (for noncommutative rings).

Solution. We have to work with noncommutative rings since, for commutative rings, "ACC on annihilators" is equivalent to "DCC on annihilators" by *LMR*-(6.57).

Suppose we have constructed a ring R that satisfies ACC on right annihilators but not DCC on right annihilators. Then by *LMR*-(6.57), R satisfies DCC on left annihilators but not ACC on left annihilators. Therefore, the opposite ring R^{op} of R satisfies DCC on right annihilators but not ACC on right annihilators.

To produce the ring R in the paragraph above, we use the construction in Exercises (12.7) and (12.8). The ring R discussed in Exercise (12.7) fails to satisfy DCC on right annihilators. By Exercise 12.8, we can construct such a ring that is in fact right noetherian. In particular, R satisfies ACC on right annihilators.

Ex. 6.24. (Extra Credit) Let ${}_{\text{pr}}\text{DCC}$ and DCC_{pr} denote the descending chain conditions for left and right *principal* ideals respectively, and define ${}_{\text{pr}}\text{ACC}$ and ACC_{pr} similarly. Jonah has proved the following criss-cross implications:

$$\begin{array}{ccc} {}_{\text{pr}}\text{DCC} & & \text{DCC}_{\text{pr}} \\ & \searrow\!\!\!\!\!\swarrow & \\ {}_{\text{pr}}\text{ACC} & & \text{ACC}_{\text{pr}} \end{array}$$

Show that there are no more implications possible among the four chain conditions above.

Solution. First, the reverse implications to those shown in the above chart are certainly not valid, as is shown by the ring of the integers. Thus, it suffices to check that there are no *horizontal* or *vertical* implications in the chart. To this end, we use an example of G. Bergman that is a modified version of an earlier example of H. Bass. Bergman's ring R has the following truth values with respect to the above chain conditions:

True	False
False	True

From this, it is clear that there can be no horizontal or vertical implications in the first chart.

Bergman's ring R is defined, as an algebra over a given field K, by generators $\{e_{ij} : 1 \leq i < j \leq \infty\}$ (note that we do allow j to be ∞) with the relations $e_{ij}e_{k\ell} = \delta_{jk}e_{i\ell}$ (for all legitimate (i,j) and (k,ℓ)), where δ_{jk} are the Kronecker deltas. We can think of the generators as "formal strictly upper triangular matrix units" (with the understanding that the $e_{i\infty}$'s are "extra" ones). From the multiplication rules, it is clear that R is spanned by 1 and $\{e_{ij} : 1 \le i < j \le \infty\}$, and a short calculation shows that these are linearly independent over K.

Let J be the K-span of the e_{ij}'s. Clearly J is an ideal of R, and we see easily that J is nil.(*) Since $R/J \cong K$, R is a *local K-algebra* with $\mathrm{rad}(R) = J$. We define two functions

$$\lambda : J\backslash\{0\} \to \mathbb{N} \quad \text{and} \quad \lambda' : J\backslash\{0\} \to \mathbb{N} \cup \{\infty\}$$

as follows. For $r \in J\backslash\{0\}$, let $\lambda(r) = i$ if i is the *largest* integer for which some e_{ij} appears in r. Similarly, let $\lambda'(r) = \ell$ if ℓ is the *smallest* "number" (possibly ∞) for which some $e_{k\ell}$ appears in r.

Lemma 1. *For r, $s \in J\backslash\{0\}$, we have*

(A) $$Rr \supsetneq Rs \implies \lambda(r) > \lambda(s), \quad \textit{and}$$

(B) $$rR \supsetneq sR \implies \lambda'(r) < \lambda'(s).$$

Proof. (A) Write $s = tr$ $(t \in R)$. Then $t \in J$ (for otherwise $t \in \mathrm{U}(R)$ and $Rr = Rs$). Any "matrix unit" appearing in s is obtained as a product $e_{k'i'}e_{i'j'} = e_{k'j'}$, where $e_{k'i'}$ appears in t and $e_{i'j'}$ appears in r. Thus, $k' < i' \le \lambda(r)$, so we get $\lambda(s) < \lambda(r)$.

The proof of (B) is similar, the only "difference" being that λ' may take the value ∞. □

We can now draw the following conclusions:

(1) *R satisfies* ${}_{\mathrm{pr}}$DCC. In fact, if $Rr_1 \supsetneq Rr_2 \supsetneq \cdots$ where each $r_i \in J\backslash\{0\}$, then (A) gives $\lambda(r_1) > \lambda(r_2) > \cdots$, so the chain must be finite.

(2) *R satisfies* ACC$_{\mathrm{pr}}$. This follows similarly, by using (B). (Of course, ACC$_{\mathrm{pr}}$ would have followed from ${}_{\mathrm{pr}}$DCC in view of Jonah's Theorem quoted earlier.)

(3) *R does not satisfy* ${}_{\mathrm{pr}}$ACC. This is clear since we have an infinite chain $Re_{1\infty} \subsetneq Re_{2\infty} \subsetneq \cdots$, due to the equation

$$e_{n\infty} = e_{n,\,n+1}e_{n+1,\infty},$$

and the obvious fact that $e_{n+1,\infty} \notin Re_{n,\infty}$.

(4) *R does not satisfy* DCC$_{\mathrm{pr}}$, since $e_{1,n+1} = e_{1n}e_{n,n+1}$ leads to an infinite chain $e_{12}R \supsetneq e_{13}R \supsetneq \cdots$. (Again, (4) would have followed from (3) in view of Jonah's Theorem.)

Comment. Note that the ring R above is neither left nor right noetherian (resp. artinian) since otherwise Levitzki's Theorem would have implied that J is nilpotent, which is not the case since $e_{12}e_{23}\cdots e_{n-1,n} = e_{1n}$ for any n.

(*) If this step does not seem easy, just read on. Some of the arguments used later will certainly clarify this.

It is useful to view the example above in the context of Bass' perfect rings (cf. *FC*-§22). By (1) above, R is *right* perfect (note the switch of side) in view of *FC*-(23.20). Similarly, by (4), R is not *left* perfect. Now the first example of a right perfect ring that is not left perfect was constructed by Bass: see *FC*-(23.22). In fact, Bass' example is precisely the K-subalgebra $R_0 \subseteq R$ generated by $\{e_{ij} : i < j < \infty\}$, which is a genuine ring of $\mathbb{N} \times \mathbb{N}$ matrices $K \cdot I + M$, where M consists of matrices with only a finite number of nonzero entries, all occurring *above* the diagonal. Bass's proof that R_0 is right but not left perfect was based on the same considerations as in (1) and (4). The novelty of Bergman example R lies in the fact (3) that it fails to satisfy $_{\text{pr}}$ACC (and *a fortiori* DCC$_{\text{pr}}$), for this shows that there are no vertical (or lower horizontal) implications in the first chart. This could not have been accomplished by Bass's ring R_0, since it turns out to have the following (asymmetric) truth values with respect to the four chain conditions (for principal 1-sided ideals):

True	False
True	True

The arguments we gave earlier are sufficient to cover all cases except the southwest corner (for $_{\text{pr}}$ACC). We can prove this chain condition for R_0 as follows (where we no longer allow the $e_{i\infty}$'s).

Let J_0 be the span of the $\{e_{ij} : i < j < \infty\}$. As before, R_0 is local with $\text{rad}(R_0) = J_0$. We make the following crucial claim:

Lemma 2. *For $r \in J_0 \backslash \{0\}$ with $\lambda'(r) = n$, $r \notin J_0^n$.*

Assuming this lemma, we can define a new function

$$\mu : \ J_0 \backslash \{0\} \longrightarrow \{1, 2, \dots\}$$

in the spirit of a p-adic valuation : $\mu(r) = k$ if k is the largest integer such that $r \in J_0^k$. If

$$0 \neq R_0 r_1 \subsetneq R_0 r_2 \subsetneq \cdots,$$

then $r_i \in J_0 r_{i+1}$ implies that $\mu(r_i) > \mu(r_{i+1})$, so we have $\mu(r_1) > \mu(r_2) > \cdots$, which then forces the chain to be finite. (This is, of course, just the idea behind the noetherian-ness of a discrete valuation ring.) Note that the same argument would have worked just as well for a chain $r_1 R_0 \subsetneq r_2 R_0 \subsetneq \cdots$.

Proof of Lemma 2. Assume, instead, that $r \in J_0^n$. Then $r = \sum s_1 \cdots s_n$ where each $s_j \in J_0$. A matrix unit appearing in r must be of the form $e_{i_1 j_1} \cdots e_{i_n j_n}$ with $j_1 = i_2$, $j_2 = i_3$, ... , $j_{n-1} = i_n$, so that it is $e_{i_1 j_n}$. But then

$$1 \leq i_1 < j_1 = i_2 < j_2 = i_3 < \cdots < j_{n-1} = i_n < j_n$$

implies that $j_n > n$, and so by the definition of λ', $\lambda'(r) > n$, a contradiction. □

Note that this argument does not apply to R, since the function μ *cannot* be defined on $J\backslash\{0\}$. In fact, the equation

$$e_{i\infty} = e_{i,i+1}e_{i+1,i+2}\cdots e_{i+(n-1),i+n}e_{i+n,\infty} \in R$$

shows that $e_{i\infty} \in \bigcap_{n=1}^{\infty} J^n$ for all i. These "infinitely deep" elements are precisely the ones needed to create the ascending chain $Re_{1\infty} \subsetneq Re_{2\infty} \subsetneq \cdots$ in (3) to defeat ${}_{\text{pr}}$ACC!

Ex. 6.25. Show that for any ring R, the set A of right annihilator ideals in R form a complete lattice with respect to the partial ordering given by inclusion. Show that A is anti-isomorphic to the lattice A' of left annihilator ideals in R. (A *complete* lattice is a partially ordered set in which any subset has a greatest lower bound, or equivalently, any subset has a least upper bound.)

Solution. First, A is a lattice with

$$\mathfrak{A} \wedge \mathfrak{A}' = \mathfrak{A} \cap \mathfrak{A}', \quad \mathfrak{A} \vee \mathfrak{A}' = \text{ann}_r(\text{ann}_\ell(\mathfrak{A} + \mathfrak{A}'))$$

for right annihilator ideals $\mathfrak{A}, \mathfrak{A}'$. Now A is closed under intersections, since

$$\bigcap_i \text{ann}_r(X_i) = \text{ann}_r\left(\bigcup_i X_i\right)$$

for arbitrary subsets $X_i \subseteq R$. Therefore, A is a complete lattice with arbitrary meet given by intersections. It is also easy to see directly the existence of arbitrary joins in A. For any family $\{\mathfrak{A}_i\}$ in A, a right annihilator $\mathfrak{A}$ contains all of $\mathfrak{A}_i$ iff it contains $\text{ann}_r(\text{ann}_\ell(\sum_i \mathfrak{A}_i))$. Therefore, the join of the family $\{\mathfrak{A}_i\}$ is simply

$$\text{ann}_r\left(\text{ann}_\ell\left(\sum_i \mathfrak{A}_i\right)\right) = \text{ann}_r\left(\bigcap_i \text{ann}_\ell(\mathfrak{A}_i)\right).$$

The map $\mathfrak{A} \mapsto \text{ann}_\ell(\mathfrak{A})$ clearly gives an anti-isomorphism from the lattice A to the lattice A' (with inverse lattice anti-isomorphism given by $\mathfrak{B} \mapsto \text{ann}_r(\mathfrak{B})$).

Comment. The following comment on our definition of a complete lattice is in order. Let S be a partially ordered set such that any subset $A \subseteq S$ has a greatest lower bound, $\inf(A)$. In particular, for $A = \emptyset$, there exists $\inf(\emptyset)$, which is then the largest element of S. On the other hand, $\inf(S)$ is clearly the smallest element of S. It follows that any subset $B \subseteq S$ has a least upper bound, $\sup(B)$, namely, the infimum of the set of all upper bounds of B. A similar argument shows that if $\sup(B)$ exists for all $B \subseteq S$, then $\inf(A)$ exists for all $A \subseteq S$.

In the above discussion, it is essential that we allow the subsets A and B to be *empty*. For instance, let U be an infinite set and let S be the set of finite subsets of U, partially ordered by inclusion. For any *nonempty* $A \subseteq S$, $\inf(A)$ is given by the intersection of all subsets that are members of A. However, the empty set $\emptyset \subseteq S$ has no infimum, since S has no largest element. In particular, S is not a complete lattice.

Ex. 6.26. (Shock) Let $S = R[X]$, where X is any (possibly infinite) set of commuting indeterminates. Show that $\text{u.dim}\, S_S = \text{u.dim}\, R_R$.

Solution. The one-variable case of this result has been proved in *LMR*-(6.65). Here, we try to generalize this result to the case of an *arbitrary* number of variables. As in the earlier proof, the following two crucial statements will give the desired conclusion:

(1) *For any right ideal* $\mathfrak{A} \subseteq_e R_R$, $\mathfrak{A}[X] \subseteq_e S_S$.
(2) *For any uniform right ideal* $\mathfrak{A} \subseteq R$, $\mathfrak{A}[X]$ *is uniform in* S_S.

The first statement has been proved in Exercise 3.30 for any set of variables X. The second was proved in *LMR*-(6.68) for the case of one variable. By induction, it also holds in the case of a finite number of variables. In the general case, we proceed as follows. Suppose $\mathfrak{A}[X]$ is not uniform. Then $fS \cap gS = 0$ for some nonzero $f, g \in \mathfrak{A}[X]$. Fix a sufficiently large set of variables $\{x_1, \ldots, x_n\} \subseteq X$ such that $f, g \in \mathfrak{A}[x_1, \ldots, x_n]$. Clearly $fT \cap gT = 0$ for $T = R[x_1, \ldots, x_n]$. This contradicts the fact that $\mathfrak{A}[x_1, \ldots, x_n]$ is uniform in T_T, so we have proved (2).

The remaining exercises in this section are devoted to the study of the properties of QI modules (and their generalizations). Exercise 27A is to be compared with Exercises (3.28) and (3.29) in §3.

Ex. 6.27A. For an R-module M_R and an ideal $J \subseteq R$, let $P = \{m \in M : mJ = 0\}$

(1) If M is a QI R-module, show that P is a QI R/J-module
(2) If $MJ = 0$, show that M is a QI R-module iff it is a QI R/J-module.

Solution. (1) Let $L \subseteq P$ be an R/J-submodule of P, and let $f \in \text{Hom}_{R/J}(L, P)$. Viewing f as an R-homomorphism from L to M, we can find $g \in \text{End}_R(M)$ extending f. Since $PJ = 0$, we have also $g(P)J = 0$, so $g(P) \subseteq P$. Thus, $g|\,P$ is an endomorphism of P extending f. This checks that P is a QI R/J-module.

(2) The "only if" part follows from (1) (since here $P = M$). Conversely, if M is QI as an R/J-module, it is clearly QI as an R-module, since any R-submodule $L \subseteq M$ is also an R/J-submodule, and $\text{Hom}_R(L, M) = \text{Hom}_{R/J}(L, M)$.

Comment. The "if" part in (2) does not work with "injective" replacing "quasi-injective" throughout, since checking M_R to be injective involves the use of R-modules that may not be R/J-modules. Nevertheless, the statement in the case of QI implies that, if $MJ = 0$ and M is an injective R/J-module, then M is at least a *quasi-injective* R/J-module, if not an injective one.

Ex. 6.27B. For an ideal J in a ring R, show that the cyclic right R-module R/J is QI iff R/J is a right self-injective ring.

Solution. Let S be the quotient ring R/J. When S is viewed as a right R-module, we have $S \cdot J = 0$. By (2) of Exercise 27A, it follows that S_R is QI iff S_S is QI, and by Baer's Criterion, the latter is the case iff S is a right self-injective ring.

Before stating and solving the next exercise, we must first correct a wrong statement about direct sums of QI modules made in the proof of *LMR*-(6.73)(2). On p. 238 of (the first edition of) *LMR*, line 7 to line 9 on the direct sum of two cyclic $\mathbb{Z}$-modules should be corrected as follows :

> *If* $N = M \oplus M'$ *where* M, M' *are the* QI *modules* $\mathbb{Z}_{p^r}$ *and* $\mathbb{Z}_{p^s}$, *with* p *prime and* $r > s$, *then* N *is not* QI.

(The proof of this is easy. If M_1 denotes the proper subgroup of M that is isomorphic to M', then an isomorphism $M_1 \to M' \subset N$ cannot be extended to an endomorphism of N.) A similar correction is also needed in the formulation of the example in *LMR*-(6.82).

Ex. 6.28. Show that a f.g. abelian group M is QI as a $\mathbb{Z}$-module iff M is finite and for any prime p, the p-primary part M_p of M is of the form $(\mathbb{Z}_{p^r})^k$ for some r, k (depending on p).

Solution. First assume M is QI. If M has rank ℓ, then it has a direct summand isomorphic to $\mathbb{Z}^\ell$. By *LMR*-(6.73)(1), $\mathbb{Z}^\ell$ must then be QI, which is possible only if $\ell = 0$. Thus, M is finite. Another application of *LMR*-(6.73)(1) also shows that each M_p is QI. In general, M_p has the form $\mathbb{Z}_{p^{r_1}} \oplus \cdots \oplus \mathbb{Z}_{p^{r_k}}$ where $r_1 \geq \ldots \geq r_k$. If the r_i's are not all equal, M_p will have a direct summand $\mathbb{Z}_{p^r} \oplus \mathbb{Z}_{p^s}$ with $r > s$, which is *not* QI according to the paragraph preceding this exercise. Therefore, we must have $r_1 = \cdots = r_k$, as desired.

Conversely, assume that each M_p has the form $(\mathbb{Z}_{p^r})^k$ for some r, k (depending on p). To show that M is QI, it suffices (according to *LMR*-(6.74)) to check that $\varphi(M) \subseteq M$ for any endomorphism φ of the injective hull $E(M)$. Now, for each prime p, φ induces an endomorphism φ_p of

$$E(M)_p = E(M_p) = E((\mathbb{Z}_{p^r})^k) = (\mathbb{Z}_{p^\infty})^k,$$

where $\mathbb{Z}_{p^\infty}$ denotes the Prüfer p-group. In $(\mathbb{Z}_{p^\infty})^k$, M_p is the subgroup of elements annihilated by p^r. This implies, of course, that $\varphi_p(M_p) \subseteq M_p$ (for each p), and hence $\varphi(M) \subseteq M$, as desired.

Comment. The argument above generalizes easily to the case of f.g. QI modules over a commutative PID.

The reader should compare this exercise with Exercise 19D, in which we determined the (larger) class of all f.g. abelian groups that are CS as $\mathbb{Z}$-modules. A direct comparison shows that the QI class is considerably more restrictive than the CS class.

Ex. 6.29. Let M_R be a QI module, and let A be an R-submodule of $E(M)$ isomorphic to a subquotient (quotient of a submodule) of M. Show that $A \subset M$.

Solution. Suppose A is isomorphic to a quotient of a submodule $L \subseteq M$. An epimorphism $f : L \to A$ extends to some endomorphism g of $E(M)$ (since $E(M)$ is an injective module). By *LMR*-(6.74), M is fully invariant in $E(M)$, so we have $A = f(L) \subseteq f(M) \subseteq M$, as desired.

Ex. 6.30. Let M, N be QI modules with $E(M) \cong E(N)$. Show that $M \oplus N$ is QI iff $M \cong N$.

Solution. If $M \cong N$, then $M \oplus N \cong M \oplus M$ is QI by *LMR*-(6.77). Conversely, assume that $M \oplus N$ is QI. Let E be a module isomorphic to $E(M) \cong E(N)$. For convenience, we may assume that M and N are embedded (as submodules) in E. Since $M \oplus N$ (external direct sum) is QI, it is fully invariant in $E(M \oplus N) = E \oplus E$. In particular, for the endomorphism $f \in \text{End}(E \oplus E)$ defined by $f(x, y) = (y, x)$, we have $f(M \oplus N) \subseteq M \oplus N$. Clearly, this implies that $M \subseteq N \subseteq M$, so in fact $M = N$.

Comment. The result in this exercise appeared in J. Ravel's paper "Sur les modules M-injectifs," Publ. Dép. Math. (Lyon) **5**(1968), fasc. 1, 63–71. Ravel's result was later extended by Goel and Jain to the somewhat more general class of π-injective modules (defined below in the *Comment* on Exercise 37) :

> *If M, N are modules such that $E(M) \cong E(N)$ and $M \oplus N$ is π-injective, then $M \cong N$.*

On the other hand, if M is π-injective, $M \oplus M$ need not be π-injective : to be more precise, $M \oplus M$ is π-injective iff M is already QI. These results appeared in the paper "π-injective modules and rings whose cyclics are π-injective," Comm. Alg. **6**(1978), 59–73.

Ex. 6.31. For any module M_R, consider the following conditions:

(1) M is Dedekind-finite;
(2) $E(M)$ is Dedekind-finite;
(3) For any module $X \neq 0$, $X \oplus X \oplus \cdots$ cannot be embedded into $E(M)$;
(4) For any module $X \neq 0$, $X \oplus X \oplus \cdots$ cannot be embedded into M.

Show that (2) $\Leftrightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (1), and that all four conditions are equivalent in case M is QI. In general, show that (1) does not imply (2), (3) or (4).

Solution. (2) $\Rightarrow$ (3). Assume, instead, that there exists $K \subseteq E(M)$ with $K \cong X \oplus X \oplus \cdots$, where $X \neq 0$. Noting that $K \oplus X \cong K$, we have

$$E(K) \cong E(K \oplus X) \cong E(K) \oplus E(X),$$

so $E(K)$ is not Dedekind-finite Since $E(K)$ may be taken as a direct summand of $E(M)$, it follows that $E(M)$ is not Dedekind-finite.

(3) $\Rightarrow$ (4) is a tautology.

(4) $\Rightarrow$ (1) (for any M). Assume M is *not* Dedekind-finite. Then $M = M_1 \oplus X_1$ for some $X_1 \neq 0$ and $M_1 \cong M$. Using the latter, we

have $M = (M_2 \oplus X_2) \oplus X_1$ where $M_2 \cong M_1$ and $X_2 \cong X_1$. Continuing in this manner, we arrive at mutually isomorphic submodules $X_i \neq 0$ with $X_1 \oplus X_2 \oplus \cdots \subseteq M$. Therefore, (4) fails to hold.

(1) $\Rightarrow$ (2) (*for quasi-injective* M). Assume M is Dedekind-finite. To show that $E(M)$ is Dedekind-finite amounts to showing that the ring $S = \text{End}_R(E(M))$ is Dedekind-finite (by Exercise (1.8)). Say $fg = 1 \in S$. Since M is QI, M is fully invariant in $E(M)$ by *LMR*-(6.74), so f, g restrict to endomorphisms f', g' of M, with $f'g' = 1 \in \text{End}_R(M)$. Now $\text{End}_R(M)$ is Dedekind-finite (since M is), so f', g' are automorphisms of M. It follows from Exercise (3.25) that f, g are automorphisms of $E(M)$, so we are done.

In particular, the above shows that (1), (2), (3), (4) are equivalent if M is QI. Using this for the injective module $E(M)$, it also follows that (2) $\Leftrightarrow$ (3) for any M.

To construct counterexamples for (1) $\Rightarrow$ (n) for $n = 2, 3, 4$, we follow a suggestion of S. López-Permouth. Let $R = \mathbb{Z}$ and consider the R-module

$$M = C_p \oplus C_{p^2} \oplus \cdots \oplus C_{p^n} \oplus \cdots,$$

where p is a prime, and $C_m = \mathbb{Z}/m\mathbb{Z}$. Let $C = C_{p^\infty}$ be the Prüfer p-group. Then $M \subseteq_e C \oplus C \oplus \cdots$, and $C \oplus C \oplus \cdots$ is divisible, so it is injective. It follows that $E(M) = C \oplus C \oplus \cdots$, so both (2) and (3) fail for $E(M)$. (4) also fails since M contains $X \oplus X \oplus \cdots$ for $X \cong C_p$. *Nevertheless, M turns out to be Dedekind-finite.* We shall show this by checking that the ring $H = \text{End}_{\mathbb{Z}}(M)$ is Dedekind-finite, using a nice argument of K. Goodearl.

Let $J = \{h \in H : \ h(M) \subseteq pM\}$, which is easily checked to be an ideal of H. Note that

(A) $h \in J \Rightarrow \ker(1 - h) \subseteq \bigcap_{n=1}^{\infty} p^n M = 0.$

This reduces our job to showing that

(B) *The ring H/J is Dedekind-finite.*

For, if $fg = 1 \in H$, then (B) implies that $gf = 1 - h$ for some $h \in J$, and (A) yields $\ker(gf) = 0$. From $gf(gf - 1) = 0$, it follows therefore that $gf = 1 \in H$. To prove (B), it suffices to show that

(C) *H/J can be embedded into the ring* $\prod_{n=1}^{\infty} \mathbb{M}_n(\mathbb{F}_p)$ (which is clearly Dedekind-finite since each $\mathbb{M}_n(\mathbb{F}_p)$ is).

First, we note that there is a natural embedding $\varepsilon : \ H/J \rightarrow \text{End}_{\mathbb{Z}}$ (M/pM). We think of M/pM as an $\mathbb{F}_p$-vector space with basis $e_1, e_2, \ldots,$ where e_i denotes the image of a generator of C_{p^i}. For each $f \in H$, we have for every n:

$$f\left(C_p \oplus C_{p^2} \oplus \cdots \oplus C_{p^n}\right) \subseteq \left(C_p \oplus C_{p^2} \oplus \cdots \oplus C_{p^n}\right) + pM,$$

so $\varepsilon(f)$ stabilizes the subspace $\mathbb{F}_p e_1 \oplus \cdots \oplus \mathbb{F}_p e_n$ for all n, inducing an endomorphism, say f_n, on this subspace. The sequence $(f_1, f_2, \ldots, f_n, \ldots)$ determines $\varepsilon(f)$ uniquely; therefore, $f \mapsto (f_1, f_2, \ldots)$ induces the desired embedding of H/J into the direct product ring $\prod_{n=1}^{\infty} \mathbb{M}_n(\mathbb{F}_p)$.

Comment. For the ideal J of H defined above, one might suspect that $J \subseteq \operatorname{rad} H$ (the Jacobson radical of H). However, this turns out to be not the case, as was pointed out by G. Marks. In fact, let $f \in H$ be the endomorphism of $M = C_p \oplus C_{p^2} \oplus \cdots$ obtained by using the natural inclusions

$$C_p \hookrightarrow C_{p^2} \hookrightarrow C_{p^3} \hookrightarrow \cdots .$$

Clearly $f(M) \subseteq pM$ so we have $f \in J$. However, $1 - f$ is *not* surjective (and hence $f \notin \operatorname{rad} H$). Indeed, if $C_p = \langle a \rangle$, then $(a, 0, 0, \ldots)$ cannot be of the form

$$\begin{aligned}(1-f)(x_1, x_2, x_3, \ldots) &= (x_1, x_2, x_3, \ldots) - (0, x_1, x_2, \ldots) \\ &= (x_1,\ x_2 - x_1,\ x_3 - x_2, \ldots)\end{aligned}$$

for $(x_1, x_2, \ldots) \in M$, for otherwise we have $x_1 = x_2 = x_3 = \cdots$ and hence $a = x_1 = 0$ since almost all x_i's are zero.

The example of the $\mathbb{Z}$-module M above showed that the implication (4) $\Rightarrow$ (1) cannot be reversed in general. It can be shown that the implication (3) $\Rightarrow$ (4) is irreversible as well (if M is not assumed to be QI). In fact, K. Goodearl has pointed out to us that, for any field k and the simple von Neumann regular ring $\varinjlim \mathbb{M}_{2^n}(k)$ (where the direct limit is formed under block diagonal embeddings), the module $M = R_R$ contains no nonzero $X \oplus X \oplus \cdots$, but $E = E(M)$ contains a submodule isomorphic to $E \oplus E \oplus \cdots$!

Ex. 6.32. For any QI module M_R, show that the following are equivalent:

(1) M is uniform;
(2) M is indecomposable;
(3) $\operatorname{End}(M_R)$ is a local ring;
(4) $E(M)$ is uniform;
(5) $E(M)$ is indecomposable;
(6) $\operatorname{End}(E(M)_R)$ is a local ring.

Under these assumptions, show that the unique maximal (left, right) ideal of $\operatorname{End}(M)$ is $\{f \in \operatorname{End}(M) : \ker(f) \neq 0\}$. Is this the same set as $\{f \in \operatorname{End}(M) : f(M) \neq M\}$?

Solution. From *LMR*-(3.52), we know already that (4), (5), (6) are equivalent, and, of course, we also have (1) $\Leftrightarrow$ (4). (So far, we do not need the assumption that M be QI.)

(6) $\Rightarrow$ (3). By *LMR*-(6.76), we have a natural surjection of rings

$$\alpha : \operatorname{End}(E(M)) \longrightarrow \operatorname{End}(M).$$

Since $\text{End}(E(M))$ is a local ring, so is $\text{End}(M)$.

$(3) \Rightarrow (2)$. If $\text{End}(M)$ is local, it has no nontrivial idempotents, so clearly M is indecomposable.

$(2) \Rightarrow (1)$. Assume M is not uniform, say $A \cap B = 0$ where $A \neq 0 \neq B$ in M. Upon taking $E(A)$ and $E(B)$ inside $E(M)$, we have $E(A) \cap E(B) = 0$. Since $E(A) + E(B) = E(A) \oplus E(B)$ is injective, we may write

$$E(M) = E(A) \oplus E(B) \oplus X$$

for some $X \subseteq E(M)$. By *LMR*-(6.79),

$$M = (M \cap E(A)) \oplus (M \oplus E(B)) \oplus (M \cap X).$$

Since $M \cap E(A) \neq 0 \neq M \cap E(B)$, this equation shows that M is decomposable.

This completes the proof for the equivalence of the six conditions. Assuming that these conditions hold, let us now compute $\text{rad}\, H$ for the local ring $H = \text{End}(M)$. Let $f \in H$ be such that $\ker(f) \neq 0$. Then f is not an automorphism of M, so $f \notin \mathrm{U}(H)$, which means that $f \in \text{rad}\, H$. Conversely, let $g \in \text{rad}\, H$. If $\ker(g) = 0$, then $g: M \longrightarrow g(M)$ is an isomorphism, and its inverse is the restriction of some $h \in H$ (since M is QI). Now $hg = 1 \in H$, and this implies that $1 \in \text{rad}\, H$, a contradiction. Therefore, $\ker(g) = 0$, as desired.

In general, the set $A = \{f \in H : f(M) \neq M\}$ lies in $\text{rad}\, H$, but equality need not hold in general. For instance, over the ring $R = \mathbb{Z}$, let M be the Prüfer p-group C_{p^∞} for a fixed prime p. Here $H = \text{End}(M)$ is the ring of p-adic integers, with $\text{rad}\, H = pH$. The endomorphism $p = p \cdot \text{Id}_M$ belongs to (and generates) $\text{rad}\, H$, but it is *onto*, and so does not lie in A. (Of course, it is not *one-one*!)

Comment. $(2) \Rightarrow (5)$ certainly need not hold if M is *not* a QI module. For instance, for $R = k[x, y]$ with the relations $x^2 = xy = yx = y^2$ (where k is any field), the module $M = R_R$ is indecomposable, but it is not uniform, and $E(M)$ is a direct sum of two copies of $\hat{R} = \text{Hom}_k(R, k)$: see *LMR*-(3.69).

For a general (not necessarily indecomposable) QI module M, a detailed analysis of the structure of $\text{End}(M)$ is given in *LMR*-(13.1). The basic case where M is indecomposable serves as a good model for this more general analysis.

Ex. 6.33. Over a right artinian ring R, show that any faithful QI module M_R is injective.

Solution. For any finite set $A \subseteq M$, $\text{ann}(A) = \{r \in R : Ar = 0\}$ is a right ideal in R. Let $B \subseteq M$ be a finite set such that $\text{ann}(B)$ is minimal among $\{\text{ann}(A) : |A| < \infty\}$. For any $a \in M$, $\text{ann}(B \cup \{a\}) \subseteq \text{ann}(B)$ implies that $\text{ann}(B \cup \{a\}) = \text{ann}(B)$. Therefore, for any $r \in R$, $Br = 0 \Rightarrow ar = 0$.

Since M is faithful, we must have $\text{ann}(B) = 0$. If $B = \{b_1, \ldots, b_n\}$, the map $1 \mapsto (b_1, \ldots, b_n)$ then defines an embedding $R \hookrightarrow M^n$. Since M is QI, so is M^n by *LMR*-(6.77). The fact that $R \hookrightarrow M^n$ now shows that M^n is injective by *LMR*-(6.71) (2). Hence M itself must be injective.

Comment. The assumption that R is right artinian is somewhat underutilized in this exercise. The property we are really using is that the module R_R is "finitely cogenerated" in the sense of *LMR*-§19. The idea in the proof above actually shows that:

> *If R_R is finitely cogenerated, then for any faithful module M_R, R embeds into M^n for some $n < \infty$.*

The converse of this statement is true also: all of this will be given more formally later as Exercise (19.9).

Ex. 6.34. (L. Fuchs) For any module M_R, show that the following are equivalent:

(1) M is QI;
(2) For any submodule $L \subseteq M$ contained in a cyclic submodule of M, any $f \in \text{Hom}_R(L, M)$ extends to an endomorphism of M;
(3) For any B_R such that $\forall\ b \in B$, $\exists\ m \in M$ with $\text{ann}(m) \subseteq \text{ann}(b)$, any R-homomorphism from a submodule of B to M extends to B;
(4) ("Quasi Baer's Test") For any right ideal $J \subseteq R$, any R-homomorphism $g: J \to M$ whose kernel contains $\text{ann}(m)$ for some $m \in M$ extends to R_R.

Solution. (3) $\Rightarrow$ (1) $\Rightarrow$ (2) are tantologies, in view of the definition of QI.

(2) $\Rightarrow$ (4). Suppose $\ker(g) \supseteq \text{ann}(m)$ as in (4). Define $f: m \cdot J \to M$ by $f(mj) = g(j)$ for every $j \in J$. Since

$$mj = 0 \Longrightarrow j \in \text{ann}(m) = 0 \Longrightarrow g(j) = 0,$$

f is a well-defined R-homomorphism. Since $m \cdot J \subseteq m \cdot R$, (2) implies that f is the restriction of some $h \in \text{End}_R(M)$. Let $m_0 := h(m) \in M$. Then, for any $j \in J$:

$$g(j) = f(mj) = h(mj) = h(m)j = m_0 j.$$

This means that $g: J \to M$ can be extended to R_R.

(4) $\Rightarrow$ (3). For B_R as in (3) and any $A_R \subseteq B$, let $f \in \text{Hom}_R(A, M)$ be given. A usual application of Zorn's Lemma enables us to assume that f cannot be extended to a submodule properly containing A. We finish by showing that $A = B$. Indeed, *suppose there exists* $b \in B \backslash A$. Then

$$J := \{j \in R : \ bj \in A\}$$

is a right ideal of R. We define $g \in \text{Hom}_R(J, M)$ by $g(j) = f(bj)$ (for every $j \in J$). By assumption, there exists $m \in M$ such that $\text{ann}(m) \subseteq \text{ann}(b)$. For this m, we have clearly $\text{ann}(m) \subseteq \ker(g)$. Therefore, (4) applies, so that there exists $m_0 \in M$ such that $g(j) = m_0 j$ for every $j \in J$. Now define $h: A + bR \to M$ by

$$h(a + br) = f(a) + m_0 r \quad (\forall\, a \in A \text{ and } r \in R).$$

To see that h is well-defined, suppose $a + br = 0$. Then $br = -a \in A$ implies that $r \in J$, so

$$f(a) = f(b(-r)) = g(-r) = m_0(-r);$$

that is, $f(a) + m_0 r = 0$. It is easy to check that h is an R-homomorphism. Since h obviously extends f, the required contradiction is at hand.

Comment. The Condition (3) brings the notion of quasi-injectivity closely in line with the usual notion of injectivity, in that it is an extension property for homomorphisms from *other* modules to M. Condition (4) in turn gives the "quasi" analogue of the classical Baer's Test. The result in this exercise comes the Fuchs' paper, "On quasi-injective modules," Annali Scuola Norm. Sup. Pisa **23** (1969), 541–546.

Ex. 6.35. An exercise in a ring theory monograph asked the reader to prove the equivalence of the following two conditions on a right ideal $I \subseteq R$:

(a) $I = eR$ for some idempotent $e \in R$;
(b) I is isomorphic to a direct summand of R.

Provide some counterexamples to this alleged equivalence.

Solution. Note that $\{eR : e = e^2 \in R\}$ gives *all* the direct summands of R_R, so clearly, (a) $\Rightarrow$ (b). The converse (b) $\Rightarrow$ (a), however, is false in general. For instance, for any domain R that is not a division ring, if $a \neq 0$ is a nonunit, then aR is isomorphic to the direct summand R_R, but aR is not of the form eR (for $e = e^2$) since the only idempotents in R are 0 and 1. For a more interesting example, let $R = \begin{pmatrix} k & k \\ 0 & k \end{pmatrix}$ for any division ring k. The two right ideals

$$I = \begin{pmatrix} 0 & k \\ 0 & 0 \end{pmatrix} \text{ and } I' = \begin{pmatrix} 0 & 0 \\ 0 & k \end{pmatrix}$$

are isomorphic (by the map $\begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix} \mapsto \begin{pmatrix} 0 & 0 \\ 0 & a \end{pmatrix}$), and I' is a direct summand of R_R (with direct complement $\begin{pmatrix} k & k \\ 0 & 0 \end{pmatrix}$). However, $I^2 = 0$, so the only idempotent in I is zero.

Comment. The equivalence (a) $\Leftrightarrow$ (b) would be the condition (C_2) for the module R_R in the following exercise. Rings satisfying this condition include the right principally injective rings (by Exercise (3.46)), which in turn include all von Neumann regular rings.

Ex. 6.36. For any module M_R, consider the following conditions, where the word "summand" means throughout "direct summand":

(C_1) M is CS (any $N \subseteq_c M$ is a summand);
(C_2) If $K \subseteq M$ is isomorphic to a summand A of M, then K itself is a summand of M;
(C_3) If A, B are summands of M and $A \cap B = 0$, then $A + B$ is a summand of M.

Show that (C_2) $\Rightarrow$ (C_3), and that any QI module M satisfies (C_1), (C_2), and (C_3). In the literature, M is called *continuous* if it satisfies (C_1), (C_2), and *quasi-continuous* if it satisfies (C_1), (C_3). With this terminology, we have the following basic implications:

$(*)$ Injective $\Rightarrow$ QI $\Rightarrow$ continuous $\Rightarrow$ quasi-continuous $\Rightarrow$ CS.

Solution. Assume (C_2), and let A, B be summands of M with $A \cap B = 0$. Pick C such that $M = A \oplus C$ and let π be the projection $M \twoheadrightarrow C$ (with $\ker(\pi) = A$). Clearly, $B \xrightarrow{\pi} \pi(B)$ is an isomorphism. Thus, to show that $A \oplus B$ is a summand, it suffices to show that $A \oplus \pi B$ is a summand (thanks to (C_2)). But (C_2) also implies that πB is a summand of M, and hence of C. We may then write $C = \pi B \oplus X$ (for some X), and get

$$M = A \oplus C = (A \oplus \pi B) \oplus X.$$

This checks (C_3).

Now consider any QI module M. By *LMR*-(6.80), M satisfies (C_1), so in view of the above, it suffices to check that M also satisfies (C_2). Let K, A be as in (C_2) and let $f: A \to K$ be an isomorphism, with inverse g. Since M is QI, there exists $h \in \text{End}_R(M)$ extending g. Let π be a fixed projection of M onto A (which exists since A is a summand of M). For any $a \in A$, we have

$$((\pi \circ h) f)(a) = \pi(gf(a)) = \pi(a) = a.$$

This means that the injection $A \overset{f}{\rightarrowtail} M$ is split by $\pi \circ h$, so $f(A) = K$ is a summand of M, as desired.

Comment. This is not the place to compile a list of detailed examples (to show, for instance, that none of the implications in $(*)$ is reversible). However, it behooves us to mention at least one concrete case in which the property (C_3) fails to hold. In $R = \mathbb{M}_2(\mathbb{Z})$, consider the right ideals

$$A = \begin{pmatrix} \mathbb{Z} & \mathbb{Z} \\ 0 & 0 \end{pmatrix}, \quad C = \begin{pmatrix} 0 & 0 \\ \mathbb{Z} & \mathbb{Z} \end{pmatrix}, \quad \text{and} \quad B = \left\{ \begin{pmatrix} x & y \\ 2x & 2y \end{pmatrix} : x, y \in \mathbb{Z} \right\}.$$

Here $R = A \oplus C = B \oplus C$, so A, B are summands of R_R. Clearly, $A \cap B = 0$, but $A \oplus B = \begin{pmatrix} \mathbb{Z} & \mathbb{Z} \\ 2\mathbb{Z} & 2\mathbb{Z} \end{pmatrix}$ is essential in R_R, and is thus not a summand. (Of course, $A \oplus B$ is not even an abelian group summand of R.)

Therefore, R_R does not satisfy (C_3), and (C_2) fails also since $2A \cong A$ but $2A$ is not a summand of R_R.

We have included this exercise because of the increasing popularity of the notions mentioned at the end of the exercise. There have been at least two recent books devoted to the study of modules satisfying some of the conditions (C_i). These are: "Continuous and Discrete Modules" by S. H. Mohamed and B. Müller, Cambridge Univ. Press, 1990, and "Extending Modules" by N. V. Dung, D. V. Huynh, P. F. Smith and R. Wisbauer, Longman Scientific & Technical, U. K., 1994. For the extensive theory of CS modules and continuous modules (and the associated rings with the same names), we refer the reader to these two monographs.

Ex. 6.37. (Goel-Jain) For any M_R, show that the following are equivalent:

(1) M is quasi-continuous (i.e. M satisfies (C_1) and (C_3));
(2) Any idempotent endomorphism of a submodule of M extends to an idempotent endomorphism of M;
(3) Any idempotent endomorphism of a submodule of M extends to an endomorphism of M;
(4) M is invariant under any idempotent endomorphism of $E(M)$;
(5) If $E(M) = \bigoplus_{i\in I} X_i$, then $M = \bigoplus_{i\in I} (M \cap X_i)$;
(6) If $E(M) = X \oplus Y$, then $M = (M \cap X) \oplus (M \cap Y)$.

Solution. (1) $\Rightarrow$ (2). Let f be an idempotent endomorphism of $N \subseteq M$. Then $N = A \oplus B$ where $A = \ker(f)$ and $B = \operatorname{im}(f)$. Let $A' \supseteq A$ be a complement of B in M. Then $A' \subseteq_c M$. Similarly, if $B' \supseteq B$ is a complement of A' in M, then $B' \subseteq_c M$. Since M is a CS module, A', B' are summands of M. Therefore, by (C_3), $M = A' \oplus B' \oplus X$ for some submodule X. Now the projection of M onto B' with respect to this decomposition is an idempotent endomorphism of M extending f.

(2) $\Rightarrow$ (3) is a tautology.

(3) $\Rightarrow$ (4). Let $e = e^2 \in \operatorname{End}_R(E(M))$. For $M_1 = M \cap \ker(e)$ and $M_2 = M \cap \operatorname{im}(e)$, we have $M_1 \cap M_2 = 0$. By (3), there exists $f \in \operatorname{End}_R(M)$ such that $f\,|M_1 = 0$ and $f|\,M_2 = \operatorname{Id}_{M_2}$. We claim that $(e-f)\,M = 0$. Indeed, if otherwise, $(e-f)\,M \cap M \neq 0$, so there exist $m, m' \in M$ with $0 \neq m' = (e-f)\,m$. Then

$$em = fm + m' \in M_2 \qquad \text{and} \qquad (1-e)\,m = m - e\,m \in M_1.$$

Therefore, $m = (1-e)\,m + em \in M_1 \oplus M_2$. Applying f, we get $fm = em$, which contradicts $m' \neq 0$. Thus, we must have $(e-f)\,M = 0$, and hence $eM = fM \subseteq M$.

(4) $\Rightarrow$ (5). Let π_i be the projection from $E(M) = \bigoplus_i X_i$ to X_i. For $m \in M$, let $m = \sum_i x_i$ (finite sum), where each $x_i \in X_i$. Then

$$x_i = \pi_i\,(x) \in \pi_i\,(M) \subseteq M$$

by (4), so $m \in \bigoplus_i (M \cap X_i)$. This shows that $M = \bigoplus_i (M \cap X_i)$.

(5) $\Leftrightarrow$ (6) is straightforward.

(5) $\Rightarrow$ (1). The argument for proving (C_1) (that M is a CS module) is the same as that for *LMR*-(6.79). (The argument there uses only the "Cutting Property" (5).)

Comment. The property (3) above is often referred to as "*π-injectivity*" ("π" here stands for "projection"). The equivalence of most of the properties in this exercise appeared in the paper of V. K. Goel and S. K. Jain, "π-injective modules and rings whose cyclics are π-injective," Comm. Algebra **6** (1978), 59–73. In this paper, it is also pointed out that any module M has a "π-injective hull", given by the smallest R-submodule of $E(M)$ containing M that is stabilized by all idempotent endomorphisms of $E(M)$.

In the work of later authors, "π-injectivity" was shown to be equivalent to "quasi-continuity" ((C_1) plus (C_3)). The use of the term "continuity" will be explained in the next exercise and the ensuing *Comment.*

Ex. 6.38. For any von Neumann regular ring R, show that the following are equivalent:

(a) R_R is continuous,
(b) R_R is quasi-continuous, and
(c) R_R is CS.

(A von Neumann regular ring R is said to be *right continuous* if it satisfies these equivalent conditions. For instance, any right self-injective von Neumann regular ring is right continuous.)

Solution. Since we have (a) $\Rightarrow$ (b) $\Rightarrow$ (c) by Exercise 36, it suffices to prove (c) $\Rightarrow$ (a). Now (a) means (c) plus the condition (C_2) for R_R (in the notation of Exercise 36). For a von Neumann regular ring R, summands of R_R are precisely the principal right ideals, so clearly R_R satisfies (C_2). (For a more general statement, see the *Comment* following Exercise 35.) This remark yields immediately (c) $\Rightarrow$ (a).

Comment. In von Neumann's original work, a von Neumann regular ring R is called *right continuous* if the lattice of principal right ideals is complete and satisfies an upper continuity axiom, and R is called *continuous* if the same lattice is complete and satisfies both the upper and lower continuity axioms. The right continuous property on the von Neumann regular ring R turns out to be equivalent to R_R being a CS module, so our definition given in this exercise is consistent with von Neumann's. For a detailed study of the theory of right continuous von Neumann regular rings, see Chapter 13 of Goodearl's book "von Neumann Regular Rings," Krieger Publ. Co., 1991. The idea of investigating the notion of continuity of modules independently of von Neumann regular rings is due to Y. Utumi.

Ex.6.39. Show that each of the four implications in $(*)$ listed at the end of Exercise 36 is irreversible.

Solution. *We first work over the ring* $R = \mathbb{Z}$. For a prime p, $M = \mathbb{Z}_p \oplus \mathbb{Z}_{p^2}$ is a CS module by Exercise (6.19C), but does not satisfy (C_3). In fact, $A = \mathbb{Z} \cdot (\overline{1}, \overline{0})$ and $B = \mathbb{Z} \cdot (\overline{1}, \overline{p})$ are both summands (with a common direct complement $\mathbb{Z} \cdot (0, \overline{1})$). We have $A \cap B = 0$, but $A \oplus B \cong \mathbb{Z}_p \oplus \mathbb{Z}_p$ is not a summand of M. Thus, M is not quasi-continuous.

Next, note that any uniform module is clearly quasi-continuous, but may not satisfy (C_2). For instance, the uniform module $\mathbb{Z}$ is quasi-continuous, but $2\mathbb{Z} \cong \mathbb{Z}$, and $2\mathbb{Z}$ is not a summand in $\mathbb{Z}$. Thus, $\mathbb{Z}$ is not continuous. And, of course, $\mathbb{Z}_n$ is QI, but not injective. Thus, it only remains to construct a module that is continuous, but not QI.

For this, consider two fields $k \subsetneq K$. Let $S = K \times K \times \cdots$, and let R be the subring of S consisting of $(a_1, a_2, \ldots)$ with all but finitely many a_i's in k. It is easy to see that R and S are commutative von Neumann regular rings. Clearly, $R_R \subsetneq_e S_R$, so R_R is not injective, which implies that R_R is not QI (by *LMR*-(6.71)(2B)). We finish by checking that any ideal $A \subseteq R$ is essential in fR for some $f = f^2 \in R$ (so that R_R is continuous, by Exercise 38). Indeed, let

$$I = \{i : \exists\, (a_1, a_2, \ldots) \in A \text{ with } a_i \neq 0\}.$$

For the idempotent $f = (f_1, f_2, \ldots)$ with $f_i = 1$ for $i \in I$ and $f_i = 0$ otherwise, we have $A \subseteq fR$. For any

$$0 \neq r = (r_1, r_2, \ldots) \in fR,$$

fix an index $i \in I$ with $r_i \neq 0$. A suitable R-multiple of r has the form $(0, \ldots, 1, 0, \ldots)$ with the "1" in the i^{th} coordinate, and this unit vector belongs to A. Therefore, we have checked that $A \subseteq_e fR$, as desired.

Ex. 6.40. For any QI module M_R, let $S = \text{End}\,(M_R)$ and $m \in M$. If $m \cdot R$ is a simple R-module, show that $S \cdot m$ is a simple S-module. From this fact, deduce that $\text{soc}\,(M_R) \subseteq \text{soc}\,({}_SM)$.

Solution. It suffices to show that, for any $s \in S$ such that $sm \neq 0$, $S \cdot sm$ contains m. Consider the R-epimorphism $\varphi : mR \to smR$ given by left multiplication by s. Since mR is simple, φ is an isomorphism. Let $\psi = \varphi^{-1}$ and extend ψ to an endomorphism $t \in S$ (using the quasi-injectivity of M_R). Now

$$tsm = \psi(sm) = \varphi^{-1}(sm) = m,$$

so $m \in S \cdot sm$, as desired.

For the last part of the exercise, note that if $x \in \text{soc}\,(M_R)$, then xR is a semisimple module, so if $x \neq 0$ we can write it in the form $m_1 + \cdots + m_n$ where m_iR is simple for each i. The above implies that each $m_i \in \text{soc}\,({}_SM)$, so $x = m_1 + \cdots + m_n \in \text{soc}\,({}_SM)$.

Comment. In *ECRT*-Ex. 3.6A, we have solved the first part of this exercise for a *semisimple* module M_R. Here we simply repeat that solution, as it works already for a QI module.

§7. Singular Submodules and Nonsingular Rings

For any module M_R, the singular submodule $\mathcal{Z}(M)$ consists of elements $m \in M$ for which $\text{ann}(m) \subseteq_e R_R$. The module is called *nonsingular* if $\mathcal{Z}(M) = 0$, and *singular* if $\mathcal{Z}(M) = M$. Note that "nonsingular" is not the same as "not singular". Any submodule of a nonsingular (resp. singular) module is nonsingular (resp. singular), and in general, there is no nonzero homomorphism from a singular module to a nonsingular module.

In any ring R, $\mathcal{Z}(R_R)$ is always an ideal, called the *right singular ideal*, and R is said to be a *right nonsingular ring* if $\mathcal{Z}(R_R) = 0$. (Left nonsingular rings are defined similarly.) Right nonsingular rings include all reduced rings, all group rings kG over formally real fields k, and all right semihereditary rings (in particular all von Neumann regular rings). Among *commutative* rings R, the nonsingular rings are just the reduced rings; however, $\mathcal{Z}(R)$ may not be the same as $\text{Nil}(R)$. For a more precise statement, see Exercise 9.

Under a suitable finiteness condition on R, one might hope for a "nil" or "nilpotent" conclusion on $\mathcal{Z}(R_R)$. For instance, if R satisfies ACC on right annihilators of elements, then $\mathcal{Z}(R_R)$ is nil, and if R satisfies ACC on right annihilators, then $\mathcal{Z}(R_R)$ is in fact nilpotent: see *LMR*-(7.15).

Section 7 also deals with the notions of Baer rings and Rickart rings, which are motivated by the study of operator algebras. A ring R is called a *right Baer ring* if every right annihilator in R has the form eR for some idempotent $e \in R$, and R is called a *right Rickart ring* if the right annihilator of any element in R has the form eR for some idempotent $e \in R$. Right Rickart rings R are characterized by the "right PP" property: every principal right ideal in R is projective (*LMR*-(7.48)). In general, right Rickart rings need not be left Rickart. However, right Baer rings turn out to be the same as left Baer rings (*LMR*-(7.46)).

In general, we have the implications:

$$\begin{array}{ccccccc} & & & & \text{Baer} & & \\ & & & & \Downarrow & \searrow & \\ \begin{array}{c}\text{von Neumann}\\ \text{regular}\end{array} & \Longrightarrow & \begin{array}{c}\text{right}\\ \text{semihereditary}\end{array} & \Longrightarrow & \begin{array}{c}\text{right}\\ \text{Rickart}\end{array} & \Longrightarrow & \begin{array}{c}\text{right}\\ \text{nonsingular}\end{array} \end{array}$$

and, for right self-injective rings, the five properties are all equivalent: see *LMR*-(7.50) and *LMR*-(7.52). The Exercises 21–29 cover various additional aspects of Baer rings and right Rickart rings. Other exercises in this section provide connections between nonsingular modules and CS modules, QI modules, uniform modules, etc. The last exercise, due to R. Shock, computes the right singular ideal of a polynomial ring, showing, in particular, that if R is right nonsingular, then so is any polynomial ring (in commuting variables) over R.

Exercises for §7

Ex. 7.1. Compute the right singular ideal $\mathcal{Z}(R_R)$ for the $\mathbb{Z}$-algebra generated by x, y with the relations $yx = y^2 = 0$.

Solution. We compute with the ring R by using the fact that any element in R is uniquely expressible in the form $f(x) + g(x)y$ where $f(x)$, $g(x) \in \mathbb{Z}[x]$. It is easy to show that $xR + yR \subseteq_e R_R$ (see *LMR*-(7.6)(4)), so from $\text{ann}_r(y) \supseteq xR + yR$, we see that $y \in \mathcal{Z}(R_R)$. Since $\mathcal{Z}(R_R)$ is an ideal, it contains

$$I = RyR = Ry = \{f(x)y : \ f(x) \in \mathbb{Z}[x]\}.$$

For any $z_0 \in R \backslash I$, we claim that $\text{ann}_r(z_0) = 0$. This will show that $z_0 \notin \mathcal{Z}(R_R)$ and therefore $\mathcal{Z}(R_R) = I$. Write $z_0 = f_0(x) + g_0(x)\,y$, where $f_0(x) \neq 0$. If $z_0 z = 0$ where $z = f(x) + g(x)\,y$, then

$$\begin{aligned} 0 &= (f_0(x) + g_0(x)\,y)(f(x) + g(x)\,y) \\ &= f_0(x) f(x) + (g_0(x) f(x) + f_0(x) g(x))\,y \end{aligned}$$

implies that $f(x) = 0$ and $g(x) = 0$ since $f_0(x)$ is not a 0-divisor in $\mathbb{Z}[x]$. Hence $z = 0 \in R$, as claimed.

Comment. Note that the ideal I above has square zero. Thus, from what we showed above, we conclude that I is also the lower and the upper nil radical of R. Since $R/I \cong \mathbb{Z}[x]$, I is the Jacobson radical of R as well. However, $\mathcal{Z}({}_RR) = 0$, as is shown in *LMR*-(7.6)(4). Thus, R is an example of a (necessarily noncommutative) ring that has a nilpotent ideal not contained in $\mathcal{Z}({}_RR)$. Finally, note that R is left noetherian but not right noetherian: see *FC*-(1.26).

Ex. 7.2. (a) Show that an R-module S is singular iff there exist two R-modules $N \subseteq_e M$ such that $S \cong M/N$. (b) Let $N \subseteq M$ be two R-modules, where M is R-free. Show that M/N is singular iff $N \subseteq_e M$.

Solution. The "if" part in (a) is proved in *LMR*-(7.6)(3). Therefore, it suffices to prove the "only if" parts in (a) and (b).

Working first with (b), we assume that M/N is singular, where M has a free R-basis $\{x_i\}$. For each i, there exists a right ideal $I_i \subseteq_e R_R$ such that $x_i I_i \subseteq N$. Clearly, we also have $x_i I_i \subseteq_e x_i R$, so by *LMR*-(3.38),

$$\bigoplus_i x_i I_i \subseteq_e \bigoplus_i x_i R = M.$$

Since $\bigoplus_i x_i I_i \subseteq N$, it follows that $N \subseteq_e M$.

Returning now to (a), consider any singular R-module S. Let M be a free R-module with an epimorphism $\varphi: M \to S$. By part (b), $N := \ker(\varphi)$ is essential in M, and we have $S \cong M/N$.

Ex. 7.3. For any submodule N in a nonsingular module M, show that M/N is singular iff $N \subseteq_e M$.

Solution. Again, the "if" part follows from *LMR*-(7.6)(3) (without any hypothesis on M). For the converse, assume that M is nonsingular and that M/N is singular. Consider any nonzero element $m \in M$. Since M/N is singular, there exists a right ideal $I \subseteq_e R_R$ such that $mI \subseteq N$. We must have $mI \neq 0$ for otherwise $m \in \mathcal{Z}(M)$. Say $mr \neq 0$, where $r \in I$. Then $0 \neq mr \in N$, and we have shown that $N \subseteq_e M$.

Comment. If M is not assumed to be nonsingular, the "only if" part of the Exercise need not hold. For instance, if $M = N \oplus S$ where S is a nonzero singular module, then $M/N \cong S$ is singular but N is *not* essential in M. Of course M is not nonsingular in this example, since $\mathcal{Z}(M) \supseteq \mathcal{Z}(S) = S \neq 0$.

Ex. 7.4. Show that an R-module M is nonsingular iff, for any singular module S, $\mathrm{Hom}_R(S, M) = 0$.

Solution. First assume M is nonsingular. Consider any R-homomorphism $f: S \to M$, where S is a singular R-module. By *LMR*-(7.2)(3),

$$f(S) = f(\mathcal{Z}(S)) \subseteq \mathcal{Z}(M) = 0,$$

so f is the zero homomorphism. Conversely, if M is not nonsingular, then $S := \mathcal{Z}(M)$ is a nonzero singular module, and the inclusion map $S \to M$ is a nonzero element in $\mathrm{Hom}_R(S, M)$.

Ex. 7.5. Let $N \subseteq M$ be R-modules. (a) If N and M/N are both nonsingular, show that M is also nonsingular. (b) Does this statement remain true if we replace the word "nonsingular" throughout by "singular"?

Solution. (a) By *LMR*-(7.2)(4), $\mathcal{Z}(M) \cap N = \mathcal{Z}(N) = 0$. Therefore, the projection map from M to M/N induces an injective homomorphism $\pi: \mathcal{Z}(M) \to M/N$. Since $\mathcal{Z}(M)$ is singular and M/N is nonsingular, we must have $\pi = 0$ by the previous exercise. This implies that $\mathcal{Z}(M) = 0$, as desired.

(b) The statement is no longer true if "nonsingular" is replaced by "singular". For instance, let $R = \mathbb{Z}/4\mathbb{Z}$. The module $M = R_R$ has singular submodule $N = \mathcal{Z}(M) = 2\mathbb{Z}/4\mathbb{Z}$, and $M/N \cong N$ is also singular. However, since $\mathcal{Z}(M) \neq M$, M itself is not singular.

Comment. In (b), the anomaly is primarily due to the fact that the ring $\mathbb{Z}/4\mathbb{Z}$ is not nonsingular. If R is a right nonsingular ring, it can be shown that the statement (a) remains true for singular R-modules. For details, see Proposition (1.23) in Goodearl's book "Ring Theory: Nonsingular Rings and Modules," Marcel-Dekker, Inc., 1976.

Ex. 7.6. Let $I \subseteq R$ be any left ideal.

(a) For any $n \geq 1$, show that $\text{ann}_\ell(I) \subseteq_e R_R$ iff $\text{ann}_\ell(I^n) \subseteq_e R_R$.
(b) If I is nilpotent, show that $\text{ann}_\ell(I) \subseteq_e R_R$.

Solution. We note first that $\text{ann}_\ell(I^n)$ is an ideal in R for any n.

(a) The "only if" part is clear, since $\text{ann}_\ell(I) \subseteq \text{ann}_\ell(I^n)$. For the "if" part, it suffices to show that

$$\text{ann}_\ell(I^2) \subseteq_e R_R \implies \text{ann}_\ell(I) \subseteq_e R_R.$$

Consider any nonzero element $r \in R$. Assuming that $\text{ann}_\ell(I^2) \subseteq_e R_R$, we have $0 \neq rs \in \text{ann}_\ell\left(I^2\right)$ for some $s \in R$. If $rsI = 0$, then $0 \neq rs \in \text{ann}_\ell(I)$ so we are done. Thus, we may assume that $rst \neq 0$ for some $t \in I$. Now $rst\, I \subseteq rsI^2 = 0$, so $r(st)$ is a nonzero element in $\text{ann}_\ell(I)$. This checks that $\text{ann}_\ell(I) \subseteq_e R_R$.

(b) Fix an integer $n \geq 1$ such that $I^n = 0$. Then $\text{ann}_\ell\left(I^n\right) = R \subseteq_e R_R$ implies that $\text{ann}_\ell(I) \subseteq_e R_R$, by (a).

Ex. 7.7. Let R be a ring for which every ideal right essential in R contains a non left-0-divisor. Show that R must be semiprime.

Solution. It suffices to show that, for any ideal $I \subseteq R$, $I^2 = 0$ implies that $I = 0$. By Exercise 6(b), $I^2 = 0$ yields $\text{ann}_\ell(I) \subseteq_e R_R$, so there exists a non left-0-divisor $r \in \text{ann}_\ell(I)$. Now $r \cdot I = 0$ implies that $I = 0$, as desired.

Comment. This exercise shows, in particular, that if every essential right ideal in R contains a regular element, then R is a semiprime ring. On the other hand, if R is a semiprime right Goldie ring, Goldie's Theorem implies that any essential right ideal $I \subseteq R$ contains a regular element. In fact, in this case, every coset $c + I$ $(c \in R)$ contains a regular element, and I is generated as a right ideal by the set of regular elements in I: see Exercises (11.26) and (11.27) below.

Ex. 7.8. (a) For any central element $x \in R$ and any $n \geq 1$, show that $\text{ann}_\ell(x) \subseteq_e R_R$ iff $\text{ann}_\ell(x^n) \subseteq_e R_R$.
(b) Use (a) to show that the center of a right nonsingular ring is reduced.
(c) Use (a) to show also that, for any commutative ring R, $R/\mathcal{Z}(R)$ is a nonsingular ring.

Solution. (a) Let I be the principal ideal $xR \subseteq R$. Since $I^n = x^n R$, we have

$$\text{ann}_\ell(I^n) = \text{ann}_\ell\left(x^n R\right) = \text{ann}_\ell(x^n).$$

Therefore, the desired conclusion follows from Exercise 6(a).

(b) Suppose R is right nonsingular. It suffices to show that, for any central element x in R such that $x^2 = 0$, we have $x = 0$. From $x^2 = 0$, we have certainly $\text{ann}_\ell(x^2) = R \subseteq_e R_R$, and therefore $\text{ann}_\ell(x) \subseteq_e R_R$ by (a) above. Since $x \cdot \text{ann}_\ell(x) = 0$, it follows that $x \in \mathcal{Z}(R_R) = 0$.

(c) Since $R/\mathcal{Z}(R)$ is a commutative ring, the desired conclusion is tantamount to $R/\mathcal{Z}(R)$ being reduced, by *LMR*-(7.12). Therefore, it suffices to show that for $x \in R$, $x^n \in \mathcal{Z}(R) \Longrightarrow x \in \mathcal{Z}(R)$. Since x is central in R, this follows from (a).

Ex. 7.9. Show that, for any commutative ring R, $\text{Nil}(R) \subseteq_e \mathcal{Z}(R)$. Give an example of a commutative ring R for which this inclusion is *not* an equality.

Solution. Recall from *LMR*-(7.11) that $\text{Nil}(R) \subseteq \mathcal{Z}(R)$. To show that this is an essential extension, consider any nonzero $a \in \mathcal{Z}(R)$, so $\text{ann}(a) \subseteq_e R$. Since $a \neq 0$, there exists $r \in R$ such that $0 \neq ar \in \text{ann}(a)$. Thus, $a^2r = 0$. In particular, $(ar)^2 = 0$, so $0 \neq ar \in \text{Nil}(R)$. This shows that $\text{Nil}(R) \subseteq_e \mathcal{Z}(R)$. As an example for the possible failure of equality, consider the commutative ring $R = \mathbb{Q}[x, z_1, z_2, \ldots]$ with the relations $x^{i+1}z_i = 0$ for all $i \geq 1$. In this ring, $\text{ann}(x)$ contains $\sum_{i=1}^{\infty} x^i z_i R$, which can be shown to be essential in R. Therefore $x \in \mathcal{Z}(R)$. On the other hand, $x \notin \text{Nil}\, R$. (In this example, $\mathcal{Z}(R) = xR$, and $\text{Nil}(R) = \sum_{i=1}^{\infty} x\, z_i R$.)

Comment. If a commutative ring R satisfies ACC for annihilators of elements, it is shown in *LMR*-(7.15)(1) that $\mathcal{Z}(R)$ is a nil ideal. In this case, we will have the equality $\text{Nil}(R) = \mathcal{Z}(R)$. In the example $R = \mathbb{Q}[x, z_1, z_2, \ldots]$ above, ACC *fails* for annihilators of elements, since $\text{ann}\left(x^{n+1}\right) = \sum_{i=1}^{n} z_i R$ and we have a strictly ascending chain

$$z_1 R \subsetneq z_1 R + z_2 R \subsetneq \cdots .$$

Indeed, the ring R was constructed precisely with this property in mind.

Ex. 7.10. Show that a commutative semihereditary ring R must be reduced.

Solution. By *LMR*-(7.7), R is nonsingular. Since R is commutative, this means that R is reduced, by *LMR*-(7.12).

Ex. 7.11. Show that, for R-modules M_i $(i \in I)$, $\mathcal{Z}(\bigoplus_i M_i) = \bigoplus_i \mathcal{Z}(M_i)$.

Solution. Let $M = \bigoplus_i M_i$. Then $\mathcal{Z}(M_i) \subseteq \mathcal{Z}(M)$ for each i, so we have $\bigoplus_i \mathcal{Z}(M_i) \subseteq \mathcal{Z}(M)$. For the reverse inclusion, consider any $(m_i)_{i \in I} \in \mathcal{Z}(M)$, where almost all $m_i = 0$. For any $i \in I$, consider the natural projection $\pi_i : M \to M_i$. By *LMR*-(7.2)(3), $\pi_i(\mathcal{Z}(M)) \subseteq \mathcal{Z}(M_i)$. Therefore, we have $m_i \in \mathcal{Z}(M_i)$ for all i, and so $(m_i)_{i \in I} \in \bigoplus_i \mathcal{Z}(M_i)$.

Ex. 7.12A. Let M_R be a simple R-module, and $S = \text{soc}(R_R)$. Show that

(a) M is either singular or projective, but not both; and
(b) M is singular iff $M \cdot S = 0$.
(c) Deduce from (a) above that a semisimple module is nonsingular iff it is projective.

Solution. We may assume that $M = R/\mathfrak{m}$, where $\mathfrak{m}$ is a maximal right ideal of R. By *LMR*-(7.2)(2), $\mathcal{Z}(M) \cdot S = 0$. Thus, *if M is singular*, we have $M \cdot S = 0$. In this case, M cannot be projective. For, if it is, then $R_R = \mathfrak{m} \oplus \mathfrak{A}$ for a minimal right ideal $\mathfrak{A}$. But $M \cdot S = 0$ means that $S \subseteq \mathfrak{m}$, contradicting the fact that $\mathfrak{A} \subseteq S$. Now assume *M is not singular*. Then, by Exercise 2, $\mathfrak{m}$ cannot be essential in R_R. Thus, $\mathfrak{m} \cap \mathfrak{A} = 0$ for some right ideal $\mathfrak{A} \neq 0$. We have then $R = \mathfrak{m} \oplus \mathfrak{A}$, so $M \cong \mathfrak{A}_R$ is projective. Also, $S \supseteq \mathfrak{A}$ implies that $S \not\subseteq \mathfrak{m}$, so $M \cdot S \neq 0$. This completes the proofs of (a) and (b).

For (c), consider a semisimple module $P = \bigoplus_i P_i$, where the P_i's are simple modules. If P is nonsingular, so is each P_i. Then P_i is not singular, and therefore projective by (a). This shows that P is projective. Conversely, if P is projective, so is each P_i. By (a), $\mathcal{Z}(P_i) \subsetneq P_i$ and hence $\mathcal{Z}(P_i) = 0$. By Exercise 11, $\mathcal{Z}(P) = 0$ too, and so P is nonsingular.

Ex. 7.12B. Let M_R be an R-module all of whose nonzero quotients have minimal submodules.(*) Show that M is nonsingular iff $P := \mathrm{soc}(M)$ is nonsingular, iff P is projective.

Solution. We need only prove the first "iff" statement, as the second one follows from (c) of the last exercise. It suffices to show the "if" part, so let us assume that P is nonsingular. Then $P \cap \mathcal{Z}(M) = 0$, so P can be enlarged to a complement Q of $\mathcal{Z}(M)$. Assume, for the moment, that $\mathcal{Z}(M) \neq 0$. Then, by the given assumption on M, there exists $T \supsetneq Q$ such that T/Q is simple. Then $T \cap \mathcal{Z}(M) \neq 0$, and we must have $T \cap \mathcal{Z}(M) \cong T/Q$. Since this is a simple module, $T \cap \mathcal{Z}(M) \subseteq \mathrm{soc}(M) = P$. This is a contradiction, since $P \cap \mathcal{Z}(M) = 0$. Thus, we must have $\mathcal{Z}(M) = 0$, as desired.

Ex. 7.12C. Let R be a right self-injective ring, and M_R be a nonsingular module with $\mathrm{u.dim}(M) < \infty$. Show that M is f.g. semisimple, and is both projective and injective.

Solution. We induct on $n = \mathrm{u.dim}(M)$, the case $n = 0$ being clear. If $n > 0$, there exists a uniform submodule $U \subseteq M$. Consider any $x \in U \backslash \{0\}$. Since M is nonsingular, $\mathrm{ann}(x)$ is not essential in the (injective) module R_R, so we have $E := E(\mathrm{ann}(x)) \subsetneq R_R$. Write $R = E \oplus I$ where I is a suitable nonzero right ideal in R. Then I_R is both projective and injective, and

$$U \supseteq x \cdot R \cong R/\mathrm{ann}\,(x) \cong E/\mathrm{ann}\,(x) \oplus I$$

implies that $U = x \cdot R \cong I$ (since U is uniform). This shows that U is generated by any of its nonzero elements, so U is simple. Writing $M = U \oplus V$, we have $\mathrm{u.dim}(V) < n$ and V is nonsingular, so the induction proceeds.

(*) Such a module M is said to be "semi-artinian" in the literature. For instance, an artinian module is always semi-artinian.

Comment. Of course, *once we know that M is semisimple*, then M is (f.g. and) projective by Exercise 12A above, and so $M \oplus M' \cong R^t$ for some $t < \infty$ shows that M is injective. The main point of the present exercise is, however, to show that M must be semisimple.

Ex. 7.13. Let M_R be any CS module over a ring R.

(1) Show that any surjection from M to a nonsingular module splits.
(2) Show that the Goldie closure 0^{**} (defined in *LMR*-(7.31)) splits in M.

Solution. (1) Let $f : M \to M'$ be a surjection, where M' is nonsingular. Let $K = \ker(f)$. If $K \subseteq_e L \subseteq M$, then L/K is singular (by *LMR*-(7.6)(3)) and it injects into the nonsingular module M'. Therefore, $L/K = 0$. This implies that $K \subseteq_c M$, and since M is CS, the closed submodule K must be a direct summand of M by *LMR*-(6.80).

(2) First let us recall the definition of the Goldie closure 0^{**} of the zero module. By 0^*, we mean the singular submodule of M, and 0^{**} is defined to be the submodule containing 0^* such that $0^{**}/0^*$ is the singular submodule of $M/0^*$. Now by Goldie's Theorem *LMR*-(7.28), $M/0^{**}$ is a nonsingular module. Applying (1), we see that 0^{**} splits in M.

Ex. 7.14. (Sandomierski) Let R be a right nonsingular ring, and let N be a quotient of a CS module. Show that

(1) $\mathcal{Z}(N)$ splits in N, and deduce that
(2) if N is indecomposable, then it is either singular or nonsingular.

Solution. Represent N as M/X where M is some CS module. Let $S \supseteq X$ be such that $S/X = \mathcal{Z}(M/X)$. Since R is a right nonsingular ring, we know that $N/\mathcal{Z}(N)$ is nonsingular by *LMR*-(7.21). This means that

$$M/S \cong \frac{M/X}{S/X} = N/\mathcal{Z}(N)$$

is nonsingular. By (1) of the last exercise, S splits in M; in particular, $\mathcal{Z}(N) = S/X$ splits in $N = M/X$.

This proves (1). For (2), assume, in addition that N is indecomposable. In view of (1), we must have either $\mathcal{Z}(N) = N$ or $\mathcal{Z}(N) = 0$. In the former case N is singular, and in the latter case, N is nonsingular.

Comment. This exercise comes from Sandomierski's paper "Semisimple maximal quotient rings," Trans. Amer. Math. Soc. **128** (1967), 112–120. Sandomierski proved this result for N being any quotient of an injective module M, but for his argument to work, the hypothesis that M be CS suffices. (The hypothesis that u.dim $R_R < \infty$ included for this result in Sandomierski's paper is not necessary.)

Ex. 7.15. Let N_1, N_2 be injective submodules of a nonsingular module M_R. Show that $N_1 + N_2$ is also injective. Give an example to show that this may not be true if M is an arbitrary module over R.

Solution. Consider the natural surjection $f: N_1 \oplus N_2 \to N_1 + N_2$. Since $N_1 \oplus N_2$ is injective and $N_1 + N_2 \subseteq M$ is nonsingular, f splits by the previous exercise. Therefore, $N_1 + N_2$ is isomorphic to a direct summand of $N_1 \oplus N_2$, so it is injective.

For a counterexample in case M is not a nonsingular module, consider the ring $R = \mathbb{Z}/4\mathbb{Z}$, which is self-injective by *LMR*-(3.13). Let $M = R_R \oplus N_R$ where N is the principal ideal $2R$, and let $N_1 = (1,0) \cdot R$, $N_2 = (1,2) \cdot R$. Then $N_1 \cong N_2 \cong R_R$ are injective R-modules. However, $N_1 + N_2 = M$ is *not* injective, since its direct summand N is not. Here, of course, M is not a nonsingular R-module. In fact, by Exercise 10,

$$\mathcal{Z}(M) = \mathcal{Z}(R_R) \oplus \mathcal{Z}(N_R) = N \oplus N.$$

Comment. Recall from Exercise 3.10 that the sum of two injective submodules of *any* module M_R is injective iff the ring R is *right semihereditary.* Therefore, an example for the second part of the Exercise can actually be found over any non right semihereditary ring.

Ex. 7.16. Let $S \subseteq R$ be rings such that $S_S \subseteq_e R_S$, and let M, N be right R-modules. If M_S is nonsingular, show that $\mathrm{Hom}_S(N, M) = \mathrm{Hom}_R(N, M)$.

Solution. It suffices to show that any $f \in \mathrm{Hom}_S(N, M)$ is an R-homomorphism. Let $n \in N$ and $r \in R$. The right ideal

$$J := \{s \in S :\ rs \in S\}$$

in S is right essential, by Ex. (3.7). For any $s \in J$, we have $f(nrs) = f(n)rs$, since $rs \in S$. On the other hand, $f(nrs) = f(nr)s$, since $s \in S$. Therefore, $(f(nr) - f(n)r)s = 0$ for all $s \in J$. Since $J \subseteq_e S_S$ and M_S is nonsingular, we must have $f(nr) = f(n)r$, so $f \in \mathrm{Hom}_R(N, M)$.

Ex. 7.17. Show that "right semihereditary" and "Baer" are independent notions.

Solution. Recall that any domain D is a Baer ring. Certainly, D need not be semihereditary. Conversely, let $A = F \times F \times \cdots$ where F is any field, and let R be the subring of A consisting of sequences $(a_1, a_2, \ldots) \in A$ that are eventually constant. It is easy to see that R is a von Neumann regular ring, so R is in particular semihereditary. However, it is shown in *LMR*-(7.54) that R is not Baer. In fact, if $e_i \in R$ denotes the i^{th} "unit vector" $(0, \ldots, 1, 0, \ldots)$ and $S = \{e_1, e_3, e_5, \ldots\} \subset R$, the annihilator $\mathrm{ann}(S)$ in R is not a finitely generated ideal.

Ex. 7.18. Let R be any right semihereditary ring and $S \subseteq R$ be a finite set. Show that $\mathrm{ann}_r(S) = eR$ for some idempotent $e \in R$.

Solution. Let $S = \{s_1, \ldots, s_n\}$, and consider the homomorphism of right R-modules

$$f:\ R \longrightarrow R^n = R \oplus \cdots \oplus R$$

defined by $f(x) = (s_1 x, \ldots, s_n x)$. The image of f is a cyclic submodule of R^n, so it is projective by *LMR*-(2.29). Therefore, $\ker(f) = \operatorname{ann}_r(S)$ is a direct summand of R_R, so $\operatorname{ann}_r(S) = eR$ for some $e = e^2 \in R$.

Ex. 7.19A. For any $a \in R$, show that the following are equivalent:

(1) $a = ava$ for some $v \in \mathrm{U}(R)$. (Such $a \in R$ is called a *unit-regular* element of R.)
(2) $a = uf$ for some $u \in \mathrm{U}(R)$ and some $f = f^2 \in R$.
(3) $a = eu$ for some $u \in \mathrm{U}(R)$ and some $e = e^2 \in R$.
(4) $a = asa$ for some $s \in R$, and $R/aR \cong \operatorname{ann}_r(a)$ as right R-modules,
(5) aR is a direct summand of R_R, and $R/aR \cong \operatorname{ann}_r(a)$ as right R-modules.

Solution. According to von Neumann, the first condition in (4) is equivalent to the first condition in (5). (The proof is easy.) Therefore, (4) $\Leftrightarrow$ (5).

(1) $\Rightarrow$ (2) Say $a = ava$ as in (1). Clearly $f = va$ is an idempotent. Then $a = uf$ for $u = v^{-1} \in \mathrm{U}(R)$.

(2) $\Rightarrow$ (3). Let $a = uf$ be as in (2). Then $e = ufu^{-1}$ is an idempotent, and we have $a = (ufu^{-1})u = eu$.

(3) $\Rightarrow$ (5). Let $a = eu$ as in (3). Then $aR = euR = eR$ is a direct summand of R_R. Moreover,

$$\begin{aligned} \operatorname{ann}_r(a) &= \{x \in R : \ 0 = ax = eux\} \\ &= \{x \in R : \ ux \in (1-e)\,R\} \\ &= u^{-1}\,(1-e)\,R. \end{aligned}$$

Therefore, $\operatorname{ann}_r(a) \cong (1-e)R \cong R/eR = R/aR$ as right R-modules.

(5) $\Rightarrow$ (1). Assume the two conditions in (4). Let Q be a right ideal such that $aR \oplus Q = R$. Since aR is projective, the exact sequence

$$0 \longrightarrow \operatorname{ann}_r(a) \longrightarrow R \xrightarrow{\varphi} aR \longrightarrow 0 \qquad (\varphi(x) = ax)$$

splits, say by a monomorphism $\psi : \ aR \to R$. Then $R = \psi(aR) \oplus \operatorname{ann}_r(a)$. By assumption, $\operatorname{ann}_r(a) \cong R/aR \cong Q$, so there exists an R-isomorphism $\psi' : \ Q \to \operatorname{ann}_r(a)$. Now (ψ, ψ') is an isomorphism from $R = aR \oplus Q$ to $R = \psi(aR) \oplus \operatorname{ann}_r(a)$, so it is given by left multiplication by some $v \in \mathrm{U}\,(R)$. Therefore,

$$a = \varphi\psi(a) = \varphi(va) = ava,$$

as desired.

Comment. This exercise is basically the same as *ECRT*-Exer. 4.14B and 4.14C, except that it is stated here for a *fixed* element a in any ring R.

More connections to von Neumann regular rings will be given in the next exercise and the ensuing *Comment.*

Ex. 7.19B. Refer to condition (5) in the list of equivalent conditions in the last exercise. Show that:

(A) If R is von Neumann regular, we can drop the first condition in (5).
(B) If R is commutative, we can drop the second condition in (5).
(C) If R is commutative and von Neumann regular, every $a \in R$ is unit-regular.
(D) In general, the two conditions in (5) are independent.

Solution. (A) is obvious in view of what we said at the beginning of the solution to the last exercise.

(B) Assume R is commutative, and that aR is a direct summand of R_R. Then we can write $aR = eR$ where $e = e^2$. By commutativity, we have $\text{ann}_r(a) = \text{ann}_r(e)$, so

$$R/aR = R/eR \cong (1-e)\, R = \text{ann}_r(e) = \text{ann}_r(a).$$

(C) follows immediately from (A) and (B).

(D) First, consider any ring R that is not Dedekind-finite, so that there exist $a, b \in R$ with $ab = 1 \neq ba$. Then $aR = R$ is certainly a direct summand of R_R. However,

$$a(1-ba) = a - (ab)a = 0 \implies \text{ann}_r(a) \neq 0.$$

Thus, $\text{ann}_r(a) \not\cong R/aR = 0$. Next, consider the ring $R = \mathbb{Z}/4\mathbb{Z}$ and take $a = \overline{2}$. We have $R/aR \cong \mathbb{Z}/2\mathbb{Z} \cong \text{ann}_r(a)$, but aR is certainly not a direct summand of R_R. (In the same spirit, we could have also taken $a = \bar{x}$ in the ring $R = \mathbb{Q}[x]\,/\,(x^2)$.)

Comment. If every element in a ring R is unit-regular, R is said to be a *unit-regular ring.* These rings form a particularly important class of von Neumann regular rings.

If R is a commutative ring, clearly the set of all unit-regular elements in R is closed under multiplication. Remarkably, the same result is true for any von Neumann regular ring, according to a result of J. Hannah and K. O'Meara, "Products of idempotents in regular rings II", J. Algebra **123** (1989), 223–239.

Consider a product of two unit-regular elements, say $x = (e_1u_1)(e_2u_2)$ where the u_i's are units and the e_i's are idempotents. As in the proof of (2) $\Rightarrow$ (3) in the last exercise, $x = e_1e_2'u_1u_2$ where $e_2' = u_1e_2u_1^{-1}$ is an idempotent. Therefore, to say that the unit-regular elements in R are closed under multiplication is equivalent to saying that the product of any two idempotents in R is unit-regular. This is, of course, not true in general. For instance, for the algebra $R = \begin{pmatrix} k & k \\ 0 & k \end{pmatrix}$ over a field k, the two idempotents

$e = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$ and $f = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ have product $a := ef = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. This element is not unit-regular in R since $aRa = 0$ does not contain a. In fact, we have a "double jeopardy" for condition (5) in Ex. 7.19A: $aR = \begin{pmatrix} 0 & k \\ 0 & 0 \end{pmatrix}$ is not a direct summand of R_R, and R/aR ($\cong k \times k$) is also not isomorphic to the indecomposable module $\text{ann}_r(a) = \begin{pmatrix} k & k \\ 0 & 0 \end{pmatrix}$.

Ex. 7.20. Let R be a ring in which all idempotents are central, and let $a \in R$. Show that aR is projective iff $a = be$ where $e = e^2$ and $\text{ann}_r(b) = 0$.

Solution. For sufficiency, assume $a = be$ where $e = e^2$ and $\text{ann}_r(b) = 0$. Then $aR = beR \cong eR$ is projective. Conversely, assume $a \in R$ is such that aR is projective. Then the short exact sequence

$$0 \longrightarrow \text{ann}_r(a) \longrightarrow R \xrightarrow{a\cdot} aR \longrightarrow 0$$

splits, so $\text{ann}_r(a) = fR$ for some $f = f^2 \in R$. Let $e = 1 - f$ and $b = a + f$. Then $a(1 - e) = 0$ yields $a = ae = be$. Finally, for $x \in \text{ann}_r(b)$,

$$ax = bex = bxe = 0$$

(since e is central), so $x \in fR$, whence

$$x = fx = (b - a)x = -ax = 0.$$

Thus, $\text{ann}_r(b) = 0$, as desired.

Comment. In the ring theory literature, rings whose idempotents are all central are called *abelian rings.* It will be convenient for us to use this terminology in the following (especially in the next two exercises).

Ex. 7.21. (Endo) Show than an abelian ring R is right Rickart iff it is left Rickart.

Solution. We need only prove the "only if" part, so assume R is right Rickart. For any $a \in R$, we want to show that Ra is projective. We do know that aR is projective, so by Exercise 20, we can write $a = be$ where $e = e^2$ and $\text{ann}_r(b) = 0$. Since $be = eb$, it suffices (by the left analogue of Exercise 20) to show that $\text{ann}_\ell(b) = 0$. Say $a'b = 0$, and write $a' = b'e'$ where $e' = e'^2$ and $\text{ann}_r(b') = 0$. Then $b'e'b = 0$ implies that $0 = e'b = be'$. Hence $e' = 0$ and $a' = b'e' = 0$, as desired. In summary, note that, in the above setting, we have

$$\text{ann}_r(a) = fR = Rf = \text{ann}_\ell(a) \qquad (\forall\, a \in R).$$

Comment. As is pointed out in the text of *LMR*, the exercise above is a "non $*$" version of the fact that the notion of Rickart *-rings is left-right symmetric. This result of S. Endo appeared in his paper "Note on p.p. rings. (A supplement to Hattori's paper)," Nagoya Math. J. **17** (1960), 167–170. For much more information on Rickart *-rings and Baer *-rings, see S. Berberian's monograph "Baer *-Rings," Grundlehren der Math. Wiss., Vol. 195, Springer-Verlag, 1972.

Ex. 7.22. (a) Show that a reduced ring is right Rickart iff it is left Rickart.
(b) Name a Rickart ring that is not reduced.

Solution. (a) According to *FC*-Exercise (12.7), any reduced ring, R is abelian. Therefore, the conclusion of the previous exercise applies to R.

(b) Let $R = \text{End}_k(V)$ where V_k is an infinite-dimensional vector space over a division ring k. Then R is von Neumann regular, and hence Rickart. However, R is clearly not reduced. The ring R here is not Dedekind-finite. For a Dedekind-finite example, take $R = \mathbb{M}_n(\mathbb{Z})$ for any n. Since $\mathbb{Z}$ is a hereditary ring, *LMR*-(7.63) implies that R is Rickart. Again, R is not reduced, if $n \geq 2$.

Ex. 7.23. Let R be a ring with exactly two idempotents 0 and 1. Show that R is right Rickart iff R is Baer, iff R is a domain.

Solution. If R is a domain, then R is Baer and hence (left and right) Rickart. Conversely, assume R is right Rickart. Let $a \neq 0$ in R. Then $\text{ann}_r(a) = eR$ for some $e = e^2 \in R$. Since $e \neq 1$, we must have $e = 0$ so $\text{ann}_r(a) = 0$. This implies that R is a domain.

Comment. The first "iff" statement above is only a special case of a much more general result due to L. Small. According to this result, if R is a ring that has no infinite orthogonal set of nonzero idempotents, then R is right Rickart iff R is Baer: see *LMR*-(7.55).

Ex. 7.24. (a) Show that a commutative Rickart ring R is always reduced.
(b) Name a commutative reduced ring that is not Rickart.

Solution. (a) Suppose $a^n = 0$. Since aR is a projective R-module, Exercise 20 above implies that we can write $a = be$ where $e = e^2$ and $\text{ann}(b) = 0$. From $0 = a^n = b^n e^n = b^n e$, we get $e = 0$ and hence $a = 0$, as desired.

(b) Take a commutative reduced ring R which has exactly two idempotents 0 and 1, but which is not an integral domain. By the previous exercise, R is not Rickart. More explicitly, take $R = \mathbb{Q}[a, a']$ with the relation $aa' = 0$. Using unique factorization in the polynomial ring $\mathbb{Q}[x, x']$, we verify easily that R is reduced, with 0 and 1 as its only idempotents. Since $aa' = 0$ and $a, a' \neq 0$, R is not an integral domain. By what we said above, R is not Rickart. Here, the principal ideal aR is not projective, and $\text{ann}(a) = a'R$ is not generated by an idempotent.

Ex. 7.25. For any domain k, and a fixed integer $n > 1$, let T be the ring of upper triangular $n \times n$ matrices over k. Show that T is a Baer ring iff T is a right Rickart ring, iff k is a division ring.

Solution. Assume first that k is a division ring. By *LMR*-(2.36), T is a hereditary ring. In particular, T is a right Rickart ring. Viewing T as a right k-vector space, we have $\dim_k T = n(n+1)/2 < \infty$. From this, it is easy to see that T has no infinite orthogonal set of nonzero idempotents. By *LMR*-(7.55), any right Rickart ring T with such a property is a Baer ring.

Conversely, assume that T is a right Rickart ring. To prove that k is a division ring, it suffices to show that every nonzero $a \in k$ has a right inverse. For ease of notation, let us write out the proof for $n = 3$. By assumption, the right annihilator $\mathfrak{A}$ of $s = \begin{pmatrix} a & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ in T has the form eT where $e = e^2 \in T$. By direct computation, we see that

$$\mathfrak{A} = \operatorname{ann}_r(s) = \left\{ \begin{pmatrix} 0 & 0 & x \\ 0 & y & z \\ 0 & 0 & -ax \end{pmatrix} : x, y, z \in k \right\}.$$

In particular, e has the form $\begin{pmatrix} 0 & 0 & x_0 \\ 0 & y_0 & z_0 \\ 0 & 0 & -ax_0 \end{pmatrix}$. Comparing the (1, 3)-entries of $e = e^2$, we get an equation $x_0 = -ax_0^2$. We must have $x_0 \neq 0$, for otherwise all matrices in eT have first row zero and eT cannot be equal to $\mathfrak{A}$. By cancellation in k, $x_0 = -ax_0^2$ leads to $a\,(-x_0) = 1$, as desired.

Ex. 7.26. Let R be a Baer ring. Show that the annihilator of a central subset $S \subseteq R$ is generated by a central idempotent. State and prove the analogue of this for a Rickart ring.

Solution. Let e, $f \in R$ be idempotents such that $\operatorname{ann}(S) = eR = Rf$. Then $f = ea$ and $e = a'f$ for some $a, a' \in R$. We have

$$ef = e(ea) = ea = f \quad \text{and} \quad ef = (a'f)f = a'f = e,$$

so $e = f$. Now for any $b \in R$, $eb \in \operatorname{ann}(S) = Rf = Re$; hence $eb = ebe$. Similarly, we can argue that $be = ebe$. Therefore, $eb = be$ $(\forall\, b \in R)$, so e is a central idempotent, as desired.

If R is a Rickart ring (instead of a Baer ring), the above argument works verbatim for a *singleton* set $S = \{s\}$ in the center of R. Therefore, the annihilator of any central element s is generated by a central idempotent in R.

Ex. 7.27. Show that the center of a Baer (resp. Rickart) ring is also a Baer (resp. Rickart) ring.

Solution. Let C be the center of a Baer ring R. For any subset $S \subseteq C$, the last exercise guarantees the existence of an idempotent $e \in C$ such that $\operatorname{ann}^R(S) = eR$. We are done if we can show that $\operatorname{ann}^C(S) = eC$. The inclusion "$\supseteq$" is clear. For "$\subseteq$", let $a \in \operatorname{ann}^C(S)$. Then $a = eb$ for some $b \in R$, and so $a = e(eb) = ea \in eC$, as desired.

The same proof works for a Rickart ring R if in the above we argue only with singleton subsets $S \subseteq C$.

Ex. 7.28. For any Baer ring R, let L be the poset (with respect to inclusion) of principal right ideals of the form eR where $e = e^2$. Show that L is a complete lattice (see Exercise 6.25), anti-isomorphic to the complete lattice L' of principal left ideals of the form Re' where $e' = e'^2$.

Solution. Note that, for any idempotent $e \in R$, $eR = \mathrm{ann}_r\,(R\,(1-e))$, and by definition of a Baer ring, any right annihilator ideal in R has the form eR for some idempotent e. Therefore, the poset L in question is just the poset of right annililator ideals (under inclusion). This poset was already shown to be a complete lattice in Exercise 6.25. In that exercise, it was also shown that L is anti-isomorphic to the lattice of left annihilator ideals, which is just the lattice L' in this exercise. The lattice anti-isomorphism $L \to L'$ is given explicitly by $eR \mapsto R\,(1-e)$ for any idempotent $e \in R$.

Ex. 7.29. Let $(R, {}^*)$ be a Rickart *-ring.

(1) For any $x \in R$, show that $\mathrm{ann}_r(x) \cap x^*R = 0$.
(2) Deduce from (1) that, for any $x \in R$, $xx^* = 0 \Longrightarrow x = 0$.

Solution. (1) By the definition of a Rickart *-ring, $\mathrm{ann}_r(x) = eR$ for some $e \in R$ such that $e^2 = e = e^*$. Let $y \in \mathrm{ann}_r(x) \cap x^*R$. Then $y = x^*s$ for some $s \in R$. We have $xe = 0$, so

$$y = ey = ex^*s = (xe)^*s = 0.$$

(2) If $xx^* = 0$, then $x^* \in \mathrm{ann}_r(x) \cap x^*R = 0$ by (1).

The next two exercises are intended for readers who are familiar with the notion of Boolean algebras. Briefly, a Boolean algebra is a distributive lattice with 0 and 1 in which every element has a complement.

Ex. 7.30. For any ring R, the set $B(R)$ of central idempotents is known to form a lattice under the following (binary) meet and join operations:

$$e \wedge f = ef, \qquad e \vee f = e + f - ef \qquad (e, f \in B(R)).$$

In fact, $B(R)$ is isomorphic to the lattice $B'(R)$ of ideal direct summands of R (by the map $e \mapsto eR$), where meet is given by intersection and join is given by sum. ($B(R)$ and $B'(R)$ are both Boolean algebras.) For any Baer ring R, show that the lattices $B(R)$ and $B'(R)$ are complete.

Solution. Let C be the center of R. By Exercise 27, C is also a Baer ring. Since $B(R) = B(C)$, we may assume now that R is *commutative*. In this case, $B'(R)$ is just the lattice L in Exercise 28, and in that exercise, we have already shown that L is complete.

It is also possible to give a proof in a less devious way. To show the completeness of $B'(R)$ (for any Baer ring R), it suffices to show that the ideals in $B'(R)$ are closed under intersection. (If so, arbitrary meets exist in $B'(R)$, and this implies the existence of arbitrary joins.) Consider any family $\{e_iR\}$ in $B'(R)$. Since $e_iR = \text{ann}(1-e_i)$ for all i, we have

$$\bigcap_i e_iR = \bigcap_i \text{ann}(1-e_i) = \text{ann}(S),$$

where $S = \{1-e_i\} \subseteq C$. By Exercise 26, $\text{ann}(S) = eR$ for some central idempotent e, as desired.

Comment. To form the join of the family $\{e_iR\}$ in $B'(R)$, we have to take

$$\text{ann}\Big(\text{ann}\Big(\sum_i e_iR\Big)\Big) = \text{ann}\Big(\bigcap_i \text{ann}(e_iR)\Big) = \text{ann}\Big(\bigcap_i (1-e_i)R\Big).$$

Thus, if $f \in B(R)$ is such that $\bigcap_i (1-e_i)R = fR$, then the join of $\{e_iR\}$ is eR where $e := 1-f$. All this is saying is that, in the Boolean algebra $B(R)$, we get the join of $\{e_i\}$ by applying De Morgan's Law:

$$\bigvee_i e_i = 1 - \bigwedge_i (1-e_i).$$

The thing to note, however, is that $\sum e_iR$ itself may not be in $B'(R)$. For instance, if $R = k \times k \times \cdots$ where k is any field, then B is a commutative Baer ring. If e_i denotes the "i^{th} unit vector" in R $(i \geq 1)$, then $\sum e_iR = k \oplus k \oplus \cdots$, which has a zero annihilator. Thus, the join of $\{e_iR\}$ in $B'(R)$ is $\text{ann}\left(\text{ann}(\sum e_iR)\right) = R$. In other words, $\bigvee_i e_i = 1$ in $B(R)$.

Ex. 7.31. For any commutative ring R and any ideal $\mathfrak{A} \subseteq R$, recall that $V(\mathfrak{A})$ denotes the Zariski closed set

$$\{\mathfrak{p} \in \text{Spec}\, R : \ \mathfrak{p} \supseteq \mathfrak{A}\} \subseteq \text{Spec}\, R.$$

Let $B\,(\text{Spec}\, R)$ be the Boolean algebra of clopen (closed and open) sets in $\text{Spec}\, R$. Show that $\varphi : \ B(R) \to B\,(\text{Spec}\, R)$ defined by $\varphi(e) = V(eR)$ is an anti-isomorphism of Boolean algebras, and $\tilde{\varphi} : \ B(R) \to B\,(\text{Spec}\, R)$ defined by $\tilde{\varphi}(e) = V((1-e)R)$ is an isomorphism of Boolean algebras.

Solution. By Exercise 2.22, every clopen set in $\text{Spec}\, R$ has the form $V(eR)$ for some $e = e^2 \in R$, so φ is *onto*. To show that φ is *one-one*, let e, e' be idempotents in R such that $V\,(eR) = V\,(e'R)$. Forming the intersection of all the prime ideals in this clopen set, we see that eR and $e'R$ have the same radical ideal. Thus, $e^m \in e'R$ and $e'^n \in eR$, for some $m, n \in \mathbb{N}$. This means that $e \in e'R$ and $e' \in eR$. Hence, $e'e = e'$ and $ee' = e$, so $e = e'$.

It remains to show that φ is a lattice anti-isomorphism. For $e, f \in B(R)$, we have

$$\begin{aligned}\varphi(e \vee f) &= V\,((e+f-ef)R) \\ &= V\,(eR + fR) \\ &= V(eR) \cap V(fR) \\ &= \varphi(e) \wedge \varphi(f),\end{aligned}$$

$$\begin{aligned}\varphi(e \wedge f) &= V(efR)\\ &= V(eR) \cup V(fR)\\ &= \varphi(e) \vee \varphi(f),\end{aligned}$$

as desired. Composing φ with the complement map on $B\,(\operatorname{Spec} R)$ (which is also an anti-automorphism of $B(\operatorname{Spec} R)$, by De Morgan's Laws), we see that $\tilde{\varphi}: B(R) \to B(\operatorname{Spec} R)$ defined by

$$\tilde{\varphi}(e) = (\operatorname{Spec} R) \setminus V(eR) = V\left((1-e)R\right)$$

is a Boolean algebra isomorphism.

Ex. 7.32. (Johnson-Wong) For any nonsingular module M_R, show that

(1) there is a canonical embedding ε of the ring $\operatorname{End}_R(M)$ into the ring $\operatorname{End}_R(E(M))$; and
(2) M is QI (quasi-injective) iff ε is an isomorphism.

Solution. (1) Let $f \in \operatorname{End}_R(M)$. Since $E(M)$ is an injective module, there exists an endomorphism $g \in \operatorname{End}_R(E(M))$ which restricts to f. We claim that such an endomorphism g is unique. In fact, suppose $g' \in \operatorname{End}_R(E(M))$ also restricts to f. Then $(g-g')\,(M) = 0$, so $g-g'$ induces a homomorphism from $E(M)/M$ to $E(M)$. Since M is nonsingular, $E(M)$ is also nonsingular (by *LMR*-(7.6)(2)). On the other hand, $M \subseteq_e E(M)$ implies that $E(M)/M$ is singular (by *LMR*-(7.6)(3)). Therefore, $\operatorname{im}(g-g') = 0$, which means that $g = g'$. The map $f \mapsto g$ clearly gives the desired ring embedding $\varepsilon: \operatorname{End}_R(M) \to \operatorname{End}_R(E(M))$.

(2) Assume that ε is onto. Then every $g \in \operatorname{End}_R(E(M))$ has the form $\varepsilon\,(f)$ for some $f \in \operatorname{End}_R(M)$, and so $g\,(M) = f\,(M) \subseteq M$. This means that M is fully invariant in $E(M)$. Therefore, by *LMR*-(6.74), M is QI. Conversely, if M is QI, then it is fully invariant in $E(M)$. For any $g \in \operatorname{End}_R(E(M))$, we have clearly $g = \varepsilon\,(g|_M)$, so ε is onto.

Comment. If M is not assumed to be nonsingular, we may not be able to embed $\operatorname{End}_R(M)$ into $\operatorname{End}_R(E(M))$ by any ring homomorphism. (The trouble is that $f \in \operatorname{End}_R(M)$ may be the restriction of two different endomorphisms of $E(M)$.) For instance, for a prime p over $R = \mathbb{Z}$, the (singular) module $M = \mathbb{Z}_p$ has injective hull $E(M)$ given by the Prüfer p-group C_{p^∞}. Here, $f = 0$ extends to both $g = 0$ and $g' =$ multiplication by p, so ε cannot be defined. In fact, $\operatorname{End}_R(M)$ is the ring $\mathbb{Z}_p$, and $\operatorname{End}_R(E(M))$ is the ring of p-adic integers; clearly, no embedding is possible from the former to the latter.

The exercise above is taken from the paper of R. E. Johnson and E. T. Wong, "Quasi-injective modules and irreducible rings," J. London Math. Soc. **36** (1961), 260–268. The next exercise is from the same source.

Ex. 7.33. (Johnson-Wong) Let M_R be a nonsingular uniform module, with $E = \text{End}_R(M)$. (1) Show that any nonzero $f \in E$ is injective, and deduce that E is a domain. (2) If M is also QI, show that E is a division ring.

Solution. (1) Assume, instead, that $K := \ker(f) \neq 0$. Then $K \subseteq_e M$. For any $y \in M$, $y^{-1}K \subseteq_e R_R$ by Exercise (3.7). Now

$$f(y) \cdot y^{-1}K \subseteq f\left(y \cdot y^{-1}K\right) \subseteq f(K) = 0,$$

so $f(y) \in \mathcal{Z}(M) = 0$, that is, $f = 0 \in E$. In particular, if $f \neq 0 \neq g$, then f, g are injective, and so is fg, which implies that $fg \neq 0$.

(2) Let $f \neq 0$ in E. Since f is injective, $f(m) \mapsto m$ defines an R-homomorphism from $f(M)$ to M. If M is assumed to be QI, this homomorphism is the restriction of some $h \in E$. Now $h(f(m)) = m$ $(\forall m \in M)$ implies that $hf = 1 \in E$. From this, we conclude that E is a division ring.

Ex. 7.34. Let R be a subring of a ring T such that $R_R \subseteq_e T_R$. Show that $\mathcal{Z}(R_R) \subseteq \mathcal{Z}(T_T)$. In particular, if T is right nonsingular, so is R.

Solution. Let $x \in \mathcal{Z}(R_R)$, so that $\text{ann}_r^R(x) \subseteq_e R_R$. If $\mathfrak{A}$ is any nonzero right ideal of T, then $\mathfrak{A} \cap R$ is a nonzero right ideal of R since $R_R \subseteq T_R$. Therefore $\mathfrak{A} \cap R \cap \text{ann}_r^R(x) \neq 0$. In particular, $\mathfrak{A} \cap \text{ann}_r^T(x) \neq 0$, so $x \in \mathcal{Z}(T_T)$, as desired.

Comment. For more conclusions concerning rings $R \subseteq T$ such that $R_R \subseteq_e T_R$, see Exercise 8.10.

Ex. 7.35. (R. Shock) Let $S = R[X]$ where X is a set of commuting indeterminates over the ring R. Show that $\mathcal{Z}(S_S) = \mathcal{Z}(R_R)[X]$. (In particular, R is right nonsingular iff S is.)

Solution. Let $I = \mathcal{Z}(R_R)$. For $a \in I$, let $\mathfrak{A} = \text{ann}_r^R(a) \subseteq_e R_R$. Then $\mathfrak{A}[X] \subseteq_e S_S$ by Exercise 3.30. Since $\mathfrak{A}[X] \subseteq \text{ann}_r^S(a)$, it follows that $\text{ann}_r^S(a) \subseteq_e S_S$, and hence $I[X] \subseteq \mathcal{Z}(S_S)$. Assuming that this is not an equality, we will seek a contradiction. Let $X = \{x_j : j \in J\}$, and let us fix a total ordering on J. With respect to this ordering, we can define a lexicographic ordering "$>$" on the set of (finite) monomials in the variables $\{x_j : j \in J\}$. Choose a polynomial

$$f(X) = b_1\beta_1 + \cdots + b_n\beta_n \in \mathcal{Z}(S_S) \backslash I[X],$$

where $b_i \in R \backslash I$, and the β_i's are monomials such that $\beta_1 > \cdots > \beta_n$. Since $b_1 \notin \mathcal{Z}(R_R)$, $\text{ann}_r^R(b_1) \cap dR = 0$ for some $d \neq 0$ in R. *We claim that*

$$(*) \qquad \text{ann}_r^S(f) \cap dS = 0.$$

Once this is established, we'll get $f \notin \mathcal{Z}(S_S)$, a contradiction. To prove $(*)$, assume instead that there exists a nonzero

$$d(c_1\gamma_1 + \cdots + c_m\gamma_m) \in \text{ann}_r^S(f) \qquad (c_i \in R, \ \gamma_i = \text{monomials}).$$

We may assume without loss of generality that $\gamma_1 > \cdots > \gamma_m$, and that $dc_1 \neq 0$. Then

$$0 = fd\,(c_1\gamma_1 + \cdots + c_m\gamma_m) = b_1 dc_1 \beta_1 \gamma_1 + \cdots,$$

where the suppressed terms on RHS involve only monomials $< \beta_1\gamma_1$. Therefore, we have $b_1 dc_1 = 0$, which is impossible since $\text{ann}_r^R(b_1) \cap dR = 0$ This proves $(*)$, and so we have $\mathcal{Z}(S_S) = I[X]$.

Comment. The result in this Exercise is from R. C. Shock's paper, "Polynomial rings over finite dimensional rings," Pacific J. Math. **42** (1972), 251–257.

Ex. 7.36. (Endo) Let R be a commutative ring with total ring of quotients $K = Q(R)$. Show that the following are equivalent:

(1) R is Rickart;
(2) K is von Neumann regular and $R_{\mathfrak{m}}$ is a domain for any maximal ideal $\mathfrak{m} \subset R$;
(3) K is von Neumann regular and every idempotent of K belongs to R.

Solution. (3) $\Rightarrow$ (1). Assuming (3), take any $a \in R$. By part (C) of Ex. 7.19B (applied to K), we can write $a = ue$ where $u \in \text{U}(K)$ and $e = e^2 \in K$. By assumption, we have $e \in R$. Since multiplication by u defines an R-isomorphism $eR \cong aR$, it follows that $(aR)_R$ is projective.

(1) $\Rightarrow$ (2). Assume R is Rickart. We see easily that the localization of R at any multiplicative set is also Rickart. In particular, for any prime ideal $\mathfrak{p} \subset R$, $R_{\mathfrak{p}}$ is Rickart, so by Ex. 23, $R_{\mathfrak{p}}$ is a domain. Next, K is also Rickart. If K is *not* von Neumann regular, then by *LMR*-(3.71), K_M is *not* a field for some maximal ideal $M \subset K$. But by the above, K_M is a domain. Let P be the kernel of the localization map $\varphi : K \to K_M$. Then, clearly, $P \subsetneq M$. Fix an element $a \in M \backslash P$. Since K is Rickart, $\text{ann}^K(a) = eK$ for some $e = e^2 \in K$. From $\varphi(ae) = 0 \neq \varphi(a)$, we see that $\varphi(e) = 0$, so $e \in P$. Let $f = 1 - e$ and identify K with the direct product ring $eK \times fK$. Since $a = a - ae = af \in fK$ and $\text{ann}^K(a) \cap fK = 0$, a is a non 0-divisor of the ring fK, from which it follows that $e+a$ is a non 0-divisor of $eK \times fK = K$. Therefore, $e + a \in \text{U}(K)$, and in particular, $a \in \text{U}(fK)$. But then $ac = f$ for some $c \in fK$. Now we have $f \in aK \subseteq M$ and $e \in P \subseteq M$, which gives the contradiction $1 = e + f \in M$.

(2) $\Rightarrow$ (3). This implication turns out to have nothing to do with K being von Neumann regular. Assuming only $R_{\mathfrak{m}}$ is a domain for every maximal ideal $\mathfrak{m} \subset R$, we shall prove that any $e = e^2 \in K$ belongs to R. Indeed, assume $e \notin R$. Then

$$I := \{r \in R :\ re \in R\} \neq R,$$

so $I \subseteq \mathfrak{m}$ for some maximal ideal $\mathfrak{m} \subset R$. Let $\mathfrak{p} = \ker(R \to R_{\mathfrak{m}})$. Since $R_{\mathfrak{m}}$ is a domain, $\mathfrak{p}$ is a prime ideal of R. On the other hand, $\mathfrak{p}$ consists of

0-divisors of R, so $\mathfrak{p}K$ is a prime ideal of K. Thus, we have either $e \in \mathfrak{p}K$ or $1-e \in \mathfrak{p}K$. If $e \in \mathfrak{p}K$, there exists $s \in R \backslash \mathfrak{m}$ such that $se = 0$. Then $s \in I \subseteq \mathfrak{m}$, a contradiction. If $1-e \in \mathfrak{p}K$, there exists $s' \in R\backslash\mathfrak{m}$ such that $s'(1-e) = 0$. But then $s'e = s' \in R$, so $s' \in I \subseteq \mathfrak{m}$, again a contradiction.

Comment. The result in this exercise comes from Proposition 1 of S. Endo's paper: "Note on p.p. rings", Nagoya Math. J. **17** (1960), 167–170. The same result is re-proved in sheaf-theoretic terms by G. Bergman in Lemma 3.1 of his paper: "Hereditary commutative rings and centers of hereditary rings", Proc. London Math. Soc. **23** (1971), 214–236.

The second halves of the conditions (2) and (3) above certainly *cannot* be omitted. A nice example is provided by the ring

$$R = \left\{(a,b) \in \mathbb{Z}^2 : \ a \equiv b \pmod 2\right\},$$

for which $K = Q(R) = \mathbb{Q} \times \mathbb{Q}$ is von Neumann regular. Here, the idempotents $(1,0)$ and $(0,1)$ do not belong to R, and the localization of R at the maximal ideal $2\mathbb{Z} \oplus 2\mathbb{Z}$ is not a domain. The ring R is *not* Rickart, since R is not a domain but has no nontrivial idempotents. On the other hand, the result in this exercise implies that any ring between $\mathbb{Z} \times \mathbb{Z}$ and $\mathbb{Q} \times \mathbb{Q}$ is Rickart.

Ex. 7.37. (Endo) With the same notations as in the last exercise, show that the following statements are equivalent:

(1) R is semihereditary;
(2) R is a flat ideal ring and K is von Neumann regular;
(3) Every torsionfree R-module is flat.

Solution. (3) $\Rightarrow$ (1). Recall that an R-module N is called torsionfree if every non 0-divisor of R acts injectively on N. Since every torsionless R-module is torsionfree (an easy fact), (3) $\Rightarrow$ (1) follows immediately from Chase's Theorem *LMR*-(4.67).

(1) $\Rightarrow$ (2). Let R be semihereditary. Then R is a flat ideal ring by *LMR*-(4.67), and a Rickart ring by *LMR*-(7.50). By the last exercise, K is von Neumann regular.

(2) $\Rightarrow$ (3). Assume (2) and let N be any torsionfree R-module. We have an injection $N \to K \otimes_R N$, which we may view as an inclusion. Since K is von Neumann regular, $K \otimes_R N$ is flat as a K-module by *LMR*-(4.21), and hence also flat as an R-module by Ex. 4.53. Using now the fact that every ideal of R is flat, it follows from *LMR*-(4.66) that the R-submodule N of $K \otimes_R N$ is also flat.

Comment. It is also possible to prove the equivalence of (1), (2), and (3) *without* appealing to Chase's Theorem, as follows. First prove (1) $\Rightarrow$ (2) and (2) $\Rightarrow$ (3) as above. Then we can deduce (2) $\Rightarrow$ (1) from Ex. 4.15B. Finally, to prove (3) $\Rightarrow$ (2), assume that every torsionfree R-module is flat. Then all ideals are flat and we have the first half of (2). To prove the second half, consider any K-module M. Clearly, as an R-module, M is torsionfree, and hence M_R is flat. But then Ex. 4.53 implies that M_K is flat. Since every K-module is flat, it follows from *LMR*-(4.21) that K is von Neumann regular, as desired.

The approach to the equivalence of (1), (2) and (3) given in the last paragraph is much closer to the original one used by S. Endo in his paper, "On semihereditary rings," J. Math. Soc. Japan **13** (1961), 109–119. In fact, Endo's results were obtained independently of those of Chase (in *LMR*-(4.67)).

Of course, in the case where R is a domain, the results in this exercise are just a part of the characterization theorem for Prüfer domains given in *LMR*-(4.69). The case of a general commutative ring is considerably harder, as we saw above.

Ex. 7.38. Show that a right self-injective ring R is right nonsingular iff it is von Neumann regular.

Solution. The "if" part is proved in *LMR*-(7.7) (for any ring R). For the "only if" part, assume R is both right self-injective and right nonsingular. For any given element $a \in R$, let $K = \text{ann}_r(a)$, and let M be a complement to K in R_R, so that $M \oplus K \subseteq_e R_R$. Since $M \cap \text{ann}_r(a) = 0$, left multiplication by a gives a right R-module isomorphism $\theta : M \to aM$. The composition

$$aM \xrightarrow{a^{-1}} M \hookrightarrow R$$

must be induced by the left multiplication of some element $b \in R$, since R_R is an injective module. Consider now the left multiplication by $a - aba$. This is zero on K, and also on M. Thus,

$$\text{ann}_r(a - aba) \supseteq M \oplus K.$$

Since R_R is a nonsingular module, this implies that $a - aba = 0$, which clinches the claim.

Comment. The solution (for the "only if" part) above is a more-or-less direct generalization of the argument proving that the endomorphism ring of a semisimple module N_R (over any ring R) is von Neumann regular; see *FC*-(4.27). Here, the module is R_R, and the semisimplicity assumption is replaced by the (quasi-)injectivity and the nonsingularity of R_R.

It would be difficult to give an accurate attribution for the result in this exercise, as many authors have proved various stronger forms of it. For instance, a rather similar argument is used to prove *LMR*-(13.1)(2). After studying the theory of maximal rings of quotients (*LMR*-Ch.5), the reader will see that, for any right nonsingular ring R, the maximal right ring of quotients Q is a von Neumann regular ring (*LMR*-(13.36)). In the case where R is also right self-injective, we have $Q = R$, so we retrieve (the "only if" part of) the present exercise.

§8. Dense Submodules and Rational Hulls

The notion of a dense submodule is a refinement of that of an essential submodule. We say that a submodule N of a module M_R is *dense* (written $N \subseteq_d M$) if, for any $x, y \in M$ with $x \neq 0$, there exists $r \in M$ such that $xr \neq 0$ and $yr \in N$. It is often convenient to express this condition in the form $x \cdot y^{-1}N \neq 0$, where, by $y^{-1}N$, we mean the right ideal $\{r \in R : \, yr \in N\}$. If $N \subseteq_d M$, we shall also say that M is a *rational extension* of N. Of course, $N \subseteq_d M \Rightarrow N \subseteq_e M$; the converse is not true in general. A useful characterization for $N \subseteq_d M$ is that

$$\mathrm{Hom}_R\,(M/N,\ E\,(M)) = 0$$

where $E(M)$ denotes the injective hull of M; see *LMR*-(8.6).

In the case where M is a *nonsingular* module, $N \subseteq_d M$ is synonymous with $N \subseteq_e M$ (*LMR*-(8.7)(3)). It follows easily that a ring R is right nonsingular iff every essential right ideal is dense in R_R (*LMR*-(8.9)).

In analogy with the existence of the injective hull, one proves that *any module M has a unique maximal rational extension*, denoted by $\tilde{E}\,(M)$. This is called the *rational hull* of M, and is constructed by first forming $I = E(M)$, $H = \mathrm{End}\,(I_R)$, and taking

$$\tilde{E}\,(M) = \{i \in I : \ \forall h \in H,\ h\,(M) = 0 \Longrightarrow h\,(i) = 0\}\,.$$

This is the largest rational extension of M inside I, as is shown in *LMR*-(8.11). In fact, any rational extension of M can be embedded into $\tilde{E}\,(M)$ over M, according to *LMR*-(8.13). In case M is a *nonsingular* module, one has always $\tilde{E}(M) = E(M)$, by *LMR*-(8.18)(5).

A module M_R is called *rationally complete* if $\tilde{E}(M) = M$, that is, if M has no proper rational extension. A characterization for rational completeness of M_R is that, for any right R-modules $A \subseteq B$ such that $\mathrm{Hom}_R\,(B/A,\ E\,(M)) = 0$, any R-homomorphism $A \to M$ can be extended to B: see *LMR*-(8.24). The first half of the exercises in this section deal with various properties of dense submodules, rationally complete modules, etc.

A byproduct of the treatment in *LMR*-§8 is the notion of a *right Kasch ring*: this is a ring R such that R_R contains an isomorphic copy of every simple right R-module. An equivalent condition is that the only dense right ideal in R is R itself, by *LMR*-(8.28). The right Kasch property is also related to the so-called "double annihilator condition". To be precise,

> *R is right Kasch iff* $\mathfrak{m} = \mathrm{ann}_r(\mathrm{ann}_\ell\, \mathfrak{m})$ *for every maximal right ideal* $\mathfrak{m}$ in R.

Later, we shall see that any QF (quasi-Frobenius) ring R satisfies this double annihilator condition for *any* right ideal $\mathfrak{m}$. In particular, QF *rings* *are right (and also left) Kasch rings.* In general, however, "right Kasch" and "left Kasch" are independent conditions, as is shown in *LMR*-(8.29)(6). Exercise 13 below offers an example of a *local* ring that is right Kasch but not left Kasch. The remaining exercises give further information and examples for 1-sided Kasch rings. For instance, if a 1-sided Kasch ring is semiprime, then it is in fact semisimple, according to Exercise 15.

Exercises for §8.

Ex. 8.1. Let M' be a submodule of M_R and $N \subseteq_d M$. For any $f \in \operatorname{Hom}_R(M', M)$, show that $f^{-1}(N) \subseteq_d M'$.

Solution. Let $N' = f^{-1}(N)$. To prove that $N' \subseteq_d M'$, it suffices to show that, for any R-module P' between N' and M', any homomorphism $g:\ P'/N' \to M'$ is zero (see *LMR*-(8.6)). Consider the surjection $P' \to (f(P') + N)/N$ induced by f. The kernel of this map is N'. Now consider

$$(f(P') + N)/N \cong P'/N' \xrightarrow{g} M' \subseteq M.$$

The composition must be zero, since $N \subseteq_d M$. But then g itself must be zero, as desired.

Comment. The assumption that $M' \subseteq M$ is essential for the proof given above. In fact, the conclusion in the exercise may not hold if M' is not assumed to be in M. To construct an example, let $M = \mathbb{Z}$ and $N = 2\mathbb{Z}$ over the ring $R = \mathbb{Z}$. Of course we have $N \subseteq_d M$. Now let $M' = \mathbb{Z} \oplus C$, where C is cyclic of order 2, generated by an element x. Consider the homomorphism $f:\ M' \to M$ that is the identity on $\mathbb{Z}$, and zero on C. Then $N' = f^{-1}(N) = 2\mathbb{Z} \oplus C$. Let $y = 1 \in \mathbb{Z} \subseteq M'$. For any $r \in R$, we have

$$yr \in N' \implies r \text{ is even} \implies xr = 0 \in M'.$$

Therefore, N' is not dense in M'. (Of course, we do have $N' \subseteq_e M'$, by Exercise 3.7.)

Ex. 8.2. Let M be an R-module containing the right regular module R_R. Show that $R_R \subseteq_d M$ iff $R_R \subseteq_e M$ and for every $y \in M$, $y^{-1}R \subseteq_d R_R$.

Solution. First assume $R_R \subseteq_d M$. Then of course $R_R \subseteq_e M$. For any $y \in M$, consider the R-homomorphism $f:\ R \to M$ given by $f(r) = yr$. Applying Exercise 1 to $R_R \subseteq_d M$, we get

$$f^{-1}(R) = y^{-1}R \subseteq_d R_R.$$

Conversely, assume that $R_R \subseteq_e M$ and that for every $y \in M$, $y^{-1}R \subseteq_d R_R$. To show that $R_R \subseteq_d M$, consider any two elements $x,\ y \in M$, with $x \neq 0$. Since $R_R \subseteq_e M$, $xr \in R \backslash \{0\}$ for some $r \in R$. Using $y^{-1}R \subseteq_d R_R$, we can therefore find $s \in R$ such that $rs \in y^{-1}R$ and $(xr)s \neq 0$. We have now $y(rs) \in R$ and $x(rs) \neq 0$, which shows that $R_R \subseteq_d M$.

Ex. 8.3. Let $C \subsetneq N \subseteq M$ be R-modules.

(a) Does $N \subseteq_e M$ imply $N/C \subseteq_e M/C$?
(b) Does $N \subseteq_d M$ imply $N/C \subseteq_d M/C$?

Solution. The answers to both questions are "no". Counterexamples can be found already over the ring $R = \mathbb{Z}$. Take

$$M = \mathbb{Z} \oplus \mathbb{Z}, \qquad N = \mathbb{Z} \oplus 2\mathbb{Z} \qquad \text{and} \qquad C = 2\mathbb{Z} \oplus 2\mathbb{Z} \subsetneq N.$$

It is easy to see that $N \subseteq_d M$, but $N/C = (\mathbb{Z}/2\mathbb{Z}) \oplus 0$ is a proper direct summand in $M/C = (\mathbb{Z}/2\mathbb{Z}) \oplus (\mathbb{Z}/2\mathbb{Z})$, so N/C is not essential (and hence also not dense) in M/C.

Counterexamples can be found just within the subcategory of torsion abelian groups also. For instance, for part (a), take $A = \mathbb{Z}/4\mathbb{Z}$, $B = \mathbb{Z}/2\mathbb{Z}$, and let

$$M = A \oplus B, \qquad N = 2A \oplus B \qquad \text{and} \qquad C = 2A \subsetneq N.$$

For any $x \in M \backslash N$, we have $2x \in N \backslash \{0\}$, so $N \subseteq_e M$. Again, N/C is a proper direct summand of M/C, so N/C is not essential in M/C.

Ex. 8.4. Let M_R be a nonsingular uniform R-module. Show that any nonzero submodule $N \subseteq M$ is dense in M.

Solution. Since M is uniform and $N \neq 0$, N is essential in M. By *LMR*-(8.7)(3), any essential submodule of a nonsingular module is dense, so we have $N \subseteq_d M$.

Ex. 8.5. Let $\mathfrak{A} \neq R$ be an ideal in a commutative ring R, and let $M_R = R/\mathfrak{A}$. Show that every nonzero submodule $N \subseteq M$ is dense in M iff $\mathfrak{A}$ is a prime ideal.

Solution. First assume $\mathfrak{A}$ is prime. Any nonzero submodule $N \subseteq M$ has the form $N = \mathfrak{B}/\mathfrak{A}$, where $\mathfrak{B}$ is an ideal properly containing $\mathfrak{A}$. Let us fix an element $b \in \mathfrak{B} \backslash \mathfrak{A}$. Consider any two elements x, $y \in R$, with $x \notin \mathfrak{A}$. Then $xb \notin \mathfrak{A}$ and $yb \in \mathfrak{B}$, so $\bar{x} \cdot b \neq 0$ in M and $\bar{y} \cdot b \in \mathfrak{B}/\mathfrak{A} = N$. This shows that $N \subseteq_d M$.

Conversely, assume $\mathfrak{A}$ is not prime. Then there are elements $x, x' \notin \mathfrak{A}$ with $x\,x' \in \mathfrak{A}$. Form the ideal

$$\mathfrak{B} = \{r \in R : \ xr \in \mathfrak{A}\},$$

which contains $\mathfrak{A}$ properly (since $x' \in \mathfrak{B}$). Then $N := \mathfrak{B}/\mathfrak{A}$ is a nonzero submodule of M. Consider the two elements $\bar{1} \in M$ and $\bar{x} \in M \backslash \{0\}$. If $r \in R$ is such that $\bar{1} \cdot r \in N$, then $r \in \mathfrak{B}$, and so $\overline{x}\, r = \overline{xr} = 0 \in M$. This shows that N is not dense in M, as desired.

Ex. 8.5A. Let M be a left module over a commutative ring R, and let $S \subseteq R$ be a multiplicative set.

(1) Show that $(\text{ann}(M))_S \subseteq \text{ann}(M_S)$, with equality if M is f.g.

(2) If $M_{\mathfrak{p}}$ is faithful (as an $R_{\mathfrak{p}}$-module) for every prime ideal $\mathfrak{p} \subset R$, show that M is faithful (as an R-module). If M is f.g., show that the converse also holds.

(3) For any ideal $I \subseteq R$, show that $I \subseteq_d R$ if $I_{\mathfrak{p}} \subseteq_d R_{\mathfrak{p}}$ for every prime ideal $\mathfrak{p} \subset R$, and conversely if I is f.g.

(4) Show that the "finitely generated" hypothesis in the conclusions (1), (2), (3) above cannot be eliminated.

Solution. (1) The inclusion in (1) is trivial (without any assumption on M). If $M = \sum_{i=1}^{n} Rm_i$, consider any $r/s \in \text{ann}(M_S)$, where $r \in R$ and $s \in S$. Then $rm_i = 0 \in M_S$ for all i, so $s'(rm_i) = 0 \in M$ for all i, where s' is a suitable element in S. But then $s'r \in \text{ann}(M)$, which implies that $r/s = s'r/s's \in (\text{ann}(M))_S$, as desired.

(2) If each $\text{ann}(M_{\mathfrak{p}}) = 0$, then $(\text{ann}(M))_{\mathfrak{p}} \subseteq \text{ann}(M_{\mathfrak{p}}) = 0$ for all prime ideals $\mathfrak{p}$, so $\text{ann}(M) = 0$. Now assume M is f.g. and faithful. Then for each prime ideal $\mathfrak{p}$, (1) implies $\text{ann}(M_{\mathfrak{p}}) = (\text{ann}(M))_{\mathfrak{p}} = 0$, so $M_{\mathfrak{p}}$ is faithful, as desired.

(3) Follows from (2), since an ideal is dense in a (commutative) ring iff it is faithful as a module over that ring (by *LMR*-(8.3)(4)).

(4) Let R be the Boolean ring $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \cdots$ (or just $k \times k \times \cdots$ for any field k). Then the ideal

$$A = \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \cdots \subseteq R$$

is dense (by *LMR*-(8.29)(4)), but not f.g. Take any maximal ideal $\mathfrak{m} \subset R$ that contains A. Since R is a von Neumann regular ring, the localization $R_{\mathfrak{m}}$ is a field. Thus, $\mathfrak{m}$ localizes to zero, and hence $A_{\mathfrak{m}} = 0$ too, which is not dense in $R_{\mathfrak{m}}$, and is not a faithful $R_{\mathfrak{m}}$-module. (On the other hand, for any prime $\mathfrak{p}$ *not* containing A, $A_{\mathfrak{p}} = R_{\mathfrak{p}}$ is both dense and faithful!) This example shows that the use of the f.g. assumption in (1), (2), (3) was essential.

Comment. The first part of (3) above can be generalized to a statement on the density of submodules also; see part (2) of the next exercise.

Ex. 8.5B. Let $N \subseteq M$ be left R-modules where R is a commutative ring.

(1) If $N_{\mathfrak{p}} \subseteq_e M_{\mathfrak{p}}$ for all prime ideals $\mathfrak{p} \subset R$, show that $N \subseteq_e M$.

(2) If $N_{\mathfrak{p}} \subseteq_d M_{\mathfrak{p}}$ for all prime ideals $\mathfrak{p} \subset R$, show that $N \subseteq_d M$.

Solution. (2) Let $x, y \in M$, where $x \neq 0$. Take a prime ideal $\mathfrak{p} \subset R$ such that $x/1 \neq 0$ in $M_{\mathfrak{p}}$. Since $N_{\mathfrak{p}} \subseteq_d M_{\mathfrak{p}}$, there exists $r \in R$ and $s \in R \setminus \mathfrak{p}$ such that $(r/s)x \neq 0 \in M_{\mathfrak{p}}$ and $(r/s)y = z/s'$ for some $z \in N$ and $s' \in R \setminus \mathfrak{p}$. Then $s'ry = sz \in M_{\mathfrak{p}}$, so $(t\,s'r)y = t\,sz \in N$ for some $t \in R \setminus \mathfrak{p}$. We must have $(t\,s'r)x \neq 0 \in M$, for otherwise $rx = 0 \in M_{\mathfrak{p}}$, which is not the case. This proves that $N \subseteq_d M$.

(1) follows from the same argument, by setting $x = y$.

Ex. 8.6. Let k be a field and $R = k\langle\{x_i : i \geq 1\}\rangle$, with relations $x_i x_j = 0$ for all unequal i, j. Let $p_i(x_i) \in k[x_i] \backslash \{0\}$. Show that the ideal $\mathfrak{A}$ generated by $\{p_i(x_i) : i \geq 1\}$ is dense in R.

Solution. Clearly R is commutative, and it is not difficult to show that R can be represented as a direct sum

$$(*) \qquad R = k \oplus x_1 k[x_1] \oplus x_2 k_1[x_2] \oplus \cdots .$$

From this, we see that R is a reduced ring, so R is nonsingular. Therefore, it is sufficient to show that $\mathfrak{A} \subseteq_e R$. (By *LMR*-(8.7)(3), $\mathfrak{A} \subseteq_e R$ implies that $\mathfrak{A} \subseteq_d R$.) Let $0 \neq r \in R$, say

$$r = a + x_1 f_1(x_1) + \cdots + x_n f_n(x_n), \quad a \in k.$$

After a reindexing, we may assume that $a + x_1 f_1(x_1) \neq 0$ in $k[x_1]$. Now

$$r x_1 p_1(x_1) = (a + x_1 f_1(x_1))\, x_1 p_1(x_1) \neq 0 \quad \text{in } k[x_1],$$

and this element lies in $p_1(x_1) R \subseteq \mathfrak{A}$. This shows that $\mathfrak{A} \subseteq_e R$.

Comment. It is easy to bypass the use of *LMR*-(8.7)(3). By *LMR*-(8.3)(4), to say that $\mathfrak{A}$ is dense amounts to $\operatorname{ann}(\mathfrak{A}) = 0$. If $r = a + \sum_j x_j f_j(x_j) \in \operatorname{ann}(\mathfrak{A})$, multiplying r with $p_i(x_i)$ shows that $a = 0$ and $f_i(x_i) = 0$ (for any $i \geq 1$), so $r = 0$, as desired.

Ex. 8.7. Show that a ring R is semisimple iff R is right nonsingular and every right ideal (resp. right R-module) is rationally complete.

Solution. First assume R is semisimple. If M, N are right R-modules such that $N \subseteq_d M$, then $N = M$ (since N is a direct summand of M). Therefore, every N_R is rationally complete. For any $a \in \mathcal{Z}(R_R)$, $\operatorname{ann}_r(a)$ is, by definition, right essential in R_R. Since $\operatorname{ann}_r(a)$ is also a direct summand in R_R, it must be the whole ring R, and so $a = 0$. This shows that R is right nonsingular.

Conversely, assume that R is right nonsingular, and every right ideal is rationally complete. We finish by showing that any right ideal $\mathfrak{A}$ is a direct summand of R_R. Let $\mathfrak{B} \subseteq R_R$ be a complement to $\mathfrak{A}$. Then $\mathfrak{A} \oplus \mathfrak{B} \subseteq_e R_R$. Since R_R is nonsingular, this implies that $\mathfrak{A} \oplus \mathfrak{B} \subseteq_d R_R$ (see *LMR*-(8.7)(3)). By hypothesis, $(\mathfrak{A} \oplus \mathfrak{B})_R$ is rationally complete. Hence $\mathfrak{A} \oplus \mathfrak{B} = R$, as desired.

Ex. 8.8. Let $f : S \to R$ be a surjective ring homomorphism, where S is a commutative ring. If N_R is rationally complete as an R-module, show that N_S is a rationally complete S-module.

Solution. Let N' be any S-module such that $N \subseteq_d N'_S$. Consider any element $a \in \ker(f)$. Multiplication by a is an S-homomorphism $N' \to N'$ that is zero on N. This induces an S-homomorphism $N'/N \to N'$, which must be zero since $N \subseteq_d N'_S$. Therefore, $N'a = 0$, so N' is in fact an R-module. From $N \subseteq_d N'_S$, we see easily that $N \subseteq_d N'_R$. Since N_R is rationally complete, we must have $N = N'$, as desired.

Ex. 8.9. Let $N \subseteq_d M$, where M, N are right R-modules. For any $y_1, \ldots, y_n \in M$ and $0 \neq x \in M$, show that there exists $r \in R$ such that $y_1 r, \ldots, y_n r \in N$ and $xr \neq 0$.

Solution. We proceed by induction on n, the case $n = 1$ being covered by the definition of denseness. Assume, by an inductive hypothesis, that there exists $s \in R$ such that $y_1 s, \ldots, y_{n-1} s \in N$ and $xs \neq 0$. By $N \subseteq_d M$, there also exists $s' \in R$ such that $(y_n s)\, s' \in N$ and $(xs)s' \neq 0$. The element $r := ss' \in R$ is what we want.

Ex. 8.10. Let $S \subseteq R$ be rings such that $S_S \subseteq_e R_S$, and let $N \subseteq M$ be right R-modules.

(1) Assume N_S is nonsingular. Show that (a) $N_R \subseteq_e M_R$ iff $N_S \subseteq_e M_S$, and (b) N_S is injective $\Rightarrow$ N_R is injective.
(2) Assume M_S is nonsingular. Show that $N_R \subseteq_d M_R$ iff $N_S \subseteq_d M_S$.
(3) If N_S is nonsingular and rationally complete, show that N_R is also rationally complete.

Solution. (2) The "if" part is trivial (and is true without any assumptions on M or on $S \subseteq R$). For the "only if" part, assume that $N_R \subseteq_d M_R$, and let $x, y \in M$ with $x \neq 0$, There exists $r \in R$ such that $xr \neq 0$ and $yr \in N$. Since $S_S \subseteq_e R_S$, $rJ \subseteq S$ for some right ideal $J \subseteq_e S_S$. Now $xr \notin \mathcal{Z}(M_S) = 0$, so $(xr)j \neq 0$ for some $j \in J$. For $s = rj \in S$, we have

$$xs \neq 0 \quad \text{and} \quad ys = (yr)j \in NJ \subseteq N.$$

This shows that $N_S \subseteq_d M_S$.

(1) For the "only if" part in (a), repeat the argument above with $y = x \neq 0$. Here $0 \neq xr = yr \in N$, so the assumption $\mathcal{Z}(N_S) = 0$ would have sufficed for the argument. Now assume N_S is injective. If $N_R \subseteq M_R$ is any essential extension, then by (a), $N_S \subseteq_e M_S$, and hence $N = M$. This shows that N_R is also injective.

(3) Suppose $N_R \subseteq_d M_R$. Then $N_R \subseteq_e M_R$. Since by assumption $\mathcal{Z}(N_S) = 0$, (1a) implies that $N_S \subseteq_e M_S$. By *LMR*-(7.6)(2), we have $\mathcal{Z}(M_S) = 0$. Therefore, by (2), $N_S \subseteq_d M_S$, and hence $N = M$, as desired.

Comment. This exercise is essentially taken from the paper of Jain-Lam-Leroy: "On uniform dimensions of ideals in right nonsingular rings," J. Pure and Applied Algebra **133** (1998), 117–139.

Ex. 8.11. Let $M = \prod_{i \in I} M_i$ where the M_i's are (right) R-modules. If each $(M_i)_R$ is rationally complete, show that so is M.

Solution. This result is stated without proof in *LMR*-(8.25). To prove it, we use the fact that M is rationally complete iff, for any R-modules. $A \subseteq B$ such that $\text{Hom}_R(B/A, E(M)) = 0$, any R-homomorphism $f: A \to M$ can be extended to B. (See *LMR*-(8.24).) Let $A \subseteq B$ be R-modules such that $\text{Hom}_R(B/A, E(M)) = 0$, and let $f: A \to M$ be any R-homomorphism. Fix any $i \in I$, and let $f_i = \pi_i f$ where π_i is the projection from M to M_i. We have $M = M_i \oplus M_i'$ where $M_i' = \prod_{j \neq i} M_i$. Therefore

$$E(M) = E(M_i) \oplus E(M_i'),$$

so $\text{Hom}_R(B/A, E(M)) = 0$ implies that $\text{Hom}_R(B/A, E(M_i)) = 0$. Since M_i is rationally complete, $f_i: A \to M_i$ extends to a homomorphism $g_i: B \to M_i$. Now $g = (g_i)_{i \in I}: B \to M$ clearly extends the given f, so we have shown that M is rationally complete.

Ex. 8.12. Let R be a local ring with a nilpotent maximal ideal $\mathfrak{m}$. Show that every module M_R is rationally complete.

Solution. We may assume $M \neq 0$. Since $\mathfrak{m}$ is nilpotent, there exists a smallest integer $r \geq 1$ such that $M\mathfrak{m}^r = 0$. Then $M\mathfrak{m}^{r-1}$ is a nonzero right vector space over the division ring $R/\mathfrak{m}$. In particular, $M\mathfrak{m}^{r-1}$ contains a 1-dimensional vector subspace N, which is therefore a simple R-submodule of M. The fact that M contains such a submodule implies that M is rationally complete, by *LMR*-(8.18)(2).

Ex. 8.13. Find an example of a local ring R that is right Kasch but not left Kasch.

Solution. Let k be any field, and S be the ring of power series in two (noncommuting) variables x, y. We take R to be $S/\left(y^2, yx\right)$, which is a local ring with unique maximal left (resp. right) ideal $\mathfrak{m} = (x, y)/\left(y^2, yx\right)$. In working with R, it is convenient to use the representation

$$R = k[[\bar{x}]] \oplus k[[\bar{x}]]\,\bar{y}.$$

From this representation, we see easily that $\text{ann}_r(\bar{x}) = 0$ and hence $\text{ann}_r(\mathfrak{m}) = 0$. This implies that the (unique) simple left R-module ${}_R(R/\mathfrak{m})$ does not embed into ${}_RR$ (see *LMR*-(8.27)). However, $0 \neq \bar{y} \in \text{ann}_\ell(\mathfrak{m})$ implies that the unique simple right R-module $(R/\mathfrak{m})_R$ embeds into R_R (via left multiplication by $\bar{y}$) as a minimal right ideal yR. In particular, R is right Kasch but not left Kasch.

Ex. 8.14. Show that a ring R is right Kasch iff, for any nonzero f.g. module M_R, $\text{Hom}_R(M, R) \neq 0$.

Solution. Suppose the latter condition holds. Then, for any simple right R-module M in particular, there exists a nonzero R-homomorphism $f: M \to R$. Since M is simple, f must be embedding, so R is right Kasch.

Conversely, assume that R is right Kasch, and consider any f.g. module $M_R \neq 0$. A simple application of Zorn's Lemma shows that M has a maximal submodule N. By the hypothesis on R, the simple right R-module M/N embeds into R_R. Composing the projection map $M \to M/N$ with such an embedding, we get a nonzero homomorphism from M to R_R. (Note that it is not necessary to exclude the zero ring in this exercise, since everything said above is vacuously true for the zero ring.)

Ex. 8.15. For any ring R, show that the following are equivalent.

(1) R is semisimple;
(2) R is von Neumann regular and right Kasch;
(3) R is Jacobson semisimple (i.e. $\text{rad}(R) = 0$) and right Kasch;
(4) R is semiprime and right Kasch.

Solution. We have clearly (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) since any semisimple ring is Kasch (*LMR*-(8.29)(1)), and

$$\text{semisimple} \Rightarrow \text{von Neumann regular} \Rightarrow \text{J-semisimple} \Rightarrow \text{semiprime}.$$

To complete the proof, assume (4). To show (1), it suffices to check that $S := \text{soc}\,(R_R)$ is the whole ring R. Assume $S \neq R$. Then there exists a maximal right ideal $\mathfrak{m} \supseteq S$. Since R is right Kasch, $R/\mathfrak{m} \cong \mathfrak{A}$ for some minimal right ideal $\mathfrak{A}$, which is necessarily in S. Now S is an ideal of R (since, for every minimal right ideal $\mathfrak{B}$ and any $r \in R$, $r\mathfrak{B}$ is either a minimal right ideal or (0)). Therefore, $\mathfrak{m} \supseteq S$ yields $(R/\mathfrak{m}) \cdot S = 0$. But then $0 = \mathfrak{A} \cdot S \supseteq \mathfrak{A}^2$, which contradicts the fact that R is semiprime.

Ex. 8.16. For $A = \mathbb{Z}/4\mathbb{Z}$, let R be Osofsky's ring $\begin{pmatrix} A & 2A \\ 0 & A \end{pmatrix}$ (defined in *LMR*-(3.45)).

(1) Show that R is a Kasch ring by finding explicit embeddings of the simple (right, left) R-modules into R.
(2) Show that the modules R_R and ${}_RR$ are not divisible (and hence not injective).

Solution. (1) Let $J = \begin{pmatrix} 2A & 2A \\ 0 & 2A \end{pmatrix}$. Since $J^2 = 0$ and $R/J \cong \mathbb{F}_2 \times \mathbb{F}_2$ is semisimple, we have $J = \text{rad}(R)$. Let $S = A/2A \cong \mathbb{F}_2$, which is the unique simple A-module. The two simple right R-modules S_1, S_2 are given by S, where $\begin{pmatrix} a_1 & b \\ 0 & a_2 \end{pmatrix}$ acts on S_i by right multiplication by a_i. We can find copies of S_1, S_2 in R_R by considering

$$I_1 = \begin{pmatrix} 2A & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad I_2 = \begin{pmatrix} 0 & 0 \\ 0 & 2A \end{pmatrix}.$$

The equations

$$\begin{pmatrix} 2a & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} a_1 & b \\ 0 & a_2 \end{pmatrix} = \begin{pmatrix} 2aa_1 & 0 \\ 0 & 0 \end{pmatrix} \text{ and } \begin{pmatrix} 0 & 0 \\ 0 & 2a \end{pmatrix}\begin{pmatrix} a_1 & b \\ 0 & a_2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 2aa_2 \end{pmatrix}$$

show that I_1, I_2 are right ideals in R, with $(I_i)_R \cong S_i$. Similar calculations show that the I_i's are also left ideals, with ${}_R(I_i)$ giving the two simple left R-modules. Therefore, R is a Kasch ring.

(2) It suffices to prove that R_R is not divisible, since the same argument will work for ${}_RR$. Let $a = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}$ and $b = \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix}$ in R. For $c = \begin{pmatrix} x & y \\ 0 & z \end{pmatrix} \in R$, we have

$$ac = 0 \implies 2z = 0 \implies bc = \begin{pmatrix} 0 & 0 \\ 0 & 2z \end{pmatrix} = 0.$$

However, $aR = \begin{pmatrix} 0 & 2A \\ 0 & 0 \end{pmatrix}$, so we have $b \notin aR$. This shows that R_R is not divisible (and hence not injective).

Comment. Of course, Osofsky's result (*LMR*-(3.45)) that $E(R_R)$ admits no ring structure compatible with its right R-module structure already implies that R_R is not injective. The above gives a little extra information. In the terminology of Nicholson and Yousif (see Exercise (3.44)), R is neither a right nor a left principally injective ring.

Ex. 8.17. Let A be the direct product $k \times k \times \cdots$ where k is a field. Let e_i $(i \geq 1)$ the i^{th} "unit vector" and S be the k-subalgebra of A generated by the e_i's (that is, $S = k \oplus ke_1 \oplus ke_2 \oplus \cdots$).

(1) Show that the simple S-modules are V_i $(i \geq 0)$ with $V_i \cong ke_i \subset S$ for $i \geq 1$, and $V_0 = S/\bigoplus_{i \geq 1} ke_i$ with all e_i's acting as zero.
(2) Show that V_0 is the only simple S-module not embeddable into S (so S is not a Kasch ring).

Solution. (1) Let $\mathfrak{m}_0 = \bigoplus_{i \geq 1} ke_i$ and for $i \geq 1$, let

$$\mathfrak{m}_i = k(e_i - 1) \oplus \bigoplus_{j \neq i} k\, e_j.$$

Each of these is an ideal of codimension 1, so they are maximal ideals in S. Let $V_i := S/\mathfrak{m}_i$ $(i \geq 0)$, which are simple S-modules. On V_0, each e_i acts as zero, and on V_i $(i \geq 1)$, e_j acts as δ_{ij}, so clearly $V_i \cong ke_i$ for $i \geq 1$. We finish by proving that each prime ideal $\mathfrak{p} \subset S$ is one of the $\mathfrak{m}_i$'s $(i \geq 0)$. Indeed, if $\mathfrak{p}$ contains all e_i, then $\mathfrak{p} = \mathfrak{m}_0$. Suppose $\mathfrak{p}$ does not contain e_i. Then $e_ie_j = 0 \Rightarrow e_j \in \mathfrak{p}$ for all $j \neq i$, and $e_i(e_i - 1) = 0 \Rightarrow e_i - 1 \in \mathfrak{p}$. Therefore, $\mathfrak{p} = \mathfrak{m}_i$, as desired.

(2) Suppose a minimal ideal $\mathfrak{A} \subseteq S$ is isomorphic to V_0. Then $\mathfrak{A} = k \cdot \alpha$ for some

$$\alpha = a_0 + a_1 e_1 + \cdots + a_r e_r, \quad a_i \in k, \ a_r \neq 0.$$

Since $\alpha e_{r+1} = 0$, we see that $a_0 = 0$. Now $\alpha e_r = 0 \Rightarrow a_r = 0$, a contradiction. Therefore, V_0 does not embed into S, as desired.

Comment. Since $S \subseteq A$, S is a reduced ring. We have shown above that S has Krull dimension 0, so S is a (commutative) von Neumann regular ring. (Of course, this is also easy to check directly from the definition of S.) By the last exercise, this actually *implies* that V_0 does not embed into S, for otherwise S would be Kasch and therefore semisimple, which it is not.

There is, of course, a "finite version" of this example. We could have started with the finite direct product $A = k \times \cdots \times k$ (say $n+1$ copies). Defining

$$e_1 = (0,1,0,\ldots,0), \ e_2 = (0,0,1, \ldots,0), \ldots, \ e_n = (0,\ldots,0,1),$$

the formation $S = k \cdot 1 \oplus ke_1 \oplus \cdots \oplus ke_n$ in A will give back A itself. Here, we can define e_0 to be $(1, 0,\ldots,0)$ to get $e_0 + e_1 + \cdots + e_n = 1$. The simple modules are $V_0, V_1, \ldots, V_n$ as defined before, except that here, $V_0 \cong ke_0 \subseteq S$ also. The ring $S = A$ here is a semisimple (Kasch) ring.

In the "infinite" case, the idempotent e_0 is curiously "missing", since we cannot talk about $1 - e_1 - e_2 - \cdots$. Nevertheless, the 0^{th} simple module $V_0 = S/\mathfrak{m}_0$ is there.

The next four Exercises arise from a correspondence between the author and Carl Faith in June, 1998.

Ex. 8.18. Show that a commutative Kasch ring R with $|\text{Ass}(R)| < \infty$ (e.g. in case u.dim $(R_R) < \infty$) must be semilocal. Then construct a commutative Kasch ring that is not semilocal.

Solution. Let $\mathfrak{m}$ be any maximal ideal of R. Since R is Kasch, $\mathfrak{m} = \text{ann}(x)$ for some $x \in R$, by *LMR*-(8.27). Therefore, $\mathfrak{m} \in \text{Ass}(R)$ by *LMR*-(3.56). By the given hypothesis, there are only finitely many such $\mathfrak{m}$'s.

In case u.dim $(R_R) < \infty$, we have $|\text{Ass}(R)| < \infty$ by Exer. 6.2, so the desired conclusion also applies.

To produce a commutative Kasch ring that is not semilocal, we can use the construction introduced in *LMR*-(8.30). Let S be any commutative ring with an infinite set of mutually nonisomorphic simple modules $\{V_i\}$ (e.g., $S = \mathbb{Z}$, and $V_i = \mathbb{Z}/p_i\mathbb{Z}$ where p_i is the i^{th} prime). We view $M = \bigoplus_i V_i$ as an (S, S)-bimodule in the natural way (with identical left, right S-actions), and form the "trivial extension" of M by S to get a (commutative) ring $R = M \oplus S$ (with $M^2 = 0$). As in *LMR*-(8.30), R is Kasch, and R has simple modules V_i (on which M acts trivially). Since R has infinitely many simple modules, R cannot be a semilocal ring, as desired.

Ex. 8.19. For a prime ideal $\mathfrak{p}$ in a commutative ring R, show that $\mathfrak{p} \in \text{Ass}(R)$ implies that the localization $R_\mathfrak{p}$ is a Kasch ring, and conversely if $\mathfrak{p}$ is a f.g. ideal (e.g. if R is noetherian). Give an example to show that the converse need not hold if $\mathfrak{p}$ is *not* a f.g. prime ideal.

Solution. First assume $\mathfrak{p} \in \text{Ass}(R)$, so $\mathfrak{p} = \text{ann}^R(r)$ for some $r \in R$. We must have $r/1 \neq 0$ in $T := R_\mathfrak{p}$, for otherwise there exists $s \in R \backslash \mathfrak{p}$ such that $sr = 0 \in R$, in contradiction to $\mathfrak{p} = \text{ann}^R(r)$. Since $(r/1) \cdot \mathfrak{p}T = 0$, *LMR*-(8.28) implies that the local ring $(T, \mathfrak{p}T)$ is Kasch.

Conversely, assume $T = R_\mathfrak{p}$ is Kasch, so that $\mathfrak{p}T = \text{ann}^T(r/s)$ for some $s \in R \backslash \mathfrak{p}$ (see *LMR*-(8.27)), which we may assume to be 1. If $\mathfrak{p}$ is f.g., say by elements $y_1, \ldots, y_n$, then $ry_i = 0 \in T$ for all $i \leq n$, so there exists $t \in R \backslash \mathfrak{p}$ such that $try_i = 0 \in R$ for all i. This gives $\mathfrak{p} \subseteq \text{ann}^R(tr)$. It suffices to show that this is an equality, for then we'll have $\mathfrak{p} \in \text{Ass}(R)$. Let $x \in \text{ann}^R(tr)$. Then $xt \cdot r/1 = 0$ in $R_\mathfrak{p}$. If $x \notin \mathfrak{p}$, x as well as t would be units in $R_\mathfrak{p}$ and we would have $r/1 = 0 \in T$, in contradiction to $\text{ann}^T(r/1) = \mathfrak{p}T \neq T$. Thus, $x \in \mathfrak{p}$, as desired.

To show that in general $R_\mathfrak{p}$ being Kasch need not imply $\mathfrak{p} \in \text{Ass}(R)$, we exploit the same idea used in the solution to Ex. 8.5A. Let R and A be as in part (4) of that solution. For any maximal ideal $\mathfrak{m}$ of R containing A, $R_\mathfrak{m}$ is certainly Kasch (as it is a field), but we knew that $\mathfrak{m} \notin \text{Ass}(R)$, by Ex. 3.40G. Here, of course, $\mathfrak{m}$ is not f.g.

Ex. 8.20. Construct a commutative noetherian Kasch ring R with a prime ideal $\mathfrak{p}$ such that the localization $R_\mathfrak{p}$ is not a Kasch ring.

Solution. In view of Exercise 19, it suffices to construct a commutative noetherian Kasch ring R with a prime $\mathfrak{p} \notin \text{Ass}(R)$. We start with a commutative noetherian local domain $(S, \mathfrak{m})$ with Krull dimension ≥ 2. Let P be a prime ideal such that $0 \neq P \neq \mathfrak{m}$, and let $V = S/\mathfrak{m}$, viewed as an (S, S)-bimodule with identical left and right S-actions. Let $R = V \oplus S$ be the trivial extension of V by S (with $V^2 = 0$). Since R is a (commutative) S-algebra that is f.g. as an S-module, R is a noetherian ring. By *LMR*-(8.30), R is a Kasch ring. Consider the ideal $\mathfrak{p} := V \oplus P$ of R. Since $R/\mathfrak{p} \cong S/P$ is a domain, $\mathfrak{p}$ is a prime ideal. We finish by showing that $\mathfrak{p}$ is *not* an associated prime of R. Indeed, if $\mathfrak{p} \in \text{Ass}(R)$, we would have $\mathfrak{p} = \text{ann}^R((v, s))$ for some nonzero $(v, s) \in R$. Taking $p \in P \backslash \{0\}$, we have

$$0 = (v, s)\,(0, p) = (vp, sp) \implies s = 0.$$

But then the annihilator of $(v, s) = (v, 0)$ contains $0 \oplus \mathfrak{m}$, since

$$(v, 0)\,(0,\, \mathfrak{m}) = (v \cdot \mathfrak{m},\, 0) = (0, 0).$$

This contradicts $\text{ann}^R((v, s)) = \mathfrak{p} = V \oplus P$.

Ex. 8.21. Let R be a commutative noetherian ring. If $R_\mathfrak{m}$ is Kasch for every maximal ideal $\mathfrak{m}$, show that R is Kasch. What if R is not assumed to be noetherian?

Solution. Let $\mathfrak{m}$ be any maximal ideal of R. Since $R_{\mathfrak{m}}$ is Kasch and $\mathfrak{m}$ is f.g., Exercise 19 shows that $\mathfrak{m} \in \mathrm{Ass}(R)$. Therefore, $\mathfrak{m} = \mathrm{ann}(x)$ for some nonzero $x \in R$. Now multiplication by x embeds $R/\mathfrak{m}$ into R, so R is a Kasch ring (and necessarily semilocal by Exercise 18).

If we do not impose the noetherian assumption, the ring $R = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \cdots$ provides a counterexample. This ring is not Kasch, but since R is von Neumann regular, the localizations $R_{\mathfrak{p}}$ at all prime (= maximal) ideals $\mathfrak{p}$ are fields, and hence Kasch rings.

Ex. 8.22. Let k be a division ring, and R be the ring of matrices

$$\gamma = \begin{pmatrix} a & 0 & b & c \\ 0 & a & 0 & d \\ 0 & 0 & a & 0 \\ 0 & 0 & 0 & e \end{pmatrix}$$

over k, as in *LMR*-(8.29)(6). Compute the left and right singular ideals of R.

Solution. The left and right socles of R have been computed in *LMR*-(8.29)(6); namely,

$$(1)\ \mathrm{soc}\,(R_R) = \{\gamma \in R:\ a = 0\}\,, \text{ and } \ \mathrm{soc}\,({}_RR) = \{\gamma \in R:\ a = e = 0\}\,.$$

Since R is artinian, *LMR*-(7.13) implies that

$$(2) \qquad \mathcal{Z}\,(R_R) = \mathrm{ann}_\ell\,(\mathrm{soc}\,(R_R))\,, \text{ and } \ \mathcal{Z}\,({}_RR) = \mathrm{ann}_r\,(\mathrm{soc}\,({}_RR))\,.$$

Computing the appropriate annihilators of the ideals in (1), we see that

$$\mathcal{Z}\,(R_R) = \begin{pmatrix} 0 & 0 & k & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \text{ and } \ \mathcal{Z}\,({}_RR) = \begin{pmatrix} 0 & 0 & k & k \\ 0 & 0 & 0 & k \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

In particular, R is neither left nor right nonsingular.

Comment. In any noetherian ring R, $\mathcal{Z}\,(R_R)$ and $\mathcal{Z}\,({}_RR)$ are known to be in $\mathrm{rad}(R)$ (the Jacobson radical of R): see *LMR*-(7.15). In the above example, we have $\mathcal{Z}\,(R_R) \subsetneq \mathcal{Z}\,({}_RR) = \mathrm{rad}(R)$.

Chapter 4
Rings of Quotients

§9. Noncommutative localization

In commutative algebra, localization of commutative rings R with respect to multiplicative sets $S \subseteq R$ provides a powerful tool for proving theorems. For noncommutative rings, however, localization is a much more difficult proposition.

For any multiplicative set S in a ring R, there does exist an "S-inverting homomorphism" ε from R to a ring R_S with an obvious universal property (*LMR*-(9.2)). However, this universal ring R_S is rather unwieldy, and its elements are difficult to express. Also, R_S may turn out to be the zero ring. Therefore, it does not seem fruitful to work with this localization R_S.

Classically, the consideration of noncommutative localization started with the problem of embedding a domain into a division ring. Mal'cev showed that a cancellative semigroup (with identity) may not be embeddable in a group, and, using the construction of semigroup algebras, he also showed that a domain may not be embeddable into a division ring (*LMR*-(9.11)). Various necessary conditions for a domain to be embeddable in a division ring are given in *LMR*-(9.13) through *LMR*-(9.16).

For a field C, the free algebra $F = C\langle u, v\rangle$ can be embedded in a division ring D. In D, we can therefore form the smallest division ring containing F; this is called a *division hull* of F. It turns out, however, that up to isomorphisms over F, F has infinitely many division hulls. This shows that it is not possible to associate a unique division ring with F in a natural way.

The exercise set in this section is very modest. Cohn's observation in Exercise 3 shows the difference between relations in a semigroup and relations in a ring, and Exercise 5 offers an explicit example of a domain B for which some localization B_S is actually a zero ring.

In order to get a more manageable localization of a ring R at a multiplicative set S, some fairly strong assumptions will have to be made on S. This leads to the Ore localization theory in the next section.

Exercises for §9.

Ex. 9.1. Show that, for any multiplicative set $S \subseteq R$, the universally S-inverting homomorphism $\varepsilon : R \to R_S$ is injective iff R can be embedded into a ring in which all elements of S have inverses.

Solution. If ε is injective, then R_S is the ring required. Conversely, assume that $i : R \to R'$ is a ring embedding such that $i(S) \subseteq \mathrm{U}(R')$. Then, by the universal property of R_S, there exists a (unique) ring homomorphism $\varphi : R_S \to R'$ such that $\varphi \circ \varepsilon = i$. Since i is injective, so is ε.

Ex. 9.2. Let S, S' be, respectively, multiplicative sets in the rings R, R', which give rise to the ring homomorphisms $\varepsilon : R \to R_S$ and $\varepsilon' : R' \to R'_{S'}$. For any ring homomorphism $f : R \to R'$ such that $f(S) \subseteq S'$, show that there is a unique ring homomorphism $f_* : R' \to R'_{S'}$ such that $f_*\varepsilon = \varepsilon' f$.

Solution. Let $g = \varepsilon' f : R \to R'_{S'}$. Since

$$g\,(S) = \varepsilon' f\,(S) \subseteq \varepsilon'\,(S') \subseteq \mathrm{U}\,(R'_{S'})\,,$$

there exists a unique ring homomorphism $f_* : R_S \to R'_{S'}$ such that $f_*\varepsilon = g := \varepsilon' f$.

Ex. 9.3. (Cohn) Let $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $N = \begin{pmatrix} x & u \\ -y & -v \end{pmatrix}$ be matrices over a ring T in which b and x are units. If $L := MN = \begin{pmatrix} 0 & 0 \\ 0 & * \end{pmatrix}$, show that $L = 0$. Does the conclusion hold if one of b, x fails to be a unit in T?

Solution. Since $b \in U$, we can perform an elementary row transformation on M to bring it to a matrix $\begin{pmatrix} a & b \\ * & 0 \end{pmatrix}$. More precisely $RM = \begin{pmatrix} a & b \\ * & 0 \end{pmatrix}$ for $R = \begin{pmatrix} 1 & 0 \\ -db^{-1} & 1 \end{pmatrix}$. Similarly, we can perform an elementary column transformation on N to bring it to a matrix $\begin{pmatrix} x & 0 \\ -y & * \end{pmatrix}$, say $NC = \begin{pmatrix} x & 0 \\ -y & * \end{pmatrix}$, where $C = \begin{pmatrix} 1 & -x^{-1}u \\ 0 & 1 \end{pmatrix}$. Now, if $L := MN$ has the form $\begin{pmatrix} 0 & 0 \\ 0 & * \end{pmatrix}$, neither of the elementary transformations above will change L. Therefore, we will have

$$L = RLC = (RM)(NC) = \begin{pmatrix} a & b \\ * & 0 \end{pmatrix} \begin{pmatrix} x & 0 \\ -y & * \end{pmatrix} = \begin{pmatrix} * & * \\ * & 0 \end{pmatrix},$$

so necessarily $L = 0$.

The conclusion $L = 0$ need not hold if one of b, x fails to be a unit in T. Indeed, if x is a unit but b is not, the product

$$MN = \begin{pmatrix} 0 & 0 \\ 0 & d \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & d \end{pmatrix} \quad (d \neq 0)$$

gives a counterexample. If, on the other hand, b is a unit but x is not, the product

$$MN = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & v \\ 0 & -v \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & -v \end{pmatrix} \quad (v \neq 0)$$

gives a counterexample.

Ex. 9.4. *True or False:* The kernel of the universal S-inverting homomorphism $\varepsilon : R \to R_S$ is generated as an ideal by the set

$$A = \{r \in R : s'rs = 0 \quad \text{for some } s, s' \in S\}$$

Solution. Let H be Mal'cev's semigroup constructed in *LMR*-(9.8). Then H is generated by a set of elements a, b, c, d, x, y, u, v, and we have

$$ax = by, \quad cx = dy, \quad au = bv \quad \text{but} \quad cu \neq dv.$$

Let k be any domain. In *LMR*-(9.11), it is shown that the semigroup ring $R = kH$ is also a domain. Now let $S \subseteq R$ be any multiplicative set containing b, d, x and u, and let $\varepsilon : R \to R_S$ be the universal S-inverting homomorphism. Then the images of b, d, x and u are units in R_S, so, repeating the proof in *LMR*-(9.5), we see that

$$\varepsilon(c)\,\varepsilon(u) = \varepsilon(d)\,\varepsilon(v) \in R_S.$$

Thus $cu - dv$ is a nonzero element in $\ker(\varepsilon)$. This shows that $\ker(\varepsilon)$ is *not* generated as an ideal by A, since the fact that R is a domain implies that $A = \{0\}$ here.

Comment. The above construction would have worked if we were to assume only that the multiplicative set $S \subseteq R$ contains b and x. For then $\varepsilon(b)$ *and* $\varepsilon(x)$ would have been units in R_S, and Exercise 3 above (applied to the ring $T = R_S$) would have implied that $\varepsilon(c)\,\varepsilon(u) = \varepsilon(d)\,\varepsilon(v) \in T$.

Ex. 9.5. Construct a domain B with a multiplicative set S such that $B_S = 0$.

Solution. Let H and $R = kH$ be as in the solution to the last exercise. Let $\mathfrak{A}$ be the ideal of R generated by $cu - dv + 1$, and let $B = R/\mathfrak{A}$. Now consider any multiplicative set $S \subseteq B$ containing $\bar{b}$ and $\bar{x}$. As we saw in the

Comment on the last exercise, $\bar{c}\bar{u} = \bar{d}\bar{v} \in B_S$. Therefore $\bar{1} = \bar{0} \in B_S$, so we have $B_S = 0$. By modifying the proof for *LMR*-(9.11), we can show that B has no zero-divisors (other than 0). To see that B is a domain, it only remains to show that $B \neq 0$. This is easy, and can be seen, for instance, by specializing a, b, c, x, y, u to 0, and specializing d and v to 1.

Ex. 9.6. (Chehata, Vinogradov) Let H be Mal'cev's semigroup (constructed in *LMR*-§9) generated by the elements $\{a, b, c, d, u, v, x, y\}$ with the relations

$$ax = by, \quad cx = dy, \quad \text{and} \quad au = bv.$$

Show that H becomes an *ordered semigroup* by the following procedure. First order the eight given generators by

$$(*) \qquad x < u < d < c < y < b < a < v.$$

Then extend this to an ordering relation "$<$" on H first according to length, and then "lexicographically" on reduced words. (As on p. 291 of *LMR*-§9, a reduced word is one that does not contain a subword ax, cx, or au. Note that, by using the three given relations, any word can be transformed into a unique reduced word.)

Solution. Let us label the eight generators in $(*)$ (from left to right) by $e_1, \ldots, e_8$, so we have $e_1 < \cdots < e_8$. Then a reduced word is one that has no subwords of the form e_7e_1, e_4e_1, or e_7e_2 (these having been replaced, respectively e_6e_5, e_3e_5, or e_6e_8). Our job is to prove that

$$(**) \qquad \alpha < \beta \text{ in } H \implies \alpha\gamma < \beta\gamma \text{ and } \gamma\alpha < \gamma\beta \quad (\forall \gamma \in H).$$

The crucial case will be that in which α, β, γ are all of length 1. We first handle this case by proving the following:

Lemma. $e_i < e_j \implies e_ie_k < e_je_k$ and $e_ke_i < e_ke_j$.

Proof. Note that $e_i < e_j$ simply means $i < j$.

Case 1. e_je_k is not reduced. There are three possibilities.

(A) $(j, k) = (7, 1)$. Here, e_je_k has the reduced form e_6e_5, and $i < j = 7$. If $i = 4$, $e_ie_k = e_4e_1$ has the reduced form e_3e_5, and both conclusions in the Lemma are seen to be true. Otherwise, $i \in \{1, 2, 3, 5, 6\}$, in which case e_ie_k and e_ke_i are reduced, and both desired conclusions are again true. (For instance, $e_ie_k < e_je_k$ amounts to $e_ie_1 < e_6e_5$, which is indeed true since $i \leq 6$.)

(B) $(j,k) = (7, 2)$. Here, e_je_k is reduced to e_6e_8, and e_ie_k is already reduced (since $i < j = 7$ and $k = 2$), as are e_ke_i and e_ke_j. Again, both desired conclusions are easily seen to be true.

(C) $(j, k) = (4, 1)$. Here, e_je_k is reduced to e_3e_5, and we have $i = 1$, 2, or 3. Since e_ke_i and e_ke_j are both reduced, $e_ke_i < e_ke_j$ is clearly true. And

$e_ie_k < e_je_k$ is also true since it amounts to $e_ie_1 < e_3e_5$, which is true since $i \le 3$.

Case 2. e_ke_j is not reduced. Since $1 = i < j$, the only possibility here is $j = 2$, $k = 7$, and $i = 1$. The desired conclusions are trivially checked in this case.

Case 3. e_je_k and e_ke_j are both reduced. Let us check, for instance, $e_ie_k < e_je_k$. This is clear if e_ie_k is reduced. Otherwise, e_ie_j will be reduced to a form $e_{i'}e_{k'}$ with $i' < i < j$, so $e_ie_k < e_je_k$ again follows. □

To check (**), it suffices to handle the case where α and β are reduced words of the same length, say

$$\alpha = a_1 \cdots a_n, \quad \beta = b_1 \cdots b_n,$$

where $a_i,\ b_j \in \{e_1, \cdots, e_8\}$. Also, we are free to assume that that $\gamma = c \in \{e_1, \cdots, e_8\}$. We shall only check $\alpha\gamma < \beta\gamma$ (as the proof for $\gamma\alpha < \gamma\beta$ is rather similar). Say $a_1 = b_1, \ldots, a_t = b_t$, but $a_{t+1} = e_i$, $b_{t+1} = e_j$, with $i < j$. If $t < n - 1$, then $\alpha\gamma < \beta\gamma$ is clear since, in bringing $\alpha\gamma$ and $\beta\gamma$ to reduced forms, we will not be changing the "front portions" $a_1 \cdots a_ta_{t+1}$ and $b_1 \cdots b_tb_{t+1}$ of these words. Finally, assume $t = n - 1$. Then

$$\alpha\gamma = a_1 \cdots a_{n-1}(a_nc), \quad \text{and} \quad \beta\gamma = a_1 \cdots a_{n-1}(e_jc).$$

By the Lemma, we know that $e_ic < e_jc$. Now replace e_ic and e_jc by their reduced forms in the expressions of $\alpha\gamma$ and $\beta\gamma$ above. Then we obtain the reduced forms of $\alpha\gamma$ and $\beta\gamma$. Since $e_ic < e_jc$, these reduced forms clearly show that $\alpha\gamma < \beta\gamma$, as desired.

Comment. The argument above is a somewhat expanded form of that in C. G. Chehata's paper: "On an ordered semigroup", J. London Math. Soc. **28**(1953), 353–356. The same result was obtained independently by A. A. Vinogradov in his paper (in Russian): "On the theory of ordered semigroups", Ivanov Gos. Ped. Inst. Ucen. Zap. Fiz.-Mat. Nauki **4**(1953), 19–21. For another construction of an ordered semigroup (with 6 generators) that cannot be embedded into a group, see R. E. Johnson's paper: "Extended Malcev domains", Proc. Amer. Math. Soc. **21**(1969), 211–213.

The main point about the semigroup H (as explained in *LMR*-§9) was that it is a cancellative semigroup that cannot be embedded into a group. The virtue of knowing that $(H, <)$ is an ordered semigroup is three-fold. First, this implies that H is cancellative. Second, it implies that $\mathbb{Q}H$ is a domain (that cannot be embedded in a division ring). Third, this will show that even an ordered semigroup may not be embeddable into a group (let alone into an ordered group with a compatible ordering). On the other hand, it is not difficult to show that a *commutative* ordered semigroup can be compatibly embedded into a commutative ordered group.

§10. Classical Ring of Quotients

In this section, instead of working with an arbitrary multiplicative set S in a ring R, we impose suitable conditions on S in order that the universal S-inverting ring R_S becomes more manageable. The conditions we require are the following:

(1) S is *right permutable*; i.e. $aS \cap sR \neq \emptyset$ for every $a \in R$ and $s \in R$;
(2) S is *right reversible*; i.e. if $s'a = 0$ for $a \in R$ and $s' \in S$, then $as = 0$ for some $s \in S$.

If S satisfies both of these conditions, we say that S is a *right denominator set.*

Under the assumption that S is a right denominator set, we can construct a ring RS^{-1} whose elements are "right fractions" of the form as^{-1} where $a \in R$ and $s \in S$. Here, of course, appropriate pairs of fractions are identified. The work involved in checking that RS^{-1} is a ring is quite tedious, but mostly routine. The ring RS^{-1} obtained in this way is called the *right ring of fractions of* R w.r.t. S. It comes with a canonical map $\varphi : R \to RS^{-1}$ defined by $\varphi(a) = a \cdot 1^{-1}$ (for $a \in R$), and the kernel of φ turns out to be the set

$$\{r \in R : \ rs = 0 \ \text{ for some } \ s \in S\}.$$

Furthermore, it may be checked that φ is just the universal S-inverting map investigated in *LMR*-§9. So the point is that, under the assumption that S is a right denominator set, all the elements of $R_S \cong RS^{-1}$ have the "right fraction" form as^{-1} and the kernel of $R \to R_S$ can be easily described as above.

The left analogues of the notions introduced above can be similarly defined, and we can form $S^{-1}R$, the left ring of fractions of R w.r.t. S, in case S is a left denominator set. In general, $S^{-1}R$ may not exist even if RS^{-1} does. However, if S in both a right and a left denominator set, then $RS^{-1} \cong S^{-1}R$ over R, according to *LMR*-(10.4).

A canonical choice for S is the set of all regular elements in R; i.e. those elements that are neither a right 0-divisor nor a left 0-divisor in R. This set S is, of course, both right and left reversible. We say that R is a *right* (*left*) *Ore ring* if S is also right (left) permutable. In this case, we speak of RS^{-1} (resp. $S^{-1}R$) as the (total) *classical right* (resp. *left*) *ring of fractions of* R, and denote it by $Q^r_{c\ell}(R)$ (resp. $Q^\ell_{c\ell}(R)$). If R is an Ore ring (that is, both right and left Ore), then $Q^r_{c\ell}(R) \cong Q^\ell_{c\ell}(R)$, and we may denote this ring by $Q_{c\ell}(R)$. If, for instance, R is a commutative ring, then it is clearly Ore, and $Q_{c\ell}(R)$ is the usual total ring of quotients of R.

In the case where R *is a domain*, the right Ore condition takes the form $aR \cap bR \neq 0$ for a, $b \in R\backslash\{0\}$. This condition amounts to u.dim $R_R = 1$, and defines a *right Ore domain.* For such a domain, $Q^r_{c\ell}(R)$ is a division ring, and, up to a unique R-isomorphism, it is the only division hull of R (see *LMR*-(10.21)). Examples of right Ore domains include:

all commutative domains, all right noetherian domains, all right Bézout domains, and all domains that are PI-algebras over a field.

The behavior of the "right Ore domain" condition under polynomial and power series extensions is studied in §10C. It turns out that the condition in question "goes up" to a polynomial extension, that does not go up to a power series extension: see *LMR*-(10.28) and *LMR*-(10.31A). If R is not assumed to be a domain, then the situation is even worse, since R being right (or even-2-sided) Ore will not imply that $R[x]$ is right Ore, according to a recent example of Cédo and Herbera (*LMR*-(10.31B)).

The terminology of "right order" introduced in *LMR*-§10 will also be useful later. For rings $R \subseteq Q$, we say that R is a right order in Q if

(1) every regular element of R is a unit in Q, and
(2) every element of Q has the form as^{-1} where $a \in R$ and s is a regular element of R.

It is easy to see that *a ring R is right Ore iff it is a right order in some ring Q, and in this case, $Q \cong Q^r_{c\ell}(R)$.*

The exercises in this section study in detail the notions introduced above, and relate them to the earlier notions of injectivity, flatness, strong rank condition, etc. The behavior of injective modules under localization in studied in some detail in Exercises 29–31.

Exercises for §10

Ex. 10.0. Let $S \subseteq R$ be a right permutable multiplicative set. Show that $\mathfrak{A} = \{a \in R : \; as = 0 \text{ for some } s \in S\}$ is an ideal in R.

Solution. Clearly, $\mathfrak{A}$ is closed under left multiplication by R. We claim that $\mathfrak{A}$ is also closed under *right* multiplication by R. Indeed, let $r \in R$ and $a \in \mathfrak{A}$, say $as = 0$ where $s \in S$. Since S is right permutable, we have $rs' = sr'$ for some $s' \in S$ and $r' \in R$. Left multiplying this equation by a, we get $(ar)s' = (as)r' = 0$, so $ar \in \mathfrak{A}$, as claimed. If a' is another element in $\mathfrak{A}$, then $(a \pm a')\, s = \pm a's \in \mathfrak{A}$ by the claim above, so for a suitable $t \in S$, we will have $0 = \pm a'st = (a \pm a')st$. This implies that $a \pm a' \in \mathfrak{A}$, so $\mathfrak{A}$ is indeed an ideal in R.

Comment. It is useful to recall that, in the case where S is a right denominator set in R, we have a ring homomorphism $\varphi : R \to RS^{-1}$ (defined by $\varphi(r) = r/1$) whose kernel is precisely the ideal $\mathfrak{A}$ above: see *LMR*-(10.9).

Ex. 10.1. Let $S \subseteq R$ be a right denominator set, and $\mathfrak{A}$ be as in the above exercise. Let $\bar{R} = R/\mathfrak{A}$ and write "bar" for the natural surjection from R to $\bar{R}$. Show that $\bar{S}$ is a right denominator set in $\bar{R}$ consisting of regular elements, and that $RS^{-1} \cong \bar{R}\bar{S}^{-1}$ over R.

Solution. We show first that $\bar{S}$ consists of regular elements of $\bar{R}$. If $\bar{x}\bar{s} = 0$ $(x \in R,\; s \in S)$, then $xs \in \mathfrak{A}$, so $xss' = 0$ for some $s' \in S$. This shows that

$x \in \mathfrak{A}$, so $\bar{x} = 0$. Next, suppose $\bar{s}\,\bar{y} = 0 \quad (y \in R,\ s \in S)$. Then $sy \in \mathfrak{A}$, so $sys_1 = 0$ for some $s_1 \in S$. Since S is right reversible, this implies that $(ys_1)s_2 = 0$ for some $s_2 \in S$. Thus, $y \in \mathfrak{A}$, so $\bar{y} = 0$. The fact that S is right permutable clearly implies that $\bar{S}$ is right permutable. Since $\bar{S}$ consists of regular elements of $\bar{R}$, it follows that $\bar{S}$ is a right denominator set in $\bar{R}$.

Let f be the composition of $R \to \bar{R} \to \bar{R}\bar{S}^{-1}$. Since $f(s) \subseteq \mathrm{U}(\bar{R}\bar{S}^{-1})$ and the natural map $\varphi\colon\ R \to RS^{-1}$ is a universal S-inverting homomorphism (*LMR*-(10.11)), there exists a (unique) ring homomorphism g: $RS^{-1} \to \bar{R}\bar{S}^{-1}$ such that $g \circ \varphi = f$. Of course, $g(rs^{-1}) = \bar{r}\,\bar{s}^{-1}$ for $r \in R$ and $s \in S$. In particular, g is onto. To see that g is one-one, take any $rs^{-1} \in \ker(g)$. Then $\bar{r}\,\bar{s}^{-1} = 0$. This implies that $r \in \mathfrak{A}$, and so $rs^{-1} = \varphi(r)\varphi(s)^{-1} = 0$ in RS^{-1}. Therefore, g is an isomorphism, and it is clear that this is an isomorphism over R.

Ex. 10.2. Let $S \subseteq R$ be a multiplicative set. (a) If $s \in S$ has a right inverse, show that $aS \cap sR \neq \emptyset$ holds for every $a \in R$. (b) If R is a reduced ring, show that S is right and left reversible.

Solution. (a) Say $st = 1$, where $t \in R$. The $a \cdot 1 = s \cdot ta$ shows that $aS \cap sR \neq \emptyset$. (b) Assume that R is a reduced ring. Then, for any $a \in R$ and $s \in S$, we have

$$sa = 0 \implies (as)^2 = a(sa)s = 0 \implies as = 0,$$

so the multiplicative set S is automatically right (and, similarly, also left) reversible.

Ex. 10.3. Let $S \subseteq R$ be any commutative multiplicative set, and let

$$A = \{a \in R:\ \ aS \cap sR \neq \emptyset \quad \text{for every } s \in S\}.$$

Show that A is a subring of R containing the centralizer of S in R.

Solution. If $a \in R$ belongs to the centralizer of S, then for any $s \in S$, $aS \cap sR$ contains the element $as = sa$, so $a \in A$. In particular, $1 \in A$. Next, we check that A is closed under multiplication. Let $a_1, a_2 \in A$, and $s \in S$. Then there exist $s_1 \in S$ and $r_1 \in R$ such that $a_1 s_1 = sr_1$, and there exist $s' \in S$ and $r \in R$ such that $a_2 s' = s_1 r$. Therefore

$$(a_1 a_2)s' = a_1(s_1 r) = (sr_1)r = s(r_1 r).$$

This shows that $a_1 a_2 \in A$. So far, we have not used the commutativity of the multiplicative set. Using this, we shall show now $a_1, a_2 \in A \implies a_1 \pm a_2 \in A$ (which then implies that A is a subring of R). For any $s \in S$, fix equations $a_i s_i = sr_i$, where $s_i \in S$ and $r_i \in R$ $(i = 1, 2)$. Then

$$(a_1 \pm a_2)s_1 s_2 = (a_1 s_1)s_2 \pm (a_2 s_2)s_1 = s(r_1 s_2 \pm r_2 s_1),$$

which shows that $a_1 \pm a_2 \in A$.

Comment. It is worth noting that, in the above exercise, an hypothesis somewhat weaker than the commutativity of S would have worked. Indeed, if we assume that the set A above contains S, then A is a subring of R (containing both S and its centralizer in R). The first part of the proof remains unchanged. To check that $a_1,\ a_2 \in A \Rightarrow a_1 \pm a_2 \in A$, take any $s \in S$, and fix an equation $a_1 s_1 = s r_1$ as before. Now $a_2 \in A$ and $s_1 \in S \subseteq A$ imply that $a_2 s_1 \in A$, so there exists an equation $(a_2 s_1) s_0 = s r_0$ where $s_0 \in S$ and $r_0 \in R$. Then

$$(a_1 \pm a_2) s_1 s_0 = (a_1 s_1) s_0 \pm (a_2 s_1) s_0 = s(r_1 s_0 \pm r_0),$$

which shows that $a_1 \pm a_2 \in A$.

Ex. 10.4. Let $x, y \in R$ be such that $xy = 1 \neq yx$, and let S be the multiplicative set $\{x^n : n \geq 0\}$. Show that

(1) S is left reversible but not right reversible;
(2) S is right permutable; and
(3) If R is generated over a central subring k by x and y, then S is also left permutable.

Solution. (1) If $ax^n = 0$ for some $n \geq 0$, then $0 = ax^n y^n = a$, so S is left reversible. On the other hand, for the element $z = yx - 1 \neq 0$, we have $xz = xyx - x = 0$, but as we saw above, $zx^n \neq 0$ for all $n \geq 0$. Therefore, S is not right reversible.

(2) Each element $x^n \in S$ has a right inverse y^n, so by (a) of Exercise 2, S is right permutable.

(3) Assume that R is generated over a central subring k by x and y. By the analogue of Exercise 3 for the other side, the set

$$A' = \{a \in R :\ Sa \cap Rs \neq \emptyset \quad \text{for every} \ \ s \in S\}$$

is a subring of R containing k and x. For any $s \in S$, the equation $(sx)y = s(xy) = 1 \cdot s$ shows that A' also contains y (since $sx \in S$). Therefore, $A' = R$, and so S is left permutable, as desired.

Comment. Without some conditions on R, the multiplicative set S is in general not left permutable. For instance, if $R = \mathbb{Z}\langle x, y, z\rangle$ with a single defining relation $xy = 1$, then one can show that $Sz \cap Rx = \emptyset$, i.e. $z \notin A'$ in the notation above. The next exercise, however, gives another example where S happens to be left permutable.

Ex. 10.5. Let V be a right vector space over a field k, with basis $\{e_1, e_2, \ldots\}$. Let $R = \text{End}(V_k)$, and let $x, y \in R$ be defined by $y(e_i) = e_{i+1}$ $(i \geq 1)$, and $x(e_1) = 0, \quad x(e_i) = e_{i-1}$ $(i \geq 2)$. Show that $S = \{x^n : n \geq 0\} \subseteq R$ is a left denominator set, but not a right denominator set.

Solution. Here we have a typical situation of a ring R with two elements x, y such that $xy = 1 \neq yx$. By (1) of Exercise 4, S is left reversible but not right reversible. In particular, S is not a right denominator set. To prove that S is a left denominator set, it remains to show that $S \subseteq R$ is left permutable (Note that (3) of Exercise 4 does not apply here since $R \neq k\langle x, y \rangle$.)

For $x^n \in S$ $(n \geq 0)$ and any $a \in R$, we would like to show that $Sa \cap Rx^n \neq \emptyset$. By a straightforward calculation, we see that $(1 - y^n x^n)(e_i)$ is e_i if $1 \leq i \leq n$, and 0 if $i > n$. Therefore, $(1 - y^n x^n)V = \sum_{i=1}^{n} e_i k$, and so

$$a(1 - y^n x^n)V \subseteq \sum\nolimits_{i=1}^{m} e_i k$$

for some integer m. Applying x^m to this inclusion relation, we see that $x^m a(1 - y^n x^n) = 0$, and so $x^m a = (x^m a y^n) x^n$, as desired.

Ex. 10.6. Let R be any ring satisfying ACC on right annihilators of elements. If a multiplicative set $S \subseteq R$ is right permutable, show that it is necessarily right reversible. Conclude from Exercise 4 that the ring R must be Dedekind-finite.

Solution. Suppose $sa = 0$, where $s \in S$ and $a \in R$. We would like to show that $as' = 0$ for some $s' \in S$. Pick an integer n such that

$$\text{ann}_r(s^n) = \text{ann}_r(s^k) \quad \text{for all} \quad k \geq n.$$

Since S is right permutable, we have $as' = s^n b$ for some $s' \in S$ and some $b \in R$. Then $s^{n+1} b = (sa)s' = 0$, so $b \in \text{ann}_r(s^{n+1})$. But then $b \in \text{ann}_r(s^n)$, so $as' = s^n b = 0$, as desired. It follows that there cannot exist $x, y \in R$ with $xy = 1 \neq yx$, for otherwise Exercise 4 would imply that $S := \{x^n : n \geq 0\}$ is right permutable but not right reversible.

Comment. The last conclusion of the exercise is usually obtained from Jacobson's construction of an infinite set of nonzero orthogonal idempotents in a non Dedekind-finite ring: see *LMR*-(6.60).

Ex. 10.7. Let R be a right Ore domain, with division ring of right fractions K. Show that, up to a unique R-isomorphism, K is the only division hull of R. (This exercise is part of *LMR*-(10.21).)

Solution. Recall (from *LMR*-§9) that a *division hull* of a diomain R is a division ring D with an inclusion map $i : R \to D$, such that D is generated as a division ring by $i(R)$ (i.e., there is no division ring D_0 such that $i(R) \subseteq D_0 \subsetneq D$). For the given $R \subseteq K$ in this exercise, certainly K is a division hull of R since every element of K has the form rs^{-1} where $r \in R$ and $s \in R \backslash \{0\}$. Now consider any division hull $i : R \to D$ (as above). Since the inclusion $R \subseteq K$ is a universal $(R\backslash\{0\})$-inverting homomorphism, i extends to a unique ring homomorphism $f : K \to D$, given by $f(rs^{-1}) = i(r)i(s)^{-1}$. This homomorphism is necessarily injective, since K is a division ring. Its image is a division ring between $i(R)$ and D, so it must be D. Thus, f is an R-isomorphism from K to D.

Ex. 10.8. Let G be the free group generated by a set X with $|X| \geq 2$. Show that the domain $R = \mathbb{Z}G$ is not right (or left) Ore.

Solution. We use the same idea for proving that the free algebra $\mathbb{Z}\langle X\rangle$ is not right (or left) Ore. If X contains two distinct elements x and y, then for the domain $R = \mathbb{Z}G$ in question, we have

$$xR \cap yR = 0 = Rx \cap Ry,$$

so R is neither right nor left Ore.

Ex. 10.9. Let R be the ring $\mathbb{Z}\langle x, y\rangle$ defined by the relations $y^2 = yx = 0$. Show that R is left Ore but not right Ore.

Solution. As in *FC*-(1.26), we express R in the form $\mathbb{Z}[x] \oplus \mathbb{Z}[x]y$. A direct calculation shows that the set of regular elements in R is given by

$$S = \{f(x) + g(x)y : \ f, g \in \mathbb{Z}[x], \ \ f(0) \neq 0\}.$$

Our job is to show that this multiplicative set S is left permutable, but not right permutable. For the latter, it suffices to show that $yS \cap (1 + x)R = \emptyset$ (noting that $1 + x \in S$). To see this, assume instead that

$$y(f(x) + g(x)y) = (1 + x)(h(x) + k(x)y),$$

where $f,\ g,\ h,\ k \in \mathbb{Z}[x]$, with $f(0) \neq 0$. Comparing the y-parts of the two sides, we get $f(0) = (1 + x)k(x)$, a contradiction.

To prove that S is left permutable, we take any $s = f(x) + g(x)y \in S$ and any $a = h(x) + k(x)y \in R$, and proceed to show that $Sa \cap Rs \neq \emptyset$. Let $q = fk - gh \in \mathbb{Z}[x]$. Then

$$\begin{aligned}(f(0)h + qy)\, s &= f(0)hf + [\, f(0)hg + f(0)q\,]\, y \\ &= f(0)hf + f(0)fky \\ &= f(0)f(h + ky) \in Sa \cap Rs,\end{aligned}$$

since $f(0)f$ (with constant term $f(0)^2 \neq 0$) belongs to S. This checks that R is a left Ore (but not right Ore) ring.

Comment. In *FC*-(1.26), it is shown that R is left noetherian but not right noetherian. The fact that R is left noetherian alone, however, does not guarantee that R is left Ore (unless R is a domain): see *LMR*-(12.27).

The choice of $\mathbb{Z}$ as coefficient ring in this problem is purely for convenience. If we define R to be $k\langle x, y\rangle$ defined by the relations $y^2 = yx = 0$ over a nonzero commutative ring k, the same arguments (with slight modifications) can be used to show that R is left Ore but not right Ore.

Ex. 10.10. Let $R = k[x; \sigma]$ where σ is an automorphism of the ring k. Show that $S = \{x^n : \ n \geq 0\}$ is a right and left denominator set of R, and that RS^{-1} and $S^{-1}R$ are both isomorphic (over R) to the ring of skew Laurent polynomials $k[x, x^{-1}; \sigma]$ (as defined in *FC*-(1.81)).

Solution. Here, S consists of regular elements of R. To show that S is a right denominator set with RS^{-1} isomorphic to $T := k[x,\ x^{-1};\ \sigma]$, it is sufficient to check the following (see the discussion at the beginning of *LMR*-§10):

(1) *The inclusion map $R \to T$ is S-inverting;*
(2) *Any element in T is of the form as^{-1} where $a \in R$ and $s \in S$.*

The first statement is clear, since $x^{-n} \in T$ for any $n \geq 0$. For the second, take any $f = \sum_{i=-n}^{n} a_i x^i \in T$. Factoring out x^{-n} on the right, we get $f = as^{-1}$ with $s = x^n \in S$ and $a = \sum_{i=-n}^{n} a_i x^{n+i} \in R$.

To see that S is a left denominator set with $S^{-1}R \cong T$, we need only prove additionally:

(3) *Any element in T is of the form $s^{-1}a$ where $a \in R$ and $s \in S$.*

This follows as before, by writing any $f \in T$ in the form $\sum_{i=-n}^{n} x^i b_i$, and factoring out x^{-n} on the left.

Comment. Of course, it is also easy to check directly that S is right (resp. left) permutable. In fact, if $a = \sum_{i \geq 0} a_i x^i \in R$ and $s = x^n \in S$, then

$$x^n \sum \sigma^{-n}(a_i) x^i = \sum a_i x^{n+i} = ax^n \in aS \cap sR.$$

Left permutability can be checked similarly.

Ex. 10.11. Let k in Exercise 10 be the polynomial ring $\mathbb{Q}[\{t_i : i \in \mathbb{Z}\}]$, with σ defined by $\sigma(t_i) = t_{i+1}$ for all $i \in \mathbb{Z}$, and let R, S be as above. Show that $\mathfrak{A} = t_1 R + t_2 R + \cdots$ is an ideal in R, but the extension $\mathfrak{A}^e := \mathfrak{A} \cdot RS^{-1}$ is not an ideal in $RS^{-1} = k[x, x^{-1}; \sigma]$.

Solution. For any $\sum_{i=0}^{n} a_i x^i \in R$ and $j \geq 1$, we have

$$\left(\sum_{i=0}^{n} a_i x^i\right) t_j = \sum_{i=0}^{n} a_i \sigma^i(t_j) x^i = \sum_{i=0}^{n} a_i t_{i+j}\, x^i \in \mathfrak{A},$$

so $\mathfrak{A}$ is an ideal in R. The extension $\mathfrak{A}^e = \mathfrak{A} \cdot RS^{-1}$ is a right ideal in RS^{-1}, given by $\{as^{-1} :\ a \in \mathfrak{A},\ s \in S\}$ (see *LMR*-(10.32)(2)). Now $t_1 \in \mathfrak{A} \subseteq \mathfrak{A}^e$, but

$$x^{-1} t_1 = \sigma^{-1}(t_1) x = t_0\, x \notin \mathfrak{A}^e,$$

so $\mathfrak{A}^e$ is *not* an ideal in RS^{-1}.

Ex. 10.12. Let R be the first Weyl algebra $A_1(k)$ over the field k (see *FC*-(1.3)(c)), identified with $k[y][x; \delta]$ where δ denotes formal differentiation on $k(y)$. Show that $S = k[y] \backslash \{0\}$ is a right and left denominator set of R, and that RS^{-1} and $S^{-1}R$ are both isomorphic (over R) to $T := k(y)[x; \delta]$.

Solution. Recall that elements of R are of the form $\sum_{i=0}^{n} a_i(y)x^i$ (where $a_i(y) \in k[y]$), with the commutation rule

$$xy = yx + \delta(y) = yx + 1.$$

Here, S consists of regular elements of R (in fact, R is a domain). As in the solution to Exercise 10, all conclusions will be proved once we show the following:

(1) *The inclusion map* $R \to T$ *is* S*-inverting*;
(2) *Any element in* T *is of the form* $s^{-1}a$ $(a \in R,\ s \in S)$;
(3) *Any element in* T *is of the form* as^{-1} $\quad (a \in R,\ s \in S)$.

(1) is clear, since $k(y) \subseteq T$.

(2) Take any $f = \sum_{i=0}^{n} f_i(y)x^i \in T$, where $f_i(y) \in k(y)$. Writing $f_i(y) = a_i(y)/s$ with a common denominator $0 \neq s \in k[y]$, we have $f = s^{-1}a$ where $a = \sum_i a_i(y)x^i \in R$.

(3) is proved similarly, since any $f \in T$ can also be written in the form

$$\sum_{i=0}^{n} x^i g_i(y) \quad (g_i(y) \in k(y)).$$

Ex. 10.13. Let R be a right Ore domain with division ring of right fractions K. Show that the center of K is given by

$$\{as^{-1} : a \in R,\ s \in R\backslash\{0\},\ ars = sra \quad \text{for all } r \in R\}.$$

Solution. Suppose as^{-1} is in the center of K, where $a \in R,\ s \in R\backslash\{0\}$. Then, for any $r \in R$, we have $as^{-1}(sr) = (sr)as^{-1}$, which simplifies to $ars = sra$. Conversely, suppose $\alpha = as^{-1}$ where a and s are such that $ars = sra$ for all $r \in R$. Letting $r = 1$, we have $as = sa$. To see that α is central in K, it suffices to show that α commutes with any $r \in R\backslash\{0\}$ (for then α also commutes with r^{-1}, and therefore with any $rt^{-1} \in K$). As we saw above, the condition imposed on a and s amounts to $(sr)\alpha = \alpha(sr)$ for any $r \in R$. Since $\alpha = as^{-1}$ commutes with s, this reduces to $sr\alpha = s\alpha r$. Cancellation of s now gives $r\alpha = \alpha r$, as desired.

Comment. The domain assumption in this exercise is not really essential. In general, if S is a right denominator set in a ring R, a similar argument can be used to show that the center of RS^{-1} consists of as^{-1} $(a \in R,\ s \in S)$ such that for any $r \in R$, there exists $t \in S$ such that $arst = srat$.

For an intriguing generalization of this exercise to prime rings due to W. S. Martindale, see Ex. 14.7.

Ex. 10.14. For R, K as in Exercise 13, show that any ring T between R and K is a right Ore domain.

Solution. Since K is a division ring, certainly any nonzero element in T is a unit in K. Also, any element in K has the form rs^{-1} where $r \in R \subseteq T$

and $0 \neq s \in R \subseteq T$. Therefore, T is a right order in K, and it follows from *LMR*-(10.21) that T is a right Ore domain.

Ex. 10.15. Let S be a right denominator set in a ring R and let $\varphi : R \to Q$ be the natural map, where $Q = RS^{-1}$.

1. If Q_R is a noetherian R-module, show that $\varphi(R) = Q$.
2. If ${}_RQ$ is a f.g. R-module, show that $\varphi(R) = Q$.

Solution. (1) Let $s \in S$ and consider the ascending chain of R-submodules $s^{-1}\varphi(R) \subseteq s^{-2}\varphi(R) \subseteq \cdots$ in Q_R. Since Q_R is noetherian, there exists an integer n such that $s^{-n}\varphi(R) = s^{-(n+1)}\varphi(R)$. Left multiplying by s^{n+1}, we get $s \cdot \varphi(R) = \varphi(R)$. In particular, $s \cdot \varphi(r) = 1 \in Q$ for some $r \in R$. Therefore, $s^{-1} \in \varphi(R)$, from which it follows that $Q = RS^{-1} = \varphi(R)$.

(2) By taking a common denominator for a finite set of generators for ${}_RQ$, we can write $Q = \sum_{i=1}^{n} \varphi(R) \cdot a_i t^{-1}$, where $a_i \in R$ and $t \in S$. Then $t^{-2} = \sum_{i=1}^{n} \varphi(r_i a_i) t^{-1}$ for suitable $r_i \in R$. Right multiplying by t, we get $t^{-1} = \sum_{i=1}^{n} \varphi(r_i a_i) \in \varphi(R)$, and so again $Q = \varphi(R)$.

Ex. 10.16. Let $f : R \to R'$ be a homomorphism between right Ore rings. Does f induce a ring homomorphism $Q_{c\ell}^r(R) \to Q_{c\ell}^r(R')$?

Solution. Let S (resp. S') be the multiplicative set of regular elements in R (resp. R'). If we are given that, under the homomorphism f, $f(S) \subseteq S'$, then indeed there is an induced ring homomorphism f_* from $Q_{c\ell}^r(R) = RS^{-1}$ to $Q_{c\ell}^r(R') = R'S'^{-1}$(cf. solution to Exercise 1), defined by

$$f_*(rs^{-1}) = f(r)f(s)^{-1} \quad (r \in R,\ s \in S).$$

However, $f(S)$ need not be in S'. Therefore, in general, there is no natural way to define an induced homomorphism from $Q_{c\ell}^r(R)$ to $Q_{c\ell}^r(R')$.

For an explicit example, take $R = \mathbb{Z}$, $R' = \mathbb{Z}/n\mathbb{Z}$ (where $n > 0$), and let f be the projection map from R to R'. Here, $n \in S$, but $f(n) = 0 \notin S'$. In this example, there is simply no homomorphism from $Q_{c\ell}^r(R) = \mathbb{Q}$ to $Q_{c\ell}^r(R') = \mathbb{Z}/n\mathbb{Z}$.

Ex. 10.17. Let S be a right denominator set in a ring R. Show that the right ring of fractions $Q = RS^{-1}$ is flat as a left R-module.

Solution. Our solution is based on the use of the Equational Criterion for Flatness in *LMR*-§4. By (the left analogue of) *LMR*-(4.24)(2), the flatness of ${}_RQ$ will follow if we can show that any R-linear relation $\sum_{i=1}^{n} r_i x_i = 0$ $(r_i \in R,\ x_i \in Q)$ is a "consequence of linear relations in R." To be more precise, this means showing that there is a formal factorization.

$$(*) \qquad \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = (b_{ij}) \begin{pmatrix} y_1 \\ \vdots \\ y_m \end{pmatrix}$$

such that $(r_1, \ldots, r_n)(b_{ij}) = 0$ over R, where $y_j \in Q$ and (b_{ij}) is an $n \times m$ matrix over R. To show this, take a "common denominator" $s \in S$ to express x_i as $a_i s^{-1}$ $(a_i \in R)$. Then

$$0 = \sum r_i x_i = \left(\sum r_i a_i\right) s^{-1} \in Q$$

implies that $\sum r_i a_i t = 0$ for some $t \in S$. Setting $y = (st)^{-1} \in Q$, we have $x_i = a_i s^{-1} = (a_i t) y \in Q$. The factorization

$$\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} a_1 t & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ a_n t & 0 & \cdots & 0 \end{pmatrix} \begin{pmatrix} y \\ \vdots \\ y \end{pmatrix}$$

clearly gives what we want, since the $n \times n$ matrix above is left annihilated by $(r_1, \ldots, r_n)$ over R.

Comment. This exercise places at our disposal the most powerful property of "localization"; namely, $- \otimes_R Q$ is an *exact* functor from $\mathfrak{M}_R$ (the category of right R-modules) to $\mathfrak{M}_Q$ (the category of right Q-modules). Another proof for this exactness property, using a somewhat different view of localization, is given in the next exercise.

In the special case where the right denominator set S consists of non 0-divisors of R, it is also possible to get the flatness of ${}_RQ$ by expressing ${}_RQ$ explicitly as a direct limit of free (and hence flat) R-modules. For s, $w \in S$, define $s \leq w$ by the relation $w \in sR$. If $w = sr$ where $r \in R$, then $s^{-1} = rw^{-1} \in Q$ implies that $Rs^{-1} \subseteq Rw^{-1}$ (in Q). It is easy to see that $(S, \leq)$ is a directed set. Indeed, let s, $t \in S$. Since S is right permutable, there exists an element $w \in sR \cap tS$. Clearly, $w \in S$, with $s, t \leq w$. The fact that S consists of non 0-divisors implies that $R \cong Rs^{-1} (\subseteq Q)$ as left R-modules. Thus,

$${}_RQ = \bigcup\nolimits_{s \in S} Rs^{-1} = \varinjlim Rs^{-1}$$

is flat by *LMR*-(4.4).

Ex. 10.18. Let $S \subseteq R$ and Q be as in Exercise 17 and let M be a right R-module.

(1) Generalizing the procedure for forming $RS^{-1} = Q$, show that there exists a "localization" MS^{-1} that is a right Q-module with elements of the form $m/s = ms^{-1} \quad (m \in M,\ s \in S)$.
(2) Show that the kernel of the natural map $M \to MS^{-1}$ is given by the set of "S-torsion elements", defined by

$$M_0 := \{m \in M : \ ms = 0 \quad \text{for some } s \in S\}.$$

(3) Show that "localization" is an exact functor from $\mathfrak{M}_R$ to $\mathfrak{M}_Q$.
(4) Show that $MS^{-1} \cong M \otimes_R Q$ in $\mathfrak{M}_Q$. Using this and (3), give another proof for the fact that ${}_RQ$ is flat.

(5) If $M_0 = 0$ in (2), show that

$$\text{u.dim}\, M_R = \text{u.dim}\left(MS^{-1}\right)_R = \text{u.dim}\left(MS^{-1}\right)_Q.$$

Solution. (1) We first define an equivalence relation "$\sim$" on $M \times S$, by taking $(m, s) \sim (m', s')$ iff there exist $b, b' \in R$ such that $sb = s'b' \in S$ and $mb = m'b' \in M$. (Note that although $sb = s'b' \in S$, b and b' need not belong to S.) The idea here is, again, that after "blowing up" s and s' to the "common denominator" $sb = s'b' \in S$, we want the numerators mb and $m'b'$ to be the same in M. After verifying that "$\sim$" is an equivalence relation, we introduce the shorthand notation $m/s = ms^{-1}$ for the equivalence class of (m, s). Using the right denominator property on S, we can then define addition and (right) scalar multiplication by Q on

$$MS^{-1} = \{ms^{-1} : \ m \in M, \ s \in S\}$$

in much the same way as we defined addition and multiplication on $Q = RS^{-1}$ in *LMR*-§10. The verification that MS^{-1} is a well-defined right Q-module is long and tedious, but basically routine. In order to save space, we omit it altogether.

(2) If $ms = 0 \in M$ where $s \in S$, then certainly $m/1 = 0 \in MS^{-1}$ since s is a unit in Q. Conversely, suppose $m/1 = 0/1$. Then by definition there exist $b, b' \in R$ such that $mb = 0b' = 0 \in M$ and $b = b' \in S$, so $m \in M_0$. This shows that $\ker\left(M \to MS^{-1}\right) = M_0$.

(3) Any R-morphism $i : \ M \to N$ clearly induces a Q-morphism $i_* : MS^{-1} \to NS^{-1}$, so we have a covariant functor $\mathfrak{M}_R \to \mathfrak{M}_Q$ by "localization". Consider any exact sequence

$$0 \to M \xrightarrow{i} N \xrightarrow{j} P \to 0 \quad \text{in } \mathfrak{M}_R,$$

and the induced zero sequence

$$0 \to MS^{-1} \xrightarrow{i_*} NS^{-1} \xrightarrow{j_*} PS^{-1} \to 0 \quad \text{in } \mathfrak{M}_Q.$$

Clearly, j_* is surjective. To show that $\ker(j_*) \subseteq \text{im}(i_*)$, take any $ns^{-1} \in NS^{-1}$ such that $0 = j_*\left(ns^{-1}\right) = j(n)\, s^{-1}$. Then $j(n)\, t = 0 \in P$ for some $t \in S$. Thus $nt = i(m)$ for some $m \in M$ and so

$$ns^{-1} = (nt)\,(st)^{-1} = i(m)\,(st)^{-1} \in \text{im}(i_*)\,.$$

Finally, changing notations, take $ms^{-1} \in \ker(i_*)$. Then $i(m)\, t = 0 \in N$ for some $t \in S$. This yields $mt = 0 \in M$, and so

$$ms^{-1} = (mt)\,(st)^{-1} = 0 \in MS^{-1}.$$

This completes the proof for the exactness of the functor $\mathfrak{M}_R \to \mathfrak{M}_Q$.

(4) We can define $\varphi : MS^{-1} \to M \otimes_R Q$ by $\varphi\left(ms^{-1}\right) = m \otimes s^{-1}$, and $\psi : M \otimes_R Q \to MS^{-1}$ by $\psi\left(m \otimes rs^{-1}\right) = (mr)\, s^{-1}$. It is easy to verify

that these are mutually inverse right Q-module homomorphisms, so we have a natural isomorphism $MS^{-1} \cong M \otimes_R Q$ in $\mathfrak{M}_Q$. From now on, it will be convenient to "identify" these two Q-modules. It follows from (3) that $\Box \otimes_R Q$ is an exact functor, so now we have a second proof for the fact that Q is a flat left R-module.

(5) *Assume now that* $M_0 = 0$. This enables us to view M as an R-submodule of MS^{-1}. If we have a direct sum of nonzero Q-submodules (or even R-submodules), $T_1 \oplus \cdots \oplus T_n \subseteq MS^{-1}$, then $M_i = T_i \cap M \neq 0$, and we have a direct sum of R-submodules $M_1 \oplus \cdots \oplus M_n \subseteq M$. Conversely, suppose $M_1 \oplus \cdots \oplus M_n$ is a direct sum of nonzero R-submodules in M. We claim that the sum

$$M_1 S^{-1} + \cdots + M_n S^{-1} \subseteq MS^{-1}$$

is also direct. In fact, suppose $x_1 + \cdots + x_n = 0$, where $x_i \in M_i S^{-1}$. Taking a common denominator, we can write $x_i = m_i s^{-1}$ for some $s \in S$, and $m_i \in M_i$. Then

$$(m_1 + \cdots + m_n)\, s^{-1} = 0 \in MS^{-1}$$

implies that $m_1 + \cdots + m_n = 0 \in M$, so all $m_i = 0$ and hence all $x_i = 0$. Since the uniform dimension of a module is the supremum of integers n for which the module contains a direct sum of n nonzero submodules. (see *LMR*-(6.6)), the conclusions in (5) follow.

Ex. 10.19. For any right R-module M, let

$$t(M) = \{m \in M : \; ms = 0 \quad \text{for some regular element} \quad s \in R\}.$$

Show that R is right Ore iff, for any right R-module M, $t(M)$ is an R-submodule of M. In this case, $t(M)$ is called the *torsion submodule* of M; M is called *torsion* if $t(M) = M$, and *torsionfree* if $t(M) = 0$. Show that, in case R is right Ore, $M/t(M)$ is always torsionfree.

Solution. In the following, let S be the multiplicative set of regular elements in R. *First assume* R *is right Ore.* Then, for any M_R, $t(M)$ is just M_0 in the last exercise. Since M_0 is the kernel of the R-homomorphism $M \to MS^{-1}$, it is an R-submodule of M. (We invite the reader to give a direct proof of this from the definition of M_0, using the fact that S is right permutable.) To see that $M/t(M)$ is torsionfree, let $m_1 \in M$ be such that $m_1 s_1 \in t(M)$, where $s_1 \in S$. Then $(m_1 s_1)\, s_2 = 0$ for some $s_2 \in S$. This implies that $m_1 \in t(M)$, and so $\overline{m}_1 = 0 \in M/t(M)$.

Now assume that $t(M)$ is a submodule in *every* right R-module M. We would like to prove that, for any $a \in R$ and $s \in S$, $aS \cap sR \neq \emptyset$. Consider the right R-module $M = R/sR$. From $\overline{1} \cdot s = \overline{s} = 0 \in M$, we see that $\overline{1} \in t(M)$. Since $t(M)$ is an R-module, we must have $t(M) = M$. In particular, there exists $t \in S$ such that $0 = \bar{a}\, t = \overline{at}$, so $at \in sR$, as desired.

Ex. 10.19A. Let x, y be non 0-divisors in a ring R such that $Rx \cap Ry = 0$, and let $S \supseteq R$ be an overring which contains elements s, t such that $sx = ty \neq 0$. For the right R-submodule $M = sR + tR$ of S, show that $\mathrm{Hom}_R(M, R) = 0$.

Solution. For any $f \in \mathrm{Hom}_R(M, R)$, let $a = f(s)$ and $b = f(t)$ in R. Then

$$ax = f(s)x = f(sx) = f(ty) = f(t)y = by$$

implies that $ax = by = 0$ (as $Rx \cap Ry = 0$), and therefore $a = b = 0$. Since M_R is generated by s and t, it follows that $f \equiv 0$.

Ex. 10.19B. Let R be a right Ore domain. Show that R is left Ore iff every f.g. torsionfree right R-module is embeddable in a free R-module.

Solution. Let K be the division ring of right fractions of R. *First suppose R is left Ore.* It is easy to see that K is also the division ring of left fractions of R. Let M_R be any f.g. torsionfree R-module, say $M = \sum_{i=1}^{n} m_i R$. By Ex. 18 above, we have an embedding $M \to M \otimes_R K$, which we may view as an inclusion map. Since $M \otimes_R K$ is K-spanned by

$$\{m_i = m_i \otimes 1 : \ 1 \le i \le n\},$$

it has a finite K-basis, say $v_1, \ldots, v_p$. Write $m_i = \sum_j v_j k_{ij}$, where $k_{ij} \in K$. As R is left Ore, we can express k_{ij} as $a^{-1} r_{ij}$ with a "common denominator" $a \in R \backslash \{0\}$, and with suitable $r_{ij} \in R$. Then

$$m_i = \sum\nolimits_j \left(v_j a^{-1}\right) r_{ij} \in \sum\nolimits_j \left(v_j a^{-1}\right) R \quad \text{(for all } i\text{)}.$$

Now the elements $u_j := v_j a^{-1}$ are right linearly independent over R, so M embeds into the free R-module $\bigoplus_{j=1}^{n} u_j R$.

Conversely, if R is not left Ore, take nonzero elements $x, y \in R$ such that $Rx \cap Ry = 0$. Consider the *torsionfree* R-submodule

$$M_R = x^{-1}R + y^{-1}R \subseteq K.$$

Applying Ex. 19A with $s = x^{-1}$ and $t = y^{-1}$ in K, we see that $\mathrm{Hom}_R(M, R) = 0$. In particular, M_R cannot be embedded in any free R-module.

Comment. This exercise is Proposition 4.1 in E. Gentile's paper, "On rings with one-sided field of quotients," Proc. A.M.S. **11**(1960), 380–384. Our solution follows closely the proof given by Gentile, except that we have generalized part of Gentile's argument into Ex. 19A.

Ex. 10.20. Let R be a subring of a division ring D. Show that ${}_RD$ is a flat left R-module iff R is a right Ore domain.

Solution. First suppose R is right Ore. Then we have $R \subseteq K \subseteq D$, where K is the division ring of right fractions of R. Then ${}_RK$ is flat by Exercise 17, and ${}_K D$ is flat since D is a K-vector space. From this, it follows that ${}_R D$ is also flat.

Now suppose the domain R is *not* right Ore. Then there exist nonzero elements $a, b \in R$ with $aR \cap bR = 0$. Since $aR \oplus bR \cong \left(R^2\right)_R$, we can construct an R-monomorphism $i : \left(R^2\right)_R \to R_R$. The functor $\square \otimes_R D$ cannot be exact, for otherwise i would induce a D-monomorphism $i_* : \left(D^2\right)_D \to D_D$, which is impossible over a division ring. Therefore, the module ${}_R D$ is not flat.

Ex. 10.21. Show that a *domain* R satisfies the (right) strong rank condition ("for any n, any set of $n+1$ elements in $(R^n)_R$ is linearly dependent": see *LMR*-(1.20)(2)) iff R is right Ore.

Solution. First assume that R is not right Ore. As in the last exercise, there exists $aR \oplus bR \subseteq R$, where $a, b \neq 0$. Then a, b are linearly independent in R_R, so R does not satisfy the strong rank condition.

Now assume R is right Ore, and let K be its division ring of right fractions. If some $(R^n)_R$ contains $n+1$ linearly independent elements, we would have an R-monomorphism $\left(R^{n+1}\right)_R \to (R^n)_R$. Localizing to K would yield a K-monomorphism $\left(K^{n+1}\right)_K \to (K^n)_K$, which is impossible. Therefore, R must satisfy the strong rank condition.

Comment. More generally, it is shown in *LMR*-(1.37) that *any nonzero ring R with* $\text{u.dim}\, R_R < \infty$ *satisfies the strong rank condition.* (For any right Ore domain R, we have u.dim $R_R = 1$.)

The "if" part of this exercise is an interesting conclusion since it means that, to guarantee that we can solve nontrivially any m homogeneous linear equations in $n > m$ unknowns over the domain R, it suffices to guarantee that we can always solve *one* such equation in *two* unknowns!

Ex. 10.22. Use Exercise 21 to give another proof for the fact that, if R is a right Ore domain, so is $R[x]$.

Solution. Assume that R is a right Ore domain, and let $T = R[x]$. By Exercise 21, R satisfies the strong rank condition. By Exercise 1.22, T also satisfies the strong rank condition. Since T is a domain, another application of Exercise 21 shows that T is a right Ore domain.

Ex. 10.23. Show that, over a right Ore domain R, any f.g. flat right R-module M is projective.

Solution. By Exercise 21, R satisfies the strong rank condition. Now by *LMR*-(4.38), over any domain with such a property, any f.g. flat right module is projective.

Ex. 10.23A. For any right Ore domain R, show that a torsionfree right R-module M is divisible iff it is injective.

Solution. We know already (from *LMR*-(3.17)′) that the "if" part is true for M_R over any ring R. Now assume R is a right Ore domain, and let M_R be torsionfree and divisible. To check that M is injective, we apply Baer's Criterion. Let $\mathfrak{A}$ be any nonzero right ideal, and let $f \in \mathrm{Hom}_R(\mathfrak{A}, M)$. For any nonzero $a \in \mathfrak{A}$, there exists a unique element $m_a \in M$ such that $f(a) = m_a a$. *We claim that all elements m_a $(a \in \mathfrak{A} \setminus \{0\})$ are equal.* Indeed, if $a,\ b \in \mathfrak{A} \setminus \{0\}$, we have $ar = bs$ for some nonzero $r, s \in R$. Then

$$\begin{aligned} f(ar) = f(bs) &\implies f(a)\,r = f(b)\,s \\ &\implies m_a ar = m_b bs \\ &\implies m_a = m_b, \end{aligned}$$

since M_R is torsionfree. If we now write m for m_a (for all nonzero $a \in \mathfrak{A}$), then $f(a) = ma$ for all $a \in \mathfrak{A}$, so f extends to the R-homomorphism $R \to M$ given by $1 \mapsto m$.

Comment. This exercise is Proposition 1.1 in E. Gentile's paper referenced in the *Comment* on Ex. 19B above. Note that, in the case where R is a commutative domain, this result has been proved in *LMR*-(3.25).

Ex. 10.23B. Show that a domain R is right Ore iff there exists a nonzero torsionfree injective right R-module.

Solution. First assume R is right Ore. Then the division ring K of right fractions of R is torsionfree and divisible as a right R-module. By the last exercise, K_R is then injective.

Conversely, suppose $M \neq 0$ is a torsionfree injective right R-module. We fix a nonzero element $m \in M$. If R is not right Ore, we have $aR \cap bR = 0$ for some nonzero $a, b \in R$. The R-homomorphism $f: \ aR \oplus bR \to M$ sending a, b to m can be extended to $g: \ R_R \to M_R$ (by the injectivity of M). For $m' = g(1) \in M$, we have

$$m = g(a) = g(1 \cdot a) = g(1)\,a = m'a,$$

and similarly, $m = m'b$. These equations imply that $m' \neq 0$ and $m'(a - b) = 0$, so the torsionfreeness of M forces $a = b$, which is not the case. Thus, R must be right Ore.

Ex. 10.23C. Let R be a right Ore domain with division ring of right fractions K. For any nonzero right ideal $\mathfrak{A} \subseteq R$, $\mathrm{Hom}_R(\mathfrak{A}, K_R)$ is a left K-vector space via the left K-action on K. As in *LMR*-(2.15), we have a K-morphism

$$\lambda: \ K \to \mathrm{Hom}_R(\mathfrak{A}, K) \quad \text{defined by} \quad \lambda(k)(a) = ka$$

for $k \in K$ and $a \in \mathfrak{A}$. Show that λ is an isomorphism.

Solution. Clearly λ is one-one, so it suffices to show that it is *onto*. As in the last exercise, we know that K_R is injective. Therefore, given any $f \in \mathrm{Hom}_R(\mathfrak{A}, K)$, there exists $g \in \mathrm{Hom}_R(R, K)$ extending f. Let $k =$

$g(1) \in K$. Then, for any $a \in \mathfrak{A}$,

$$f(a) = g(1 \cdot a) = g(1)a = ka = \lambda(k)(a).$$

Therefore, $f = \lambda(k)$, as desired.

Comment. The main idea of the proof here is basically the same as that in the proof of *LMR*-(2.15) in the case where R is a commutative domain. It is interesting to see how, in our new arguments, the commutativity of R is replaced by the right Ore property.

Ex. 10.24. Let $R \subseteq L$ be domains. Show that L is injective as a right R-module iff R is right Ore and L contains the division ring of right fractions K of R.

Solution. First suppose R is right Ore, with $R \subseteq K \subseteq L$. It is easy to check that L_R is torsionfree and divisible, so by Exer. 23A, it is injective.

Conversely, suppose L_R is injective. Since L_R is torsionfree, Exer. 23B implies that R is right Ore. If we can show that every nonzero $a \in R$ has an inverse in L, then we'll have $R \subseteq K \subseteq L$, as desired. The R-homomorphism $f: aR \to L$ defined by $f(a) = 1$ extends to some $g: R \to L$, by the injectivity of L_R. Now $1 = f(a) = g(1)a$ shows that a^{-1} exists in L.

Ex. 10.25. Show that a domain R is right Ore iff it has a nonzero right ideal of finite uniform dimension.

Solution. The "only if" part is clear since a right Ore domain has right uniform dimension 1. Conversely, assume that R has a nonzero right ideal C with $\text{u.dim}(C_R) < \infty$. Then C contains a right ideal D with $\text{u.dim}(D_R) = 1$. Fix a nonzero element $d \in D$. Then for any $a, b \in R\backslash\{0\}$, the two right ideals daR and dbR in D must intersect nontrivially, i.e. $dar = dbs \neq 0$ for suitable $r, s \in R$. Cancelling d, we get $ar = bs \neq 0$ so $aR \cap bR \neq 0$. This proves that R is right Ore.

Comment. A somewhat different way to formulate the proof of the "if" part is the following. Fix $0 \neq d \in C$. Then

$$R_R \cong dR_R \subseteq C_R \implies \text{u.dim}(R_R) \leq \text{u.dim}(C_R) < \infty,$$

so by *LMR*-(10.22), R is a right Ore domain. In comparison, the proof given in the solution above is a bit more straightforward.

Ex. 10.26. Let R be a domain. Show that R is a PRID iff R is right Ore and all right ideals are free, iff R is right noetherian and all right ideals are free.

Solution. If R is a PRID, then all nonzero right ideals are isomorphic to R_R (and therefore free), and certainly R is right noetherian.

Recall from *LMR*-(10.23) that any right noetherian domain is right Ore. Therefore, to complete the proof, it suffices to show that, if R is right Ore and all right ideals of R are free, then R is a PRID. Let $\mathfrak{A}$ be any nonzero right ideal. By assumption, there exists a free R-basis $\{a_i : i \in I\}$, so that $\mathfrak{A} = \bigoplus_{i\in I} a_iR$. If $|I| \geq 2$, then for $i \neq j$ in I, we have $a_iR \cap a_jR = 0$, in contradiction to the fact that R is a right Ore domain. Therefore, we must have $|I| = 1$, which means that $\mathfrak{A}$ is principal.

Ex. 10.27. Let $\mathfrak{A} \neq 0$ be a right ideal in a domain R, and $b \in R\backslash\{0\}$. If $\mathfrak{A} \cap bR = 0$, show that $\mathfrak{A} + bR$ is a nonprincipal right ideal. Deduce from this that any right Bézout domain is a right Ore domain.

Solution. Assume that $\mathfrak{A} + bR = cR$ for some $c \in R$. Then $b = cd$ for some $d \in R\backslash\{0\}$. We have right R-module isomorphisms

$$\mathfrak{A} \cong (\mathfrak{A} \oplus bR)/bR = cR/cdR \cong R/dR.$$

Here, $\mathfrak{A}$ is a torsionfree module, but R/dR has a nonzero element $\bar{1}$ killed by d, a contradiction.

A right Bézout domain is a domain in which every finitely generated right ideal is principal. If a, b are nonzero elements in such a domain R, the above implies that $aR \cap bR \neq 0$ (for otherwise $aR + bR$ is nonprincipal). Hence R is a right Ore domain.

Comment. A somewhat different proof for the fact that any right Bézout domain is right Ore is given in *LMR*-(10.24).

Ex. 10.28. Let $S \subseteq R$ be a right denominator set consisting of regular elements, and let $Q = RS^{-1}$.

(1) True or False: R is simple iff Q is.
(2) True or False: R is reduced iff Q is.

Solution. (1) The "if" part is false (in general). For instance, $R = \mathbb{Z}$ is not simple, but for $S = \mathbb{Z}\backslash\{0\}$, $Q = RS^{-1} = \mathbb{Q}$ is simple. The "only if" part happens to be true. Indeed, assume R is simple. Let I be a nonzero ideal in Q, say $0 \neq as^{-1} \in I$, where $a \in R$ and $s \in S$. Then I contains $(as^{-1})\, s = a \neq 0$, so $I_0 := I \cap R$ is a nonzero ideal in R. Since R is simple, $1 \in I_0$, and hence $I = Q$. This shows that Q is also simple.

(2) The "if" part is clearly true, since $R \subseteq Q$. The "only if" part turns out to be true also. Indeed, assume R is reduced. To show that Q is reduced, it suffices to check that, for any $x \in Q$, $x^2 = 0 \Rightarrow x = 0$. Write $x = as^{-1}$, where $a \in R$ and $s \in S$. Since S is right permutable, there exists an equation $at = sr$ where $r \in R$ and $t \in S$. From

$$0 = as^{-1}as^{-1} = art^{-1}s^{-1},$$

we get $ar = 0$. Thus, $(ra)^2 = 0$ and so $ra = 0$. It follows that $(asr)^2 = 0$, which in turn implies that $0 = asr = aat$. Since $t \in S$ is regular, we have now $a^2 = 0$, and hence $a = 0$, $x = as^{-1} = 0$.

Comment. The affirmative answer to (2) is certainly folklore in the subject of rings of quotients. For an explicit reference, see G. Mason's paper "Prime ideals and quotient rings of reduced rings," Math. Japonica **34**(1989), 941–956.

Ex. 10.29. Let $Q = RS^{-1}$, where S is a right denominator set in R. For any right Q-module N, show that N_Q is injective iff N_R is injective.

Solution. First assume N_R is injective. We show that N_Q is injective by proving that any inclusion $N_Q \subseteq X_Q$ splits. Fix an R-submodule $Y \subseteq X$ such that $N \oplus Y = X$. We claim that Y must be a Q-submodule of X (which will then do the job). Let $y \in Y$ and $rs^{-1} \in Q$, where $r \in R$ and $s \in S$. Then $yrs^{-1} = n + y'$ where $n \in N$ and $y' \in Y$. But then

$$yr = ns + y's \in N \oplus Y$$

implies that $yr = y's$, so $yrs^{-1} = y' \in Y$ as claimed.

Conversely, assume that N_Q is injective. To check that N_R is injective, consider an inclusion of R-modules $A \subseteq B$ and $f \in \mathrm{Hom}_R(A, N)$. We can "extend" f easily to an $f' \in \mathrm{Hom}_{RS^{-1}}(AS^{-1}, N)$. Since $A \subseteq B$ induces an inclusion $AS^{-1} \subseteq BS^{-1}$, we can further extend f' to a $g' \in \mathrm{Hom}_{RS^{-1}}(BS^{-1}, N)$. Composing g' with the natural map $B \to BS^{-1}$, we obtain a $g \in \mathrm{Hom}_R(B, N)$ extending the original f.

Comment. A more abstract way to formulate the second part of the solution is as follows. The fact that $A \subseteq B \Rightarrow AS^{-1} \subseteq BS^{-1}$ (used in a crucial way above) amounts to the flatness of Q as a left R-module (cf. Exercise 17). That said, the desired conclusion that N_Q injective $\Rightarrow$ N_R injective follows more generally from *LMR*-(3.6A).

Some concrete cases of this exercise have already appeared in *LMR*-§3. For instance, if $\mathfrak{p}$ is a prime ideal in a commutative ring R, the injective hull $E(R/\mathfrak{p})$ has a natural structure as an $R_{\mathfrak{p}}$-module, and as such, it remains injective (*LMR*-(3.77)). For the other direction, it was shown in *LMR*-(3.9) that, if R is a commutative domain with quotient field K, then any K-module is injective over R.

Ex. 10.30. In two different graduate algebra texts, the following exercise appeared: "Let S be a multiplicative subset of the commutative ring R. If M is an injective (right) R-module, show that MS^{-1} is an injective (right) RS^{-1}-module." Find a counterexample.

Solution. Since this wrong exercise appeared in two algebra texts, we offer two counterexamples below!

(1) We shall construct a commutative self-injective ring R for which some localization RS^{-1} is not self-injective. Over such a ring R, the free module of rank 1, $M = R_R$, will provide the counterexample we need. Start with any commutative noetherian complete local domain $(A, \mathfrak{m})$ of dimension > 1 (e.g. $\mathbb{Q}[[x_1, \ldots, x_n]]$ with $n > 1$), and let $E = E(A/\mathfrak{m})$ (the injective hull of $A/\mathfrak{m}$). Then E is a faithful injective A-module (by *LMR*-(3.76)), and by Matlis' Theorem *LMR*-(3.84), the natural map $A \to \mathrm{End}_A(E)$ is an

isomorphism. These properties of the module E_A guarantee that the "trivial extension" $R = A \oplus E$ (constructed with $E = {}_A E_A$ and $E^2 = 0$ in R) is a commutative self-injective ring: see *LMR*-(19.22). Now let $S = \langle s \rangle \subseteq A \subseteq R$ where s is any nonzero element in $\mathfrak{m}$. By *LMR*-(3.78), any element in E is killed by some power of $\mathfrak{m}$, in particular by s^n for some n. From this, we see easily that

$$RS^{-1} \cong AS^{-1} = A\left[\frac{1}{s}\right] \subsetneq K \text{ (quotient field of } A),$$

since $\dim (A) > 1$. By Exercise (3.2), the integral domain $A\left[\frac{1}{s}\right]$ is not self-injective, as desired.

Note that the self-injective ring R constructed above is actually a *local* ring. This follows from the fact that A itself is local, and that E is an ideal of square zero in R.

The second example to be presented below is due to G. Bergman. We thank him for permitting us to include his example here.

(2) In this example, let k be any infinite field. View $k[x, y]$ as a subring of $M := k^{k \times k}$ by identifying any polynomial in $k[x, y]$ with the function $k \times k \to k$ that it defines. Let

$$R = k\,[x, y] \oplus I \subseteq M = k^{k \times k},$$

where $I = \left\{g \in k^{k\times k} : \ |\text{supp}\,(g)| < \infty\right\}$. It is easy to see that R is a subring of M. Let us first show that M_R is injective. Since $M = \prod M_{(a,b)}$ where

$$M_{(a,b)} = \{g \in M : \ \text{supp}(g) \subseteq \{(a, b)\}\},$$

it suffices to show that *each* $M_{(a,b)}$ is injective over R. Let $e = e_{(a,b)} \in I$ be the characteristic function on $\{(a, b)\}$. Then $e^2 = e$ and $R = eR \oplus (1 - e)\,R$. Upon localization at the multiplicative set $\langle e \rangle = \{1, e\}$, $(1 - e)R$ becomes zero, so $R_{\langle e \rangle} \cong k$, and $M_{(a,b)}$ is just the $R_{\langle e \rangle}$-module k viewed as an R-module. Since k is an injective $R_{\langle e \rangle}$-module, Exercise 29 guarantees that it is an injective R-module, as desired.

Let $S = k\,[x] \setminus \{0\} \subseteq R$. We finish by showing that MS^{-1} *is not an injective* RS^{-1}*-module.* First, every function in I is killed by some $s \in S$, so I localizes to zero. This shows that $RS^{-1} \cong k\,[y]\,S^{-1}$. Next, we claim that

$$(*) \quad MS^{-1} \cong M/M_0, \text{ where } M_0 = \{g \in M : \ \text{supp}\,(g) \subseteq L_{a_1} \cup \cdots \cup L_{a_n} \text{ for suitable } a_i\}.$$

Here, L_a denotes the "vertical line" $x = a$ in $k \times k$. To prove $(*)$, note that each g above is killed by $(x - a_1) \cdots (x - a_n) \in S$. It is also easy to check that each $x - a$ acts as a bijection on M/M_0, so $(*)$ follows. [*Note.* Each element in S has the form

$$(x - a)\,(x - b) \cdots (x - c)\,h\,(x)\,,$$

where $h(x)$ has no zeros and hence is a unit in R.]

The final step is to show that $MS^{-1} \cong M/M_0$ is not injective over $RS^{-1} \cong k[y]S^{-1}$. We do this by checking that M/M_0 *is not even a divisible module over* $k[y]S^{-1}$. Since $k[y]S^{-1}$ is a domain, it suffices to check that $y\cdot(M/M_0) \subsetneq M/M_0$. This is clear, since the constant function 1 (on $k \times k$) does not belong to $yM + M_0$. (Any function in $yM + M_0$ is zero on all but a finite number of points on the x-axis.) Note that the infinitude of k is used only in this very last step!

Comment. The first counterexamples recorded in the literature seem to be those of E. C. Dade, in his paper "Localization of injective modules," J. Algebra **69**(1981), 416–425. For an injective R-module M, Dade found a criterion for MS^{-1} to be an injective RS^{-1}-module in terms of the "Ext" functor over R. Dade also developed necessary and sufficient conditions for *all* injective R-modules to localize to injective RS^{-1}-modules (for a given S). His counterexamples for the present exercise are based on these criteria. The counterexamples presented in (1) and (2) of our solution are different, and seem to be a bit simpler than Dade's. For more information on the injectivity of MS^{-1} (or the lack thereof), see the next exercise.

Ex. 10.31. Show that the statement in quotes in the last exercise is true under either one of the following assumptions:

(1) R is noetherian (or more generally, R is right noetherian and S is a central multiplicative set in R);
(2) M is S-torsionfree (i.e. for $s \in S$ and $m \in M$, $ms = 0 \Rightarrow m = 0$). Your proof in this case should work under the more general assumption that S is a right denominator set in a possibly noncommutative ring R.

Solution. (1) It will be sufficient to assume that R is right noetherian and that the multiplicative set S is central in R. We shall check that MS^{-1} is an injective RS^{-1}-module by applying Baer's Criterion *LMR*-(3.7). Take any right ideal in RS^{-1}, which we may assume to be of the form $\mathfrak{A}S^{-1}$ for some right ideal $\mathfrak{A} \subseteq R$. We need to show that the natural map

$$\mathrm{Hom}_{RS^{-1}}\left(RS^{-1}, MS^{-1}\right) \xrightarrow{\varphi} \mathrm{Hom}_{RS^{-1}}\left(\mathfrak{A}S^{-1}, MS^{-1}\right)$$

defined by restriction is onto. By a standard localization fact (see, e.g. the author's "Serre's Problem on Projective Modules," Prop. I.2.13, Springer Monographs in Math., 2006), for any *finitely presented* right R-module N, the natural map

$$\sigma : \ (\mathrm{Hom}_R(N, M))\, S^{-1} \to \mathrm{Hom}_{RS^{-1}}\left(NS^{-1}, MS^{-1}\right)$$

is an isomorphism. Using σ for $N = R$ and $N = \mathfrak{A}$ respectively (R and $\mathfrak{A}$ both being finitely presented since R is right noetherian), we may "identify" φ with the map

$$(\mathrm{Hom}_R(R, M))\, S^{-1} \to (\mathrm{Hom}_R(\mathfrak{A}, M))\, S^{-1}.$$

Since $\mathrm{Hom}_R(R, M) \to \mathrm{Hom}_R(\mathfrak{A}, M)$ is onto by the injectivity of M_R, it follows that f^* is also onto, as desired.

(2) We assume here that S is a right denominator set, and that M is S-torsionfree. For $s \in S$, $r \in R$ and $m \in M$, we claim that

$$(*) \qquad sr = 0 \in R \implies mr = 0 \in M.$$

In fact, from $sr = 0$, the "right reversible" condition on S implies that $rs' = 0$ for some $s' \in S$ (see *LMR*-(10.4)). Therefore, $mrs' = 0 \in M$, whence $mr = 0$ since M is S-torsionfree. Now the injective R-module M is divisible, so $(*)$ implies that $m \in Ms$. It follows that each $s \in S$ acts as a *bijection* on M, so the R-module structure on M can be extended to an RS^{-1}-module structure. (In particular, $MS^{-1} = M$.) Since M is an injective module over R, it follows from Exercise 29 that M is also an injective module over RS^{-1}.

Comment. Part (1) of this exercise is due to H. Bass; see his paper "Injective dimension in noetherian rings," Trans. A.M.S. **102**(1962), 18–29. Part (2) is more or less folklore.

If R (possible noncommutative) is noetherian and "right fully bounded", and S is a right and left denominator set, the injectivity of MS^{-1} has also been proved by K. A. Brown (partly using (2) above). For this and other positive results in the noncommutative case, see the article of K. R. Goodearl and D. A. Jordan: "Localizations of injective modules," Proc. Edinburgh Math. Soc. **28**(1985), 289–299. Other examples on the failure of injectivity on MS^{-1} in the noncommutative case (using differential polynomial ring constructions) can also be found in this paper.

§11. Right Goldie Rings and Goldie's Theorem

In the context of the theory of rings of quotients, "regular elements" of a ring usually mean elements that are neither left nor right zero divisors. These are not to be confused with the von Neumann regular elements. In this section, we write $\mathcal{C}_R$ for the multiplicative set of regular elements in a ring R. Recall that, for rings $R \subseteq Q$, we say that R is a *right order in* Q if $\mathcal{C}_R \subseteq \mathrm{U}(Q)$, and every element in Q has the form as^{-1} where $a \in R$ and $s \in \mathcal{C}_R$. There are two main points here:

(1) a ring R is a right order in some ring Q iff R is right Ore, and $Q \cong Q^r_{c\ell}(R)$ over R;
(2) a ring Q has a right order in it iff $\mathcal{C}_Q = \mathrm{U}(Q)$.

Note that the latter condition is left-right symmetric; if it holds, Q is said to be a *classical ring*. For instance, any von Neumann regular ring is a classical ring, and so is any right self-injective ring, by Exercise 8.

Right orders in division rings are exactly the right Ore domains. Goldie's Theorem purports to answer the more general question: *what kinds of rings are right orders in semisimple rings?*

We say that a ring R is *right Goldie* if $\text{u.dim}\, R_R < \infty$ and R has ACC on right annihilator ideals. (Left Goldie rings are defined similarly.) Goldie's famous theorem (*LMR*-(11.13)), in its most basic form, tells us that *a ring R is a right order in a semisimple ring Q iff R is semiprime and right Goldie, in which case $Q \cong Q^r_{c\ell}(R)$*. Two more significant characterizations for such rings R are the following:

(A) *R is semiprime, right nonsingular, and* $\text{u.dim}\, R_R < \infty$.
(B) *For any right ideal $\mathfrak{A}$ in R, $\mathfrak{A} \subseteq_e R$ iff $\mathfrak{A} \cap \mathcal{C}_R \neq \emptyset$.*

These characterizations of right orders in semisimple rings constitute what is sometimes called "Goldie's Second Theorem." Goldie's First Theorem, also obtained by Lesieur and Croisot, is best viewed as a special case of the Second Theorem: it states that *R is an order in a simple artinian ring iff it is a prime right Goldie ring.*

The point about Goldie's Theorem is that it provides us a way to check "internally" within R when it has a classical right ring of fractions that is semisimple. Since the two conditions defining a right Goldie ring are both weaker than ACC on right ideals, it follows that any right noetherian ring is right Goldie. Thus, Goldie's theorem implies that:

> *Any semiprime right noetherian ring R has a semisimple classical right ring of fractions.*

Another easy example of a semiprime Goldie ring is any ring R between $\mathbb{Z}G$ and $\mathbb{Q}G$ for a finite group G. Here, R is a (left, right) order in the semisimple ring $\mathbb{Q}G$; in fact we can get $\mathbb{Q}G$ already by inverting the regular central elements $\{1, 2, 3, \dots\}$.

Using Goldie's Theorem, it is often possible to prove results about right noetherian rings that may not be easy to prove otherwise. Some illustrations of these are given in *LMR*-§11C.

In formulating Goldie's Theorem, we can also start with a semiprime ring R with only finitely many minimal prime $\mathfrak{p}_1, \dots, \mathfrak{p}_t$. In this case, the $\mathfrak{p}_i$'s are precisely the "maximal annihilators" in R, and t equals the "2-sided uniform dimension" of R (*LMR*-(11.41), *LMR*-(11.43)), and R is right Goldie iff each $R/\mathfrak{p}_i$ is (*LMR*-(11.44)).

In the commutative case, a semisimple ring means simply a finite direct product of fields, so the purpose of Goldie's Theorem in this case would be to characterize (commutative) rings that are orders in a finite direct product of fields. This characterization is easy to achieve by using the standard results of commutative algebra:

> *A commutative ring R is an order in a finite direct product of fields iff R is semiprime* (i.e. *reduced*) *and has only a finite number of minimal primes, iff R is semiprime and every essential ideal contains a regular element of R.*

(See LMR-(11.46).) Note that the latter condition is a simplification of the condition (B) given before.

Most exercises for this section are on the theme of Goldie's Theorem and its variations. As a whole, they give a lot of additional information about regular elements, ideals, right annihilators, and minimal primes. There are also a couple of exercises ((12) and (13)) concerning the 1-*sided artinian radicals* in a ring R. By definition, the right artinian radical $A^r(R)$ is the sum of all right ideals of R that are artinian as right R-modules. The left artinian radical $A^\ell(R)$ is defined similarly. These artinian radicals are *ideals* in R, and in the case when R is noetherian, we have $A^\ell(R) = A^r(R)$, according to *LMR*-(11.35).

Exercises for §11

Ex. 11.0. Let $R_i \subseteq Q_i \, (i \in I)$ be rings. Show that $\prod_i R_i$ is a right order in $\prod_i Q_i$ iff each R_i is a right order in Q_i.

Solution. Let $R = \prod_i R_i$ and $Q = \prod_i Q_i$. First assume each R_i is a right order in Q_i. If $r = (r_i)_{i \in I} \in \mathcal{C}_R$, then $r_i \in \mathcal{C}_{R_i} \subseteq \mathrm{U}(Q_i)$ for each $i \in I$, so $r \in \mathrm{U}(Q)$. For any $q = (q_i)_{i \in I} \in Q$, we can write $q_i = s_i r_i^{-1}$ where $s_i \in R_i$ and $r_i \in \mathcal{C}_{R_i}$ for each i. Therefore, $q = sr^{-1}$ for $s = (s_i)_{i \in I}$ and $r = (r_i)_{i \in I} \in \mathcal{C}_R$, so we have checked that R is a right order in Q. The converse is proved similarly.

Ex. 11.1. Let $a \in R$ be right regular (i.e. $\mathrm{ann}_r(a) = 0$), and let $I \subseteq R$ be a right ideal such that $aR \cap I = 0$.

(a) Show that the sum $\sum_{i \geq 0} a^i I$ is direct.
(b) From (a), deduce that if $\mathrm{u.dim}\, R_R < \infty$, we must have $aR \subseteq_e R_R$.
(c) Give an example to show that $aR \subseteq_e R_R$ need not hold if $\mathrm{u.dim}\, R_R = \infty$.

Solution. (a) Suppose $b_0 + ab_1 + \cdots + a^n b_n = 0$, where $b_i \in I$. Then $b_0 \in aR \cap I = 0$, and so $a\left(b_1 + ab_2 + \cdots + a^{n-1}b_n\right) = 0$. Since $\mathrm{ann}_r(a) = 0$, we have $b_1 + ab_2 + \cdots + a^{n-1}b_n = 0$. Repeating this argument, we get $b_i = 0$ for all i.

(b) Assume that $\mathrm{u.dim}\, R_R < \infty$. For any right ideal $I \neq 0$, we must have $aR \cap I \neq 0$, for otherwise, by (a), R would contain an infinite direct sum $\bigoplus_{i \geq 0} a^i I$ (where each $a^i I \neq 0$), in contradiction to $\mathrm{u.dim}\, R_R < \infty$. Therefore, $aR \subseteq_e R_R$.

(c) A typical example here is $R = \mathbb{Q}\langle a, b\rangle$. Since R is a domain, every nonzero element is regular. A principal right ideal such as aR is clearly *not* essential in R_R. For instance, $aR \cap I = 0$ for $I = bR$ (and R contains the infinite direct sum $\bigoplus_{i \geq 0} a^i I = \bigoplus_{i \geq 0} a^i bR$).

Comment. It is instructive to recall another (shorter) proof of (*b*) given in *LMR*-(11.14). If aR was not essential in R_R, we would have $\text{u.dim}\, aR < \text{u.dim}\, R_R < \infty$. However, $\text{ann}_r(a) = 0$ implies that $aR \cong R_R$, and so $\text{u.dim}\, aR = \text{u.dim}\, R_R$. If a is assumed regular (instead of just right regular), other conditions may also guarantee $aR \subseteq_e R_R$; see, for instance, (a) in the exercise below.

Ex. 11.2. Let R be a right Ore ring.

(a) Show that any right ideal $\mathfrak{A}$ containing a regular element is essential (in R_R).
(b) Show that R is semiprime right Goldie iff any essential right ideal of R contains a regular element.
(c) Give an example of a commutative (hence Ore) ring with an essential ideal $\mathfrak{A} \subseteq R$ not containing any regular elements.

Solution. (a) Suppose $\mathfrak{A}$ contains a regular element a. For any $b \neq 0$ in R, there exists an equation $ar = bs$ where $s \in R$ is regular. Therefore, bs is a nonzero element in $aR \subseteq \mathfrak{A}$, and we have shown that $\mathfrak{A} \subseteq_e R_R$.

(b) follows from (a) and the characterization (see *LMR*-(11.13)) of semiprime right Goldie rings as rings in which a right ideal is essential iff it contains a regular element.

(c) Let $\mathfrak{A}$ be the ideal $\mathbb{Z} \oplus \mathbb{Z} \oplus \cdots$ in the commutative ring $R = \mathbb{Z} \times \mathbb{Z} \times \cdots$. Then $\mathfrak{A} \subseteq_e R$. In fact, for any $b = (b_1, b_2, \ldots)$ with $b_n \neq 0$, the principal ideal bR contains the nonzero element $(0, \ldots, b_n, 0, \ldots) \in \mathfrak{A}$ (with b_n in the nth coordinate). However, every element

$$(r_1, \ldots, r_m, 0, \ldots) \in \mathfrak{A}$$

is killed by any $(s_1, s_2, \ldots)$ with $s_1 = \cdots = s_m = 0$, so $\mathfrak{A}$ does not contain any regular elements of R.

In this example, $\text{u.dim}\, R_R = \infty$. It is just as easy to give an example for which $\text{u.dim}\, R_R < \infty$. For instance, the 2-dimensional $\mathbb{Q}$-algebra $R = \mathbb{Q}[u]$ with the relation $u^2 = 0$ has an essential ideal $\mathfrak{A} = Ru = \mathbb{Q}u$ containing no regular elements. Here, $\text{u.dim}\, R_R = 1$, but R is not semiprime since $\mathfrak{A}^2 = 0$.

Ex. 11.3. (Goldie) For any element a in a right Goldie ring R, show that there exists an integer $n \geq 1$ such that $\mathfrak{A} := a^n R + \text{ann}_r(a^n)$ is a direct sum and $\mathfrak{A} \subseteq_e R_R$.

Solution. Since right annihilators in R satisfy ACC, there exists $n \geq 1$ such that $\text{ann}_r(a^n) = \text{ann}_r(a^{n+1}) = \cdots$. If $a^n b \in \text{ann}_r(a^n)$, then

$$b \in \text{ann}_r\left(a^{2n}\right) = \text{ann}_r(a^n),$$

so $a^n b = 0$. This shows that $\mathfrak{A} = a^n R + \text{ann}_r(a^n)$ is a *direct* sum. To show that $\mathfrak{A} \subseteq_e R_R$, we use a method similar to that used in the solution to Exercise 1(a). Let I be a right ideal such that $I \cap \mathfrak{A} = 0$. *We claim that the sum* $\sum_{i \geq 0} a^{in} I$ *is direct.* Indeed, suppose $b_0 + a^n b_1 + \cdots + a^{mn} b_m = 0$ with $b_i \in I$. Then $b_0 \in I \cap a^n R = 0$, and so

$$a^n \left(b_1 + a^n b_2 + \cdots + a^{(m-1)n} b_m \right) = 0,$$

which implies that

$$b_1 \in I \cap (a^n R + \text{ann}_r(a^n)) = I \cap \mathfrak{A} = 0.$$

Since $\text{ann}_r(a^n) = \text{ann}_r\left(a^{2n}\right) = \cdots$, we can repeat this argument to show that all b_i=0, thus proving our claim. Invoking now the assumption that $\text{u.dim}\, R_R < \infty$, we see that $a^{in} I = 0$ for large i and so $a^n I = 0$. Therefore, $I \subseteq \text{ann}_r(a^n) \subseteq \mathfrak{A}$, whence $I = 0$. We have thus proved that $\mathfrak{A} \subseteq_e R_R$.

Comment. The result above was used by Goldie in his original proof of Goldie's Theorem (*LMR*-(11.13)) characterizing right orders in semisimple rings. In *LMR*, a different proof was given that was independent of the present exercise. The result in this exercise bears a remarkable resemblance to some results on strongly π-regular rings, that is, rings in which every descending chain $aR \supseteq a^2 R \supseteq \cdots$ $(a \in R)$ stabilizes. According to a theorem of Armendariz, Fisher and Snider:

> *A ring R is strongly π-regular iff, for any $a \in R$, there exists an integer $n \geq 1$ such that $R = a^n R \oplus \text{ann}_r(a^n)$.*

This equality relation is, of course, much stronger than the essentiality relation $a^n R \oplus \text{ann}_r(a^n) \subseteq_e R_R$.

Ex. 11.4. Let R be a right Ore domain that is not left Ore, say, $Ra \cap Rb = 0$, where $a, b \in R \backslash \{0\}$. The ring $A = \mathbb{M}_2(R)$ is prime right Goldie by *LMR*-(11.18), so right regular elements of A are automatically regular by *LMR*-(11.14)(a). Show, however, that $\sigma = \begin{pmatrix} a & a \\ b & b \end{pmatrix} \in A$ is left regular but not regular.

Solution. Suppose $x, y, z, w \in R$ are such that $\begin{pmatrix} x & y \\ z & w \end{pmatrix} \sigma = 0$. Then

$$xa + yb = za + wb = 0.$$

Since $Ra \cap Rb = 0$, we have $x = y = z = w = 0$, so σ is left regular. However, $\sigma \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} = 0$, so σ is not regular.

Ex. 11.5. Let R be a semiprime right Goldie ring with

$$Q = Q_{c\ell}^r(R) \cong \mathbb{M}_{n_1}(D_1) \times \cdots \times \mathbb{M}_{n_t}(D_t)$$

where the D_i's are division rings. Show that $\text{u.dim}\, R_R = n_1 + \cdots + n_t$, and that a right ideal $\mathfrak{A} \subseteq R$ is uniform iff $\mathfrak{A}^e$ is a minimal right ideal in Q. If, in addition, R is prime, show that $\text{u.dim}\, R_R$ is the largest index of nilpotency of nilpotent elements in R.

Solution. First, by *LMR*-(10.35), $\text{u.dim}\, R_R = \text{u.dim}\, Q_Q$. For $Q_i = \mathbb{M}_{n_i}(D_i)$, we have $\text{u.dim}\,(Q_i)_{Q_i} = n_i$, so it follows that

$$\text{u.dim}\, R_R = \sum\nolimits_i \text{u.dim}\,(Q_i)_{Q_i} = n_1 + \cdots + n_t.$$

For a right ideal $\mathfrak{A} \subseteq R$, *LMR*-(10.35) also gives $\text{u.dim}\, \mathfrak{A}_R = \text{u.dim}\,(\mathfrak{A}^e)_Q$. Therefore, $\mathfrak{A}_R$ is uniform iff $(\mathfrak{A}^e)_Q$ is uniform. Since Q is semisimple, the latter simply means that $\mathfrak{A}^e$ is a minimal right ideal in Q.

Now assume R is prime. Then $t = 1$ in the above, and we have $R \subseteq Q^r_{c\ell}(R) = \mathbb{M}_{n_1}(D_1)$.

Ex. 11.6. Let $x, y \in R$ where R is a semiprime right Goldie ring. If $xy \in \mathcal{C}_R$ (the set of regular elements in R), show that $x, y \in \mathcal{C}_R$.

Solution. Since $\text{ann}_r(y) \subseteq \text{ann}_r(xy) = 0$, y is right regular. Since R is semiprime right Goldie, $y \in \mathcal{C}_R$. To show that x is also (right) regular, choose $b \in Q^r_{c\ell}(R)$ such that $b(xy) = 1$. Then, for $a \in R$:

$$xa = 0 \implies 0 = b\,(xa) = y^{-1}a \quad \text{in} \quad Q^r_{c\ell}(R),$$

so $a = 0$. Thus, $\text{ann}_r(x) = 0$, as desired.

Ex. 11.7. Let Q be an algebraic algebra over a field k. Show that any $q \in \mathcal{C}_Q$ is a unit by considering the minimal polynomial of q over k. (Thus, Q is a classical ring.)

Solution. Let $q^n + a_{n-1}q^{n-1} + \cdots + a_1 q + a_0 = 0$ $(a_i \in k)$ with n chosen smallest. If $a_0 = 0$, then, since $q \in \mathcal{C}_Q$, we'll have

$$q^{n-1} + a_{n-1}q^{n-2} + \cdots + a_1 = 0,$$

a contradiction. Therefore, $a_0 \neq 0$, and we see that

$$-a_0^{-1}\left(q^{n-1} + a_{n-1}q^{n-2} + \cdots + a_1\right)$$

is an inverse for q.

Comment. In *LMR*-(11.6)(2), it is shown that any strongly π-regular ring is classical. The above result is a special case of this since, in fact, any algebraic k-algebra is strongly π-regular: see *ECRT*-(23.6).

Ex. 11.8. Show that any right self-injective ring Q is a classical ring.

Solution. Let $q \in \mathcal{C}_Q$. Then $qQ \cong Q_Q$ is injective, so qQ is a direct summand of Q_Q. Write $qQ = eQ$, where $e = e^2 \in Q$. Then

$$(1-e)\,q \in (1-e)\,eQ = 0,$$

so $e = 1$. This gives $qQ = Q$, so $qq' = 1$ for some q'. Now $q\,(q'q - 1) = 0$ leads to $q'q = 1$, showing that $q \in \mathrm{U}\,(Q)$.

Comment. The above argument made use of the fact that q is neither a left nor a right 0-divisor. Note, however, that if q is only assumed to be not a left 0-divisor, one can still show that q has a left inverse in Q: see Exercise 3.2.

An important example to keep in mind for the result in this Exercise is $Q = \mathrm{End}(V_k)$, where V is a right vector space over a division ring k. By (the right analogue of) *LMR*-(3.74b), Q is a right (although not necessarily left) self-injective ring. An element $q \in Q$ is not a left (resp. right) 0-divisor iff the linear map $q : V \to V$ is injective (resp. surjective). Thus, if $q \in \mathcal{C}_Q$, $q : V \to V$ will be *bijective*, and hence indeed $q \in \mathrm{U}\,(Q)$. Note that the ring Q here is *not* Dedekind-finite, unless $\dim_k V < \infty$.

Ex. 11.9. Using Ex. 5.23A, show that a f.g. projective right module M over a commutative classical ring Q is "cohopfian", in the sense that any injective Q-endomorphism $\varphi: M \to M$ is an automorphism. Is this still true if Q is not commutative?

Solution. After adding a suitable f.g. module N to M and replacing φ by $\varphi \oplus 1_N$, we may assume that M is a free module Q^n. Represent the elements of Q^n as column vectors and let $A = (a_{ij}) \in \mathbb{M}_n(Q)$ be such that $\varphi\,(x) = Ax$ for any

$$x = (x_1, \ldots, x_n)^T \in Q^n.$$

The injectivity of φ means that the homogeneous system of linear equations $Ax = 0$ has only the trivial solution. By part (3) of Ex. 5.23A, this implies that the McCoy rank of A is n; that is, $\det(A)$ (the unique $n \times n$ minor of A) is a non 0-divisor in Q. Since Q is a classical ring, we have $\det(A) \in \mathrm{U}(Q)$. Thus, $A \in \mathrm{GL}_n(Q)$, so φ is an automorphism.

If Q is a *noncommutative* classical ring, the conclusion in the Exercise fails in general, even for $M = Q_Q$. In fact, if Q is not Dedekind-finite (e.g. $Q = \mathrm{End}\,(V_k)$ for an infinite dimensional vector space V over a division ring k), say with $ab = 1 \neq ba$ in A, then $\varphi : Q_Q \to Q_Q$ defined by $\varphi\,(x) = bx$ is an injective Q-endomorphism that is not an automorphism.

Ex. 11.10. Let $\bar{R} = R/\mathfrak{A}$ where $\mathfrak{A}$ is an ideal of R. (a) If $\mathrm{u.dim}\, R_R < \infty$, is $\mathrm{u.dim}\, \bar{R}_{\bar{R}} < \infty$? (b) Exhibit a right Goldie ring R with a factor ring $\bar{R}$ that is not right Goldie.

Solution. The ring $R = \mathbb{Z}\,[x_1, x_2, \ldots]$ is a commutative domain, so R is a Goldie ring with $\mathrm{u.dim}\, R_R = 1$. Now consider the subring $S \subset \mathbb{Z} \times \mathbb{Z} \times \cdots$

consisting of sequences $(a_1, a_2, \ldots)$ that are eventually constant. Clearly, S is the subring of $\mathbb{Z} \times \mathbb{Z} \times \cdots$ generated by the unit vectors $e_1, e_2, \ldots$, so the ring homomorphism $\varphi: R \to S$ defined by $\varphi(x_i) = e_i$ $(\forall i)$ is a surjection. For $\mathfrak{A} = \ker(\varphi)$, we have $S \cong R/\mathfrak{A}$, and $\text{u.dim}\, S_S = \infty$ since S contains $\mathbb{Z} \oplus \mathbb{Z} \oplus \cdots$, which is an infinite direct sum of ideals $\bigoplus_{i=1}^{\infty} Se_i$. In particular, $R/\mathfrak{A} \cong S$ is *not* a Goldie ring.

Ex. 11.11. *True or False*: Every right Goldie ring is stably finite?

Solution. The statement is, in fact, true for any ring R with $\text{u.dim}\, R_R < \infty$. For, by *LMR*-(6.62), this implies that $\text{u.dim}\, S_S < \infty$ for $S = \mathbb{M}_n(R)$, and by *LMR*-(6.60), $\text{u.dim}\, S_S < \infty$ implies that S is Dedekind-finite. Since this holds for all (finite) n, R is stably finite.

Ex. 11.12. The ring $R = \begin{pmatrix} \mathbb{Z} & \mathbb{Q} \\ 0 & \mathbb{Q} \end{pmatrix}$ is right noetherian but not left noetherian, by *FC*-(1.22). Show that the right artinian radical $A^r(R) = \begin{pmatrix} 0 & \mathbb{Q} \\ 0 & \mathbb{Q} \end{pmatrix}$, but the left artinian radical $A^\ell(R) = 0$.

Solution. The right (resp. left) artinian radical of R is, by definition, the sum of all artinian right ideals in R. To compute $A^r(\text{R})$, we recall from *FC*-(1.17)(2) that any right ideal $\mathfrak{A} \subseteq R$ has the form $J_1 \oplus J_2$ where J_1 is a right ideal of $\mathbb{Z}$, and J_2 is a right $\mathbb{Q}$-submodule of $\mathbb{Q} \oplus \mathbb{Q}$ containing $J_1\mathbb{Q} \oplus 0$. Assume now $\mathfrak{A}_R$ is artinian. Since the ideal $\begin{pmatrix} 0 & \mathbb{Q} \\ 0 & \mathbb{Q} \end{pmatrix}$ acts as zero on $(\mathfrak{A}/J_2)_R$, we see that $(J_1)_{\mathbb{Z}}$ must be artinian, and so $J_1 = 0$. Therefore, $\mathfrak{A}$ is just any right $\mathbb{Q}$-submodule of $\mathbb{Q} \oplus \mathbb{Q}$, and of course, any such $\mathbb{Q}$-submodule is artinian over $\mathbb{Q}$, and hence artinian over R. This shows that $A^r(R)$ is just $\begin{pmatrix} 0 & \mathbb{Q} \\ 0 & \mathbb{Q} \end{pmatrix}$.

To compute $A^\ell(R)$, recall from *FC*-(1.17)(1) that any left ideal $\mathfrak{B} \subseteq R$ has the form $I_1 \oplus I_2$, where I_2 is a left ideal in $\mathbb{Q}$, and I_1 is a left $\mathbb{Z}$-submodule of $\mathbb{Z} \oplus \mathbb{Q}$ containing $0 \oplus \mathbb{Q}I_2$. Here, the ideal $\begin{pmatrix} 0 & \mathbb{Q} \\ 0 & \mathbb{Q} \end{pmatrix}$ acts as zero on ${}_R(I_1)$, so if ${}_R\mathfrak{B}$ is artinian, ${}_{\mathbb{Z}}(I_1)$ must be artinian. We finish by showing that the only artinian subgroup of $\mathbb{Z} \oplus \mathbb{Q}$ is zero, for then, we have $I_1 = 0$ and hence $A^\ell(R) = 0$. Let G be any nonzero subgroup of $\mathbb{Z} \oplus \mathbb{Q}$. For $0 \neq g \in G$, we have $\mathbb{Z} \cdot g \cong \mathbb{Z}$ since $\mathbb{Z} \oplus \mathbb{Q}$ is torsionfree. Since ${}_{\mathbb{Z}}(\mathbb{Z})$ is not artinian, it follows that ${}_{\mathbb{Z}}G$ is also not artinian.

Ex. 11.13. Find $A^r(R)$ and $A^\ell(R)$ for $R = \begin{pmatrix} \mathbb{Q} & \mathbb{Q}(t) \\ 0 & \mathbb{Q}(t) \end{pmatrix}$.

Solution. Since $\mathbb{Q}$ and $\mathbb{Q}(t)$ are artinian rings and $\mathbb{Q}(t)_{\mathbb{Q}(t)}$ is an artinian module, *FC*-(1.22) implies that the triangular ring R here is right artinian. In particular, $A^r(R) = R$. To compute $A^\ell(R)$, consider any left artinian ideal $\mathfrak{B} \subseteq R$. By *FC*-(1.17)(1) again, we have $\mathfrak{B} = I_1 \oplus I_2$, where I_2 is a left ideal in $\mathbb{Q}(t)$, and I_1 is a left $\mathbb{Q}$-submodule of $\mathbb{Q} \oplus \mathbb{Q}(t)$ containing

$0 \oplus \mathbb{Q}(t)\, I_2$. There are two possibilities: $I_2 = \mathbb{Q}(t)$, or $I_2 = 0$. If $I_2 = \mathbb{Q}(t)$, then $I_1 \supseteq 0 \oplus \mathbb{Q}(t)$. But then ${}_R(I_1)$ is not artinian since $\begin{pmatrix} 0 & \mathbb{Q}(t) \\ 0 & \mathbb{Q}(t) \end{pmatrix}$ acts as zero on ${}_R(I_1)$ so the left R-structure on I_1 is essentially its left $\mathbb{Q}$-structure, and $\dim_{\mathbb{Q}} I_1 \geq \dim_{\mathbb{Q}} \mathbb{Q}(t) = \infty$. Thus, we must have I_2=0, in which case I_1 can be any finite-dimensional left $\mathbb{Q}$-subspace of $\mathbb{Q} \oplus \mathbb{Q}(t)$. Summing these, we get $A^\ell(R) = \begin{pmatrix} \mathbb{Q} & \mathbb{Q}(t) \\ 0 & 0 \end{pmatrix}$.

Ex. 11.14. Use *LMR*-(11.43) to show that, if R has ACC on ideals, then R has only finitely many minimal primes.

Solution. According to *LMR*-(11.43), if a semiprime ring S has ACC on annihilator ideals, then S has only finitely many minimal primes. We apply this result to the semiprime ring $S := R/\operatorname{Nil}_* R$, where $\operatorname{Nil}_* R$ is the lower nil radical (i.e. intersection of all prime ideals) of R. If R has ACC on ideals, then so does S, so by the result quoted above, S has only finitely many minimal primes. Since all prime ideals of R contain $\operatorname{Nil}_* R$, this implies that R also has only finitely many minimal primes.

Comment. This exercise already appeared in *ECRT*-(10.15), but the solution there is different. In that solution, one uses "noetherian induction" to show that any ideal $\mathfrak{A} \subseteq R$ contains a finite product of prime ideals. The case $\mathfrak{A} = 0$ leads easily to the desired conclusion.

Ex. 11.15. Let R be a semiprime (i.e. reduced) commutative ring with finitely many minimal primes $\mathfrak{p}_1, \ldots, \mathfrak{p}_t$. Show by a localization argument that the set $\mathcal{C}_R$ of regular elements in R is given by $R \backslash (\mathfrak{p}_1 \cup \cdots \cup \mathfrak{p}_t)$.

Solution. Let $a \in \mathcal{C}_R$. If $a \in \mathfrak{p}_i$ for some i, then a localizes to a nilpotent element in $R_{\mathfrak{p}_i}$ (since $R_{\mathfrak{p}_i}$ has only one prime ideal, $\mathfrak{p}_i R_{\mathfrak{p}_i}$). Therefore, $ra^n = 0$ for some $n \geq 1$ and some $r \in R \backslash \mathfrak{p}_i$. This contradicts $a \in \mathcal{C}_R$. Thus, $\mathcal{C}_R \subseteq R \backslash (\mathfrak{p}_1 \cup \cdots \cup \mathfrak{p}_t)$. Conversely, for any $a \notin \mathfrak{p}_1 \cup \cdots \cup \mathfrak{p}_t$,

$$ab = 0 \implies b \in \mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_t = \operatorname{Nil}(R) = 0,$$

so $a \in \mathcal{C}_R$.

Comment. The equation $\mathcal{C}_R = R \backslash (\mathfrak{p}_1 \cup \cdots \cup \mathfrak{p}_t)$ (for a semiprime commutative ring R) appeared in *LMR*-(11.47). There , it is deduced from *LMR*-(11.42), which is a more general result characterizing the regular elements in *any* (possibly noncommutative) semiprime ring with finitely many minimal prime ideals. The present exercise is the commutative version. Note that there is no noetherian assumption on the ring R in this exercise!

Ex. 11.16. Let R be a semiprime ring.

(a) Show that an ideal $\mathfrak{B} \subseteq R$ is an annihilator (in the sense of *LMR*-(11.37)) iff $\mathfrak{B}$ is a right annihilator.

(b) If R is also right Goldie and $Q = Q^r_{c\ell}(R)$ has t Wedderburn components, show that the annihilator ideals in R are exactly the contractions to R of the 2^t ideals in Q.

Solution. (a) The "only if" part is clear. For the converse, suppose the ideal $\mathfrak{B}$ is a right annihilator. Then $\mathfrak{B} = \text{ann}_r(\text{ann}_\ell(\mathfrak{B}))$. Since $\mathfrak{B}$ is an ideal, $\mathfrak{A} := \text{ann}_\ell(\mathfrak{B})$ is also an ideal (by *LMR*-(11.36)). Thus, $\mathfrak{B} = \text{ann}_r(\mathfrak{A})$ is an annihilator ideal in the sense of *LMR*-(11.37).

(b) Let $Q = Q_1 \times \cdots \times Q_t$ where the Q_i's are the Wedderburn components, and let

$$\mathfrak{q}_i = \sum\nolimits_{j \neq i} Q_j, \quad \mathfrak{p}_i = \mathfrak{q}_i \cap R \quad (1 \leq i \leq t).$$

By *LMR*-(11.22), $\mathfrak{p}_1, \ldots, \mathfrak{p}_t$ are all the minimal primes of R. Now, by *LMR*-(11.40) and (11.41), the annihilator ideals in R are precisely intersections of subsets of $\{\mathfrak{p}_1, \cdots, \mathfrak{p}_t\}$. An intersection $\mathfrak{p}_{i_1} \cap \cdots \cap \mathfrak{p}_{i_k}$ is the contraction of $\mathfrak{q}_{i_1} \cap \cdots \cap \mathfrak{q}_{i_k}$ to R, and $\mathfrak{q}_{i_1} \cap \cdots \cap \mathfrak{q}_{i_k}$ is an arbitrary ideal of Q. Thus, the annihilator ideals of R are precisely the contractions to R of the 2^t ideals in Q.

Ex. 11.17. Let R be a semiprime ring with finitely many minimal prime ideals $\mathfrak{p}_1, \ldots, \mathfrak{p}_t$. Let $\mathfrak{B}_i = \prod_{j \neq i} \mathfrak{p}_j$, and let

$$\bar{\mathfrak{B}}_i = (\mathfrak{B}_i + \mathfrak{p}_i)/\mathfrak{p}_i \text{ in } \bar{R}_i = R/\mathfrak{p}_i.$$

After identifying R with a subring of $S := \prod_i \bar{R}_i$, show that $\mathfrak{B} := \prod_i \bar{\mathfrak{B}}_i$ is an ideal of S lying in R, and that $\text{ann}^S_\ell(\mathfrak{B}) = 0$.

Solution. The natural map $\varphi: R \to S$ given by $\varphi(b) = (\bar{b}, \ldots, \bar{b}) \in S$ is an injection, since $\bigcap_i \mathfrak{p}_i$ is the prime radical of R, which is (0) in the semiprime ring R. We shall henceforth identify R with $\varphi(R) \subseteq S$. Since $\bar{\mathfrak{B}}_i$ is an ideal in $\bar{R}_i$ for each i, $\mathfrak{B} = \prod_i \bar{\mathfrak{B}}_i$ is an ideal in S. To show that $\mathfrak{B} \subseteq R$, consider any element $\beta = (\bar{b}_1, \cdots, \bar{b}_t) \in \mathfrak{B}$, where $b_i \in \mathfrak{B}_i$. For $b = b_1 + \cdots + b_t \in R$, we have

$$\beta = (\bar{b}_1, \ldots, \bar{b}_t) = (\bar{b}, \cdots, \bar{b}) \in R,$$

as desired. For the last part of the exercise, note that $\mathfrak{p}_i \not\supseteq \prod_{j \neq i} \mathfrak{p}_j$ (for otherwise $\mathfrak{p}_i$ contains $\mathfrak{p}_j$ for some $j \neq i$). This implies that $\bar{\mathfrak{B}}_i$ is a *nonzero* ideal in $\bar{R}_i$. Since $\bar{R}_i$ is a prime ring, $\bar{\mathfrak{B}}_i$ has zero left annihilator in $\bar{R}_i$, for each i. From this, it follows immediately that $\text{ann}^S_\ell(\mathfrak{B}) = 0$.

Ex. 11.18. Keep the notations in Exercise 17, and assume that each $\bar{R}_i = R/\mathfrak{p}_i$ is right Goldie, with $Q_i := Q^r_{c\ell}(\bar{R}_i)$. Independently of *LMR*-(11.44), show that R is also right Goldie, with $Q^r_{c\ell}(R) \cong Q := \prod_i Q_i$.

Solution. By Goldies's First Theorem *LMR*-(11.16), each Q_i is a simple artinian ring, so $Q = \prod_i Q_i$ is a semisimple ring. To get the desired conclusion, it thus suffices to show that R is a right order in Q. For any

$q = (q_1, \ldots, q_t) \in Q$, let $q_i b_i \in \bar{R}_i$ where b_i is regular in $\bar{R}_i$. Since $\bar{R}_i$ is prime and $\bar{\mathfrak{B}}_i \subseteq \bar{R}_i$ is a nonzero ideal, *LMR*-(8.4)(3) implies that $\bar{\mathfrak{B}}_i$ is essential as a right ideal in $\bar{R}_i$. By *LMR*-(11.13), $\bar{\mathfrak{B}}_i$ contains a regular element c_i of $\bar{R}_i$. Then $b_i c_i$ and $q_i b_i c_i$ are both in $\bar{\mathfrak{B}}_i$. By Exercise 17 above,

$$s := (b_1 c_1, \ldots, b_t c_t) \in R \cap \mathrm{U}(Q),$$

and also $qs \in R$. Since Q is semisimple, this is sufficient ground for us to conclude that R is a right order in Q, by *LMR*-(11.7)(3).

Comment. This result was already proved in *LMR*-(11.44), by using right uniform dimensions. The above solution offers an interesting alternative approach.

Ex. 11.19. (See Ex. 8.6.) Let R be the (commutative) ring $\mathbb{Q}\langle\{x_i : i \geq 1\}\rangle$ with relations $x_i x_j = 0$ for all unequal i, j.

(a) Show that R is semiprime (i.e. reduced).
(b) R does not satisfy ACC on annihilators.
(c) $\mathrm{u.dim}\, R_R = \infty$.
(d) Let $p_i(x_i)$ $(i \geq 1)$ be nonzero polynomials without constant terms. Then the ideal $\mathfrak{A}$ generated by $\{p_i(x_i) : i \geq 1\}$ is dense, but contains no regular elements of R.
(e) Show that the minimal primes of R are given precisely by $\mathfrak{p}_i = \sum_{j \neq i} R x_j$ $(i \geq 1)$.

Solution. (a) is already observed in the solution to Exercise 8.6, as is the denseness of $\mathfrak{A}$. Since, for any n, x_{n+1} kills each of $p_1(x_1), \ldots, p_n(x_n)$, the ideal $\mathfrak{A}$ consists of 0-divisors, so we have (d). Next, the direct sum

$$\bigoplus_{i \geq 1} R x_i = x_1 \mathbb{Q}[x_1] \oplus x_2 \mathbb{Q}[x_2] \oplus \cdots$$

shows (c), and (b) follows from Exercise 6.22 by choosing $s_i = t_i = x_i$. To prove (e), note first that $R/\mathfrak{p}_i \cong \mathbb{Q}[x_i]$, so each $\mathfrak{p}_i$ is a prime ideal. Clearly, there is no inclusion relation among the $\mathfrak{p}_i$'s. Therefore, (e) will follow if we can show that any prime ideal $\mathfrak{p} \subset R$ contains some $\mathfrak{p}_i$. This is clear if $x_i \in \mathfrak{p}$ for all i, so assume that $x_i \notin \mathfrak{p}$ for some i. Then, for any $j \neq i$, $x_i x_j = 0 \in \mathfrak{p}$ implies that $x_j \in \mathfrak{p}$, so $\mathfrak{p} \supseteq \mathfrak{p}_i$, as desired.

Comment. The ring R in this exercise is a typical commutative semiprime ring that fails to be Goldie. By *LMR*-(11.46), for a commutative semiprime ring R, any of the properties (b), (c), (d), (e) above is tantamount to the fact that R is *not* Goldie. The point of this exercise is to provide an example of a commutative semiprime ring in which *each* of (b), (c), (d), (e) can be easily checked to hold.

Ex. 11.20. Show that R is a reduced right Goldie ring iff R is a right order in a finite direct product of division rings.

Solution. Suppose R is a right order in a ring $Q = Q_1 \times \cdots \times Q_t$, where the Q_i's are division rings. Since Q is semisimple, Goldie's Theorem *LMR*-(11.13) implies that R is a semiprime right Goldie ring. The fact that Q is reduced implies that R is also a reduced ring. Conversely, assume that R is a reduced (hence semiprime) right Goldie ring. By *LMR*-(11.22), R has only finitely many minimal prime ideals, say $\mathfrak{p}_1, \ldots, \mathfrak{p}_t$, and each $R/\mathfrak{p}_i$ is a prime right Goldie ring. Also,

$$Q^r_{c\ell}(R) \cong Q_1 \times \cdots \times Q_t,$$

where $Q_i = Q^r_{c\ell}(R_i)$. Since R is a reduced ring, *FC*-(12.6) implies that each $R/\mathfrak{p}_i$ is a domain, and therefore a right Ore domain, by *LMR*-(11.20). Thus, each Q_i is a division ring, and R is a right order in the finite direct product of division rings $Q_1 \times \cdots \times Q_t$.

The following four exercises are taken from a paper of C. Procesi and L. Small (J. Algebra **2**(1965), 80–84), where they used these results to give an alternative proof for the main equivalence (1) $\Leftrightarrow$ (2) in Goldie's Theorem in *LMR*-(11.13). *We assume, in these four exercises, that R is a semiprime ring satisfying* ACC *on right annihilators.*

Ex.11.21. Let $B \subseteq A$ be right ideals in R such that $\mathrm{ann}_\ell(A) \subsetneq \mathrm{ann}_\ell(B)$. Show that there exists $x \in A$ such that $xA \neq 0$ and $xA \cap B = 0$. In particular, B cannot be essential in A.

Solution. Since R satisfies ACC on right annihilators, it satisfies DCC on left annihilators (by *LMR*-(6.57)). Therefore, there exists a left annihilator U minimal with respect to $\mathrm{ann}_\ell(A) \subsetneq U \subseteq \mathrm{ann}_\ell(B)$. Then $UA \neq 0$, and so $UA\,UA \neq 0$ since R is semiprime. Pick elements $a \in A$ and $u \in U$ such that $UauA \neq 0$. It suffices to show that

$$(*) \qquad auA \cap B = 0,$$

for then the desired element $x \in A$ in the exercise can be chosen to be au. To prove $(*)$, assume, instead, that $0 \neq aua' \in B$ for some $a' \in A$. Then $U' := \mathrm{ann}_\ell(a') \cap U$ is a left annihilator with $\mathrm{ann}_\ell(A) \subseteq U' \subseteq U \subseteq \mathrm{ann}_\ell(B)$. But

$$Uaua' \subseteq UB = 0 \implies Uau \subseteq U', \quad \text{and}$$
$$UauA \neq 0 \implies Uau \nsubseteq \mathrm{ann}_\ell(A),$$

so $\mathrm{ann}_\ell(A) \neq U'$. The minimality of U then implies that $U' = U$; that is, $U \subseteq \mathrm{ann}_\ell(a')$. But then $Ua' = 0$, which contradicts $aua' \neq 0$ (since $au \in U$).

Ex.11.22. Deduce from Exercise 21 that any chain of right annihilators in R has length $\leq \mathrm{u.dim}\, R_R$.

Solution. Suppose $A_0 \supsetneqq A_1 \supsetneqq \cdots \supsetneqq A_n$ is a chain of right annihilators of length n. Since $\text{ann}_r(\text{ann}_\ell A_i) = A_i$, we have

$$\text{ann}_\ell(A_0) \subsetneqq \text{ann}_\ell(A_1) \subsetneqq \cdots \subsetneqq \text{ann}_\ell(A_n).$$

Applying Exercise 21, we see that each A_{i-1} contains a nonzero right ideal T_i such that $T_i \cap A_i = 0$ $(1 \leq i \leq n)$. We have then $T_1 \oplus \cdots \oplus T_n \subseteq A_0$, which implies that $n \leq \text{u.dim}\, R_R$.

Comment. The conclusion in this exercise could have been deduced easily from the existing results in *LMR*. We may assume that $\text{u.dim}\, R_R < \infty$, so R is a semiprime right Goldie ring. By *LMR*-(11.13), R is right nonsingular, so the desired result is contained in *LMR*-(7.51)$'$.

Ex. 11.23. Let $x, y \in R$. If $xR \subseteq_e R$ and $yR \subseteq_e R$, show that $xyR \subseteq_e R$.

Solution. For any right ideal $C \neq 0$, we would like to show that $C \cap xyR \neq 0$. Consider right ideals A, B defined by

$$A := \{s \in R : xs \in C\} \supseteq B := \text{ann}_r(x).$$

Then $xB = 0$, but $xA \neq 0$ (since $C \cap xR \neq 0$). Therefore, we are in the situation $\text{ann}_\ell(A) \subsetneqq \text{ann}_\ell(B)$ as in Exercise 21. By that exercise, there exists a nonzero right ideal $T \subseteq A$ such that $T \cap B = 0$. This T must contain an element $yz \neq 0$ $(z \in R)$ since $yR \subseteq_e R$. Now $yz \notin B \Rightarrow x(yz) \neq 0$, and $yz \in A \Rightarrow x(yz) \in C$. Therefore, $C \cap xyR$ contains the nonzero element xyz, as desired.

Ex.11.24. Let $a \in R$. If $aR \subseteq_e R$, show that a is regular in R.

Solution. Since $aR \subseteq_e R$, the last part of Exercise 21 implies that $\text{ann}_\ell(aR) = \text{ann}_\ell(R) = 0$. Therefore, $\text{ann}_\ell(a) = 0$. To show that $\text{ann}_r(a)$ is also zero, pick an integer n such that $\text{ann}_r(a^n) = \text{ann}_r(a^{n+1})$. Then

$$(*) \qquad a^n R \cap \text{ann}_r(a) = 0.$$

In fact, for $b \in a^n R \cap \text{ann}_r(a)$, we have $b = a^n s$ for some $s \in R$. Then $0 = ab = a^{n+1}s$ implies that $b = a^n s = 0$. On the other hand, by Exercise 23, $aR \subseteq_e R \Rightarrow a^n R \subseteq_e R$, so $(*)$ gives $\text{ann}_r(a) = 0$, as desired.

Ex. 11.25. Let R be a prime right Goldie ring, and I be an essential right ideal in R. Show that any coset $c + I$ $(c \in R)$ contains a regular element of R.

Solution. Since the right annihilators in R satisfy the DCC (by *LMR*-(7.51)$'$), there exists an element $a \in c + I$ with $\text{ann}_r(a)$ minimal. *We claim that a is regular.* Indeed, assume it is not. By the last exercise, aR cannot be essential in R_R, so there exists a nonzero right ideal B with $B \cap aR = 0$. For any $b \in B \cap I$, we have $aR \cap bR = 0$, which implies that $\text{ann}_r(a + b) \subseteq \text{ann}_r(a)$. Since

$$a + b \in c + I + b = c + I,$$

the minimal choice of $\text{ann}_r(a)$ implies that $\text{ann}_r(a+b) = \text{ann}_r(a)$. For any $x \in \text{ann}_r(a)$, we have then $0 = (a+b)x = bx$, which shows that $(B \cap I) \cdot \text{ann}_r(a) = 0$. Now, $B \cap I \neq 0$ (since $I \subseteq_e R_R$), and $\text{ann}_r(a) \neq 0$ (since a not regular implies that a is not right regular, by *LMR*-(11.14)(a)). This contradicts the fact that R is a prime ring.

Ex. 11.26. Prove the result in Exercise 25 for any *semiprime* right Goldie ring (by using a reduction to the prime case).

Solution. Let $\mathfrak{p}_1, \ldots, \mathfrak{p}_t$ be the minimal primes of R. Since R is semiprime, $\mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_t = 0$, and by *LMR*-(11.41), the $\mathfrak{p}_i$'s are annihilator ideals (in the sense of *LMR*-(11.37)). Let $A_i := \text{ann}(\mathfrak{p}_i)$, which is given by $\bigcap_{j \neq i} \mathfrak{p}_j$ according to *LMR*-(11.40). Consider the given right ideal $I \subseteq_e R_R$, and let $B_i = I \cap A_i$. Writing "bar" for the quotient map $R \to R/\mathfrak{p}_i$, we claim that $\bar{B}_i \subseteq_e \bar{R}$. Indeed, let $X \supseteq \mathfrak{p}_i$ be a right ideal such that $\bar{X} \cap \bar{B}_i = 0$, that is, $X \cap (B_i + \mathfrak{p}_i) = 0$. Since

$$B_i \cap \mathfrak{p}_i \subseteq \mathfrak{p}_1 \cap \cdots \cap \mathfrak{p}_t = 0,$$

this means that $X \cap B_i = 0$. Thus, $(X \cap A_i) \cap I = 0$, and so $X \cap A_i = 0$. It follows that $X \cdot A_i \subseteq X \cap A_i = 0$. Therefore,

$$X \subseteq \text{ann}(A_i) = \text{ann}(\text{ann}(\mathfrak{p}_i)) = \mathfrak{p}_i,$$

since $\mathfrak{p}_i$ is an annihilator ideal. This checks that $\bar{B}_i \subseteq_e \bar{R}$. Let $c \in R$ be a given element. Applying Exercise 25 to $\bar{R}$ (which is a prime right Goldie ring by *LMR*-(11.22)), we find an element $b_i \in B_i$ such that $c + b_i$ maps to a regular element of $R/\mathfrak{p}_i$. Now consider the element

$$a := c + b_1 + \cdots + b_t \in c + I.$$

For any $j \neq i$, we have $b_j \in A_j \subseteq \mathfrak{p}_i$, so the image of a in $R/\mathfrak{p}_i$ is the same as the image of $c + b_i$, which is regular in $R/\mathfrak{p}_i$. By *LMR*-(11.42), this implies that $a \in c + I$ is regular in R, as desired.

Comment. The result in this exercise actually holds under somewhat weaker hypotheses on the ring R. As long as R is a semiprime ring with ACC and DCC on right annihilators (or equivalently with ACC on both left and right annihilators), it can be shown that the conclusion of this exercise holds. The proof for this more general case requires a couple of extra steps, which the reader can find on pp. 14–15 of the book "Rings with Chain Conditions" by A.W. Chatters and C.R. Hajarnavis, Pitman, London, 1980.

Ex. 11.27. (Robson) Let R be any semiprime right Goldie ring, and J be an essential right ideal in R. Show that J is generated as a right ideal by the regular elements in J.

Solution. Since $J \subseteq_e R_R$, J does contain a regular element. Let I be the right ideal generated by the regular elements in J. Then $I \subseteq_e R_R$ by *LMR*-(11.13). Consider any $c \in J$. By the last exercise, there exists a regular element $a \in c + I \subseteq J$. Then $a \in I$, and we have $c \in a + I = I$. This shows that $J = I$, as desired.

Ex. 11.28. (Small) If a ring R is right perfect and right Rickart, show that it is semiprimary.

Solution. The desired result is a combination of a number of theorems. First, by Bass' Theorem (*FC*-(24.25)), the right perfect ring R satisfies DCC on principal left ideals, so by *LMR*-(6.59), R has no infinite set of nonzero orthogonal idempotents. Since R is also right Rickart, *LMR*-(7.55) implies that R satisfies ACC on *left* as well as on *right* annihilators. Finally, note that the Jacobson radical $\operatorname{rad} R$ is nil (since R is right perfect). Therefore, we can apply *LMR*-(11.49) to deduce that $\operatorname{rad} R$ is nilpotent. Since $R/\operatorname{rad} R$ is semisimple by assumption (that R is right perfect), it follows that R is semiprimary.

Comment. The above result of L. Small comes from his paper "Semihereditary rings" in Bull. Amer. Math. Soc. **73**(1967), 656–658.

Ex. 11.29. For any idempotent e in a semiprime ring R, show that the following are equivalent:

(a) e is central in R;
(b) $(1 - e)\, Re = 0$;
(c) eR is an ideal in R.

Solution. $(a) \Rightarrow (c)$ is clear. If (c) holds, then $Re \subseteq eR$, so

$$(1 - e)Re \subseteq (1 - e)\, eR = 0.$$

This shows that $(c) \Rightarrow (b)$. Finally, assume (b). Let $A = (1 - e)\, R$ and $B = eR$. Then $AB = (1 - e)\, ReR = 0$, so $(BA)^2 = B(AB)A = 0$. Since R is semiprime, $BA = 0$. From $(1 - e)\, Re = 0$, we have $re = ere$ for any $r \in R$; and from $eR(1 - e) = 0$, we have $er = ere$ for any $r \in R$. This yields (a).

Comment. If R is *not* semiprime, neither (b) nor (c) is sufficient to give (a). For instance, in the ring $R = \begin{pmatrix} k & k \\ 0 & k \end{pmatrix}$ where k is any nonzero ring, the idempotent $e = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ is easily seen to satisfy (b), (c), but e is not central in R since it does not commute with $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$.

Ex. 11.30. Let R be a semiprime ring with $\operatorname{u.dim} R_R < \infty$.

(1) Show that $\operatorname{soc}(R_R) = eR$ for a central idempotent $e \in R$.
(2) There exists a direct product decomposition $R \cong S \times T$ where S is a semisimple ring, and T is a semiprime ring with $\operatorname{soc}(T_T) = 0$.
(3) If $\operatorname{soc}(R_R) \subseteq_e R_R$, show that R is a semisimple ring.

Deduce from the above that a prime ring R is simple artinian iff u. dim $R_R < \infty$ and R has a minimal right ideal.

Solution. (1) Let $S = \mathrm{soc}(R_R)$. We have u. dim $S_R \leq$ u. dim $R_R < \infty$, so the semisimple module S_R must be f.g. By *ECRT*-(10.9), $S = eR$ for a suitable idempotent $e \in R$. Now $S = \mathrm{soc}(R_R)$ is an ideal of R, so by the last exercise, e is central in R.

For the reader's convenience, we recall briefly the proof for the existence of e. If $S \neq 0$, it contains a minimal right ideal A_1 which, by Brauer's Lemma (*FC*-(10.22)), has the form $e_1 R$ for an idempotent e_1. Then $S = A_1 \oplus B_1$ where $B_1 = S \cap (1 - e_1) R$. If $B_1 \neq 0$, then, again, B_1 contains a minimal right ideal $A_2 = e_2 R$ where $e_2^2 = e_2$. Then $A_1 \oplus A_2 = e_2' R$ for the idempotent $e_2' := e_1 + e_2 (1 - e_1)$, and

$$B_2 := S \cap (1 - e_2') R \subseteq B_1.$$

Continuing like this (and using u. dim $S_R < \infty$), we get eventually

$$S = A_1 \oplus \cdots \oplus A_n = e_n' R$$

for some n, where $e_n' = e_n'^2 \in R$.

(2) Since $S = eR$ with $e = e^2$ central, we have $R = S \times T$ where $T = (1 - e) R$. Then $\mathrm{soc}(R_R) = \mathrm{soc}(S_S) \times \mathrm{soc}(T_T)$ implies that $\mathrm{soc}(S_S) = S$ and $\mathrm{soc}(T_T) = 0$. The former implies that S_S is a semisimple module, so S is a semisimple ring; and clearly T is a semiprime ring.

(3) Since S_R is a direct summand of R_R, $S \subseteq_e R_R \Rightarrow S = R$, so R itself is a semisimple ring.

For the last statement in the Exercise, the "only if" part is, of course, clear. Conversely, let R be a prime ring with a minimal right ideal, and such that u. dim $R_R < \infty$. Note that $S := \mathrm{soc}(R_R)$ is an ideal. If A is any right ideal such that $S \cap A = 0$, then $A \cdot S \subseteq A \cap S = 0$. Since R is prime and $S \neq 0$, we must have $A = 0$. This shows that $S \subseteq_e R_R$, and so (3) above implies that R is semisimple. Since R is prime, it must be a simple artinian ring.

Comment. For a nice application of part (3), see Exercise (19.9′), where it is shown that a semiprime ring R with R_R "finitely cogenerated" must be semisimple.

Concerning part (1), it is worth pointing out that, *for any semiprime ring* R, $\mathrm{soc}(R_R) = \mathrm{soc}({}_R R)$. (This follows easily from *FC*-(11.9).) The hypothesis that u. dim $R_R < \infty$ is crucial for the validity of (1) in this Exercise. Here is an example. Let V_k be an infinite-dimensional right vector space over a division ring k, and let $R = \mathrm{End}(V_k)$. This is a prime ring, so the only central idempotents in R are 0 and 1. It is easy to see that $\mathrm{soc}(R_R) \neq R$ (since R is not semisimple) and $\mathrm{soc}(R_R) \neq 0$ (fR is a minimal right ideal for any $f : V \to V$ of rank 1). Therefore, the conclusion

(1) does not hold for R. In fact, by *ECRT*-(11.18),

$$S := \operatorname{soc}(R_R) = \operatorname{soc}({}_R R) = \{g \in R : \dim_k g(V) < \infty\},$$

and an easy application of linear algebra shows that S is both right and left essential in R. Thus, R yields a nice example of a prime (non-Goldie) ring in which a right and left essential ideal contains *no* regular elements.

Ex. 11.31. (Attarchi) Let R be a ring such that $dR \subseteq_e R_R$ whenever $\operatorname{ann}_r(d) = 0$, and $Rd \subseteq_e {}_R R$ whenever $\operatorname{ann}_\ell(d) = 0$. Let S be the ring of 2×2 upper triangular matrices over R. Show that $\alpha = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix}$ is regular in S iff a and c are both regular in R.

Solution. First, assume that a, c are regular in R, and say $\alpha\alpha' = 0 \in S$, where $\alpha' = \begin{pmatrix} a' & b' \\ 0 & c' \end{pmatrix}$. Then we have $aa' = cc' = 0$ and $ab' + bc' = 0$. It follows that $a' = c' = 0$, and $0 = ab' + bc' = ab'$ implies that $b' = 0$. Thus $\alpha' = 0$. A similar argument shows that $\alpha'\alpha = 0 \Rightarrow \alpha' = 0$.

Conversely, assume that α is regular in S. We would like to show that a and c are both regular in R. If $aa' = 0$, then

$$\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}\begin{pmatrix} a' & 0 \\ 0 & 0 \end{pmatrix} = 0 \implies \begin{pmatrix} a' & 0 \\ 0 & 0 \end{pmatrix} = 0 \implies a' = 0,$$

so $\operatorname{ann}_r(\mathrm{a}) = 0$. In a similar vein, if $c'c = 0$, then

$$\begin{pmatrix} 0 & 0 \\ 0 & c' \end{pmatrix}\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} = 0 \implies \begin{pmatrix} 0 & 0 \\ 0 & c' \end{pmatrix} = 0 \implies c' = 0.$$

Thus, we have $\operatorname{ann}_\ell(c) = 0$, and so by hypothesis $Rc \subseteq_e {}_R R$.

By Exercise 3.7, we then have

$$(*) \qquad (Rc)\, b^{-1} := \{t \in R : tb \in Rc\} \subseteq_e {}_R R.$$

We claim that $\operatorname{ann}_\ell(a) \cap (Rc)\, b^{-1} = 0$. In fact, for any

$$x \in \operatorname{ann}_\ell(a) \cap (Rc)\, b^{-1},$$

we have $xb = yc$ for some $y \in R$, and so

$$\begin{pmatrix} x & -y \\ 0 & 0 \end{pmatrix}\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} = \begin{pmatrix} xa & xb - yc \\ 0 & 0 \end{pmatrix} = 0.$$

Since $\alpha = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix}$ is regular in S, we have $\begin{pmatrix} x & -y \\ 0 & 0 \end{pmatrix} = 0$ and so $x = 0$, as claimed. From $(*)$, $\operatorname{ann}_\ell(a) \cap (Rc)\, b^{-1} = 0$ implies $\operatorname{ann}_\ell(a) = 0$, so $a \in R$ is regular. A similar argument (using the hypothesis that $Rd \subseteq_e R_R$ whenever $\operatorname{ann}_\ell(d) = 0$) implies that $\operatorname{ann}_r(c) = 0$, so $c \in R$ is also regular.

Comment. Recall from Exercise 1(b) that, under the assumption that $\mathrm{u.dim}\,(R_R) < \infty$, we will have $\mathrm{ann}_r\,(d) = 0 \Rightarrow dR \subseteq_e R_R$; and a similar remark holds for left ideals. Therefore, if R is a ring with u. dim $(R_R) < \infty$ and $\mathrm{u.dim}\,({}_RR) < \infty$, the conclusion of the present exercise applies.

H. Attarchi's result in this exercise appeared in the paper of A. W. Chatters, "Three examples concerning the Ore condition in noetherian rings," Proc. Edinburgh Math. Soc. **23**(1980), 187–192.

Ex. 11.32. Let R and S be as in the last exercise. If R is right Ore, with $Q = Q^r_{c\ell}\,(R)$, show that S is also right Ore, with $Q^r_{c\ell}\,(S)$ given by the ring T of 2×2 upper triangular matrices over Q.

Solution. It suffices to check that (1) every regular element of S is a unit in T, and (2) every element of T has the form $\beta\alpha^{-1}$ where $\beta \in S$, and α is a regular element of S.

To check (1), let $\alpha = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix}$ be regular in S. By the last exercise, a, c are both regular in R, so a^{-1}, c^{-1} exist in Q. A direct calculation now shows that $\begin{pmatrix} a^{-1} & -a^{-1}bc^{-1} \\ 0 & c^{-1} \end{pmatrix} \in T$ is an inverse of α.

To check (2), consider any $\gamma = \begin{pmatrix} q_1 & q_2 \\ 0 & q_3 \end{pmatrix} \in T$, where all $q_i \in Q$. Since R is right Ore and $Q = Q^r_{c\ell}\,(R)$, we can express the q_i's with a "common denominator," say $q_i = a_i r^{-1}$ where $a_i \in R$ $(i = 1, 2, 3)$, and r is a regular element in R. Then

$$\gamma = \begin{pmatrix} a_1r^{-1} & a_2r^{-1} \\ 0 & a_3r^{-1} \end{pmatrix} = \begin{pmatrix} a_1 & a_2 \\ 0 & a_3 \end{pmatrix} \begin{pmatrix} r & 0 \\ 0 & r \end{pmatrix}^{-1},$$

so we have checked (2) with $\beta = \begin{pmatrix} a_1 & a_2 \\ 0 & a_3 \end{pmatrix} \in S$ and $\alpha = \begin{pmatrix} r & 0 \\ 0 & r \end{pmatrix}$ is a regular element of S.

Comment. This exercise is taken from the paper of A. W. Chatters referenced in the *Comment* on the last exercise. Chatters also pointed out that the hypotheses imposed on R at the beginning of Exercise 31 cannot be removed from the present exercise. In fact, Chatters constructed a right noetherian ring R that is right Ore, but the ring S of 2×2 upper triangular matrices over R is *not* right Ore. For such a ring R, we have $\mathrm{u.dim}\,(R_R) < \infty$, so $dR \subseteq_e R_R$ whenever $\mathrm{ann}_r\,(d) = 0$. This shows that the other hypothesis ($Rd \subseteq_e {}_RR$ whenever $\mathrm{ann}_\ell(d) = 0$) cannot be removed from Exercise 31. For the ring R constructed by Chatters, it is also true that the full matrix ring $\mathbb{M}_2\,(R)$ is not right Ore.

§12. Artinian Rings of Quotients

Having characterized the class of rings that are right orders in semisimple rings, we proceed to characterize more generally those rings that are right orders in right artinian rings.

The key object to work with here is $\mathcal{C}(N)$, which denotes the multiplicative set of elements $r \in R$ such that $\bar{r} = r + N$ is a regular element in R/N. For this notation to be meaningful, N can be any ideal in R. Of course, $\mathcal{C}(0)$ is just the multiplicative set of regular elements of R, which was also denoted earlier by $\mathcal{C}_R$.

Suppose a ring R is such that its prime radical $N := \text{Nil}_* R$ is nilpotent, and $S := R/N$ is semiprime right Goldie. Then it is possible to define a *Goldie ρ-rank* for any right R-module M by taking any filtration

$$0 = M_n \subseteq \cdots \subseteq M_0 = M \text{ with } M_i N \subseteq M_{i+1} \quad (\forall\, i),$$

and setting $\rho_R(M) = \sum \text{rank}_S(M_i/M_{i+1})$. Here, $\text{rank}_S A$ denotes Goldie's reduced rank of an S-module A_S defined in *LMR*-§7C. The Goldie ρ-rank has the advantage of being additive over short exact sequences of right R-modules, and *we have*

> $\rho_R(M) = 0$ *iff, for every* $m \in M$, *there exists* $r \in \mathcal{C}(N)$ *such that* $mr = 0$ (*LMR*-(12.9)).

We note in particular, that $\rho_R(M)$ is defined over any right noetherian ring R, with $0 \le \rho_R(M) < \infty$ whenever M is f.g. (*LMR*-(12.8)).

The criterion for a ring R to be a right order in a right artinian ring is stated in terms of $\mathcal{C}(0)$ and $\mathcal{C}(N)$, where $N := \text{Nil}_* R$. The necessary and sufficient condition is given by the following (by *LMR*-(12.10)):

(A) N *is nilpotent.*
(B) $S = R/N$ *is* (*semiprime*) *right Goldie.*
(C) $\rho_R(R) < \infty$.
(D) $\mathcal{C}(N) \subseteq \mathcal{C}(0)$.

If these statements all hold, then in fact $\mathcal{C}(N) = \mathcal{C}(0)$. In case R is right noetherian, (A), (B), (C) are all automatic, so the criterion for R to be a right order in a right artinian ring boils down to the single condition $\mathcal{C}(N) \subseteq \mathcal{C}(0)$ for $N = \text{Nil}_* R$. If, moreover, R is commutative, the condition means that all the associated primes of R_R are minimal primes (*LMR*-(12.21)).

In the commutative case, the above theory is strong enough to imply that *any commutative noetherian ring can be embedded in a commutative artinian ring* (*LMR*-(12.25)). However, Exercise 8 below shows that not every right noetherian ring can be embedded in a 1-sided artinian ring or even just a left noetherian ring.

If a right noetherian ring R satisfies the condition $\mathcal{C}(N) \subseteq \mathcal{C}(0)$, then $Q^r_{c\ell}(R)$ exists and is right artinian, according to the above theory. In particular, R is a right Ore ring. However, without the $\mathcal{C}(N) \subseteq \mathcal{C}(0)$ condition, a right noetherian ring R need not be right Ore. An example due to L. Small is given in *LMR*-(12.27). But of course, a right noetherian domain is always a right Ore domain, by *LMR*-(10.23).

The exercises in this section focus on the relations between $\mathcal{C}(0)$, $\mathcal{C}(N)$, and on the interplay between right noetherian and right artinian rings. The last exercise (Ex. 12.11) offers a nice application of Goldie's ρ-rank to the theory of Euler characteristics for modules with finite free resolutions.

Exercises for §12

In the following Exercises, $N = \mathrm{Nil}_* R$ denotes the lower nilradical of R.

Ex. 12.1. Name a ring R that is not right noetherian, but has the following properties:

(A) N is nilpotent;
(B) R/N is (semiprime) right Goldie;
(C) $\rho_R(R_R) < \infty$; and
(D) $\mathcal{C}(N) \subseteq \mathcal{C}(0)$.

Solution. Let R be any commutative non noetherian domain. Then $N = 0$, so (A), (B), (D) are trivially satisfied. For $Q = Q^r_{c\ell}(R)$ (quotient field of R), the ρ-rank of R_R is just

$$\mathrm{rank}_R(R) = \mathrm{length}_Q(R \otimes_R Q) = \mathrm{length}_Q(Q) = 1,$$

so we have (C).

Ex. 12.2. Show that a ring R is right artinian iff it is right noetherian and $\mathcal{C}(N) \subseteq \mathrm{U}(R)$.

Solution. First assume R is right artinian. Then R is also right neotherian by the Hopkins-Levitzki Theorem (*FC*-(4.15)). Since N is nil, *LMR*-(12.1) implies that $\mathcal{C}(N) = \mathcal{C}(0)$, and by *LMR*-(11.6)(2), $\mathcal{C}(0) = \mathrm{U}(R)$.

Conversely, assume that R is right noetherian with $\mathcal{C}(N) \subseteq \mathrm{U}(R)$. Then, of course, $\mathcal{C}(N) \subseteq \mathcal{C}(0)$, so Small's Theorem *LMR*-(12.15) implies that $Q = Q^r_{c\ell}(R)$ exists, is right artinian, and R is a right order in Q. By *LMR*-(12.15), we also have $\mathcal{C}(0) \subseteq \mathcal{C}(N)$, so now $\mathcal{C}(0) \subseteq \mathrm{U}(R)$. Therefore, $R = Q$ is right artinian.

Ex. 12.3. Let $\mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z}$, where $n \geq 2$. Compute $\mathcal{C}(0)$ and $\mathcal{C}(N)$ for the ring $R = \begin{pmatrix} \mathbb{Z} & 0 \\ \mathbb{Z}_n & \mathbb{Z}_n \end{pmatrix}$, and show that R is a 2-sided order in a noetherian, nonartinian ring.

Solution. Consider the ideal $I = \begin{pmatrix} 0 & 0 \\ \mathbb{Z}_n & \mathrm{Nil}(\mathbb{Z}_n) \end{pmatrix} \subseteq R$. A direct computation shows that I is a nilpotent ideal. The quotient ring

$$R/I \cong \mathbb{Z} \times (\mathbb{Z}_n/\mathrm{Nil}(\mathbb{Z}_n))$$

is semiprime, so we must have $I = N$. From the above computation of $R/I = R/N$, we see that

$$\mathcal{C}\,(N) = \left\{ \begin{pmatrix} x & 0 \\ \bar{y} & \bar{z} \end{pmatrix} : \; x, y, z \in \mathbb{Z}; \;\; x \neq 0; \;\; p_i \nmid z \;\; (\forall\, i) \right\},$$

where $n = p_1^{a_1} \dots p_k^{a_k}$ is the complete factorization of n. We claim that

$$(*) \qquad \mathcal{C}\,(0) = \left\{ \begin{pmatrix} x & 0 \\ \bar{y} & \bar{z} \end{pmatrix} : \;\; p_i \nmid x \;\; (\forall\, i)\,, \text{ and } \; p_i \nmid z \;\; (\forall\, i) \right\}.$$

Indeed, if, say $p_i | x$, then

$$\begin{pmatrix} 0 & 0 \\ \bar{m} & 0 \end{pmatrix} \begin{pmatrix} x & 0 \\ \bar{y} & \bar{z} \end{pmatrix} = 0 \;\text{ for }\; m = n/p_i.$$

And, if $p_i |\, z$, then

$$\begin{pmatrix} x & 0 \\ \bar{y} & \bar{z} \end{pmatrix} \begin{pmatrix} 0 & 0 \\ \bar{m} & 0 \end{pmatrix} = 0 \;\text{ for }\; m = n/p_i.$$

This proves the inclusion "$\subseteq$" in $(*)$. Conversely, if neither x nor z is divisible by any p_i, then $\begin{pmatrix} x & 0 \\ \bar{y} & \bar{z} \end{pmatrix}$ is left regular. In fact, if

$$0 = \begin{pmatrix} u & 0 \\ \bar{v} & \bar{w} \end{pmatrix} \begin{pmatrix} x & 0 \\ \bar{y} & \bar{z} \end{pmatrix} = \begin{pmatrix} ux & 0 \\ \overline{vx} + \overline{wy} & \overline{wz} \end{pmatrix},$$

then we must have $u = 0$, $\bar{w} = 0$, and $0 = \overline{vx} + \overline{wy}$ implies $\bar{v} = 0$. Similarly, we can show that $\begin{pmatrix} x & 0 \\ \bar{y} & \bar{z} \end{pmatrix}$ is right regular, so we have established $(*)$. In particular, we have $\mathcal{C}\,(0) \subsetneq \mathcal{C}\,(N)$.

Let S be the multiplicative set $\mathbb{Z} \setminus \bigcup_{i=1}^{k} p_i \mathbb{Z}$, and write $\mathbb{Z}_S$ for the localization of $\mathbb{Z}$ at S. Since any $m \in S$ acts invertibly on $\mathbb{Z}_n$, $\mathbb{Z}_n$ is an $\mathbb{Z}_S$-module, so that we can form the triangular ring

$$Q := \begin{pmatrix} \mathbb{Z}_S & 0 \\ \mathbb{Z}_n & \mathbb{Z}_n \end{pmatrix} \supseteq R.$$

It is easy to show that $\mathcal{C}\,(0) \subseteq \mathrm{U}\,(Q)$, and that any element of Q has the form rt^{-1} where $r \in R$ and

$$t \in T := \left\{ \begin{pmatrix} s & 0 \\ 0 & \bar{s} \end{pmatrix} : \; s \in S \right\} \subseteq \mathcal{C}\,(0)\,.$$

Since T is a *central* multiplicative set in R, we see that R is a 2-sided order in Q. By *FC*-(1.22), Q is a noetherian ring, but is neither a left nor a right artinian ring.

Comment. $\mathcal{C}(0) \subseteq \mathcal{C}(N)$ is an expected property of the (right) noetherian ring R: see *LMR*-(12.15). By the same result, the failure of $\mathcal{C}(0) \subseteq \mathcal{C}(N)$ to be an equality implies that R *cannot* be a right order in a right artinian ring. This exercise is a generalization of *LMR*-(12.19)(2), which is the case where n is a prime.

Ex. 12.4. Let k be a field.

(1) Compute $\mathcal{C}(0)$ and $\mathcal{C}(N)$ for the ring $R = \begin{pmatrix} k & k[x] \\ 0 & k[x] \end{pmatrix}$.

(2) Show that R is right noetherian and is a right order in the right artinian ring $Q = \begin{pmatrix} k & k(x) \\ 0 & k(x) \end{pmatrix}$.

(3) Show that every right regular element of R is (right) regular in Q, but a left regular element of R need not be left regular in Q.

Solution. (1) Let N_0 be the ideal $\begin{pmatrix} 0 & k[x] \\ 0 & 0 \end{pmatrix}$ in R. Since $N_0^2 = 0$ and $R/N_0 \cong k \times k[x]$ is semiprime, it follows that N_0 is precisely the lower nilradical N. From the structure of R/N, it is clear that

$$\mathcal{C}(N) = \left\{ \begin{pmatrix} a & f \\ 0 & g \end{pmatrix} : ag \neq 0 \right\}.$$

On the other hand, an easy computation shows that $\mathcal{C}(0)$ is also given by the RHS of the above equation. Therefore, $\mathcal{C}(0) = \mathcal{C}(N)$.

(2) By *FC*-(1.22), R is right noetherian, since k and $k[x]$ are noetherian, and $k[x]_{k[x]}$ is a noetherian module. In view of $\mathcal{C}(0) = \mathcal{C}(N)$ (proved above), Small's Criterion *LMR*-(12.15) applies to show that R is a right order in a right artinian ring. In fact, in *LMR*-(10.27)(f), it is already shown that R is a right order in $Q := \begin{pmatrix} k & k(x) \\ 0 & k(x) \end{pmatrix}$. And, by *FC*-(1.22) again, Q is a right artinian ring.

(3) Let $\mathcal{C}'(0)$ be the set of right regular elements. We claim that the inclusion $\mathcal{C}(0) \subseteq \mathcal{C}'(0)$ is an equality. Indeed, let $\alpha = \begin{pmatrix} a & f \\ 0 & g \end{pmatrix} \in R \backslash \mathcal{C}(0)$. Then we have either $a = 0$ or $g = 0$. If $a = 0$, then $\alpha\beta = 0$ for $\beta = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$; if $g = 0$ but $a \neq 0$, then $\alpha\beta' = 0$ for $\beta' = \begin{pmatrix} 0 & a^{-1}f \\ 0 & -1 \end{pmatrix}$. In any case, we'll have $\alpha \notin \mathcal{C}'(0)$. Therefore, $\mathcal{C}'(0) = \mathcal{C}(0) \subseteq \mathrm{U}(Q)$. Now consider the element $\gamma = \begin{pmatrix} 0 & 1 \\ 0 & x \end{pmatrix} \in R$. We claim that $\gamma \in {}'\mathcal{C}(0)$ (the set of left regular elements

in R). Indeed, if

$$0 = \begin{pmatrix} a & f \\ 0 & g \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & x \end{pmatrix} = \begin{pmatrix} 0 & a + fx \\ 0 & gx \end{pmatrix},$$

clearly $a = 0 \in k$ and $f = g = 0 \in k[x]$. However, γ is *not* left regular in Q, since

$$\begin{pmatrix} -1 & x^{-1} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & x \end{pmatrix} = 0 \in Q.$$

Comment. (a) In (1) and (3), the equation $\mathcal{C}(0) = \mathcal{C}'(0) = \mathcal{C}(N)$ is to be expected of *any* right noetherian ring: see *LMR*-(12.15).

(b) In *LMR*-(10.27)(f), it was shown that the ring R is not left Ore. Therefore, R is not a left order in any ring.

Ex 12.5. Prove or disprove: every left regular element in a right artinian ring R is a unit.

Solution. If $a \in R$ is right regular, consideration of left multiplication by a on R_R shows quickly that $a \in \mathrm{U}(R)$. If $a \in R$ is *left* regular instead, it is still true that $a \in \mathrm{U}(R)$, but the proof is a bit more complicated. For the details, see *ECRT*-(21.23). For another proof based on mathematical induction on length (R_R), see Lemma 25.6 in Passman's book "A course in Ring Theory," Wadsworth & Brooks/Cole, 1991 (reprinted by Chelsea-AMS, 2004).

Comment. The right noetherian ring R and the right artinian ring Q in Exercise 4 provide good illustrative examples. In R, the element $\gamma = \begin{pmatrix} 0 & 1 \\ 0 & x \end{pmatrix}$ is not right regular but is left regular. In contrast, in Q, the fact that γ is not right regular *guarantees* that γ is also not left regular.

Ex. 12.6. Name a ring R for which N is nilpotent, R/N is noetherian, R_R has finite ρ-rank, but ${}_RR$ has infinite ρ-rank.

Solution. For the reader's convenience, we recall (from *LMR*-(12.10)) the necessary and sufficient conditions for a ring R to be a right order in a right artinian ring. These are the following (collectively):

(A) *N is nilpotent.*
(B) *R/N is (semiprime) right Goldie.*
(C) $\rho_R(R_R) < \infty$.
(D) $\mathcal{C}(N) = \mathcal{C}(0)$.

Now consider the ring R constructed in Exercise 4. In that exercise, we have already checked the truth of (A), (B) and (D), and have also checked directly that R is a right order in the right artinian ring $Q = \begin{pmatrix} k & k(x) \\ 0 & k(x) \end{pmatrix}$. Therefore, the condition (C) must hold. On the other hand, the conditions

(A), (B) and (D) being left-right symmetric, if $\rho_R({}_RR)$ was finite, R would have been also a left order in some left artinian ring. Now, in the *Comment* following Exercise 4, we have already pointed out that R is not a right order in any ring. Therefore, we must have $\rho_R\left({}_RR\right) = \infty$.

Comment. Of course, it will be more pleasing to be able to compute $\rho_R(R_R)$ and $\rho_R({}_RR)$ directly. We start with the former. Since R is a right order in the right artinian ring Q, *LMR*-(12.10) implies that $\rho_R(R_R) =$ length(Q_Q). Now the Jacobson (or prime) radical for Q is given by $J = \begin{pmatrix} 0 & k\,(x) \\ 0 & 0 \end{pmatrix}$, and we have $Q/J \cong k \times k\,(x)$. Therefore, Q has exactly two simple right modules V_1 and V_2, given respectively by k and $k(x)$, with Q acting via the projections of Q (modulo J) to these two fields. In particular, length $(Q/J)_Q = 2$. Observing that

$$\begin{pmatrix} 0 & h \\ 0 & 0 \end{pmatrix} \begin{pmatrix} a & f \\ 0 & g \end{pmatrix} = \begin{pmatrix} 0 & hg \\ 0 & 0 \end{pmatrix} \quad (\forall\, f,\, g,\, h \in k\,(x)\,;\ a \in k)\,,$$

we see further that $J_Q \cong V_2$. Therefore, length $(J_Q) = 1$, and

$$\text{length}\,(Q_Q) = \text{length}\,(J_Q) + \text{length}\,(Q/\,J)_Q = 1 + 2 = 3,$$

so we have $\rho_R(R_R) = 3$. To compute $\rho_R({}_RR)$, we can use the Loewy series $0 \subseteq N \subseteq R$ in ${}_RR$. The left R/N-action on N is given by

$$\begin{pmatrix} a & f \\ 0 & g \end{pmatrix} \begin{pmatrix} 0 & h \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & ah \\ 0 & 0 \end{pmatrix} \quad (\forall\, f,\, g,\, h \in k\,[x]\,;\ a \in k)\,.$$

This is just the k-action on $k[x]$, pulled back along the maps

$$R \longrightarrow R/N = k \times k[x] \xrightarrow{\pi_1} k.$$

Since $\dim_k k[x] = \infty$, we have $\rho_R\left({}_RN\right) = \infty$, and so

$$\rho_R\left({}_RR\right) = \rho_R\left({}_R\left(R/\,N\right)\right) + \rho_R\left({}_RN\right) = \infty$$

as well.

Ex. 12.7. Let $R \neq (0)$ be a ring with a faithful, singular right module M, and let A be the triangular ring $\begin{pmatrix} \mathbb{Z} & M \\ 0 & R \end{pmatrix}$, where R is viewed as a $(\mathbb{Z},\, R)$-bimodule.

(1) Show that, for any finite set $N \subseteq M$, $\text{ann}_r^A \begin{pmatrix} 0 & N \\ 0 & 0 \end{pmatrix} \supseteq \begin{pmatrix} 0 & 0 \\ 0 & \mathfrak{A} \end{pmatrix}$ for some essential right ideal $\mathfrak{A} \subseteq R$.

(2) Show that $\text{ann}_r^A \begin{pmatrix} 0 & M \\ 0 & 0 \end{pmatrix} \neq \text{ann}_r^A \begin{pmatrix} 0 & N \\ 0 & 0 \end{pmatrix}$ for any finite subset $N \subseteq M$. Deduce from Exercise (6.21) that A does not satisfy DCC on right annihilators.

(3) Using (2), show that A cannot be embedded in a right artinian or a left noetherian ring.

Solution. (1) Let $N = \{m_1, \ldots, m_n\} \subseteq M$. Since M_R is a singular module, $\text{ann}\,(m_i) \subseteq_e R_R$ for every i, and therefore $\mathfrak{A} = \bigcap_{i=1}^n \text{ann}\,(m_i) \subseteq_e R_R$ by *LMR*-(3.6)(a). By the definition of $\mathfrak{A}$, we have $\begin{pmatrix} 0 & N \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & \mathfrak{A} \end{pmatrix} = 0$ in A, so (1) follows.

(2) Suppose $\text{ann}_r^A \begin{pmatrix} 0 & M \\ 0 & 0 \end{pmatrix} = \text{ann}_r^A \begin{pmatrix} 0 & N \\ 0 & 0 \end{pmatrix}$ for some finite set $N \subseteq M$. Then, for the right ideal $\mathfrak{A} \subseteq R$ constructed above, we will have

$$0 = \begin{pmatrix} 0 & M \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & \mathfrak{A} \end{pmatrix} = \begin{pmatrix} 0 & M\mathfrak{A} \\ 0 & 0 \end{pmatrix}.$$

Since M_R is also faithful, this implies that $\mathfrak{A} = 0$, which contradicts $\mathfrak{A} \subseteq_e R_R$. Therefore, the finite subset $N \subseteq M$ above cannot exist. It follows from Exercise (6.21) that the ring A does not satisfy DCC on right annihilators.

(3) Suppose $A \subseteq B$, where B is a ring that is either right artinian or left noetherian. In the former case, B clearly satisfies DCC on the right annihilators. In the latter case, B satisfies ACC on left annihilators, and therefore satisfies DCC on right annihilators (by *LMR*-(6.57)). But then by *LMR*-(6.61), the subring $A \subseteq B$ also satisfies DCC on right annihilators, in contradiction to (2). Therefore, (3) follows.

Ex. 12.8. Construct a right noetherian ring A that cannot be embedded in a right artinian or a left noetherian ring.

Solution. Of course, the idea is to use (3) in the Exercise above. We have to make sure that A is *right noetherian.* According to *FC*-(1.22), this will follow if R itself is right neotherian, and M_R is a f.g. (and hence noetherian) R-module. However, we also need M to be *faithful* and *singular* as a (right) R-module.

Let R be any right noetherian simple domain that is not a division ring. Let $\mathfrak{B}$ be any nonzero right ideal $\subsetneq R$, and let M be the cyclic right R-module $R/\mathfrak{B}$. Since the annihilator of M in R is an ideal $\subsetneq R$ and R is simple, M_R must be a faithful module. On the other hand, by *LMR*-(10.23), R is a right Ore domain, so necessarily $\mathfrak{B} \subseteq_e R_R$, and therefore, by *LMR*-(7.6)(3), $M = R/\mathfrak{B}$ is a singular right R-module, as desired.

Comment. A good choice for the ring R above is the first Weyl algebra $A_1(k)$ over a field k of characteristic zero. It is well-known that such a ring R is a (2-sided) noetherian simple domain.

Ex. 12.9. For an ideal $\mathfrak{A} \subseteq R$, let $\mathcal{C}(\mathfrak{A})$ (resp. $\mathcal{C}'(\mathfrak{A})$) be the set of elements $r \in R$ such that $r + \mathfrak{A}$ is regular (resp. right regular) in $R/\mathfrak{A}$. If R is right noetherian and $\mathfrak{A}$ is a semiprime ideal, show that $\mathcal{C}(\mathfrak{A}) = \mathcal{C}'(\mathfrak{A})$. Exhibit an example to show that the hypothesis that $\mathfrak{A}$ be semiprime cannot be removed.

Solution. Since R is right noetherian and $\mathfrak{A}$ is semiprime, the factor ring $\bar{R} = R/\mathfrak{A}$ is a semiprime right Goldie ring. If $r + \mathfrak{A}$ is right regular in

$\bar{R}$, *LMR*-(11.14)(a) implies that $r + \mathfrak{A}$ is in fact regular. This shows that $\mathcal{C}(\mathfrak{A}) = \mathcal{C}'(\mathfrak{A})$.

To construct a counterexample in the case where $\mathfrak{A}$ is *not* semiprime, consider the triangular ring $R = \begin{pmatrix} \mathbb{Z} & 0 \\ \mathbb{Z}/2\mathbb{Z} & \mathbb{Z} \end{pmatrix}$. The ideal $\mathfrak{A} = 0$ is not semiprime, since $\begin{pmatrix} 0 & 0 \\ \mathbb{Z}/2\mathbb{Z} & 0 \end{pmatrix} \subseteq R$ is an ideal of square zero. The element $r = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$ is in $\mathcal{C}'(0)$ since

$$0 = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x & 0 \\ z & y \end{pmatrix} = \begin{pmatrix} 2x & 0 \\ z & y \end{pmatrix}$$

implies that $x = y = 0 \in \mathbb{Z}$ and $z = 0 \in \mathbb{Z}/2\mathbb{Z}$. However, $r \notin \mathcal{C}(0)$, since $\begin{pmatrix} 0 & 0 \\ \bar{1} & 0 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} = 0$. Therefore, $\mathcal{C}(0) \subsetneq \mathcal{C}'(0)$.

Ex. 12.10 . For any commutative ring R, show that $Q_{c\ell}(R)$ is a semilocal ring iff the set of 0-divisors of R is a finite union of prime ideals.

Solution. Let T be the set of 0-divisors of R, and $S = R \backslash T$. If

$$T = \mathfrak{p}_1 \cup \cdots \cup \mathfrak{p}_n$$

where the $\mathfrak{p}_i$'s are prime ideals, we may assume that $\mathfrak{p}_i \nsubseteq \mathfrak{p}_j$ for $i \neq j$. By a standard localization argument (as that in the proof of *LMR*-(8.31)(2)), the maximal ideals of $K := Q_{c\ell}(R)$ are precisely $(\mathfrak{p}_i)_S$ $(1 \leq i \leq n)$, so K is semilocal.

Conversely, assume K is semilocal, say with maximal ideals $M_1, \ldots, M_n$. Let $\mathfrak{p}_i = M_i \cap R$, which are prime ideals in R. Clearly $\mathfrak{p}_i \subseteq T$ for otherwise M_i would contain a unit of K. Therefore $\mathfrak{p}_1 \cup \cdots \cup \mathfrak{p}_n \subseteq T$. On the other hand, if $x \in T$, then $xK \neq K$ implies that $xK \subseteq M_i$ for some i, and so $x \in M_i \cap R = \mathfrak{p}_i$. Therefore, we have $T = \mathfrak{p}_1 \cup \cdots \cup \mathfrak{p}_n$, as desired.

Comment. The result in this exercise was first observed by E. D. Davis in "Overrings of commutative rings II", Trans. Amer. Math. Soc. **110**(1964), 196–212. Commutative rings R for which the set of 0-divisors is a finite union of prime ideals have now been christened. They are known as "commutative rings with few 0-divisors." For many interesting facts about such rings, see J. Huckaba's book "Commutative Rings with Zero Divisors," Monographs in Pure and Applied Math., Vol. 117, Marcel Dekker, New York-Basel-Hong Kong, 1988. See also the following two related papers of C. Faith: "Annihilator ideals, associated primes, and Kasch-McCoy commutative rings," Comm. Alg. **19**(1991), 1867–1892, and "Rings with few zero divisors are those with semilocal Kasch quotient rings," Houston J. Math. **22**(1996), 687–670. However, readers of these two papers must exercise extreme caution, since Faith used the term "associated prime" in a non-standard sense—meaning maximal point annihilators.

Ex. 12.11. For a ring R with IBN (the "invariant basis number" property, let χ (M) denote the Euler characteristic of a right R-module with a finite free resolution (FFR); see Ex. 5.21A.

(A) Give an example of M_R for which $\chi(M) < 0$.
(B) Let R be a nonzero right noetherian ring. For a right R-module M with FFR, use the ρ-rank to show that $\chi(\mathrm{M}) \geq 0$, with equality iff for every $m \in M$, $mr = 0$ for some $r \in \mathcal{C}(N)$, where $N = \mathrm{Nil}_*(R)$ is the lower nilradical of R.

Solution. (A) If R fails to satisfy the (right) "Strong Rank Condition" (in *LMR*-(1.20)(2)), then there exists an injection $R^\ell \xrightarrow{\alpha} R^k$ for some $k < \ell < \infty$. In this case, $M = \mathrm{coker}(\alpha)$ has FFR, with $\chi(M) = k - \ell < 0$. In particular, if R fails to satisfy the "Rank Condition" (in *LMR*-(1.20)(1)), then there exists a surjection $R^k \xrightarrow{\beta} R^\ell$ for some $k < \ell < \infty$. In this case, a splitting of β gives an injection $R^\ell \xrightarrow{\alpha} R^k$, and

$$M = \mathrm{coker}\,(\alpha) \cong \ker\,(\beta)$$

is a f.g. *stably free* right R-module with $\chi(M) = k - \ell < 0$.

(B) For R, N as in (B), let $\rho\,(M) = \rho_R\,(M) \in [0, \infty)$ denote the ρ-rank of a f.g. R-module M_R. Note that $\rho(R) > 0$. Indeed, if B is the classical right ring of quotients of the semiprime right noetherian ring $A = R/N$, then

$$\rho\,(R) \geq \rho\,(R/N) = \mathrm{length}_B\,(A \otimes_A B) = \mathrm{length}_B\,(B) > 0.$$

If M has a FFR, say

$$0 \to R^{r_n} \to \cdots \to R^{r_0} \to M \to 0,$$

the additivity of ρ over (short) exact sequences gives

$$\begin{aligned} \rho\,(M) &= \rho\,(R^{r_0}) - \rho\,(R^{r_1}) + \cdots \\ &= r_0\;\rho\,(R) - r_1\;\rho\,(R) + \cdots \\ &= \chi\,(M)\;\;\rho\,(R)\;\;\in \mathbb{Z}. \end{aligned}$$

Since $\rho(M) \geq 0$ and $\rho(R) > 0$, this gives $\chi(M) \geq 0$, as desired. The above equation also implies that $\chi(M) = 0$ iff $\rho(M) = 0$, and, according to *LMR*-(12.9), the latter is equivalent to the condition that, for every $m \in M$, one has $mr = 0$ for some $r \in \mathcal{C}\,(N)$.

Comment. R.G. Swan has pointed out to us a second proof for (B) in which the ρ-rank used above is replaced by a ρ_P-rank (for f.g. R-modules) constructed for any given minimal prime ideal P of R. This ρ_P-rank is still additive over short exact sequences, so it suffices to define $\rho_P(M)$ for f.g. R-modules M that are killed by *some* prime ideal $P' \subseteq R$ (depending on M): see Ex. 3.40F(2), which applies well here. If $P' \neq P$, we define $\rho_P(M) = 0$. If $P' = P$, M may be viewed as a (f.g.) R/P-module, and we can define

$$\rho_P(M) = \text{length}_C \left(M \otimes_{R/P} C\right),$$

where $C := Q^r_{c\ell}(R/P)$ (which is a simple artinian ring). We have again $\rho_P(R) > 0$, so the above proof for $\chi(M) \geq 0$ works also with ρ replaced by ρ_P. (However, this new proof does not seem to give directly a criterion for $\chi(M) = 0$.) One advantage of the P-rank ρ_P is that it has a very nice interpretation in the case where R is a *commutative* (noetherian) ring. In this case, it is easy to check that, for any f.g. R-module M,

$$\rho_P(M) = \text{length}_{R_P}(M_P),$$

where M_P is the localization of M at P, viewed as a (f.g.) module over R_P, which is a (commutative) local artinian ring.

In the case where $R \neq 0$ is commutative and noetherian, the last part of (B) in this exercise amounts to the fact that $\chi(M) = 0$ *iff* $Mr = 0$ *for some* $r \in \mathcal{C}(N)$. This characterization for $\chi(M) = 0$ in the commutative noetherian case is, of course, already contained in the conclusions of Exercises 5.22C and 5.22D, noting that $\mathcal{C}(0) \subseteq \mathcal{C}(N)$ (as per *LMR*-(12.20)).

Ex.12.12. Let $R \neq 0$ be a right noetherian ring or a commutative ring, and let M be a right R-module with a FFR.

(A) If there is a surjection $R^m \to M$, show that $\chi(M) \leq m$.
(B) If there is an injection $M \to R^n$, show that $\chi(M) \leq n$.

Solution. (A) has been proved in the commutative case in the solution to Ex. 5.22 F. This proof also works in the right noetherian case since all we needed there was the fact that $\chi(K) \geq 0$ (in the display $(*)$), which follows here from Ex. 12.11(B).

(B) A "dual" argument works in this case. Fix a FFR:

$$0 \to F_k \to \cdots \to F_0 \xrightarrow{\beta} M \to 0,$$

and compose β with the injection $M \to R^n$ to get

$$0 \to F_k \to \cdots \to F_0 \xrightarrow{\gamma} R^n \to C \to 0,$$

where $C =$coker (γ). Then

$$\chi(C) = n - \text{rk } F_0 + \text{rk } F_1 - \cdots = n - \chi(M) \geq 0$$

gives $\chi(M) \leq n$.

Chapter 5

More Rings of Quotients

§13. Maximal Rings of Quotients

While the classical right ring of quotients $Q^r_{c\ell}(R)$ exists (if and) only if R is a right Ore ring, a *maximal right ring of quotients* $Q^r_{\max}(R)$ exists for any ring R. To define $Q^r_{\max}(R)$, one must first understand the structure of the endomorphism ring of a QI (quasi-injective) module.

Let $H = \text{End}(I_R)$ where I is QI. Then rad H, the Jacobson radical of H, is given by the following ideal

$$N := \{f \in H : \ker(f) \subseteq_e I\},$$

and we have the following basic information about H and $\bar{H} = H/N$:

(A) *$\bar{H}$ is a right self-injective von Neumann regular ring.*
(B) *Any idempotent in $\bar{H}$ can be lifted to an idempotent of H.*
(C) *If I_R is nonsingular or semisimple, then $N = \text{rad}\, H = 0$.*

In case R is a right self-injective ring, we can apply the above to $I = R_R$ and get several important conclusions on $\text{End}(R_R) \cong R$.

To construct $Q^r_{\max}(R)$, we apply the above results to the endomorphism ring H of the injective module $I := E(R_R)$. We let $Q = \text{End}({}_H I)$, so that $I = {}_H I_Q$. We can embed Q in I by the map $\varepsilon : Q \to I$ sending $q \in Q$ to $1 \cdot q$ (for any $q \in Q$). The embedded image $\varepsilon(Q)$ turns out to be exactly $\tilde{E}(R_R)$, the rational hull of R_R. Thus, $\tilde{E}(R_R) \cong Q$ has a natural ring structure extending its R-module structure: this ring is defined to be $Q^r_{\max}(R)$. The maximal left ring of quotients, $Q^\ell_{\max}(R)$, is defined similarly.

From a more axiomatic viewpoint, we can define a "(general) right ring of quotients" of R to be any overring T such that $R_R \subseteq_d T_R$

(i.e. R is dense as an R-submodule of T_R). Then any such T admits a unique ring homomorphism g into $Q^r_{\max}(R)$ extending the identity map on R, and g is necessarily an embedding (*LMR*-(13.11)). This justifies the name "maximal right ring of quotients" for $Q^r_{\max}(R)$. In the case where $Q^r_{c\ell}(R)$ exists (i.e. when R is right Ore), it can be embedded (over R) as a subring of $Q^r_{\max}(R)$ (*LMR*-(13.12)), but these two rings of quotients need not be equal.

For the purposes of working with $Q^r_{\max}(R)$, we may think of its elements as "classes" of R-homomorphism $f: \mathfrak{A} \to R$ where $\mathfrak{A}$ is any dense right ideal of R. The addition and multiplication of the classes of such f's are described in *LMR*-(13.21). In the case where R has a minimal dense right ideal D (e.g. R is right artinian), it follows that $Q^r_{\max}(R) \cong \text{End}(D_R)$. This gives a nice way to compute $Q^r_{\max}(R)$ (when D exists). In particular, for any right Kasch ring R, we have just $Q^r_{\max}(R) = R$.

The passage from R to $Q^r_{\max}(R)$ is, as expected, a "closure operation", in the sense that $Q^r_{\max}(Q^r_{\max}(R))$ is just $Q^r_{\max}(R)$ itself. This is proved in *LMR*-(13.31), and Exercises 10–11 below provide more information and perspective on this situation.

Johnson's Theorem (*LMR*-(13.36)) gives several fundamental characterizations for right nonsingular rings :

> *A ring R is right nonsingular iff $H := \text{End}\,(E(R_R))$ is Jacobson semisimple, iff $Q := Q^r_{\max}(R)$ is von Neumann regular.*

In this case, we have $Q = E(R_R)$, *and* $Q \cong H$ are right self-injective rings. Note that the equation $Q = E(R_R)$ means that $E(R_R)$ has here the structure of a von Neumann regular right self-injective ring. In case R is a domain, Q is also a simple ring, by *LMR*-(13.38′).

While Johnson's Theorem characterizes the rings R for which $Q = Q^r_{\max}(R)$ is von Neumann regular, one may further ask for a characterization of rings R for which Q is semisimple. This is accomplished by Gabriel's Theorem (*LMR*-(13.40)), which states that

> *Q is semisimple iff R is right nonsingular and $u.\dim R_R < \infty$.*

These conditions imply that R is right Goldie; if R is semiprime, the converse holds, and we have $Q = Q^r_{c\ell}(R)$. These results thus integrate Goldie's Theorem into the theory of maximal right ring of quotients.

The exercises in this section deal with various aspects of $Q^r_{\max}(R)$, and offer several computations of this ring. Exercise 7 characterizes rings of the form $\text{End}(V_D)$ where V is a right vector space over a division ring D, and Exercises 18–19 contain partial information on $Q^r_{\max}(R)$ of a Boolean ring R.

Exercises for §13

In the following exercises, R denotes a ring, $I = E(R_R)$ denotes the injective hull of R_R, and $H = \operatorname{End}(I_R)$. The maximal right ring of quotients, $Q^r_{\max}(R)$, is denoted by Q throughout.

Ex. 13.1. (Utumi) Show that $Q^r_{\max}(\prod_j R_j) \cong \prod_j Q^r_{\max}(R_j)$ for any family of rings $\{R_j : j \in J\}$.

Solution. Let $Q_j = Q^r_{\max}(R_j)$ for any $j \in J$. A coordinate-wise check shows that $R := \prod_j R_j \subseteq_d \prod_j Q_j$ as right R-modules. Therefore, by *LMR*-(13.11), we may assume that $\prod_j Q_j$ lies inside $Q^r_{\max}(R)$. We'll be able to conclude that these two rings are equal if we can show that, for any dense right ideal $A \subseteq R$, any $f \in \operatorname{Hom}_R(A, R_R)$ is realizable as left multiplication by some element from $\prod_j Q_j$. (See the description of $Q^r_{\max}(R)$ in *LMR*-(13.21).) Let $\pi_j : R \to R_j$ be the natural projection, and $\varepsilon_j : R_j \to R$ be the natural injection (for every j). The projection $A_j := \pi_j(A)$ is easily checked to be a dense right ideal in R_j. Now let $f_j \in \operatorname{Hom}_{R_j}(A_j, R_j)$ be defined as $\pi_j f \varepsilon_j$. (We omit the routine check that f_j is indeed an R_j-module homomorphism.) Since $Q_j = Q^r_{\max}(R_j)$, f_j is realized as left multiplication (on A_j) by a suitable element $q_j \in Q_j$. We have now an element

$$q := (q_j) \in \prod_j Q_j.$$

For any element $a = (a_j) \in A$, we have $a_j \in A_j$, and $f_j(a_j) = q_j a_j$ by the choice of q_j. To compute $(b_j) := f((a_j))$, we shall make use of the "unit vectors" $e_i \in R$ given by $(\delta_{ij})_{j \in J}$, where δ_{ij} are the Kronecker deltas. We have

$$\begin{aligned} b_i &= \pi_i((b_j)e_i) = \pi_i(f((a_j))e_i) = \pi_i(f((a_j)e_i)) \\ &= \pi_i(f(\varepsilon_i(a_i))) = f_i(a_i) = q_i a_i. \end{aligned}$$

Since this holds for all i, we see that

$$f((a_j)) = (q_j a_j) = q \cdot (a_j),$$

as desired.

Comment. The fact that the maximal right ring of quotients construction respects *arbitrary* direct products of rings was first observed by Y. Utumi in his paper "On quotient rings," Osaka Math. J. **8**(1956), 1–18.

Ex. 13.2. Let $b \in R$ be right regular in R, i.e. $\operatorname{ann}^R_r(b) = 0$. Show that

(1) b remains right regular in Q; and
(2) if R is right nonsingular, b has a left inverse in Q.
(3) If b is regular in R, is it necessarily regular in Q?

Solution. (1) Say $bq = 0$ where $q \in Q$. There exists a right ideal $A \subseteq_d R_R$ such that $qA \subseteq R$. Then $0 = b(qA)$ implies that $qA = 0$ (since $\text{ann}_r^R(b) = 0$), and hence $q = 0$.

(2) Since R is right nonsingular, Q is right self-injective by Johnson's Theorem (*LMR*-(13.36)). The rest is now a familiar argument. By (1), the map $f: \ bq \mapsto q$ is a well-defined Q-homomorphism $bQ \to Q_Q$, This must be given by left multiplication by some $c \in Q$, so $cb = f(b) = 1$.

(3) The answer is "no" in general, even in the case where R is a domain. In *LMR*-(13.28), it is shown that, for $R = \mathbb{Q}\langle a, b \rangle$, there exists $q \in Q$ such that $qa = 1$ and $qb = 0$. Therefore, although b remains right regular in Q (according to (1)), it is not left regular in Q.

Ex. 13.3. Show that an element $b \in R$ is a unit in Q iff b is right regular in R and $bR \subseteq_d R_R$.

Solution. First assume $b \in \text{U}(Q)$, say $bq = 1$ where $q \in Q$. Of course b is (right) regular in R. We shall check that $bR \subseteq_d Q_R$ (clearly equivalent to $bR \subseteq_d R_R$) by applying the denseness criterion in *LMR*-(13.18). According to this criterion, we need only check that

$$h \in H, \ h\,(bR) = 0 \implies h\,(1) = 0.$$

Now from $h(bR) = 0$, we have $0 = h(b)q = h(bq) = h(1)$, so we are done.

Conversely, let us assume that $\text{ann}_r^R(b) = 0$ and $bR \subseteq_d R_R$. Then the well-defined R-homomorphism $f : br \mapsto r$ from bR *to* R_R is induced by left multiplication by some $q_1 \in Q$, by *LMR*-(13.20). Then $q_1 b = f(b) = 1$ and so

$$(bq_1 - 1)bR = (bq_1 b - b)R = 0.$$

Since $bR \subseteq R_R$, this implies (again by *LMR*-(13.20)) that $bq_1 - 1 = 0$. Hence $b \in \text{U}(Q)$ (with inverse q_1).

Ex. 13.4. For $q \in Q$, show that the following are equivalent:

(1) $q \in \text{U}\,(Q)$;
(2) q is right regular in Q and $qQ \subseteq_d Q_Q$;
(3) For $i \in I$, $iq = 0 \Rightarrow i = 0$, and, for $r \in R$, $qr = 0 \Rightarrow r = 0$.

Solution. (1) $\Leftrightarrow$ (2) follows from the last exercise (applied to the case $R = Q$). (1) $\Rightarrow$ (3) is clear, so it suffices to prove (3) $\Rightarrow$ (1). Assume (3) holds. Then $qr \mapsto r$ ($r \in R$) is a well-defined R-homomorphism from qR to I_R. Since I_R is injective, there exists $h \in H = \text{End}(I_R)$ such $h(qr) = r$ for all $r \in R$. Let $q' := h(1) \in I$. Then

$$q'q = h(1)q = h(1q) = h(q) = 1.$$

Suppose we know that $q' \in Q$. Then qq' makes sense (in Q) and $(qq' - 1)q = q - q = 0$ implies that $qq' = 1$ by the first condition on q in (3). This shows that $q \in \mathrm{U}(Q)$. Finally, we have to show $q' \in Q$. By *LMR*-(13.7), this will follow if we can show that, for $k \in H$, $k(R) = 0 \Rightarrow k(q') = 0$. Now $k(R) = 0$ implies that

$$0 = k(1) = k(q'q) = k(q')q,$$

so indeed $k(q') = 0$ by our assumption(s) on q.

Ex. 13.5. Show that an element $q \in Q$ is central in Q iff it commutes with all elements of R. Deduce from this that, if R is commutative, so is Q.

Solution. The first part is proved eventually in *LMR*-(14.15), so we won't repeat the proof here. For the second part, assume that R is commutative. For $r \in R$, certainly r commutes with all elements of R, so by the first part, $r \in Z(Q)$. Therefore, for any $q \in Q$, q commutes with all $r \in R$, and hence, by the first part again, $q \in Z(Q)$.

Ex. 13.6. (Utumi) Let R be a prime ring with $S = \mathrm{soc}(R_R) \neq 0$. Show that $Q = Q^r_{\max}(R)$ is isomorphic to $\mathrm{End}(V_D)$ for a suitable right vector space V over some division ring D.

Solution. It is well known that the right socle S is an ideal, so by *LMR*-(8.4)(3), the fact that $S \neq 0$ implies that $S \subseteq_d R_R$. Now any essential right ideal must contain any minimal right ideal and hence contain S. This shows that S is the smallest dense right ideal. Applying *LMR*-(13.22)(3), we see that $Q \cong \mathrm{End}(S_R)$. Now note that any two minimal right ideals $\mathfrak{A}$, $\mathfrak{B}$ in R are isomorphic. In fact, since R is prime, $\mathfrak{A}\mathfrak{B} \neq 0$ so $a\mathfrak{B} \neq 0$ for some $a \in \mathfrak{A}$. Therefore, left multiplication by a defines an R-isomorphism from $\mathfrak{B}$ to $\mathfrak{A}$.

Now identify S_R with $\bigoplus_{j \in I} \mathfrak{A}$, and let $D = \mathrm{End}(\mathfrak{A}_R)$, which is a division ring by Schur's Lemma. Then each R-endomorphism φ of S_R is determined by a column-finite "matrix" (φ_{ij}), where $\varphi_{ij} \in D$ is the composition of

$$(j^{\text{th}} \text{ copy of } \mathfrak{A}) \xrightarrow{\varphi} S \xrightarrow{i^{\text{th}} \text{ projection}} \mathfrak{A}.$$

Note that for a given j, the φ_{ij}'s are almost all zero, since the cyclic R-module φ (j^{th} copy of $\mathfrak{A}$) is contained in a direct sum of a finite number of copies of $\mathfrak{A}$. Therefore, $\mathrm{End}(S_R)$ is isomorphic to the ring of $I \times I$ column-finite matrices over D, which is just the endomorphism ring of a right D-vector space $V = \bigoplus_{j \in I} e_j D$.

Comment. For readers who prefer a proof not relying on the use of matrices, we can offer the following alternative approach. Write a minimal right ideal $\mathfrak{A}$ as eR where $e = e^2$. Then $D = \mathrm{End}(eR)_R \cong eRe$ by *FC*-(21.7), and $V := Re$ is a right vector space over D. We can define a map

$$\varphi : \mathrm{End}(S_R) \to \mathrm{End}(V_D)$$

as follows. For $f \in \mathrm{End}(S_R)$, take $\varphi(f)(re) = f(re)$ (noting that $re \in Re \subseteq S$ and that $f(re) = f(re^2) = f(re)e \in Re$). It is not difficult to check that φ is a ring isomorphism, which then gives what we want.

Ex. 13.7. (1) Show that a ring R has the form $\mathrm{End}(V_D)$ where V is a right vector space over a division ring D iff R is prime, right self-injective, and with $\mathrm{soc}(R_R) \neq 0$.

(2) Show that a prime, self-injective ring R with $\mathrm{soc}(R_R) \neq 0$ must be semisimple.

Solution. First suppose $R = \mathrm{End}(V_D)$. Easy linear algebra considerations show that R is prime, and *LMR*-(3.74B) (applied with "left" and "right" reversed) shows that R is right self-injective. Finally, by *ECRT*-Ex. 11.17, if $e \in R$ is any D-endomorphism of rank 1, eR is a minimal right ideal of R, so $\mathrm{soc}(R_R) \neq 0$.

Conversely, let R be a ring that is prime, right self-injective, and with $\mathrm{soc}(R_R) \neq 0$. The second of these conditions implies that $R = Q^r_{\max}(R)$, so the last exercise yields $R \cong \mathrm{End}(V_D)$ for some right vector space V over some division ring D. *Now assume, in addition, that R is left self-injective.* Then by *LMR*-(3.74B) (applied with "left" and "right" reversed), we must have $n := \dim_D(V) < \infty$, and therefore $R \cong \mathrm{End}(V_D) \cong \mathbb{M}_n(D)$ is semisimple.

Ex. 13.8. Show that, if R is simple (resp. prime, semiprime), so is every general right ring of quotients of R.

Solution. Let $S \supseteq R$ be any general right ring of quotients of R. This means that $R \subseteq_d S_R$. First assume that R is simple. Let A be a nonzero ideal of S. Then $A \cap R$ is a nonzero ideal in R, so $1 \in A \cap R$, whence $A = S$. This shows that S is also simple. Next, assume that R is prime. Suppose $qSq' = 0$, where $q,\ q' \in S$. Take dense right ideals A, A' in R such that $qA,\ q'A' \subseteq R$. Then $(qA)(q'A') = (qAq')A' = 0$ implies that $qA = 0$ or $q'A' = 0$, since R is prime. Therefore, using $R \subseteq_d S_R$ once more, we have $q = 0$ or $q' = 0$. This shows that S is prime, and the semiprime case is similar (by letting $q = q'$).

Ex. 13.9. Let $\mathfrak{B} \subseteq R \subseteq S$, where R, S are rings, and $\mathfrak{B}$ is a left ideal of S with $\mathrm{ann}^S_\ell(\mathfrak{B}) = 0$. Let $I = E(S_S)$. Show that

(1) For $i \in I,\ i\,\mathfrak{B} = 0 \Rightarrow i = 0$;
(2) $I = E(R_R)$;
(3) $\mathrm{End}(I_R) = \mathrm{End}(I_S)$; and finally,
(4) $Q^r_{\max}(R) = Q^r_{\max}(S)$.

Solution. (1) If $i \neq 0$, we have $i\,s \in S \backslash \{0\}$ for some $s \in S$. But then $is\,\mathfrak{B} \subseteq i\,\mathfrak{B} = 0$, contradicting $\mathrm{ann}^S_\ell(\mathfrak{B}) = 0$.

(2) By *LMR*-(3.42), I_R is an injective module, so it suffices to show that $R \subseteq_e I_R$. Let $i \in I \backslash \{0\}$ and fix $s \in S.$ with $i\,s \in S \backslash \{0\}$. There exists

$b \in \mathfrak{B}$ with $isb \neq 0$, while we have $(is)\,b \in S\mathfrak{B} \subseteq \mathfrak{B} \subseteq R$. This checks that $R \subseteq_e I_R$.

(3) For $f \in \text{End}(I_R)$, we must show that $f(i\,s) = f(i)s$ for $i \in I$ and $s \in S$. This follows from (1) since

$$[f(i\,s) - f(i)s]\,b = f(isb) - f(isb) = 0 \quad (\forall\, b \in \mathfrak{B}).$$

(4) This follows from (3) since both maximal right rings of quotients are obtained from $H = \text{End}(I_R) = \text{End}(I_S)$ by taking the H-endomorphism ring of ${}_HI$.

Comment. The point of this exercise is that if we know one of the maximal right quotient rings, then we know the other. This technique can be used to compute the maximal right quotient rings for many rings, for instance those in *LMR*-(3.43) (Examples (A) through (F)).

Ex. 13.10. (This exercise provides a more general view, and a new proof, for the fact that $R \mapsto Q^r_{\max}(R)$ is a "closure operation": cf. *LMR*-(13.31)(3).) Let $R \subseteq S \subseteq T$ be rings such that S (resp. T) is a general right ring of quotients of R (resp. S). Show that T is a general right ring of quotients of R.

Solution. If we can show that $S_R \subseteq_d T_R$, then we can use the transitivity property of denseness (*LMR*-(8.7)(2)) to deduce that $R_R \subseteq_d T_R$. Let x, $y \in T$, where $x \neq 0$. Choose $s \in S$ such that $xs \neq 0$ and $ys \in S$. Then choose $s' \in S$ such that $0 \neq xss' \in S$. Finally (using $R \subseteq_d S_R$), choose $r \in R$ such that $(xss')\,r \neq 0$ and $(ss')\,r \in R$. Then, for $r' = ss'r \in R$, we have $xr' \neq 0$, and $yr' = (ys)\,(s'r) \in S$.

Ex. 13.11. Let T be a general right ring of quotients of R.

(1) Show that $T = Q^r_{\max}(R)$ iff $Q^r_{\max}(T) = T$.
(2) In general, $Q^r_{\max}(T) = Q^r_{\max}(R)$.

Solution. (1) The "only if" part is the "closure" property of the $Q^r_{\max}$-formation (implied by the last exercise and also proved in *LMR*-(13.31)(3)). For the "if" part, assume that $Q^r_{\max}(T) = T$. Since $R \subseteq_d T_R$, we may assume (by *LMR*-(13.11) that $T \subseteq Q := Q^r_{\max}(R)$. Then $R \subseteq_d Q_R$ implies that $T_R \subseteq_d Q_R$, which in turn implies that $T_T \subseteq_d Q_T$. Since $Q^r_{\max}(T) = T$, we must have $T = Q$.

(2) Let $Q' = Q^r_{\max}(T)$. Then, by the last exercise, Q' is a general right ring of quotients of R. Since $Q' = Q^r_{\max}(Q')$ by the closure property, part (1) above (applied to Q') implies that $Q' = Q^r_{\max}(R)$.

Ex. 13.12. Let $k \subsetneq K$ be fields and let $R = \begin{pmatrix} k & K \\ 0 & K \end{pmatrix} \subseteq S = \mathbb{M}_2(K)$. Show that $S = Q^r_{\max}(R)$, but S is not a general *left* ring of quotients of R.

Solution. Note that $\mathfrak{B} = \begin{pmatrix} 0 & K \\ 0 & K \end{pmatrix}$ is a left ideal of S that is contained in R. An easy computation shows that $\text{ann}_\ell^S(\mathfrak{B}) = 0$. Therefore, Exercise 9 applies to give $Q^r_{\max}(R) = Q^r_{\max}(S)$. The latter is just S, since S is a semisimple ring.

Next, we check that R is not dense as a left submodule of ${}_RS$ (so S is not a general left ring of quotients of R). Take $x = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \neq 0$ and $y = \begin{pmatrix} d & 0 \\ 0 & 0 \end{pmatrix} \in S$, where d is any element in $K \backslash k$. If $r = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \in R$ is such that $ry = \begin{pmatrix} ad & 0 \\ 0 & 0 \end{pmatrix} \in R$, we must have $a = 0$. But then

$$rx = \begin{pmatrix} 0 & b \\ 0 & c \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = 0.$$

This shows that R is not dense in ${}_RS$.

Comment. It is worth double-checking that R is dense in S_R by a direct calculation, so that we can see the difference between the left and the right structures for the pair $R \subseteq S$. Take any $x,\ y \in S$ with $x \neq 0$. We have $yr \in R$ for any $r = \begin{pmatrix} 0 & b \\ 0 & c \end{pmatrix} \in R$. So, choosing (b, c) to be either $(1, 0)$ or $(0, 1)$, we will have $xr \neq 0$. This checks that $R \subseteq_d S_R$!

Ex. 13.13. For the ring $R = \begin{pmatrix} \mathbb{Z} & \mathbb{Q} \\ 0 & \mathbb{Q} \end{pmatrix}$, compute $Q^r_{c\ell}(R)$ and $Q^r_{\max}(R)$.

Solution. The same procedure used in the last exercise shows that $Q^r_{\max}(R) = \mathbb{M}_2(\mathbb{Q})$. On the other hand, $Q^r_{c\ell}(R)$ is given by $T : \begin{pmatrix} \mathbb{Q} & \mathbb{Q} \\ 0 & \mathbb{Q} \end{pmatrix}$. To see this, first observe that every element in T has the form an^{-1} where $a \in R$ and n is a positive integer in R. Secondly, every regular element of R has the form $\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}$ where $ac \neq 0$; such elements are clearly invertible in T. These observations are sufficient to show that $Q^r_{c\ell}(R)$ exists and is equal to T. So in this example, $Q^r_{c\ell}(R)$ is a proper subring of $Q^r_{\max}(R)$.

Ex. 13.14. Let $R = \begin{pmatrix} k & k & k \\ 0 & k & 0 \\ 0 & 0 & k \end{pmatrix}$ where k is a semisimple ring. Determine $Q^\ell_{\max}(R)$ and $Q^r_{\max}(R)$ directly by computing the injective hulls of ${}_RR$ and R_R.

Solution. By the analogue of *LMR*-(3.43B) for left injective hulls, we know that $E({}_RR) = \mathbb{M}_3(k)$, which contains R as a subring. Since R is *nonsingular* by *LMR*-(7.14b), it follows from *LMR*-(13.39)(1) that $Q^\ell_{\max}(R) = \mathbb{M}_3(k)$. We can compute $Q^r_{\max}(R)$ similarly if we can find a ring $A \supseteq R$

such that $E(R_R) = A_R$. Take A to be the ring $\mathbb{M}_2(k) \times \mathbb{M}_2(k)$ and define $\varphi : R \to A$ by

$$\varphi \begin{pmatrix} a & b & c \\ 0 & d & 0 \\ 0 & 0 & e \end{pmatrix} = \left(\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}, \begin{pmatrix} a & c \\ 0 & e \end{pmatrix} \right).$$

This is easily verified to be a ring embedding, so we may view R as a subring of A via φ. We shall apply *LMR*-(3.42) by taking $\mathfrak{B} = \begin{pmatrix} 0 & k & k \\ 0 & k & 0 \\ 0 & 0 & k \end{pmatrix} \subseteq R$. It is routine to check that $\mathfrak{B}$ is a left ideal of A, and that for $a \in A$, $a\mathfrak{B} = 0$ implies $a = 0$. In particular, $R_R \subseteq_e A_R$. Since A is a semisimple ring, A_A is injective. It follows from *LMR*-(3.42) that A_R is injective, and so $E(R_R) = A_R$. Therefore, we can take $Q^r_{\max}(R)$ to be the (semisimple) ring A. (Alternatively, once we know that $R_R \subseteq_e A_R$ and A_A is injective, then $A = Q^r_{\max}(R)$ follows also from *LMR*-(13.39)(2).)

Comment. Here again, $T = \mathbb{M}_3(k) \supseteq R$ fails to be the maximal right ring of quotients of R since R is *not essential in* T_R. Indeed, since

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & x \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} a & b & c \\ 0 & d & 0 \\ 0 & 0 & e \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & xe \\ 0 & 0 & 0 \end{pmatrix},$$

kE_{23} is an R-submodule of T with zero intersection with R.

Ex. 13.15. Show that any automorphism of a ring extends uniquely to an automorphism of the maximal right ring of quotients $Q = Q^r_{\max}(R)$.

Solution. We first show the *uniqueness*. For this it suffices to show that, if an automorphism Φ of Q restricts to the identity on R, then Φ itself is the identity map. Consider any $q \in Q$. Fix a right ideal $A \subseteq_d R_R$ such that $qA \subseteq R$. For any $a \in A$, we have

$$\Phi(q)a = \Phi(q)\Phi(a) = \Phi(qa) = qa.$$

Therefore, $(\Phi(q) - q)A = 0$; this implies that $\Phi(q) = q$ (by *LMR*-(13.22) or *LMR*-(13.23)).

Next we show how to extend a given automorphism φ of R to Q. We think of elements of Q as classes $[A,\ f]$ where A is a dense right ideal of R and $f \in \mathrm{Hom}_R(A_R, R_R)$. Let us define $\Phi[A, f] = [\varphi A, f']$ where $f' : \varphi A \to R$ is defined by $f'(\varphi(a)) = \varphi(f(a))$, for any $a \in A$. We make the following two observations:

(1) $f' \in \mathrm{Hom}_R((\varphi A)_R, R_R)$. Indeed, for $s \in R$,

$$\begin{aligned} f'(\varphi(a)s) &= f'\left(\varphi\left(a\varphi^{-1}s\right)\right) = \varphi f\left(a\varphi^{-1}s\right) = \varphi\left(f(a)\varphi^{-1}s\right) \\ &= \varphi(f(a))s = f'(\varphi(a))s. \end{aligned}$$

(2) The right ideal φA is dense in R_R. To see this, let $x, y \in R$ be given, with $x \neq 0$. Since $A \subseteq_d R_R$, there exists $s \in R$ such that $\varphi^{-1}(x)\, s \neq 0$ and $\varphi^{-1}(y)\, s \in A$. Applying φ, we see that $x \cdot \varphi(s) \neq 0$ and $y \cdot \varphi(s) \in A$. This shows that $\varphi A \subseteq_d R_R$.

The formula $\Phi[A,\ f] = [\varphi A,\ f']$ defines, therefore, an action Φ on Q. If f is just left multiplication by some $b \in R$, then

$$f'(\varphi(a)) = \varphi(f(a)) = \varphi(ba) = \varphi(b)\varphi(a)$$

shows that f' is left multiplication by $\varphi(b)$. Therefore, the Φ-action on Q extends the φ-action on R. Finally, a routine calculation shows that the Φ-action on Q is an automorphism of the ring Q.

Ex. 13.16. (R. E. Johnson) Let $Q = Q^r_{\max}(R)$, where R is a right nonsingular ring. Show that any closed R-submodule of Q_R is a principal right ideal of Q (and conversely). Using this, show that there is a one-one correspondence between the closed right ideals of R and the principal right ideals of Q.

Solution. First note that, since R_R is nonsingular, Q_R is also nonsingular, by *LMR*-(7.6)(2). For any R-submodule $N \subseteq Q$, we know there is a largest essential extension of N in Q_R, defined by

$$N^* = \{q \in Q : \text{ there exists } I \subseteq_e R_R \text{ such that } qI \subseteq N\}.$$

This is always a right ideal in Q. To see this, let $q_1 \in Q$. By Exercise (3.7), $I \subseteq_e Q_R$ implies that $q_1^{-1}I \subseteq_e R_R$. Now $qq_1\left(q_1^{-1}I\right) \subseteq qI \subseteq N$, so $qq_1 \in N^*$. In particular, if N is closed in Q_R, then $N = N^*$ is a right ideal in Q. Since Q_R is injective, $Q = N \oplus N'$ for some N'_R. Then N' is also a right ideal in Q, so N, N' are both *principal* right ideals of Q. Conversely, since Q is von Neumann regular by *LMR*-(13.36), any principal right ideal of Q is a direct summand of Q_Q, and is, in particular, a closed R-submodule of Q_R.

By *LMR*-(7.44′), there is a one-one correspondence between the closed R-submodules of R_R and those of Q_R. Now, closed R-submodules of R_R are closed right ideals of R, and closed R-submodules of Q_R are principal right ideals of Q; hence the last conclusion in the exercise.

Ex. 13.17. (1) Show that a commutative ring R is reduced iff $Q_{\max}(R)$ is reduced, iff $Q_{\max}(R)$ is von Neumann regular.
(2) Exhibit a reduced commutative ring R which $Q_{c\ell}(R)$ is not von Neumann regular (and in particular $R \subseteq Q_{c\ell}(R) \subsetneq Q_{\max}(R)$).

Solution. (1) Recall that, for R commutative, $Q^r_{\max}(R) = Q^\ell_{\max}(R)$ over R, so we can write $Q = Q_{\max}(R)$ for either ring. By *LMR*-(13.34), Q is also commutative. If Q is reduced (which will be the case if Q is von Neumann regular), of course so is R. Conversely, assume R is reduced. Then R is nonsingular by *LMR*-(7.12). By Johnson's Theorem (*LMR*-(13.36)), Q is von Neumann regular, and therefore reduced.

(2) Let $R = \mathbb{Q}[x_1, x_2, \ldots]$, with relations given by $x_i x_j = 0$ for all $i \neq j$. Every element has a unique form

$$(*) \qquad \alpha = a + f_1(x_1) + \cdots + f_n(x_n),$$

where $a \in \mathbb{Q}$, $f_i(x_i) \in x_i\mathbb{Q}[x_i]$, and $n \geq 1$ is some integer. From this, it is clear that $\alpha \neq 0 \Rightarrow \alpha^m \neq 0$ for any $m \geq 1$, so R is reduced. Next, note that if $a = 0$ in $(*)$, then α is a 0-divisor of R (since $\alpha x_{n+1} = 0$). On the other hand, if $a \neq 0$, an easy argument shows that α is *not* a 0-divisor. Therefore, $Q_{c\ell}(R) = S^{-1}R$ where S is the complement of the maximal ideal $\mathfrak{p} = \sum_{j=1}^{\infty} x_j R$. In particular, $Q_{c\ell}(R)$ is a local ring with a maximal ideal of $\mathfrak{p}R_{\mathfrak{p}} \neq 0$. It follows that $Q_{c\ell}(R)$ is not von Neumann regular (since any von Neumann regular ring must have a zero Jacobson radical).

Comment. We can get more insight into the example above by computing min-Spec(R), the minimal prime spectrum of R equipped with the induced Zariski topology from the prime spectrum Spec(R). If a prime does not contain $\mathfrak{p}$, then it misses some x_i and therefore must contain x_j for all $j \neq i$. This shows that the minimal primes of R are precisely the ideals $\mathfrak{p}_i = \sum_{j \neq i} x_j R \ (i \in \mathbb{N})$. The fact that x_i is contained in all $\mathfrak{p}_j \ (j \neq i)$ but not in $\mathfrak{p}_i$ shows that min-Spec(R) is a *discrete* space.

The above computation of min-Spec(R) implies easily that R and $Q_{c\ell}(R)$ have Krull dimension 1. In particular, $Q_{c\ell}(R)$ cannot be a von Neumann regular ring. It turns out that the lack of compactness of min-Spec(R) in general is also sufficient to imply that $Q_{c\ell}(R)$ is not von Neumann regular; for more details, see J. Huckaba's book "Commutative Rings with Zero Divisors," pp. 18–19, Marcel-Dekker, 1988.

It is also of interest to note that $\mathfrak{p} = \bigcup_{i=1}^{\infty} \mathfrak{p}_i$ (and $\mathfrak{p} \not\subseteq \mathfrak{p}_i$ for all i) provides a counterexample to the Principle of Prime Avoidance in the case of an *infinite* union of prime ideals.

Ex. 13.18. Show that R is a Boolean ring iff $Q^r_{\max}(R)$ is a Boolean ring.

Solution. If $Q^r_{\max}(R)$ is a Boolean ring, so is R since R is a subring of $Q^r_{\max}(R)$. Conversely, let R be a Boolean ring. We think of the elements of $Q_{\max}(R) = Q^r_{\max}(R)$ as "classes" of R-homomorphisms $f : \mathfrak{A} \to R$ where $\mathfrak{A}$ denotes any dense ideal of R (see *LMR*-(13.21)). It suffices to show that the square of the class of $f : \mathfrak{A} \to R$ is the same as the class of f. Since

$$f(\mathfrak{A}) = f(\mathfrak{A}^2) \subseteq f(\mathfrak{A})\mathfrak{A} \subseteq \mathfrak{A},$$

the square of the class of f is given by the class of the composition of

$$(*) \qquad \mathfrak{A} \xrightarrow{f} \mathfrak{A} \xrightarrow{f} R.$$

Now for $a \in \mathfrak{A}$,

$$\begin{aligned} f^2(a) &= f(f(a)) = f(f(a^2)) = f(f(a)a) \\ &= f(af(a)) = f(a)f(a) = f(a), \end{aligned}$$

so the composition in $(*)$ is just f itself, as desired.

Comment. Since the only unit in a Boolean ring R is $\{1\}$, R is a classical ring, so $Q_{c\ell}(R) = R$. However, the Boolean ring $Q^r_{\max}(R)$ is usually larger. In fact, $Q^r_{\max}(R)$ is exactly what is known as the *Dedekind-MacNeille completion* of the Boolean ring R: see p. 45 of Lambek's "Lectures on Rings and Modules." While we won't prove (or assume) this result, the next exercise offers an explicit computation of such a completion.

Ex. 13.19. A subset X in a set W is said to be *cofinite* if $W\backslash X$ is finite. Show that

$$R = \{X \subseteq W : \ X \text{ is either finite or cofinite } \}$$

is a Boolean subring of the Boolean ring S of all subsets of W, and show that $Q_{\max}(R) = S$.

Solution. The crucial step is to check that R is dense as an R-submodule in S_R. To see this, consider two elements of S, given by $A,\ A' \subseteq W$, where $A \neq \emptyset$. Pick $x \in A$ and let $X = \{x\} \in R$. Then the product $A \cdot X = A \cap X = \{x\}$ is nonzero in S; also, the product $B \cdot X = B \cap X$ is in R since $B \cap X \subseteq X$ has at most one element. This checks that $R \subseteq_d S_R$. Therefore, by *LMR*-(13.11), we have

$$R \subseteq S \subseteq Q := Q_{\max}(R) .$$

Now recall that the ring S is self-injective by *LMR*-(3.11D). Therefore $Q = S \oplus M$ for a suitable S-submodule M of Q. Since $R \subseteq_e Q_R$, we must have $M = 0$, and hence $S = Q = Q_{\max}(R)$.

Comment. After showing that $R \subseteq_d S_R$, we can also complete the solution slightly differently as follows. Since S is self-injective, we have $Q_{\max}(S) = S$. Then by Ex. 11(1), $Q_{\max}(R) = S$.

Ex. 13.20. Let R be a domain and $Q = Q^r_{\max}(R)$.

(1) For any nonzero idempotent $e \in Q$, show that $eQ \cong Q_Q$.
(2) Show that any nonzero f.g. right ideal $\mathfrak{A} \subseteq Q$ is isomorphic to Q_Q.
(3) If Q is Dedekind-finite, show that R is a right Ore domain and Q is its division ring of right fractions.

Solution. (1) Since R is right nonsingular, Q is a right self-injective von Neumann regular ring (by *LMR*-(13.36)). Being a direct summand of Q_Q, $(eQ)_Q$ is therefore injective. If we can show that eQ contains a copy of Q_Q, Bumby's Theorem (Exercise 3.31) will then give $eQ \cong Q_Q$. Now, from $R \subseteq_e Q_R$, there exists $s \in R$ such that $s_1 := es \in R\backslash\{0\}$. By Exercise 2,

$$\operatorname{ann}^R_r(s_1) = 0 \implies \operatorname{ann}^Q_r(s_1) = 0.$$

Therefore, $s_1 Q \cong Q_Q$, and we have $eQ \supseteq esQ = s_1 Q \cong Q_Q$, as desired.

(2) Since Q is von Neumann regular, $\mathfrak{A} = eQ$ for some (nonzero) idempotent $e \in Q$. By (1), $\mathfrak{A} \cong Q_Q$.

(3) For any nonzero idempotent $e \in Q$, consider the decomposition $Q_Q = eQ \oplus (1-e)Q$. Since $eQ \cong Q_Q$ (by (1)) and Q_Q is Dedekind-finite, we must have $(1-e)Q = 0$; that is, $e = 1$. For any nonzero $q \in Q$, there exists $q' \in Q$ such that $q = qq'q$. Then qq' is a nonzero idempotent in Q, and so $qq' = 1$ by the above. By *ECRT*-(1.2), this implies that Q is a division ring. It follows now from *LMR*-(13.43) that R is a right Ore domain, so by *LMR*-(13.41), $Q = Q^r_{c\ell}(R)$.

Comment. The Exercise has the following remarkable consequence:

> *If R is a domain that is not right Ore, then $Q = Q^r_{\max}(R)$ is right self-injective but not left self-injective.*

In fact, if Q was also left self-injective, then *LMR*-(6.49) would imply that Q is Dedekind-finite, and this Exercise would imply that R is right Ore!

Ex. 13.21. Let R be a semiprime ring with only finitely many minimal prime ideals $\mathfrak{p}_1, \dots, \mathfrak{p}_t$, and let $\overline{R}_i = R/\mathfrak{p}_i$. Show that $Q^r_{\max}(R) \cong \prod_i Q^r_{\max}(\overline{R}_i)$.

Solution. We shall make good use of the constructions in Exercise (11.17). In that exercise, we formed the ideals $\mathfrak{B}_i = \prod_{j \neq i} \mathfrak{p}_j$ and let

$$\bar{\mathfrak{B}}_i = (\mathfrak{B}_i + \mathfrak{p}_i)/\mathfrak{p}_i$$

in $\overline{R}_i$. Since $\bigcap_i \mathfrak{p}_i = \text{Nil}_*(R) = 0$, we identify R with its image under the natural map R $\to S = \prod_i \overline{R}_i$. In Exercise (11.17), we have shown that $\mathfrak{B} := \prod_i \overline{\mathfrak{B}}_i$ is an ideal of S lying in R, with $\text{ann}^S_\ell(\mathfrak{B}) = 0$. Therefore, Exercise 9 applies, and we have

$$Q^r_{\max}(R) \cong Q^r_{\max}(S) \cong \prod_{i=1}^t Q^r_{\max}(\overline{R}_i),$$

where the last isomorphism is justified by (a special case of) Exercise 1.

Ex. 13.22. (Cf. Exercise (11.19)) Let R be the (commutative) ring $\mathbb{Q}\langle\{x_i : i \geq 1\}\rangle$ with the relations $x_i x_j = 0$ for all unequal i, j.

(1) Show that $Q_{\max}(R)$ is isomorphic to the direct product $T = \prod_{i \geq 1} \mathbb{Q}(x_i)$, where R is embedded in T by sending $a \in \mathbb{Q}$ to $(a, a, \dots)$ and sending x_i to $(0, \dots, x_i, 0, \dots)$ with x_i in the i^{th} position.

(2) Show that $Q_{c\ell}(R) \subsetneq Q_{\max}(R)$.

Solution. According to Ex. (11.19), R is semiprime, and the minimal prime ideals of R are $\mathfrak{p}_i = \sum_{j \neq i} Rx_j$ $(i \geq 1)$. Each factor ring $\overline{R}_i = R/\mathfrak{p}_i$ is isomorphic to $\mathbb{Q}[x_i]$, and we can identify R with its image in $S = \prod_i \bar{R}_i \subseteq T$ under the natural embedding $\varphi : R \to S$. If Ex. 21 was applicable, we would get

$$Q_{\max}(R) \cong \prod_i Q_{\max}(\mathbb{Q}[x_i]) = \prod_i \mathbb{Q}(x_i) = T,$$

as desired. However, R has infinitely many minimal primes, so we cannot apply Ex. 21 here.

Using another strategy, let us check that $R \subseteq_d T_R$. This means that T is a general (right) ring of quotients of R. Since T (a direct product of fields) is self-injective by *LMR*-(3.11B), we have $Q_{\max}(T) = T$, and so Ex. 11 will give $Q_{\max}(R) = T$.

To check $R \subseteq_d T_R$, we use the fact that, under the embedding φ above, x_i "becomes" $(0, \dots, x_i, 0, \dots)$, with x_i in the i^{th} coordinate. Take elements $f = (f_i)$, $g = (g_i)$ in T, with $f \neq 0$. We may assume that $f_1 \neq 0$. Write $f_1 = h(x_1)/k(x_1)$ and $g_1 = h'(x_1)/k'(x_1)$, where h, h', k, k' are polynomials, with h, k, $k' \neq 0$. Multiplying f and g by

$$r = (x_1 k(x_1) k'(x_1), 0, 0, \dots) \in R,$$

we get

$$fr = (h(x_1) x_1 k'(x_1), 0, 0, \dots) \neq 0 \text{ and } gr = (h'(x_1) x_1 k(x_1), 0, 0, \dots) \in R.$$

This checks that $R \subseteq_d T_R$.

To see that $Q_{c\ell}(R) \subsetneq Q_{\max}(R)$, it suffices to check that $x = (x_1, x_2, \dots) \in T$ cannot be multiplied by a regular element r of R to get into R. Assume, on the contrary that such a regular element r exists. Then

$$r = a + x_1 f_1(x_1) + \cdots + x_n f_n(x_n), \text{ with } a \in \mathbb{Q} \backslash \{0\}.$$

As an element of T, r is expressed in the form

$$(a + x_1 f_1(x_1), \dots, a + x_n f_n(x_n), a, a, \dots),$$

and so $xr \in T$ has the form

$$(x_1(a + x_1 f_1(x_1)), \dots, x_n(a + x_n f_n(x_n)), x_{n+1}a, x_{n+2}a, \dots),$$

which clearly cannot lie in R.

Ex. 13.23. Let $T = k \times k \times \cdots$ where k is any right self-injective ring, and let R be the subring of T consisting of all sequences $(a_1, a_2, \dots) \in T$ that are eventually constant. Show that $T = Q^r_{\max}(R)$.

Solution. By *LMR*-(3.11B), the ring T is right self-injective. Therefore, $Q^r_{\max}(T) = T$. If we can show that $R \subseteq_d T_R$, it will follow from Ex. 11(1) that $T = Q^r_{\max}(R)$. To see that $R \subseteq_d T_R$, consider two elements

$$x = (x_1, x_2, \dots), \quad y = (y_1, y_2, \dots)$$

in T, where $x \neq 0$. Say $x_i \neq 0$. Let e_i be the i^{th} "unit vector" $(0, \dots, 1, 0, \dots)$, which clearly lies in R. Since

$$xe_i = (0, \dots, x_i, 0, \dots) \neq 0 \text{ and } ye_i = (0, \dots, y_i, 0, \dots) \in R,$$

we have checked that $R \subseteq_d T_R$.

Ex. 13.24. Give an example of a pair of commutative local artinian rings $R \subseteq S$ such that $R \subseteq_e S_R$, but S is *not* a general (right) ring of quotients of R.

Solution. We use the same commutative example in the solution to Ex. 3.36. Let k be a field and $S = k[t]$ with relation $t^4 = 0$, and let R be the k-subalgebra $k \oplus kt^2 \oplus kt^3$. Clearly, S and R are local algebras. We have already checked that $R \subseteq_e S_R$ in the solution to Ex. 3.36. On the other hand, R is a Kasch ring, so by *LMR*-(13.24), $Q_{\max}(R) = R$. In particular, $S \supsetneq R$ cannot be a general (right) ring of quotients of R.

Comment. The example above appeared in Y. Utumi's paper "On quotient rings", Osaka Math. J. **8**(1956), 1–18. Utumi observed that for the two elements t and $t^3 \in S$, if $r \in R$ is such that $tr \in R$, then r must lie in $kt^2 + kt^3$ and therefore $t^3 r$ must be zero. This shows explicitly that R is not dense in S_R.

The injective hull $I = E(R_R)$ turns out to be 6-dimensional over k, and is a direct sum of two copies of $E(k_R)$, where k_R is the unique simple R-module: see the *Comment* on Ex. 3.36. It follows that the ring $H = \mathrm{End}(I_R)$ is a certain 2×2 matrix ring, and is therefore not commutative.

Ex. 13.25. Recall that a ring R is *right principally injective* if, for any $a \in R$, any $f \in \mathrm{Hom}_R(aR, R)$ extends to an endomorphism of R_R (see Exercise (3.44)). Show that, for such a ring R, $\mathrm{rad}\, R = \mathcal{Z}(R_R)$.

Solution. Let $a \in \mathcal{Z}(R_R)$. For any $x \in R$, we have clearly

$$\mathrm{ann}_r(1 - xa) \cap \mathrm{ann}_r(xa) = 0.$$

Since $xa \in \mathcal{Z}(R_R), \mathrm{ann}_r(xa) \subseteq_e R_R$. Therefore, $\mathrm{ann}_r(1 - xa) = 0$. By Exercise (3.45)(1), we have $R \cdot (1 - xa) = R$. Since this holds for all $x \in R$, we have $a \in \mathrm{rad}\, R$ (by *FC*-(4.1)). Conversely, let $a \in \mathrm{rad}\, R$. Suppose $b \in R$ is such that $bR \cap \mathrm{ann}_r(a) = 0$. Consider the map $\varphi : abR \to R$ given by $\varphi(aby) = by$. Since

$$aby = 0 \implies by \in bR \cap \mathrm{ann}_r(a) = 0,$$

φ is a well-defined R-homomorphism. Therefore, φ is given by left multiplication by some element $x \in R$. We have then $b = \varphi(ab) = xab$, so $(1-xa)b = 0$. Since $1-xa \in \mathrm{U}(R)$, $b = 0$. This shows that $\mathrm{ann}_r(a) \subseteq_e R_R$, so by definition $a \in \mathcal{Z}(R_R)$.

Comment. In *LMR*-(13.2)(1), the equation $\mathrm{rad}\, R = \mathcal{Z}(R_R)$ was proved for any right self-injective ring R. This exercise provides a generalization, due to W. K. Nicholson and M. F. Yousif; see their paper "Principally injective rings," J. Algebra **174**(1995), 77–93.

Ex. 13.26. Show that a ring R is semisimple iff R has the following three properties:

(1) R is semiprime,
(2) R right principally injective, and
(3) R satisfies ACC on right annihilators of elements.

Solution. It is well-known that any semisimple ring has the properties (1), (2) and (3). Conversely, let R be a ring satisfying (1), (2) and (3). By Exercise 25, (2) implies that $\text{rad}(R) = \mathcal{Z}(R_R)$ (the right singular ideal), and by *LMR*-(7.15)(1), (3) implies that $\mathcal{Z}(R_R)$ is a nil ideal. Moreover, by *FC*-(10, 29), (1) and (3) imply that every nil (1-sided) ideal in R is zero. Therefore, we have $\text{rad } R = 0$, and so *LMR*-(13.2)(5) shows that R is a von Neumann regular ring. Now any f.g. right ideal in R has the form eR for some idempotent $e \in R$, and we have $eR = \text{ann}_r(1-e)$. Thus, another application of (3) shows that the f.g. right ideals satisfy ACC. This means that R is right noetherian, from which we conclude that R_R is a semisimple module, as desired.

Comment. The result in this exercise appeared in M. Satyanarayana's paper "A note on a self-injective ring", Canad. Math. Bull. **14**(1971), 271–272, except that in that paper, Satyanarayana proved the "if" part under somewhat stronger assumptions, namely, (2) is replaced by R being right self-injective, and (3) is replaced by ACC on all right annihilators in R. If we further specialize the condition (1) to R being prime, then R must be a simple artinian ring: this is an earlier result of K. Koh.

It follows from this exercise that *a right principally injective semiprime right Goldie ring must be semisimple.*

Ex. 13.27. For any right principally injective ring R, show that $\text{soc}(R_R) \subseteq \text{soc}({}_RR)$. (In particular, equality holds for any principally injective ring R.)

Solution. Note that the module R_R has endomorphism ring R acting on the left of R. As in the solution to Exercise (6.40), it suffices to show that, for any $m \in R$:

$$(*) \qquad (mR)_R \text{ simple} \implies {}_R(Rm) \text{ simple}.$$

Using the notations in the solution to Ex. (6.40), we need to extend the R-homomorphism $\psi: \ smR \to mR \ (s \in R)$ to an endomorphism of R_R. What we need for this is precisely the right principal injectivity of R, so $(*)$ holds. The statement in parentheses in the exercise now follows immediately from this.)

Ex. 13.28. Let I_R be a QI module, and $I' = E(I)$. Let $H = \text{End}(I_R)$, $N = \text{rad } H$, and $H' = \text{End}(I'_R)$, $N' = \text{rad } H'$. Show that there is a natural ring isomorphism $H'/N' \cong H/N$.

Solution. By *LMR*-(6.76), we have a natural surjection $\alpha: \ H' \to H$ defined by restriction of endomorphisms (I being a fully invariant submodule of I'). For $f \in H'$, we have $f \in N'$ iff $\ker(f) \subseteq_e I'$, iff $\ker(f) \cap I \subseteq_e I$ (by *LMR*-(13.1)(1)). This shows that $N' = \alpha^{-1}(N)$, so α induces a ring isomorphism $H'/N' \to H/N$.

Ex. 13.29. Let R be a simple ring. For any nonzero $q \in Q^r_{\max}(R)$, show that $R \subseteq RqR$.

Solution. Since R is right essential in $Q^r_{\max}(R)$, there exists $r \in R$ such that $0 \neq qr \in R$. The simplicity of R then implies that

$$R = R(qr)R \subseteq Rq(rR) \subseteq RqR.$$

Comment. This exercise is a replacement for that with the same numbering in *LMR*-p. 383. The latter, a result due to M. Ikeda and T. Nakayama, is already fully covered in (3.18) of *LMR*. The author apologizes for the repetition.

The following two additional exercises arose from a conversation I had with S. K. Jain.

Ex. 13.30. Let R be a right nonsingular ring, and $Q = Q^r_{\max}(R)$. For $a \in R$, show that a has a left inverse in Q iff $\operatorname{ann}_r^R(a) = 0$.

Solution. If there exists $q \in Q$ such that $qa = 1$, then for $s \in \operatorname{ann}_r^R(a)$, we have $s = (qa)s = q(as) = 0$. Conversely, assume that $\operatorname{ann}_r^R(a) = 0$. Since $R_R \subseteq_e Q_R$, we also have $\operatorname{ann}_r^Q(a) = 0$. Now by *LMR*-(13.36), Q is a right self-injective ring. Recalling Ex. (3.2)(1), we see that $\operatorname{ann}_r^Q(a) = 0 \Rightarrow Qa = Q$.

Ex. 13.31. Let R be a domain, and $Q = Q^r_{\max}(R)$. For any left ideal A of Q, show that either $A = Q$ or $A \cap R = 0$. From this, retrieve the fact that Q is a simple domain (a fact already proved in *LMR*-(13.38)′).

Solution. Suppose $A \cap R \neq 0$. Fix a nonzero element $a \in A \cap R$. Since R is a domain, $\operatorname{ann}_r^R(a) = 0$. By the last exercise, a has a left inverse in Q, and so $A \supseteq Qa = Q$.

To see that Q is simple, consider any nonzero ideal A of Q. Since $R_R \subseteq_e Q_R$, we have $A \cap R \neq 0$. By what is proved above, we must have $A = Q$.

§14. Martindal's Ring of Quotients

Martindale's theory of rings of quotients is a theory specifically designed for semiprime rings, i.e. rings R in which $\mathfrak{A}^2 = 0 \Rightarrow \mathfrak{A} = 0$ for any ideal $\mathfrak{A}$. In such a ring, any ideal A has the property that $\operatorname{ann}_\ell(A) = \operatorname{ann}_r(A)$, so we may write $\operatorname{ann}(A)$ for this common annihilator. The family of ideals

$$\mathcal{F} = \mathcal{F}(R) = \{A \text{ (ideal)}: \ \operatorname{ann}(A) = 0\}$$

will play a large role in Martindale's theory. In *LMR*-(14.1), it is shown that $\mathcal{F}$ is just the family of ideals A such that $A \subseteq_e {}_RR_R$ (2-sided essential in R), or equivalently, $A \subseteq_e R_R$ (right essential in R), assuming that R is semiprime. In the case where R is a prime ring, $\mathcal{F}(R)$ is just the family of all nonzero ideals.

In the following, we shall assume that R is semiprime, and shall work within $Q := Q^r_{\max}(R)$, the maximal right ring of quotients of R. The two subrings

$$\begin{aligned} Q^r(R) &= \{q \in Q: \ q\mathrm{A} \subseteq R \text{ for some } A \in \mathcal{F}\}, \\ Q^s(R) &= \{q \in Q: \ q\mathrm{A},\, Bq \subseteq R \text{ for some } A,\ B \in \mathcal{F}\} \end{aligned}$$

are called *Martindale's right* (*respectively symmetric*) *ring of quotients of* R. These rings and Q all have the same center, which is called the *extended centroid* of R (*LMR*-(14.14)). This is always a (commutative) von Neumann regular ring, and is a field in case R is a prime ring (*LMR*-(14.20), *LMR*-(14.22)).

A convenient way to work with $Q^r(R)$ is to think of its elements as "classes" $[A,\, f]$ where $A \in \mathcal{F} = \mathcal{F}(\mathcal{R})$ and $f \in \mathrm{Hom}_R(A_R, R_R)$. Here, we get the "classes" by identifying $(A,\, f)$ with $(A',\, f')$ if $f = f'$ on $A \cap A'$. Addition and multiplication of classes are defined naturally (*LMR*-(14.9)). Using this description of $Q^r(R)$, the extended centroid may be described as the subring of $Q^r(R)$ consisting of classes $[A,\, f]$ where $f: \ A \to R$ is an $(R,\, R)$-homomorphism (*LMR*-(14.19)).

The ring $Q^r(R)$ and its left analogue can also be defined axiomatically as overrings of R with certain special properties. The same is true for the symmetric Martindale rings of quotients within them. This axiomatic approach suffices to show that the two symmetric rings of quotients are, in fact, isomorphic over R.

For rings $R \subseteq S$ with the same identity, an element $x \in S$ is said to be *R-normalizing* if $xR = Rx$. The set N of R-normalizing elements in S is always closed under multiplication, so the set $R \cdot N$ consisting of finite sums $\sum r_i x_i$ ($r_i \in R,\ x_i \in N$) is a subring of S containing R, called the *normal closure* of R in S. In the case where R is semiprime and $S = Q$, this normal closure is a subring of $Q^s(R)$ (*LMR*-(14.31)). Moreover, if $x \in N$ is such that, for $c \in R$, $xc = 0 \Rightarrow c = 0$, then x must be a unit in $Q^s(R)$ (*LMR*-(14.32)).

For any automorphism $\varphi \in \mathrm{Aut}(R)$, we define

$$X(\varphi) = \{x \in Q: \ xa = \varphi(a)x \ \text{ for all } \ a \in R\}.$$

This is a module over the extended centroid $Z(Q)$, consisting of R-normalizing elements. By definition, φ is called an X-*inner automorphism* if $X(\varphi) \neq 0$. The set of such automorphisms is denoted by X-Inn(R). The "X" here is taken from the first letter of Kharchenko's Russian name, since Kharchenko's work first called attention on the set $X(\varphi)$. Kharchenko showed that $X(\varphi)$ is always a *cyclic* module over $Z(Q)$, although this is only stated without proof in *LMR*-§14.

In the case where R is a prime ring, $N^* = N \backslash \{0\}$ is a group under multiplication. Here, X-Inn(R) is a group, consisting of automorphisms of R induced by inner automorphisms of Q defined by the elements of N^*. In particular, we have $X\text{-Inn}(R) \cong N^*/C^*$, where C is the field $Z(Q)$.

The exercises in this section give additional properties of the Martindale rings of quotients $Q^r(R)$ and $Q^s(R)$, and provide some computational examples not yet covered in the text. The last few exercises are devoted to a further study of X-inner automorphisms φ and their Kharchenko sets $X(\varphi)$ over semiprime rings R.

Exercises for §14

Throughout the following exercises, $\mathcal{F} = \mathcal{F}(R)$ denotes the family of ideals with zero annihilators in a semiprime ring R.

Ex. 14.1. For any semiprime ring R, show that $\mathcal{F}$ contains any prime ideal A of R that is not a minimal prime ideal.

Solution. Let $B = \text{ann}(A)$, which is an ideal in R. Consider any minimal prime ideal $\mathfrak{p} \subset$ R. Since $AB = \{0\} \subseteq \mathfrak{p}$, we have either $A \subseteq \mathfrak{p}$ or $B \subseteq \mathfrak{p}$. If $A \subseteq \mathfrak{p}$, then $A = \mathfrak{p}$ (since $\mathfrak{p}$ is a minimal prime). This contradicts our assumption on A, so we must have $B \subseteq \mathfrak{p}$ instead. Therefore, B is contained in the intersection of *all* minimal primes of R, which is the prime radical of R (see *FC*-Exer. (10.14)). Since R is semiprime, its prime radical is zero, so we have $B = 0$; that is, $A \in \mathcal{F}$.

Ex. 14.2. Let $S = Q^r(R)$, where R is a prime ring. If $I,\ I' \subseteq S$ are nonzero right (resp. left) R-submodules of S, show that $I\,I' \neq 0$. (In particular, S is also a prime ring.)

Solution. Pick nonzero elements $q \in I$ and $q' \in I'$. There exist A, $A' \in \mathcal{F}(\mathcal{R})$ such that $qA,\ q'A' \subseteq R$. By *LMR*-(14.9), $qA \neq 0 \neq q'A'$. Since R is prime, we have $(qA)(q'A') \neq 0$, so $qAq' \neq 0$. If I, I' are right R-modules, then $I\,I' \supseteq (qR)q' \supseteq qAq' \neq 0$. If I, I' are left R-modules, then $I\,I' \supseteq q(Rq') \supseteq qAq' \neq 0$.

Ex. 14.3. Let $S = Q^r(R)$, where R is a semiprime ring.

(1) If $I \subseteq S$ is a nonzero right or left R-submodule of S, show that $I^2 \neq 0$. (In particular, S is also a semiprime ring.)
(2) If $J \subseteq S$ is an (R, R)-subbimodule of S, show that $\text{ann}_r^S(J) = \text{ann}_\ell^S(J)$.

Solution. (1) follows by specializing the solution of Exercise 2 to the case $q = q'$. For (2), let $x \in \text{ann}_r^S(J)$, i.e. $Jx = 0$. Then $(xJ)^2 = xJxJ = 0$. Since xJ is a right R-submodule of S, (1) implies that $xJ = 0$, so $x \in \text{ann}_\ell^S(J)$. Similarly, $y \in \text{ann}_\ell^S(J) \Rightarrow y \in \text{ann}_r^S(J)$.

Comment. Of course, the conclusions of this and the last exercise also hold *verbatim* if we replace $Q^r(R)$ by $Q^s(R)$, the symmetric Martindale ring of quotients, or by the normal closure $R \cdot N$ of R (where N denotes the set of R-normalizing elements in $Q^r_{\text{max}}(R)$), or by the central closure $R \cdot C$ of R (where C denotes the extended centroid of R).

Ex. 14.4. Let R be any reduced ring. Show that $Q^s(R)$ is also a reduced ring. How about $Q^r(R)$?

Solution. Suppose $q \in Q^s(R)$ is such that $q^2 = 0$. To show that $q = 0$, choose an ideal $A \subseteq R$ with zero annihilator such that $qA,\ Aq \subseteq R$. Then $qA^2q = (qA)(Aq) \subseteq R$. Since qA^2q consists of nilpotent elements, we have $qA^2q = 0$. This implies that $qA^2 \subseteq R$ consists of nilpotent elements, so now $qA^2 = 0$, whence $q = 0$.

Next, we consider Martindale's right ring of quotients $Q^r(R)$. We claim that, even for domains R, $Q^r(R)$ may not be reduced. For this, we use the same example as in *LMR*-(14.13). Let $R = k\langle x, y\rangle$, where k is any field. Consider the ideal $A \subseteq R$ consisting of all (noncommuting) polynomials with zero constant terms. As a right R-module, $A = xR \oplus yR$ is free with basis $\{x, y\}$. Let $f : A_R \to R_R$ be defined by $f(x) = y$ and $f(y) = 0$. Then $q = [A, f]$ is a nonzero element in $Q^r(R)$. To compute q^2, let us use the domain

$$A^2 = x^2R + y^2R + xyR + yxR.$$

We have

$$\begin{aligned} f(x^2) &= f(x)x = yx, & f(y^2) &= f(y)y = 0,\\ f(xy) &= f(x)y = y^2, & f(yx) &= f(y)x = 0. \end{aligned}$$

Therefore,

$$\begin{aligned} f^2(x^2) &= f(yx) = 0, & f^2(y^2) &= f(0) = 0,\\ f^2(xy) &= f(y^2) = 0, & f^2(yx) &= f(0) = 0. \end{aligned}$$

It follows that $q^2 = [A^2, f^2] = 0$, so $Q^r(R)$ is not reduced.

Ex.14.5. Let R be a semiprime ring. If the extended centroid C of R is a field, show that R must be a prime ring.

Solution. Clearly, $R \neq 0$. Assume that R is not prime. Then there exist nonzero ideals $A,\ A' \subseteq R$ such that $AA' = 0$. Then $B := \text{ann}(A) \neq 0$, and we have $E := A \oplus B \subseteq_e {}_RR_R$ by *LMR*-(11.38), and so $E \in \mathcal{F}(R)$ by (14.2). Now consider $f, g :\ E \to E \subseteq R$ defined respectively, by $\text{Id}_A \oplus 0$ and $0 \oplus \text{Id}_B$. Clearly, $f,\ g$ are nonzero $(R,\ R)$-bimodule homomorphisms from E to R, so they define nonzero elements $\alpha,\ \beta$ of the extended centroid C (see *LMR*-(14.19)). Now

$$\alpha\beta = [E, f]\,[E, g] = [E,\ fg] = 0 \in C,$$

contradicting the assumption that C is a field.

Ex. 14.6. Let $\mathbb{M}_\infty(k)$ be the additive group of $\mathbb{N} \times \mathbb{N}$ matrices over a field k, and let E be the ring of matrices in $\mathbb{M}_\infty(k)$ that are both row finite and column finite. Show that E is a prime ring, and determine the Martindale rings of quotients $Q^r(E)$, $Q^\ell(E)$ and $Q^s(E)$.

Solution. Let S (resp. S') be the ring of all column (resp. row) finite matrices in $\mathbb{M}_\infty(k)$, so that $E = S \cap S'$. Also, let A be the set of all finite matrices in $\mathbb{M}_\infty(k)$. By an easy calculation, we see that A is a *left* ideal in S, and a *right* ideal in S'. Therefore, A is an ideal in $E = S \cap S'$.

Let $R = k + A$. In *LMR*-(14.39), it is shown that R is a prime ring. Since R contains the nonzero ideal A of E, *LMR*-(14.14) applies to show that E is also a prime ring. Moreover, *LMR*-(14.14) shows that E and R have the same Martindale rings of quotients. Therefore, by *LMR*-(14.39), we have

$$Q^r(E) = Q^r(R) = S, \quad Q^\ell(E) = Q^\ell(R) = S', \quad \text{and} \quad Q^s(E) = Q^s(R) = E.$$

Ex. 14.7. (Martindale) Let R be a prime ring and let $a, b, c, d \in R$, with $a, c \neq 0$. Show that the following are equivalent:

(1) $arb = crd$ for all $r \in R$.
(2) There exists a unit q in the extended centroid $C = Z(Q^s(R))$ such that $c = qa$ and $d = q^{-1}b$.

Solution. (2) $\Rightarrow$ (1). Suppose (2) holds. Then, for any $r \in R$, $crd = qarq^{-1}b = arb$ since q is central in $Q^s(R)$.

(1) $\Rightarrow$ (2). Suppose (1) holds. Consider the two nonzero ideals $A = RaR$ and $A' = RcR$ in R, and define the two maps $f: A \to A' \subseteq R$ and $g: A' \to A \subseteq R$ by

$$f\left(\sum x_i a y_i\right) = \sum x_i c y_i \quad \text{and} \quad g\left(\sum x_i c y_i\right) = \sum x_i a y_i,$$

where $x_i, y_i \in R$. To check that f and g are well-defined, suppose that $\sum x_i a y_i = 0$. Then, for any $r \in R$,

$$0 = \left(\sum x_i a y_i\right) rb = \sum x_i a (y_i r) b = \sum x_i c (y_i r) d = \left(\sum x_i c y_i\right) rd.$$

Since R is prime and $d \neq 0$, this implies that $\sum x_i c y_i = 0$. Similarly, we also have $\sum x_i c y_i = 0 \Rightarrow \sum x_i a y_i = 0$. Thus f, g are well-defined, and clearly they are (R, R)-bimodule homomorphisms. According to *LMR*-(14.19), these define elements q, q' in the extended centroid C. Since $fg = \mathrm{Id}_{A'}$ and $gf = \mathrm{Id}_A$, we have $qq' = q'q = 1$ in C. Now $c = f(a) = qa$, so for any $r \in R$,

$$arb = crd = qard = arqd;$$

that is, $aR(b - qd) = 0$. Since R is prime and $a \neq 0$, this yields $b = qd$, and so $d = q^{-1}b$.

Comment. Note that the present exercise subsumes Ex. 10.13, which computes the center of the division hull K of a right Ore domain R. In fact, for given $c, d \in R \backslash \{0\}$, let $a = d$ and $b = c$ in this exercise. If $q := cd^{-1}$ is central in K, then $c = qd = qa$ and $d = q^{-1}c = q^{-1}b$, so this exercise yields $crd = drc$ for all $r \in R$. Conversely, if $crd = drc$ for all $r \in R$, then by this exercise there exists $q \in C = Z(K)$ such that $c = qd$, and so $cd^{-1} = q \in Z(K)$.

Back to prime rings R, it is relevant to recall that the extended centroid $C = Z(Q^s(R))$ is, in fact, a field (see *LMR*-(14.22)). The conclusion $c = qa$ in this exercise, therefore, expresses the linear dependence of a and c over the field C. More generally, Martindale proved that if a_i, b_i $(1 \le i \le n)$ are nonzero elements in the prime ring R such that $\sum_{i=1}^{n} a_i r b_i = 0$ for all $r \in R$, then the set $\{a_1, \ldots, a_n\}$ must be linearly dependent over C. Martindale used this as a tool in handling linear identities on prime rings; see his paper "Prime rings satisfying a generalized polynomial identity," J. Algebra **12**(1969), 576–584.

There is also a more general version of this exercise where a, b, c, d are allowed to be elements of $Q^r(R)$ (instead of R), and (1) is replaced by $arb = c\varphi(r)b$ $(\forall\ r \in R)$, with φ a given automorphism of R. The equivalent condition (2) now reads: there exists a nonzero $q \in X(\varphi^{-1})$ such that $c = aq$ and $d = q^{-1}b$. The proof uses almost exactly the same calculations: see p. 105 of Passman's book "Infinite Crossed Products," Academic Press, Inc., N.Y., 1989.

Ex. 14.8. Let $R \subseteq S$ be rings, and $N \subseteq S$ be the set of R-normalizing elements in S. Show that if $e = e^2 \in N$, then e commutes with every element of R. Deduce that any R-normalizing idempotent in $Q^r_{\max}(R)$ is central.

Solution. The proof of the first statement is based on the argument for the fact that any idempotent in a right duo ring is central: see *ECRT*-Ex. 22.4A. Let $e = e^2 \in N$ and let $a \in R$. Write $ea = a'e$. Then $ea = (a'e)e = eae$. By symmetry, we also have $ae = eae$ and so e commutes with every $a \in R$. In particular, if $e = e^2 \in Q^r_{\max}(R)$ is R-normalizing, then $ea = ae$ (for every $a \in R$) implies that $e \in Z(Q^r_{\max}(R))$ by *LMR*-(14.15).

Ex. 14.9. Let $\varphi \in \operatorname{Aut}(R)$ where R is a semiprime ring. By Ex. (13.15), φ extends uniquely to an automorphism of $Q^r_{\max}(R)$, which we denote by Φ. Show that

(1) $\Phi \mid Q^r(R)$ (resp. $\Phi \mid Q^s(R)$) is the unique extension of φ to $Q^r(R)$ (resp. $Q^s(R)$).
(2) For every $\sigma \in \operatorname{Aut}(R)$, $\Phi(X(\sigma)) = X(\varphi\sigma\varphi^{-1})$. (In particular, $\Phi(X(\varphi)) = X(\varphi)$.)
(3) The set X-Inn(R) is closed under conjugation in Aut(R).
(4) $X(\varphi) = X(\Phi)$. (Basically, this requires proving that, whenever $x \in X(\varphi)$, $xq = \Phi(q)x$ for any $q \in Q^r_{\max}(R)$.)

Solution. (1) The *uniqueness* of the extension (to $Q^r(R)$ or to $Q^s(R)$) is checked in the same way as in the solution to Ex. (13.22). Thus, it suffices to show that Φ restricts to an automorphism of $Q^r(R)$ (resp. $Q^s(R)$). Let $q \in Q^r(R)$. Then $qA \subseteq R$ for an ideal $A \subseteq R$ with zero annihilator.

Applying Φ, we have

$$\Phi(q)\varphi(A) \subseteq \varphi(R) = R.$$

Since $\varphi(A)$ is clearly also an ideal in R with zero annihilator, we have $\Phi(q) \in Q^r(R)$. The same argument applied to φ^{-1} shows that $\Phi^{-1}(q) \in Q^r(R)$, so Φ restricts to an automorphism of $Q^r(R)$. The argument for $Q^s(R)$ is similar.

(2) Let $y \in X(\sigma)$. Then $ya = \sigma(a)y$ for all $a \in R$. Applying Φ to this equation, we get

$$\Phi(y)\varphi(a) = \varphi\sigma(a)\Phi(y).$$

Replacing a by $\varphi^{-1}(a)$ gives

$$\Phi(y)(a) = \varphi\sigma\varphi^{-1}(a)\Phi(y).$$

Since this holds for all $a \in R$, we conclude that $\Phi(y) \in X\left(\varphi\sigma\varphi^{-1}\right)$. This shows that $\Phi(X(\sigma)) \subseteq X\left(\varphi\sigma\varphi^{-1}\right)$. The other inclusion follows by reversing this argument.

(3) If $\sigma \in X\text{-Inn}(\mathrm{R})$, then $X(\sigma) \neq 0$, so by (2),

$$X\left(\varphi\sigma\varphi^{-1}\right) = \Phi(X(\sigma)) \neq 0.$$

Therefore, we have $\varphi\sigma\varphi^{-1} \in X\text{-Inn}(\mathrm{R})$ too.

(4) Since Φ is an automorphism of $Q = Q^r_{\max}(R)$, when we form $X(\Phi)$ we have to look at elements in $Q^r_{\max}(Q)$. Since the latter is just Q itself (by Ex. (13.10), or *LMR*-(13.31)(3)), we need only work with elements of Q. If $x \in Q$ belongs to $X(\Phi)$, then $xq = \Phi(q)x$ for every $q \in Q$. In particular, $xa = \varphi(a)x$ for every $a \in R$, so $x \in X(\varphi)$. This shows that $X(\Phi) \subseteq X(\varphi)$. Conversely, let $x \in X(\varphi)$ and $q \in Q$. There exists a dense right ideal $A \subseteq R$ such that $qA \subseteq R$. For any $a \in A$, we have

$$x(qa) = \varphi(qa)x = \Phi(qa)x = \Phi(q)\varphi(a)x = \Phi(q)xa.$$

Therefore, $(xq - \Phi(q)x)A = 0$. Since $A \subseteq_d R_R$, this implies that $xq = \Phi(q)x \quad (\forall\, q \in Q)$, and hence $x \in X(\Phi)$.

Ex. 14.10. Keep the notations in the last exercise, and let $x \in X(\varphi)$.

(1) Show that $\Phi\left(x^2\right) = x^2$.
(2) Show that $\Phi(x) = x$ if R is either commutative or prime.

Solution. (1) Let $z = \Phi(x)$. Then, by (4) of the last exercise, $x^2 = \Phi(x)x = zx$ and also $x \cdot \Phi^{-1}(x) = \Phi\left(\Phi^{-1}(x)\right)x = x^2$. Applying Φ to the latter, we get $zx = z^2$. Therefore

$$\Phi\left(x^2\right) = z^2 = zx = x^2.$$

(2) First assume R is commutative. Then

$$\left(x - z^2\right) = x^2 - 2xz + z^2 = 0.$$

Since $Q_{\max}(R) = Q^r(R)$ is (commutative and) semiprime by Exercise 3, we must have $\Phi(x) = z = x$.

Next, assume that R is prime. If $x = 0$, there is nothing to prove, so let us assume $x \neq 0$. Then, by *LMR*-(14.32), the R-normalizing element x must be a unit in $Q^s(R)$. Therefore, the automorphism φ on R is induced by the inner automorphism $q \mapsto xqx^{-1}$ of $Q^r_{\max}(R)$. In particular, by the uniqueness of the extension of φ, we have $\Phi(q) = xqx^{-1}$ for every $q \in Q^r_{\max}(R)$. In particular, $\Phi(x) = xxx^{-1} = x$.

Ex. 14.11. Keeping the notations in the last exercise, show that $\Phi(x) = x$ always holds (for every $x \in X(\varphi)$).

Solution. Let $z = \Phi(x)$. Consider any element $q \in Q := Q^r_{\max}(R)$. Using (4) of Exercise 9, we have

(a) $xqx = xx\Phi^{-1}(q) = \Phi(x)\, x\Phi^{-1}(q) = zqx.$

On the other hand, using $\Phi(x^2) = x^2$ obtained in the last exercise, we have

(b) $xqx = \Phi(q)x^2 = \Phi(q)\Phi(x^2) = \Phi\left(qx^2\right) = \Phi\left(x\Phi^{-1}(q)x\right) = zqz.$

From (a), we have $(z - x)Qx = 0$. Since Q is semiprime (by Ex. 13.8), this implies that $xQ(z - x) = 0$. On the other hand, (a) and (b) give $zqz = zqx$, so we also have $zQ(z - x) = 0$. Therefore,

(c) $(z - x)\, Q\, (z - x) \subseteq zQ\,(z - x) - xQ\,(z - x) = 0,$

whence $x = z = \Phi(x)$.

Comment. In 1996, I obtained a solution to this Exercise by using Kharchenko's result (*LMR*-(14.40)) that $X(\varphi)$ is a cyclic module over the extended centroid $Z(Q)$. This result implies that any two elements in $X(\varphi)$ commute. Since (by Ex. 9(2)) $z \in \Phi(X(\varphi)) = X(\varphi)$, we have in particular $xz = zx$. From this equation, I derived (c). After seeing my solution, A. Leroy suggested the above alternative solution, which avoids the use of Kharchenko's result.

Ex. 14.12. Let C denote the extended centroid of a semiprime ring R, and let G be any subgroup of $\mathrm{Aut}(R)$. Recall that, for any $\varphi \in G$:

$$X(\varphi) = \{x \in Q^r_{\max}(R) : \quad xa = \varphi(a)\, x \quad \forall a \in R\},$$

and let $X(G) := \sum_{\varphi \in G} X(\varphi)$. Show that $X(G)$ is a C-subalgebra of the normal closure of R in $Q^r_{\max}(R)$, and deduce that $X(G) \subseteq Q^s(R)$.

Solution. First note that each $X(\varphi)$ is an additive group. For $x \in X(\varphi)$ and $x' \in X(\varphi')$ where φ, $\varphi' \in G$, we have

$$xx'a = x\varphi'(a)\,x' = \varphi\varphi'(a)\,x \quad (\forall\, a \in R),$$

so $xx' \in X(\varphi\varphi')$. This shows that $X(\varphi)\,X(\varphi') \subseteq X(\varphi\varphi')$. Also

$$X(\mathrm{Id}_R) = \{r \in Q^r_{\max}(R) : \; xa = ax \;\; \forall\, a \in R\} = C.$$

Therefore, $X(G) = \sum_{\varphi \in G} X(\varphi)$ is a C-algebra (associated with G). We know (by *LMR*-(14.35)) that each $X(\varphi)$ consists entirely of R-normalizing elements, so $X(G)$ lies in the normal closure of R (in $Q^r_{\max}(R)$). Now, by *LMR*-(14.31), this normal closure lies in $Q^s(R)$, so we have $X(G) \subseteq Q^s(R)$.

Comment. Let $G_0 = G \cap X\text{-Inn}(G)$ in $\mathrm{Aut}(R)$. If $\varphi \notin X\text{-Inn}(R)$, then by definition, $X(\varphi) = 0$. Therefore, we have $X(G) = \sum_{\varphi \in G_0} X(\varphi)$. Thus, we could have replaced G by G_0, except that G_0 may not be a group. If R is a prime ring, then indeed G_0 is a group by *LMR*-(14.39), and it is safe to write $X(G) = X(G_0)$. In this case, let us fix a nonzero element $x_\varphi \in X(\varphi)$ for each $\varphi \in G_0$. Then, by the proof of *LMR*-(14.40), $X(\varphi) = C \cdot x_\varphi$, so we have $X(G) = \sum_{\varphi \in G_0} C \cdot x_\varphi$.

Ex. 14.13. Compute $\mathrm{Inn}(\bar{R})$, $X\text{-Inn}(\bar{R})$ and $X\text{-Inn}(\bar{R})/\mathrm{Inn}(\bar{R})$ for the Hurwitz ring of quaternions $\bar{R}$. Also, determine the group of nonzero normalizing elements for $\bar{R}$ in its division ring of quotients.

Solution. The Hurwitz ring of quaternions $\bar{R}$ consists of

$$\left\{ \frac{1}{2}(\pm a \pm bi \pm cj \pm dk) : \quad a, b, c, d \in \mathbb{Z} \;\text{ all even or all odd} \right\},$$

with unit group

$$\mathrm{U}(\bar{R}) = \{\pm 1,\ \pm i,\ \pm j,\ \pm k,\ (\pm 1 \pm i \pm j \pm k)/2\}$$

where the signs are arbitrarily chosen. (This is called the *binary tetrahedral group.*) Let R be the usual ring of integer quaternions, with $\mathbb{Z}$-basis $\{1,\ i,\ j,\ k\}$, and unit group $\mathrm{U}(R) = \{\pm 1, \pm i, \pm j, \pm k\}$ (the quaternion group). It is observed in *LMR*-(14.43) that the natural map

$$\mathrm{Aut}(R) \longrightarrow \mathrm{Aut}(\mathrm{U}(R))$$

is an isomorphism, and both groups are isomorphic to S_4. Now $\mathrm{U}(R)$ is a normal 2-Sylow subgroup in $\mathrm{U}(\bar{R})$, so it is a characteristic subgroup. Therefore, any $\varphi \in \mathrm{Aut}(\bar{R})$ defines an automorphism of $\mathrm{U}(R)$, which in turn determines an automorphism of R. This shows that

$$\mathrm{Aut}(\bar{R}) \cong \mathrm{Aut}(R) \cong S_4.$$

Since any automorphism of R is X-inner by *LMR*-(14.43), the same is true for $\bar{R}$. Therefore, we have

$$X\text{-Inn}\left(\bar{R}\right) \cong S_4, \quad \text{Inn}\left(\bar{R}\right) \cong \mathrm{U}\left(\bar{R}\right)/\{\pm 1\} \cong A_4,$$

and $X\text{-Inn}\left(\bar{R}\right)/\text{Inn}\left(\bar{R}\right)$ is just $\mathbb{Z}/2\mathbb{Z}$.

It is now easy to determine $\bar{N}$, the set of normalizing elements for $\bar{R}$ in $Q^r_{\max}\left(\bar{R}\right)$ (which is the division ring of rational quaternions). The center of the latter is $\mathbb{Q}$ so $\bar{N}^*/\mathbb{Q}^* \cong X\text{-Inn}\left(\bar{R}\right)$. Now by the calculations in *LMR*-(14.43), the X-inner automorphisms of $\bar{R}$ are induced by conjugations by elements in $\mathrm{U}\left(\bar{R}\right)$ or in $y\cdot\mathrm{U}\left(\bar{R}\right)$ where $y = 1+i$. Therefore, $\bar{N}^*$ has $\mathrm{U}\left(\bar{R}\right)\cdot\mathbb{Q}^*$ as a subgroup of index 2, with a nontrivial coset given by the representative y. This is exactly the same as the group of nonzero elements normalizing the ring of integer quaternions $\mathbb{Z}1 \oplus \mathbb{Z}i \oplus \mathbb{Z}j \oplus \mathbb{Z}k$, as determined in *LMR*-(14.43).

Ex. 14.14. Let A be a commutative unique factorization domain with quotient field K, and let $R = \mathbb{M}_n(A)$. Show that $Q^r_{\max}(R) = \mathbb{M}_n(K)$ and that the group of nonzero R-normalizing elements N^* in $Q^r_{\max}(R)$ is exactly $K^* \cdot \mathrm{U}(R)$. What are the normal and central closures of R in this example?

Solution. By *LMR*-(13.15), we know that $Q^r_{\max}(R) = \mathbb{M}_n(K)$. (See also Ex. 17.15.) Since clearly $K^* \subseteq N^*$ and $\mathrm{GL}_n(A) \subseteq K^*$, and N^* is a group (by *LMR*-(14.33)), we have $K^* \cdot \mathrm{GL}_n(A) \subseteq N^*$. Now consider any $x = (x_{ij}) \in N^*$. To show that $x \in K^* \cdot \mathrm{GL}_n(A)$, we may assume (after scaling x by an element of K^*) that all $x_{ij} \in R$ and that there is no common prime factor to all x_{ij}. Let $y = (y_{ij}) := x^{-1}$ and write $y_{ij} = a_{ij}/a$, where $a_{ij} \in A$ and $a \in A\backslash\{0\}$ has no prime factor dividing all a_{ij}. From $x^{-1}Rx = R$, we see that a divides all entries of $(a_{ij})\,E_{pq}\,(x_{k\ell})$ for all matrix units E_{pq}. Now

$$(a_{ij})\,E_{pq}\,(x_{k\ell}) = \left(\sum_{i,j} a_{ij}E_{ij}\right) E_{pq} \left(\sum_{k,\ell} x_{k\ell}E_{k\ell}\right) = \sum_{i,\ell} a_{ip}x_{q\ell}E_{i\ell},$$

so $a | a_{ip}x_{q\ell}$ for all i, p, q, ℓ. If a is not a unit in A, there would exist a prime element $\pi | a$ in A. Choosing i, p such that $\pi \nmid a_{ip}$, we would have $\pi | x_{q\ell}$ for all q and ℓ, which is not the case. Therefore, we must have $a \in \mathrm{U}(A)$. This gives $x^{-1} \in \mathbb{M}_n(A)$, and hence $x \in \mathrm{GL}_n(A) = \mathrm{U}(R)$, as desired.

The central closure of R is the subring of $Q^r_{\max}(R)$ generated by R and the extended centroid C of R. Since

$$C = Z\left(Q^r_{\max}(R)\right) = Z\left(\mathbb{M}_n(K)\right) = K,$$

the central closure of R is $R \cdot C = \mathbb{M}_n(A) \cdot K = \mathbb{M}_n(K)$. It follows that the normal closure $R \cdot N$ is also $\mathbb{M}_n(K)$. Note that these calculations can be made without first knowing the exact structure of N^*.

Comment. Since the X-inner automorphisms of R are induced by the inner automorphisms arising from the elements of N^*, it follows from the equation $N^* = K^* \cdot \mathrm{U}(R)$ that all X-inner automorphisms of R are inner automorphisms.

Ex. 14.15. Show that the conclusion $N^* = K^* \cdot \mathrm{U}(R)$ in the last exercise may not hold if the commutative domain A there is *not* a unique factorization domain.

Solution. Take any commutative domain A with a nonprincipal ideal $J = aA + bA$ such that J^2 is principal, say $J^2 = sA$. Write $s = ad - bc$ with $c,\ d \in J$. Consider the matrix $x = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbb{M}_2(J)$ with determinant s. Then $x^{-1} = s^{-1}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$. Now any matrix in

$$x\,\mathbb{M}_n(A)\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

has entries in J^2, which are all divisible by s. Therefore,

$$x\,\mathbb{M}_n(A)\,x^{-1} \subseteq \mathbb{M}_n(A), \quad \text{and similarly,} \quad x^{-1}\mathbb{M}_n(A)\,x \subseteq \mathbb{M}_n(A).$$

This shows that $x \in N^*$.

We claim that $x \notin K^* \cdot \mathrm{U}(R)$. Indeed, assume that x has the form $k \cdot \begin{pmatrix} t & u \\ v & w \end{pmatrix}$ where $k \in K^*$ and $\begin{pmatrix} t & u \\ v & w \end{pmatrix} \in \mathrm{U}(R) = \mathrm{GL}_n(A)$. Then

$$kI_2 = x\begin{pmatrix} t & u \\ v & w \end{pmatrix}^{-1} \in \mathbb{M}_2(J)$$

shows that $k \in J$, and $a = kt$, $b = ku$ show that $J \subseteq kA$. Therefore, we get $J = kA$, which is a contradiction.

The conclusion that $x \notin K^* \cdot \mathrm{U}(R)$ implies that the map $\alpha \mapsto x\alpha x^{-1}$ is an X-inner automorphism of $R = \mathbb{M}_n(A)$ that is *not* an inner automorphism.

For some concrete examples of A, take K to be any number field of class number 2 and take A to be the ring of algebraic integers in K. We can then choose J to be any nonprincipal ideal of A (which can always be generated by two elements since A is a Dedekind domain). An explicit example is $A = \mathbb{Z}[\theta]$ with $\theta = \sqrt{-5}$, and $J = 2A + (1+\theta)A$: see *LMR*-(2.19D). Here, we can take

$$x = \begin{pmatrix} 2 & 1+\theta \\ -1+\theta & -2 \end{pmatrix}$$

with $\det(x) = 2$. Since $x^2 = -2I_2$, the noninner X-inner automorphism on $\mathbb{M}_2(\mathbb{Z}[\theta])$ given by conjugation by x has order 2.

Ex. 14.16. Let A be a commutative domain with a nonidentity automorphism φ_0, and let φ be the automorphism on $R = \mathbb{M}_n(A)$ defined by $\varphi((a_{ij})) = (\varphi_0(a_{ij}))$. Show that φ is not an X-inner automorphism.

Solution. As in the solution to Ex. 14, $Q^r_{\max}(R) = \mathbb{M}_n(K)$ where K is the quotient field of A. Since an X-inner automorphism on $R = \mathbb{M}_n(A)$ is induced by a conjugation in $\mathbb{M}_n(K)$, it must be determinant-preserving. However,

$$\det(\varphi((a_{ij}))) = \det(\varphi_0(a_{ij})) = \varphi_0\big(\det((a_{ij}))\big).$$

Since φ_0 is not the identity, φ is not determinant-preserving. Therefore, φ cannot be X-inner.

Ex. 14.17. Let R be a semiprime ring with extended centroid C, and let $R \cdot C$ be its central closure. Show that the central closure of $R \cdot C$ is itself.

Solution. We know that $S := R \cdot C$ is semiprime by the *Comment* on Ex. 2. Our job is to prove that any element q in the extended centroid of S is already in S. Think of q as an (S, S)-bimodule homomorphism $f : A \to S$, where A is a suitable ideal of S such that $A \subseteq_d S_S$. *We claim that* $A \subseteq_d S_R$. To see this, let $x, y \in S$, with $x \neq 0$. Fix $s \in S$ such that $xs \neq 0$ and $ys \in A$. Since $S \subseteq Q^r_{\max}(R)$, there exists a dense right ideal B of R such that $sB \subseteq R$. Since $xs \neq 0$, we have $xsB \neq 0$, and $ysB \subseteq yR \subseteq A$. This checks that $A \subseteq_d S_R$. Combining this with $R \subseteq_d S_R$, we get

$$A_0 := A \cap R \subseteq_d R_R.$$

Now f restricts to an (R, R)-bimodule homomorphism $f_0 : A_0 \to S$. Since A_0 is a submodule of S_R, Ex. 8.1 applies to show that

$$A_1 := f_0^{-1}(R) \subseteq_d (A_0)_R.$$

By the transitivity of denseness, we get $A_1 \subseteq_d R_R$, and so $A_1 \in \mathcal{F}(\mathcal{R})$. The (R, R)-homomorphism $f_1 = f_0| A_1$ is given by left multiplication by some $c \in C$ on A_1, so we have now $(q - c) A_1 = 0$, which implies that $q = c \in C$, as desired.

Chapter 6

Frobenius and Quasi-Frobenius Rings

§15. Quasi-Frobenius Rings

A *quasi-Frobenius* (*or* QF) *ring* is a ring R that is right injective and right (or equivalently, left) noetherian. Such a ring is always 2-sided artinian and 2-sided injective. In particular, "QF" is a left-right symmetric condition.

There is also a characterization of a QF ring that is independent of the injectivity conditions:

> *R is* QF *iff R is right noetherian and satisfies the double annihilator conditions for both left and right ideals.*

All of these results are contained in *LMR*-(15.1), which, of course, requires considerable work for its proof.

The following are some of the properties of the (left, right) modules over a QF ring R:

(A) *A module over R is projective iff it is injective.*
(B) *Any module can be embedded into a free module.*
(C) *Any f.g. module is reflexive.*
(D) *Taking R-duals defines a "perfect duality" between f.g. right R-modules and f.g. left R-modules.*
(E) *Let A, $B \subseteq M$ where M is a f.g. projective R-module. Then $A \cong B$ iff $M/A \cong M/B$.*

These results reveal the beginning of an unusually rich module theory for QF rings. Some examples of QF rings are collected in *LMR*-(15.26); more examples are given in *LMR*-§16.

In the commutative category, the structure of the QF rings is easy to describe. A commutative artinian ring R decomposes into $R_1 \times \cdots \times R_s$

where each R_i is a (commutative) local artinian ring. With respect to this decomposition, *R is QF iff each R_i has a simple socle.* These "building blocks" R_i are known as *zero-dimensional Gorenstein rings* in commutative algebra.

The exercises in this section offer further properties of (and characterizations for) QF rings. Exercise 10, due to Nakayama, says that $\text{ann}_\ell(J^n) = \text{ann}_r(J^n)$ for $J = \text{rad}\, R$ for any QF ring R; this generalizes the fact that $\text{soc}(R_R) = \text{soc}({}_RR)$, proved independently in *LMR*-(15.8). Exercises 14–15 lead to new examples of noncommutative QF rings, and Exercises 21–25 offer a general method for constructing commutative local Frobenius algebras. Exercises 17 and 18 provide a link between this section and the earlier section on homological dimensions.

Exercises for §15

Ex. 15.1. Redo Exercise (3.2)(3) ("A right self-injective domain R is a division ring") using the general facts on right self-injective rings proved in *LMR*-§15.

Solution. Let R be a right self-injective domain, and let $c \in R$ be a nonzero element. Using right self-injectivity alone (see Step 2 in the proof of *LMR*-(15.1)), we see that Rc is a left annihilator. Since R is a domain, this left annihilator can only be R itself, so c has a left inverse. This shows that R is a division ring.

Comment. Yet other arguments are possible. For instance, by *LMR*-(13.2)(4), a right self-injective right nonsingular ring R is always von Neumann regular. If R is a domain, this will make R a division ring.

Ex. 15.2. Give an example of a commutative ring R with two ideals A, B such that $\text{ann}(A) + \text{ann}(B)$ is properly contained in $\text{ann}\,(A \cap B)$.

Solution. Let $R = k[x, y]$ be defined by the relations $x^2 = y^2 = xy = 0$, where k is any field. Then $R = k \oplus kx \oplus ky$, and for the (minimal) ideals $A = kx$, $B = ky$, we have $A \cap B = 0$ so $\text{ann}\,(A \cap B) = R$. On the other hand,

$$\text{ann}\,(A) = \text{ann}\,(B) = kx + ky,$$

so $\text{ann}\,(A) + \text{ann}\,(B)$ is also $kx + ky$, which is properly contained in R.

Comment. In Step 1 in the proof of (2) $\Rightarrow$ (4) in *LMR*-(15.1), it is shown that $\text{ann}_\ell\,(A \cap B) = \text{ann}_\ell\,(A) + \text{ann}_\ell\,(B)$ for any right ideals A, B in a right self-injective ring R. Of course, the commutative ring we considered in the above solution is *not* self-injective, since the R-homomorphism $\varphi:\ kx \to ky$ defined by $\varphi\,(x) = y$ cannot be realized as multiplication by an element of R.

Ex. 15.3. Show that if R has ACC on left annihilators and is right self-injective, then R is QF.

Solution. For any right self-injective ring R, it is shown in Step 2 in the proof of (2) $\Rightarrow$ (4) in *LMR*-(15.1) that any f.g. left ideal in R is a left annihilator. Therefore, if R has ACC on left annihilators, it will have ACC on f.g. left ideals, and this is tantamount to R being left noetherian. Therefore, R is QF by *LMR*-(15.1).

Comment. It might have been somewhat more natural to work with conditions imposed on just one side, assuming, say, that R has ACC on *right* annihilators and is *right* self-injective. In this case, it can be shown again that R is QF. The proof of this is, however, quite a bit harder, requiring a theorem of Bass (*FC*-(23.20)) for right perfect rings. For the details, we refer the reader to C. Faith's "Algebra II", Grundlehren der Math. Wissenschaften, Vol. 191, Springer-Verlag, 1976, pp. 208–209.

Ex. 15.4. Show that R is QF iff every right ideal is the right annihilator of a finite set and every left ideal is the left annihilator of a finite set.

Solution. We use the idea in the solution of Exercise 6.21, although we cannot utilize this exercise directly.

First assume R is QF and let A be a right ideal. Then $\text{ann}_\ell(A) = Rx_1 + \cdots + Rx_n$ for some $x_1, \ldots, x_n \in R$, since R is left noetherian. Now

$$A = \text{ann}_r(\text{ann}_\ell(A)) = \text{ann}_r(Rx_1 + \cdots + Rx_n) = \text{ann}_r(X)$$

for the finite set $X := \{x_1, \ldots, x_n\}$. By symmetry, any left ideal $B \subseteq R$ has the form $\text{ann}_\ell(Y)$ for some finite set Y.

Conversely, assume that every right ideal is the right annihilator of a finite set and every left ideal is the left annihilator of a finite set. In particular, every 1-sided ideal satisfies the double annihilator condition. In view of *LMR*-(15.1), R is QF if we can show that it is artinian. Consider any chain of right ideals $A_1 \supseteq A_2 \supseteq \ldots$, and let $S_i = \text{ann}_\ell(A_i)$. Let S be the union of the ascending left ideals $S_1 \subseteq S_2 \subseteq \ldots$. By assumption, there exists a finite set S_0 such that $\text{ann}_r(S) = \text{ann}_r(S_0)$. Since S is a left ideal, we have

$$S = \text{ann}_\ell(\text{ann}_r(S)) = \text{ann}_\ell(\text{ann}_r(S_0)) \supseteq S_0.$$

For sufficiently large i, we have therefore $S_i \supseteq S_0$ and so $\text{ann}_r(S_i) = \text{ann}_r(S)$. But

$$\text{ann}_r(S_i) = \text{ann}_r(\text{ann}_\ell(A_i)) = A_i,$$

so $A_i = \text{ann}_r(S)$ for large i. This checks that R is right artinian, and a similar argument checks that R is left artinian.

Ex. 15.5. Show that a quotient R/I of a QF ring R need not be QF.

Solution. Consider $R = k[x,y]/(x^2,y^2)$ where k is any field. This is a commutative 4-dimensional algebra over k, and, by *LMR*-(3.15B), it is a Frobenius k-algebra. In particular, R is a self-injective ring and hence a QF ring. For the ideal

$$I = \left(x^2, xy, y^2\right)/\left(x^2, y^2\right),$$

we have a quotient $\bar{R} := R/I \cong k\,[x,y]/\left(x^2,\ xy,\ y^2\right)$. This is *not* a self-injective ring (and hence not QF) according to *LMR*-(3.69). In fact, for the two ideals $A = k \cdot \bar{x}$ and $B = k \cdot \bar{y}$ in $\bar{R}$, the $\bar{R}$-isomorphism $A \to B$ sending $\bar{x}$ to $\bar{y}$ obviously *cannot* be extended to an endomorphism of $\bar{R}_{\bar{R}}$.

Comment The main point here is that $\mathrm{soc}(R)$, generated over k by the image of xy, is a minimal ideal in R, while $\mathrm{soc}\left(\bar{R}\right)$, generated over k by the images of x and y, is not a minimal ideal in $\bar{R}$. (See Exercise 3.14.)

Ex. 15.6. Let C be a cyclic right R-module, say $C = R/A$ where A is a right ideal in R.

(1) Show that $C^* \cong \mathrm{ann}_\ell\,(A)$ as left R-modules.
(2) Show that C is torsionless (i.e. the natural map $\varepsilon:\ C \to C^{**}$ is an injection) iff A is a right annihilator.
(3) Show that C is reflexive iff A is a right annihilator and every left R-homomorphism $\mathrm{ann}_\ell\,(A) \to {}_RR$ is given by right multiplication by an element of R.

Solution. (1) The right R-homomorphisms from $C \to R$ are left multiplications by elements of R which left annihilate A. More formally, we have an isomorphism $f:\ C^* \to \mathrm{ann}_\ell\,(A)$ defined by $f\,(\varphi) = \varphi\,(1 + A)$ for $\varphi \in C^*$. (This is a special case of the first isomorphism in *LMR*-(15.14).)

(2) From (1) we have an isomorphism $f^* : (\mathrm{ann}_\ell\,(A))^* \to C^{**}$. Let δ be the composition of

$$C \xrightarrow{\ \varepsilon\ } C^{**} \xrightarrow{\ f^{*-1}\ } (\mathrm{ann}_\ell\,(A))^*\,.$$

What is $\delta\,(x + A)$ as a functional on $\mathrm{ann}_\ell\,(A)$? We "identify" a general element $b \in \mathrm{ann}_\ell\,(A)$ with $\varphi = f^{-1}\,(b)$, which is left multiplication by b (mapping R/A to R). Then

$$(*) \qquad \delta\,(x + A)\,(b) = \varepsilon\,(x + A)\,(\varphi) = \varphi\,(x + A) = bx.$$

Now, ε is one-one iff δ is one-one. In view of $(*)$, the latter is the case iff, for any $x \in R$, $\mathrm{ann}_\ell\,(A)\,x = 0 \Rightarrow x \in A$. This amounts precisely to $A = \mathrm{ann}_r\,(\mathrm{ann}_\ell\,(A))$, i.e. A is a right annihilator.

(3) ε is onto iff δ is onto. In view of $(*)$, the latter is the case iff every $\alpha \in (\mathrm{ann}_\ell\,(A))^*$ is given by right multiplication by an element $x \in R$. The desired conclusion follows by combining this with (2).

Comment. If R is a *left self-injective* ring, the last condition in (3) is automatic. In this case, it follows that *a cyclic right R-module R/A is reflexive iff A is a right annihilator.*

Ex. 15.7. Show that a right noetherian ring R is QF iff every 1-sided cyclic R-module is torsionless.

Solution. If R is QF, we know, in fact, that *every* 1-sided R-module is torsionless (*LMR*-(15.11)(1)). Conversely, assume that every 1-sided cyclic R-module is torsionless. By Exercise 6, it follows that any right (resp. left) ideal in R is a right (resp. left) annihilator. Since R is right noetherian, R is QF by *LMR*-(15.1).

Ex. 15.8. Let $e = e^2 \in R$, and $J \subseteq R$ be a right ideal such that $eJ \subseteq J$. Use Exercise 6 to show that $(eR/eJ)^* \cong \text{ann}_\ell(J) \cdot e$ as left R-modules.

Proof. Let $C = eR/eJ$ which is a cyclic right R-module generated by $\bar{e} = e + eJ$. We represent C in the form R/A by taking $A = \ker(\pi)$ where $\pi : R \to C$ is induced by left multiplication by e. An easy computation shows that

$$A = \{x \in R : \ ex \in eJ\} = J + (1 - e)R.$$

By Exercise 6, C^* is given by

$$\text{ann}_\ell(A) = \text{ann}_\ell(J) \cap \text{ann}_\ell(1 - e)R = \text{ann}_\ell(J) \cap Re.$$

Clearly, this left ideal is contained in $\text{ann}_\ell(J) \cdot e$. Conversely, if $x \in \text{ann}_\ell(J) \cdot e$, say $x = ye$ where $y \in \text{ann}_\ell(J)$, then $xJ = y(eJ) \subseteq yJ = 0$ implies that $x \in \text{ann}_\ell(J) \cap Re$. Therefore,

$$C^* \cong \text{ann}_\ell(A) = \text{ann}_\ell(J) \cdot e,$$

as desired.

Comment. The isomorphism σ from $\text{ann}_\ell(J) \cdot e$ to $(eR/eJ)^*$ may be defined by

$$\sigma(ye)(ez + J) = yez \in R$$

for $y \in \text{ann}_\ell(J)$ and $z \in R$. It is, of course, also possible to verify directly that σ defines a left R-module isomorphism. The assumption that $eJ \subseteq J$ is needed to show that σ is well-defined. Otherwise, no assumptions are needed on R. In the special case where R is a 1-sided artinian ring and $J = \text{rad}(R)$ (the Jacobson radical of R), we can replace $\text{ann}_\ell(J)$ by the right socle $\text{soc}(R_R)$, according to *FC*-Exer. (4.20). This yields the isomorphism $(eR/eJ)^* \cong \text{soc}(R_R) \cdot e$.

Ex. 15.9. Let R be a QF ring and $J = \text{rad}(R)$. In *LMR*-(15.7), a proof is given for $\text{ann}_\ell(J) = \text{ann}_r(J)$ using the equality of the right and left socles of R. Give another proof for $\text{ann}_\ell(J) = \text{ann}_r(J)$ by using the last exercise.

Solution. Consider any primitive idempotent $e \in R$. Then eR/eJ is a simple right R-module (by *FC*-(25.2)). Since R is QF, *LMR*-(15.13) guarantees that $(eR/eJ)^*$ is a simple left R-module. Invoking Exercise 8, we see that $\text{ann}_\ell(J) \cdot e$ is simple, and so

$$J \cdot (\text{ann}_\ell(J) \cdot e) = 0.$$

Since 1_R is a sum of primitive idempotents, it follows that $J \cdot \text{ann}_\ell(J) = 0$; that is, $\text{ann}_\ell(J) \subseteq \text{ann}_r(J)$. By symmetry, we also have $\text{ann}_r(J) \subseteq \text{ann}_\ell(J)$.

Ex. 15.10. For (R, J) as in the last exercise, show that $\text{ann}_\ell(J^n) = \text{ann}_r(J^n)$ for any positive integer n.

Solution. The last exercise shows that the conclusion holds for $n = 1$. In general, for any ring R and any subset $J \subseteq R$ such that $\text{ann}_\ell(J) = \text{ann}_r(J)$, it can be shown, by induction on n, that $\text{ann}_\ell(J^n) = \text{ann}_r(J^n)$. Indeed, suppose this holds for a given n. Then, for any $x \in \text{ann}_r\left(J^{n+1}\right)$, we have $J^n \cdot (Jx) = 0$, so

$$Jx \subseteq \text{ann}_r(J^n) = \text{ann}_\ell(J^n).$$

Thus, $JxJ^n = 0$, which gives

$$xJ^n \subseteq \text{ann}_r(J) = \text{ann}_\ell(J),$$

and so $xJ^{n+1} = 0$. This shows that $\text{ann}_r\left(J^{n+1}\right) \subseteq \text{ann}_\ell\left(J^{n+1}\right)$, and the reverse inclusion follows from symmetry.

Comment. The result in this exercise is due to T. Nakayama, and is Theorem 6 in his paper "On Frobeniusean algebras, I", Annals of Math. **40** (1939), 611–633.

In general, a QF ring R may have ideals $A \subseteq R$ such that $\text{ann}_r(A) \neq \text{ann}_\ell(A)$. For such examples, see *LMR*-(16.66) and Ex. (16.19) below. However, if R is a *symmetric* algebra over a field, then for *any* ideal $A \subseteq R$, $\text{ann}_r(A) = \text{ann}_\ell(A)$: see *LMR*-(16.65).

Ex. 15.11. Show that a QF ring is right semihereditary iff it is semisimple.

Solution. The "if" part is clear. For the converse, assume that R is QF and right semihereditary. For any right ideal $I \subseteq R$, I is f.g. (since R is right noetherian), so I_R is projective. By *LMR*-(15.9), I_R is injective, so it is a direct summand of R_R. This shows that R_R is a semisimple module, so R is a semisimple ring.

Comment. This exercise is only a small part of *LMR*-(7.52). Part (1) of this result applies to rings without chain conditions, giving a list of five equivalent conditions (von Neumann regular, right semihereditary, right Rickart, right nonsingular, and Baer) for any right self-injective ring R. If we add a right noetherian condition to R, then each of the five conditions is equivalent to R being semisimple.

Ex. 15.12. Assuming the Faith-Walker Theorem (*LMR*-(15.10)), show that a ring R is QF iff every module M_R embeds into a free R-module.

Solution. The "only if" part is already proved in *LMR*-(15.11). We repeat the proof here: M_R embeds into its injective hull $E(M)$, and $E(M)$ is projective by *LMR*-(15.9). Therefore, $E(M)$ embeds into a suitable free R-module.

Conversely, assume that any right R-module embeds into a free R-module. Then, for any injective module I_R, we have $I \subseteq F$ for some free module F. But then I is a direct summand of F, and so I_R is projective. Since all injective right R-modules are now projective, the Faith-Walker Theorem guarantees that R is a QF ring.

Ex. 15.13. For any QF ring R, show that:

(1) For any simple module S_R, the injective hull $E(S)$ is a principal indecomposable R-module;
(2) For any f.g. module M_R, $E(M)$ is also f.g.

Solution. (1) We know (from *LMR*-(15.1)) that R is right Kasch, so S embeds into R_R. Since R_R is a finite direct sum of principal indecomposables, it follows that there exists an embedding $S \hookrightarrow U$ where U is a suitable principal indecomposable. Now by *LMR*-(15.9), U_R is *injective*, so we may assume that $S \subseteq E(S) \subseteq U$. But then $E(S)$ is a direct summand of U, so we must have $E(S) = U$.

(2) By Exercise 12, $M \subseteq F$ for some free module F. Since M is f.g., we have $M \subseteq F_0 \subseteq F$ for some free module F_0 of finite rank. By *LMR*-(15.9) again, F_0 is an injective module, so (a copy of) $E(M)$ can be found inside F_0. Then $E(M)$ is a direct summand of F_0, and so is also f.g. (Alternatively, since R is a right noetherian ring, the finite generation of F_0 implies that of any of its submodules.)

Ex. 15.14. (Nakayama, Connell) Let R be a group ring AG where A is a ring and G is a finite group. Show that R is right self-injective (resp. QF) iff A is.

Solution. If A is right artinian, then R_A is artinian, so R_R is also artinian. Conversely, if R_R is artinian, then A_A must be artinian since $A \cong R/I$ where I is the augmentation ideal of R. Since "QF" is "right artinian" together with "right self-injective," it is enough to handle the "right self-injective" case in this exercise.

We have an A-homomorphism $t: \; R_A \to A_A$ defined by $t\left(\sum a_g g\right) = a_1$. Let P be any (R, R)-bimodule. Then P is also an (R, A)-bimodule, and we have an additive group homomorphism

$$(*) \qquad \varphi: \; \mathrm{Hom}_R\left({}_RP_R, \; R_R\right) \longrightarrow \mathrm{Hom}_A\left({}_RP_A, \; A_A\right)$$

defined by $\varphi(f) = t \circ f$. Both groups are right R-modules (the right R-structures being induced by the *left* action of R on P), and the following check shows that φ is a right R-homomorphism:

$$\varphi(fr)(p) = t((fr)p) = t(f(rp)) = (\varphi(f)r)(p),$$

where $f \in \operatorname{Hom}_R(P, R)$, $r \in R$ and $p \in P$. *We claim that φ is an isomorphism.* Indeed, if $f \neq 0$, then $f(p) = \sum a_g g \neq 0$ for some $p \in P$. Say $a_{g_0} \neq 0$. Then

$$\varphi(f)\left(pg_0^{-1}\right) = t\left(f\left(pg_0^{-1}\right)\right) = t\left(f(p)\,g_0^{-1}\right) = a_{g_0} \neq 0$$

implies that $\varphi(f) \neq 0$, so φ is *injective.* To show that φ is *surjective*, take any $h \in \operatorname{Hom}_A(P, A)$. Define $f : P \to R$ by

$$f(p) = \sum\nolimits_{g \in G} h(pg)\,g^{-1} \quad (\forall\, p \in P).$$

An easy calculation shows that $f \in \operatorname{Hom}_R(P_R, R_R)$, and we have

$$\varphi(f)(p) = t(f(p)) = h(p \cdot 1) = h(p) \quad (\forall\, p \in P),$$

so $h = \varphi(f)$.

Now assume A is right self-injective, and let P above be the (R, R)-bimodule R. Then $(*)$ yields a right R-module isomorphism.

$$\operatorname{Hom}_R({}_RR_R,\ R_R) \cong \operatorname{Hom}_A({}_RR_A,\ A_A).$$

The right R-module on the LHS is canonically isomorphic to R_R. On the other hand, since ${}_RR$ is projective and A_A is injective, the Injective Producing Lemma (*LMR*-(3.5)) implies that $\operatorname{Hom}_R({}_RR_A, A_A)$ is an injective right R-module. Therefore, R_R is injective, as desired.

Conversely, assume that R is right self-injective. To check that A_A is injective, we apply Baer's Test. Let $h : J \to A$ be a right A-homomorphism, where $J \subseteq A$ is a right ideal. We can extend h to $h' : JG \to R$ by defining

$$h'\left(\sum a_g g\right) = \sum h(a_g)\,g \quad (a_g \in J).$$

Clearly, JG is a right ideal in R, and h' is a right R-homomorphism. Since R_R is injective, there exists $\beta = \sum b_g g \in R$ such that $h'(\alpha) = \beta\alpha$ for every $\alpha \in JG$. Letting $\alpha = a \cdot 1$ for $a \in J$, we have, in particular, $h(a) = b_1 a$ (by comparing the coefficients of 1). Thus, Baer's Criterion implies the injectivity of A_A.

Comment. The result in this exercise comes from I. Connell's paper "On the group ring," Canad. J. Math. **15**(1963), 650–685, although the QF case was done earlier by T. Nakayama. Connell and subsequent authors have also dealt with the case of infinite groups G. As it turns out, if G is infinite, a group ring AG is *never* right self-injective, unless $A = \{0\}$.

Ex. 15.15. Let R be a QF ring. Show that, for any central multiplicative set $S \subseteq R$, the localization RS^{-1} is also a QF ring.

Solution. Since R is right noetherian, *LMR*-(10.32)(6) implies that $T := RS^{-1}$ is also right noetherian. Next, the fact that R_R is injective implies, upon localization, that T_T is injective, according to Exercise 10.31(1). Therefore, T is a right noetherian, right self-injective ring, so by *LMR*-(15.1), it is a QF ring.

Comment. If R is only right self-injective (and not right noetherian), a central localization RS^{-1} need not be right self-injective. In the solution (1) to Exercise 10.30, we have constructed a commutative self-injective local ring R for which some localization RS^{-1} is *not* self-injective.

Ex. 15.16. Show that an idempotent e in a QF ring R is central iff eR is an ideal of R.

Solution. The "only if" part is trivial. Assuming now that eR is an ideal, we shall prove that e is central in two different ways below.

First Proof. It suffices to check that $Re \subseteq eR$ is an equality. Indeed, if this holds, then for any $r \in R$, we have

$$er(1-e) = 0 = (1-e)re,$$

which implies that $er = re$.

Let $J = eR$, and let $\bar{R}$ be the factor ring R/J. Since $\bar{R}_R \cong (1-e)\,R_R$, $\bar{R}_R$ is injective. From $\bar{R}\cdot J = 0$, it follows that $\bar{R}_{\bar{R}}$ is also injective, according to Ex. 3.28. Therefore, the artinian ring $\bar{R}$ is QF (by *LMR*-(15.1)), and thus

$$\text{(1)} \qquad \text{length}\left({}_{\bar{R}}\bar{R}\right) = \text{length}\left(\bar{R}_{\bar{R}}\right)$$

(for instance, by *LMR*-(15.6)). The same equation also holds with $\bar{R}$ replaced by R, so it follows that

$$\text{(2)} \qquad \text{length}\,({}_R J) = \text{length}\,(J_R).$$

Let $J^* = \text{Hom}_R\,(J_R, R_R)$ be the dual of J_R, viewed in the usual way as a left R-module (via the left action of R on itself). By the duality between f.g. right and left R-modules (in *LMR*-(15.12)), we have length $(J_R) =$ length $({}_R J^*)$. But ${}_R J^*$ is easily seen to be $\cong {}_R(Re)$. (This uses the right self-injectivity of R; for more details, see *LMR*-(16.13).) Therefore,

$$\text{length}\ (J_R) = \text{length}\ {}_R(Re).$$

Combining this with (2), we have length $({}_R J) = \text{length}_R(Re)$. Since $Re \subseteq J$, this implies that $Re = J = eR$, as desired.

Second Proof. Since eR is an ideal, so is its left annihilator Rf where $f = 1-e$. Thus, $Rf \supseteq fR$. Taking a Krull-Schmidt decomposition of $R_R = eR \oplus fR$, we get a complete set of principal indecomposables

$$\{U_i = e_i R : 1 \le i \le n\} \quad (e_i^2 = e_i)$$

such that (say)

$$e_1, \dots, e_m \in eR \quad \text{and} \quad e_{m+1}, \dots, e_n \in fR.$$

Then $S_i = U_i/e_i\text{rad}\,(R)$ $(1 \le i \le n)$ provide a complete set of the simple right R-modules. Let $T_i = \text{soc}(U_i)$, which are also simple modules since R is QF (see *LMR*-(16.4)). Moreover, there is a Nakayama permutation π of $\{1, \dots, n\}$ such that $T_i \cong S_{\pi(i)}$ for all i. We claim that:

(3) *If $X \subseteq Rf$ is a minimal right ideal, then $X \cong S_j$ for some $j > m$.*

Indeed, if $X \cong S_k$ for some $k \le m$, then $Xe_k \ne 0$. However,

$$Xe_k \subseteq Rfe_k \subseteq Rf\,eR = 0,$$

a contradiction.

Note that (3) shows, in particular, that $i > m \Rightarrow \pi\,(i) > m$. Therefore, π permutes $\{m+1, \dots, n\}$ and $\{1, \dots, m\}$ separately. Using this information, we shall check that:

(4) $\qquad Rf \supseteq fR$ *is an equality.*

For, if otherwise, we would have $eR \cap Rf \ne 0$. Let X be a minimal right ideal in $eR \cap Rf$. Then $X \cong S_j$ for some $j > m$ by (3). On the other hand, $X \subseteq \text{soc}(eR)$ forces $X \cong T_k$ for some $k \le m$, and hence $X \cong S_{\pi(k)}$ with $\pi(k) \le m$, a contradiction.

Now that we know $Rf = fR$, the beginning part of the first proof shows that f is central. Thus, so is e. (Of course, this also follows from the fact that we have an *ideal* decomposition $R = eR \oplus fR$.

Comment. This exercise is taken from Kasch's book "Modules and Rings," p. 362, Academic Press, N.Y., 1982. It was probably known to Nakayama, the inventor of the notion of QF rings. The earliest explicit reference I can find in the literature is Proposition 4.4 in F.W. Anderson's paper, "Endomorphism rings of projective modules," Math. Zeit. **111**(1969), 322–332. The second proof given above follows closely Anderson's. However, this proof uses the idea of the Nakayama permutation π, which is only developed in *LMR*-§16. The first proof is thus a little more appropriate, since it uses (in the main) only material in *LMR*-§15. I learned this proof from Carl Faith.

Note that the conclusion of this exercise is false for general artinian rings. For instance, in the ring R of 2×2 upper triangular real matrices, the idempotent $e = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ generates $eR = \left\{ \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \right\}$, which is an ideal, but e is *not* central in R.

Some generalizations of this exercise are possible. Most notably, Faith has shown that this exercise is valid over any right pseudo-Frobenius ring; see Theorem 3.9 of his preprint, "Factor rings of pseudo-Frobenius rings." On the other hand, W. K. Nicholson and E. Sánchez Campos have shown that this exercise also holds over the so-called left (or right) morphic rings. (A ring R is called *left morphic* if, for any $a \in R$, $R/Ra \cong \text{ann}_\ell^R(a)$ as left R-modules.) This, however, *does not* imply the present exercise, since QF rings need not be left or right morphic. The Nicholson–Campos paper, "Rings with the dual of the isomorphism theorem," appeared in J. Algebra **271**(2004), 391–406.

Ex. 15.17. For any module M_R over a QF ring R, show that $\text{pd}(M)$ (the projective dimension of M) is either 0 or ∞. Prove the same thing for $\text{id}(M)$ (the injective dimension of M).

Solution. To get the conclusion on projective dimensions, it suffices to show that there is no module M_R with $\text{pd}(M) = 1$. In fact, if such a module M exists, then there is a short exact sequence

$$0 \to P_1 \to P_0 \to M \to 0$$

where P_0, P_1 are projective right R-modules. By *LMR*-(15.9), P_1 is also an injective module, so the exact must splits. This shows that M_R is projective, a contradiction to $\text{pd}(M) = 1$. Similarly, we can show that there is no module M_R with $\text{id}(M) = 1$, so for any M_R, $\text{id}(M)$ can only be 0 or ∞.

Ex. 15.18. (Bass) Let R be a left noetherian ring such that, for any f.g. module M_R, $\text{pd}(M)$ is either 0 or ∞. Show that R is a left Kasch ring.

Solution. Using only the assumption on right projective dimensions, we shall show that *any f.g. left ideal* $A = \sum_{i=1}^n Ra_i \subsetneq R$ *has a nonzero right annihilator in* R. In case R is left noetherian, this will show that $\text{ann}_r \mathfrak{m} \neq 0$ for any maximal left ideal $\mathfrak{m}$, and hence by *LMR*-(8.28), R is left Kasch.

Given A as above, let F be the free module $\bigoplus_{i=1}^n e_i R$, and let $\alpha = \sum e_i a_i \in F$. Assume, for the moment, that $\text{ann}_r A = 0$. Then clearly $\alpha \cdot R$ is free on $\{\alpha\}$, and the exact sequence

$$0 \to \alpha \cdot R \to F \to F/\alpha \cdot R \to 0$$

shows that $\text{pd}(F/\alpha \cdot R)_R \leq 1$. By the assumption on projective dimensions of f.g. right R-modules, we have $\text{pd}(F/\alpha \cdot R) = 0$, that is, $F/\alpha \cdot R$ is projective. Therefore, the above exact sequence splits, so $\alpha \cdot R$ is a direct summand of F. Since $\alpha \cdot R$ is free on $\{\alpha\}$, the Unimodular Column Lemma (Ex. (1.34)) implies that $A = \sum_{i=1}^n Ra_i = R$, a contradiction. Therefore, we must have $\text{ann}_r A \neq 0$, as desired.

Comment. This result, and several refinements thereof, appeared in Bass' paper referenced in the *Comment* on Exercise (1.32).

Ex. 15.19. Let R be a commutative noetherian ring in which the ideal (0) is meet-irreducible. Show that $Q_{c\ell}(R)$ is a (commutative) local QF ring.

Solution. By Noether's Theorem, (0) is a primary ideal. Therefore, $\mathfrak{p} := \text{rad}\,(0) = \text{Nil}\,(R)$ is a prime ideal, and it is easy to check that $\mathfrak{p}$ consists precisely of all 0-divisors of R. The ring $T := R_{\mathfrak{p}}$ is local, noetherian, of Krull dimension 0, so T is an artinian ring. We need to show that T has a *simple* socle, for this will imply, by *LMR*-(15.27), that T is a QF ring. Since T is artinian, we know $\text{soc}(T) \neq 0$. If $\text{soc}(T)$ is not simple, there would exist minimal ideals X, Y of T such that $X \cap Y = (0)$. But then $X' = X \cap R$ and $Y' = Y \cap R$ are nonzero ideals of R with

$$X' \cap Y' = X \cap Y \cap R = (0),$$

which contradicts the fact that (0) is meet-irreducible in R.

Ex. 15.20. For any field k, let $R = k[u,v]$, with the relations $u^2 = v^2 = 0$, and $S = k[x,y,]$ with the relations $xy = x^2 - y^2 = 0$. It is known that R and S are (commutative) 4-dimensional local Frobenius k-algebras (see the examples (4) and (5) at the end of *LMR*-§15D). Show that $R \cong S$ as k-algebras iff $-1 \in k^2$ and $\text{char}(k) \neq 2$. (In particular, $R \cong S$ if $k = \mathbb{C}$ and $R \ncong S$ if $k = \mathbb{R}$.)

Solution. A basis for R is $\{1, u, v, uv\}$. Since $x^3 = x \cdot x^2 = xy^2 = 0$ and $y^3 = y \cdot y^2 = yx^2 = 0$, a basis for S is $\{1, x, y, x^2\}$. The unique maximal ideals for R and S are respectively, $\mathfrak{m}_R = (u, v)$ and $\mathfrak{m}_S = (x, y)$.

If $-1 \in k^2$ and $\text{char}(k) \neq 2$, let $i = \sqrt{-1} \in k$. It is easy to check that $\varphi: R \to S$ defined by

$$\varphi(u) = x + iy \quad \text{and} \quad \varphi(v) = x - iy$$

is a k-algebra homomorphism, with an inverse $\psi : S \to R$, defined by $\psi(x) = (u+v)/2$ and $\psi(y) = (u-v)/2i$. Therefore, $R \cong S$ as k-algebras.

Conversely, assume $R \cong S$, and let $\varphi : R \to S$ be a k-algebra isomorphism. Since $\varphi(\mathfrak{m}_R) \subseteq \mathfrak{m}_S$, we have $\varphi(u) = ax + by + cx^2$ for some a, b, $c \in k$. From

$$0 = \varphi(u^2) = \left(ax + by + cx^2\right)^2 = a^2x^2 + b^2y^2 = \left(a^2 + b^2\right)x^2,$$

we have $a^2 + b^2 = 0$. If $a = 0$, then $b = 0$ too and so $\varphi(u) = cx^2 \in \mathfrak{m}_S^2$. This is impossible since $u \notin \mathfrak{m}_R^2$. Therefore $a \neq 0$, so $-1 = (b/a)^2 \in k^2$. We finish by showing that $\text{char}(k) \neq 2$. If otherwise, $a^2 + b^2 = 0 \Rightarrow a = b$, and hence $\varphi(u) = a(x+y) + cx^2$. A similar argument shows that $\varphi(v) = d(x+y) + ex^2$ for some $d, e \in k$. Now

$$\varphi(uv) = \left(a(x+y) + cx^2\right)\left(d(x+y) + ex^2\right) = ad\left(x^2 + y^2\right) = 0.$$

This is impossible since $uv \neq 0$ in R.

Comment. If $\text{char}(k) = 2$, the k-algebra map $\varphi : R \to S$ above is still defined. But now $\varphi(u) = \varphi(v) = x + y$, so φ is not injective (and we have in fact $\dim_k \varphi(R) = 2$). Note that, in case $\text{char}(k) = 2$, $R = k[u,v]$ above is isomorphic to the group algebra kG, where G is the Klein 4-group.

The next exercise describes a general method for constructing quotients of polynomial algebras that are local Frobenius algebras. The remaining *exercises amplify this point, and provide further explicit computational examples for this interesting construction of* QF *algebras.*

Ex. 15.21. Let $A = k[x_1, \ldots, x_r]$ (where k is a field), and let $\mathfrak{m} = (x_1, \ldots, x_r)$. Let $\lambda : A \to k$ be a k-linear functional with $\ker(\lambda) \supseteq \mathfrak{m}^n$ for some $n \geq 1$ and J_λ be the largest ideal of A that is contained in $\ker(\lambda)$. If $\lambda \neq 0$, show that $J_\lambda \subseteq \mathfrak{m}$ and that A/J_λ is a local Frobenius k-algebra. Conversely, if $J \subseteq \mathfrak{m}$ is an ideal in A such that A/J is a local Frobenius k-algebra, show that J has the form J_λ for some nonzero functional λ as described above.

Solution. Let $f \in J_\lambda$. Then $Af \subseteq J_\lambda$ and so $\lambda(Af) = 0$. Let $n \geq 1$ be minimal such that $\lambda(\mathfrak{m}^n) = 0$. Then there exists $g \in \mathfrak{m}^{n-1}$ such that $\lambda(g) \neq 0$. Writing $f = a + f_0$ where $a \in k$ and $f_0 \in \mathfrak{m}$, we have

$$0 = \lambda(gf) = \lambda(ag + gf_0) = a\lambda(g) \quad (\text{since } \lambda(gf_0) \in \lambda(\mathfrak{m}^n) = 0).$$

Therefore, $a = 0$, and we have $f \in \mathfrak{m}$. This shows that $J_\lambda \subseteq \mathfrak{m}$.

Since $\mathfrak{m}^n \subseteq J$, $\mathfrak{m}/J \subseteq A/J$ is a maximal ideal with $(\mathfrak{m}/J)^n = 0$. Hence A/J is a (finite-dimensional) local k-algebra. The hyperplane $\ker(\lambda)/J$ in it contains no nonzero ideals, so by *LMR*-(3.15), A/J is a Frobenius k-algebra.

Conversely, let $J \subseteq \mathfrak{m}$ be an ideal in A such that A/J is a local Frobenius k-algebra. Then $\text{rad}(A/J) = \mathfrak{m}/J$ is nilpotent, so $\mathfrak{m}^n \subseteq J$ for some $n \geq 1$. By *LMR*-(3.15) again, there exists a hyperplane

$$\ker(\overline{\lambda}) \subseteq A/J$$

(for some nonzero k-linear functional $\overline{\lambda} : A/J \to k$) which contains no nonzero ideal of A/J. Let $\lambda : A \to k$ be the functional on A induced by $\overline{\lambda}$. Clearly, J is the largest ideal of A contained in $\ker(\lambda)$, so $J = J_\lambda$. Here,

$$\mathfrak{m}^n \subseteq J \subseteq \ker(\lambda),$$

so λ is the functional we want.

Comment. Recall, from the proof of *LMR*-(3.15), that a nonsingular "associative" bilinear pairing on the Frobenius algebra A/J_λ is given by $(\bar{x}, \bar{y}) \mapsto \lambda(\bar{x}\bar{y}) \in k$.

Ex.15.22. Keep the notations in the last exercise, and let N be the space of functionals

$$\{\lambda \in \text{Hom}_k(A, k) : \lambda(\mathfrak{m}^n) = 0 \text{ for some } n \geq 1\},$$

viewed as an A-submodule of the A-module $\text{Hom}_k(A, k)$. For any $\lambda \in N$, let $\text{ann}^A(A\lambda)$ denote the A-annihilator of the cyclic submodule $A\lambda \subseteq {}_AN$, and let $\text{Ann}^A\,(A\lambda)$ denote the space of common zeros of the linear functionals in $A\lambda$. Show that

$$\text{ann}^A(A\lambda) = \text{Ann}^A(A\lambda) = J_\lambda,$$

and deduce that $A\lambda \cong A/J_\lambda$ as A-modules.

Solution. Note that the A-action on N is defined by

$$(*) \qquad (f \cdot \lambda)(g) = \lambda(gf) \qquad (f,\, g \in A,\ \ \lambda \in N).$$

In particular, $(f \cdot \lambda)(g) = (g \cdot \lambda)(f)$. Therefore,

$$\begin{aligned} f \in \mathrm{ann}^A(A\lambda) &\iff f \cdot \lambda = 0 \in N \\ &\iff (f \cdot \lambda)(g) = 0 \quad \forall g \in A \\ &\iff (g \cdot \lambda)(f) = 0 \quad \forall g \in A \\ &\iff f \in \mathrm{Ann}^A(A\lambda). \end{aligned}$$

This establishes the first desired equation, and the second one follows from:

$$\begin{aligned} f \in \mathrm{Ann}^A(A\lambda) &\iff (g \cdot \lambda)(f) = 0 \quad \forall g \in A \\ &\iff \lambda(Af) = 0 \\ &\iff Af \subseteq \ker(\lambda) \\ &\iff f \in J_\lambda. \end{aligned}$$

From the equation $J_\lambda = \mathrm{ann}^A(A\lambda)$, it follows immediately that $A\,\lambda \cong A/J_\lambda$ as A-modules.

Comment. If we denote the actions of functional by using the notation $\langle g,\, \lambda \rangle = \lambda(g)$ for $g \in A$ and $\lambda \in N$, the formula $(*)$ above can be transcribed into an associativity formula

$$\langle g,\, f\lambda \rangle = \langle gf,\, \lambda \rangle \quad \text{for} \quad f,\, g \in A, \quad \text{and} \quad \lambda \in N.$$

Ex. 15.23. In the notations of the last two exercises, let $\mathrm{ann}^N(J_\lambda)$ be the annihilator of J_λ in the A-module N, and let $\mathrm{Ann}^N(J_\lambda)$ be the space of functionals in N vanishing on the ideal J_λ. Show that

$$\mathrm{ann}^N(J_\lambda) = \mathrm{Ann}^N(J_\lambda) = A\lambda,$$

and deduce that, for any $\lambda,\, \mu \in N$, $A\lambda = A\mu$ iff $J_\lambda = J_\mu$.

(Comment. Combining this with Exercise 21, we get a one-one correspondence between the nonzero cyclic A-submodules of N and the ideals $J \subseteq (x_1, \ldots, x_r)$ of A for which A/J is a local Frobenius algebra.)

Solution. Since $\dim_k A\lambda = \dim_k A/J_\lambda < \infty$, vector space duality shows that

$$\mathrm{Ann}^N(J_\lambda) = \mathrm{Ann}^N\big(\mathrm{Ann}^A(A\lambda)\big) = A\lambda.$$

Secondly, the formula $(f \cdot \lambda)(g) = \lambda(gf)$ from the last exercise shows that $\mathrm{ann}^N(J) = \mathrm{Ann}^N(J)$ for any ideal $J \subseteq A$.

If $A\lambda = A\mu$, taking annihilators (in either sense) in A gives $J_\lambda = J_\mu$. Conversely, if $J_\lambda = J_\mu$, taking annihilators (in either sense) in N gives $A\lambda = A\mu$.

Comment. More generally, $J \mapsto \text{ann}^N(J) = \text{Ann}^N(J)$ gives a one-one correspondence between the ideals $J \subsetneq (x_1, \ldots, x_r)$ of A such that A/J is a finite-dimensional local k-algebra and the nonzero f.g. A-submodules M of N. The proof of this requires just a couple more steps, using some facts about injective hulls. See, for instance, Theorem 21.6 in Eisenbud's book "Commutative Algebra, with a View toward Algebraic Geometry", Graduate Texts in Math., Vol. **150** (1995), Springer-Verlag, 1995. The correspondence

$$J \leftrightarrow M = \text{ann}^N(J)$$

was deeply rooted in the classical mathematics on ideals in polynomial rings over a field, and goes back to the work of F.S. Macaulay. In Macaulay's book "The Algebraic Theory of Modular Systems" (Cambridge Univ. Press, Cambridge, 1916), the A-module $\text{ann}^N(J)$ was called an "inverse system" of the polynomial ideal $J \subsetneq A$.

Ex. 15.24. Keeping the notations in the above exercises, identify the space of functionals N with the A-module of "inverse polynomials" $T = k\left[x_1^{-1}, \ldots, x_r^{-1}\right]$, as in *LMR*-(3.91)(1). For the "functionals"

$$\lambda_1 = x^{-1}y^{-2}, \quad \lambda_2 = x^{-2} + y^{-2}, \quad \lambda_3 = x^{-1}y^{-1} + z^{-2}$$

in $T = k\left[x^{-1}, y^{-1}\right]$ and $T = k\left[x^{-1}, y^{-1}, z^{-1}\right]$ respectively, show that

$$J_{\lambda_1} = \left(x^2,\, y^3\right), \quad J_{\lambda_2} = \left(xy,\, x^2 - y^2\right), \quad \text{and}$$
$$J_{\lambda_3} = \left(x^2,\, y^2,\, xz,\, yz,\, xy - z^2\right).$$

State a generalization for each of these three cases.

Solution. We shall compute J_λ by using the two expressions obtained for it in Exercise 22. First consider $\lambda_1 = x^{-1}y^{-2}$. Recall that this is the functional on $A = k[x, y]$ that takes xy^2 to 1 and all other monomials in x, y to zero. Clearly, $A\lambda_1 \subseteq k\left[x^{-1}, y^{-1}\right]$ has the k-basis

$$\left\{1,\, x^{-1},\, y^{-1},\, x^{-1}y^{-1},\, y^{-2},\, x^{-1}y^{-2}\right\}.$$

The common null space of these six functionals is our J_{λ_1}, so $\dim_k A/J_{\lambda_1} = 6$. Now clearly, x^2 and y^3 are in this null space, so J_{λ_1} contains the ideal $\left(x^2, y^3\right) \subseteq A$. It is easy to see that $\dim_k A/(x^2,\, y^3) = 6$, so we must have $J_{\lambda_1} = \left(x^2, y^3\right)$. This method of computation clearly generalizes to show that, if $\lambda_1 = x^{-m}y^{-n}$ where $m,\, n \geq 0$, then $J_{\lambda_1} = \left(x^{m+1}, y^{n+1}\right)$.

Next, we take the functional $\lambda_2 = x^{-2} + y^{-2}$ in $k\left[x^{-1}, y^{-1}\right]$. Iterating the x and y actions on λ_2, we see that $A\lambda_2$ has a k-basis $\left\{1, x^{-1}, y^{-1}, x^{-2} + y^{-2}\right\}$. Therefore, $\dim_k A/J_{\lambda_2} = 4$. Letting "$\cdot$" denote the A-action on $k\left[x^{-1}, y^{-1}\right]$,

$$xy \cdot \lambda_2 = xy \cdot \left(x^{-2} + y^{-2}\right) = 0, \quad \text{and}$$
$$\left(x^2 - y^2\right) \cdot \lambda_2 = \left(x^2 - y^2\right) \cdot \left(x^{-2} + y^{-2}\right) = 1 - 1 = 0,$$

so we have

$$xy,\ x^2 - y^2 \in \operatorname{ann}^A(A\lambda_2) = J_{\lambda_2}.$$

Since $A/\left(xy,\ x^2 - y^2\right)$ have dimension 4 as we have seen in Exercise 19, it follows that

$$J_{\lambda_2} = \left(xy,\ x^2 - y^2\right) \subseteq A.$$

The method of computation here generalizes easily to show that, if $\lambda_2 = x^{-n} + y^{-m}$ where $n,\ m \geq 1$, then $J_{\lambda_2} = (xy,\ x^n - y^m)$, with $\dim_k A/J_{\lambda_2} = n + m$.

Finally, take $\lambda_3 = x^{-1}y^{-1} + z^{-2}$ in $T = k\left[x^{-1},\ y^{-1}, z^{-1}\right]$. Applying the x, y, z actions again (repeatedly) to λ_3, we see that $A\lambda_3$ has a basis

$$(*) \qquad \left\{1,\ x^{-1},\ y^{-1},\ z^{-1},\ x^{-1}y^{-1} + z^{-2}\right\},$$

and so $\dim_k A/J_{\lambda_3} = 5$. We can use the previous method to determine a set of ideal generators for J_{λ_3}, but let us solve the problem by another method for illustration. Thinking of J_{λ_3} as $\operatorname{Ann}^A(A\lambda_3)$, we see that each of the six "functionals" in $(*)$ imposes a linear condition on the coefficients of a polynomial $f \in J_{\lambda_3}$. In fact, these conditions are, that f has zero constant term, zero coefficients for the x, y, z terms, and coefficients for the xy and z^2 terms summing to zero. Therefore, the most general form for a polynomial f in J_{λ_3} is

$$f = ax^2 + by^2 + cxz + dyz + e\left(xy - z^2\right) + g,$$

where $g \in (x, y, z)^3$. Therefore, if

$$J := \left(x^2,\ y^2,\ xz,\ yz,\ xy - z^2\right),$$

we have $J_{\lambda_3} = J + (x, y, z)^3$. But $(x, y, z)^3 \subseteq J$ (noting, especially, that $z^3 = x(yz) - z(xy - z^2)$). Hence $J_{\lambda_3} = J$, as desired. This method of computation generalizes easily to $\lambda_3 = x^{-1}y^{-1} + z^{-n}$ (with $n \geq 1$), for which we'll have

$$J_{\lambda_3} = \left(x^2,\ y^2,\ xz,\ yz,\ xy - z^n\right).$$

The case $n = 0$ is a bit exceptional. A straightforward computation along the same lines gives here $J_{\lambda_3} = \left(x^2, y^2, z\right)$.

Comment. In all cases, it is easy to see directly that, as predicted by Exercise 21, A/J_{λ_i} is a local Frobenius k-algebra. Indeed, the socles for A/J_{λ_i}, in the generalized cases, are the simple modules

$$k \cdot \bar{x}^m \bar{y}^n,\quad k \cdot \bar{x}^n,\quad \text{and}\quad k \cdot \overline{xy},$$

for $i = 1, 2,$ and 3.

Ex. 15.25. For the ideal $J := (y^3, x^2 - xy^2)$ in $A = k[x, y]$, show that $R = A/J$ is a 6-dimensional local Frobenius k-algebra, and find a linear functional $\lambda \in T = k\left[x^{-1}, y^{-1}\right]$ such that $J = J_\lambda$.

Solution. Note that in R,

$$\bar{x}^2\,\bar{y} = \bar{x}\,\bar{y}^2 \cdot \bar{y} = 0 \quad \text{and} \quad \bar{x}^3 = \bar{x} \cdot \bar{x}^2 = \bar{x} \cdot \bar{x}\,\bar{y}^2 = 0.$$

From this, it follows easily that R has a basis $1, \bar{x}, \bar{y}, \bar{x}^2, \bar{x}\,\bar{y}, \bar{y}^2$ (where, of course, $\bar{x}^2 = \bar{x}\,\bar{y}^2$). A routine computation shows that $\mathrm{soc}(R) = k\,\bar{x}^2$, so R is local with maximal ideal $(\bar{x}, \bar{y})$, and has a simple socle. By *LMR*-(15.27), R is a 6-dimensional local Frobenius k-algebra.

The linear functional $\lambda := x^{-2} + x^{-1}y^{-2} \in T$ has the property that, under the A-action on T, $y^3 \cdot \lambda = (x^2 - xy^2) \cdot \lambda = 0$. Therefore,

$$(y^3,\, x^2 - xy^2) \subseteq \mathrm{ann}^A(A\lambda) = J_\lambda.$$

Now, by applying x and y to λ repeatedly, we see that $A\lambda$ has basis

$$\left\{1,\, x^{-1},\, y^{-1},\, x^{-1}y^{-1},\, x^{-1} + y^{-2},\, x^{-2} + x^{-1}y^{-2}\right\}.$$

Therefore, $J_\lambda = \mathrm{Ann}^A(A\lambda)$ has codimension 6 in A. Since $(y^3,\, x^2 - xy^2)$ also has codimension 6, the inclusion $(y^3,\, x^2 - xy^2) \subseteq J_\lambda$ must be an equality.

§16. Frobenius Rings and Symmetric Algebras

The characterization of a commutative QF ring as a finite direct product of commutative artinian local rings with simple socles has a nice generalization to the noncommutative setting. For any (possibly noncommutative) artinian ring R,

> *R is QF iff R is Kasch and every 1-sided principal indecomposable R-module has a simple socle.*

This description of a QF ring R leads to the definition of the Nakayama permutation of R. Let $U_i = e_iR$ $(1 \le i \le s)$ be a complete set of right principal indecomposables over R, and let S_i be the unique top composition factor of U_i. Then we have a permutation π of $\{1, 2, \ldots, s\}$ such that $\mathrm{soc}(U_i) \cong S_{\pi(i)}$; π is called the *Nakayama permutation* of R. Of course, π is only defined up to a conjugation in the symmetric group on s letters, since the labelling of the U_i's is completely arbitrary.

It is shown that in *LMR*-(15.8) that, in any QF ring R, $\mathrm{soc}(R_R) = \mathrm{soc}({}_RR)$, so we may write $\mathrm{soc}(R)$ for the common socle. Let $\bar{R} = R/\mathrm{rad}\, R$, which is an (R, R)-bimodule. By definition, a QF ring R is called a *Frobenius ring* if $\mathrm{soc}(R) \cong \bar{R}$ as right (or equivalently, left) R-modules. It is also possible to recognize a Frobenius ring by using the multiplicity numbers n_i defined in the following Krull-Schmidt decomposition:

$$R_R \cong n_1 U_1 \oplus \cdots \oplus n_s U_s.$$

In *LMR*-(16.14), it is shown that

> *A* QF *ring* R *is a Frobenius ring iff* $n_i = n_{\pi(i)}$ $(1 \le i \le s)$ *for the Nakayama permutation* π *of* R.

Note that the n_i's can also be defined by the following Wedderburn decomposition of the semisimple ring $\bar{R}$:

$$\bar{R} \cong \mathbb{M}_{n_1}(D_1) \times \cdots \times \mathbb{M}_{n_s}(D_s),$$

where the D_i's are division rings. In particular, if, in this decomposition, all the n_i's happen to be equal (e.g. $\bar{R}$ is a commutative ring or a simple ring), then the QF ring R will be Frobenius.

In *LMR*-§3C, we have introduced a class of finite-dimensional algebras over a field k called *Frobenius algebras.* It can be shown without much difficulty that, for any finite-dimensional k-algebra R, *R is a Frobenius k-algebra iff R is a Frobenius ring.* Another useful characterization, due to Nakayama, is the following equation

$$\dim_k A + \dim_k \operatorname{ann}_\ell(A) = \dim_k R = \dim_k \mathfrak{A} + \dim_k \operatorname{ann}_r(\mathfrak{A}),$$

imposed on all right ideals $A \subseteq R$ and all left ideals $\mathfrak{A} \subseteq R$.

Any Frobenius k-algebra R is equipped with a *Nakayama automorphism* σ, which is defined up to an inner automorphism of R. We first fix a nonsingular, bilinear pairing $B : R \times R \to k$ with the associativity property $B(xy,\, z) = B(x,\, yz)$. Then σ is defined by the equation

$$B(a,\, x) = B(x,\, \sigma(a)) \quad (\forall\ a,\ x \in R)\,.$$

The presence of this Nakayama automorphism σ makes it possible to formulate many additional properties of a Frobenius algebra. We note also that the Nakayama automorphism σ "effects" the Nakayama permutation π, in the sense that $R\sigma(e_i) \cong Re_{\pi(i)}$ for all i, where $U_i = e_i R$ are the distinct right principal indecomposables.

We can define QF algebras too: these are (finite-dimensional) k-algebras R over a field k for which R_R and $\left(\hat{R}\right)_R$ have the same distinct indecomposable components, where $\hat{R}$ denotes the (R,R)-bimodule $\operatorname{Hom}_k(R,k)$. Again, these turn out to be the k-algebras that are QF as rings. A more significant step is the introduction of *symmetric algebras*: a k-algebra R is said to be symmetric if $R \cong \hat{R}$ as (R,R)-bimodules. This condition is equivalent to the existence of a nonsingular bilinear, associative pairing $B : R \times R \to k$ that is also *symmetric.* Among Frobenius algebras, the symmetric algebras are precisely the ones whose Nakayama automorphisms are *inner.* We have the hierarchy:

$$\begin{pmatrix}\text{semisimple}\\ \text{algebras}\end{pmatrix} \subset \begin{pmatrix}\text{symmetric}\\ \text{algebras}\end{pmatrix} \subset \begin{pmatrix}\text{weakly}\\ \text{symmetric}\\ \text{algebras}\end{pmatrix} \subset \begin{pmatrix}\text{Frobenius}\\ \text{algebras}\end{pmatrix}$$
$$\subset \begin{pmatrix}\text{QF}\\ \text{algebras}\end{pmatrix},$$

where "weakly symmetric algebras" are defined to be the QF algebras whose Nakayama permutations are the identity.

Many properties of the above classes of algebras are covered by the exercises in this section. There are also various examples of these algebras, including a construction (Exercise 3) showing that any permutation of $\{1, 2, \ldots, s\}$ can be realized as the Nakayama permutation of a suitable QF algebra. Other exercises touch upon the notions of finite representation types, invariant subalgebras, Cartan matrices and parastrophic matrices, etc. (see *LMR*-§16 for the appropriate definitions).

Exercises for §16.

Ex. 16.0. Show that, for a QF ring, two principal indecomposable right R-modules U, U' are isomorphic iff $\text{soc}\,(U) \cong \text{soc}\,(U')$.

Solution. Let $S = \text{soc}\,(U)$ and $S' = \text{soc}\,(U')$. Then S, S' are simple right R-modules by *LMR*-(16.4). If $U \cong U'$, certainly $S \cong S'$. Conversely, suppose $S \cong S'$. As in the first part of the proof of *LMR*-(16.4), $U = E(S)$ and $U' = E(S')$ (where E denotes taking the injective hull), so we have $U \cong U'$.

Comment. Of course, the conclusion of the exercise also follows by applying the Nakayama permutation to the principal indecomposables. The above solution is more direct.

Ex. 16.1. Let $(R, \mathfrak{m})$ be a local artinian ring with $K = R/\mathfrak{m}$. Show that the following are equivalent:

(1) R is QF;
(1)′ R is Frobenius;
(2) soc(R_R) is a simple right R-module and soc$({}_RR)$ is a simple left R-module;
(2)′ $\text{ann}_\ell(\mathfrak{m})$ is a 1-dimensional right K-vector space and $\text{ann}_r(\mathfrak{m})$ is a 1-dimensional left K-vector space;
(3) R_R and ${}_RR$ are uniform R-modules;
(4) $E\left((R/\mathfrak{m})_R\right) \cong R_R$;
(4)′$E\left({}_R(R/\mathfrak{m})\right) \cong {}_RR$.

Show that these conditions imply each of the following:

(5) $E((R/\mathfrak{m})_R)$ is a cyclic R-module;
(5)$'E({}_R(R/\mathfrak{m}))$ is a cyclic R-module;
(6) R is a subdirectly irreducible ring in the sense of FC-(12.2) (i.e. R has a smallest nonzero ideal).

If R is commutative, show that all ten conditions above are equivalent.

Solution. Since $R/\text{rad } R = K$ is a division ring, *LMR*-(16.18) implies that (1) $\Leftrightarrow$ (1)$'$. Also, since R is artinian, we have $\text{soc}(R_R) = \text{ann}_\ell(\mathfrak{m})$ and $\text{soc}({}_RR) = \text{ann}_r(\mathfrak{m})$, so (2) $\Leftrightarrow$ (2)$'$; and the fact that any nonzero left (right) ideal contains a minimal left (right) ideal yields the equivalence (2) $\Leftrightarrow$ (3).

(1) $\Rightarrow$ (4). Since R_R is injective and indecomposable, we have

$$E((R/\mathfrak{m})_R) \cong E(\text{soc}(R_R)) = R_R.$$

(4) $\Rightarrow$ (1). By (4), R_R is injective, so R is QF by *LMR*-(15.1).

(1) $\Leftrightarrow$ (4)$'$ follows now from left-right symmetry.

(1) $\Leftrightarrow$ (2). Since R is artinian and there is only one simple (left, right) R-module, R is Kasch. Also, there is only one principal indecomposable (left, right) R-module, namely, R itself. Therefore, (1) $\Leftrightarrow$ (2) follows from *LMR*-(16.4).

(4) $\Rightarrow$ (5) and (4)$'$ $\Rightarrow$ (5)$'$ are clear.

(2) $\Rightarrow$ (6). We know that $\text{soc}(R_R)$ is an ideal (for any ring R). Since R is artinian, $\text{soc}(R_R)$ is clearly the smallest nonzero ideal under the assumption (2), so R is subdirectly irreducible (and we have $\text{soc}(R_R) = \text{soc}({}_RR)$).

Finally, we assume that R is commutative.

(6) $\Rightarrow$ (2). Let L be the smallest nonzero ideal in R. Clearly L is the unique minimal ideal in R, and hence $\text{soc}(R) = L$ is simple.

(5) $\Rightarrow$ (4). If $E(R/\mathfrak{m})$ is cyclic, it is a quotient of R. Since $E(R/\mathfrak{m})$ is a *faithful* R-module by *LMR*-(3.76), it follows that $E(R/\mathfrak{m}) \cong R$.

Comment. An example to keep in mind for this exercise is $R = KG$ where K is a field of characteristic $p > 0$, and G is a finite p-group. Here, R is a Frobenius K-algebra, and is local with Jacobson radical $\mathfrak{m}$ given by the augmentation ideal. It is easy to check that

$$\text{soc}(R) = \text{ann}_\ell(\mathfrak{m}) = \text{ann}_r(\mathfrak{m})$$

is given by $K \cdot \sum_{g \in G} g$. Another family of (noncommutative) local Frobenius algebras is given in Exercise 22 below.

In the commutative case, the rings in question are known to commutative algebraists and algebraic geometers as zero-dimensional local Gorenstein rings.

The next exercise shows that, in this exercise the condition (2) has to be imposed on both $\mathrm{soc}(R_R)$ and $\mathrm{soc}({}_RR)$, and that (6) $\Rightarrow$ (1) need not hold if R is not assumed to be commutative.

Ex. 16.2. Construct a local artinian (necessarily Kasch) ring R such that (1) R is subdirectly irreducible, and (2) $\mathrm{soc}({}_RR)$ is simple but $\mathrm{soc}(R_R)$ is not. (Such a ring R is, in particular, *not* QF.)

Solution. Let K be a field with an endomorphism σ such that, for $L = \sigma(K)$, $[K : L]$ is an integer $n > 1$. Let $K[x;\sigma]$ be the skew polynomial ring consisting of $\sum a_i x^i$ ($a_i \in K$) with the multiplication rule $xa = \sigma(a)x$ for $a \in K$. Let $(x^2) = \left\{\sum_{i\geq 2} a_i x^i\right\}$ be the ideal generated by x^2, and let

$$\text{(A)} \qquad R = K\,[x;\sigma]\,/\,(x^2) = K \bigoplus K\bar{x}.$$

Clearly, $\mathfrak{m} = K\bar{x} \subset R$ is an ideal of square zero, and $R/\mathfrak{m} \cong K$. Therefore, R is a local ring with maximal ideal $\mathfrak{m}$. Let $\{a_i, \ldots, a_n\}$ be a basis of K over L. Then

$$\text{(B)} \qquad \mathfrak{m} = K\,\bar{x} = \bigoplus a_i\, L\,\bar{x} = \bigoplus a_i \sigma(K)\,\bar{x} = \bigoplus a_i \bar{x}\, K.$$

From (A) and (B), we see that R has left dimension 2 and right dimension $n+1$ over K. In particular, we see that R is a left and right artinian ring.

Since $\mathfrak{m}^2 = 0$, we have $\mathrm{soc}({}_RR) = \mathrm{soc}(R_R) = \mathfrak{m}$. From (A), we see that $\mathrm{soc}({}_RR)$ is simple (as left R-module), and from (B), we see that $\mathrm{soc}(R_R)$ is semisimple of length n (as right R-module). (Note that, since $\bar{x}^2 = 0$,

$$a_i\bar{x}\,R = a_i\bar{x}\,(K \oplus K\,\bar{x}) = a_i\bar{x}\,K,$$

so, each $a_i\bar{x}\,K$ is a simple right R-module.) Since $n > 1$, $\mathrm{soc}(R_R)$ is not simple so R is not QF. From the left structure of R, it is clear that R has exactly three left ideals, (0), $\mathfrak{m}$, and R. In particular, $\mathfrak{m}$ is the smallest nonzero ideal of R, so R is subdirectly irreducible.

Note that, in this example, $\mathrm{u.dim}({}_RR) = 1$ but $\mathrm{u.dim}(R_R) = n > 1$, so R is left uniform but not right uniform (as R-module). Also, since R is its only (left, right) principal indecomposable module, we see that R has left Cartan matrix (2), and right Cartan matrix $(n+1)$.

Comment. The example R constructed above is an artinian (local) ring. If one tries to find a *finite* example, one is doomed to fail: Weimin Xue has shown that:

> *For a finite local ring R, $\mathrm{soc}({}_RR)$ is simple iff $\mathrm{soc}(R_R)$ is simple (so such rings are Frobenius);*

see his paper "A note on finite local rings," Indag. Math. **9**(1998), 627–628. (This answered a question of R.W. Goldbach and H.L. Classen ("A field-like property of finite rings," Indag. Math. **3**(1992), 11–26.) Xue's result was later generalized to arbitrary rings by T. Honold, who proved that:

A finite ring R is Frobenius if (and only if) $_R(R/\text{rad}R) \cong \text{soc}(_RR)$;

see his paper "Characterization of finite Frobenius rings," Arch. Math. **76**(2001), 406–415. Honold's result was motivated by his work on coding theory over finite chain rings. In fact, finite chain rings and Frobenius rings have come to play rather substantial roles in the modern coding theory developed over rings.

Ex. 16.3. Let $R = R_1 \times \cdots \times R_r$, where each R_i is a QF ring. Describe the Nakayama permutation of R in terms of the Nakayama permutations π_i of R_i ($1 \le i \le r$). (Note that R is a QF ring by *LMR*-(15.26)(3).) Using this result and the computation in *LMR*-(16.19)(4), show that there exist Frobenius algebras over any given field whose Nakayama permutation π is any prescribed permutation on a finite number of letters.

Solution. For ease of notation, let us work with the case $r = 2$, and write R as $A \times B$ where A, B are QF rings. Say $e_1A, \ldots, e_sA$ are the distinct right principal indecomposables of A, and $e_1'B, \ldots, e_t'B$ are the distinct principal indecomposables of B. Then

$$e_1R, \ldots, e_sR, \quad e_1'R, \ldots, e_t'R$$

are the distinct principal indecomposables of R, where e_i means $(e_i, 0)$, and e_j' means $(0, e_j')$. Let S_i (resp. T_j) be the unique simple quotient of e_iA (resp. $e_j'B$), so that $\{S_i, T_j\}$ is the complete set of simple right modules of $R = A \times B$, with B (resp. A) acting trivially on S_i (resp. T_j). Then $\text{soc}(e_iR) = \text{soc}(e_iA)$ is $S_{\pi(i)}$ with trivial B-action where π is the Nakayama permutation of A, and a similar statement holds for $\text{soc}(e_j'R)$. Therefore, the Nakayama permutation of R permutes separately the sets of indices

$$\{1, \ldots, s\} \text{ and } \{1, \ldots, t\}$$

(treated as disjoint sets), the former by π (the Nakayama permutation of A), and the latter by π' (the Nakayama permutation of B).

In *LMR*-(16.19)(4), it is shown that, for any field k, and any integer $n \ge 1$, there exists a Frobenius k-algebra with Nakayama permutation $(12 \cdots n)$. Since any permutation factors into a product of disjoint cycles, the above work implies that the Nakayama permutation of a Frobenius k-algebra can be any prescribed permutation on any (finite) number of letters.

Ex. 16.4. Show that the Nakayama permutation of a commutative QF ring R is the identity.

Solution. By (15.27), we have $R \cong R_1 \times \cdots \times R_r$, where each R_i is a commutative *local* QF ring. Then R_i has a unique principal indecomposable module (namely R_i itself), so the Nakayama permutation of R_i is the identity permutation (of a singleton set). By the last exercise, it follows that the Nakayama permutation of $R \cong R_1 \times \cdots \times R_r$ is also the identity permutation (on r letters).

Comment. In the case where R is a finite-dimensional algebra over some field k, R being QF implies that R is a Frobenius algebra, and thus a symmetric algebra by *LMR*-(16.55). Therefore, the present exercise also follows from *LMR*-(16.64), which guarantees that the Nakayama permutation of any symmetric algebra is the identity.

Ex. 16.5. For a QF ring R, it is shown in *LMR*-(15.25) that there is an (R, R)-bimodule isomorphism $\operatorname{soc}(R_R) \cong \left({}_R\bar{R}\right)^*$, where $\bar{R} = R/\operatorname{rad}(R)$. Confirm this as a right R-module isomorphism by using the Nakayama permutation π for R.

Solution. Label the principal indecomposables and simple R-modules as U_i, U_i', S_i and S_i', as in *LMR*-(16.8). Using the decomposition $R_R \cong \bigoplus_i n_i \cdot U_i$, we have by *LMR*-(16.9):

$$\operatorname{soc}(R_R) \cong \bigoplus\nolimits_i n_i \cdot \operatorname{soc}(U_i) \cong \bigoplus\nolimits_i n_i \cdot S_{\pi(i)}.$$

On the other hand,

$$\left({}_R\bar{R}\right)^* \cong \left(\bigoplus\nolimits_i n_i \cdot S_i'\right)^* \cong \bigoplus\nolimits_i n_i \cdot (S_i')^* \cong \bigoplus\nolimits_i n_i \cdot S_{\pi(i)}$$

by *LMR*-(16.10). This shows that $\operatorname{soc}(R_R) \cong \left({}_R\bar{R}\right)^*$ as right R-modules.

Comment. The isomorphism $\operatorname{soc}(R_R) \cong \left({}_R\bar{R}\right)^*$ (for a QF ring R) in this exercise is not to be confused with $\operatorname{soc}(R_R) \cong \bar{R}_R$, which holds if and only if R is a Frobenius ring, according to *LMR*-(16.14).

Ex. 16.6. For any primitive idempotent f in a QF ring R, it is shown in *LMR*-(16.6) that $\operatorname{soc}(Rf) \cong \left(\bar{f}\bar{R}\right)^*$ as left R-modules (where $\bar{R} = R/J$, $J = \operatorname{rad}(R)$). Give a direct proof of this by using Exercise (15.8), assuming only that R is 1-sided artinian and that $\operatorname{soc}(R_R) = \operatorname{soc}({}_RR)$.

Solution. By the *Comment* following Exercise (15.8), the dual module $\left(\bar{f}\,\bar{R}\right)^* \cong (fR/fJ)^*$ is given by $\operatorname{soc}(R_R) \cdot f$. By the given hypothesis, this equals $\operatorname{soc}({}_RR) \cdot f$, which is in turn equal to $\operatorname{soc}(Rf)$ according to Exercise (6.12)(7).

Ex. 16.7. (This exercise is stated in *LMR*-(16.37).) Show that an artinian ring R is QF iff $\operatorname{length}(N) = \operatorname{length}(N^*)$ for any f.g. module N_R and ${}_RN$.

Solution. Suppose the length equation holds. Then, clearly the dual of a simple (left or right) module is simple. By *LMR*-(16.2), R is QF. Conversely, assume R is QF, and let N be a f.g. left R-module. Then N has a composition series so $\operatorname{length}(N)$ makes sense. Proceeding as in the proof of (16.34), let $M \subseteq N$ be any simple submodule. Then M^* is a simple right R-module, and for

$$M^{\perp} := \{f \in N^* : \; f(M) = 0\},$$

we have by *LMR*-(15.14):

$$M^* \cong N^*/M^{\perp} \quad \text{and} \quad M^{\perp} \cong (N/M)^* \quad \text{(as right } R\text{-module).}$$

The former implies that $M^{\perp}$ is a maximal submodule of N^*, so

$$\text{length}\,(N^*) = 1 + \text{length}\,(M^{\perp}).$$

Invoking an inductive hypothesis, we have

$$\text{length}\left(M^{\perp}\right) = \text{length}\,(N/M)^* = \text{length}\,(N/M)\,.$$

Therefore, $\text{length}\,(N^*) = 1 + \text{length}\,(N/M) = \text{length}\,(N)$. The same argument works for right R-modules N_R.

Ex. 16.8. Let $a \in R$ where R is a QF ring. Show that length (Ra) = length(aR). If R is, in fact, a Frobenius algebra over a field k, show that $\dim_k Ra = \dim_k aR$. Does this equation hold over a QF algebra?

Solution. As long as R is a right self-injective ring, we have $(aR)^* \cong Ra$ as left R-modules, according to *LMR*-(16.13). If R is QF, then any f.g. R-module has finite length, and by Exercise 7, this length is unchanged upon taking the R-dual. Therefore, we have

$$\text{length}\,(Ra) = \text{length}\,(aR)^* = \text{length}\,(aR)\,.$$

Now assume R is a Frobenius k-algebra. Then, by *LMR*-(16.34):

$$\dim_k (Ra) = \dim_k (aR)^* = \dim_k (aR)\,.$$

Alternatively, by *LMR*-(16.40)(2):

$$\text{(1)} \qquad \dim_k R = \dim_k (Ra) + \dim_k \text{ann}_r(Ra);$$

and the standard short exact sequence

$$0 \longrightarrow \text{ann}_r\,(Ra) \longrightarrow R \xrightarrow{\ f\ } aR \longrightarrow 0 \quad (f(x) = ax)$$

shows that

$$\text{(2)} \qquad \dim_k R = \dim_k (aR) + \dim_k \text{ann}_r(Ra).$$

A comparison of (1) and (2) gives $\dim_k (Ra) = \dim_k (aR)$.

For a general QF algebra R, Ra and aR may have different dimensions. For instance, let R be the k-algebra consisting of 6×6 matrices over k of the following form:

$$\begin{pmatrix} a & b & p & 0 & 0 & 0 \\ c & d & q & 0 & 0 & 0 \\ 0 & 0 & r & 0 & 0 & 0 \\ 0 & 0 & 0 & r & s & t \\ 0 & 0 & 0 & 0 & a & b \\ 0 & 0 & 0 & 0 & c & d \end{pmatrix}.$$

In *LMR*-(16.19)(5), it is shown that R is a QF algebra, but not a Frobenius algebra. Choosing $a \in R$ to be the matrix unit E_{13}, we compute easily that

$$aR = kE_{13} \quad \text{and} \quad Ra = kE_{13} + kE_{23},$$

so we have $\dim_k aR = 1$ while $\dim_k Ra = 2$. Here, aR is a simple right R-module and Ra is a simple left R-module: this is consistent with the first conclusion of the Exercise.

Ex. 16.9. Let a, b, c be elements in a QF ring R such that $a = b + c$. If $aR = bR + cR$ and $bR \cap cR = 0$, show that $Ra = Rb + Rc$ and $Rb \cap Rc = 0$.

Solution. Since $Ra \subseteq Rb + Rc$, we have by the last exercise

$$\begin{aligned} \text{length}\,(Rb + Rc) &\geq \text{length}\,(Ra) \\ &= \text{length}\,(aR) \\ &= \text{length}\,(bR) + \text{length}\,(cR) \\ &= \text{length}(Rb) + \text{length}\,(Rc)\,. \end{aligned}$$

Therefore, we must have $Rb \cap Rc = 0$, and the first inequality above must be an equality, which means that $Rb + Rc = Ra$.

Ex. 16.10. Let R be a QF ring with $J = \text{rad}(R)$ such that $J^n = 0 \neq J^{n-1}$. Show that if M_R is an indecomposable module such that $MJ^{n-1} \neq 0$, then M is isomorphic to a principal indecomposable module.

Solution. Since $MJ^{n-1} \neq 0$, it contains a nonzero cyclic submodule, and therefore a simple submodule, say S. Let $E = E(S)$ be the injective hull of S. By Exercise 15.13(1), E is a principal indecomposable module. The inclusion map $S \to E$ can be extended to a homomorphism $f : M \to E$. Then $f(M) \nsubseteq EJ$, for otherwise

$$f\,(S) \subseteq f\left(MJ^{n-1}\right) \subseteq f\,(M)\,J^{n-1} \subseteq EJ^n = 0,$$

which is not the case. Since EJ is the unique maximal submodule of E, it follows that $f(M) = E$. Thus, f is a split surjection, and the indecomposability of M implies that $f : \ M \to E$ is an isomorphism.

Ex. 16.11. A ring R is said to have (right) *finite representation type* if R has only finitely many isomorphism types of f.g. indecomposable right R-modules. Let R be a QF ring as in Exercise 10.

(1) Show that R has finite representation type iff R/J^{n-1} does.
(2) If each R/J^i is QF, show that R has finite representation type.

Solution. (1) The "only if" part is trivial, since each indecomposable module over R/J^{n-1} may be viewed as one over R. For the converse, assume that R/J^{n-1} has finite representation type. Consider any f.g. indecomposable right R-module M. If $MJ^{n-1} = 0$, then M may be viewed as a (f.g. indecomposable) right R/J^{n-1}-module, so there are only a finite number of possibilities for the isomorphism type of M. If $MJ^{n-1} \neq 0$, then by the last exercise, M is a principal indecomposable R-module, and again there are only a finite number of possibilities for its isomorphism type. The two cases combine to show that the ring R itself has finite representation type.

(2) We induct here on n, the index of nilpotency of the Jacobson radical J of the QF ring R in question. For $n = 1$, R is just a semisimple ring. Here the indecomposable modules are just the simple modules, of which there are only a finite number (up to isomorphism). For $n > 1$, note that, for the ring S, $\text{rad}(S) = J/J^{n-1}$ and $S/\text{rad}\,(S)^i \cong R/J^i$ is QF for all i. Since the index of nilpotency for $\text{rad}(S)$ is $n-1$, the inductive hypothesis implies that S has finite representation type. From Exercise 10, it follows therefore that R also has finite representation type.

Ex. 16.12. Let M be a right module over a QF ring R. Show that M_R is faithful iff M has a direct summand isomorphic to $N = e_1R \oplus \cdots \oplus e_sR$, where $\{e_1R, \ldots, e_sR\}$ is a complete set of principal indecomposable right R-modules.

Solution. $k \cdot N$ for a sufficiently large integer k, $k \cdot N$ contains a copy of R_R. Therefore, $k \cdot N$ is faithful, and this implies that N is faithful. It follows that any M_R containing a copy of N is faithful. Conversely, let M be any f.g. faithful right R-module. Assume, by induction, that M has a direct summand

$$K \cong e_1R \oplus \cdots \oplus e_kR \quad \text{where} \quad 0 \le k < s.$$

Say $M = K \oplus K'$. Let $S := \text{soc}(e_{k+1}R)$. *We claim that* $K \cdot S = 0$. Indeed, if $x \cdot S \neq 0$ for some $x \in K$, consider the right R-homomorphism

$$\varphi:\ e_{k+1}R \to K \ \text{ defined by } \ \varphi(e_{k+1}r) = x(e_{k+1}r)$$

for any $r \in R$. Since $S \not\subseteq \ker(\varphi)$ and S is the unique minimal right ideal in $e_{k+1}S$ (by *LMR*-(16.4)), we must have $\ker(\varphi) = 0$. Therefore K contains a copy of $e_{k+1}R$, necessarily as a direct summand since $e_{k+1}R$ is injective (by *LMR*-(15.9)). This contradicts the Krull-Schmidt Theorem (*FC*-(19.23)) since $e_{k+1}R$ is not isomorphic to any of $e_1R, \ldots, e_kR$. Having shown that $K \cdot S = 0$, we see that $K' \cdot S \neq 0$ by the faithfulness of M. Repeating the above argument for K', we have $K' = K_1 \oplus K_2$ where $K_1 \cong e_{k+1}R$. Therefore

$$M = K \oplus (K_1 \oplus K_2) \cong (e_1R \oplus \cdots \oplus e_{k+1}R) \oplus K_2,$$

so we are done by induction.

Comment. In view of *LMR*-(18.8), this Exercise implies that, over a QF ring R, a module M_R is faithful iff it is a generator. A ring R for which every faithful right module is a generator is called a *right* PF (*pseudo-Frobenius*) *ring*. It can be shown that QF rings are precisely the right or left noetherian right PF rings.

Unlike the situation with QF rings, however, the PF notion is *not* left-right symmetric, i.e., there exist right PF rings that are not left PF, and vice versa. The 2-sided PF rings are known as *cogenerator rings.* Such rings play a substantial role in the study of duality theory: see *LMR*-§19, especially the concluding remarks of the subsection §19B.

Ex. 16.13. For the algebra $R = k[x,y]/(x,y)^{n+1}$ over a field k, show that the m^{th} R-dual of the unique simple R-module V is isomorphic to $(n+1)^m \cdot V$.

Solution. The ideal $(x,y)^{n+1}$ is generated by all the monomials of degree $n+1$, so $R = k\,[\bar{x},\, \bar{y}]$ is defined by the relations $\bar{x}^i\, \bar{y}^j = 0$ (for all i, j with $i+j = n+1$). This algebra R has been studied in *LMR*-(3.69). It is a local algebra with a unique simple module V, where V is just k with trivial $\bar{x}, \bar{y}$ actions. It is easy to see that

$$\operatorname{soc}(R) = \bigoplus_{i+j=n} \bar{x}^i\bar{y}^j k \cong V \oplus \cdots \oplus V \quad (n+1 \text{ copies}).$$

To determine the R-dual V^*, note that the R-homomorphisms $\varphi : V \to R$ are determined by $\varphi(1)$, which can be any element of $\operatorname{soc}(R)$. We have $(\bar{x} \cdot \varphi)\,(1) = \bar{x} \cdot \varphi\,(1) = 0$ and similarly $(\bar{y} \cdot \varphi)\,(1) = 0$, so $\bar{x}$, $\bar{y}$ also act trivially on V^*. From these remarks, we see that

$$\dim_k V^* = \dim_k \operatorname{soc}(R) = n+1,$$

and $V^* \cong (n+1) \cdot V$ as R-modules. Repeating this calculation, we see that the m^{th} R-dual of V is given by $(n+1)^m \cdot V$!

Ex. 16.14. Let R be the algebra $k[t]/(f(t))$ over a field k, where $f(t)$ is a nonconstant polynomial. Verify explicitly that R is a Frobenius k-algebra by applying Nakayama's "dimension characterizations" for such algebras (as in *LMR*-(16.40)(2)).

Solution. Since R is a finite-dimensional commutative k-algebra, all we need to show is that, for any ideal $\mathfrak{A} \subseteq R$, we have

$$\dim_k R = \dim_k \mathfrak{A} + \dim_k \operatorname{ann}(\mathfrak{A}),$$

for, if so, Nakayama's characterization theorem cited above will imply that R is a Frobenius algebra. Let us express the ideal $\mathfrak{A}$ in the form $(g(t))/(f(t))$, and write $f(t) = g(t)h(t)$. Say f, g, h have degrees d, m, n, so that $d = m+n$. Then $\operatorname{ann}(\mathfrak{A}) = (h(t))/(f(t))$. We have

$$\dim_k \mathfrak{A} = \dim_k k\,[t]\,/\,(f\,(t)) - \dim_k k\,[t]\,/\,(g(t)) = d - m = n,$$

and similarly $\dim_k \operatorname{ann}(\mathfrak{A}) = m$. Therefore,

$$\dim_k \mathfrak{A} + \dim_k \operatorname{ann}(\mathfrak{A}) = n + m = d = \dim_k R,$$

as desired.

Ex. 16.15. For any division ring k, show that the set of matrices of the form

$$\gamma = \left(\begin{array}{cc|cc|cc} a & 0 & b & 0 & 0 & 0 \\ 0 & a & 0 & b & p & 0 \\ \hline c & 0 & d & 0 & 0 & 0 \\ 0 & c & 0 & d & q & 0 \\ \hline 0 & 0 & 0 & 0 & r & 0 \\ s & 0 & t & 0 & 0 & r \end{array}\right)$$

over k forms a QF ring that is not a Frobenius ring.

Solution. We think of γ as the matrix of a linear transformation g on a right k-vector space V with basis vectors $e_1, \ldots, e_6$, so that

$$g(e_1) = e_1 a + e_3 c + e_6 s, \quad g(e_2) = e_2 a + e_4 c, \quad g(e_3) = e_1 b + e_3 d + e_6 t,$$
$$g(e_4) = e_2 b + e_4 d, \quad g(e_5) = e_2 p + e_4 q + e_5 r, \quad g(e_6) = e_6 r.$$

Let us rename the basis vectors by defining

$$e_1' = e_2, \quad e_2' = e_4, \quad e_3' = e_5, \quad e_4' = e_6, \quad e_5' = e_1, \quad e_6' = e_3.$$

With this new (ordered) basis, we have

$$g(e_1') = e_1' a + e_2' c, \quad g(e_2') = e_1' b + e_2' d, \quad g(e_3') = e_1' p + e_2' q + e_3' r,$$
$$g(e_4') = e_4' r, \quad g(e_5') = e_5' a + e_6' c + e_4' s, \quad g(e_6') = e_5' b + e_6' d + e_4' t,$$

so the matrix of g now has the form

$$\gamma' = \begin{pmatrix} a & b & p & 0 & 0 & 0 \\ c & d & q & 0 & 0 & 0 \\ 0 & 0 & r & 0 & 0 & 0 \\ 0 & 0 & 0 & r & 0 & 0 \\ 0 & 0 & 0 & 0 & a & b \\ 0 & 0 & 0 & 0 & c & d \end{pmatrix}.$$

Therefore, R can be conjugated by a change-of-basis matrix to the set R' of matrices γ' above. Now in *LMR*-(16.19)(5), it has been shown that R' is a QF ring but not a Frobenius ring. Therefore, the same is true for R. The change-of-basis matrix is the permutation matrix

$$\alpha = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}.$$

The reader can check easily that, indeed, $\alpha \gamma \alpha^{-1} = \gamma'$.

Comment. The ring R above appeared in (4.28) of Jay Wood's paper, "Characters and codes over finite rings," Purdue University preprint, 1995. Wood attributed this example to David Benson, but, as we showed above, R is just a conjugate of Nakayama's original example (of a QF ring that is not Frobenius) presented in *LMR*-(16.19)(5).

Ex. 16.16. Let $k \subseteq K$ be a field extension of degree $n > 1$. In *LMR*-(16.19)(4), it is shown that the ring S of matrices

$$(*) \qquad \gamma = \begin{pmatrix} a & x & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & b & y \\ 0 & 0 & 0 & a \end{pmatrix} \quad (a, b, x, y \in K)$$

is a QF (in fact Frobenius) ring. For the subring $R = \{\gamma \in S : a \in k\}$ of S, show that R is Kasch but not QF, and compute the right Cartan matrix of R.

Solution. Clearly, rad(R) is

$$J = \{\gamma \in R : a = b = 0\},$$

since J is nilpotent, and $R/J \cong k \times K$, where the isomorphism is given by $\bar{\gamma} \mapsto (a, b)$. The two simple right R-modules are $S_1 = k$ and $S_2 = K$, where γ acts on S_1 (resp. S_2) by right multiplication by a (resp. b). The two simple left R-modules S_1' and S_2' can be described similarly, and they "correspond" to the orthogonal primitive idempotents $e_1 = E_{11} + E_{44}$ and $e_2 = E_{22} + E_{33}$. An easy computation shows that

$$U_1 := e_1 R = \{\gamma \in R :\ b = y = 0\}, \quad U_1 J = E_{23} K,$$
$$U_2 := e_2 R = \{\gamma \in R :\ a = x = 0\}, \quad U_2 J = E_{34} K.$$

As always, $U_i / U_i J \cong S_i$. To compute $U_i\, J$, note that

$$(E_{23} x')\,\gamma = E_{23}\,(x' b) \quad \text{and} \quad (E_{34} y')\,\gamma = E_{34}\,(y' a)$$

for any $\gamma \in R$ as in $(*)$. Since $[K : k] = n$, we see that $U_1 J \cong S_2$ and $U_2 J \cong n \cdot S_1$. From these, we deduce the following:

(1) Each S_i embeds in R_R, so R *is right Kasch.*

(2) soc$\,(U_1) = U_1 J \cong S_2$ and *soc*$\,(U_2) = U_2 J \cong n \cdot S_1$. Since $n > 1$, the latter implies that R *is not* QF, by *LMR*-(16.4).

(3) The Cartan matrix of R is $\begin{pmatrix} 1 & 1 \\ n & 1 \end{pmatrix}$.

A similar calculation shows that soc$\,(Re_1) \cong S_2'$ and soc$\,(Re_2) \cong n \cdot S_1'$. Thus each S_i' also embeds in ${}_R R$, and we see that R is a Kasch ring.

Comment. The exercise comes from Nakayama's paper "On Frobeniusean algebras, I", Ann. Math. **40**(1939), 611–633, although Nakayama only considered the case where $[K : k] = n = 2$. In this case, Nakayama observed that the (2-sided) ideals $I \subseteq R$ satisfy the double-annihilator conditions

$$\operatorname{ann}_\ell(\operatorname{ann}_r(I)) = I = \operatorname{ann}_r(\operatorname{ann}_\ell(I)).$$

The point is, therefore, that these conditions imposed on 2-sided ideals alone will *not* guarantee a finite-dimensional algebra to be QF (even if it is given to be Kasch).

Ex. 16.17. Let R be the ring of matrices

$$\gamma = \begin{pmatrix} a & x & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & b & y \\ 0 & 0 & 0 & c \end{pmatrix}$$

over a division ring k. Show that the socle of any 1-sided principal indecomposable R-module is simple, and compute the (left, right) Cartan matrices of R. Is R a QF ring?

Solution. First, $J := \operatorname{rad} R = \{\gamma \in R : a = b = c = 0\}$. Since

$$R/\operatorname{rad}R \cong k \times k \times k$$

with the isomorphism given by $\bar{\gamma} \mapsto (a, b, c)$, there are three distinct simple right R-modules $S_i \cong k$ $(i = 1, 2, 3)$, where $\gamma \in R$ acts on S_1, S_2, S_3 by right multiplication of a, b, c respectively. The three simple left R-modules S_i' $(i = 1, 2, 3)$ can be described similarly.

Let $e_1, e_2, e_3 \in R$ be the orthogonal idempotents defined by

$$e_1 = \operatorname{diag}(1,0,0,0), \quad e_2 = \operatorname{diag}(0,1,1,0), \quad e_3 = \operatorname{diag}(0,0,0,1).$$

Clearly, these are mutually nonisomorphic primitive idempotents with $e_1 + e_2 + e_3 = 1$. Let $U_i = e_iR$ and $U_i' = Re_i$. A straightforward computation shows that

$$\begin{aligned} &U_1 = kE_{11} + kE_{12}, \quad && U_2 = k(E_{22} + E_{33}) + kE_{34}, \quad && U_3 = kE_{44}, \\ &e_1J = kE_{12}, && e_2J = kE_{34}, && e_3J = 0. \end{aligned}$$

Therefore, $\operatorname{soc}(U_1) = e_1J$, $\operatorname{soc}(U_2) = e_2J$, $\operatorname{soc}(U_3) = U_3$, and these are simple. By computing the right action of $\gamma \in R$ on these socles, we see that $\operatorname{soc}(U_1) \cong S_2$, $\operatorname{soc}(U_2) \cong S_3$, and $\operatorname{soc}(U_3) \cong S_3$. Thus, the right Cartan matrix is given by $\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$.

The left structure of R is computed similarly: we have

$$\begin{aligned} &U_1' = kE_{11}, \quad && U_2' = kE_{12} + k(E_{22} + E_{33}), \quad && U_3' = kE_{34} + kE_{44}, \\ &Je_1 = 0, && Je_2 = kE_{12}, && Je_3 = kE_{34}. \end{aligned}$$

This shows that $\operatorname{soc}(U_1') = U_1'$, $\operatorname{soc}(U_2') = Je_2$, $\operatorname{soc}(U_3') = Je_3$, and a computation of the left action of $\gamma \in R$ on these socles shows that

$\mathrm{soc}\,(U_1') \cong S_1'$, $\mathrm{soc}\,(U_2') \cong S_1'$, and $\mathrm{soc}\,(U_3') \cong S_2'$, each of which is simple. Thus, the left Cartan matrix of R is $\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$. Finally, since

$$\mathrm{soc}\,(R_R) \cong S_2 \oplus 2S_3 \quad \text{and} \quad \mathrm{soc}\,({}_RR) \cong 2\,S_1' \oplus S_2',$$

R is neither right Kasch nor left Kasch. In particular, R is not QF. (Alternatively,

$$\mathrm{soc}\,(R_R) = kE_{12} + kE_{34} + kE_{44}, \quad \mathrm{soc}\,({}_RR) = kE_{11} + kE_{12} + kE_{34}.$$

Since these are not equal, *LMR*-(15.8) shows that R is not QF.)

Ex. 16.18. In a ring theory text, the following statement appeared: "If R is QF, then R is the injective hull of $(R/\mathrm{rad}\,R)_R$." Find a counterexample; then suggest a remedy.

Solution. In *LMR*-(16.19)(5), we have constructed a QF ring R with principal indecomposables U_1, U_2 such that $R_R \cong U_1 \oplus 2\cdot U_2$, and that $\mathrm{soc}\,(U_1) \cong S_2$, $\mathrm{soc}\,(U_2) \cong S_1$, where S_i is the unique simple quotient of U_i. For this ring R, we have then $E\,(S_2) \cong U_1$ and $E\,(S_1) \cong U_2$. On the other hand $\bar{R} = R/\mathrm{rad}\,R \cong S_1 \oplus 2 \cdot S_2$ as right R-modules, so

$$E\left(\bar{R}_R\right) \cong E\,(S_1 \oplus 2 \cdot S_2) \cong U_2 \oplus 2 \cdot U_1.$$

By the Krull-Schmidt Theorem, this is not isomorphic to R_R!

The ring R above provided a counterexample because it is QF but not Frobenius. This remark suggests the necessary remedy, which we formulate as follows.

Proposition. *A right artinian ring R is a Frobenius ring iff $E\left(\bar{R}_R\right) \cong R_R$ (where $\bar{R} = R/\mathrm{rad}\ R$).*

Proof. First assume R is Frobenius. Then $\bar{R}_R \cong S := \mathrm{soc}(R_R)$. Since R is right artinian and right self-injective, we have $S \subseteq_e R_R$ and so

$$R_R = E\,(S_R) \cong E\left(\bar{R}_R\right).$$

Conversely, assume that $E\left(\bar{R}_R\right) \cong R_R$. Then R is right self-injective and therefore QF. To show that it is Frobenius, we take the standard decomposition of R_R into principal indecomposables:

$$R_R \cong n_1 U_1 \oplus \cdots \oplus n_s U_s$$

and try to prove that $n_i = n_{\pi(i)}$ where π is the Nakayama permutation of R. By definition, $\mathrm{soc}(U_i) \cong S_{\pi(i)}$, where S_j denotes the unique simple quotient of U_j. Therefore, $E\left(S_{\pi(i)}\right) \cong U_i$, and we have

$$\begin{aligned} E\left(\bar{R}_R\right) &\cong E\,(n_1 S_1 \oplus \cdots \oplus n_s S_s) \\ &\cong n_1 \cdot E\,(S_1) \oplus \cdots \oplus n_s \cdot E\,(S_s) \end{aligned}$$

$$\cong n_1 \cdot U_{\pi^{-1}(1)} \oplus \cdots \oplus n_s \cdot U_{\pi^{-1}(s)}.$$
$$\cong n_{\pi(1)} U_1 \oplus \cdots \oplus n_{\pi(s)} U_s.$$

Since $E\left(\bar{R}_R\right) \cong R_R$ (and the U_i's are indecomposable), the Krull-Schmidt Theorem implies that $n_{\pi(i)} = n_i$ for every i, as desired. $\square$

Ex. 16.19. In *LMR*-(16.19)(4), the k-algebra R consisting of matrices

$$\gamma = \begin{pmatrix} a & x & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & b & y \\ 0 & 0 & 0 & a \end{pmatrix}$$

over a field k is shown to be a Frobenius algebra. Find an ideal A in R for which $\text{ann}_r(A) \neq \text{ann}_\ell(A)$.

Solution. The set $A = \{\gamma \in R: \ a = b = y = 0\}$ is easily checked to be an ideal in R. By a direct calculation, we get

$$\text{ann}_\ell(A) = \{\gamma \in R: \ a = 0\} \quad \text{and} \quad \text{ann}_r(A) = \{\gamma \in R: \ b = 0\}.$$

These two annihilators are certainly not equal.

Comment. By *LMR*-(16.65), $\text{ann}_r(A) = \text{ann}_\ell(A)$ holds for any ideal A in a *symmetric* algebra. Therefore, the algebra R above is not a symmetric algebra. This is not surprising since we have seen in *LMR*-(16.19)(4) that R has a Nakayama permutation (12): this says that R is not even a weakly symmetric algebra. As a matter of fact, examples of $\text{ann}_r(A) \neq \text{ann}_\ell(A)$ can already be found over a weakly symmetric algebra: see *LMR*-(16.66).

Ex. 16.20. Show that if R and S are symmetric algebras over a field k, then so are $R \times S$, $R \otimes_k S$, and $\mathbb{M}_n(R)$.

Solution. Most of the work is already done in the solution to Exercise 3.12, where R, S were assumed to be Frobenius algebras. Referring to the notations used in that solution, all we need to observe here is the fact that, if $B: \ R \times R \to k$ and $C: \ S \times S \to k$ are both symmetric, then the pairing $D: \ T \times T \to k$ on $T = R \times S$ and the pairing $E: \ W \times W \to k$ on $W = R \otimes_k S$ constructed in the solution to Exercise (3.12) are also symmetric.

Finally, we have to show that $\mathbb{M}_n(R)$ is a symmetric. By *LMR*-(16.57), $\mathbb{M}_n(k)$ is a symmetric k-algebra. Since

$$\mathbb{M}_n(R) \cong R \otimes_k \mathbb{M}_n(k),$$

it follows by taking $S = \mathbb{M}_n(k)$ above that $\mathbb{M}_n(R)$ is a symmetric k-algebra.

Ex. 16.21. Let K/k be a field extension, and let R be a finite-dimensional k-algebra. Show that R is a symmetric algebra over k iff $R^K = R \otimes_k K$ is a symmetric algebra over K.

Solution. First assume R is a symmetric k-algebra. Then, by *LMR*-(16.54), these exists a nonsingular, symmetric and associative pairing $B : R \times R \to k$. The scalar extension of this pairing:

$$B^K : R^K \times R^K \longrightarrow K,$$

defined by $B^K(r \otimes a, r' \otimes a') = B(r, r')aa'$, is easily seen to be also nonsingular, symmetric and associative. Therefore, R^K is a symmetric algebra over K (loc. cit.).

Conversely, assume that R^K is a symmetric K-algebra. We proceed as in the solution to Exercise (3.16). Consider the (R, R)-bimodules R and $\widehat{R}$; we may view these as right A-modules where $A = R \otimes_k R^{op}$. Our job is to prove that these A-modules are isomorphic. By the Noether-Deuring Theorem (*FC*-(19.25)), it suffices to prove that they are isomorphic after scalar extension from k to K. Since

$$A^K = (R \otimes_k R^{op})^K \cong R^K \otimes_K \left(R^K\right)^{op},$$

the assumption that R^K is a symmetric K-algebra implies that R^K and $\left(R^K\right)^{\wedge} \cong \left(\widehat{R}\right)^K$ are isomorphic as right A^K-modules, as desired.

Ex. 16.22. Let $K \supseteq k$ be a finite field extension with a non-identity k-automorphism τ on K. With the multiplication

$$(a, b)(c, d) = (ac, ad + b\tau(c)) \quad (a, b, c, d \in K),$$

$R := K \oplus K$ is a k-algebra of dimension $2[K : k]$. Show that R is a weakly symmetric, but not symmetric, local k-algebra.

Solution. First note that K is a (K, K)-bimodule with the left action $x \cdot a = xa$ and right action $a \cdot y = a\tau(y)$ (where $a, x, y \in K$). The ring R is simply the "trivial extension" formed from this (K, K)-bimodule K.

Let $J := (0) \oplus K$, which is an ideal of square zero in R with $R/J \cong K$. Therefore, R is a local k-algebra with the unique maximal ideal J. As in *LMR*-(3.15C), we can check that the only left ideals in R are (0), J and R, so R is a Frobenius k-algebra. Since R is local, it is, of course, a weakly symmetric k-algebra.

We claim that R is *not* a symmetric k-algebra. Indeed, if it is, there would exist a nonzero k-linear functional $\lambda : R \to k$ whose kernel contains no nonzero left ideals of R, but contains all additive commutators $\alpha\beta - \beta\alpha$ for $\alpha, \beta \in R$. Now for $\alpha = (a, 0)$ and $\beta = (0, d)$ we have

$$\alpha\beta - \beta\alpha = (0, ad) - (0, d\tau(a)) = (0, (a - \tau(a))d).$$

Fixing $a \in K$ such that $\tau(a) \neq a$ and varying d, we see that $\alpha\beta - \beta\alpha$ ranges over J. Therefore $\lambda(J) = 0$, a contradiction.

Comment. A nonsingular k-bilinear associative pairing $B : R \times R \to k$ is easy to construct. In fact, taking any nonzero k-linear functional $t : K \to k$, we can take B to be the following:

$$B\left((a,\ b),\ (c,\ d)\right) = t\left(ad + b\tau(c)\right).$$

This is, of course, not a symmetric pairing.

Ex. 16.23. (Nakayama-Nesbitt). In the last exercise, assume that $[K : k] = 2$, char $k \neq 2$, and let R be the weakly symmetric k-algebra defined there. Show that the scalar extension $R^K := R \otimes_k K$ is a K-algebra isomorphic to

$$S = \left\{ \begin{pmatrix} a & x & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & b & y \\ 0 & 0 & 0 & a \end{pmatrix} : a,\ b,\ x,\ y \in K \right\}.$$

Deduce that R^K is not a weakly symmetric K-algebra. (This contrasts with the conclusions of Exercise 21.)

Solution. Since $[K : k] = 2$, the τ in the last exercise generates the Galois group of K/k. It is easy to check that $\varphi :\ R \to S$ defined by

$$\varphi\left((a, x)\right) = \begin{pmatrix} a & x & 0 & 0 \\ 0 & \bar{a} & 0 & 0 \\ 0 & 0 & \bar{a} & \bar{x} \\ 0 & 0 & 0 & a \end{pmatrix} \quad \text{(where } \ \bar{b} := \tau(b))$$

is a k-algebra embedding. Let $R' = \varphi(R)$. Since R' commutes elementwise with $K = K \cdot I_4 \subseteq S$, we have a K-algebra homomorphism

$$\Phi :\ R \otimes_k K \to S \quad \text{defined by} \quad \Phi\left((a,\ x) \otimes c\right) = \varphi\left((a,\ x)\right) c.$$

A routine check shows that $R' \cdot K = S$, so Φ is an epimorphism. Since

$$\dim_K \left(R \otimes_k K\right) = \dim_k R = 4 = \dim_K S,$$

we see that Φ is a K-algebra isomorphism. In *LMR*-(16.19)(4), it is shown that S is a Frobenius K-algebra with a Nakayama permutation (12), so S (and hence R^K) is *not* a weakly symmetric K-algebra.

Comment. The conclusion that R^K is not a weakly symmetric K-algebra actually gives another way for us to see that R itself is not a symmetric k-algebra. For, if R is a symmetric k-algebra, Exercise 21 would have implied that R^K is a symmetric K-algebra, but R^K is not even a weakly symmetric K-algebra.

This exercise is based on the paper of T. Nakayama and C. Nesbitt, "Note on symmetric algebras," Annals of Math. **39**(1938), 659–668.

Ex. 16.24. If R and S are both symmetric algebras over a field k, show that $R \otimes_k S$ is also a symmetric k-algebra.

Solution. Let $B: R \times R \to k$ be a nonsingular k-bilinear pairing with the associative and symmetric properties, and let $C: S \times S \to k$ be a corresponding object for S. We have checked before (in the solution to Exercise 3.12) that

$$E(r \otimes s,\ r' \otimes s') = B(r,\ r')\ C(s,\ s') \quad (r,\ r' \in R;\ s,\ s' \in S)$$

induces a nonsingular k-bilinear pairing with the associative property. Since

$$\begin{aligned} E(r' \otimes s',\ r \otimes s) &= B(r',\ r)\, C(s',\ s) \\ &= B(r,\ r')\, C(s,\ s') \\ &= E(r \otimes s,\ r' \otimes s'), \end{aligned}$$

E also has the symmetric property. Therefore, $R \otimes_k S$ is a symmetric k-algebra by *LMR*-(16.54).

Ex. 16.25. If R is symmetric k-algebra over a field k and $0 \neq e = e^2 \in R$, show that eRe is also a symmetric k-algebra with $\text{soc}(eRe) = e(\text{soc}(R))e$. Using this, show that, for any nonzero f.g. projective right R-module P, $\text{End}_R(P)$ is also a symmetric k-algebra.

Solution. Let $\lambda: R \to k$ be a nonzero k-linear functional such that $\lambda(xy) = \lambda(yx)$ for all $x,\ y \in R$, and that $\ker(\lambda)$ contains no nonzero right ideal of R. Let $\lambda_0: eRe \to k$ be the restriction of λ to eRe. Of course we still have $\lambda_0(xy) = \lambda_0(yx)$ for all $x, y \in eRe$. Also, $\lambda_0 \not\equiv 0$. For if otherwise, we would have

$$0 = \lambda(eR \cdot e) = \lambda(e \cdot eR) = \lambda(eR),$$

which would imply that $e = 0$, a contradiction. We show next that $\ker(\lambda_0)$ contains no nonzero right ideal of eRe, for then *LMR*-(16.54) shows that eRe is a symmetric k-algebra. Suppose $x \in eRe$ is such that $\lambda_0(x \cdot eRe) = 0$. Noting that $x = exe$, we have

$$0 = \lambda(xeR \cdot e) = \lambda((exe)R) = \lambda(xR).$$

This implies that $x = 0$, as desired.

To prove the equation $\text{soc}(eRe) = e(\text{soc}(R))e$, let $J = \text{rad}(R)$. We shall use the fact that $\text{soc}(R) = \text{ann}_r(J) = J^0$, where $J^0 = \{x \in R : \lambda(Jx) = 0\}$, and that $\text{rad}(eRe) = eJe$ (see *FC*-(21.10)). Let $x \in \text{soc}(eRe)$. Then

$$0 = \lambda_0(eJe \cdot x) = \lambda(Jexe) = \lambda(Jx)$$

implies that $x \in \text{soc}(R)$, so $x = exe \in e(\text{soc}(R))e$. Conversely,

$$\lambda_0(eJe \cdot e(\text{soc}(R))e) \subseteq \lambda(e \cdot eJe \cdot \text{soc}(R)) \subseteq \lambda(J \cdot \text{soc}(R)) = 0$$

implies that $e(\text{soc}(R))e \subseteq \text{soc}(eRe)$.

For the last part of the Exercise, take a R-module Q such that $P \oplus Q \cong R^n$ for some $n < \infty$. By Exercise 20,

$$S := \text{End}_R(R^n) \cong \mathbb{M}_n(R)$$

is a symmetric k-algebra. Let $e \in S$ be the projection of R^n onto P with respect to the direct sum decomposition $R^n \cong P \oplus Q$. (We think of the isomorphism as an equality.) It is easy to see that $\text{End}_R(P) \cong eSe$, which is a symmetric k-algebra by what we have proved above.

Comment. In contrast to this exercise, if R is a QF algebra over k, $eRe = \text{End}_R(eR)$ need not be a QF algebra. For an explicit counterexample, see the paper of Pascaud and Valette referenced in the *Comment* on Exercise 32 below. To complete the picture, however, we should mention the following. Let P_R be a *faithful* f.g. projective right module over a QF ring R. Then P is a generator according to Exercise 12. By the Morita Theory in *LMR*-§18, $\text{End}_R(P)$ is Morita equivalent to R, and therefore is also QF by Exercise 18.7A below. This result was proved (independently) by C.W. Curtis and K. Morita.

Ex. 16.26. Let R be a symmetric k-algebra with center $Z(R)$, and let $B : R \times R \longrightarrow k$ be a nonsingular k-bilinear form that is both symmetric and associative. For any subset $A \subseteq R$, let $A^0 = \{x \in R : \ B(A, x) = 0\}$. Show that:

(1) $z \in (xRy)^0 \iff yzx = 0$.
(2) $xRy = 0 \iff yRx = 0$.
(3) $Z(R) = [R, R]^0$, where $[R, R]$ denotes the additive subgroup of R generated by $xy - yx \quad (x, y \in R)$.

Solution. (1) For any $r \in R$, we have

$$B(xry,\ z) = B(xr,\ yz) = B(yz,\ xr) = B(yzx,\ r).$$

Since B is nonsingular, (1) follows.

(2) From (1), we have

$$xRy = 0 \iff (xRy)^0 = R \iff yRx = 0.$$

(3) For any x, y, $z \in R$, we have

$$\begin{aligned} B(xy - yx,\ z) &= B(x,\ yz) - B(z,\ yx) \\ &= B(x,\ yz) - B(zy,\ x) \\ &= B(x,\ yz - zy). \end{aligned}$$

Therefore, $z \in [R,\ R]^0$ iff $yz - zy \in R^0 = \{0\}$ for all $y \in R$, iff $z \in Z(R)$.

Ex. 16.27. For a field k, compute the parastrophic determinants of the commutative k-algebras

$$R = k[x,\ y]/\left(x^2,\ y^2\right), \quad S = k[x,\ y]/\left(xy,\ x^2 - y^2\right),$$
$$T = k[x,\ y]/\left(x^2,\ xy,\ y^2\right)$$

and apply Frobenius' Criterion (*LMR*-(16.82)) to determine which of these is a Frobenius algebra.

Solution. Recall that, for a k-basis $\{\epsilon_i\}$ in a k-algebra R, the structure constants $\{c_{\ell ij}\}$ are defined via the equations $\epsilon_i\epsilon_j = \sum_\ell c_{\ell ij}\epsilon_\ell$. A parastrophic matrix is a matrix P_α (with parameter $\alpha = (\alpha_1, \ldots, \alpha_n) \in k^n$) defined by

$$(P_\alpha)_{ij} = \sum\nolimits_\ell c_{\ell ij}\alpha_\ell.$$

Note that the space of parastrophic matrices is spanned by $P_1, \ldots, P_n$ ($n = \dim_k R$), where P_ℓ means the parastrophic matrix corresponding to the ℓ^{th} unit vector $(0, \ldots, 1, \ldots 0)$.

For the k-algebra $T = k[x, y]/\left(x^2, xy, y^2\right)$, we use the basis

$$\epsilon_1 = 1, \;\; \epsilon_2 = x, \;\; \epsilon_3 = y$$

(where x means $\bar{x}$, etc.). Since $\epsilon_1\epsilon_\ell = \epsilon_\ell\epsilon_1 = \epsilon_\ell$, and $\epsilon_i\epsilon_j = 0$ for $i, j \geq 2$, simple inspection shows

$$P_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad P_2 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad P_3 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

Therefore, a general parastrophic matrix has the form

$$P_\alpha = \alpha_1 P_1 + \alpha_2 P_2 + \alpha_3 P_3 = \begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \alpha_2 & 0 & 0 \\ \alpha_3 & 0 & 0 \end{pmatrix},$$

so the parastrophic determinant is 0. According to Frobenius' Criterion, T is *not* a Frobenius algebra.

For the k-algebra $R = k\,[x, y]/\left(x^2, y^2\right)$, we use the k-basis

$$\epsilon_1 = 1, \;\; \epsilon_2 = x, \;\; \epsilon_3 = y, \;\; \epsilon_4 = xy.$$

Here, for $i, j \geq 2$, we have $\epsilon_i\epsilon_j = 0$, with the exception that $\epsilon_2\epsilon_3 = \epsilon_3\epsilon_2 = \epsilon_4$. By inspection, we have

$$P_1 = \begin{pmatrix} 1&0&0&0 \\ 0&0&0&0 \\ 0&0&0&0 \\ 0&0&0&0 \end{pmatrix}, \quad P_2 = \begin{pmatrix} 0&1&0&0 \\ 1&0&0&0 \\ 0&0&0&0 \\ 0&0&0&0 \end{pmatrix},$$

$$P_3 = \begin{pmatrix} 0&0&1&0 \\ 0&0&0&0 \\ 1&0&0&0 \\ 0&0&0&0 \end{pmatrix}, \quad P_4 = \begin{pmatrix} 0&0&0&1 \\ 0&0&1&0 \\ 0&1&0&0 \\ 1&0&0&0 \end{pmatrix},$$

so a general parastrophic matrix has the form

$$P_\alpha = \alpha_1 P_1 + \alpha_2 P_2 + \alpha_3 P_3 + \alpha_4 P_4 = \begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 \\ \alpha_2 & 0 & \alpha_4 & 0 \\ \alpha_3 & \alpha_4 & 0 & 0 \\ \alpha_4 & 0 & 0 & 0 \end{pmatrix}.$$

The parastrophic determinant is therefore $\alpha_4^4 \neq 0$. By Frobenius' Criterion, R is a Frobenius k-algebra.

Finally, for the algebra $S = k[x,y]/(xy,\ x^2 - y^2)$, we have $x^3 = x{\cdot}x^2 = xy^2 = 0$ and similarly $y^3 = 0$. Thus, a k-basis for S is given by

$$\epsilon_1 = 1, \quad \epsilon_2 = x, \quad \epsilon_3 = y, \quad \text{and} \quad \epsilon_4 = x^2.$$

Since $\epsilon_1 \epsilon_\ell = \epsilon_\ell \epsilon_1 = \epsilon_\ell$, $\epsilon_2^2 = \epsilon_3^2 = \epsilon_4$, $\epsilon_4^2 = 0$, and $\epsilon_i \epsilon_j = 0$ for $i, j \geq 2$ with $i \neq j$, simple inspection shows

$$P_1 = \begin{pmatrix} 1&0&0&0 \\ 0&0&0&0 \\ 0&0&0&0 \\ 0&0&0&0 \end{pmatrix}, \quad P_2 = \begin{pmatrix} 0&1&0&0 \\ 1&0&0&0 \\ 0&0&0&0 \\ 0&0&0&0 \end{pmatrix},$$

$$P_3 = \begin{pmatrix} 0&0&1&0 \\ 0&0&0&0 \\ 1&0&0&0 \\ 0&0&0&0 \end{pmatrix}, \quad P_4 = \begin{pmatrix} 0&0&0&1 \\ 0&1&0&0 \\ 0&0&1&0 \\ 1&0&0&0 \end{pmatrix}.$$

Therefore, a general parastrophic matrix has the form

$$P_\alpha = \alpha_1 P_1 + \alpha_2 P_2 + \alpha_3 P_3 + \alpha_4 P_4 = \begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 \\ \alpha_2 & \alpha_4 & 0 & 0 \\ \alpha_3 & 0 & \alpha_4 & 0 \\ \alpha_4 & 0 & 0 & 0 \end{pmatrix}.$$

The parastrophic determinant is now $-\alpha_4^4 \neq 0$. By Frobenius' Criterion, S is again a Frobenius k-algebra.

Comment. The purpose of this exercise is just to illustrate the workings of the parastraphic determinant. Of course, it is much easier to decide which of R, S, T is a Frobenius algebra by computing their socles.

Ex. 16.28. (Nakayama-Nesbitt) Show that a finite-dimensional k-algebra R (over a field k) is a symmetric k-algebra iff there exists a *symmetric* nonsingular parastrophic matrix.

Solution. According to *LMR*-(16.54), R is a symmetric k-algebra iff there exists a nonzero k-linear functional $\lambda: R \to k$ such that $\lambda(xy) = \lambda(yx)$ for all $x, y \in R$ such that $\ker(\lambda)$ contains no nonzero left (resp. right) ideal of R. The hyperplane $\ker(\lambda)$ has the form

$$H_\alpha = \left\{ \sum x_\ell \epsilon_\ell : \sum \alpha_\ell x_\ell = 0 \right\}, \qquad \alpha = (\alpha_1, \ldots, \alpha_n) \in k^n \backslash \{0\},$$

where $\{\epsilon_\ell\}$ is a fixed basis of R. By *LMR*-(16.83), H_α contains no nonzero left ideal of R iff the parastrophic matrix P_α associated with $\alpha = (\alpha_1, \ldots, \alpha_n)$ (see the last exercise for notations) is nonsingular. Therefore, we are done if we can show that

(∗) *H_α contains all $xy - yx$ ($x,\ y \in R$) iff P_α is a symmetric matrix.*

Now, the additive group $[R, R]$ generated by all $xy-yx$ is already generated by all $\epsilon_i\epsilon_j - \epsilon_j\epsilon_i$. For the structure constants $\{c_{\ell ij}\}$ of the algebra R, we have

$$\epsilon_i\epsilon_j - \epsilon_j\epsilon_i = \sum\nolimits_\ell (c_{\ell ij} - c_{\ell ji})\,\epsilon_\ell.$$

Therefore, $H_\alpha \supseteq [R, R]$ amounts to

$$0 = \sum\nolimits_\ell \alpha_\ell\,(c_{\ell ij} - c_{\ell ji}) = (P_\alpha)_{ij} - (P_\alpha)_{ji}\,,$$

which amounts to the symmetry of the parastrophic matrix P_α.

Ex. 16.29. Let k be a field with $u,\ v \in k$ (possibly zero), and let R be the k-algebra consisting of matrices

$$\begin{pmatrix} a & b & c & d \\ 0 & a & 0 & uc \\ 0 & 0 & a & vb \\ 0 & 0 & 0 & a \end{pmatrix}$$

over k (see *LMR*-(16.66)). Using Frobenius' Criterion (*LMR*-(16.82)) and the last exercise, show that
(1) R is a Frobenius k-algebra iff $uv \neq 0$;
(2) R is a symmetric k-algebra iff $u = v \neq 0$.

Solution. We choose the following k-basis for R:

$$\epsilon_1 = I_4, \quad \epsilon_2 = \begin{pmatrix} 0&1&0&0 \\ 0&0&0&0 \\ 0&0&0&v \\ 0&0&0&0 \end{pmatrix}, \quad \epsilon_3 = \begin{pmatrix} 0&0&1&0 \\ 0&0&0&u \\ 0&0&0&0 \\ 0&0&0&0 \end{pmatrix}, \quad \epsilon_4 = \begin{pmatrix} 0&0&0&1 \\ 0&0&0&0 \\ 0&0&0&0 \\ 0&0&0&0 \end{pmatrix}.$$

The products $\epsilon_i\epsilon_j$ $(i, j \geq 2)$ are easily computed as follows:

$$\epsilon_2\epsilon_3 = u\,\epsilon_4, \quad \epsilon_3\epsilon_2 = v\,\epsilon_4; \quad \text{all other} \quad \epsilon_i\epsilon_j = 0.$$

Therefore, the spanning parastrophic matrices are:

$$P_1 = \begin{pmatrix} 1&0&0&0 \\ 0&0&0&0 \\ 0&0&0&0 \\ 0&0&0&0 \end{pmatrix}, \quad P_2 = \begin{pmatrix} 0&1&0&0 \\ 1&0&0&0 \\ 0&0&0&0 \\ 0&0&0&0 \end{pmatrix},$$

$$P_3 = \begin{pmatrix} 0&0&1&0\\ 0&0&0&0\\ 1&0&0&0\\ 0&0&0&0 \end{pmatrix}, \quad P_4 = \begin{pmatrix} 0&0&0&1\\ 0&0&u&0\\ 0&v&0&0\\ 1&0&0&0 \end{pmatrix},$$

and a general parastrophic matrix has the form

$$P_\alpha = \begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4\\ \alpha_2 & 0 & u\alpha_4 & 0\\ \alpha_3 & v\alpha_4 & 0 & 0\\ \alpha_4 & 0 & 0 & 0 \end{pmatrix}, \quad \text{where} \quad \alpha = (\alpha_1,\ \alpha_2,\ \alpha_3,\ \alpha_4) \in k^4.$$

The parastrophic determinant is $\det(P_\alpha) = uv\alpha_4^4$. This vanishes identically on $(\alpha_1,\ \alpha_2,\ \alpha_3,\ \alpha_4)$ iff $uv = 0$. Therefore, by Frobenius' Criterion *LMR*-(16.82), R is a Frobenius k-algebra iff $uv \neq 0$. This proves (1).

To prove (2), first assume $uv \neq 0$ and $u = v$. Then *any* parastrophic matrix P_α is symmetric, so by the last exercise, R is a symmetric k-algebra. Conversely, if R is a symmetric k-algebra, *some* P_α is nonsingular and symmetric. If $\alpha = (\alpha_1, \alpha_2,\ \alpha_3, \alpha_4)$, this means that $uv\alpha_4^4 \neq 0$ and $u\alpha_4 = v\alpha_4$. Therefore, we must have $uv \neq 0$, $\alpha_4 \neq 0$, and $u = v$.

Ex. 16.30. For any field k and any finite group G, compute the parastrophic matrix of the group algebra $R = kG$. Using this computation and Exercise 28, give another proof for the fact that R is a symmetric k-algebra.

Solution. We can take the elements $\{\epsilon_i\}$ of the group G to be a k-basis of $R = kG$. The structure constants $\{c_{\ell ij}\}$ of R with respect to this canonical basis (defined in general by the equations $\epsilon_i\epsilon_j = \sum_\ell c_{\ell ij}\,\epsilon_\ell$) are just

(1) $\qquad c_{\ell ij} = 1 \quad \text{if} \quad \epsilon_i\epsilon_j = \epsilon_\ell, \quad \text{and} \quad c_{\ell ij} = 0 \quad \text{otherwise.}$

A general parastrophic matrix P_α for the parameter $\alpha = (\alpha_1, \dots, \alpha_n) \in k^n$ is defined by $(P_\alpha)_{ij} = \sum_\ell \alpha_\ell c_{\ell ij}$. By the description of the $c_{\ell ij}$'s given above, we have then

(2) $\qquad (P_\alpha)_{ij} = \alpha_\ell \quad \text{where} \quad \epsilon_i\epsilon_j = \epsilon_\ell.$

In particular, each row and each column of P_α is a permutation of $\alpha_1, \dots, \alpha_n$, where $n = |G|$.

For simplicity, let us assume that $\epsilon_1 = 1$. The parastrophic matrix P corresponding to the parameter $(1,\ 0, \dots,\ 0)$ is then given by $P_{ij} = 1$ if $\epsilon_i\epsilon_j = 1$ and $P_{ij} = 0$ otherwise. Thus, P is just a permutation matrix, with determinant ± 1. Also, since $\epsilon_i\epsilon_j = 1$ iff $\epsilon_j\epsilon_i = 1$, P is a *symmetric* (nonsingular) matrix. Therefore, by Exercise 28, R is a symmetric k-algebra.

Comment. The parastrophic matrix P_α for the group G above is of great historical significance. By permuting the columns of P_α, we obtain another matrix, say P'_α, whose $(i,j)^{\text{th}}$ entry is $\alpha_{\ell'}$, where $\epsilon_i\epsilon_j^{-1} = \epsilon_{\ell'}$. The matrix P'_α is called the "group matrix" of G, and was introduced by Dedekind around 1880. The "group determinant" $\det\ P'_\alpha = \pm\det\ P_\alpha$ for an arbitrary finite group G was investigated at great length by Frobenius. With the parameters $\{\alpha_i\}$ viewed as "variables", the study of the factorization of the group determinant into irreducible factors over the complex numbers (c. 1896) eventually led Frobenius to the discovery of the character theory (and later the full representation theory) of finite groups in characteristic 0. By the time when Frobenius proved his criterion on the equivalence of the first and second regular representations of an algebra in terms of parastrophic matrices (c. 1903), the theory of group determinants was already fully developed.

Ex. 16.31. (Theorem on Structure Constants) Let R be an algebra over a field k with basis $\{\epsilon_1, \dots, \epsilon_n\}$ and let $\epsilon_i\epsilon_j = \sum_\ell c_{\ell ij}\epsilon_\ell$. For $\alpha_1, \dots, \alpha_n \in k$, show that:

(1) $\sum_\ell \alpha_\ell c_{\ell ij} = 0 \quad (\forall\, i, j) \Rightarrow \alpha_\ell = 0 \quad (\forall\, \ell)$.
(2) $\sum_\ell \alpha_\ell c_{i\ell j} = 0 \quad (\forall\, i, j) \Rightarrow \alpha_\ell = 0 \quad (\forall\, \ell)$.
(3) $\sum_\ell \alpha_\ell c_{ij\ell} = 0 \quad (\forall\, i, j) \Rightarrow \alpha_\ell = 0 \quad (\forall\, \ell)$.

Solution. For $r \in R$, let $\epsilon_i r = \sum_j a_{ij}^{(r)}\epsilon_j$ and $r\epsilon_i = \sum_j b_{ji}^{(r)}\epsilon_j$, and define the $n \times n$ matrices $A(r)$, $B(r)$ by $A(r)_{ij} = a_{ij}^{(r)}$ and $B\,(r)_{ji} = b_{ji}^{(r)}$. Then $r \mapsto A(r)$ and $r \mapsto B(r)$ are the first and second regular representations of R (see *LMR*-§16G). For $\alpha_1, \dots, \alpha_n \in k$, write $r := \sum \alpha_i \epsilon_i \in R$.

(2) Note that

$$r\epsilon_j = \left(\sum\nolimits_\ell \alpha_\ell\epsilon_\ell\right)\epsilon_j = \sum\nolimits_i \left(\sum\nolimits_\ell \alpha_\ell c_{i\ell j}\right)\epsilon_i,$$

so $b_{ij}^{(r)} = \sum_\ell \alpha_\ell c_{i\ell j}$. Therefore, (2) amounts to $B(r) = 0 \Rightarrow r = 0$. This follows from the faithfulness of the second regular representation B.

(3) As in (2), we note that

$$\epsilon_j r = \epsilon_j \left(\sum\nolimits_\ell \alpha_\ell\epsilon_\ell\right) = \sum\nolimits_i \left(\sum\nolimits_\ell \alpha_\ell c_{ij\ell}\right)\epsilon_i$$

implies that $a_{ji}^{(r)} = \sum_\ell \alpha_\ell c_{ij\ell}$. Therefore, (3) simply expresses the faithfulness of the first regular representation A.

(1) Recall that, for $\alpha = (\alpha_1, \dots, \alpha_n) \in k^n$, the parastrophic matrix P_α is defined by $(P_\alpha)_{ij} = \sum_\ell \alpha_\ell c_{\ell ij}$. Let P_ℓ $(1 \le \ell \le n)$ be the parastrophic matrix P_α when $\alpha = (0, \dots 1, 0, \dots 0)$ (the ℓ^{th} unit vector). Then $(P_\ell)_{ij} = c_{\ell ij}$, and (3) amounts to the linear independence of $P_1, \dots, P_n$ over k. This is proved in *LMR*-(16.85). Another proof can be given as follows. From the equation $a_{ji}^{(r)} = \sum_\ell \alpha_\ell c_{ij\ell}$, we have $A\,(\epsilon_\ell)_{ji} = c_{ij\ell}$. Therefore, from $\sum_\ell \alpha_\ell c_{\ell ij} = 0 \ (\forall\, i, j)$, we get, for any scalars $x_1, \dots, x_n \in k$:

$$0 = \sum_j \left(\sum_\ell \alpha_\ell c_{\ell ij}\right) x_j = \sum_\ell \alpha_\ell \sum_j A\,(x_j\epsilon_j)_{i\ell} \quad (\forall\, i).$$

For $x := \sum_j x_j \epsilon_j \in R$, we have then $\sum_\ell \alpha_\ell A(x)_{i\ell} = 0$. For suitable choices of $x_1, \ldots, x_n$, we have $x = 1$, for which $A(x) = I_n$. Therefore, $0 = \sum_\ell \alpha_\ell \delta_{i\ell} = \alpha_i$ for every i, as desired.

Comment. The second argument given for (3) above lends itself to another proof of the fact (*LMR*-(16.85)) that $P_1, \ldots, P_n$ form a k-basis for the space S of matrices intertwining the representations A and B. First, a direct calculation using $(\epsilon_i r)\,\epsilon_j = \epsilon_i\,(r\epsilon_j)$ shows that

$$A(r)\,P_\ell = P_\ell B(r) \quad (\forall\, r \in R),$$

so $P_\ell \in S$ for every ℓ. Since $P_1, \ldots, P_n$ are k-linearly independent and

$$\dim_k S = \dim_k \operatorname{Hom}_R\big(R_R,\ (\hat{R})_R\big) = \dim_k \hat{R} = n,$$

it follows that $P_1, \ldots, P_n$ form a k-basis of S.

Ex. 16.32. (Pascaud-Valette) For any field k of characteristic $\neq 2$, show that there exists a symmetric k-algebra R with a k-automorphism α of order 2 such that the fixed ring $R^\alpha = \{x \in R : \ \alpha(x) = x\}$ is any prescribed finite-dimensional k-algebra. (In particular, R^α need not be QF.)

Solution. For any finite-dimensional k-algebra A, use the (A, A)-bimodule $\hat{A} = \operatorname{Hom}_k(A, k)$ to form the "trivial extension" $R = A \oplus \hat{A}$. According to *LMR*-(16.62), R is always a symmetric k-algebra. Consider the k-linear map $\alpha : R \to R$ defined by $\alpha(a, \varphi) = (a, -\varphi)$ for $a \in A$, $\varphi \in \hat{A}$. Since

$$\begin{aligned} \alpha\,(a,\ \varphi)\ \alpha\,(a',\ \varphi') &= (a,\ -\varphi)\,(a',\ -\varphi') = (aa',\ -a\varphi' - \varphi a') \\ &= \alpha\,(aa',\ a\varphi' + \varphi a') = \alpha\,((a,\ \varphi)\,(a',\ \varphi')), \end{aligned}$$

α is a k-algebra automorphism of R. In view of char $k \neq 2$, α has order 2, and $R^\alpha = \{(a, 0) : \ a \in A\} \cong A$.

Comment. The paper of J. Pascaud and J. Valette, entitled "Group actions on QF rings," appeared in Proc. Amer. Math. Soc. **76**(1979), 43–44. The present exercise is a slight variant of their result. There were earlier results in the literature to the effect that, if R is QF and if G is a finite group of automorphisms on R with $|G| \in \mathrm{U}(R)$, then R^G is also QF. The construction of Pascaud and Valette showed that this is not the case. For a positive result and further comments in this direction, see the next exercise.

Ex. 16.33. (K. Wang) Let R be a Frobenius algebra over a field k with a nonsingular associative k-bilinear form $B : R \times R \to k$. Let G be a finite group of k-automorphisms of R such that B is G-invariant (that is, $B(gr, gr') = B(r, r')$ for every $g \in G$). If $|G|$ is not divisible by the characteristic of k, show that the fixed ring R^G is also a Frobenius k-algebra. Prove the same result for symmetric algebras.

Solution. Let $S = R^G$. The restriction of B to $S \times S$ is clearly an associative k-bilinear form on S. If this pairing is nonsingular, then S will be a Frobenius k-algebra, as desired.

Let $s \in S$ be such that $B(s, S) = 0$. For any element $r \in R$, we have $|G|^{-1} \sum_{g \in G} gr \in S$, so

$$\begin{aligned} 0 &= B\left(s,\, |G|^{-1} \sum\nolimits_{g \in G} gr\right) = |G|^{-1} \sum\nolimits_{g \in G} B\left(s,\, gr\right) \\ &= |G|^{-1} \sum\nolimits_{g \in G} B\left(g^{-1}s,\, r\right) = |G|^{-1} \sum\nolimits_{g \in G} B\left(s,\, r\right) \\ &= B\left(s,\, r\right). \end{aligned}$$

This shows that $B(s, R) = 0$, so $s = 0$, as desired.

In the case where B is a *symmetric* form, the same proof works for symmetric algebras.

Comment. The above proof is a simplified version of that of K. Wang in his paper "Fixed subalgebra of a Frobenius algebra," in Proc. Amer. Math. Soc. **87**(1983), 576–578. As observed by Wang, the following is a special case of the result in this exercise:

> *Let A be a Frobenius algebra over a field k of characteristic $\neq 2$, and let α be the k-automorphism of $A \otimes_k A$ given by $\alpha\left(a \otimes a'\right) = a' \otimes a$. Then the fixed algebra $\left(A \otimes_k A\right)^{\alpha}$ is also a Frobenius k-algebra.*

(Apply the exercise to the Frobenius algebra $R = A \otimes_k A$: see Exercise 3.12.) This result was first obtained by G. Azumaya. Subsequently, Wang's result was generalized to a Hopf algebra setting; see M. Ouyang's paper "A note on Frobenius algebras," in Comm. Algebra **25**(1997), 2557–2567.

Ex. 16.34. (Rim, Giorgiutti) Let R be a right artinian ring with (right) Cartan matrix (c_{ij}). If $\det(c_{ij}) \neq 0$, show that two f.g. projective right R-modules are isomorphic iff they have the same composition factors (counted with multiplicities).

Solution. Let $\{U_i : 1 \leq i \leq s\}$ be a full set of principal indecomposable right R-modules, and $\{S_i : 1 \leq i \leq s\}$ be a full set of simple right R-modules, with $S_i = U_i / U_i \cdot \text{rad}\, R$. By definition, $c_{ij} \in \mathbb{Z}$ is the number of composition factors of U_i that are isomorphic to S_j, so (c_{ij}) is an $s \times s$ matrix over $\mathbb{Z}$. We assume here that $\det(c_{ij}) \neq 0$.

Consider any f.g. projective R-module P_R. By the Krull-Schmidt Theorem, $P \cong \bigoplus_i a_i U_i$ for uniquely determined integers $a_i \geq 0$. Accordingly, the number b_j of composition factors of P that are isomorphic to S_j is computed by $\sum_i a_i c_{ij}$. This can be expressed in a matrix equation

$$(b_1, \ldots, b_s) = (a_1, \ldots, a_s) \cdot (c_{ij}).$$

Since (c_{ij}) is a *nonsingular* matrix over $\mathbb{Q}$, the integer vector $(a_1, \ldots, a_s)$ is uniquely determined by the vector of composition factor multiplicities $(b_1, \ldots, b_s)$. The desired conclusion follows immediately from this observation.

Comment. Of course, the assumption that $\det(c_{ij}) \neq 0$ is crucial here. Without this assumption, for instance, the Cartan matrix (c_{ij}) may have two identical rows (see the example mentioned after *LMR*-(16.20)). This means that two *different* principal indecomposables may have exactly the same composition factors (counted with multiplicities).

Chapter 7

Matrix Rings, Categories of Modules and Morita Theory

§17. Matrix Rings

The introduction to the abstract Morita theory is preceded by a more down to earth discussion of the properties of matrix rings. Throughout, the E_{ij}'s denote the matrix units in a matrix ring $\mathbb{M}_n(S)$, and the base ring S is usually thought of as embedded in $\mathbb{M}_n(S)$, with the identification $s = \text{diag}(s, \ldots, s)$ for $s \in S$.

Three different criteria are presented in *LMR*-§17 for recognizing a ring R as a matrix ring $\mathbb{M}_n(S)$ for some ring S. The first is the existence of a set "abstract matrix units" (satisfying conditions). The second is the criterion that R_R be the direct sum of n mutually isomorphic right ideals. The third is the Agnarsson-Amitsur-Robson criterion, that there exist natural numbers p, q with $p + q = n$, and elements a, b, $f \in R$ such that

$$f^{p+q} = 0, \quad \text{and} \quad af^p + f^q b = 1.$$

Under this last criterion, a base ring S for the matrix ring R can be constructed explicitly as some sort of "eigenring": see *LMR*-(17.15) for details.

Of these three criteria, the last is the deepest and the most recent. It can be used to give various nonobvious examples of $n \times n$ matrix rings.

Given $R = \mathbb{M}_n(S)$, it is quite easy to "connect" the right S-modules to the right R-modules, and vice-versa. As usual we write $\mathfrak{M}_R$ and $\mathfrak{M}_S$ for the categories of right modules over the rings R and S. For $V \in \mathfrak{M}_S$, we let $G(V) = V^{(n)}$, which may be thought of as an object in $\mathfrak{M}_R$. On the other hand, for $U \in \mathfrak{M}_R$, we set $F(U) = UE_{11} \in \mathfrak{M}_S$. The construction of F and G leads quickly to a category equivalence between

$\mathfrak{M}_R$ and $\mathfrak{M}_S$. This is the easiest and the most concrete instance of a *Morita equivalence.*

Along with the study of recognition criteria for matrix rings, it is also of interest to investigate the uniqueness question for the base ring of a matrix ring. We say that a ring S is $\mathbb{M}_n$-*unique* if, for any ring S', $\mathbb{M}_n(S) \cong \mathbb{M}_n(S')$ implies that $S \cong S'$. The fact that we introduce such a definition means, of course, that not all rings are $\mathbb{M}_n$-unique. However, there is no lack of $\mathbb{M}_n$-unique rings; for instance, these include all commutative rings (*LMR*-(17.26)) and all semilocal rings (*LMR*-(17.31)).

The study of $\mathbb{M}_n$-uniqueness is seen to be related to certain cancellation properties. We briefly summarize the requisite definitions. Let $\mathcal{C}$ be a class of modules over a ring. We say that $\mathcal{C}$ *satisfies n-cancellation* if, for any P, $P' \in \mathcal{C}$,

$$n \cdot P \cong n \cdot P' \implies P \cong P'.$$

To weaken this condition somewhat, we say that $\mathcal{C}$ *satisfies weak n-cancellation* if, for any P, $P' \in \mathcal{C}$,

$$n \cdot P \cong n \cdot P' \implies \mathrm{End}_R(P) \cong \mathrm{End}_R(P').$$

The precise relationship between these two cancellation properties and the question of $\mathbb{M}_n$-uniqueness is rather subtle; we refer the reader to *LMR*-(17.29) for the details. From the discussion in *LMR*-§17C, we see that some very standard (and "nice") noncommutative rings can fail to be $\mathbb{M}_n$-unique. For instance, although commutative rings are $\mathbb{M}_n$-unique for any n, there exist commutative rings A for which some matrix ring $\mathbb{M}_r(A)$ fails to be $\mathbb{M}_n$-unique for some n (*LMR*-(17.35)).

The exercises in this section elaborate on the recognition criteria for matrix rings, and offer a couple of applications of the category equivalence between $\mathfrak{M}_S$ and $\mathfrak{M}_R$ ($R = \mathbb{M}_n(S)$) to the consideration of lattices of submodules in free modules.

Exercises for §17

Ex. 17.1. Let $R = \mathbb{M}_n(S)$, where $n \geq 1$. Show that R satisfies IBN (resp. the rank condition in *LMR*-§1) iff S does.

Solution. First, suppose R satisfies IBN. Since $S \subseteq R$ (by identifying S with the ring of scalar matrices in R), S also satisfies IBN by *LMR*-(1.5). Now, suppose S satisfies IBN. To show that R also does, assume that $p \cdot R_R \cong q \cdot R_R$. Then $p \cdot R_S \cong q \cdot R_S$. Since $R_S \cong n^2 \cdot S_S$, we have $pn^2 \cdot S_S \cong qn^2 \cdot S_S$. It follows that $pn^2 = qn^2$, and hence $p = q$.

Next, suppose R satisfies the rank condition. Exploiting the inclusion $S \subseteq R$ again and using *LMR*-(1.23), we see as above that S satisfies the rank condition. Now assume that S satisfies the rank condition, and suppose there is a surjection $\varphi : p \cdot R_R \twoheadrightarrow q \cdot R_R$. Using again the fact that $R_S \cong n^2 \cdot S_S$, we see that φ gives a surjection $pn^2 \cdot S_S \twoheadrightarrow qn^2 \cdot S_S$. It follows that $pn^2 \geq qn^2$, and hence $p \geq q$.

Ex. 17.2. Let R be a ring with a full set of $n \times n$ matrix units

$$\{e_{ij} : 1 \leq i, j \leq n\},$$

and let S be the centralizer ring of this set in R. Show that the e_{ij}'s are left (and right) linearly independent over S.

Solution. Suppose $\sum s_{ij} e_{ij} = 0$ where $s_{ij} \in S$. Left multiplying by $e_{i_0 i_0}$ and right multiplying by $e_{j_0 j_0}$, we get $s_{i_0 j_0} e_{i_0 j_0} = 0$. Therefore, it suffices to show that, whenever $se_{ij} = 0$ where $s \in S$, we have $s = 0$. Now, from $se_{ij} = 0$, we have

$$se_{kk} = e_{ki}(se_{ij})e_{jk} = 0.$$

Therefore, $s = s \sum e_{kk} = 0$.

Ex. 17.3. Suppose a ring R has three elements a, b, f such that $f^2 = af + fb = 1$. Show that R also has an element c such that $cf + fc = 1$.

Solution. Left multiplying $af + fb = 1$ by f, we get $faf = f$. By symmetry, we also have $fbf = f$. Therefore, for $c := afb \in R$, we have

$$cf + fc = afbf + fafb = af + fb = 1.$$

Ex. 17.4. (Agnarsson) Suppose a ring R has two elements a, f such that $f^{p+q} = 0$ and $af^p + f^q a = 1$, where $p, q \geq 1$ and $p \neq q$. Show that $R = 0$.

Solution. We may assume, without loss of generality, that $p > q$. Left (resp. right) multiplying $af^p + f^q a = 1$ by f^p (resp. f^q), we get $f^p a f^p = f^p$ and $f^q a f^q = f^q$. Therefore,

$$\begin{aligned} &f^p = f^{p-q}\left(f^q a f^q\right) f^{p-q} = f^{p-q} f^q f^{p-q} = f^{2p-q}, \quad \text{so} \\ &f^{p+q-1} = f^{2p-q+q-1} = f^{2p-1} = 0. \end{aligned}$$

By *LMR*-(17.13), $af^p + f^q a = 1$ implies that $1 \in Rf^{p+q-1} + fR$, so we have $1 \in fR$. Since f is nilpotent, this implies that $1 = 0 \in R$, so $R = 0$.

Alternatively, Mark S. Davis has given the following solution to the exercise that is easier and also independent of *LMR*-(17.13). We assume as before that $p > q$, and let m be the smallest natural number for which $f^m = 0$. If $p \geq m$, then $f^p = 0$, and we have $f^q a = 1$. Since f^q is nilpotent, this implies that $1 = 0 \in R$, so $R = 0$. We may thus assume that $m > p$. Then

$$\begin{aligned} 0 = f^m &= f^{p-q} \cdot 1 \cdot f^{m-p+q} \\ &= f^{p-q}\left(af^p + f^q a\right) f^{m-p+q} \\ &= f^{p-q} a f^{m+q} + f^p a f^{m-p+q} \\ &= f^p a f^{m-p+q}, \end{aligned}$$

so in particular, $f^p a f^{m-1} = 0$. But then

$$\begin{aligned} f^{m-1} &= f^p \cdot 1 \cdot f^{m-p-1} \\ &= f^p \left(af^p + f^q a\right) f^{m-p-1} \\ &= f^p a f^{m-1} + f^{p+q} a f^{m-p-1} \\ &= 0, \end{aligned}$$

which contradicts the minimal choice of m.

Comment. Agnarsson's result appeared in his paper "On a class of presentations of matrix algebras," Comm. Algebra **24**(1996), 4331–4338. The next exercise will show that the assumption $p \neq q$ above is indispensable.

Ex. 17.5. Let k be a commutative ring, and R be the k-algebra with generators a, f and relations $f^{2p} = 0$, $af^p + f^p a = 1$, where $p \geq 1$. Show that $R \cong \mathbb{M}_{2p}(S)$ for a suitable k-algebra $S \supseteq k$. (In particular, if $k \neq 0$, then $R \neq 0$.)

Solution. By *LMR*-(17.10), $R \cong \mathbb{M}_{2p}(S)$ for a suitable k-algebra S. As in the proof of *LMR*-(17.10), the matrices

$$A = \begin{pmatrix} 0 & I_p \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad F = E_{21} + E_{32} + \cdots + E_{2p,2p-1}$$

in $\mathbb{M}_{2p}(k)$ satisfy the relations $F^{2p} = 0$, $AF^p + F^p A = I_{2p}$. Therefore, there exists a k-algebra homomorphism from R into $\mathbb{M}_{2p}(k)$. From this, it is clear that the natural map $k \to S$ is a monomorphism.

Ex. 17.6. (Agnarsson) (1) Let k be a commutative ring, and R be the k-algebra with generators a, f, and relations

$$a^n = f^n = 0, \quad a^{n-1} f^{n-1} + fa = 1.$$

Show that $R \cong \mathbb{M}_n(k)$ as k-algebras.
(2) Show that a ring A is an $n \times n$ matrix ring iff there exist a_0, $f_0 \in A$ such that $a_0^n = f_0^n = 0$ and $a_0^{n-1} f_0^{n-1} + f_0 a_0 = 1$.

Solution. (1) By the theorem of Agnarsson-Amitsur-Robson (see *LMR*-(17.10) and *LMR*-(17.15)), R can be realized as a matrix ring $\mathbb{M}_n(S)$ (for a suitable k-algebra S) with matrix units

$$E_{ij} = f^{i-1} \left(a^{n-1} f^{n-1}\right) a^{j-1} \quad (1 \leq i, j \leq n),$$

and moreover with

$$f = E_{21} + E_{32} + \cdots + E_{n,n-1}.$$

Let $k_0 = k \cdot 1 \subseteq R$, and let $R_0 = \sum k_0 \cdot E_{ij}$ be the k-subalgebra of R generated by $\{E_{ij}\}$. Then R_0 contains

$$\begin{aligned} E_{12} + E_{23} + \cdots + E_{n-1,n} &= a^{n-1} f^{n-1} a + f a^{n-1} f^{n-1} a^2 \\ &\quad + \cdots + f^{n-2} a^{n-1} f^{n-1} a^{n-1} \end{aligned}$$

$$= (1 - fa)\,a + f\,(1 - fa)\,a + \cdots + f^{n-2}\,(1 - fa)\,a^{n-1} = a - f^{n-1}a^n = a.$$

Since R_0 also contains f, we have $R_0 = R$. We are done if we can show that $k \to k_0$ is an isomorphism. This is clear since there is a well-defined k-algebra homomorphism $\varphi : R \to \mathbb{M}_n(k)$ given by

$$\varphi\,(a) = E_{12} + \cdots + E_{n-1,n} \in \mathbb{M}_n\,(k) \quad \text{and}$$
$$\varphi\,(f) = E_{21} + \cdots + E_{n,n-1} \in \mathbb{M}_n\,(k)\,.$$

(Note that the two matrices in $\mathbb{M}_n(k)$ above satisfy the given relations between a and f in R.)

(2) Now follows easily from the above by taking $k = \mathbb{Z}$.

Ex. 17.7. (Agnarsson-Amitsur-Robson) Show that, for $n \geq 3$, the existence of c, d, $f \in R$ such that $f^n = 0$ and $cf + fd = 1$ need not imply that R is an $n \times n$ matrix ring.

Solution. First assume $n \neq 5$. Let $R = \mathbb{M}_5(\mathbb{Q})$, with elements

$$c = E_{13} + E_{24}, \quad d = E_{13} + E_{24} + E_{35}, \quad \text{and} \quad f = E_{31} + E_{42} + E_{53}.$$

Then $f^2 = E_{53}E_{31} = E_{51}$, so $f^n = 0$ for $n \geq 3$. Also,

$$cf + fd = (E_{13}+E_{24})(E_{31}+E_{42}+E_{53}) + (E_{31}+E_{42}+E_{53})(E_{13}+E_{24}+E_{35})$$
$$= E_{11} + E_{22} + E_{33} + E_{44} + E_{55} = I_5.$$

However, an easy application of the Wedderburn-Artin Theorem shows that R is not an $n \times n$ matrix ring.

Finally, for $n = 5$, let $R = \mathbb{M}_4(\mathbb{Q})$. For the matrices

$$a = E_{12} + E_{23} + E_{34}, \quad b = E_{14}, \quad \text{and} \quad f = E_{21} + E_{32} + E_{43}$$

in R, we have (as in the proof of *LMR*-(17.10)):

$$f^4 = 0 \quad \text{and} \quad af^3 + fb = I_4.$$

Therefore, $cf + fb = I_4$ for $c = af^2$ (and of course $f^5 = 0$). However, as before, we can show easily that R is not a 5×5 matrix ring.

Ex. 17.8. Let $R = \mathbb{M}_n(S)$ and $R' = \mathbb{M}_n(S')$, where S, S' are rings.

(1) For any $(R,\ R')$-bimodule M, show that the triangular ring $T = \begin{pmatrix} R & M \\ 0 & R' \end{pmatrix}$ is an $n \times n$ matrix ring.

(2) Let N be an $(S,\ S')$-bimodule, and let $M = \mathbb{M}_n(N)$, viewed as an $(R,\ R')$-bimodule in the obvious way. According to (1), $T = \begin{pmatrix} R & M \\ 0 & R' \end{pmatrix}$ is an $n \times n$ matrix ring. Determine a base ring for T.

Solution. (1) By *LMR*-(17,10), there exist elements c, d, $f \in R$ such that $f^n = 0$ and $cf^{n-1} + fd = 1$. Similarly, there exist elements c', d', $f' \in R'$ such that $f'^n = 0$ and $c'f'^{n-1} + f'd' = 1$. If we set

$$C = \begin{pmatrix} c & 0 \\ 0 & c' \end{pmatrix}, \quad D = \begin{pmatrix} d & 0 \\ 0 & d' \end{pmatrix}, \quad \text{and} \quad F = \begin{pmatrix} f & 0 \\ 0 & f' \end{pmatrix}$$

in T, then clearly $F^n = 0$, and $CF^{n-1} + FD = 1_T$. By *LMR*-(17.10) again, we see that T is an $n \times n$ matrix ring.

(2) Let T_0 be the triangular ring $\begin{pmatrix} S & N \\ 0 & S' \end{pmatrix}$. Then $T \cong \mathbb{M}_n(T_0)$, since every element of T may be thought of as an $n \times n$ matrix whose entries are 2×2 matrices from the triangular ring T_0. For instance, if $n = 3$, the isomorphism $\varphi : T \to \mathbb{M}_3(T_0)$ is given by

$$\begin{pmatrix} \begin{bmatrix} s_1 & s_2 & s_3 \\ s_4 & s_5 & s_6 \\ s_7 & s_8 & s_9 \end{bmatrix} & \begin{bmatrix} n_1 & n_2 & n_3 \\ n_4 & n_5 & n_6 \\ n_7 & n_8 & n_9 \end{bmatrix} \\ \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} & \begin{bmatrix} s_1' & s_2' & s_3' \\ s_4' & s_5' & s_6' \\ s_7' & s_8' & s_9' \end{bmatrix} \end{pmatrix} \mapsto \begin{pmatrix} \begin{bmatrix} s_1 & n_1 \\ 0 & s_1' \end{bmatrix} & \begin{bmatrix} s_2 & n_2 \\ 0 & s_2' \end{bmatrix} & \begin{bmatrix} s_3 & n_3 \\ 0 & s_3' \end{bmatrix} \\ \begin{bmatrix} s_4 & n_4 \\ 0 & s_4' \end{bmatrix} & \begin{bmatrix} s_5 & n_5 \\ 0 & s_5' \end{bmatrix} & \begin{bmatrix} s_6 & n_6 \\ 0 & s_6' \end{bmatrix} \\ \begin{bmatrix} s_7 & n_7 \\ 0 & s_7' \end{bmatrix} & \begin{bmatrix} s_8 & n_8 \\ 0 & s_8' \end{bmatrix} & \begin{bmatrix} s_9 & n_9 \\ 0 & s_9' \end{bmatrix} \end{pmatrix}.$$

Note that this map can be realized formally as the conjugation $\alpha \mapsto \pi^{-1}\alpha\pi$ where π is the permutation matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

so φ is indeed a ring isomorphism.

Ex. 17.9. For s, $t \in F^*$ where F is a field of characteristic $\neq 2$, let R be the F-quaternion algebra generated by two elements i, j with the relations

$$i^2 = s, \quad j^2 = t \quad \text{and} \quad ij = -ji.$$

Assume that there exist u, $v \in F$ such that $su^2 + tv^2 = 1$. Using the Recognition Theorem *LMR*-(17.10) for $p = q = 1$, show that $R \cong \mathbb{M}_2(F)$ as F-algebras.

Solution. The elements u, v cannot be both zero, so let us assume $v \neq 0$. Let $k = ij \in R$. Then $k^2 = -st$ and the elements $\{i, j, k\}$ pairwise anticommute. For $f = tvi + suj + k \in R$, we have

$$\begin{aligned} f^2 &= (tv)^2 i^2 + (su)^2 j^2 + k^2 \\ &= st^2v^2 + ts^2u^2 - st \\ &= st\left(su^2 + tv^2\right) - st = 0. \end{aligned}$$

On the other hand, since i anticommutes with j, k:

$$if + fi = i\,(tvi) + (tvi)\,i = 2\,stv \neq 0.$$

Therefore, for $a = i/(2stv) \in R$, we have $af + fa = 1$. By the Recognition Theorem *LMR*-(17.10), $R \cong \mathbb{M}_2(S)$ for some F-algebra S. Since

$$4 \geq \dim_F R = \dim_F \mathbb{M}_2\,(S) = 4\dim_F S,$$

we conclude that $S = F$, so that $R \cong \mathbb{M}_2(F)$.

Comment. What we gave above is a somewhat unusual proof for the fact that $R \cong \mathbb{M}_2(F)$. The "usual" proof assumes the fact that R is always a *simple* F-algebra. Since $f^2 = 0$ and $f \neq 0$, R cannot be a division algebra. By Wedderburn's Theorem, $R \cong \mathbb{M}_n(D)$ for some division F-algebra D and some integer $n \geq 2$. By counting dimensions as before, we see that $n = 2$ and $D = F$.

There is also a strong converse to the Exercise; namely, *if the equation $su^2 + tv^2 = 1$ has no solutions in F, then R is a division F-algebra.* This follows by first showing that $x^2 - su^2 - tv^2 + stw^2 = 0$ has only the trivial solution in F. Then, for any $\alpha = x + ui + vj + wk \neq 0$ in R, we have

$$N(\alpha) := \alpha\,\bar{\alpha} = x^2 - su^2 - tv^2 + stw^2 \neq 0,$$

where $\bar{\alpha} = x - ui - vj - wk$, so $N(\alpha)^{-1}\bar{\alpha}$ gives an inverse for α in R.

Ex. 17.10. (Fuchs-Maxson-Pilz) Show that a ring R is a 2×2 matrix ring iff there exists $f, g \in R$ such that $f^2 = g^2 = 0$ and $f + g \in \mathrm{U}(R)$.

Solution. If R is a 2×2 matrix ring, we can choose $f = E_{21}$ and $g = E_{12}$, for which $(f + g)^2 = 1$. Conversely, assume that there exists $f, g \in R$ such that $f^2 = g^2 = 0$ and $b := f + g \in \mathrm{U}(R)$, with inverse a. Left-multiplying by f and right-multiplying by a, we get $fga = f$. Similarly, left-multiplying by a and right-multiplying by g, we get $afg = g$, so

$$af \;=\; a\,(fga) \;=\; (afg)\,a \;=\; ga.$$

Now right multiplication of $g + f = b$ by a gives

$$1 \;=\; ga + fa \;=\; af + fa,$$

so the Recognition Theorem *LMR*-(17.10) implies that R is a 2×2 matrix ring.

Comment. The above result appeared in the last section of a paper on near-rings by Fuchs, Maxson and Pilz in Proc. Amer. Math. Soc. **112**(1991), 1–7. It was subsequently generalized to $n \times n$ matrix rings by Fuchs: see his paper in Bull. Austral. Math. Soc. **43**(1991), 265–267. Fuchs' result states that

> *R is an $n \times n$ matrix ring iff there exists $f, g \in R$ such that* $f^n = g^2 = 0$, $f + g \in \mathrm{U}(R)$, and $Rg \cap \mathrm{ann}_\ell(f^{n-1}) = 0$.

Ex. 17.11. Let $R = \mathbb{M}_n(S)$, which may be viewed as a left S-module. Let V, W be right S-modules.

(1) Show that $V \otimes_S R \cong n \cdot V^{(n)}$, where $V^{(n)} = (V, \ldots, V)$ is viewed as a right R-module in the natural way.
(2) Show that $\mathrm{End}_R(V \otimes_S R) \cong \mathbb{M}_n(\mathrm{End}_S V)$.
(3) Show that $V \otimes_S R \cong W \otimes_S R$ as R-modules iff $n \cdot V \cong n \cdot W$ as S-modules.

Solution. (1) First there is a natural isomorphism

$$V \otimes_S R = V \otimes_S \mathbb{M}_n(S) \cong \mathbb{M}_n(V).$$

The right R-module $\mathbb{M}_n(V)$ is the direct sum of its n "rows," each of which is isomorphic to $V^{(n)}$. Therefore, $V \otimes_S R \cong n \cdot V^{(n)}$ as R-modules.

(2) Recall that there is a category equivalence $G : \mathfrak{M}_S \to \mathfrak{M}_R$ that sends any right S-module V to the right R-module $V^{(n)}$. Using (1), we have therefore

$$\begin{aligned}\mathrm{End}_R(V \otimes_S R) &\cong \mathrm{End}_R\left(n \cdot V^{(n)}\right) \cong \mathbb{M}_n\left(\mathrm{End}_R\left(V^{(n)}\right)\right)\\ &\cong \mathbb{M}_n(\mathrm{End}_R\, G(V)) \cong \mathbb{M}_n(\mathrm{End}_S V),\end{aligned}$$

where the last isomorphism results from the fact that G is a category equivalence.

(3) using (1) again, we have

$$\begin{aligned}V \otimes_S R \cong W \otimes_S R &\iff n \cdot V^{(n)} \cong n \cdot W^{(n)}\\ &\iff n \cdot G(V) \cong n \cdot G(W)\\ &\iff n \cdot V \cong n \cdot W,\end{aligned}$$

since G is a category equivalence.

Ex. 17.12. For any S-module M_S, let $\mathrm{Lat}_S(M)$ denote the lattice of S-submodules of M. For any ring S and $R = \mathbb{M}_n(S)$, let $M = S_S^n$ be identified with RE_{11}. Define

$$\mathrm{Lat}_R(R_R) \underset{g}{\overset{f}{\rightleftarrows}} \mathrm{Lat}_S(M)$$

by $f(U) = UE_{11}$ for $U \in \mathrm{Lat}_R(R_R)$ and $g(V) = \sum_{j=1}^n VE_{1j}$ for $V \in \mathrm{Lat}_S(M)$. Show that f and g are mutually inverse lattice isomorphisms. (Note, in particular, that this "classifies" the right ideals in the matrix ring R.)

Solution. The map f above is just the "restriction" (to subobjects of R_R) of the category equivalence F from $\mathfrak{M}_R$ to $\mathfrak{M}_S$. Therefore, it is enough to check that the map g is also the restriction (to subobjects of $\mathfrak{M}_S$) of the "inverse" category equivalence G from $\mathfrak{M}_S$ to $\mathfrak{M}_R$ constructed in the proof of *LMR*-(17.20). Let V be an S-submodule of $M = S^n_S = RE_{11}$. By definition,

$$G(V) = V^{(n)} := \{(v_1, \ldots, v_n) : v_i \in S^n_S\} \subseteq R.$$

This is easily seen to be $\sum_{j=1}^n VE_{1j}$ when we think of V as sitting in RE_{11}. Since F, G are inverse category equivalences, they induce mutually inverse lattice isomorphisms between $\text{Lat}_R(R_R)$ and $\text{Lat}_S(M)$.

Comment. It is quite easy to see directly that, for $V \subseteq RE_{11}$ as above, $g(V) = \sum_{j=1}^n VE_{1j}$ is a right ideal in R. In fact, for any $s \in S$:

$$g(V)\, sE_{pq} = \sum_{j=1}^{n} VE_{1j}\,(sE_{pq}) = (Vs)\, E_{1q} \subseteq g(V).$$

Since $R = \sum SE_{pq}$, we have $g(V)R \subseteq g(V)$. An easy computation will show that fg and gf are identity maps. However, it is much more informative to ascertain that f and g are "induced" by the inverse category equivalences F and G.

Ex. 17.13. If S, T are two rings such that $\mathbb{M}_n(S) \cong \mathbb{M}_m(T)$ as rings, show that $\text{Lat}_S(S^n_S) \cong \text{Lat}_T(T^m_T)$ as lattices.

Solution. By the last exercise, $\text{Lat}_S(S^n_S)$ and $\text{Lat}_T(T^m_T)$ are both isomorphic to the lattice of right ideals of $\mathbb{M}_n(S) \cong \mathbb{M}_m(T)$!

Ex. 17.14. Let S, T be nonzero commutative rings, and n, $m \geq 1$ be integers. If $\mathbb{M}_n(S) \cong \mathbb{M}_m(T)$, show that $n = m$ and $S \cong T$.

Solution. Let us identify $\mathbb{M}_n(S)$ with $\mathbb{M}_m(T)$ (using a fixed isomorphism), and call the resulting ring A. Let $\mathfrak{m}$ be a maximal ideal of A. Then there exist an ideal $I \subseteq S$ and an ideal $J \subseteq T$ such that $\mathfrak{m} = \mathbb{M}_n(I) = \mathbb{M}_m(J)$. Since

$$\mathbb{M}_n(S/I) \cong \mathbb{M}_n(S)/\mathbb{M}_n(I) = A/\mathfrak{m}$$

is a simple ring, so is S/I, and hence S/I is a field. Similarly, T/J is a field, so by Wedderburn's Theorem, $\mathbb{M}_n(S/I) \cong \mathbb{M}_m(T/J)$ implies that $n = m$. Therefore, we have $\mathbb{M}_n(S) \cong \mathbb{M}_n(T)$. Taking the centers of both sides, we conclude that $S \cong T$.

Comment. There is another quite well-known proof of the result in this exercise by using the theory of rings with polynomial identities. Let

$$\Gamma_r(y_1, \ldots, y_r) = \sum_{\sigma \in S_r} (\text{sgn}\,\sigma) y_{\sigma(1)} \cdots y_{\sigma(r)},$$

where S_r denotes the symmetric group on $\{1, \ldots, r\}$. It is known that, for any nonzero commutative ring S, the matrix ring $A = \mathbb{M}_n(S)$ satisfies the "standard identity" $\Gamma_{2n} \equiv 0$, but not the identity $\Gamma_{2n-2} = 0$. From this, it follows immediately that, for S, T as in the exercise, $\mathbb{M}_n(S) \cong \mathbb{M}_m(T)$ implies that $n = m$ (and hence $S \cong T$ by taking centers).

Of course, this proof is much more sophisticated than the one given in our solution above, in that it made crucial use of the theory of PI rings. However, the structural simplicity and conceptual elegance of this second proof make it quite compelling.

Ex. 17.15. Let S, T be two rings such that $\mathbb{M}_n(S) \cong \mathbb{M}_m(T)$ as rings, where n, $m \geq 2$ are given integers. If S is commutative and n is prime, show that we must have $n = m$ and $S \cong T$.

Solution. Our solution here is modeled upon the proof of the $\mathbb{M}_n$-uniqueness of a commutative ring S given in *LMR*-(17.26). Let $\varphi : \mathbb{M}_n(S) \to \mathbb{M}_m(T)$ be a ring isomorphism. Then φ maps the center of $\mathbb{M}_n(S)$ isomorphically to that of $\mathbb{M}_m(T)$, so φ induces an isomorphism of S with $Z(T)$ (the center of T). To simplify the notations, let us identify S with $Z(T)$. Since φ is a ring isomorphism, we have $\mathbb{M}_n(S)_S \cong \mathbb{M}_m(T)_S$. On the other hand,

$$\mathbb{M}_n(S)_S \cong S_S^{n^2} \quad \text{and} \quad \mathbb{M}_m(T)_S \cong (T_S)^{m^2}.$$

Therefore, $(T_S)^{m^2} \cong S_S^{n^2}$. This shows that T_S is a f.g. projective right S-module of constant rank, say t. Then, taking the ranks from $(T_S)^{m^2} \cong S_S^{n^2}$, we get $tm^2 = n^2$. Since n is a prime and $m \geq 2$, we must have $m = n$ and $t = 1$. But by *LMR*-(2.50), $T_S = S \oplus X$ for some (f.g. projective) right S-module X. Taking ranks again shows that $X = 0$ and so $S = T$, as desired.

Comment. If we do not assume n to be prime, then the equation $tm^2 = n^2$ will only imply $t = r^2$ for some integer r with $rm = n$. We will then have

$$\mathbb{M}_m(T) \cong \mathbb{M}_n(S) \cong \mathbb{M}_m\left(\mathbb{M}_r(S)\right).$$

However, this need not imply that $T \cong \mathbb{M}_r(S)$, since we know (from *LMR*-(17.35)) that $\mathbb{M}_r(S)$ may not be $\mathbb{M}_m$-unique.

Ex. 17.16. Show that $Q^r_{\max}(\mathbb{M}_n(R)) \cong \mathbb{M}_n(Q^r_{\max}(R))$. If R is a semiprime ring, prove the same results for Martindale's right (resp. symmetric) ring of quotients.

Solution. Write $S = \mathbb{M}_n(R)$ and $Q = Q^r_{\max}(R)$. Our goal is to show that $Q^r_{\max}(S) = \mathbb{M}_n(Q)$. We first check that $S \subseteq_d \mathbb{M}_n(Q)_S$. Take two matrices $x = (x_{ij})$, $y = (y_{ij})$ over Q, with (say) $x_{i_0 j_0} \neq 0$. By Ex.8.9, there exists $a \in R$ such that $x_{i_0 j_0} a \neq 0$, and $y_{ij} a \in R$ for all i, j. Therefore, for the scalar matrix $a \cdot I_n \in S$, we have

$$x \cdot aI_n \neq 0 \quad \text{and} \quad y \cdot aI_n \in \mathbb{M}_n(R) = S.$$

Having checked that $S \subseteq_d \mathbb{M}_n(Q)_S$, we may assume (by *LMR*-(13.11)) that $\mathbb{M}_n(Q) \subseteq Q^r_{\max}(S)$. Now, by *LMR*-(17.7), we know that $Q^r_{\max}(S)$ must have the form $\mathbb{M}_n(T)$, where T is a suitable ring containing Q. Whatever T is, $S = \mathbb{M}_n(R) \subseteq_d \mathbb{M}_n(T)_S$. From this, it is easy to check that $R \subseteq_d T_R$. Since $Q \subseteq T$ and $Q = Q^r_{\max}(R)$, we must then have $Q = T$, and so $Q^r_{\max}(S) = \mathbb{M}_n(Q)$, as asserted.

To compute the Martindale rings of quotients of S, let us write Q^r (resp. Q^s) for $Q^r(R)$ (resp. $Q^s(R)$), the Martindale right (resp. symmetric) ring of quotients of R. Consider the rings

$$S = \mathbb{M}_n(R) \subseteq \mathbb{M}_n(Q^r) \subseteq \mathbb{M}_n(Q) = Q^r_{\max}(S).$$

For any $q = (q_{ij}) \in \mathbb{M}_n(Q^r)$, we have $q_{ij}A_{ij} \subseteq R$ for suitable ideals $A_{ij} \subseteq R$ with zero annihilators. By *LMR*-(14.4), $A = \bigcap_{i,j} A_{ij} \subseteq R$ also has zero annihilator. This implies that $\mathbb{M}_n(A)$ has zero annihilator in $S = \mathbb{M}_n(R)$, and we have clearly $q \cdot \mathbb{M}_n(A) \subseteq S$. This shows that $\mathbb{M}_n(Q^r) \subseteq Q^r(S)$. Now consider any $q = (q_{ij}) \in \mathbb{M}_n(Q)$ that belongs to $Q^r(S)$. Then $qB \subseteq \mathbb{M}_n(R)$ for some ideal $B \subseteq \mathbb{M}_n(R)$ with zero annihilator. By *LMR*-(17.8), B must have the form $\mathbb{M}_n(A)$ for some ideal $A \subseteq R$, which necessarily has a zero annihilator. From $q \cdot \mathbb{M}_n(A) \subseteq \mathbb{M}_n(R)$, we clearly have $q_{ij}A \subseteq R$ for all i, j, and so $q_{ij} \in Q^r$. This shows that $Q^r(S) \subseteq \mathbb{M}_n(Q^r)$; hence equality holds. The equation $Q^s(S) = \mathbb{M}_n(Q^s)$ is proved by the same argument, applied to both sides of an element $q = (q_{ij}) \in \mathbb{M}_n(Q)$.

§18. Morita Theory of Category Equivalences

An entering wedge to this section on Morita Theory is the idea of "categorical properties" of modules (and their morphisms). A property $\mathcal{P}$ on objects (resp. morphism) in a module category $\mathfrak{M}_R$ is said to be a *categorical property* if, for any category equivalence $F : \mathfrak{M}_R \to \mathfrak{M}_S$, whenever $M \in \mathfrak{M}_R$ (resp. $g \in \operatorname{Hom}_R(M, N)$) satisfies $\mathcal{P}$, so does $F(M)$ (resp. $F(g)$). For instance, being a projective or injective module is a categorical property, and being a monomorphism or epimorphism is likewise a categorical property. Some module-theoretic properties initially defined by using elements may be checked to be categorical properties too; two examples for this are the finite generation and the faithfulness of a module.

Two rings R, S are said to be *Morita equivalent* ($R \approx S$ for short) if there exists a category equivalence $F : \mathfrak{M}_R \to \mathfrak{M}_S$. A ring-theoretic property $\mathcal{P}$ is said to be *Morita invariant* if, whenever R has the property $\mathcal{P}$, so does every $S \approx R$. A large number of ring-theoretic properties turn out to be Morita invariant, e.g., being semisimple, right noetherian (artinian), right (semi)-hereditary, von Neumann regular, etc. are such properties.

Consideration of categorical properties of modules and Morita invariant properties of rings leads naturally to the question: *How do category equivalences arise between two module categories* $\mathfrak{M}_R$ and $\mathfrak{M}_S$? The Morita Theorems (I, II, III in *LMR*-§18) provide full fledged answers to this question (and much more).

Basically, for a given R, *all rings* $S \approx R$ *arise as endomorphism rings of progenerators of* R. Here, a module P_R is a *generator* if R_R is a direct summand of some direct sum $n \cdot P$, and P is a *progenerator* if it is a f.g. projective generator.

Given a module P_R over a ring R, let $S = \text{End}(P_R)$ and $Q = P^* = \text{Hom}_R(P, R)$. Then $P = {}_QP_R$ and $Q = {}_RQ_S$, and we obtain the *Morita Context* $(R,\ P,\ Q,\ S;\ \alpha,\ \beta)$. Here,

$$\alpha: \ Q \otimes_S P \to R \quad \text{and} \quad \beta: \ P \otimes_R Q \to S$$

are naturally defined $(R,\ R)$- and $(S,\ S)$-homomorphisms respectively. This Morita Context can also be encoded into a formal *Morita ring* $M = \begin{pmatrix} R & Q \\ P & S \end{pmatrix}$. The many formal properties of a Morita Context are given in *LMR*-§18C.

In the case where P_R is a progenerator, α, β above are isomorphisms, and we have the functors

$$- \otimes_R Q: \ \mathfrak{M}_R \to \mathfrak{M}_S \quad \text{and} \quad - \otimes_S P: \ \mathfrak{M}_S \to \mathfrak{M}_R$$

that are mutually inverse category equivalences. The first functor is also equivalent to the functor $\text{Hom}_R(P, -)$, and the second to $\text{Hom}_S(Q, -)$. These facts essentially constitute Morita I (*LMR*-(18.24)), and Morita II says that any pair of mutually inverse category equivalences between $\mathfrak{M}_R$ and $\mathfrak{M}_S$ arises essentially in the above fashion (*LMR*-(18.26)).

A special example of a Morita equivalence is given by taking $P = eR$ where e is a *full* idempotent, that is, an idempotent with $ReR = R$. The fullness condition amounts to the fact that P is a progenerator. In this case, Morita *I* applies, and the net result is $R \approx eRe$.

The significance of the notion of full idempotents lies in the fact that it leads to a description of the rings $S \approx R$. In fact, these rings S arise precisely as $e\,\mathbb{M}_n(R)e$ where e is a full idempotent in a matrix ring $\mathbb{M}_n(R)$ (*LMR*-(18.33)). A consequence of this is that:

> *A ring-theoretic property $\mathcal{P}$ is Morita invariant iff, whenever a ring R satisfies $\mathcal{P}$, so do eRe (for any full idempotent $e \in R$) and $\mathbb{M}_n(R)$ (for any $n \geq 2$).*

If $R \approx S$, then R and S have certain common features (besides sharing all Morita invariant properties). For instance, they have isomorphic centers, and isomorphic ideal lattices (*LMR*-(18.42), *LMR*-(18.44)).

Section 18 concludes with a discussion of a useful generalization of Morita I where the role of $\mathfrak{M}_R$ is replaced by that of a full subcategory $\sigma[M]$. Here, M is a fixed right R-module, and $\sigma[M]$ is the subcategory of $\mathfrak{M}_R$ whose objects are the subquotients of arbitrary direct sums of copies of M. In case there exists a $\sigma[M]$-progenerator P (defined in a natural way), we get a category equivalence $\sigma[M] \to \mathfrak{M}_S$ for $S = \text{End}_R(P)$, defined by $\text{Hom}_R(P,\ -)$ (*LMR*-(18.57)).

The exercises in this section are devoted to various aspects of Morita's Theory of equivalences. Many categorical properties of modules and Morita invariant properties of rings are catalogued. Morita's characterization of a generator (Exercise 17) is a standard result, and so is Camillo's characterization of pairs of Morita equivalent rings in terms of column-finite matrix rings (Exercise 30). The three parts of Exercise 7 provide a nice connection between Morita's theory and the study of Frobenius and quasi-Frobenius rings.

Exercises for §18

Ex. 18.0. Prove the following characterization of f.g. modules stated in *LMR*-(18.3): A module M_R is f.g. iff, for any family of submodules $\{N_i : i \in I\}$ in M which form a chain, $N_i \neq M$ for all $i \in I$ implies that $\bigcup_{i \in I} N_i \neq M$.

Solution. First assume M is f.g., say $M = m_1R + \cdots + m_kR$. If $\{N_i\colon i \in I\}$ is a chain of submodules with $\bigcup_{i \in I} N_i = M$, then each m_j lies in some N_{i_j} $(1 \leq j \leq k)$. There exists an $i \in \{i_1, \ldots, i_k\}$ such that each $N_{i_j} \subseteq N_i$. Then $m_j \in N_i$ for $1 \leq j \leq k$ and so $N_i = M$.

Now suppose M has the stated property on chains of submodules, but is *not* f.g. Then we have a nonempty family $\mathcal{B} = \{B \subseteq M : M/B \text{ is not f.g.}\}$. Consider any chain $\{B_i\colon i \in I\}$ in $\mathcal{B}$ and let $B = \bigcup_i B_i$. *We claim that* $B \in \mathcal{B}$. Indeed, if $B \notin \mathcal{B}$, then $M = B + x_1R + \cdots + x_kR$ for some $x_1, \ldots, x_k \in M$. We have now a chain

$$\{N_i = B_i + x_1R + \cdots + x_kR \neq M\}_{i \in I}$$

whose union is $B + x_1R + \cdots + x_kR = M$, a contradiction. Therefore, $B \in \mathcal{B}$, and this provides an upper bound for $\{B_i : i \in I\}$. By Zorn's Lemma, $\mathcal{B}$ has a maximal element, say N. Clearly $N \neq M$, and for any $x \notin N$, $N + xR \in \mathcal{B}$, contradicting the maximality of N.

Ex. 18.1A. In an additive category $\mathfrak{M}$ with arbitrary direct sums, an object M is said to be *small* if every morphism $f : M \to \bigoplus_{i \in I} A_i$ factors through a finite direct sum of the A_i's. Show that

(1) Every quotient object of a small object is small.
(2) $M = N \oplus N'$ is small iff N, N' are small.

Solution. (1) Suppose M is small and $N = M/K$ is a quotient of M. For any $f : N \to \bigoplus_{i \in I} A_i$, the composition $M \to N \xrightarrow{f} \bigoplus_{i \in I} A_i$ factors through a morphism $g : M \to \bigoplus_{i \in J} A_i$, for some finite $J \subseteq I$. Since $g(K) = 0$, g induces a morphism $\bar{g} : N \to \bigoplus_{i \in J} A_i$. Now f factors through $\bar{g}$, as desired.

(2) First suppose $M = N \oplus N'$ is small. Then, by (1), so are $N \cong M/N'$ and $N' \cong M/N$. Conversely, assume N, N' are small, and consider any morphism $f : M \to \bigoplus_{i \in I} A_i$. Then f induces $g : N \to \bigoplus_{i \in I} A_i$ and $g' : N' \to \bigoplus_{i \in I} A_i$. For a sufficiently large finite subset $J \subseteq I$, g and g' both factor through $\bigoplus_{i \in J} A_i$. It follows that f also factors through $\bigoplus_{i \in J} A_i$, as desired.

Comment. The notion of small objects in categories with arbitrary direct sums is standard in category theory; see, e.g., p. 74 of B. Mitchell's book "Theory of Categories," Academic Press, 1965. This notion played a substantial role in the work in the 1960s on the characterization of the categories of modules over rings.

Some caution is necessary in distinguishing "small modules" from "small submodules" as defined in *LMR*-p. 74. A small submodule $N \subseteq_s M$ may not be small as a module on its own! The small confusion inherent in this choice of terminology is, fortunately, seldom fatal.

Ex. 18.1B. For any module category $\mathfrak{M}_R$, prove the following:

(1) A module M_R is small iff there does not exist an infinite family of submodules $M_j \subsetneq M$ $(j \in J)$ such that every $m \in M$ lies in almost all M_j's.
(2) Every f.g. R-module M_R is small in the category $\mathfrak{M}_R$.
(3) M_R is noetherian iff every submodule of M is small in $\mathfrak{M}_R$.
(4) If R is right noetherian, M_R is small iff M is f.g.
(5) If M_R is projective, M is small iff M is f.g.
(6) Give an example of a module that is small but not f.g.

Solution. (1) Assume that there exists an infinite family of submodules $M_j \subsetneq M$ $(j \in J)$ such that every $m \in M$ lies in almost all M_j. Let $A_j = M/M_j$ and let $f : M \to \bigoplus_{j \in J} A_j$ be defined by

$$f(m) = (m + M_j)_{j \in J} \quad \text{where} \quad m + M_j \in A_j \text{ for every } j.$$

Since $m + M_j$ is zero for almost all j, $f(m)$ is indeed in $\bigoplus_{j \in J} A_j$. Since each $M_j \subsetneq M$, clearly f does not factor through a finite direct sum of the A_j's. Hence M is not small.

Conversely, assume M is not small. Then there exists $f : M \to \bigoplus_{i \in I} A_i$ which does not factor through a finite direct sum of the A_i's. Let $\pi_i : M \to A_i$ be the obvious maps such that $f(m) = (\pi_i(m))_{i \in I}$ for every $m \in M$, and let $M_i = \ker(\pi_i)$. The set

$$J := \{j \in I : \;\; M_j \subsetneq M\}$$

is infinite, for otherwise $f(M)$ would lie in the finite direct sum $\bigoplus_{j \in J} A_j$. If there exists $m \in M$ such that $m \notin M_j$ for infinitely many $j \in J$, then $f(m) = (\pi_i(m))_{i \in I}$ has infinitely many nonzero coordinates, in contradiction to $f(m) \in \bigoplus_{i \in I} A_i$. Thus, each $m \in M$ lies in M_j for almost all $j \in J$.

(2) Let $M_R = \sum_{k=1}^{n} m_k R$ and consider any homomorphism $f : M \to \bigoplus_{i\in I} A_i$. Since each $f(m_k)$ lies in a finite direct sum $\bigoplus_{i\in J_k} A_i$, $f(M)$ lies in the finite direct sum $\bigoplus_{i\in J} A_i$ where $J = \bigcup_{k=1}^{n} J_k$. This checks that M is small.

(3) First assume that M is noetherian. Then every submodule $N \subseteq M$ is f.g. and hence small in $\mathfrak{M}_R$ by (2). Conversely, assume that every submodule of M is small in $\mathfrak{M}_R$. Consider any ascending chain of submodules $M_1 \subseteq M_2 \subseteq \cdots$ in M. By assumption, their union N is small. Each $n \in N$ clearly lies in almost all M_j. By (1), *some* M_j must be equal to N, so we have $M_j = M_{j+1} = \cdots$. This checks the ACC for submodules of M.

(4) It suffices to prove the "only if" part. Let R be right noetherian, and M_R be small. By (3), it suffices to show that any submodule $N \subseteq M$ is also small. Consider any homomorphism $f : N \to \bigoplus_{i\in I} A_i$. Let $E_i = E(A_i)$ be the injective hull of A_i. By the Bass-Papp Theorem *LMR*-(3.46), $\bigoplus_{i\in I} E_i$ is injective, so we have a commutative diagram

$$\begin{array}{ccc} N & \xrightarrow{\quad f \quad} & \bigoplus_{i\in I} A_i \\ \downarrow & & \downarrow \\ M & \xrightarrow{\quad g \quad} & \bigoplus_{i\in I} E_i \end{array}$$

for a suitable homomorphism g. Since M is small, there exists a finite subset $J \subseteq I$ such that $g(M) \subseteq \bigoplus_{i\in J} E_i$. Therefore, $f(N) \subseteq \bigoplus_{i\in J} A_i$, as desired.

(5) Again, it suffices to prove the "only if" part. Since M is projective, there exists a module M' such that $M \oplus M' = \bigoplus_{i\in I} R_i$, where $R_i \cong R_R$ for all i. If M is small, we have $M \subseteq \bigoplus_{i\in J} R_i$ for some finite set $J \subseteq I$. But then

$$\bigoplus_{i\in J} R_i = M \oplus \Big(M' \cap \bigoplus_{i\in J} R_i\Big)$$

shows that M is f.g.

(6) Following a suggestion of Peter Vámos, we look for an example among the uniserial modules. (Recall that a module is *uniserial* if its submodules form a chain with respect to inclusion.) The crucial observation for solving (6) is the following lemma, which may seem, at first sight, a little counter-intuitive.

Lemma. *Let M_R be a uniserial right R-module. If M is not countably generated, then M is small.*

Proof. If M is not small, there exist proper submodules $\{M_j\}_{j\in J}$ (J an infinite set) such that every element $m \in M$ lies in almost all M_j's. Without loss of generality, we may assume that $J = \{1, 2, 3, \ldots\}$. For any $j \in J$, let $x_j \in M \backslash M_j$, and let $N = \sum_{j=1}^{\infty} x_j R$. Then $N \not\subseteq M_j$ for each j, so we must have $M_j \subseteq N$. If there exists an element $m \in M \backslash N$, then m lies in none of the M_j's, a contradiction. Therefore, $M = N$, which is countably generated. □

To construct the example needed for (6), it suffices to produce an ordered abelian group $(G, <)$ with the property that, for any *countable* set $A > 0$ in G, there exists an element $g \in G$ such that $0 < g < A$. Once such an ordered group $(G, <)$ is found, we can take R to be the valuation ring in any field K with a Krull valuation $v : K^* \twoheadrightarrow G$. The maximal ideal M of R is (uniserial and) not countably generated as an R-module, since any countable subset of M generates an ideal I with

$$v(I \backslash \{0\}) \subsetneqq \{g \in G : \; g > 0\},$$

which implies that $I \subsetneqq M$. By the Lemma above, M is small, and provides the example we want.

To produce the desired ordered abelian group $(G, <)$, let $(S, <)$ be a well-ordered set in which no countable subset is cofinal. (For instance, take S to be the first uncountable ordinal.) Now let $G = \bigoplus_{s \in S} \mathbb{Z} \cdot s$ (the free abelian group generated by S), and order G "lexicographically"; that is, we declare $g = \sum_s g(s)s > 0$ if $g(s_0) > 0$ for $s_0 = \min \{\text{supp}(g)\}$. To check that $(G, <)$ has the required property, consider any countable set

$$A = \left\{ g_i = \sum\nolimits_s g_i(s)s : \; i \in \mathbb{N} \right\} \subseteq G.$$

Since $\bigcup_i \text{supp}(g_i) \subseteq S$ is countable, it cannot be cofinal in S, so there exists an $s > \text{supp}(g_i)$ for all i. For $g := 1 \cdot s \in G$, we have

$$\min \{\text{supp}(g_i - g)\} = \min \{\text{supp}(g_i)\} < s \quad (\text{for every } i),$$

so $0 < g < A$, as desired.

Comment. The notion of small modules can be extended as follows. For any cardinal number $\aleph$, an R-module M is said to be $\aleph$-*small* if every morphism $f : M \to \bigoplus_{i \in I} M_i$ factors through $\bigoplus_{i \in J} M_i$ for some subset $J \subseteq I$ with $|J| \leq \aleph$. The following are immediate observations:

(1) *If $\aleph$ is a finite cardinal, then M being $\aleph$-small means that $M = 0$.*
(2) *If M is small, then M is $\aleph_0$-small (where $\aleph_0 = |\mathbb{N}|$).*
(3) *Any uniserial module M_R is $\aleph_0$-small.*

To see (3), note that if M_R is *not* countably generated, then M is small by the Lemma, so we are done by (2). Thus, we may assume that $M = \sum_{j=1}^{\infty} m_j R$. Given any morphism $f : M \to \bigoplus_{i \in I} M_i$, the set $\{f(m_j) : j \in \mathbb{N}\}$ involves only countably many "coordinates," so $f(M) \subseteq \bigoplus_{i \in J} M_i$ for a suitable $J \subseteq I$ with $|J| \leq \aleph_o$. This proves (3), and the same proof shows that the uniserial module M_R can be written as an ascending union $\bigcup_{i=1}^{\infty} M_i$, where each $M_i \subseteq M$ is small.

For more information on $\aleph$-small modules and the role they play in the general direct sum decomposition theory of modules, see Section 2.9 of A.

Facchini's book "Module Theory," Birkhäuser Verlag, Basel-Boston-Berlin, 1998.

Ex. 18.2. Show that M_R being a singular (resp. nonsingular) R-module is a categorical property.

Solution. First note that, for $B \subseteq A$ in $\mathfrak{M}_R$, B being *essential* in A (written $B \subseteq_e A$) is a categorical property. Now, by Exercise 7.2, a module M_R is singular iff $M \cong A/B$ for some $B \subseteq_e A$. This shows that M_R being singular is a categorical property. Next, recall from Exercise 7.4 that a module N_R is nonsingular iff $\text{Hom}_R(M, N) = 0$ for any singular M_R. Therefore, N_R being nonsingular is also a categorical property.

Ex. 18.3. Characterize right nonsingular rings R by the property that every right projective R-module is nonsingular, and deduce that R being right nonsingular is a Morita invariant property.

Solution. First we check the stated characterization of a right nonsingular ring. If R is right nonsingular, then by Exercise 7.11, any free right R-module is nonsingular, and so any projective right R-module is also nonsingular. Conversely, if any projective right R-module is nonsingular, then R_R is nonsingular, so R is a right nonsingular ring.

Since $M_R \in \mathfrak{M}_R$ being a nonsingular (resp. projective) module is a categorical property by Exercise 2, the characterization for R to be a right nonsingular ring given above can be stated in terms of the category $\mathfrak{M}_R$ alone. Therefore, R being a right nonsingular ring is a Morita invariant property.

Ex. 18.4. Show that R being semiperfect (resp. right perfect) is a Morita invariant property.

Solution. First, $N \subseteq_s M$ (N_R being *superfluous* in M_R, meaning that $N + X = M \Rightarrow X = M$) is clearly a categorical property. Therefore, $\theta : P \twoheadrightarrow M$ being a *projective cover* for M_R (meaning that P is projective and $\ker(\theta) \subseteq_s P$) is also a categorical property. Now, by *FC*-(24.18), R is *right perfect* iff every $M \in \mathfrak{M}_R$ has a projective cover. Since the latter is a categorical property, it follows that R being right perfect is a Morita invariant property.

Next, by *FC*-(24.16), R is *semiperfect* iff every f.g. $M \in \mathfrak{M}_R$ has a projective cover. Since "f.g." is also a categorical property (by Exercise 1, or, more simply, by *LMR*-(18.2)), it follows that R being semiperfect is also a Morita invariant property.

Ex. 18.5. Show that *finiteness* of a ring is a Morita invariant property without using *LMR*-(18.35).

Solution. The finiteness of a ring R can be characterized by the property that, for any f.g. module $M \in \mathfrak{M}_R$, $\text{End}_R(M)$ is finite. Since this is a categorical property of $\mathfrak{M}_R$, it follows that R being finite is a Morita invariant property.

Ex. 18.6. Show that "u.dim $R_R < \infty$" is a Morita invariant property.

Solution. For $M \in \mathfrak{M}_R$, u.dim(M) can be characterized as the supremum ($\leq \infty$) of the integers n for which there exists a direct sum of n nonzero submodules in M. Therefore, u.dim(M) is an invariant of M as an object in the category $\mathfrak{M}_R$. Now u.dim$(R_R) < \infty$ iff u.dim$(P) < \infty$ for every f.g. projective R-module in $\mathfrak{M}_R$. The latter being a categorical property of $\mathfrak{M}_R$, we see that "u.dim$(R_R) < \infty$" is a Morita invariant property.

Ex. 18.7A. Show that the property of R being right self-injective or quasi-Frobenius can be characterized by suitable categorical properties of $\mathfrak{M}_R$. Deduce that "right self-injective" and "QF" are Morita invariant properties of rings.

Solution. We claim that *R is right self-injective iff every f.g. projective $M \in \mathfrak{M}_R$ is injective.* The "if" part is clear. For the "only if" part, assume R_R is injective. If $P \in \mathfrak{M}_R$ is f.g. projective, then $P \oplus Q \cong R_R^n$ for some $Q \in \mathfrak{M}_R$ and some integer n. Since $R_R^n = R_R \oplus \cdots \oplus R_R$ is injective, so is its direct summand P, as desired. This proves our claim, which shows that R being right self-injective is a Morita invariant property.

A *quasi-Frobenius* (or QF) *ring* is a ring R that is both right noetherian and right self-injective. Since each of these is a Morita invariant property, it follows that R being QF is also a Morita invariant property.

Comment. Although "right self-injective" is a Morita invariant property, in general R being right self-injective does not imply that eRe is, for an idempotent $e \in R$. For an explicit counterexample, see the paper of Pascaud and Valette referenced in the *Comment* on Ex. 16.32.

The fact that, for a fixed n, R is right self-injective iff $S = \mathbb{M}_n(R)$ is can be seen directly from the fact that $M \mapsto ME_{11}$ gives a category equivalence from $\mathfrak{M}_S$ to $\mathfrak{M}_R$. Under this equivalence, $S_S \mapsto R_R^n$, so S_S is injective iff R_R is injective.

Ex. 18.7B. (1) Show that the basic ring of a QF ring R is always a Frobenius ring.

(2) Let k be a division ring, and R be the ring of matrices

$$\begin{pmatrix} a & b & p & 0 & 0 & 0 \\ c & d & q & 0 & 0 & 0 \\ 0 & 0 & r & 0 & 0 & 0 \\ 0 & 0 & 0 & r & s & t \\ 0 & 0 & 0 & 0 & a & b \\ 0 & 0 & 0 & 0 & c & d \end{pmatrix}$$

over k. This is shown to be a QF ring in *LMR*-(16.19)(5). Compute the basic ring of R.

(3) Show that being a Frobenius ring is *not* a Morita invariant property.

Solution. (1) Let $e_1R, \ldots, e_sR$ be a complete set of right principal indecomposables, where the e_i's are mutually orthogonal primitive idempotents. Then $e = e_1 + \cdots + e_s$ is a basic idempotent, and we can take the basic ring of R to be $S := eRe$. Since $S \approx R$, the last exercise guarantees that S is QF. Clearly, the e_i's remain pairwise nonisomorphic in S, so each e_iS appears with multiplicity 1 in the Krull-Schmidt decomposition of S_S. According to *LMR*-(16.14), this implies that R is a Frobenius ring.

(2) We use here the notations in *LMR*-(16.19)(4). In particular, the full set of right principal indecomposables for R is $\{e_1R, e_2R\}$ where $e_1 = E_{33}+E_{44}$ and $e_2 = E_{11} + E_{55}$. Therefore, a basic idempotent for R is

$$e = e_1 + e_2 = E_{11} + E_{33} + E_{44} + E_{55}.$$

By direct matrix multiplication, we see that the basic ring eRe consists of matrices of the form

$$\begin{pmatrix} a & 0 & p & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & r & 0 & 0 & 0 \\ 0 & 0 & 0 & r & s & 0 \\ 0 & 0 & 0 & 0 & a & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

After dropping the 2nd and 6th rows and columns, we can represent eRe as the ring of 4×4 matrices

$$\begin{pmatrix} a & p & 0 & 0 \\ 0 & r & 0 & 0 \\ 0 & 0 & r & s \\ 0 & 0 & 0 & a \end{pmatrix}.$$

This is exactly the ring in *LMR*-(16.9)(4), which was shown there to be Frobenius (confirming (1) above).

(3) Since eRe is the basic ring of R, we have $R \approx eRe$. While eRe is Frobenius, we know from *LMR*-(16.19)(5) that R is *not* Frobenius. Therefore, being a Frobenius ring is not a Morita invariant property.

Ex. 18.7C. Show that being a symmetric algebra over a field k is a Morita invariant property.

Solution. Let R be a symmetric k-algebra. By *LMR*-(18.35), it suffices to check that $\mathbb{M}_n(R)$ is a symmetric k-algebra for every n, and that eRe is a symmetric k-algebra for every full idempotent $e \in R$. The former follows from Exercise 16.20, and the latter follows from Exercise 16.25 (even for *any* idempotent $e \in R$).

Ex. 18.8. Show that a necessary and sufficient condition for a ring-theoretic property $\mathcal{P}$ to be Morita invariant is that, for any full idempotent e in a ring R, R satisfies $\mathcal{P}$ iff eRe does.

Solution. Recall *LMR*-(18.35), which states that $\mathcal{P}$ is Morita invariant iff, whenever a ring R satisfies $\mathcal{P}$, so do eRe (for any full idempotent $e \in R$) and $\mathbb{M}_n(R)$ (for any $n \geq 2$). The new condition given in the present exercise is a slight variation of this.

Suppose the property $\mathcal{P}$ is Morita invariant. Let R be a ring and $e \in R$ be a full idempotent. Since $R \approx eRe$, R satisfies $\mathcal{P}$ iff eRe does. Conversely, suppose the property $\mathcal{P}$ is such that, for any full idempotent e in a ring S, S satisfies $\mathcal{P}$ iff eSe does. To check that $\mathcal{P}$ is Morita invariant, we apply the criterion *LMR*-(18.35) recalled in the paragraph above. Let R be a ring satisfying $\mathcal{P}$. By assumption, eRe also satisfies $\mathcal{P}$ for any full idempotent $e \in R$. Now consider $S = \mathbb{M}_n(R)$ for any $n \geq 2$. Let e be the matrix unit E_{11}. Since $(aE_{i1})E_{11}(E_{1j}) = aE_{ij}$ for any $a \in R$, we have $SeS = S$, so e is a full idempotent in S. Now

$$eSe = \{aE_{11} : a \in R\} \cong R \quad \text{(as rings)}.$$

Therefore, eSe satisfies $\mathcal{P}$. By assumption, $S(= \mathbb{M}_n(R))$ also satisfies $\mathcal{P}$. Thus, *LMR*-(18.35) applies, and we conclude that $\mathcal{P}$ is a Morita invariant property.

Ex. 18.9. Use *LMR*-(18.35) (instead of *LMR*-(18.45) and *LMR*-(18.50)) to show that semiprimitivity, primeness and semiprimeness are Morita invariant properties.

Solution. Let $\mathcal{P}$ be the property of being semiprimitive, prime, or semiprime. By *FC*-p. 61 and *FC*-p. 172, if R satisfies $\mathcal{P}$, so does $\mathbb{M}_n(R)$. And by *FC*-p. 322, if R satisfies $\mathcal{P}$, so does eRe for *any* (nonzero) idempotent $e \in R$. Therefore, by *LMR*-(18.35), $\mathcal{P}$ is a Morita invariant property.

Ex. 18.10. Let $S = \text{End}(P_R)$ where P_R is a progenerator over the ring R. Show that the ring S has IBN iff $P^n \cong P^m$ in $\mathfrak{M}_R$ implies $n = m$.

Solution. Since P_R is a progenerator, we have a category equivalence $\mathfrak{M}_S \to \mathfrak{M}_R$ which, on the level of objects, is defined by $-\otimes_S P$. In particular, the right regular module S_S in $\mathfrak{M}_S$ corresponds to $S \otimes_S P \cong P_R$ in $\mathfrak{M}_R$. Therefore, the IBN property for the ring S translates into $P^n \cong P^m$ (in $\mathfrak{M}_R$) $\Rightarrow n = m$.

Ex. 18.11. (Bergman) For any ring R, let $\mathcal{P}(R)$ be the monoid of isomorphism classes of f.g. projective right R-modules (under the direct sum operation). Using the technique of coproducts (see *Comment* below), it can be shown that there exists a ring R for which $\mathcal{P}(R)$ is generated as a monoid by $[R]$ together with $[M]$, $[N]$, with the defining relations

$$[M] + [N] = [R] = [R] + [R].$$

Show that $S = \text{End}_R(M \oplus R)$ is Morita equivalent to R and has IBN, but R does not have IBN. (This shows that IBN is *not* a Morita invariant property.)

Solution. Since M_R is f.g. projective, $P := M \oplus R$ is a progenerator. Therefore, $S = \text{End}_R(M \oplus R) \approx R$. Since $\mathcal{P}(R)$ has the defining relations $[M] + [N] = [R] = [R] + [R]$, there exists a monoid homomorphism $\varphi : \mathcal{P}(R) \to \mathbb{Z}$ defined by

$$\varphi[R] = 0, \quad \varphi[M] = 1 \quad \text{and} \quad \varphi[N] = -1.$$

Then, $\varphi[P] = \varphi[M] + \varphi[R] = 1$, so $P^n \cong P^m$ will imply that $n = m$. By Exercise 10, this guarantees that S has IBN. On the other hand, we have $R \oplus R \cong R$ in $\mathfrak{M}_R$, so R does not have IBN.

Comment. Bergman's work referred to above is: "Coproducts and some universal ring constructions," Trans. Amer. Math. Soc. **200**(1974), 33–88.

Ex. 18.12. Suppose a ring-theoretic property $\mathcal{P}$ is such that, for any $n \geq 2$ and any ring R, R satisfies $\mathcal{P}$ iff $\mathbb{M}_n(R)$ does. Is $\mathcal{P}$ necessarily a Morita invariant property?

Solution. The answer is "no." In fact, let $\mathcal{P}$ be the "IBN" property. By Exercise 17.1, R satisfies $\mathcal{P}$ iff $\mathbb{M}_n(R)$ does (for a given n). By the exercise above, $\mathcal{P}$ is *not* a Morita invariant property.

Ex. 18.13. Show that a projective module P_R is a generator iff every simple module M_R is an epimorphic image of P.

Solution. Assume P_R is a generator. By *LMR*-(18.8), there exists a surjection $\bigoplus_i P_i \to M$ where each $P_i = P$. Clearly some $P_i \to M$ is nonzero, so this is the surjection we want. Conversely, suppose every simple right R-module is an epimorphic image of P. *We claim that* $\text{tr}(P)$ (the trace ideal of P_R) *is equal to* R. Once this is proved, *LMR*-(18.8) implies that P is a generator. Assume, instead, that $\text{tr}(P) \neq R$. Then there exists a maximal right ideal $\mathfrak{m} \supseteq \text{tr}(P)$. By assumption, there exists a surjection $\theta : P \twoheadrightarrow R/\mathfrak{m}$. Since P_R is projective, θ "lifts" to a homomorphism $\varphi : P \to R$ (so that $\pi\varphi = \theta$ for the projection map $\pi : R \to R/\mathfrak{m}$). But $\varphi(P) \subseteq \text{tr}(P) \subseteq \mathfrak{m}$ implies that $\pi\varphi = 0$, a contradiction.

Comment. This result is probably folklore in the subject. The earliest reference I can find for it is T. Kato's paper, "Self-injective rings," Tôhoku Math. J. **19**(1967), 485–495.

The notion of a cogenerator, dual to that of a generator, is introduced in *LMR*-§19. The above exercise has a well-known dual version: *An injective module U is a cogenerator iff every simple module embeds into U*: see *LMR*-(19.9).

Ex. 18.14. Find the flaw in the following argument: "Let R, S be division rings. Construct functors $F : \mathfrak{M}_R \to \mathfrak{M}_S$ and $G : \mathfrak{M}_S \to \mathfrak{M}_R$ by: $F(U_R) = \bigoplus_{i \in I} S_S$ where $|I| = \dim_R U$, and $G(V_S) = \bigoplus_{j \in J} R_R$ where $|J| = \dim_S V$. Then F, G are inverse category equivalences and hence $R \approx S$."

Solution. Defining a functor $F : \mathfrak{M}_R \to \mathfrak{M}_S$ involves defining F on *objects* of $\mathfrak{M}_R$ and on *morphisms* of $\mathfrak{M}_R$. The rule $F(U_R) = \bigoplus_{i\in I} S_S$ given above defines F on the objects of $\mathfrak{M}_R$, but no definition is given for F on the morphisms of F. Therefore, F is not a functor, nor is G. Since no functor is given from $\mathfrak{M}_R$ to $\mathfrak{M}_S$ (or from $\mathfrak{M}_S$ to $\mathfrak{M}_R$), we are in no position to conclude that $\mathfrak{M}_R$ and $\mathfrak{M}_S$ are equivalent.

Indeed, we know that, if R and S are nonisomorphic division rings, then the module categories $\mathfrak{M}_R$ and $\mathfrak{M}_S$ *cannot* be equivalent.

Ex. 18.15. Let $e = e^2 \in R$, and let

$$\alpha : \ Re \otimes_{eRe} eR \to R, \qquad \beta : \ eR \otimes_R Re \to eRe$$

be defined by $\alpha(re \otimes er') = rer'$, $\beta(er \otimes r'e) = err'e$ (as in *LMR*-(18.30A)).

(1) Give a direct proof for the fact that β is an isomorphism by explicitly constructing an inverse for β.
(2) Show that $eR \cdot (\ker \alpha) = 0$. Using this, give another proof for the fact that α is an isomorphism if e is a full idempotent.

Solution. (1) Define $\gamma : eRe \to eR \otimes_R Re$ by

$$\gamma\,(ere) = er \otimes_R e = e \otimes_R re \ \in \ eR \otimes_R Re.$$

Note that if $ere = 0$, then $er \otimes_R e = er \otimes_R ee = ere \otimes_R e = 0$, so γ is well-defined. We have

$$\begin{aligned} \gamma\beta\,(er \otimes_R r'e) &= \gamma(err'e) = err' \otimes_R e = er \otimes_R r'e, \\ \beta\gamma(ere) &= \beta(er \otimes_R e) = ere, \end{aligned}$$

so β, γ are mutually inverse isomorphisms.

(2) Note that the domain of α, $Re \otimes_{eRe} eR$, is an (R, R)-bimodule, so the equation $(eR) \cdot (\ker \alpha) = 0$ makes sense. To check that this equation holds, we take $x = \sum_i r_i e \otimes er'_i \in \ker(\alpha)$. Then $0 = \alpha(x) = \sum_i r_i er'_i$, and so for any $r \in R$:

$$\begin{aligned} er \cdot x &= er \cdot \sum\nolimits_i r_i e \otimes_{eRe} \ er'_i = \sum\nolimits_i e\,(err_i e) \otimes_{eRe} \ er'_i \\ &= e \otimes_{eRe} \sum\nolimits_i err_i er'_i = 0, \end{aligned}$$

as desired.

Now assume e is a *full* idempotent. Then there is an equation $1 = \sum_j a_j eb_j$, for suitable a_j, $b_j \in R$. We have

$$\alpha\left(\sum\nolimits_j a_j e \otimes eb_j\right) = \sum\nolimits_j a_j eb_j = 1,$$

so α is onto. Also $eR \cdot (\ker \alpha) = 0$ implies that

$$\ker\ \alpha = (ReR)\,(\ker\ \alpha) = R(eR \cdot \ker\ \alpha) = 0,$$

so α is also one-one. Thus, we have shown that α is an isomorphism.

Ex. 18.16. (Partial refinement of *LMR*-(18.22)) Let P_R be a right R-module, with associated Morita context $(R,\ P,\ Q,\ S;\ \alpha,\ \beta)$.

(1) If P_R is a generator, show that ${}_SP$ is f.g. projective.
(2) If P_R is f.g. projective, show that ${}_SP$ is a generator.

Solution. (1) Say $P^n \cong R \oplus M$ in $\mathfrak{M}_R$. Applying the functor

$$\Phi = \mathrm{Hom}_R(-,\ {}_SP_R) : \mathfrak{M}_R \to {}_S\mathfrak{M},$$

and noting that $\Phi(P^n) \cong {}_SS^n$, $\Phi(R) \cong {}_SP$, we get an isomorphism

$${}_SS^n \cong {}_SP \oplus X \ \text{ in } \ {}_S\mathfrak{M}$$

where $X = \Phi(M)$. Therefore, ${}_SP$ is f.g. projective in ${}_S\mathfrak{M}$.

(2) Assume now P_R is f.g. projective, so that $R_R^n \cong P \oplus N$ for some N_R. Applying the same functor Φ above, and noting that $\Phi(R_R^n) \cong {}_SP^n$, $\Phi(P) \cong {}_SS$, we get an isomorphism

$${}_SP^n \cong {}_SS \oplus Y \ \text{ in } \ {}_S\mathfrak{M}$$

where $Y = \Phi(N)$. Therefore, ${}_SP$ is a generator in ${}_S\mathfrak{M}$.

Comment. It is possible to apply "abstract nonsense" to get some of the conclusions above. For instance, assume that P_R is a generator, as in (1). From the given Morita context $(R,\ P,\ Q,\ S;\ \alpha,\ \beta)$ of P_R, we can determine the Morita context of ${}_SP$. By *LMR*-(18.17) (2), $\mathrm{End}({}_SP) \cong R$ and $\mathrm{Hom}_S({}_SP,\ {}_SS) \cong Q$. Therefore, the Morita context of ${}_SP$ is $(S,\ P,\ Q,\ R;\ \beta,\ \alpha)$. Since P_R is a generator, $\alpha : Q \otimes_S P \to R$ is *onto* by *LMR*-(18.17) (1). Applying this fact to the Morita context of ${}_SP$, we see by *LMR*-(18.19) (1) that ${}_SP$ is f.g. projective. This method actually yields a little more in the case of (1). In fact, applying *LMR*-(18.17)(2) again, we can likewise determine the Morita context of Q_S to be $(S,\ Q,\ P,\ R;\ \beta,\ \alpha)$. Since α is onto (as above), *LMR*-(18.19)(1) implies that Q_S is f.g. projective. This method, however, does not apply so well in the case (2).

Ex. 18.17. (Morita) Show that P_R is a generator iff, for $S = \mathrm{End}(P_R)$, ${}_SP$ is f.g. projective and the natural map $\lambda : R \to \mathrm{End}({}_SP)$ is an isomorphism.

Solution. The "only if" part follows from (1) of the above exercise and the *Comment* made thereafter. Conversely, assume ${}_SP$ is f.g. projective and λ is an isomorphism. Using λ, we identify $\mathrm{End}({}_SP)$ with R. Since ${}_SP$ is f.g. projective with endomorphism ring R, the left module analogue of (2) of the above exercise implies that P_R is a generator.

Comment. The above characterization of a generator module P_R appeared already in Morita's famous paper, "Duality for modules and its applications to the theory of rings with minimum condition," Science Reports, Tokyo Kyoiku Daigaku Sec. A (1958), 83–142. See also Theorem 2 in Carl Faith's paper "A general Wedderburn Theorem," Bull. Amer. Math. Soc. **73**(1967), 65–67. Faith made the observation that, if R is a simple ring and $P \subseteq R$ is a nonzero right ideal, then (by *LMR*-(18.9)(E)) P_R is a generator, and hence by the above result, the map $\lambda : R \to \text{End}({}_SP)$ is an isomorphism for $S = \text{End}(P_R)$. This was a result of M. Rieffel (see *FC*-(3.11)), published in his paper with the same title as Faith's, in Proc. Nat'l Acad. Sci. USA **54**(1965), 1513. Refining Rieffel's result, Faith showed moreover that, in this context, S is a simple ring iff the right ideal P_R is a f.g. projective R-module.

For an interesting account on the circumstances surrounding the writing and publication of Faith's paper in Proc. Nat'l. Acad. Sci. USA, see the section on "Marc Rieffel, Serge Lang, Steve Smale and me" (pp. 319–320) in the "Snapshots" chapter of Faith's recent book "Rings and Things and a Fine Array of Twentieth Century Associative Algebra," 2nd Ed., Math. Monographs and Surveys, Vol. 65, Amer. Math. Soc., 2004.

Ex. 18.18. Let R, S be rings, and ${}_SP_R$, ${}_RQ_S$ be bimodules. Let $\alpha : Q \otimes_S P \to R$ be an (R, R)-homomorphism, and $\beta : P \otimes_R Q \to S$ be an (S, S)-homomorphism. Define

$$pq = \beta(p \otimes q) \in S \quad \text{and} \quad qp = \alpha(q \otimes p) \in R,$$

and let $M = \begin{pmatrix} R & Q \\ P & S \end{pmatrix}$ (formally). Show that M is a ring under formal matrix multiplication (as in *LMR*-(18.16)) iff

$$q'(pq) = (q'p)q \quad \text{and} \quad p(qp') = (pq)p'$$

hold for all $p, p' \in P$ and $q, q' \in Q$. (Note that, in the special case where $P = 0$, the additional conditions are vacuous, and we get back the "triangular ring" construction $M = \begin{pmatrix} R & Q \\ 0 & S \end{pmatrix}$ in *FC*-(1.14).)

Solution. First assume M is a ring under formal matrix multiplication. Noting that

$$\begin{pmatrix} 0 & q' \\ 0 & 0 \end{pmatrix} \left[\begin{pmatrix} 0 & 0 \\ p & 0 \end{pmatrix} \begin{pmatrix} 0 & q \\ 0 & 0 \end{pmatrix} \right] = \begin{pmatrix} 0 & q' \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & pq \end{pmatrix} = \begin{pmatrix} 0 & q'(pq) \\ 0 & 0 \end{pmatrix},$$

$$\left[\begin{pmatrix} 0 & q' \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ p & 0 \end{pmatrix} \right] \begin{pmatrix} 0 & q \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} q'p & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & q \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & (q'p)q \\ 0 & 0 \end{pmatrix},$$

we see that $q'(pq) = (q'p)q$, and a similar calculation with the matrices $\begin{pmatrix} 0 & 0 \\ p & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & q \\ 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & 0 \\ p' & 0 \end{pmatrix}$ shows that $p\,(qp') = (pq)\,p'$. Conversely, if $q'(pq) = (q'p)q$ and $p(qp') = (pq)p'$ for all $p, p' \in P$ and $q, q' \in Q$, a direct formal matrix multiplication shows that $A(BC) = (AB)C$ for any three matrices A, B, $C \in M$. The distributive laws are also easy to verify, so M is indeed a ring (called the *Morita ring* associated with $\{P, Q\,;\, \alpha, \beta\}$).

Ex. 18.19. Suppose the "associativity" conditions in the above exercise are satisfied, so M is a ring. Let $e = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $f = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$. Show that $R = eMe$, $S = fMf$, $P = fMe$ and $Q = eMf$.

Solution. We have

$$eMe = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} r & q \\ p & s \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} r & q \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} r & 0 \\ 0 & 0 \end{pmatrix},$$

so $eMe = R$. (Here, we identify R with $\begin{pmatrix} R & 0 \\ 0 & 0 \end{pmatrix} \subseteq M$.) Similarly, we have $fMf = S$. Also,

$$eMf = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} r & q \\ p & s \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} r & q \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & q \\ 0 & 0 \end{pmatrix},$$

so $eMf = Q$, and similarly, $fMe = P$.

Comment. In view of the above, the "Peirce Decomposition"

$$M = eMe \oplus fMf \oplus eMf \oplus fMe$$

is just the decomposition of $M = \begin{pmatrix} R & Q \\ P & S \end{pmatrix}$ into the four pieces: $R \oplus S \oplus Q \oplus P$.

Ex. 18.20. Let M be a ring with idempotents e, f such that $e + f = 1$. Let

$$R = eMe, \quad S = fMf, \quad P = fMe \quad \text{and} \quad Q = eMf.$$

Show that the natural maps $\alpha : Q \otimes_S P \to R$, and $\beta : P \otimes_R Q \to S$ satisfy the "associativity" conditions in Exercise 18, and that the formal Morita ring $\begin{pmatrix} R & Q \\ P & S \end{pmatrix}$ constructed there is isomorphic to the original ring M.

Solution. Here, α, β are defined by

$$\alpha\,(emf \otimes fm'e) = e\,(mfm')\,e \quad \text{and} \quad \beta\,(fme \otimes em'f) = f\,(mem')\,f$$

for any m, $m' \in M$. To see that α is well-defined, note that

$$\alpha\,(emf \otimes (fxf)\,(fm'e)) = \alpha\,(emf \otimes f\,(xfm')\,e) = e\,(mfxfm')\,e,$$
$$\alpha\,((emf)\,(fxf) \otimes (fm'e)) = \alpha\,(e\,(mfx)\,f \otimes fm'e) = e\,(mfxfm')\,e.$$

Similarly, we can check that β is well-defined. In fact, the "products" $Q \times P \to R$ and $P \times Q \to S$ induced by α and β are exactly the products between the elements of P and Q in M. Therefore, the associativity conditions in Exercise 18 are automatic, and α is an (R, R)-homomorphism, β is an (S, S)-homomorphism. According to Exercise 18, we can form a Morita ring $\tilde{M} = \begin{pmatrix} R & Q \\ P & S \end{pmatrix}$.

We claim that $\varphi\colon \tilde{M} \to M$ defined by $\varphi\begin{pmatrix} r & q \\ p & s \end{pmatrix} = r+s+p+q$ is a ring isomorphism. By the Peirce Decomposition Theorem, we know that φ is an additive group isomorphism, so it remains only to check that φ respects multiplication.

Since $ef = fe = 0$, we have $RS = RP = 0$, $SR = SQ = 0$, $PP = PS = 0$ and $QQ = QR = 0$. Therefore,

$$\begin{aligned} \varphi\begin{pmatrix} r & q \\ p & s \end{pmatrix} \varphi\begin{pmatrix} r' & q' \\ p' & s' \end{pmatrix} &= (r+s+p+q)\,(r'+s'+p'+q') \\ &= (rr'+rq') + (ss'+sp') + (pr'+pq') + (qs'+qp') \\ &= (rr'+qp') + (pq'+ss') + (pr'+sp') + (rq'+qs') \\ &= \varphi\left(\begin{pmatrix} r & q \\ p & s \end{pmatrix}\begin{pmatrix} r' & q' \\ p' & s' \end{pmatrix}\right), \end{aligned}$$

as desired.

Ex. 18.21. Suppose, in Exercise 18, the associativity conditions are satisfied, and that α, β are both *onto*. Show that

(1) α, β are isomorphisms;
(2) P_R is a progenerator; and
(3) The Morita Context for P_R is $(R,\ P,\ Q,\ S;\ \alpha,\ \beta)$.

Solution. (1) Since α is onto, there exists an equation $1_R = \sum_j q_j p_j$ where $p_j \in P$, $q_j \in Q$. Suppose $x = \sum_i b_i \otimes_S a_i \in \ker(\alpha)$ $(a_i \in P,\ b_i \in Q)$; that is, $0 = \alpha(x) = \sum_i b_i a_i$. Then

$$x = \left(\sum\nolimits_i b_i \otimes_S a_i\right) 1_R = \sum\nolimits_{i,j} b_i \otimes_S (a_i q_j p_j) = \sum\nolimits_j \sum\nolimits_i b_i a_i q_j \otimes_S p_j = 0.$$

This shows that α is one-one, so it is an isomorphism. Similarly, we can show that if β is onto, it is an isomorphism.

(2) Write $1_R = \sum_j q_j p_j$ as above, and similarly, write $1_S = \sum_k p'_k q'_k$. Define $\lambda : Q \to P^* = \mathrm{Hom}_R(P, R)$ by $\lambda(q)(p) = qp$. Then λ *is one-one.* In fact, if $\lambda(q) = 0$, then

$$q = q \cdot 1_S = q\sum\nolimits_k p'_k q'_k = \sum\nolimits_k (qp'_k)q'k = 0.$$

We claim that λ *is also onto.* To see this, let $f \in \mathrm{Hom}_R(P, R)$. For $q := \sum f(p'_k)q'_k \in Q$, we have

$$\lambda(q)\,(p) = \sum\nolimits_k f(p'_k)\,(q'_k p) = f\left(\sum\nolimits_k p'_k q'_k p\right) = f(p) \quad (\forall p \in P),$$

so $f = \lambda(q)$. Therefore, we have $Q \cong P^*$. For any $p \in P$, we have

$$p = 1_S\, p = \sum\nolimits_k p'_k\,(q'_k p) = \sum\nolimits_k p'_k\ f_k(p),$$

where $f_k := \lambda(q'_k) \in P^*$. Applying the Dual Basis Lemma, we see that P_R is f.g. projective. Also, $\mathrm{tr}(P)$ contains $\lambda(q_j)p_j = q_j{\cdot}p_j$ for each j, so it contains $\sum_j q_j p_j = 1_R$. Thus, $\text{tr}(P) = R$, so by *LMR*-(18.8), P is a generator in $\mathfrak{M}_R$. This proves (2).

(3) By Exercise 2.20(4), $\text{End}_R(P) \cong P \otimes_R P^*$. Identifying P^* with Q and using the isomorphism $\beta : P \otimes_R Q \to S$, we see that $\text{End}_R(P) \cong S$. It follows readily that the Morita Context associated with the progenerator P_R is precisely $(R, P, Q, S; \alpha, \beta)$.

Ex. 18.22. Using the category equivalence between $\mathfrak{M}_S$ and $\mathfrak{M}_R$ where $R = \mathbb{M}_n(S)$, show that there is an isomorphism from the lattice of right ideals of R to the lattice of S-submodules of the free module S^n_S. From this, deduce that, if two rings S, T are such that $\mathbb{M}_n(S) \cong \mathbb{M}_m(T)$, then the free modules S^n_S and T^m_T have isomorphic lattices of submodules.

Solution. Recall that a category equivalence from $\mathfrak{M}_R$ to $\mathfrak{M}_S$ is given (on objects) by $F(U) = U \cdot E_{11}$ where $U = U_R$ and E_{11} is the matrix unit with 1 in the (1, 1)-position and 0's elsewhere. Under this category equivalence, the free module R_R corresponds to $F(R_R) = R \cdot E_{11}$, which can be identified with the space S^n_S of column n-tuples over S. A right ideal $\mathfrak{A} \subseteq R$ then corresponds to

$$\mathfrak{A} \cdot E_{11} = (\mathfrak{A} R) E_{11} = \mathfrak{A} (R E_{11}) = \mathfrak{A} S^n_S,$$

which is an S-submodule of S^n_S. Since F is a category equivalence, $\mathfrak{A} \mapsto \mathfrak{A} S^n_S$ gives the desired isomorphism from the lattice of right ideals of R to the lattice of S-submodules of S^n_S.

For the last part of the exercise, let $R = \mathbb{M}_n(S)$ and $A = \mathbb{M}_m(T)$. If $R \cong A$, then R and A have isomorphic lattices of right ideals. Using the first part of the exercise, we see that S^n_S and T^m_T have isomorphic lattices of submodules.

Comment. There is a converse to the last part of the above exercise. If $n \geq 3$, a theorem of J. von Neumann (from his book in Continuous Geometry) implies that, if the free modules S^n_S and T^m_T have isomorphic submodule lattices, then in fact, one has a ring isomorphism $\mathbb{M}_n(S) \cong \mathbb{M}_m(T)$. J. von Neumann's result was later generalized by W. Stephenson, who proved that, if P_S, Q_T are two modules (over S and T) with isomorphic submodule lattices, and P has the form of a direct sum $X^{(I)}$ where $|I| \geq 3$ (for instance, $P = S^n_S$ with $n \geq 3$), then there is a ring isomorphism $\text{End}_S P \cong \text{End}_T Q$; see his paper "Lattice isomorphisms between modules, (1) Endomorphism rings," J. London Math. Soc. **1** (1969), 177–183.

Ex. 18.23. Show that, under the ideal correspondence between S and $\mathbb{M}_n(S)$ established in *LMR*-(18.44), an ideal $\mathfrak{B} \subseteq S$ corresponds to the ideal $\mathbb{M}_n(\mathfrak{B}) \subseteq \mathbb{M}_n(S)$. Deduce from *LMR*-(18.50) that

$$\text{rad } \mathbb{M}_n(S) = \mathbb{M}_n(\text{rad } S).$$

Solution. As in the last exercise, let $R = \mathbb{M}_n(S)$. The category equivalence $G : \mathfrak{M}_S \to \mathfrak{M}_R$ is given (on objects) by

$$G(V) = (V, \ldots, V) \quad \text{for any} \quad V = V_S,$$

where $(V, \ldots, V)$ is viewed as a right R-module with the obvious matrix action. In the standard notation of Morita equivalence, G is given by $-\otimes_S P$ where P is the (S, R)-bimodule $(S, \ldots, S)$. By *LMR*-(18.44), an ideal $\mathfrak{B} \subseteq S$ and an ideal $\mathfrak{A} \subseteq R$ correspond under the canonical ideal correspondence iff $\mathfrak{B}P = P\mathfrak{A}$. Since this equation obviously holds for the choice $\mathfrak{A} = \mathbb{M}_n(\mathfrak{B})$, we conclude that the one-one correspondence is given by $\mathfrak{B} \leftrightarrow \mathbb{M}_n(\mathfrak{B})$. (This implies, in particular, that any ideal in $\mathbb{M}_n(S)$ is of the form $\mathbb{M}_n(\mathfrak{B})$, for a unique ideal $\mathfrak{B} \subseteq S$.)

According to *LMR*-(18.50), under a category equivalence of $\mathfrak{M}_S$ and $\mathfrak{M}_R$, the Jacobson radicals of S and R correspond to each other under the canonical ideal correspondence. In our special case, therefore, rad S corresponds to rad $\mathbb{M}_n(S)$. But by the above work, we know that rad S corresponds to $\mathbb{M}_n(\text{rad } S)$. It follows that rad $\mathbb{M}_n(S) = \mathbb{M}_n(\text{rad } S)$.

Comment. A direct proof for the equation rad $\mathbb{M}_n(S) = \mathbb{M}_n(\text{rad } S)$ (without using category theory) is available in *FC*-p. 61.

Ex. 18.24. Let e be a full idempotent in a ring R, so that we have $R \approx eRe$. Show that the ideal correspondence between R and eRe is as given in *FC*-(21.11); that is, $\mathfrak{A} \subseteq R$ goes to $e\mathfrak{A}e \subseteq eRe$, and $\mathfrak{B} \subseteq eRe$ goes to $R\mathfrak{B}R \subseteq R$.

Solution. Let $S = eRe$. The category equivalence $\mathfrak{M}_S \to \mathfrak{M}_R$ is given (on objects) by $M_S \mapsto M \otimes_S P$ where $P = eR$ is viewed as an (S, R) bimodule. By *LMR*-(18.44), an ideal $\mathfrak{B} \subseteq S$ and an ideal $\mathfrak{A} \subseteq R$ correspond under the ideal correspondence iff $\mathfrak{B}P = P\mathfrak{A}$.

(1) For any ideal $\mathfrak{B}$ of S, we have

$$P \cdot R\mathfrak{B}R = eR\mathfrak{B}R = (eRe)\mathfrak{B}R = \mathfrak{B}R,$$

so $\mathfrak{B} \subseteq S$ corresponds to $R\mathfrak{B}R \subseteq R$.

(2) For any ideal $\mathfrak{A}$ of R, we have

$$e\mathfrak{A}e \cdot P = e\mathfrak{A}eR = e\mathfrak{A}(ReR) = e\mathfrak{A} = P\mathfrak{A},$$

so $\mathfrak{A} \subseteq R$ corresponds to $e\mathfrak{A}e$. (We have to know, of course, that $e\mathfrak{A}e$ is an ideal in S. This is straightforward.)

Comment. Of course, the first part of Exercise 23 is only a special case of this exercise. If we take $R = \mathbb{M}_n(S)$ and take e to be the full idempotent $E_{11} \in R$, then $eRe = S \cdot E_{11}$ can be identified with S as a ring. For an ideal $\mathfrak{B} \subseteq S$, $R\,\mathfrak{B}R$ is just $\mathbb{M}_n(\mathfrak{B})$.

Ex. 18.25. Under a category equivalence $F : \mathfrak{M}_R \to \mathfrak{M}_S$, show that the ideal correspondence between R and S sends an ideal $\mathfrak{A} \subseteq R$ to the annihilator (in S) of the right S-module $F(R/\mathfrak{A})$.

Solution. Say $M_R = R/\mathfrak{A}$ corresponds to $N := F(R/\mathfrak{A})$. By *LMR*-(18.47), $\text{ann}_R(M) \subseteq R$ corresponds to $\text{ann}_S(N) \subseteq S$ under the ideal correspondence between R and S. Since

$$\text{ann}_R(M) = \text{ann}_R(R/\mathfrak{A}) = \mathfrak{A},$$

it follows that $\mathfrak{A}$ corresponds to $\text{ann}_S(F(R/\mathfrak{A}))$.

Ex. 18.26. *True or False*: "If $R \approx S$, then there is a one-one correspondence between the subrings of R and those of S"?

Solution. False, of course. For instance, with respect to the Morita equivalence

$$R = \mathbb{Z} \approx \mathbb{M}_n(\mathbb{Z}) = S \quad (n > 1),$$

R has only one subring (namely itself), while S has infinitely many subrings. (Of course, by "subring" here, we mean a subring containing the identity element of the ambient ring.)

Ex. 18.27. *True or False*: "If $R \approx S$, then there exist natural numbers n and m such that $\mathbb{M}_n(R) \cong \mathbb{M}_m(S)$?"

Solution. False again! For instance, let k be a division ring and let $R = k \times k$, $S = k \times \mathbb{M}_2(k)$. Since $k \approx \mathbb{M}_2(k)$, we have $R \approx S$. For n, $m \geq 1$, we see easily that

$$\mathbb{M}_n(R) \cong \mathbb{M}_n(k) \times \mathbb{M}_n(k), \quad \mathbb{M}_m(S) \cong \mathbb{M}_m(k) \times \mathbb{M}_{2m}(k).$$

These cannot be isomorphic, by the uniqueness part of Wedderburn's Theorem on semisimple rings.

Ex. 18.28. For any (right) R-module P, let $P^{(\mathbb{N})}$ denote the direct sum $P \oplus P \oplus \cdots$, and let $S = \text{End}(P_R)$. Show that, if P_R is f.g., then $\text{End}_R(P^{(\mathbb{N})}) \cong \text{CFM}(S)$, the ring of column-finite $\mathbb{N} \times \mathbb{N}$ matrices over S.

Solution. First, it is easy to see that $\text{CFM}(S)$ is indeed a well-defined ring. Write $P^{(\mathbb{N})}$ as $P_1 \oplus P_2 \oplus \cdots$ where each P_i is a copy of P. Let $\varepsilon_j : P_j \to P$ be the j^{th} inclusion map, and $\pi_i : P \to P_i$ be the i^{th} projection map. For any $f \in \text{End}_R(P^{(\mathbb{N})})$, let $f_{ij} = \pi_i f \varepsilon_j$. This is a homomorphism from P_j to P_i, so it is an element of $S = \text{End}_R(P)$. Since P_j is f.g., $f\varepsilon_j(P_j)$ lies in a finite direct sum $P_1 \oplus \cdots \oplus P_m$ for some $m = m(j)$. Therefore, $f_{ij} = \pi_i f \varepsilon_j = 0$ for any $i > m(j)$ and so $(f_{ij}) \in \text{CFM}(S)$. We claim that

$$\varphi : \text{End}_R(P^{(\mathbb{N})}) \longrightarrow \text{CFM}(S) \text{ defined by } \varphi(f) = (f_{ij})$$

is a ring homomorphism. Indeed, for f, $g \in \text{End}_R(P^{(\mathbb{N})})$, we have

$$(gf)_{ij} = \pi_i (gf) \varepsilon_j = \sum\nolimits_k (\pi_i g \varepsilon_k)(\pi_k f \varepsilon_j).$$

Here, the sum is *finite* since, for a fixed j, $\pi_k f \varepsilon_j = 0$ for almost all k. Therefore, $(gf)_{ij} = \sum_k g_{ik} f_{kj}$, which is the ij^{th} entry of $\varphi(g)\varphi(f)$. This shows that $\varphi(gf) = \varphi(g)\varphi(f)$, and it follows readily that φ is a ring isomorphism.

Ex. 18.29. Show that a f.g. right R-module P is a progenerator iff $P^{(\mathbb{N})} \cong R^{(\mathbb{N})}$ as right R-modules. Is this still true if P is not assumed f.g.?

Solution. First assume $P^{(\mathbb{N})} \cong R^{(\mathbb{N})}$. Then P is isomorphic to a direct summand of the free module $R^{(\mathbb{N})}$ so P is projective. Since R_R is isomorphic to a direct summand of $P^{(\mathbb{N})}$, P is also a generator. Therefore, P is a progenerator. Conversely, suppose P is a progenerator. Say

$$(1) \qquad R^n \cong P \oplus Q \quad \text{and} \quad P^m \cong R \oplus Q',$$

where Q, Q' are suitable right R-modules. From the first isomorphism in (1), we have (by "Eilenberg's Trick"):

$$(2) \quad R^{(\mathbb{N})} \cong (R^n)^{(\mathbb{N})} \cong P^{(\mathbb{N})} \oplus Q^{(\mathbb{N})} \cong P^{(\mathbb{N})} \oplus \left(P^{(\mathbb{N})} \oplus Q^{(\mathbb{N})}\right) \cong P^{(\mathbb{N})} \oplus R^{(\mathbb{N})}.$$

Similarly, from the second isomorphism in (1), we have

$$(3) \qquad P^{(\mathbb{N})} \cong R^{(\mathbb{N})} \oplus P^{(\mathbb{N})}.$$

Comparing (2) and (3), we see that $P^{(\mathbb{N})} \cong R^{(\mathbb{N})}$.

The assumption that P is f.g. is essential in the above. For instance, let $P := R^{(\mathbb{N})}$ over any nonzero ring R. Then certainly

$$P^{(\mathbb{N})} \cong \left(R^{(\mathbb{N})}\right)^{(\mathbb{N})} \cong R^{(\mathbb{N}\times\mathbb{N})} \cong R^{(\mathbb{N})}.$$

Indeed, P_R is projective; but P is not f.g., so P cannot be a progenerator.

Ex. 18.30. (Camillo) Show that two rings R and S are Morita equivalent iff $\mathrm{CFM}(R) \cong \mathrm{CFM}(S)$.

Solution. First assume $R \approx S$. Then by Morita Theory, we have $S \cong \mathrm{End}(P_R)$ where P_R is a suitable progenerator in $\mathfrak{M}_R$. By the last exercise, we have an R-isomorphism $P^{(\mathbb{N})} \cong R^{(\mathbb{N})}$. Taking the endomorphism rings of these modules and applying Exercise 28, we get a sequence of ring isomorphisms:

$$\mathrm{CFM}(R) \cong \mathrm{End}_R\left(R^{(\mathbb{N})}\right) \cong \mathrm{End}_R\left(P^{(\mathbb{N})}\right) \cong \mathrm{CFM}(S),$$

as desired.

To show the converse, we first do some preparatory work. Recall that, for two idempotents e, f in a ring E, we have $eE \cong fE$ iff $Ee \cong Ef$, iff there exist $a, b \in E$ such that $e = ab$ and $f = ba$ (see *FC*-(21.20)). In this case, we say that e and f are *isomorphic idempotents*, and write $e \cong f$. For readers not familar with this result, it will be all right to take "$e = ab$, $f = ba$ for some $a, b \in E$" as the definition of $e \cong f$. In fact, the following lemma (in a special case) will, among other things, yield the other characterizations of isomorphic idempotents.

Lemma 1. *Let* $E = \text{End}(V_R)$ *where* V *is a right module over a ring* R, and let $e, f \in E$ *be idempotents. Then* $e \cong f$ *in* E *iff* $eV \cong fV$ *as (right)* R*-modules.*

Proof. Say $e = ab$, $f = ba$ for some $a, b \in E$. Then

$$b: \ eV \to fV \quad \text{and} \quad a: \ fV \to eV$$

make sense since $b(eV) = babV \subseteq fV$ and $a(fV) = abaV \subseteq eV$. The compositions of these maps are identities, since $ab(ev) = e^2v = ev$ and similarly $ba(fv) = fv$ for any $v \in V$. Conversely, assume $eV \cong fV$ as R-modules, and let

$$b: \ eV \to fV \quad \text{and} \quad a: \ fV \to eV$$

be mutually inverse R-isomorphisms. We may assume a, b to be in E by taking b to be zero on $(1-e)V$, and a to be zero on $(1-f)V$. An easy calculation shows that $e = ab$ and $f = ba$ in E, so $e \cong f$ as desired. □

Now let R, S *be rings such that* $\text{CFM}(R) \cong \text{CFM}(S)$. Then, by Exercise 28, there exists a ring isomorphism

$$\sigma: \ \text{End}\left(S^{(\mathbb{N})}\right) \longrightarrow \text{End}\left(R^{(\mathbb{N})}\right).$$

Let us write $V = S^{(\mathbb{N})} = \bigoplus_{i=1}^{\infty} v_iS$ and $U = R^{(\mathbb{N})} = \bigoplus_{i=1}^{\infty} u_iR$. Let $e_{ij} \in \text{End}_S(V)$ and $f_{ij} \in \text{End}_R(U)$ be the matrix units (with $e_{ij}\ (v_k) = \delta_{jk}v_i$, etc.), and write $e'_{ij} = \sigma(e_{ij})$. Consider the idempotent $e'_{11} \in \text{End}_R(U)$, and let $P = e'_{11}(U) \subseteq U$. The crucial step is to prove the following.

Lemma 2. *The* R*-module* $P = e'_{11}(U)$ *above is f.g. projective.*

Proof. First, P is a direct summand of U_R, so P_R is countably generated and projective. Note that $e_{11}, e_{22}, \ldots$ are mutually isomorphic orthogonal idempotents; therefore, so are $e'_{11}, e'_{22}, \ldots$. From the orthogonality of the e'_{ii}'s, it follows that

$$e'_{11}(U) + e'_{22}(U) + \cdots$$

is a *direct* sum in U_R, and Lemma 1 shows that each $e'_{ii}(U) \cong P$. Assume, for now, that P is *not* f.g. Then, by Exercise (2.32) (and its proof), there exists an R-embedding

$$g: \ P \to e'_{11}(U) \oplus e'_{22}(U) \oplus \cdots \subseteq U$$

such that $g(P)$ is not contained in any *finite* direct sum on the RHS. We may assume that $g \in \text{End}_R\ (U)$ by taking g to be zero on $(1-e'_{11})U$. Now go back to V and consider its endomorphism $\sigma^{-1}(g)e_{11}$. We have

$$\sigma^{-1}(g)e_{11}V = \sigma^{-1}(g)e_{11}\,(v_1S + v_2S + \cdots) = \sigma^{-1}(g)v_1S \subseteq \sum_{j=1}^{m} v_jS$$

for some $m < \infty$. Thus, $e_{ii}\sigma^{-1}(g)e_{11} = 0$ for any $i > m$. Applying σ, we get $e'_{ii}ge'_{11} = 0$ for $i > m$, that is, $e'_{ii}g(P) = 0$ for $i > m$. This implies that

$$g(P) \subseteq e'_{11}(U) + \cdots + e'_{mm}(U),$$

a contradiction to the choice of g. Thus, P_R must be f.g. □

Finally, we need the following self-strengthening of Lemma 2.

Lemma 3. *P_R is a progenerator.*

Proof. It suffices to prove that

$$(*) \qquad U = e'_{11}(U) \oplus e'_{22}(U) \oplus \cdots,$$

for, if so, then $R^{(\mathbb{N})} \cong U \cong P^{(\mathbb{N})}$, and we are done by Exercise 29. Now we already know that the sum $\sum e'_{ii}(U)$ is direct, so it only remains to show that each basis vector $u_i \in U = \bigoplus_{j=1}^{\infty} u_j R$ lies in that sum. Consider $\sigma^{-1}(f_{ii})$. Applying Lemma 2 to σ^{-1}, we know that $\sigma^{-1}(f_{ii})V$ is f.g., so it is contained in $v_1 R \oplus \cdots \oplus v_n R$ for some large n. For any $v \in V$,

$$\begin{aligned}(e_{11} + \cdots + e_{nn})\sigma^{-1}(f_{ii})(v) &= (e_{11} + \cdots + e_{nn})(v_1 a_1 + \cdots + v_n a_n) \\ &\quad (\text{for some } a_i \in S) \\ &= v_1 a_1 + \cdots + v_n a_n = \sigma^{-1}(f_{ii})(v),\end{aligned}$$

so $(e_{11} + \cdots + e_{nn})\sigma^{-1}(f_{ii}) = \sigma^{-1}(f_{ii}) \in \mathrm{End}_S(V)$. Applying σ, we get

$$(e'_{11} + \cdots + e'_{nn})f_{ii} = f_{ii} \in \mathrm{End}_R(U).$$

Evaluating the two sides on $u_i \in U$, we conclude that

$$u_i \in e'_{11}(U) + \cdots + e'_{nn}(U),$$

as desired. □

To conclude the proof, note that $P = e'_{11}(U)$ implies that

$$\mathrm{End}_R(P) \cong e'_{11} \cdot \mathrm{End}_R(U) \cdot e'_{11} \cong e_{11} \cdot \mathrm{End}_S(V) \cdot e_{11} \cong S.$$

Since P_R is a progenerator by Lemma 3, Morita's Theorem now gives $R \approx S$.

Comment. The "only if" part of this exercise is due to S. Eilenberg; a statement of it appeared in the book of F. Anderson and K. Fuller. Eilenberg's observation inspired V. Camillo to the full version of the theorem in this exercise; see Camillo's paper, "Morita equivalence and infinite matrix rings," Proc. Amer. Math. Soc. **90**(1984), 186–188. In Camillo's paper, it was stated that the result was "conjectured by W. Stephenson (around 1965). It turned out that this statement was only prompted by the referee's report; there seemed to be no evidence that Stephenson had made such a conjecture. Instead, in his thesis, Stephenson worked with rings of matrices with only finitely many nonzero entries. The above "history" on the present exercise is gleaned from a letter from V. Camillo to C. Faith quoted on

p. 283 of Faith's recent book, "Rings and Things and a Fine Array of Twentieth Century Associative Algebra," 2nd Ed., Math. Surveys and Monographs, Vol. 65, Amer. Math. Soc., 2005. In his letter, Camillo concluded; "The credit for first looking at infinite matrix rings as characterizing Morita Equivalence certainly belongs to Stephenson."

A few more results on rings of column-finite $\mathbb{N} \times \mathbb{N}$ matrices have appeared recently. For instance, J.J. Simón has shown that Morita equivalent column-finite matrix rings must be isomorphic; see his paper in J. Alg. **173**(1995), 390–393. Extending Camillo's techniques, J. Haefner, A. del Río and J.J. Simón have also shown that $R \approx S$ iff $\text{RCFM}(R) \cong \text{RCFM}(S)$, where $\text{RCFM}(A)$ denotes the ring of *row and column finite* $\mathbb{N} \times \mathbb{N}$ matrices over a ring A; see their paper in Proc. Amer. Math. Soc. **125**(1997), 1651–1658.

Ex. 18.31. (Hattori-Stallings Trace) For any ring R, let $\bar{R}$ be the additive group $R/[R, R]$, where $[R, R]$ denotes the additive subgroup $\left\{ \sum(ab - ba) : a, b \in R \right\}$ of R.

(1) Show that the projection map "bar": $R \to \bar{R}$ is a universal group homomorphism with respect to the "trace property" $\overline{ab} = \overline{ba}$ (for all a, $b \in R$).
(2) Show that the group $\bar{R}$ is uniquely determined by the Morita equivalence class of R.

Solution. (1) Let $\varphi : R \to G$ be a homomorphism from R to any additive group G with the property that $\varphi(ab) = \varphi(ba)$ for all $a, b \in R$. Then φ vanishes on the subgroup $\{\sum(ab - ba) : a, b \in R\}$, so there exists a unique group homomorphism $\psi : \bar{R} \to G$ such that $\varphi(r) = \psi(\bar{r})$ for every $r \in R$. This checks the universal property of the "bar" map.

(2) Given $R \approx S$, we must show that $\bar{R} \cong \bar{S}$ as abelian groups. We may assume that R and S are related to each other as in "Morita I" in *LMR*-(18.24), so in particular we have a Morita context $(R, P, Q, S; \alpha, \beta)$. Define a group homomorphism $f : Q \otimes_S P \to \bar{S}$ by $f(q \otimes_S p) = \overline{pq} \in \bar{S}$. This homomorphism is well-defined since

$$f(qs \otimes_S p) = \overline{p(qs)} = \overline{(pq)s} = \overline{s(pq)} = \overline{(sp)q} = f(q \otimes_S sp).$$

Identifying $Q \otimes_S P$ with R, we have then a homomorphism $g : R \to \bar{S}$ given by $g(qp) = \overline{pq}$. Since

$$\begin{aligned} g(qp \cdot q'p') &= g(qpq' \cdot p') = \overline{p'(qpq')} = \overline{(p'qp)q'} \\ &= g(q' \cdot p'qp) = g(q'p' \cdot qp), \end{aligned}$$

g induces a homomorphism $\bar{g} : \bar{R} \to \bar{S}$. Similarly, we can define a homomorphism $\bar{h} : \bar{S} \to \bar{R}$ by $\bar{h}(\overline{pq}) = \overline{qp} \in \bar{R}$. Then

$$\bar{h}\,\bar{g}(\overline{qp}) = \bar{h}(\overline{pq}) = \overline{qp} \quad \text{and} \quad \bar{g}\,\bar{h}(\overline{pq}) = \bar{g}(\overline{qp}) = \overline{pq},$$

so $\bar{h}$, $\bar{g}$ are mutually inverse isomorphisms. In particular, $\bar{R} \cong S$ as abelian groups.

Comment. The exercise only barely scratched the surface of the construction of the Hattori-Stallings trace. After obtaining the additive group $\bar{R} = R/[R, R]$, one constructs, for any f.g. projective right R-module P, an additive homomorphism

$$\mathrm{tr}_P : \mathrm{End}_R(P) \to \bar{R},$$

generalizing the trace on $\mathrm{End}_R(P)$ defined in Exer. (2.28) in the commutative case. In particular, $\mathrm{tr}_P\ (\mathrm{Id}_P) \in \bar{R}$ may be viewed as the "rank" of the f.g. projective module P_R.

For the relevant literature, see A. Hattori, "Rank element of a projective module," Nagoya J. Math. **25**(1965), 113–120, and J. Stallings, "Centerless groups — an algebraic formulation of Gottlieb's Theorem," Topology **4** (1965), 129–134.

Ex. 18.32. For any right R-module M, let $\sigma[M]$ be the full subcategory of $\mathfrak{M}_R$ whose objects are subquotients of direct sums of copies of M. Show that $\sigma[M] = \mathfrak{M}_R$ iff R_R can be embedded into M^n for some integer n.

Solution. As is observed in *LMR*-§18F, $\sigma[M]$ is closed with respect to submodules, quotient modules, and arbitrary direct sums. If R_R can be embedded into some direct sum $M^{(I)}$, then $R \in \sigma[M]$, and hence $\mathfrak{M}_R = \sigma[M]$. Conversely, assume that $\mathfrak{M}_R = \sigma[M]$. Then R is a quotient module of some module $Q \subseteq M^{(I)}$. Since R_R is projective, this implies that R can be embedded in Q, and hence in $M^{(I)}$. But then R can be embedded in M^n for some $n < \infty$.

Ex. 18.33. Let M be a right R-module with $A = \mathrm{ann}^R(M)$. Let $S = \mathrm{End}_R(M)$ (operating on the left of M). If M is f.g. as a left S-module, show that $\sigma[M] = \mathfrak{M}_{R/A}$. Deduce that, if M is a f.g. (right) module over a commutative ring R, then $\sigma[M] = \mathfrak{M}_{R/A}$ (for $A = \mathrm{ann}^R(M)$).

Solution. Clearly, $\sigma[M]$ is a (full) subcategory of $\mathfrak{M}_{R/A}$. According to the last exercise, $\sigma[M] = \mathfrak{M}_{R/A}$ will follow if we can show that $(R/A)_R$ can be embedded into M^n for some integer n. Say

$$M = Sm_1 + \cdots + Sm_n \quad (m_i \in M).$$

Then $A = \bigcap_{i=1}^n \mathrm{ann}^R(m_i)$. Hence, in the direct sum M^n, the cyclic R-submodule $(m_1, \ldots, m_n) \cdot R$ is isomorphic to $(R/A)_R$, as desired.

Now assume R is commutative, and M_R is f.g. We have a natural ring homomorphism $R \to S = \mathrm{End}_R(M)$ sending $r \in R$ to the right multiplication by r on M. Thus, M_R being f.g. implies that ${}_SM$ is f.g., so the first part of this exercise applies.

Ex. 18.34. For $R = \mathbb{Z}$ and $M = \mathbb{Q}/\mathbb{Z}$, show that the only $\sigma[M]$-projective module is (0).

Solution. We shall use the fact that any direct summand of a $\sigma[M]$-projective module is $\sigma[M]$-projective. Here, we know that $\sigma[M]$ is the category of all torsion abelian groups, by *LMR*-(18.52)(4).

Suppose P is a nonzero $\sigma[M]$-projective. By replacing P by one of its primary components, we may assume that P is p-primary for some prime p. We argue in the following two cases.

Case 1. P contains a copy of the Prüfer p-group C_{p^∞}. Since C_{p^∞} is a divisible abelian group, it is isomorphic to a direct summand of P, and is therefore $\sigma[M]$-projective. This is impossible since the surjection $C_{p^\infty} \to C_{p^\infty}$ defined by multiplication by p is nonsplit.

Case 2. P does not contain a copy of C_{p^∞}. By a theorem in abelian group theory (see Theorem 9 in Kaplansky's "Infinite Abelian Groups"), any nonzero p-primary group with this property must contain a nonzero direct summand isomorphic to a cyclic group C_{p^n}. But then C_{p^n} would be $\sigma[M]$-projective. This is impossible, since a surjection $C_{p^{n+1}} \to C_{p^n}$ obviously does not split.

Ex. 18.35. (Blackadar) In this exercise, we write $r \cdot P$ for the direct sum of r copies of a module P. Let P, Q, X be right modules over a ring R such that $P \oplus X \cong Q \oplus X$. If X embeds as a direct summand in $r \cdot P$ and in $r \cdot Q$ (for some integer r), show that $n \cdot P \cong n \cdot Q$ for all $n \geq 2r$. Deduce that if P, Q are generators over R and X is a f.g. projective R-module such that $P \oplus X \cong Q \oplus X$, then $n \cdot P \cong n \cdot Q$ for all sufficiently large n.

Solution. As is observed by K. Goodearl, the proof of the first claim in this exercise can be formulated quite generally in the language of semigroups. Let $(S, +)$ be a commutative semigroup with elements p, q, x such that $r \cdot p = x + a$ and $r \cdot q = x + b$ for some $a, b \in S$ and some natural number r. We claim that

$$(1) \qquad p + x = q + x \implies n \cdot p = n \cdot q \in S \quad \text{for all} \quad n \geq 2r.$$

Indeed, adding p to $r \cdot p = x + a$, we have

$$(2) \qquad p + rp = (p + x) + a = (q + x) + a = q + rp.$$

We can express this equation informally by saying that rp "converts" p into q (upon addition). By symmetry, we see that rq also "converts" q into p, that is,

$$(3) \qquad q + rq = p + rq.$$

Thus, for any $n = 2r + \varepsilon$ with $\varepsilon \geq 0$, we have by (2), (3):

$$\begin{aligned} np &= rp + rp + \varepsilon p = rp + rq + \varepsilon q \\ &= rq + rq + \varepsilon q = nq. \end{aligned}$$

The module-theoretic claim now follows by applying (1) to the semigroups S of isomorphism classes of R-modules, with the sum in S induced by the direct sum of modules.

For the second claim in the exercise, let P, Q be generators over R such that $P \oplus X \cong Q \oplus X$, where X is a f.g. projective R-module. Then X is a direct summand of some free module $s \cdot R$. For sufficiently large r, $r \cdot P$ and $r \cdot Q$ will have surjections onto $s \cdot R$. Therefore, $s \cdot R$ (and hence X) embeds as a direct summand in both $r \cdot P$ and $r \cdot Q$, and the first part of the Exercise applies.

Comment. B. Blackadar's paper, "Rational C^*-algebras and non-stable K-theory," appeared in Rocky Mountain J. Math. **20**(1990), 285–316. Our formulation of Blackadar's result follows that of K. Goodearl in "von Neumann regular rings and direct sum decomposition problems," in Abelian Groups and Modules, Padova 1994 (A. Facchini and C. Menini, eds.), Math. Appl. **343**, Kluwer Acad. Publ., Dordrecht, 1995.

§19. Morita Duality Theory

Before taking up the study of Morita's duality theory, it is convenient to introduce first the notion of *finitely cogenerated* (f.cog.) *modules.* The definition of a f.cog. module is dual to one of the known descriptions of a f.g. module: we say that M_R is f.cog. if, for any family of submodules $\{N_i : i \in I\}$ in M, if $\bigcap_{i \in I} N_i = 0$, then $\bigcap_{i \in J} N_i = 0$ for some finite subset $J \subseteq I$. There are a few equivalent descriptions for M being f.cog., the most useful of which is that $S := \mathrm{soc}(M)$ is f.g. and $S \subseteq_e M$ (*LMR*-(19.1)). Exercise 7 below gives Vámos' original description:

> *M is f.cog. iff $E(M) \cong E(V_1) \oplus \cdots \oplus E(V_n)$ for suitable simple modules $V_1, \ldots, V_n$.*

We also need the notion of a *cogenerator* (*module*), which is dual to that of a generator: a module U_R is called a cogenerator if, for any N_R and $0 \neq x \in N$, there exists $g \in \mathrm{Hom}_R(N, U)$ such that $g(x) \neq 0$. A useful characterization is that U contains a copy of $E(V)$ for every simple module V_R. Just as projective generators are important, we can expect that injective cogenerators are important.

A *cogenerator ring* is a ring R for which both R_R and ${}_RR$ are cogenerators. QF rings are examples of cogenerator rings; in fact, they are precisely the cogenerator rings that are left (or right) noetherian. Regardless of chain conditions, cogenerator rings are precisely the self-injective Kasch rings, according to *LMR*-(19.18).

A "duality" between two categories $\mathcal{A}, \mathcal{B}$ means a pair of contravariant functors $\mathcal{A} \to \mathcal{B}$, $\mathcal{B} \to \mathcal{A}$, whose compositions are equivalent to the identity functors. Classical examples include vector space duality, Galois duality, Pontryagin duality, Gel'fand-Naimark duality, etc. A nice ring-theoretic example is the duality between f.g. right R-modules and f.g. left R-modules over a QF ring R; here, the contravariant functors are both given by the R-dual $\mathrm{Hom}_R(-, R)$.

More generally, given an (S, R)-bimodule U, we have the contravariant functors $F : \mathfrak{M}_R \to \mathfrak{M}_S$, $G : \mathfrak{M}_S \to \mathfrak{M}_R$ defined by forming the U-duals:

$$F(M_R) = \mathrm{Hom}_R(M_R,\ {}_SU_R) \quad \text{and} \quad G({}_SN) = \mathrm{Hom}_S({}_SN,\ {}_SU_R).$$

Writing $\mathfrak{M}_R[U]$ and ${}_S\mathfrak{M}[U]$ for the categories of U-reflexive modules (in $\mathfrak{M}_R$ and ${}_S\mathfrak{M}$ respectively), we say that U *defines a Morita duality* (from R to S) if both $\mathfrak{M}_R[U]$ and ${}_S\mathfrak{M}[U]$ are Serre subcategories containing R_R and ${}_SS$ respectively. Of course, the duality here is really between the two categories $\mathfrak{M}_R[U]$ and ${}_S\mathfrak{M}[U]$. The conditions imposed above on the Morita duality ensures that these are sufficiently nice and sufficiently big categories. (A Serre subcategory of $\mathfrak{M}_R$ means a subcategory $\mathfrak{M}$ with the property that, for any exact sequence $0 \to K \to L \to M \to 0$ in $\mathfrak{M}_R$, we have $L \in \mathfrak{M}$ iff $K, M \in \mathfrak{M}$.)

The theorem Morita I for duality gives necessary and sufficient conditions for a bimodule ${}_SU_R$ to define a Morita duality. For instance, one set of necessary and sufficient condition is that:

> *U_R and ${}_SU$ both be injective cogenerators, and ${}_SU_R$ be faithfully balanced* (i.e. $\mathrm{End}(U_R) = S$ and $\mathrm{End}({}_SU) = R$).

The existence of such a Morita duality automatically imposes various conditions on R; foremost is the fact that R must be a *semiperfect ring* (i.e. $R/\mathrm{rad}R$ is semisimple, and idempotents of $R/\mathrm{rad}R$ can be lifted to R). *The cogenerator rings are precisely the rings R for which $U = {}_RR_R$ defines a Morita duality from R to itself.* Here we get a duality between the left and right reflexive modules given by the usual R-dual functors. It follows from what we said above that *any cogenerator ring is a semiperfect ring.*

Another important example of a Morita duality from R to itself is given by the case where R is a commutative complete noetherian local ring $(R, \mathfrak{m})$. Here, Matlis' theory predicts that $U := E(R/\mathfrak{m})$ defines a Morita duality from R to R. Here, the U-reflexive objects include all noetherian and all artinian R-modules (*LMR*-(19.56)).

The notion of a linearly compact module turns out to be important for the study of Morita dualities. A module M_R is said to be *linearly compact* (or l.c. for short) if, for any system of submodules $\{M_i\}_{i\in I}$ and elements $m_i \in M$, $\bigcap_{j\in J}(m_j + M_j) \neq \emptyset$ for any finite $J \subseteq I$ implies that $\bigcap_{i\in I}(m_i + M_i) \neq \emptyset$. The category $\mathfrak{M}_R^{\ell c}$ of l.c. R-modules is always a Serre subcategory of $\mathfrak{M}_R$, by a result of Zelinsky (Exercise 35 below).

The relvance of l.c. modules to Morita duality is seen from the following result of Müller (*LMR*-(19.68)):

If a bimodule ${}_SU_R$ *defines a Morita duality from* R *to* S, *then* $\mathfrak{M}_R[U] = \mathfrak{M}_R^{\ell c}$.

This surprising result says that the category of U-reflexive objects in R is uniquely determined (as the category of l.c. modules), independently of U and S! For a given $U_R \in \mathfrak{M}_R$ and $S := \text{End}(U_R)$, there is also a characterization for ${}_SU_R$ to define a Morita duality from R to S in terms of linear compactness: the necessary and sufficient condition is that U_R be a f.cog. injective cogenerator, and R_R, U_R both be linearly compact *LMR*-(19.70).

The case of Morita dualities between artinian rings was the primary motivation of the classical work of Morita and Azumaya. If R is a right artinian ring with simple right modules $V_i, \ldots, V_n$, a necessary and sufficient condition for R to admit a Morita duality (to some other ring S) is that each $E(V_i)$ be f.g. In this case, the minimal injective cogenerator $U^0 = E(V_1) \oplus \cdots \oplus E(V_n)$ defines a Morita duality from R to $S = \text{End}(U_R^0)$. (*LMR*-(19.74)).

The first part of this set of exercises deals with the notion of f.cog. modules, and is followed by exercises on cogenerator modules and cogenerator rings (and their 1-sided versions, the right/left PF rings). The remaining exercises deal with Morita dualities and linearly compact modules.

Exercises for §19

Ex. 19.0. Prove the following two facts stated in *LMR*-(19.4):

(1) A module M_R is artinian iff every quotient of M is f.cog.
(2) A ring R is right artinian iff every cyclic (resp. f.g.) module M_R is f.cog.

Solution. (1) For the "only if" part, it suffices to show that M artinian $\Rightarrow$ M is f.cog. Let $\{N_i : i \in I\}$ be a family of submodules of M such that any finite subfamily has nonzero intersection. We may assume that the family is closed under finite intersections. Since M is artinian, there is a minimal N_{i_0} in the family. For any $i \in I$, $N_i \cap N_{i_0} \neq 0$ implies then $N_i \cap N_{i_0} = N_{i_0}$. Therefore, $\bigcap_{i \in I} N_i \supseteq N_{i_0} \neq 0$. This checks that M is f.cog.

For the "if" part in (1), assume that every quotient of M is f.cog. If M is not artinian, there would exist submodules $N_1 \supsetneq N_2 \supsetneq \cdots$ in M. Let $N = \bigcap_{i=1}^{\infty} N_i$ and consider the submodules $\overline{N}_i := N_i/N$ in M/N. Clearly, any finite family of these has a nonzero intersection, but $\bigcap_{i=1}^{\infty} \overline{N}_i = 0$. Therefore, M/N is *not* f.cog., contrary to our assumption.

(2) First assume R is right artinian. Then any f.g. M_R is also artinian, and hence f.cog. by (1). Conversely, assume that every cyclic M_R is f.cog. Then every quotient of R_R is f.cog., and hence by (1) R_R is artinian.

Comment. The characterization of artinian modules in (1) of this exercise is, of course, dual to the following well-known characterization of noetherian modules: M_R *is noetherian iff every submodule of* M *is* f.g.

Ex. 19.1. In *LMR*-(19.1), the following two conditions on a module M_R were shown to be equivalent:

(1) For any family of submodules $\{N_i : i \in I\}$ in M, if $\bigcap_{i\in I} N_i = 0$, then $\bigcap_{i\in J} N_i = 0$ for some finite subset $J \subseteq I$.
(2) For any family of submodules $\{N_i : i \in I\}$ in M which form a chain, if each $N_i \neq 0$, then $\bigcap_{i\in I} N_i \neq 0$.

The proof of the equivalence depends on using another condition on the socle of M. Give a *direct* proof for (1) $\Leftrightarrow$ (2) without using the socle.

Solution. (1) $\Rightarrow$ (2) is clear, so we need only prove (2) $\Rightarrow$ (1). Assume (2) holds but (1) does not. Then there exists a family of submodules $\{N_i : i \in I\}$ such that $\bigcap_{i\in J} N_i \neq 0$ for any finite subset $J \subseteq I$, but $\bigcap_{i\in I} N_i = 0$. By Zorn's Lemma, there exists such a family that is maximal (w.r.t. inclusion). We may thus assume that $\{N_i : i \in I\}$ is such a maximal family. Clearly, *this family must be closed with respect to finite intersections.* Now, by Hausdorff's maximal principle, there exists a maximal chain, say $\{A_\ell : \ell \in L\}$, in the family $\{N_i\}$. By (2), $A := \bigcap_{\ell\in L} A_\ell \neq 0$. There are the following two possibilities:

(a) $A = N_{i_0}$ *for some* $i_0 \in I$. For any $j \in I$, $A \cap N_j = N_{i_0} \cap N_j$ still belongs to the family $\{N_i\}$. By the maximality of the chain $\{A_\ell\}$, we must have $A \cap N_j = A$. This yields $A \subseteq N_j$ for all j and so $\bigcap\{N_j : j \in I\} \supseteq A \neq 0$, a contradiction.

(b) A *is not in the family* $\{N_i : i \in I\}$. Since the intersection of $\{A, N_i\ (i \in I)\}$ is obviously zero, the maximal choice of $\{N_i : i \in I\}$ implies that $A \cap N_{i_1} \cap \cdots \cap N_{i_n} = 0$ for suitable $i_1, \ldots, i_n \in I$. But

$$N_{i_1} \cap \cdots \cap N_{i_n} = N_{i_0} \quad \text{for some} \quad i_0 \in I,$$

so we have $A \cap N_{i_0} = 0$. Applying (2) to the chain $\{N_{i_0} \cap A_\ell : \ell \in L\}$, however, we have

$$0 \neq \bigcap_{\ell\in L} (N_{i_0} \cap A_\ell) = N_{i_0} \cap \bigcap\nolimits_{\ell\in L} A_\ell = N_{i_0} \cap A,$$

a contradiction.

Comment. The argument above involves some of the most tricky applications of maximal principles I have seen. It is based on a hint to the same exercise given on p. 131 of the text of Anderson-Fuller, "Rings and Categories of Modules," (2nd edition), Graduate Texts in Math., Vol. 13, Springer-Verlag, N.Y., 1992.

Ex. 19.2. Let $N \subseteq M$ be R-modules. If M is f.cog., it is clear that N is also f.cog. Show that the converse holds if $N \subseteq_e M$.

Solution. Assume that $N \subseteq_e M$ and N is f.cog. Let $\{X_i : i \in I\}$ be submodules of M such that $\bigcap_{i\in J} X_i \neq 0$ for every finite subset $J \subseteq I$. Consider the family of submodules $Y_i = X_i \cap N \subseteq N$. For any finite $J \subseteq I$, $N \subseteq_e M$ implies that

$$\bigcap\nolimits_{i\in J} Y_i = N \cap \left(\bigcap\nolimits_{i\in J} X_i\right) \neq 0.$$

Since N is f.cog., this gives $\bigcap_{i\in I} Y_i \neq 0$. In particular, $\bigcap_{i\in I} X_i \neq 0$, so we have checked that M is f.cog.

Alternatively, we can solve this exercise by using the fact (*LMR*-(19.1)) that a module is f.cog. iff it has a f.g. and essential socle. Since $N \subseteq_e M$ implies that $\operatorname{soc}(N) = \operatorname{soc}(M)$ by Exercise (6.12)(4), the transitivity of essential extensions show that N being f.cog. implies M is f.cog.

Comment. The conclusion of this exercise is motivated by its dual statement in the f.g. case: *If N is small in M* (that is, for any submodule $X \subseteq M$, $N + X = M \Rightarrow X = M$), and M/N is f.g., then M is f.g.

Ex. 19.3. For any module M, show that the following are equivalent:

(1) M is semisimple and f.g.;
(2) M is semisimple and f.cog.;
(3) rad $M = 0$ and M is f.cog.

Show that these statements imply, but are not equivalent to:

(4) rad $M = 0$ and M is f.g.

Solution. (1) $\Rightarrow$ (2). Under the assumptions of (1), M has finite length, and is, in particular, artinian. By Exercise 0, every quotient of M is f.cog.

(2) $\Rightarrow$ (3). Since M is semisimple, $M = N \oplus \text{rad } M$ for some submodule N. If rad $M \neq 0, N$ is contained in a maximal submodule P of M. But then $P \supseteq \text{rad } M$ and hence $P = M$, a contradiction. This shows that rad $M = 0$.

(3) $\Rightarrow$ (1). Since rad M is the intersection of all maximal submodules of M, (3) implies that $(0) = M_1 \cap \cdots \cap M_n$ for suitable maximal submodules $M_1, \ldots, M_n$ of M. Therefore, M embeds into the semisimple module $\bigoplus_{i=1}^n M/M_i$, which implies that M is f.g. semisimple.

Clearly, the equivalent statements (1), (2), (3) imply (4). However, in general, (4) does not imply (1), (2) or (3). For instance, the $\mathbb{Z}$-module $M = \mathbb{Z}$ satisfies (4), but is not semisimple.

Comment. This exercise serves to warn us that not every valid proposition has a valid dual!

Here is a nice application of the exercise: *If* rad M *is small in* M *and* $M/\text{rad } M$ *is f.cog., then* M *is* f.g. Indeed, since $\text{rad}(M/\text{rad } M) = 0$, the assumption that $M/\text{rad}\,M$ is f.cog. implies that $M/\text{rad}\,M$ is f.g., by (3) $\Rightarrow$ (4) above. Since $\text{rad}\,M$ is small in M, this implies that M is f.g. (see the remarks made after *LMR*-(19.2)).

Ex. 19.4. True or False: "For any exact sequence $0 \to K \to L \to M \to 0$, if K, M are f.cog., so is L"?

Solution. It is well-known that this is a true statement if "f.cog." is replaced by "f.g." Therefore, it is reasonable to expect the truth of this statement for f.cog. modules.

Assume that K and M are f.cog. Consider any chain of nonzero submodules $\{N_i : i \in I\}$ in L. We claim that $\bigcap_{i\in I} N_i \neq 0$. (By *LMR*-(19.1), this would imply that L is f.cog.) For $K_i = N_i \cap K$ $(i \in I)$, we have two cases.

Case 1. Every $K_i \neq 0$. Since K is f.cog., $\bigcap_{i\in I} K_i \neq 0$. In particular, $\bigcap_{i\in I} N_i \neq 0$, as desired.

Case 2. There exists $i_0 \in I$ such that $K_{i_0} = 0$. Then $N_{i_0} \cap K = 0$. Consider now the chain $\{N_j : j \in J\}$ $(J \subseteq I)$ of modules $N_j \subseteq N_{i_0}$. For each $j \in J$, the image $\bar{N}_j$ of N_j in $L/K \cong M$ is nonzero. Since M is f.cog., $\bigcap_{j\in J} \bar{N}_j \neq 0$. This implies that $\bigcap_{j\in J} N_j \neq 0$. For each $i \in I$, we have either $N_i \supseteq N_{i_0}$ or $N_i \subseteq N_{i_0}$. It follows that $\bigcap_{i\in I} N_i = \bigcap_{j\in J} N_j \neq 0$.

Ex. 19.5. Show that $M = M_1 \oplus \cdots \oplus M_n$ is f.cog. iff each M_i is.

Solution. If M is f.cog., clearly each submodule M_i is also f.cog. Conversely, if $M_1, \ldots, M_n$ are f.cog., then by Exercise 4 and induction on n, the direct sum $M_1 \oplus \cdots \oplus M_n$ is also f.cog.

Ex. 19.6. Show that any f.cog. module has finite uniform dimension. Is the converse true, at least for injective modules?

Solution. Let M be f.cog. By *LMR*-(19.1), $\text{soc}(M)$ is f.g. and essential in M. Since $\text{soc}(M)$ is semisimple, we have

$$\text{u}\cdot\dim M = \text{u}\cdot\dim\left(\text{soc}(M)\right) = \text{length}\left(\text{soc}(M)\right) < \infty.$$

The converse is not true, even for injective modules M. For instance, over the ring $\mathbb{Z}$, the $\mathbb{Z}$-module $M = \mathbb{Q}$ is injective and uniform. But $\mathbb{Q}_{\mathbb{Z}}$ is not f.cog. since $\mathbb{Z}_{\mathbb{Z}}$ is not.

Ex. 19.7. For any module M_R, show that the following are equivalent:

(1) M is f.cog.
(2) $E(M) \cong E(V_1) \oplus \cdots \oplus E(V_n)$ for suitable simple modules $V_1, \ldots, V_n$.
(3) $M \subseteq E(V_1) \oplus \cdots \oplus E(V_r)$ for suitable simple modules $V_1, \ldots, V_r$.

Solution. (2) $\Rightarrow$ (3) is trivial.

(3) $\Rightarrow$ (1). Clearly, $\text{soc}(E(V_i)) = V_i$. Since this socle is simple and essential in $E(V_i)$, $E(V_i)$ is f.cog. by *LMR*-(19.1). Therefore, by Exercise 5, $E(V_1) \oplus \cdots \oplus E(V_r)$ is f.cog., and so its submodule M is also f.cog.

(1) ⇒ (2). By *LMR*-(19.1) again, $S := \text{soc}(M) \subseteq_e M$ and $S = V_1 \oplus \cdots \oplus V_n$ for suitable simple modules $V_1, \ldots, V_n$. Therefore,

$$E(M) = E(S) = E(V_1 \oplus \cdots \oplus V_n) \cong E(V_1) \oplus \cdots \oplus E(V_n).$$

Comment. In Vámos' paper "The dual of the notion of 'finitely generated'", J. London Math. Soc. **43**(1968), 643–646, the condition (2) above was taken to be the definition of M being f.cog. (Actually, Vámos used the term "finitely embedded", instead of "f.cog.", for this property.)

Ex. 19.8. (Matlis) For any commutative noetherian ring R, show that a module M_R is artinian iff it is f.cog.

Solution. The "only if" part follows from Exercise 0, so we need only prove the "if" part. Let M_R be f.cog. By the last exercise, there exists an embedding

$$M \subseteq E(V_1) \oplus \cdots \oplus \ E(V_r)$$

where each V_i is a simple R-module. Therefore, it suffices to show that $E := E(V)$ *is artinian for any simple module* V_R. Say $V = R/\mathfrak{m}$ where $\mathfrak{m} \subset R$ is a maximal ideal. By *LMR*-(3.77), we can identify E with the injective hull $E(R_\mathfrak{m}/\mathfrak{m}R_\mathfrak{m})$ (formed over $R_\mathfrak{m}$). We claim that:

(1) *Any R-submodule $X \subseteq E$ is an $R_\mathfrak{m}$-submodule of E.*

It suffices to show that, for any $s \in R\backslash\mathfrak{m}$ and any $x \in X$, $xs^{-1} \in xR$. By *LMR*-(3.78), $x\mathfrak{m}^n = 0$ for some n. Since $\bar{s}$ is clearly a unit in $R/\mathfrak{m}^n$, $st \equiv 1 \pmod{\mathfrak{m}^n}$ for some $t \in R$. Then $x(st-1) = 0$ implies that

$$xs^{-1} = xt \in xR,$$

as desired.

In view of (1), we may replace R by $R_\mathfrak{m}$ to assume that R is *local.* By Matlis' Theorem *LMR*-(3.84), $\text{End}_R E$ is isomorphic to $\tilde{R}$, the $\mathfrak{m}$-adic completion of $(R, \mathfrak{m})$. Now, by Exercise (3.49), E is also the injective hull of the unique simple module of $\tilde{R}$, and the solution to that exercise shows that any R-submodule of E is also an $\tilde{R}$-submodule. Therefore, it suffices to check that E is an artinian $\tilde{R}$-module. Since $\tilde{R}$ is complete, local, and noetherian, this follows from *LMR*-(19.56).

Comment. This exercise is based on Matlis' paper "Modules with descending chain condition," Trans. Amer. Math. Soc. **97**(1960), 495–508. Note that the above proof works as long as each localization $R_\mathfrak{m}$ is a noetherian ring. It turns out this is exactly what we need, in view of the following result of P. Vámos:

> *Over a commutative ring R, f.cog. R-modules coincide with the artinian ones iff the localization of R at every maximal (or prime) ideal is noetherian.*

See his paper in J. London Math. Soc. cited in the *Comment* on the last exercise.

Ex. 19.9. A module M_R is said to be *cofaithful* if R_R embeds into M^n for some $n < \infty$.

(1) Show that a cofaithful module M_R is always faithful.
(2) If R is commutative, show that a f.g. M_R is faithful iff it is cofaithful.
(3) Show that R_R is f.cog. iff all faithful right R-modules are cofaithful. (In particular, the latter condition holds over any right artinian ring, and any cogenerator ring.)

Solution. (1) Say $R \hookrightarrow M^n$. Since R_R is faithful, M^n is also faithful, and this implies that M is faithful.

(2) In view of (1), we need only prove the "only if" part. Write $M = \sum_{i=1}^{n} m_i R$ and assume it is faithful. If $r \in R$ is such that $m_i r = 0$ for all i, then by commutativity,

$$Mr = \sum_{i=1}^{n} m_i\ Rr = \sum_{i=1}^{n} m_i\, rR = 0,$$

and so $r = 0$. The map $1 \mapsto (m_1, \ldots, m_n)$ then defines an embedding of R_R into M^n. Hence M is cofaithful.

(3) First assume R_R is f.cog., and let M_R be any faithful module. Then $\{\text{ann}(m)\colon m \in M\}$ is a family of right ideals in R with intersection (0). Since R_R is f.cog., there exists $m_1, \ldots, m_n \in M$ $(n < \infty)$ such that $\bigcap_{i=1}^{n} \text{ann}(m_i) = 0$. From this we see as in (2) that M is cofaithful.

Conversely, assume that any faithful M_R is cofaithful. Consider the canonical cogenerator M (direct sum of the injective hulls of a complete set of simple right R-modules). Certainly M_R is faithful (by *LMR*-(19.7)), so R_R embeds into some M^n. Since R_R is cyclic, this implies that R_R embeds into $E(V_1) \oplus \cdots \oplus E(V_r)$ for a finite set of simple modules $V_1, \ldots, V_r$. By Exercise 7 above, R_R is f.cog.

Comment. The result (3) in this exercise appeared in J. A. Beachy's paper "On quasi-artinian rings," J. London Math. Soc. **3**(1971), 449–452. In the literature, cofaithful modules have appeared under other names. For instance, in C. Faith's "Algebra II" (Springer-Verlag, 1976), they are called "compactly faithful" modules.

Ex. 19.9′. If R is a semiprime ring such that R_R is f.cog., show that R is a semisimple ring.

Solution. By Exercise 6, $\text{u.dim}(R_R) < \infty$, and by *LMR*-(19.1), $\text{soc}(R_R) \subseteq_e R_R$. Therefore, by Exercise (11.30)(3), R must be a semisimple ring.

Ex. 19.10. For any ring R, $R' := \text{Hom}_{\mathbb{Z}}\,(R, \mathbb{Q}/\mathbb{Z})$ is an (R, R)-bimodule. Show that $(R')_R$ is an injective cogenerator (and similarly for ${}_R(R')$).

Solution. Since ${}_RR$ is flat, Lambek's Theorem (*LMR*-(4.9)) implies that $(R')_R$ is injective. To check that $(R')_R$ is a cogenerator, it thus suffices to show that any simple module S_R embeds into $(R')_R$ (see *LMR*-(19.9)). Fix a nonzero $\mathbb{Z}$-homomorphism $g : S \to \mathbb{Q}/\mathbb{Z}$ (which exists by *LMR*-(4.7)). Then define $f : S \to R'$ by $f(s)(x) = g(sx)$ for $s \in S$ and $x \in R$. For $r \in R$, we have

$$f(sr)\,(x) = g\,(srx) = f(s)(rx) \;= (f(s) \cdot r)\,(x),$$

so $f(sr) = f(s) \cdot r$, i.e., f is a right R-homomorphism. Since $f(s)(1) = g(s) \neq 0$ for some s, f is not zero. Therefore, f gives an embedding of S_R into $(R')_R$, as desired.

Ex. 19.11. Show that $R = \prod_{j=1}^{n} R_j$ is a cogenerator ring iff each R_j is.

Solution. Recall that R is a self-injective ring iff each R_j is (by *LMR*-(3.11B)). Next, a simple (left, right) R-module is just a simple R_i-module for some i with the other R_j's acting as zero. From this, we see easily that R is Kasch iff each R_j is Kasch. Since being a cogenerator ring amounts to being self-injective and Kasch (by *LMR*-(19.18)), the conclusion of the exercise follows.

Ex. 19.12. Let R be a QF ring. In *LMR*-(15.11)(1), it is shown that all left/right R-modules are torsionless, from which it follows that R is a cogenerator ring. Give another proof of this result without using *LMR*-(15.11)(1).

Solution. By symmetry, it suffices to prove that R_R is a cogenerator. Consider any simple module S_R. By Exercise (15.13)(1), $E(S)$ is isomorphic to a principal indecomposable right R-module. In particular, $E(S)$ embeds into R_R. By *LMR*-(19.8), this implies that R_R is a cogenerator, as desired.

Ex. 19.13. Show that over $R = \mathbb{Z}$, a module U_R is a cogenerator iff every nonzero M_R admits a nonzero homomorphism into U.

Solution. We need only prove the "if" part. Assume that U_R has the given property. Our job is to prove that, for any simple module S_R, $E(S)$ embeds into U. Now $S \cong \mathbb{Z}/p\mathbb{Z}$ for some prime p, and $E(S)$ is (isomorphic to) the Prüfer p-group C_{p^∞}. By assumption, there exists a nonzero homomorphism $f : C_{p^\infty} \to U$. We have $K := \ker(f) \neq C_{p^\infty}$ and C_{p^∞}/K embeds into U. Now, C_{p^∞}/K is just another copy of C_{p^∞}, so C_{p^∞} embeds into U, as desired.

Ex. 19.14. Let $R = S \oplus M$ be a trivial extension of a bimodule ${}_SM_S$ by the ring S, where ${}_SM$ is faithful. According to *LMR*-(19.22),

> R_R *is injective iff* M_S *is injective and the natural map* $S \to \mathrm{End}(M_S)$ (*giving the left* S*-action on* M) *is onto* (*hence an isomorphism*).

The proof for the "if" part is given in *LMR*-(19.22). Supply here a proof for the "only if" part.

Solution. The following proof of the "only if" part does not require the faithfulness of ${}_SM$.

Assume that R_R is injective. Let $I \subseteq S$ be a right ideal, and $f \in \text{Hom}_S(I, M_S)$. For $J := I \oplus M$ (a right ideal in R), we can define $F : J \to R$ by

$$F(i,\, m) = (0,\, f(i)) \quad \text{where} \quad i \in I \text{ and } m \in M.$$

A routine calculation shows that $F \in \text{Hom}_R(J, R_R)$. Since R_R is injective, F is the left multiplication by some $(s_0, m_0) \in R$. Thus,

$$(0,\, f(i)) = F(i,\, 0) = (s_0,\, m_0)\,(i,\, 0) = (s_0 i,\, m_0 i),$$

and so $f(i) = m_0 i$ for all $i \in I$. By Baer's Criterion, we see that M_S is injective.

To show that $S \to \text{End}(M_S)$ is onto, consider any $g \in \text{End}(M_S)$. Viewing g as a map from M to R, we check easily that $g \in \text{Hom}_R(M_R, R_R)$. Again by the injectivity of R_R, g must be the left multiplication by some $(s_1,\, m_1) \in R$. This time,

$$(0,\, g(m)) = (s_1,\, m_1)\,(0,\, m) = (0,\, s_1 m)$$

shows that g is left multiplication by s_1 on M, as desired.

Comment. Mark Davis has observed that the first part of this exercise can also be solved by invoking Ex. 3.28(1). In fact, if R is injective, consider the ideal M in R. The annihilator of M taken in R_R has the form $X \oplus M$, where $X = \text{ann}_\ell^S(M)$ is the kernel of the left action of S on M. By Ex. 3.28(1), $X \oplus M$ is injective when it is viewed as a right module over the ring $R/M \cong S$. From this, it follows that M_S is injective.

Ex. 19.15. Keep the notations in the last exercise, and assume that ${}_SM$ and M_S are faithful, $M = E(V)$ for some module V_S, and that the map $S \to \text{End}(M_S)$ is onto. Show that, upon viewing V as a right ideal of $R = S \oplus M$, we have $R = E(V_R)$. (A special case of this appeared in the arguments in *LMR*-(19.24).)

Solution. First note that, since $M^2 = 0$ in the ring R, M is an ideal and V is a right ideal of R. By the last exercise, R_R is injective, so it only remains to check that $V_R \subseteq_e R_R$. Since $V_S \subseteq_e M_S$, it suffices to show that $M_R \subseteq_e R_R$. Consider any $(s_0, m_0) \in R \backslash M$. Then $s_0 \neq 0$, and the faithfulness of ${}_SM$ implies that $s_0 m \neq 0$ for some $m \in M$. But then

$$(s_0,\, m_0)\,(0,\, m) = (0,\, s_0 m) \in M \backslash \{0\},$$

as desired.

Ex. 19.16A. Suppose R_R is a nonzero cogenerator.

(1) If R has no nontrivial idempotents, show that R is a right self-injective local ring.
(2) If R is a domain, show that R is a division ring.

Solution. (1) Fix a simple right R-module V. Since R_R is a cogenerator, we have an embedding $E(V) \hookrightarrow R_R$ by *LMR*-(19.8). Therefore, $E(V)$ is a direct summand of R_R, so the assumption on R implies that $R_R = E(V)$ is injective. By Exercise 3.2(2), R is a local ring.

(2) R is right self-injective by the above, so R must be a division ring by Exercise 3.2(3).

Ex. 19.16B. For any ring R, show that $E(R_R)$ is an (injective) cogenerator iff R is a right Kasch ring.

Solution. First assume R is right Kasch. Then every simple module V_R embeds into $R_R \subseteq E := E(R_R)$. By *LMR*-(19.9), E is an (injective) cogenerator. Conversely, assume E is a cogenerator. Let V_R be any simple right R-module. By *LMR*-(19.9) again, we may assume that $V \subseteq E$. Since $R_R \subseteq_e E$, we must have $V \cap R \neq 0$ and so $V \subseteq R$. This checks that R is right Kasch.

Ex. 19.16C. (Kato) For any right self-injective ring R, show that the following are equivalent:

(1) R_R is a cogenerator.
(2) R is right Kasch.
(3) $\mathrm{ann}_r(\mathrm{ann}_\ell(A)) = A$ for any right ideal $A \subseteq R$.

Solution. (1) $\Rightarrow$ (3). Since R_R is a cogenerator, *LMR*-(19.6) implies that every module N_R is torsionless (that is, $N \to N^{**}$ is injective). In particular, $(R/A)_R$ is torsionless for any right ideal A. By *LMR*-(19.15), (3) follows.

(3) $\Rightarrow$ (2). Consider any maximal right ideal $\mathfrak{m} \subset R$. Since $\mathrm{ann}_r(\mathrm{ann}_\ell(\mathfrak{m})) = \mathfrak{m}$, we have $\mathrm{ann}_\ell(\mathfrak{m}) \neq 0$. By *LMR*-(8.28), (2) follows.

(2) $\Rightarrow$ (1). Let S be any simple right R-module. By (2), S is isomorphic to a minimal right ideal S_0. Since R_R is injective, R contains a copy of $E(S_0) \cong E(S)$. By *LMR*-(19.8), (1) follows.

Comment. The result in this exercise appeared in T. Kato's paper "Self-injective rings," Tôhoku Math. J. **16**(1967), 485–495.

A right self-injective ring R satisfying any (and hence all) of the conditions (1), (2), (3) above is called a *right* PF *(pseudo Frobenius) ring.* For a more extensive list of conditions characterizing right PF rings, see *LMR*-(19.25).

While an arbitrary right self-injective ring R may not satisfy (3), it is worthwhile to recall (from Step 2 of (2) $\Rightarrow$ (4) in the proof of *LMR*-(15.1)) that $\mathrm{ann}_\ell(\mathrm{ann}_r(B)) = B$ always holds for any f.g. *left* ideal B in R.

Ex. 19.16.D. Let R be a commutative ring that is subdirectly irreducible (i.e. R has a smallest nonzero ideal L). If R is self-injective, show that R is a local cogenerator ring.

Solution. Clearly, L is a simple R-module, and we have $L = \text{soc}(R)$. Also, since every nonzero ideal contains L, we have $L \subseteq_e R$. Using the fact that R is self-injective, we infer from *LMR*-(13.1)$'$ that $R/\text{rad}\ R \cong \text{End}_R(L)$, which is a division ring by Schur's Lemma. Therefore, R is a local ring. The unique simple R-module must be isomorphic to $L \subseteq R$. Since R_R is injective, it follows from *LMR*-(19.9) that R_R is a cogenerator.

Comment. For a commutative subdirectly irreducible ring R, it follows from the above that R is self-injective $\Rightarrow$ R is locally compact (see Müller's First Theorem in *LMR*-(19.68)). The converse is true too, but the proof is harder: we refer the reader to Prop. 6.7 in Xue's "Rings with Morita Duality", Springer Lecture Notes in Math., Vol. **1523**, Springer-Verlag, 1992.

The next four exercises purport to show that "R_R being injective" and "R_R being a cogenerator" are, in general, independent conditions.

Ex. 19.17. Let $R = \prod_{j\in J} A_j$, where the A_j's are division rings, and J is infinite. By *LMR*-(3.11B), R_R is injective. Show that R_R is not a cogenerator, and that R is not right Kasch.

Solution. Let $A = \bigoplus_j A_j$, which is a proper ideal in R. We claim that any $g \in \text{Hom}_R((R/A)_R,\ R_R)$ is zero. Indeed, consider $g(\overline{1}) = r = (r_j)$. For the j^{th} idempotent e_j in R, we have

$$re_j = g(\overline{1})\, e_j = g(\overline{1}\ e_j) = 0.$$

Hence $r = 0$ and so $g = 0$. This shows that R_R is not a cogenerator (see *LMR*-(19.6)). The fact that $\text{Hom}_R((R/A)_R,\ R_R) = 0$ also shows that $R/\mathfrak{m}$ cannot be embedded into R_R for any maximal right ideal $\mathfrak{m} \supseteq A$, so R is not right Kasch. (Alternatively, the fact that A has zero left annihilator implies that R is not right Kasch, as in *LMR*-(8.29)(4).)

Comment. Since R_R is injective, we know from Exercise 16C that the two conclusions in this exercise are actually equivalent. Incidentally, the fact that R here is not a cogenerator ring also shows that the conclusion of Exercise 11 does not hold for *infinite* direct products.

Ex. 19.18. Let k be a field, and S be the commutative k-algebra $k \oplus \bigoplus_{i\geq 1} ke_i$ constructed in Exercise (8.17), with $e_ie_j = \delta_{ij}e_i$. Let $V_i = kv_i\ (i \geq 0)$ be the simple right S-modules constructed in that exercise, with the S-action

(A) $$v_ie_j = \delta_{ij}v_i \quad (i \geq 0,\ j \geq 1).$$

Let M be the right S-module $\bigoplus_{i\geq 0} V_i = kv_0 \oplus kv_1 \oplus \ldots$, and define a left S-action on M by

(B) $$e_jv_i = \delta_{j-1,i}v_i \ \ (i \geq 0,\ j \geq 1).$$

(1) Check that, under the above actions, M is an (S, S)-bimodule, faithful on both sides.
(2) Let $R = S \oplus M$ be the trivial extension of ${}_SM_S$ by S. Show that R is right Kasch but not left Kasch.
(3) Show that R is neither right nor left self-injective.

Solution. (1) First, it is easy to check that the left S-action on M is well-defined, and makes M into a left S-module. Next, a quick computation shows that $(e_j v_i)e_\ell$ and $e_j(v_i e_\ell)$ are both equal to $\delta_{j-1,i}\delta_{i,\ell}v_i$, so M is an (S, S)-bimodule. To show that ${}_SM$ is faithful, take any nonzero

$$\alpha = a_0 + a_1 e_1 + \cdots + a_n e_n \in S,$$

with $a_i \in k$, and $a_n \neq 0$. If $a_0 \neq 0$, then $\alpha v_n = a_0 v_n \neq 0$. We may thus assume $a_0 = 0$, in which case $\alpha v_{n-1} = a_n v_{n-1} \neq 0$. The faithfulness of M_S can be shown similarly.

(2) Since $M^2 = 0$ in R, R has the same (left, right) simple modules as $S \cong R/M$. Now each $V_i = kv_i = v_i R \subseteq R_R$ $(i \geq 0)$, so R is right Kasch. To analyse the left structure, let us label the simple left S-modules by V_i' $(i \geq 0)$, where $\dim_k V_i' = 1$ for all i, and e_j acts as δ_{ij} on V_i' for $j \geq 1$. By (B), we have ${}_S(kv_i) \cong V_{i+1}'$, so ${}_R(V_{i+1}')$ embeds into ${}_RR$ for $i \geq 0$. We claim that ${}_R(V_0')$ *does not* embed into ${}_RR$ (so R is not left Kasch). The idea of proof here is similar to that used in the solution to Exer. (8.17)(2). Indeed, assume that $k \cdot \alpha \cong {}_R(V_0')$ for some

$$\alpha = a_0 + a_1 e_1 + \cdots + a_n e_n + b_0 v_0 + \cdots + b_n v_n \in R.$$

Then $0 = e_{n+2}\,\alpha = a_0 e_{n+2}$ yields $a_0 = 0$. Now

$$0 = e_i\,\alpha = a_i e_i + b_{i-1} v_{i-1}$$

yields $a_i = 0 = b_{i-1}$ for all $i \geq 1$. Thus, $\alpha = 0$, a contradiction.

(3) We know from (1) that ${}_SM$ is faithful. We claim that *the natural map* $\lambda : S \to \mathrm{End}(M_S)$ *is not onto.* By Exercise 18, this would imply that R_R is not injective. Consider $\lambda(\alpha)$ for

$$\alpha = a_0 + a_1 e_1 + \cdots + a_n e_n.$$

For $i \geq n$, we have $\lambda(\alpha)(v_i) = a_0 v_i$. Thus, if we define an S-endomorphism β on $M_S = \bigoplus_{i \geq 0} V_i$ by taking β to be 0 on $V_0, V_2, V_4, \ldots,$ and the identity on $V_1, V_3, \ldots,$ then β cannot be of the form $\lambda(\alpha)$, so $\beta \notin \mathrm{im}(\lambda)$. The proof that ${}_RR$ is not injective is similar.

Ex. 19.19. Keep the above notations and work in the ring R. Show that:

(1) $e_i R = ke_i + V_{i-1}$ $(i \geq 1)$, with $e_i R / V_{i-1} \cong V_i$ as R-modules;
(2) $\mathrm{soc}(e_i R) = V_{i-1} \subseteq_e e_i R$;

(3) For any right ideal $A \subseteq R$ and any $i,\ j \geq 1$, show that any R-homomorphism $f:\ A \to e_iR$ can be extended to an R-homomorphism $g:\ A + e_jR \to e_iR$.
(4) Using (3) and Baer's Criterion, show that $e_iR = E((V_{i-1})_R)$ for any $i \geq 1$.

Solution. Right multiplying e_i by the e_j's and v_j's, we see that

$$e_iR = ke_i + kv_{i-1} = ke_i + V_{i-1} \quad \text{for } i \geq 1.$$

The quotient R-module e_iR/V_{i-1} is generated by $\overline{e}_i$, on which e_i acts as identity, and other e_j's act as zero. Therefore, the quotient module is isomorphic to V_i.

(2) For any $ae_i + bv_{i-1}$ with $a,\ b \in k,\ a \neq 0$, we have

$$(ae_i + bv_{i-1})\, v_{i-1} = av_{i-1} \neq 0.$$

Therefore, $V_{i-1} = kv_{i-1} \subseteq_e e_iR$, which implies that $\text{soc}(e_iR) = V_{i-1}$.

(3) In general, if $B \subseteq R$ is any right ideal, to extend $f:\ A \to e_iR$ to $A + B$ amounts to extending $f|A \cap B$: $A \cap B \to e_iR$ to B. We shall use this observation for $B = e_jR$ $(j \geq 1)$. We need to consider the extension problem only in the case $\dim_k(A \cap B) = 1$, in other words when $A \cap B = \text{soc}(e_jR) = V_{j-1}$. If $j \neq i$,

$$f|A \cap B:\ V_{j-1} \longrightarrow e_iR$$

is the zero map (since $\text{soc}(e_iR) = V_{i-1}$), so there is no problem. If $j = i$, $f|A \cap B$ maps V_{i-1} to itself by the multiplication of a scalar $c \in k$, so obviously $f|A \cap B$ extends to $c \cdot \text{Id}_B$ on B.

(4) In view of $V_{i-1} \subseteq_e e_iR$, we need only check that e_iR is injective for all $i \geq 1$. To do this, we apply Baer's Criterion. Let f: $A \to e_iR$ be an R-homomorphism, where $A \subseteq R$ is a right ideal. We wish to extend f to R_R. Applying (3) repeatedly, we may enlarge the domain A of f to assume that it contains

$$I = e_1R \oplus e_2R \oplus \cdots = \bigoplus_{j \geq 1} (ke_j + kv_{j-1}),$$

which has codimension 1 in R. If $A = R$ we are done; otherwise $A = I$. Consider the restriction:

$$f_j = f|e_jR\ :\ e_jR \longrightarrow e_iR.$$

If $j \neq i,\ i-1$, we have $f_j = 0$ since $\text{soc}(e_jR) = V_{j-1}$ and $e_jR/V_{j-1} \cong V_j$ which admit only zero homomorphisms into e_iR. For $j = i$, f_j is scalar multiplication by some $c \in k$ since $\text{End}_R\ (e_iR) \cong e_iRe_i = ke_i$. For $j = i-1$, f_j is zero on V_{j-1} and $f_j(e_j) = dv_{i-1}$ for some $d \in k$. [*Note.* In case $i = 1$, interpret e_0 as 0 and take $d = 0$.] Let h: $R \to e_iR$ be the

R-homomorphism given by left multiplication by $dv_{i-1} + ce_i$. Then

$$\begin{aligned} h(e_{i-1}) &= (dv_{i-1} + ce_i)\, e_{i-1} = dv_{i-1} = f(e_{i-1}), \\ h(e_i) &= (dv_{i-1} + ce_i)\, e_i = ce_i = f(e_i), \\ h(e_j) &= (dv_{i-1} + ce_i)\, e_j = 0 = f(e_j) \qquad \text{for } j \neq i,\, i-1, \end{aligned}$$

so $h: R \to e_i R$ extends f, as desired.

Comment. It is easy to see that I_R is not a direct summand of R_R. In particular, I_R is not an injective module. This gives an explicit example of a direct sum of injective modules, $\bigoplus_{i\geq 1} e_i R$, which fails to be injective. Of course, the ring R here is neither right nor left noetherian.

Ex. 19.20. For the ring R in the last two exercises, show that:

(1) R_R is a cogenerator, but ${}_RR$ is not a cogenerator;
(2) R is neither right PF nor left PF.

Solution. We continue to use the notations introduced in Exercises 18 and 19.

(1) The V_{i-1}'s ($i \geq 1$) constitute a complete set of simple right R-modules (by Exercise (8.17)). And by Exercise 19(4), R_R contains a copy of the injective hull of each V_{i-1}. According to *LMR*-(19.8), this implies that R_R is a cogenerator. On the other hand, the fact that R is not left Kasch (Exercise 18(2)) implies that ${}_RR$ is not a left cogenerator.

(2) On the right side, R_R is a cogenerator but is not injective by Exercise 18(3), so R is not right PF. On the left side, ${}_RR$ is neither injective nor a cogenerator, so R is not left PF.

Comment. In *LMR*-(19.25), it is mentioned (without proof) that

> *R is right* PF *iff R_R is a cogenerator and there are only finitely many isomorphism classes of simple right R-modules.*

Since there are infinitely many such isomorphism classes V_i ($i \geq 0$), it is to be expected that R is not right PF. Similarly, we could have predicted that R is not left PF.

Exercises 18–20 are adapted from B. Osofsky's paper, "A generalization of quasi-Frobenius rings," J. Algebra 4(1966), 373–387. (The slight difference in notations is due to a shift in the subscript of the e_i's.)

Ex. 19.21. Let U_R be the minimal injective cogenerator over a commutative ring R. View U as a bimodule ${}_RU_R$ and let $F: \mathfrak{M}_R \to {}_R\mathfrak{M}$, $G: {}_R\mathfrak{M} \to \mathfrak{M}_R$ be the U-dual functors, denoted as usual by *. Show that for any simple V_R, V^* is a simple (left) R-module. Using this, show that F, G define a self-duality on the Serre subcategory of R-modules of finite length. (This generalizes the usual self-duality of finite abelian groups noted in *LMR*-(19.29).)

Solution. Let $\{V_i : i \in I\}$ be a complete set of simple right R-modules, so that $U := E(\bigoplus_i V_i)$. Let $V = V_j$ (for a fixed $j \in I$), and consider any nonzero $f \in V^* = \mathrm{Hom}_R(V, U)$. Since $f(V) \cong V$ is simple and $\bigoplus_i V_i \subseteq_e U$, we must have $f(V) \subseteq \bigoplus_i V_i$, and so necessarily $f(V) = V_j$. If g is another nonzero element in V^*, then $g(V) = V_j$ too, so $g = f \circ \alpha$ for a suitable $\alpha \in \mathrm{Aut}_R(V)$. Now, the commutativity of R implies that $\mathrm{End}_R(V)$ consists of multiplications by elements of R. (This is clear if we view V as $R/\mathfrak{m}$ for some maximal ideal $\mathfrak{m}$.) Therefore, $g = r \cdot f \in V^*$ for some $r \in R$. Since this relationship holds for any nonzero $f, g \in V^*$, we see that V^* is simple.

For convenience, let us "identify" the two categories $\mathfrak{M}_R$, ${}_R\mathfrak{M}$, and just write $\mathfrak{M}$ for them. This enables us to also "identify" F and G. Let $\mathfrak{M}^\circ$ be the Serre subcategory (of $\mathfrak{M}$) given by the R-modules of finite length. Since U is injective, F is an exact functor from $\mathfrak{M}^\circ$ to $\mathfrak{M}$. By what we have done above and induction, we see that

$$\mathrm{length}(M^*) = \mathrm{length}(M) \quad \text{for any} \quad M \in \mathfrak{M}^\circ.$$

Therefore, we may view F as a functor from $\mathfrak{M}^\circ$ to $\mathfrak{M}^\circ$. Since U is a cogenerator, $\theta_M : M \to M^{**}$ is injective. If $M \in \mathfrak{M}^\circ$, length $(M^{**}) =$ length (M) implies that θ_M is an isomorphism. Therefore, F defines a self-duality on $\mathfrak{M}^\circ$, as desired.

Comment. Although ${}_RU_R$ defines a duality from $\mathfrak{M}^\circ$ to itself, it does not give a Morita duality from R to R in the sense of *LMR*-(19.41). This is because a Morita duality (from R to R) requires, among other things, that R_R be reflexive with respect to U, i.e. that the natural map $R \to \mathrm{End}_R(U)$ be an isomorphism. This is, in general, not the case for a commutative ring R. (It fails, for instance, for $R = \mathbb{Z}$!) If, indeed, $R \to \mathrm{End}_R(U)$ is an isomorphism, then ${}_RU_R$ is faithfully balanced, and the fact that U_R is an injective cogenerator implies that U defines a Morita duality from R to R, according to Morita I: *LMR*-(19.43).

Ex. 19.22. Let U be an (S, R)-bimodule, and let * denote the U-dual as usual. Let $P_R = \bigoplus_{i \in I} P_i$. We identify P^* with $\prod P_i^*$ and $(\bigoplus P_i^*)^*$ with $\prod P_i^{**}$. Let $\varepsilon : \bigoplus P_i^* \to \prod P_i^*$ be the inclusion map.

(1) Show that $P \in \mathfrak{M}_R[U]$ iff each $P_i \in \mathfrak{M}_R[U]$, ε^* is injective, and $\mathrm{im}(\varepsilon^*) = \bigoplus P_i^{**}$ (in $\prod P_i^{**}$).
(2) Show that ε^* is injective iff $(\prod P_i^* / \bigoplus P_i^*)^* = 0$.
(3) Show that, if ${}_SU$ is injective or a cogenerator, then $P \in \mathfrak{M}_R[U]$ implies that $P_i = 0$ for almost all i.

Solution. (1) We have the following commutative diagram

$$\begin{array}{ccccc} P & \xrightarrow{\theta_P} & \left(\prod P_i^*\right)^* & \xrightarrow{\varepsilon^*} & \left(\bigoplus P_i^*\right)^* \\ & & \| & & \| \\ \bigoplus P_i & \xrightarrow{\oplus\,\theta_i} & \bigoplus P_i^{**} & \hookrightarrow & \prod P_i^{**} \end{array}$$

where θ_P is the natural map from P to P^{**}, and $\theta_i := \theta_{P_i}$. If $P \in \mathfrak{M}_R[U]$, it is easy to see that each direct summand $P_i \in \mathfrak{M}_R[U]$. The diagram above then shows that ε^* is injective, with image exactly equal to $\bigoplus P_i^{**}$. Conversely, if each $P_i \in \mathfrak{M}_R[U]$, then θ_P is injective. If, moreover, ε^* is injective with image $\bigoplus P_i^{**}$, then θ_P is surjective, so $P \in \mathfrak{M}_R[U]$. This proves (1).

(2) Consider the short exact sequence induced by ε:

$$0 \longrightarrow \bigoplus P_i^* \xrightarrow{\varepsilon} \prod P_i^* \longrightarrow \left(\prod P_i^*\right) / \bigoplus P_i^* \longrightarrow 0.$$

Applying the left exact functor *, we get

$$0 \longrightarrow \left(\prod P_i^* / \bigoplus P_i^*\right)^* \longrightarrow \left(\prod P_i^*\right)^* \xrightarrow{\varepsilon^*} \left(\bigoplus P_i^*\right)^*.$$

Therefore, ε^* is injective iff $(\prod P_i^* / \bigoplus P_i^*)^* = 0$.

(3a) Assume that ${}_SU$ is injective. Since ε is an inclusion in the category of left S-modules, ε^* is surjective. If $P = \bigoplus_i P_i \in \mathfrak{M}_R[U]$, $\mathrm{im}(\varepsilon^*) = \bigoplus P_i^{**}$ by (1). Therefore, $\bigoplus P_i^{**} = \prod P_i^{**}$. This implies that $P_i^{**} = 0$ for almost all i. But each $P_i \in \mathfrak{M}_R[U]$, so $P_i = 0$ for almost all i.

(3b) Assume now that ${}_SU$ is a cogenerator, and that $P = \bigoplus P_i \in \mathfrak{M}_R[U]$. Then ε^* is injective by (1) and so $(\prod P_i^* / \bigoplus P_i^*)^* = 0$ by (2). The fact that ${}_SU$ is a cogenerator then implies that the left S-module $\prod P_i^* / \bigoplus P_i^*$ is zero. This means that $P_i^* = 0$ for almost all i, and therefore $P_i \cong P_i^{**} = 0$ for almost all i.

Comment. The theme that, under suitable hypotheses, a reflexive object cannot be an infinite direct sum of nonzero objects is due to B. Osofsky. The formulation of the exercise above was suggested to me by I. Emmanouil; see Kasch's book "Modules and Rings", p. 329, for (3b) in the special case $U = {}_RR_R$.

Ex. 19.23. Show that the hypothesis on U in (3) of the above exercise cannot be dropped.

Solution. In the above Exercise, let $R = S = \mathbb{Z}$, and ${}_SU_R = \mathbb{Z}$ also. We know from Exercise 2.8′ that $P = \mathbb{Z} \oplus \mathbb{Z} \oplus \cdots$ is a *reflexive* $\mathbb{Z}$-module. This shows that, in (3) of the above exercise, the hypothesis on U cannot be dropped. Of course, in the present case, ${}_SU = {}_{\mathbb{Z}}\mathbb{Z}$ is neither injective nor a cogenerator.

Actually, Exercise 22 does shed some light on the reflexivity of $P = \mathbb{Z} \oplus \mathbb{Z} \oplus \cdots$ (and vice versa). In the notation in that exercise, we have $P_i = \mathbb{Z}$ (for $i \geq 1$) with $P_i^* \cong \mathbb{Z}$, and

$$\left(\prod_i P_i^* / \bigoplus_i P_i^*\right)^* = \mathrm{Hom}_{\mathbb{Z}}\left(\left(\prod_i \mathbb{Z}\right) / \bigoplus_i \mathbb{Z},\ \mathbb{Z}\right) = 0$$

according to LMR-(2.8)$'$. Thus, the map g^* in Exercise 22 is injective here. Hence the reflexivity of P hinges upon the equation

$$\mathrm{im}(g^*) = \bigoplus_i P_i^{**} = \bigoplus_i \mathbb{Z}.$$

This translates into the statement that, for any homomorphism $f : \prod_{i=1}^{\infty} \mathbb{Z} \to \mathbb{Z}$, $f(e_i)$ are almost all zero (where the e_i's are the unit vectors). This is precisely the "slenderness" property we verified in Exercise 2.8$'$.

Ex. 19.24. Suppose ${}_SU_R$ defines a Morita duality from R to S. Show that any $M \in \mathfrak{M}_R^{fg}$ has a projective cover; that is, there exists an epimorphism $\pi : P \to M$ with P_R projective and $\ker(\pi)$ small (= superfluons) in P.

Solution. Fix a surjection $R^n \to M$. This induces an injection $M^* \hookrightarrow U^n$ in ${}_S\mathfrak{M}$, so an injective hull $E = E(M^*)$ can be found as a direct summand of ${}_SU^n$. Since ${}_SU \in {}_S\mathfrak{M}[\mathrm{U}]$, we have $E \in {}_S\mathfrak{M}[U]$. Dualizing, we get surjections $R^n \twoheadrightarrow E^* \overset{\pi}{\twoheadrightarrow} M$ in $\mathfrak{M}_R\,[U]$, where $(E^*)_R$ is projective. We finish by showing that $\ker(\pi)$ is small. Let $Q_R \subseteq E^*$ be such that $Q + \ker(\pi) = E^*$. Then we have $Q \twoheadrightarrow M$, which dualizes to $M^* \hookrightarrow Q^*$. Since $M^* \subseteq_e E$, the surjection $E \twoheadrightarrow Q^*$ (dual to $Q \hookrightarrow E^*$) must be an isomorphism. This implies that $Q = E^*$, as desired.

Comment. It is shown in LMR-(19.50) that:

> *Whenever ${}_SU_R$ defines a Morita duality from R to S, the two rings R, S must be semiperfect.*

This exercise gives another view of this basic result. Indeed, the fact that any $M \in \mathfrak{M}_R^{fg}$ has a projective cover is tantamount to R being semiperfect, according to FC-(24.16). With a Morita duality given from R to S, it is quite natural to deduce the existence of a projective cover of $M \in \mathfrak{M}_R^{fg}$ from the existence of an injective hull of M^* in ${}_S\mathfrak{M}$.

Ex. 19.25. Suppose ${}_SU_R$ defines a Morita duality from R to S. If R is right artinian, show that S is left artinian, and U gives a duality between $\mathfrak{M}_R^{fg} = \mathfrak{M}_R[U]$ and ${}^{fg}_S\mathfrak{M} = {}_S\mathfrak{M}[U]$.

Solution. We have $\mathfrak{M}_R^{fg} = \mathfrak{M}_R^{\ell c}$ by LMR-(19.65), and $\mathfrak{M}_R^{\ell c} = \mathfrak{M}_R[U]$ by Müller's First Theorem (LMR-(16.68)). Therefore, $\mathfrak{M}_R^{fg} = M_R[U]$. Also, any f.g. M_R is f.cog. (since M_R is artinian), and conversely, any f.cog. M'_R is in $\mathfrak{M}_R[U]$ (by Morrita I: LMR-(19.43)), and is therefore f.g.

It remains to show that S is left artinian. Once this is proved, we will have ${}^{fg}_S\mathfrak{M} = {}_S\mathfrak{M}[U]$ as in the last paragraph, and U gives a duality between $\mathfrak{M}_R[U] = \mathfrak{M}_R^{fg}$ and ${}_S\mathfrak{M}[U] = {}^{fg}_S\mathfrak{M}$.

To see that S is left artinian, consider any f.g. S-module ${}_SN$. By the given duality, $(N^*)_R$ is f.cog., and therefore f.g. by the first paragraph. Taking U-dual again, ${}_S(N^{**})$ is f.cog. But ${}_S(N^{**}) \cong {}_SN$ by Morita I (LMR-(19.43)), so ${}_SN$ is f.cog. Having now shown that f.g. left S-modules are f.cog., we conclude from LMR-(19.4) that S is left artinian.

Comment. In the context of this exercise, one can ask the following question: *If R is artinian, is S also necessarily artinian*? The answer is apparently unknown. (This exercise only guarantees that S is left artinian.)

Ex. 19.26. Suppose ${}_SU_R$ defines a Morita duality from R to S. If R is QF, show that S is also QF.

Solution. First, since R is right artinian, the last exercise implies that S is left artinian. Next, we use the fact that, under the given duality, U_R corresponds to ${}_SS$. By Morita I (*LMR*-(19.43)), U_R is injective, so U_R is projective. (since R is QF). As we saw in the last exercise, U_R is also f.g. Therefore, the corresponding module ${}_SS$ is injective. Since S is left artinian, this implies that S is QF, by *LMR*-(15.1).

Comment. There is an extension of this exercise to the more general class of cogenerator rings. The proof is similar, but depends on Onodera's 1-sided characterization of cogenerator rings (*LMR*-(19.69)): see Exercise 32 below.

Ex. 19.27. In general, an artinian module N_R is l.c. (*LMR*-(19.64)). Show that the converse holds if every nonzero right R-module has a simple submodule.

Solution. Let N_R be l.c. We propose to show that any quotient M of N is f.cog. According to Exercise 1, this will imply that N is artinian.

The important fact here is that $\mathfrak{M}_R^{\ell c}$ is a Serre subcategory of $\mathfrak{M}_R$. In view of this, the quotient module M must be l.c., and so is its socle soc(M). In particular, soc(M) is f.g. (Argue directly from the definition of linear compactness, or use the fact that l.c. modules have finite uniform dimension: *LMR*-(19.62).) Finally, since every nonzero submodule of M has a simple submodule, soc$(M) \subseteq_e M$. Combining this with the finite generation of soc(M), we conclude from *LMR*-(19.1) that M is f.cog., as desired.

Ex. 19.28. Let R be a *left* perfect ring (i.e. $R/\text{rad } R$ is semisimple, and for any sequence $\{a_1, a_2, \ldots\} \subseteq \text{rad } R$, $a_1a_2 \cdots a_n = 0$ for some n).

(1) Show that every nonzero right R-module has a simple submodule;
(2) For any M_R, show that soc$(M) \subseteq_e M$;
(3) Deduce from the last exercise that a right R-module is l.c. iff it is artinian.
(4) Show that a noetherian right R-module is artinian.

Solution. (1) Let $J = \text{rad } R$, and consider any $N_R \neq 0$. We claim that *there exists* $0 \neq x \in N$ *such that* $xJ = 0$. For, if otherwise, for a fixed nonzero element $y \in N$, we would have $ya_1 \neq 0$ for some $a_1 \in J$, and $(ya_1)a_2 \neq 0$ for some $a_2 \in J$, etc. This would give a sequence $\{a_1, a_2, \ldots\} \subseteq J$ with $a_1a_2 \cdots a_n \neq 0$ for any n, contrary to our assumption on J.

For the element $x \in N\backslash\{0\}$ above with $xJ = 0$, xR is a nonzero (right) module over the semisimple ring R/J, so it contains a simple R/J-submodule. It follows that N contains a simple R-submodule, as desired.

(2) Given any right R-module M, consider $S = \text{soc}(M)$. For any nonzero submodule $N \subseteq M$, (1) shows that N contains a simple submodule V. Then $V \subseteq S$, and so $N \cap S \supseteq V \neq 0$. This checks that $S \subseteq_e M$.

(3) The conclusion here follows from (1), the last exercise, and *LMR*-(19.64).

(4) Suppose M_R is noetherian and consider any quotient module N of M. Then N is also noetherian, so $S := \text{soc}(N)$ is certainly f.g. By (2), we have also $S \subseteq_e N$. Therefore, N is f.cog. (by the characterization of f.cog. modules in *LMR*-(19.1)). Since this is true for every quotient N of M, it follows that M is artinian by *LMR*-(19.4).

Comment. In *LMR*-(19.65), the statement "a right R-module is l.c. iff it is artinian" is proved for any right artirian ring R. Part (3) of this exercise is a generalization, since any right artinian ring is (right and left) perfect.

In parallel to (4), it is also true that, for the left perfect ring R, any artinian *right* R-module is noetherian. This will follow from the next exercise.

Ex. 19.29. Let R be a right perfect ring (i.e. $R/\text{rad } R$ is semisimple, and for any sequence $\{a_1, a_2, \dots\} \subseteq \text{rad } R$, $a_n \cdots a_2 a_1 = 0$ for some n). Show that if a module M_R is l.c., then it is noetherian. Deduce that, over a perfect ring, a (left or right) module is l.c. iff it has finite length.

Solution. Let $J = \text{rad }(R)$ and consider any submodule $N \subseteq M$. Then N and hence N/NJ are l.c. as R-modules. Since N/NJ is an R/J-module, it is semisimple, so N/NJ must be f.g. Finally, the hypothesis on J implies that NJ is superfluous in N (see *FC*-(23.16)), so N is f.g. Since this holds for every submodule $N \subseteq M$, M is noetherian.

Now assume R is (left and right) perfect, and let M be a l.c. (say right) R-module. By Exercise 28 and the above, M is both artinian and noetherian. Therefore, M has finite length.

Ex. 19.30. (Osofsky, Sandomierski) Let R be a ring such that R_R is l.c. If R is 1-sided perfect, show that R must be right artinian.

Solution. The idea here is to apply the last two exercises to the module $M = R_R$.

First assume R is *left* perfect. By Exercise 28, R is right artinian. Now assume R is *right* perfect. Then, by Exercise 29, R is right noetherian. The ideal $J = \text{rad }(R)$ is clearly nil, so it is nilpotent by Levitzki's Theorem (*FC*-(10.30)). Therefore, R is a semiprimary ring. Since R is right noetherian, the Hopkins-Levitzki Theorem (*FC*-(4.15)) implies that R is right artinian.

Comment. The conclusion of this exercise applies, in particular, to any ring R that admits a Morita duality (into another ring S). In this case, the result is due to B. Osofsky; note that once we know R is right artinian, Exercise 25 implies that S is left artinian.

Ex. 19.31. (Osofsky) Let R be a cogenerator ring. If R is 1-sided perfect, show that R is QF.

Solution. Since R is a cogenerator ring, the R-dual functor defines a Morita duality from reflexive right R-modules to reflexive left R-modules. If R is also 1-sided perfect, the *Comment* following the last exercise implies also that R is right artinian. This fact, coupled with the right self-injectivity of R, shows that R is QF.

Comment. A more general result is true, according to B. Osofsky; namely:

> *If R is a self-injective ring, and R is left perfect, then R is already* QF.

This can be proved as follows, assuming the theorem *LMR*-(19.25) characterizing the 1-sided PF (pseudo-Frobenius) rings.

Assume first R is right self-injective and left perfect. Then, by *FC*-(23.20), every nonzero right R-module M has a simple submodule. It follows that every such M_R has an essential socle. Therefore, by *LMR*-(19.25)(5), R is right PF, and therefore a Kasch ring. If R is *also* assumed to be left self-injective, then *LMR*-(19.25)(2) (applied to the left side) shows that R is left PF. Now R is a cogenerator ring, and the exercise solved above implies that R is QF.

Recall that a QF ring may be defined as a right (or left) artinian right self-injective ring. In attempts to relax the very strong artinian (or noetherian) condition, there are various results of the form

$$\text{“Right self-injective} + X \implies \text{QF”}$$

in the literature, where X is a suitable condition (or combination of conditions) weaker than the original chain conditions. For a survey of results of this nature, see C. Faith's article, "When self-injective rings are QF: a report on a problem," Centre Recerca Matematica, Institut d'Estudis Catalans, Spain, 1990. An outstanding problem in this area of investigation is whether

$$\text{“Right self-injective} + \text{perfect} \implies \text{QF.”}$$

Even the weaker problem, whether a right self-injective semiprimary ring need be QF, is unsettled. The truth of this statement is referred to by some authors as "Faith's Conjecture."

Ex. 19.32. Onodera's 1-sided characterization of cogenerator rings states that *a ring R is a cogenerator ring iff R_R is a l.c. cogenerator* (see *LMR*-(19.69)). Suppose ${}_SU_R$ defines a Morita duality from R to S. Using Onodera's result, show that if R is a cogenerator ring, then so is S.

Solution. Under the duality assumption, we know that R_R and ${}_SS$ are l.c. by Müller's First Theorem (*LMR*-(19.68)). It suffices to show that ${}_SS$ is a cogenerator, for then Onodera's result cited above will imply that S is a cogenerator ring. Now under the given duality, U_R in $\mathfrak{M}_R[U]$ corresponds to ${}_SS$ in ${}_S\mathfrak{M}[U]$. By Morita I, U_R is a cogenerator, and hence a generator by *LMR*-(19.19) (since R is assumed to be a cogenerator ring). Therefore, ${}_SS$ is a cogenerator, as desired.

Comment. This exercise may be viewed as an extension of Exercise 26 to rings possibly without noetherian or artinian conditions.

Ex. 19.33. (Leptin) Let $N \subseteq M$ be R-modules where N is l.c. Let $\{A_i\colon i \in I\}$ be an inverse system of submodules in M, in the sense that, for any finite $J \subseteq I$, there exists $j \in I$ such that $A_j \subseteq A_i$ for all $i \in J$. Show that $\bigcap_{i\in I}(N + A_i) = N + \bigcap_{i\in I} A_i$.

Solution. We need only prove "$\subseteq$." Let $m \in \bigcap_i(N + A_i)$, and write $m = n_i + a_i$ where $n_i \in N$ and $a_i \in A$ (for each $i \in I$). Then the family $\{n_i,\ N\cap A_i\}_{i\in I}$ is finitely solvable. Indeed, if $J \subseteq I$ is finite, there exists $j \in I$ such that $A_j \subseteq A_i$ for all $i \in J$. Then, for any $i \in J$,

$$n_j - n_i = a_i - a_j \in N \cap A_i \implies n_j \equiv n_i \ (\text{mod}\ N \cap A_i).$$

Since N is l.c., there exists $n \in N$ such that $n \equiv n_i$ (mod$N \cap A_i$) for all $i \in I$. Therefore,

$$m - n = (n_i - n) + a_i \in A_i \quad (\forall\, i \in I),$$

so $m = n + (m - n) \in N + \bigcap_{i\in I} A_i$, as desired.

Comment. For more general results along this direction, see H. Leptin: "Linear kompakte moduln und ringe," Math. Zeit. **63**(1955), 241–267.

Ex. 19.34. (Sandomierski) Let $N \subseteq M$ be R-modules where N is l.c. Show that the family $\mathcal{F}$ of submodules $A \subseteq M$ such that $N + A = M$ has a minimal member.

Solution. We define a partial ordering on $\mathcal{F}$ by taking $A_1 \leq A_2$ iff $A_1 \subseteq A_2$. We are done if we can apply Zorn's Lemma to $\{\mathcal{F}, \leq\}$. Let $\{A_i\colon i \in I\}$ be a totally ordered family of submodules in $\mathcal{F}$. Then $\{A_i\colon i \in I\}$ is an inverse system of submodules in the sense of the last exercise, so by that exercise,

$$N + \bigcap\nolimits_{i\in I} A_i = \bigcap\nolimits_i (N + A_i) = \bigcap\nolimits_i M = M.$$

This means that $A := \bigcap_{i\in I} A_i \in \mathcal{F}$, so A is a lower bound of $\{A_i\colon i \in I\}$, and Zorn's Lemma applies.

Comment. A minimal member of the family $\mathcal{F}$ above is called an *addition complement* of N in M. Sandomierski's result above shows that a l.c. module N has an addition complement in any module M containing it. A module M is called *complemented* if every submodule of M has an addition complement. It follows from this exercise that *any l.c. module M is complemented* (since any submodule $N \subseteq M$ is also l.c.) Sandomierski's paper "Linearly compact modules and local Morita duality" appeared in Proc. Conf. Ring Theory Utah 1971 (R. Gordon, ed.), pp. 333–346, Academic Press, London/New York, 1972.

The notion of a complemented module has various other useful connections. For instance, *a ring R is semiperfect iff the module R_R (resp. ${}_RR$) is complemented.* This result appeared in the paper of F. Kasch and E. A. Mares, "Eine Kennzeichnung semiperfekter Moduln," Nagoya Math. J. **27**(1966), 525–529. The notion of a semiperfect module, generalizing Bass' notion of a semiperfect ring, is due to Mares; see her paper "Semiperfect modules," Math. Zeit. **82**(1963), 347–360.

Combining the present exercise with the result of Kasch and Mares, we see that *if R_R is l.c., then R is semiperfect.* This was proved in the special case of rings admitting Morita dualities in *LMR*-(19.50).

Ex. 19.35. It is stated in *LMR*-(19.58) that $\mathfrak{M}_R^{\ell c}$ is a Serre subcategory of $\mathfrak{M}_R$, i.e. for any short exact sequence $0 \to K \to L \to M \to 0$ in $\mathfrak{M}_R$, L is l.c. iff K and M are l.c. Prove the "if" part of this statement.

Solution. Assume that K and M are l.c., and let $\{x_i,\, N_i\}_{i\in I}$ be a finitely solvable system in L, where the N_i's are submodules of L. To solve this system, we may assume that the family $\{N_i\}$ is closed under finite intersections. (In fact, for any finite set $J \subseteq I$, let $x_J \in L$ be such that $x_J \equiv x_i \pmod{N_i}$ for any $i \in I$. Letting $N_J = \bigcap_{i\in J} N_i$, we check easily that $\{x_J, N_J\}$ (J ranging over finite subsets of I) is still finitely solvable, so it suffices to solve this system, where the N_J's are now closed under finite intersections.)

Using the linear compactness of L/K on the system $\{\bar{x}_i,\, \bar{N}_i\}$, we first find $\ell \in L$ such that $\bar{\ell} \equiv \bar{x}_i \pmod{\bar{N}_i}$ for all i. Then $\ell - x_i = n_i + k_i$ where $n_i \in N_i$, $k_i \in K$. We claim that $\{k_i, K \cap N_i\}_{i\in I}$ is finitely solvable in the submodule K. In fact, let $J \subseteq I$ be finite. By assumption $\bigcap_{i\in J} N_i = N_j$ for some index j, so

$$k_j - k_i = (n_i - n_j) + (x_i - x_j) \in N_i \cap K \quad (\forall\, i \in J).$$

Therefore, by the linear compactness of K, there exists $k \in K$ such that $k \equiv k_i \pmod{K \cap N_i}$ for all $i \in I$. Now

$$(\ell - k) - x_i = n_i + (k_i - k) \in N_i \quad (\forall\, i \in I),$$

so $\ell - k \in L$ solves the given system $\{x_i,\, N_i\}_{i\in I}$ in L.

Comment. The above result appeared in D. Zelinsky's paper "Linearly compact modules and rings," Amer. J. Math. **75**(1953), 79–90. Zelinsky's inspiration came from Lefshetz' result that if H is closed subgroup of a topological group, and if any two of the groups H, G, G/H are compact, then so is the third: see (II. 5.5) in Lefshetz' book "Algebraic Topology," Colloq. Publications, Vol. 27, Amer. Math. Soc., 1942.

Ex. 19.36. (Essentially Grothendieck) Let K/k be a finite Galois field extension with Galois group G. Let $\mathcal{B}$ be the category of finite G-sets, and $\mathcal{A}$ be the category of finite-dimensional commutative étale k-algebras that are split over K (i.e. algebras A such that $A \otimes_k K \cong K \times \cdots \times K$). Show that there are natural contravariant functors $F : \mathcal{A} \to \mathcal{B}$ and $F' : \mathcal{B} \to \mathcal{A}$ defining a duality between $\mathcal{A}$ and $\mathcal{B}$ such that the transitive G-sets in $\mathcal{B}$ correspond to field extensions of k that are embeddable into K.

Solution. To solve this problem, it is convenient to use an equivalent definition for the algebras in $\mathcal{A}$, namely, $A \in \mathcal{A}$ iff $A \cong A_1 \times \cdots \times A_n$ where each A_i is a field extension of k embeddable into K. (Needless to say, all embeddings and algebra homomorphisms considered below are over k.) This and other characterizations of étale algebras can be found in Bourbaki.

In this problem, the field K plays the role of the object U in the duality set-up of "Morita I." Notice that K is both a G-set and an étale algebra split by K. For a G-set $B \in \mathcal{B}$, let

$$B^* = \mathrm{Hom}_G(B, K).$$

Adding and multiplying functions pointwise makes B^* into a commutative ring. This is in fact a k-algebra if we view the constant functions into k as the scalars in B^*. To see that $B^* \in \mathcal{A}$, we break up B into transitive G-sets to reduce to the case where B is transitive. In this case, if a function $\beta : B \to K$ is nonzero at one point, then it is nonzero at all points, and so β^{-1} exists in B^*. This shows B^* is a field, and evaluating functions on a given point provides a field embedding of B^* into K. This shows that $B^* \in \mathcal{A}$ for any $B \in \mathfrak{B}$. Defining $F' : \mathcal{B} \to \mathcal{A}$ by $F'(B) = B^*$ (and making the obvious definition on morphisms), we get a contravariant functor F'.

Now consider an étale algebra $A \in \mathcal{A}$: we define

$$A^* := \mathrm{Hom}_{k\text{-alg}}\,(A,\ K).$$

Since $\dim_k A < \infty$, A is semilocal (i.e. with only finitely many maximal ideals), so $|A^*| < \infty$. The G-action on K makes A^* into a G-set, so $A^* \in \mathcal{B}$. With again the obvious definition on morphisms, $F : \mathcal{A} \to \mathcal{B}$ given by $F(A) = A^*$ defines a contravariant functor from $\mathcal{A}$ to $\mathcal{B}$. The remaining job is to show that $F' \circ F$ and $F \circ F'$ are naturally equivalent to the identity functors on $\mathcal{A}$ and $\mathcal{B}$.

To treat $F \circ F'$, note that we have a natural map $ev_B : B \to B^{**}$ defined in the usual way ("by evaluation"):

$$(ev_B(b))\ \ (\beta) = \beta(b) \ \text{ for } \ \beta \in B^* \ \text{ and } \ b \in B.$$

To see that ev_B is bijective, it suffices to consider the case where B is a *transitive* G-set. Represent B in the form G/H (coset space with respect to a subgroup H) and let $A = K^H$ (the fixed field of H). By separability, $A = k(\theta)$ for a suitable primitive element $\theta \in A$. Galois theory implies that the fixed group of θ is exactly H, so there is a G-set isomorphism

$$\beta : G/H \longrightarrow G \cdot \theta \subseteq K.$$

In particular, $b \neq b'$ in $B = G/H$ implies that $\beta(b) \neq \beta(b')$. This checks the injectivity of ev_B. For surjectivity, note that, in the above notation, B^* is isomorphic to the field A. By Galois theory again, the various embeddings of A into K are given by applying elements of the Galois group G. Therefore, B^* is also a *transitive* G-set. It follows that the injective G-map $ev_B : B \to B^{**}$ must be an isomorphism.

To treat $F' \circ F$, consider $A \in \mathcal{A}$. Again we have an algebra homomorphism $ev_A : A \to A^{**}$, defined by evaluation:

$$(ev_A(a))(\alpha) = \alpha(a) \text{ for any } a \in A \text{ and } \alpha \in A^*.$$

To show that ev_A is bijective, we go into the crucial case where A is a field. In this case, the injectivity of ev_A is automatic. For convenience, let us represent A as a k-subfield of K. For $H = \text{Gal}(K/A)$, we may identify A^* with the transitive G-set G/H. As we have seen in the last paragraph, $A^{**} \cong (G/H)^*$ is an algebra of k-dimension $[G : H]$. Since

$$\dim_k A = \dim_k K / \dim_A K = |G|/|H| = [G : H]$$

too, it follows that the injection $ev_A : A \to A^{**}$ is a (k-algebra) isomorphism.

Comment. The duality treated in this exercise is a variant of the classical Galois duality (cf. *LMR*-(19.27)) between subgroups of G and k-subfields of K. But the categories involved in these dualities are different. In the category of subgroups of G, the morphisms are taken to be inclusion maps only (and similarly for subfields of K). In the category of G-sets, the morphisms are arbitrary G-maps (and similarly for the category of étale algebras). There are "fewer" G-sets than subgroups of G since two subgroups H, $H' \subseteq G$ give rise to the "same" G-set if (and only if) H and H' are conjugate in G. A similar remark can be made about subfields of K and étale algebras split by K.

There is also a profinite version of this exercise, although we will not give the details here. In this profinite version, K/k is an infinite Galois extension, and G is the profinite Galois group $\text{Gal}(K/k)$. The definition for the category $\mathcal{A}$ is unchanged, and we take $\mathcal{B}$ to be the category of "continuous" finite G-sets (finite sets on which G acts via a suitable discrete quotient). The definitions for the functors F and F' are unchanged, and the proof for the duality is basically reduced to the case of $[K : k] < \infty$.

There are other variants of the Galois duality too. For instance, the Jacobson-Bourbaki correspondence is such an example. For the details, see Jacobson's Basic Algebra II, pp. 468–471.

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