POINCARÉ DUALITY AND COMMUTATIVE DIFFERENTIAL GRADED ALGEBRAS

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Abstract. We prove that every commutative differential graded algebra whose cohomology is a simply-connected Poincaré duality algebra is quasi-isomorphic to one whose underlying algebra is simply-connected and satisfies Poincaré duality in the same dimension. This has applications in rational homotopy, giving Poincaré duality at the cochain level, which is of interest in particular in the study of configuration spaces and in string topology.

Dualité de Poincaré et algèbres différentielles graduées commutatives.

Résumé. Nous démontrons que toute algèbre différentielle graduée commutative (ADGC) dont la cohomologie est une algèbre simplement connexe à dualité de Poincaré est quasi-isomorphe à une ADGC dont l'algèbre sous-jacente est à dualité de Poincaré dans la même dimension. Ce résultat a des applications en théorie de l'homotopie rationnelle, permettant d'obtenir la dualité de Poincaré au niveau des cochaines, entre autres dans l'étude des espaces de configurations et en topologie des cordes.

1. Introduction

The first motivation for the main result of this paper comes from rational homotopy theory. Recall that Sullivan [15] has constructed a contravariant functor

$$\mathrm{A_{PL}}\colon \mathrm{Top} \to \mathrm{CDGA}_{\mathbb{Q}}$$

from the category of topological space to the category of commutative differential graded algebras over the field $\mathbb{Q}$ (see Section 2 for the definition). The main feature of $\mathrm{A_{PL}}$ is that when X is a simply-connected space with rational homology of finite type, then the rational homotopy type of X is completely encoded in any CDGA (A,d) weakly equivalent to $\mathrm{A_{PL}}(X)$. By *weakly equivalent* we mean that (A,d) and $\mathrm{A_{PL}}(X)$ are connected by a zig-zag of CDGA morphisms inducing isomorphism in homology, or *quasi-isomorphisms* for short,

$$(A,d) \xleftarrow{\simeq} \cdots \xrightarrow{\simeq} \mathrm{A_{PL}}(X).$$

We then say that (A,d) is a *CDGA-model* of X (see [4] for a complete exposition of this theory). We are particularly interested in the case when X is a simply-connected closed manifold of dimension n, since then $\mathrm{H}^*(A,d)$ is a simply-connected

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Poincaré duality algebra of dimension n (a graded algebra A is said to be *simply-connected* if A^0 is isomorphic to the ground field and $A^1 = 0$; see also Definition 2.1).

Our main result is the following:

Theorem 1.1. *Let* $\mathbf{k}$ *be a field of any characteristic and let* (A, d) *be a CDGA over* $\mathbf{k}$ *such that* $\mathrm{H}^*(A, d)$ *is a simply-connected Poincaré duality algebra in dimension* n*. Then there exists a CDGA* (A', d') *weakly equivalent to* (A, d) *and such that* A' *is a simply-connected algebra satisfying Poincaré duality in dimension* n*.*

Our theorem was conjectured by Steve Halperin over 20 years ago. The A' of the theorem is called a *differential Poincaré duality algebra* or *Poincaré duality CDGA* (Definition 2.2). In particular any simply-connected closed manifold admits a Poincaré duality CDGA-model.

Notice that the theorem is even valid for a field of non-zero characteristic. Also our proof is very constructive: Starting from a finite-dimensional CDGA (A, d), it shows how to compute explicitly a weakly equivalent differential Poincaré duality algebra (A', d'). We will also prove in the last section that under some extra connectivity hypotheses, any two such weakly equivalent differential Poincaré duality algebra can be connected by a zig-zag of quasi-isomorphisms between differential Poincaré duality algebras.

Aubry, Lemaire, and Halperin [1] and Lambrechts [8, p.158] prove the main result of this paper in some special cases. Also in [13] Stasheff proves some chain level results about Poincaré duality using Quillen models. An error in Stasheff's paper was corrected in [1].

Before giving the idea of the proof of Theorem 1.1, we describe a few applications of this result.

1.1. **Applications.** There should be many applications of this result to constructions in rational homotopy theory involving Poincaré duality spaces. We consider here two: The first is to the study of configuration spaces over a closed manifold, and the second to string topology.

Our first application is to the determination of the rational homotopy type of the configuration space

$$F(M, k) := \{(x_1, \ldots, x_k) \in M^k : x_i \neq x_j \text{ for } i \neq j\}$$

of k points in a closed manifold M of dimension n. When $k = 2$ and M is 2-connected, we showed in [9, Theorem 1.2] that if A is a Poincaré duality CDGA-model of M then a CDGA-model of $F(M, 2)$ is given by

$$A \otimes A/(\Delta) \tag{1.1}$$

where (Δ) is the differential ideal in $A \otimes A$ generated by the so-called *diagonal class* $\Delta \in (A \otimes A)^n$.

For $k \geq 2$ we have constructed in [10] an explicit CDGA

$$F(A, k)$$

generalizing (1.1) and which is an $A^{\otimes k}$*-DGmodule model* of $F(M, k)$. Poincaré duality of the CDGA A is an essential ingredient in the construction of $F(A, k)$. If M is a smooth complex projective variety then we can use $H^*(M)$ as a model for M, and in this case $F(H^*(M), k)$ is exactly the model of Kriz and Fulton-Mac

Pherson [7][6] for $F(M,k)$. However we do not know in general if $F(A,k)$ is also a CDGA-model although it seems to be the natural candidate.

A second application is to string topology, a new field created by Chas and Sullivan [3]. They constructed a product, a bracket and a Δ operator on the homology of the free loop space $LM = M^{S^1}$ of a closed simply-connected manifold M, that turned it into Gerstenhaber algebra and even a BV algebra. On the Hochschild cohomology $\mathrm{HH}^*(A,A)$ of a (differential graded) algebra A, there are the classical cup product and Gerstenhaber bracket, and Tradler [14] showed that for $A = C^*(M)$ there is also a Δ operator on Hochschild homology making it into a BV algebra. Menichi [11] later reproved this result and showed that the Δ can be taken to be the dual of the Connes boundary operator. Recently Felix and Thomas [5] have shown that over the rationals the Chas-Sullivan BV structure on the homology of LM is isomorphic to the BV structure on $\mathrm{HH}^*(C^*(M), C^*(M))$. Their proof uses the main result of this paper. Yang [16] also uses our results to give explicit formulas for the BV-algebra structure on Hochschild cohomology.

1.2. **Idea of proof.** The proof is completely constructive. We start with a CDGA (A,d) and an orientation $\epsilon\colon A^n \to \mathbf{k}$ (Definition 2.3). We consider the pairing at the chain level

$$\phi\colon A^k \otimes A^{n-k} \to \mathbf{k},\ a \otimes b \mapsto \epsilon(ab)$$

We may assume that ϕ induces a non degenerate bilinear form on cohomology making $\mathrm{H}^*(A)$ into a Poincaré duality algebra. The problem is that ϕ itself may be degenerate; there may be some orphan elements (see Definition 3.1) a with $\epsilon(ab) = 0$ for all b. Quotienting out by the orphans $\mathcal{O}$ we get a differential Poincaré duality algebra $A/\mathcal{O}$, and a map $f\colon A \to A/\mathcal{O}$ (Proposition 3.3). With this observation the heart of the proof begins.

Now the problem is that f might not be a quasi-isomorphism - this happens whenever $\mathrm{H}^*(\mathcal{O}) \neq 0$. The solution is to add generators to A to get a quasi-isomorphic algebra $\hat{A}$ with better properties. An important observation is that $\mathrm{H}^*(\mathcal{O})$ satisfies a kind of Poincaré duality so it is enough to eliminate $\mathrm{H}^*(\mathcal{O})$ starting from about half of the dimension and working up from there. In some sense we perform something akin to surgery by eliminating the cohomology of the orphans in high dimensions and having the lower dimensional cohomology naturally disappear at the same time. In the middle dimension, the extra generators have the effect of turning orphans which represent homology classes into non orphans. In higher dimensions some of the new generators become orphans whose boundaries kill elements of $\mathrm{H}^*(\mathcal{O})$. In both cases the construction introduces no new orphan homology between the middle dimension and the dimension where the elements of $\mathrm{H}^*(\mathcal{O})$ are killed. This together with the duality in $\mathrm{H}^*(\mathcal{O})$ is enough to get an inductive proof of Theorem 1.1.

2. Some terminology

Just for the record we introduce the terms CDGA, Poincaré duality algebra and differential Poincaré duality algebra.

We fix once for all a ground field $\mathbf{k}$ of any characteristic. So tensor product, algebras, etc., will always be over that field. A commutative differential graded algebra, or CDGA, (A,d) is a non-negatively graded commutative algebra, together with a differential d of degree $+1$. If an element $a \in A$ is in degree n, we write $|a| = n$. The set of elements of degree n in A is denoted A^n. Since A is graded commutative we have the formula $ab = (-1)^{|a||b|}ba$ and $a^2 = 0$ when $|a|$ is odd, including when $\mathbf{k}$ is of characteristic 2. Also d satisfies the graded Leibnitz formula $d(ab) = (da)b+(-1)^{|a|}adb$. CDGA over the rationals are of particular interest since they are models of rational homotopy theory. For more details see [4].

Convention: All of the CDGA we consider in this paper will be connected, in other words $A^0 = \mathbf{k}$, and of finite type.

Note that every simply connected CW-complex of finite type admits such a CDGA model of its rational homotopy type.

Poincaré duality is defined as follows:

Definition 2.1. An *oriented Poincaré duality algebra* of dimension n is a pair (A, ϵ) such that A is a connected graded commutative algebra and $\epsilon\colon A^n \to \mathbf{k}$ is a linear map such that the induced bilinear forms

$$A^k \otimes A^{n-k} \to \mathbf{k},\ a \otimes b \mapsto \epsilon(ab)$$

are non-degenerate.

The following definition comes from [9]:

Definition 2.2. An *oriented differential Poincaré duality algebra* or *oriented Poincaré CDGA* is a triple (A, d, ϵ) such that

(i) (A, d) is a CDGA,
(ii) (A, ϵ) is an oriented Poincaré duality algebra,
(iii) $\epsilon(dA) = 0$

An oriented differential Poincaré duality algebra is essentially a CDGA whose underlying algebra satisfies Poincaré duality. The condition $\epsilon(dA) = 0$ is equivalent to $\mathrm{H}^*(A, d)$ being a Poincaré duality algebra in the same dimension [9, Proposition 4.8].

For convenience we make the following:

Definition 2.3. An *orientation* of a CGDA (A, d) is a linear map

$$\epsilon\colon A^n \to \mathbf{k}$$

such that $\epsilon(dA^{n-1}) = 0$ and there exists a cocycle $\mu \in A^n \cap \ker d$ with $\epsilon(\mu) = 1$.

Recall that $s^{-n}\mathbf{k}$ is the chain complex which is non-trivial only in degree n where it is $\mathbf{k}$. Notice that the above definition is equivalent to the fact that $\epsilon\colon (A, d) \to s^{-n}\mathbf{k}$ is a chain map that induces an epimorphism $\mathrm{H}^n(\epsilon)\colon\ \mathrm{H}^n(A, d) \to \mathbf{k}$. We will use this alternative definition of orientation interchangeably with the first without further comment. The definition of differential Poincaré duality algebra can be thought of as a combination of Definitions 2.3 and 2.1.

If V is a vector space and $v_1, \dots, v_l$ are elements of V, we let $\langle v_1, \dots, v_l\rangle$ or $\langle\{v_i\}\rangle$ denote the linear subspace spanned by these elements.

3. The set of orphans

In this section we consider a fixed CDGA (A, d) such that $\mathrm{H}^*(A, d)$ is a connected Poincaré duality algebra in dimension n.

The proof of our main theorem will be based on the study of orphans, which is the main topic of this section.

Definition 3.1. If ϵ is an orientation on (A,d) then the *set of orphans* of (A,d,ϵ) is the set

$$\mathcal{O} := \mathcal{O}(A,d,\epsilon) := \{a \in A | \forall b \in A, \epsilon(a \cdot b) = 0\}.$$

Proposition 3.2. *The set of orphans $\mathcal{O}$ is a differential ideal in (A,d)*

Proof. $\mathcal{O}$ is clearly a vector space since $\mathcal{O} = \cap_{b\in A} \ker(\epsilon(b \cdot -))$.

If $a \in \mathcal{O}$ and $\xi \in A$ then for any $b \in A$ we have $\epsilon((a\xi)b) = \epsilon(a(\xi b)) = 0$. Therefore $\mathcal{O}$ is an ideal.

If $a \in \mathcal{O}$ then for any $b \in A$ we have, using the fact that $\epsilon(dA) = 0$,

$$\epsilon((da)b) = \pm\epsilon(d(ab)) \pm \epsilon(a(db)) = 0.$$

Therefore $d\mathcal{O} \subset \mathcal{O}$. □

Clearly $\mathcal{O} \subset \ker \epsilon$ since $\epsilon(\mathcal{O} \cdot 1) = 0$. Thus the orientation $\epsilon\colon A \to s^{-n}\mathbf{k}$ extends to a chain map $\bar\epsilon\colon \bar A := A/\mathcal{O} \to s^{-n}\mathbf{k}$ that also induces an epimorphism in H^n, and so $\bar\epsilon\colon \bar A \to s^{-n}\mathbf{k}$ is itself an orientation.

Proposition 3.3. *Let (A,d) be a CDGA such that $\mathrm{H}^*(A,d)$ is a Poincaré duality algebra in dimension n and let $\epsilon\colon A^n \to \mathbf{k}$ be an orientation. Assume that A is connected and of finite type. Let $\mathcal{O}$ be the set of orphans of (A,d,ϵ), let $(\bar A, \bar d) := (A,d)/\mathcal{O}$, and let $\bar\epsilon\colon \bar A \to s^{-n}\mathbf{k}$ be the induced orientation.*

Then $(\bar A, \bar d, \bar\epsilon)$ is an oriented differential Poincaré duality algebra and $H(\bar A, \bar d)$ is a Poincaré duality algebra in degree n.

Proof. We know that $\bar\epsilon(d\bar A) = 0$ since $\bar\epsilon$ is a chain map. As in [9, Definition 4.1] consider the bilinear form

$$\langle -, - \rangle\colon \bar A \otimes \bar A \to \mathbf{k}\,,\ \bar a \otimes \bar b \mapsto \bar\epsilon(\bar a.\bar b)$$

and the induced map

$$\theta\colon \bar A \to \hom(\bar A, \mathbf{k})\,,\ \bar a \mapsto \langle \bar a, - \rangle.$$

Let $\bar a = a \mod \mathcal{O} \in \bar A \setminus \{0\}$. Then $a \in A \setminus \mathcal{O}$ and there exists $b \in A$ such that $\epsilon(a.b) \neq 0$. Set $\bar b = b \mod \mathcal{O} \in \bar A$. Then $\theta(\bar a) \neq 0$ because $\theta(\bar a)(\bar b) \neq 0$. Thus θ is injective and since $\bar A$ and $\hom(\bar A, \mathbf{k})$ have the same dimension this implies that θ is an isomorphism and $(\bar A, \bar d, \bar\epsilon)$ is a differential Poincaré duality algebra in the sense of 2.2. By [9, Proposition 4.7] $\mathrm{H}^*(\bar A, \bar d)$ is a Poincaré duality algebra in dimension n. □

Lemma 3.4. *$\mathcal{O} \cap \ker d \subset d(A)$.*

Proof. Let $\alpha \in \ker d$ of degree k. If $\alpha \notin \operatorname{im} d$ then $[\alpha] \neq 0$ in $\mathrm{H}^*(A,d)$ and by Poincaré duality there exists $\beta \in A \cap \ker d$ of degree $n-k$ such that $[\alpha].[\beta] \neq 0$. Therefore $\epsilon(\alpha.\beta) \neq 0$ and $\alpha \notin \mathcal{O}$. □

Consider the following short exact sequence

$$0 \longrightarrow \mathcal{O} \hookrightarrow A \xrightarrow{\ \pi\ } \bar A = A/\mathcal{O} \longrightarrow 0. \tag{3.1}$$

Notice that, in spite of Lemma 3.4, the differential ideal $\mathcal{O}$ is in general not acyclic. When it is then the map π is a quasi-isomorphism and Proposition 3.3 shows that $\bar A$ is the desired differential Poincaré duality model of A. The idea of the proof of our main theorem will be to modify A in order to turn the ideal of orphans into an acyclic ideal. Actually a Poincaré duality argument shows that its enough to get the acyclicity of $\mathcal{O}$ in a range of degrees above half the dimension. In order to make this statement precise we introduce the following definition:

Definition 3.5. The set of orphans $\mathcal{O}$ is said to be *k-half-acyclic* if $\mathcal{O}^i \cap \ker d \subset d(\mathcal{O}^{i-1})$ for $n/2+1 \leq i \leq k$.

In other words $\mathcal{O}$ is k-half-acyclic iff $\mathrm{H}^i(\mathcal{O}, d) = 0$ for $n/2+1 \leq i \leq k$. Clearly this condition is empty for $k \leq n/2$. Therefore an orphan set is always $(n/2)$-half-acyclic.

Proposition 3.6. *If $\mathcal{O}$ is $(n+1)$-half-acyclic and A is connected and of finite type then $\pi\colon A \to \bar{A} := A/\mathcal{O}$ is a quasi-isomorphism.*

Proof. By hypothesis $\mathrm{H}^*(A)$ is a Poincaré duality algebra in dimension n and by Proposition 3.3 the same is true for $\mathrm{H}^*(\bar{A})$. Moreover since $A^0 = \mathbf{k}$, these cohomologies are connected and $\pi^* = \mathrm{H}^*(\pi)$ sends the fundamental class of $\mathrm{H}^n(A)$ to the fundamental class of $\mathrm{H}^n(\bar{A})$. All of this implies that π^* is injective.

Thus the short exact sequence (3.1) gives us short exact sequences

$$0 \longrightarrow \mathrm{H}^i(A) \xrightarrow{\pi^*} \mathrm{H}^i(\bar{A}) \longrightarrow (\mathrm{coker}\pi^*)^i \cong \mathrm{H}^{i+1}(\mathcal{O}) \longrightarrow 0.$$

By $(n+1)$-half-acyclicity, $\mathrm{H}^i(\mathcal{O}) = 0$ for $n/2+1 \leq i \leq n+1$. Also $(\mathrm{coker}\pi^*)^{>n} = 0$. Thus $(\mathrm{coker}\pi^*)^{\geq n/2} = 0$. By Poincaré duality of $\mathrm{H}^*(A)$ and $\mathrm{H}^*(\bar{A})$ we deduce that $(\mathrm{coker}\pi^*)^{\leq n/2} = 0$. Therefore $\mathrm{coker}\pi^* = 0$ and π is a quasi-isomorphism. □

4. A certain extension of a given oriented CDGA

The aim of this section is, given an integer $k \geq n/2+1$ and an oriented CDGA (A, d, ϵ), to construct a certain quasi-isomorphic oriented CDGA $(\hat{A}, \hat{d}, \hat{\epsilon})$. In the next section we will prove that if the set $\mathcal{O}$ of orphans of A is $(k-1)$-half-acyclic then the set $\hat{\mathcal{O}}$ of orphans of $\hat{A}$ is k-half-acyclic.

In this section we will always suppose that (A, d) is a CDGA equipped with a chain map $\epsilon\colon (A, d) \to s^{-n}\mathbf{k}$ satisfying the following hypotheses:

$$(4.1) \qquad \begin{cases} \text{(i) } A \text{ is of finite type} \\ \text{(ii) } A^0 \cong \mathbf{k},\ A^1 = 0,\ A^2 \subset \ker d \\ \text{(iii) } \mathrm{H}^*(A, d) \text{ is a Poincaré duality algebra in dimension } n \geq 7 \\ \text{(iv) } \epsilon\colon (A, d) \to s^{-n}\mathbf{k} \text{ is an orientation.} \end{cases}$$

We also suppose given a fixed integer $k \geq n/2+1$.

Next we start the construction of the oriented CDGA $(\hat{A}, \hat{d}, \hat{\epsilon})$. Set $l := \dim(\mathcal{O}^k \cap \ker d) - \dim(d(\mathcal{O}^{k-1}))$. Choose l linearly independent elements $\alpha_1, \ldots, \alpha_l \in \mathcal{O}^k \cap \ker d$ such that

$$(4.2) \qquad \mathcal{O}^k \cap \ker d = d(\mathcal{O}^{k-1}) \oplus \langle \alpha_1, \ldots, \alpha_l \rangle.$$

In a certain sense the α_i's are the obstruction to $\mathrm{H}^k(\mathcal{O})$ being trivial. By Lemma 3.4 there exist $\gamma'_1, \ldots, \gamma'_l \in A^{k-1}$ such that $d\gamma'_i = \alpha_i$.

Choose a family $h_1, \ldots, h_m \in A \cap \ker d$ such that $\{[h_i]\}$ is a homogeneous basis of $\mathrm{H}^*(A, d)$. Using the Poincaré duality of $\mathrm{H}^*(A, d)$ there exists another family $\{h_i^*\} \subset A \cap \ker d$ such that $\epsilon(h_j^*.h_i) = \delta_{ij}$, where δ_{ij} is the Kronecker symbol. We set

$$\gamma_i := \gamma'_i - \sum_j \epsilon(\gamma'_i.h_j).h_j^*$$

and

$$\Gamma := \langle \gamma_1, \dots, \gamma_l \rangle \subset A^{k-1}. \tag{4.3}$$

The two main properties of this family are the following:

Lemma 4.1. *$d(\gamma_i) = \alpha_i$ and $\epsilon(\Gamma \cdot \ker d) = 0$.*

Proof. The first equation is obvious since $d\gamma_i' = \alpha_i$ and h_j^* are cocycles.

A direct computation shows that $\epsilon(\gamma_i \cdot h_j) = 0$. On the other hand using the facts that $\epsilon(\operatorname{im} d) = 0$ and $\alpha_i \in \mathcal{O}$ we have that for $\xi \in A$,

$$\epsilon(\gamma_i \cdot d\xi) = \pm\epsilon(d(\gamma_i \cdot \xi)) \pm \epsilon(\alpha_i \cdot \xi) = 0.$$

Since $\ker d = \langle h_1, \dots, h_m \rangle \oplus \operatorname{im} d$, the lemma has been proven. □

Next using the above data we construct a relative Sullivan algebra $(\hat{A}, \hat{d})$ that is quasi-isomorphic to (A, d) and with some new generators c_i that bound the α_i. To define $(\hat{A}, \hat{d})$ properly we distinguish two cases:
Case 1: when $char(\mathbf{k}) = 0$ or k is odd,

$$(\hat{A}, \hat{d}) := (A \otimes \wedge(c_1, \dots, c_l, w_1, \dots, w_l); \hat{d}(c_i) = \alpha_i, \hat{d}(w_i) = c_i - \gamma_i) \tag{4.4}$$

Case 2: when $char(\mathbf{k})$ is a prime p and k is even,

$$(\hat{A}, \hat{d}) := (A \otimes \wedge(\{c_i, w_i, u_{i,j}, v_{i,j}\}_{1 \le i \le l, j \ge 1})$$

with differential given by:

$$\hat{d}(c_i) = \alpha_i, \hat{d}(w_i) = c_i - \gamma_i, \hat{d}(u_{i,1}) = w_i^p, \hat{d}u_{i,j} = v_{i,j-1}^p,$$
$$\hat{d}v_{i,1} = (c_i - \gamma_i) w_i^{p-1}, \hat{d}v_{i,j} = v_{i,j-1}^{p-1} \hat{d}v_{i,j-1}$$

Notice that $\deg(c_i) = k - 1$, $\deg(w_i) = k - 2$, $\deg(u_{i,1}) = p(k - 2) - 1$ and $\deg(v_{i,1}) = p(k - 2)$. All the other generators $u_{i,j}$ and $v_{i,j}$ have degree larger than n. It will turn out that only $c_i, w_i, u_{i,1}$ and $v_{i,1}$ will be relevant and the last two only when k is small and $p = 2$.

Lemma 4.2. *The injection $j \colon (A, d) \to (\hat{A}, \hat{d})$ is a quasi-isomorphism.*

Proof. The lemma follows since the cofibre $\hat{A} \otimes_A \mathbf{k}$ of j is $\wedge(c_1, \dots, c_l, w_1, \dots, w_l)$ or $\wedge(\{c_i, w_i, u_{i,j}, v_{i,j}\}_{1 \le i \le l, j \ge 1}$ which are acyclic. □

Our next step is to build a suitable orientation $\hat{\epsilon}$ on $\hat{A}$ that extends ϵ. We construct this orientation so that the c_i are orphans (except when k is about half the dimension which requires a special treatment). This will prevent the α_i from obstructing the set of orphans from having trivial cohomology in degree k. In order to define this orientation $\hat{\epsilon}$ we first need to define a suitable complementary subspace of $\operatorname{im} d$ in A.

Next we choose a complement Z of $\mathcal{O} \cap d(A)$ in $\mathcal{O}$.

Lemma 4.3. *$d(Z) = d(\mathcal{O})$, $Z \cap \Gamma = 0$ and $(Z \oplus \Gamma) \cap d(A) = 0$.*

Proof. The proof that $d(Z) = d(\mathcal{O})$ is straightforward.

Let $\gamma = \sum_i r_i \gamma_i \in Z \cap \Gamma$. Since $Z \subset \mathcal{O}$, $\alpha := \sum r_i \alpha_i = d\gamma \in d(\mathcal{O})$. By equation (4.2) this implies that each $r_i = 0$, hence $\gamma = 0$ and $Z \cap \Gamma = 0$.

Let $z \in Z$ and $\gamma = \sum_i r_i \gamma_i \in \Gamma$. Suppose that $z + \gamma \in \operatorname{im} d$. Then $d(z + \gamma) = 0$, hence $\alpha := \sum r_i \alpha_i = d\gamma = -dz \in d(\mathcal{O})$. Again this implies that each $r_i = 0$ and $\gamma = 0$. Therefore $z \in \operatorname{im} d$. By the definition of Z this implies that $z = 0$. □

Choose a complement U of $Z \oplus \Gamma \oplus d(A)$ in A. Set

$$T := Z \oplus \Gamma \oplus U$$

which is a complement of $d(A)$ in A.

We are now ready to define our extension $\hat{\epsilon}$ on $\hat{A}$. For $\xi \in A^+$ and $t \in T$ we set

$$\begin{cases} \text{(i) } \hat{\epsilon}(\xi) = \epsilon(\xi) \\ \text{(ii) } \hat{\epsilon}(w_i d(\xi)) = (-1)^k \epsilon(\gamma_i \xi) \\ \text{(iii) } \hat{\epsilon}(c_i c_j) = -\epsilon(\gamma_i \gamma_j) \\ \text{(iv) } \hat{\epsilon}(w_i) = \hat{\epsilon}(w_i t) = \hat{\epsilon}(c_i) = \hat{\epsilon}(c_i \xi) = \hat{\epsilon}(c_i c_j \xi) = \hat{\epsilon}(c_i w_j) = \\ \qquad \hat{\epsilon}(c_i w_j \xi) = \hat{\epsilon}(w_i w_j) = \hat{\epsilon}(w_i w_j \xi) = \hat{\epsilon}(u_{i,1}) = \hat{\epsilon}(w_i u_{j,1}) = \\ \qquad \hat{\epsilon}(v_{i,1}) = \hat{\epsilon}(u_{i,1}\xi) = \hat{\epsilon}(v_{i,1}\xi) = 0 \\ \text{(v) } \hat{\epsilon}(x) = 0 \text{ if } \deg(x) \neq n. \end{cases} \tag{4.5}$$

Lemma 4.4. *The formulas (4.5) define a unique linear map* $\hat{\epsilon}\colon \hat{A} \to s^{-n}\mathbf{k}$.

Proof. Let $x \in \hat{A}^n$. Since $n \geq 7$, and $|w_i|, |c_i| \geq n/2 - 1$, for degree reasons x is of length at most 2 in the w_i and c_i. Similarly x is of length at most one in $v_{i,1}$ and $u_{i,1}$. Moreover for $j > 1$, $|v_{i,j}| > |u_{i,j}| > n$, and $v_{i,1}w_j$, $v_{i,1}c_j$, and $u_{i,1}c_j$ all have degree $> n$. Also $A = T \oplus d(A)$. From these facts it follows that (4.5) defines $\hat{\epsilon}$ on each monomial of $\hat{A}$. We can extend linearly to all of $\hat{A}$.

Notice that (4.5)(ii) is well defined since $\epsilon(\Gamma \cdot \ker d) = 0$ by Lemma 4.1. Again using the fact that $A = T \oplus d(A)$, the well definedness of $\hat{\epsilon}$ follows. □

Lemma 4.5. $\hat{\epsilon}\colon \hat{A} \to s^{-n}\mathbf{k}$ *is an orientation.*

Proof. We need to check that $\hat{\epsilon}(d(\hat{A}^{n-1})) = 0$. Using the fact that $\hat{A}^{\leq 1} = \hat{A}^0 = \mathbf{k}$ and that $k \geq n/2 + 1$ we get that every element of $\hat{A}^{n-1}$ is a linear combinations of terms of the form ξ, $w_i\xi$, $c_i\xi$ for some $\xi \in A$ and possibly terms of the form w_iw_j, w_ic_j, c_ic_j, $u_{i,1}\xi$ and $v_{i,1}$. Using the definition (4.5) of $\hat{\epsilon}$ we compute:

- $\hat{\epsilon}(d\xi) = \epsilon(d\xi) = 0$.
- $\hat{\epsilon}(d(w_i\xi)) = \hat{\epsilon}(c_i\xi) - \hat{\epsilon}(\gamma_i\xi) + (-1)^{\deg(w_i)}\hat{\epsilon}(w_i d\xi) = 0$ by formulas (iv) and (ii) of (4.5).
- $\hat{\epsilon}(d(c_i\xi)) = \hat{\epsilon}(\alpha_i\xi) \pm \hat{\epsilon}(c_i d\xi) = 0$ because $\alpha_i \in \mathcal{O}$.
- $\hat{\epsilon}(d(w_ic_j)) = \hat{\epsilon}(c_ic_j) - \hat{\epsilon}(\gamma_ic_j) + (-1)^k\hat{\epsilon}(w_i\alpha_j) = -\hat{\epsilon}(\gamma_i\gamma_j) + (-1)^k\hat{\epsilon}(w_i\alpha_j) = -\hat{\epsilon}(\gamma_i\gamma_j) + \hat{\epsilon}(\gamma_i\gamma_j) = 0$.
- $\hat{\epsilon}(d(c_ic_j)) = \hat{\epsilon}(\alpha_ic_j) \pm \hat{\epsilon}(c_i\alpha_j) = 0$.
- $\hat{\epsilon}(d(w_iw_j)) = \hat{\epsilon}(c_iw_j) - \hat{\epsilon}(\gamma_iw_j) \pm \hat{\epsilon}(w_ic_j) \pm \hat{\epsilon}(w_i\gamma_j) = 0$ because $\gamma_i, \gamma_j \in T$.
- $\hat{\epsilon}(du_{i,1}\xi) = \hat{\epsilon}(w_i^p\xi) \pm \hat{\epsilon}(u_{i,1}d\xi) = 0$.
- $\hat{\epsilon}(dv_{i,1}) = \hat{\epsilon}((c_i - \gamma_i)w_i^{p-1}) = 0$ because $\gamma_i \in T$.

This proves that $\hat{\epsilon}(d\hat{A}) = 0$, in other words $\hat{\epsilon}$ is a chain map. That it induces an epimorphism in cohomology in degree n follows immediately from the facts that ϵ does and that $\epsilon = \hat{\epsilon}j$. □

This completes our construction of an oriented CDGA $(\hat{A}, \hat{d}, \hat{\epsilon})$ quasi-isomorphic to (A, d, ϵ).

5. Extending the range of half-acyclicity

The aim of this section is to prove that the construction of the previous section increases the range in which the set of orphans is half-acyclic. More precisely we will prove the following:

Proposition 5.1. *Let (A, d, ϵ) be an oriented CDGA satisfying the assumptions (4.1) and let $k \geq n/2 + 1$. Then the CDGA $(\hat{A}, \hat{d}, \hat{\epsilon})$ constructed in the previous section also satisfies the assumptions (4.1).*

Moreover if the set $\mathcal{O}$ of orphans of (A, d, ϵ) is $(k-1)$-half-acyclic, then the set $\hat{\mathcal{O}}$ of orphans of $(\hat{A}, \hat{d}, \hat{\epsilon})$ is k-half-acyclic.

The proof of this proposition consists of a long series of lemmas. Recall the spaces Γ from equation (4.3) and Z from above Lemma 4.3.

Notice that by assumption 4.1(iii), $n \geq 7$ and hence $k \geq 5$.

Lemma 5.2. *If $i > n - k + 2$ then $\mathcal{O}^i \subset \hat{\mathcal{O}}^i$.*

Proof. Since $n - i < k - 2$ we have that $\hat{A}^{n-i} = A^{n-i}$. Therefore $\hat{\epsilon}(\mathcal{O}^i.\hat{A}^{n-i}) = \epsilon(\mathcal{O}^i.A^{n-i}) = 0$. So $\mathcal{O}^i \subset \hat{\mathcal{O}}^i$. □

Lemma 5.3. *For $i = k-2$, $k-1$ or k, we have $\hat{\mathcal{O}}^i \cap \ker d \subset \mathcal{O}^i \cap \ker d$.*

Proof. Case 1: $i = k - 2$.
We have $\hat{A}^{k-2} = A^{k-2} \oplus \langle\{w_i\}\rangle$. Let $\omega = \xi + \sum_i r_i w_i \in \hat{A}^{k-2}$ with $\xi \in A^{k-2}$ and $r_i \in \mathbf{k}$. Then

$$d\omega = (d\xi - \sum r_i\gamma_i) + \sum r_i c_i \in A^{k-1} \oplus \langle\{c_i\}\rangle.$$

Therefore if $d\omega = 0$ then we must also have $r_i = 0$ for each i. This implies that $\hat{A}^{k-2} \cap \ker d \subset A^{k-2} \cap \ker d$. Thus $\hat{\mathcal{O}}^{k-2} \cap \ker d \subset \mathcal{O}^{k-2} \cap \ker d$.

Case 2: $i = k - 1$.
Since $A^1 = 0$ and $A^0 = \mathbf{k}$, $\hat{A}^{k-1} = A^{k-1} \oplus \langle\{c_i\}\rangle$. Let $\omega = \xi + \sum r_i c_i \in \hat{A}^{k-1}$ with $\xi \in A^{k-1}$. Suppose that $\omega \in \hat{\mathcal{O}}^{k-1} \cap \ker d$. Then $\xi \in \mathcal{O}^{k-1}$ because otherwise there would exist $\xi^* \in A$ such that $\epsilon(\xi\xi^*) \neq 0$, and since $\hat{\epsilon}(c_i.A) = 0$ we would have $\hat{\epsilon}(\omega.\xi^*) \neq 0$.

Also $d\omega = 0$ implies that $\sum r_i\alpha_i = d(-\xi) \in d(\mathcal{O}^{k-1})$. But by definition of $\{\alpha_i\}$ we have $\langle\{\alpha_i\}\rangle \cap d(\mathcal{O}^{k-1}) = 0$. Therefore $r_i = 0$ for each i, hence $\omega = \xi \in \mathcal{O}^{k-1} \cap \ker d$.

Case 3: $i = k$.
Let $\{\lambda_j\}$ be a basis of A^2. By assumption 4.1(ii) this basis consists of cocycles. Since $n \geq 7$, we have $k - 2 > 2$ and $\hat{A}^k = A^k \oplus \langle\{w_i.\lambda_j\}\rangle$. Let $\omega = \xi + \sum r_{ij} w_i \lambda_j \in \hat{A}^k$ with $\xi \in A^k$. Then

$$d\omega = (d\xi - \sum r_{ij}\gamma_i\lambda_j) + \sum r_{ij} c_i \lambda_j \in A^{k+1} \oplus \langle\{c_i \cdot \lambda_j\}\rangle.$$

Therefore $d\omega \neq 0$ unless $r_{ij} = 0$ for all i, j. This implies that $\hat{A}^k \cap \ker d \subset A^k \cap \ker d$, hence $\hat{\mathcal{O}}^k \cap \ker d \subset \mathcal{O}^k \cap \ker d$ □

Now the rest of the proof of Proposition 5.1 splits into three cases: $k = n/2 + 1$ and n even, $k = (n+1)/2 + 1$ and n odd, and $k \geq n/2 + 2$.

5.1. **The case n even and $k = n/2+1$.**

Lemma 5.4. *Let*

$$0 \longrightarrow A \xrightarrow{\ i\ } B \underset{p}{\overset{r}{\rightleftarrows}} C \longrightarrow 0$$

be a short exact sequence of vector spaces, $r\colon C \to B$ be a linear spliting of p and $\langle \_, \_\rangle\colon B\otimes B \to \mathbf{k}$ be a non-degenerate bilinear form on B. If $\langle \mathrm{im}\, r, \mathrm{im}\, i\rangle = 0$, then $\langle r\_, r\_\rangle\colon C\otimes C \to \mathbf{k}$ is a non-degenerate bilinear form on C.

Proof. For any $\gamma \in C\setminus\{0\}$, there is a $b\in B$ such that $\langle r\gamma, b\rangle \neq 0$. Thus $\langle r\gamma, rpb\rangle \neq 0$, since $(rpb) - b \in \mathrm{im}\, i$. □

Recall the space $\Gamma = \langle\{\gamma_i\}\rangle$ defined in (4.3).

Lemma 5.5. *If n is even and $k = n/2+1$ then the bilinear form*

$$\Gamma\otimes\Gamma \to \mathbf{k}\,,\ \gamma\otimes\gamma' \mapsto \epsilon(\gamma.\gamma')$$

is non degenerate.

Proof. Set $n = 2m$ and $k = m+1$. As in the proof of Proposition 3.6, the short exact sequence $0 \to \mathcal{O} \to A \to \bar{A} := A/\mathcal{O} \to 0$ induces a short exact sequence

$$0 \longrightarrow \mathrm{H}^m(A) \xrightarrow{\ \pi^*\ } \mathrm{H}^m(\bar{A}) \xrightarrow{\ \delta\ } \mathrm{H}^{m+1}(\mathcal{O}) \longrightarrow 0$$

where δ is the connecting homomorphism. Since $\mathcal{O}^{m+1}\cap \ker d = d(\mathcal{O}^m)\oplus\langle\{\alpha_i\}\rangle$, we get that $\mathrm{H}^{m+1}(\mathcal{O}) = \langle\{[\alpha_i]\}\rangle$. Let $[\bar\gamma_i]\in \mathrm{H}^m(\bar{A})$ be the cohomology classes represented by $\bar\gamma_i = \gamma_i \mod \mathcal{O} \in A^m/\mathcal{O}$.

By Proposition 3.3, ϵ induces a non-degenerate pairing $\langle\cdot,\cdot\rangle$ on $\mathrm{H}^n(\overline{A})$. Let $[\alpha_i] \mapsto [\bar\gamma_i]$ define a linear section r of δ. By Lemma 4.1 we have $\epsilon(\Gamma\cdot\ker d) = 0$, and hence $\langle \mathrm{im}\, r, \mathrm{im}\, \pi^*\rangle = 0$. Thus by Lemma 5.4 the pairing restricts to a non-degenerate pairing on $\mathrm{im}\, r$. Finally observe that under the identification of $\mathrm{im}\, r$ with Γ which sends $[\bar\gamma_i]$ to γ_i the restricted pairing is sent to the pairing given in the statement of the lemma. □

Lemma 5.6. *If n is even and $k = n/2+1$ then $\langle\{\alpha_i\}\rangle \cap \hat{\mathcal{O}}^k = 0$.*

Proof. Let $\alpha := \sum r_i\alpha_i \in \langle\{\alpha_i\}\rangle$. If the r_i are not all zero then by Lemma 5.5 there exist $r_j^*\in\mathbf{k}$ such that $\epsilon\left((\sum r_i\gamma_i)(\sum r_j^*\gamma_j)\right) \neq 0$. Then

$$\hat\epsilon\left((\sum r_i\alpha_i)(\sum r_j^* w_j)\right) = \sum r_i r_j^* \hat\epsilon((d\gamma_i).w_j) = \pm \sum r_i r_j^*\hat\epsilon(\gamma_i.\gamma_j) \neq 0.$$

Hence $\alpha \notin \hat{\mathcal{O}}$ if $\alpha\neq 0$. □

Lemma 5.7. *If n is even and $k = n/2+1$ then $\mathcal{O}^{k-1}\subset\hat{\mathcal{O}}^{k-1}$*

Proof. Let $\beta \in \mathcal{O}^{k-1}$. Then $\hat{A}^{n-(k-1)} = A^{k-1}\oplus\langle\{c_i\}\rangle$. Let $\omega = \xi + \sum r_i c_i$ with $\xi \in A^{k-1}$. Then $\hat\epsilon(\beta\omega) = \epsilon(\beta\xi) + \sum r_i\hat\epsilon(\beta c_i) = 0$. Therefore $\beta\in\hat{\mathcal{O}}^{k-1}$. So $\mathcal{O}^{k-1}\subset\hat{\mathcal{O}}^{k-1}$ and we are done. □

Lemma 5.8. *If n is even and $k = n/2+1$ then $\hat{\mathcal{O}}$ is k-half-acyclic.*

Proof. We only need to check that $\hat{\mathcal{O}}^k \cap \ker d \subset d(\hat{\mathcal{O}}^{k-1})$. By Lemma 5.3

$$\hat{\mathcal{O}}^k \cap \ker d \subset \mathcal{O}^k \cap \ker d = d(\mathcal{O}^{k-1}) \oplus \langle\{\alpha_i\}\rangle.$$

By Lemma 5.7 this implies that

$$\hat{\mathcal{O}}^k \cap \ker d \subset d(\hat{\mathcal{O}}^{k-1}) \oplus \langle\{\alpha_i\}\rangle. \tag{5.1}$$

Since the set of orphans is a differential ideal, we also have $d(\hat{\mathcal{O}}^{k-1}) \subset \hat{\mathcal{O}}^k \cap \ker d$. This combined with Lemma 5.6 and inclusion (5.1) implies that $\hat{\mathcal{O}}^k \cap \ker d \subset d(\hat{\mathcal{O}}^{k-1})$. □

5.2. **The case n odd and $k = (n+1)/2 + 1$.** Recall the space Z defined before Lemma 4.3

Lemma 5.9. *If n is odd and $k = (n+1)/2 + 1$ then $Z^{k-1} \oplus \langle\{c_i\}\rangle \subset \hat{\mathcal{O}}^{k-1}$.*

Proof. Notice that $n - (k-1) = k-2$ and $\hat{A}^{k-2} = A^{k-2} \oplus \langle\{w_i\}\rangle$. It is immediate to check, using the definition (4.5) of $\hat{\epsilon}$ and the fact that $Z \subset T \cap \mathcal{O}$, that $\hat{\epsilon}(\hat{A}^{k-2} \cdot Z^{k-1}) = 0$. Also $\hat{\epsilon}(\hat{A}^{k-2} \cdot c_i) = 0$ since $c_j \notin \hat{A}^{k-2}$. □

Lemma 5.10. *If n is odd and $k = (n+1)/2 + 1$ then $\hat{\mathcal{O}}$ is k-half-acyclic.*

Proof. We only need to check that $\hat{\mathcal{O}}^k \cap \ker d \subset d(\hat{\mathcal{O}}^{k-1})$. Using Lemmas 5.3 and 4.3 we have that

$$\hat{\mathcal{O}}^k \cap \ker d \subset \mathcal{O}^k \cap \ker d = d(\mathcal{O}^{k-1}) \oplus \langle\{\alpha_i\}\rangle = d(Z^{k-1}) \oplus d(\langle\{c_i\}\rangle)$$

By Lemma 5.9 the last set is included in $d(\hat{\mathcal{O}}^{k-1})$. □

5.3. **The case $k \geq n/2 + 2$.**

Lemma 5.11. *If $n/2 \leq i \leq k-3$ then $\hat{\mathcal{O}}^i = \mathcal{O}^i$.*

Proof. If $i \leq k-3$ then $\hat{A}^i = A^i$, so $\hat{\mathcal{O}}^i \subset \mathcal{O}^i$.

If $n/2 \leq i \leq k-3$ then

$$i \geq n/2 = n - n/2 \geq n - (k-3) > n - k + 2$$

and by Lemma 5.2 $\mathcal{O}^i \subset \hat{\mathcal{O}}^i$. □

Lemma 5.12. *If $k \geq n/2 + 2$ then $Z^{k-2} \subset \hat{\mathcal{O}}^{k-2}$.*

Proof. First suppose that n is odd or that $k \geq n/2 + 3$. In these cases $2k > n + 4$, hence $k - 2 > n - k + 2$ which implies by Lemma 5.2 that $\mathcal{O}^{k-2} \subset \hat{\mathcal{O}}^{k-2}$. Since $Z^{k-2} \subset \mathcal{O}^{k-2}$, this completes the proof of the lemma in these cases.

Now suppose that n is even and $k = n/2+2$, then $n-(k-2) = k-2$. Since $Z \subset \mathcal{O}$ we have $\epsilon(Z^{k-2} \cdot A^{k-2}) = 0$. Also by definition of $\hat{\epsilon}$ since $Z \subset T$, $\hat{\epsilon}(Z^{k-2} \cdot w_i) = 0$. Since $\hat{A}^{n-(k-2)} = A^{n-(k-2)} \oplus \langle\{w_i\}\rangle$ this implies that $Z^{k-2} \subset \hat{\mathcal{O}}^{k-2}$. □

Lemma 5.13. *If $k \geq n/2 + 2$ and $\mathcal{O}$ is $(k-1)$-half-acyclic then so is $\hat{\mathcal{O}}$.*

Proof. For $n/2 + 1 \leq i \leq k-3$, using Lemma 5.11 twice, we get that

$$\hat{\mathcal{O}}^i \cap \ker d = \mathcal{O}^i \cap \ker d \subset d(\mathcal{O}^{i-1}) = d(\hat{\mathcal{O}}^{i-1}).$$

By Lemmas 5.3, 4.3 and 5.12 we have

$$\hat{\mathcal{O}}^{k-1} \cap \ker d \subset \mathcal{O}^{k-1} \cap \ker d \subset d(\mathcal{O}^{k-2}) = d(Z^{k-2}) \subset d(\hat{\mathcal{O}}^{k-2}).$$

Suppose that $k \geq n/2+3$ (otherwise there is no need to check $(k-2)$-half-acyclicity.) By Lemma 5.11 $\hat{\mathcal{O}}^{k-3} = \mathcal{O}^{k-3}$ and so by Lemma 5.3 we have

$$\hat{\mathcal{O}}^{k-2} \cap \ker d \subset \mathcal{O}^{k-2} \cap \ker d \subset d(\mathcal{O}^{k-3}) = d(\hat{\mathcal{O}}^{k-3}).$$

□

Lemma 5.14. *If $k \geq n/2+2$ then $\langle\{c_i\}\rangle \subset \hat{\mathcal{O}}^{k-1}$.*

Proof. By definition of $\hat{\epsilon}$ the only products with c_i which could prevent them from being orphans are

$$\hat{\epsilon}(c_i c_j) = -\epsilon(\gamma_i \gamma_j)$$

but those are zeros for degree reasons. □

Lemma 5.15. *If $k \geq n/2+2$ and $\mathcal{O}$ is $(k-1)$-half-acyclic then $\hat{\mathcal{O}}$ is k-half-acyclic.*

Proof. We already know by Lemma 5.13 that $\hat{\mathcal{O}}$ is $(k-1)$-half-acyclic. Since $k \geq n/2+2$ we have $k-1 > n-k+2$ and Lemma 5.2 implies that $\mathcal{O}^{k-1} \subset \hat{\mathcal{O}}^{k-1}$.

By Lemma 5.14 and the definitions of dc_i, we have

$$\langle\{\alpha_i\}\rangle = d(\langle\{c_i\}\rangle) \subset d(\hat{\mathcal{O}}^{k-1}).$$

Using Lemma 5.3 and the definition of $\{\alpha_i\}$ we get

$$\hat{\mathcal{O}}^{k} \cap \ker d \subset \mathcal{O}^{k} \cap \ker d = d(\mathcal{O}^{k-1}) \oplus \langle\{\alpha_i\}\rangle \subset d(\hat{\mathcal{O}}^{k-1}).$$

This proves that $\hat{\mathcal{O}}$ is k-half-acyclic. □

5.4. **End of the proof of Proposition 5.1.**

Proof of Proposition 5.1. Since $n \geq 7$ we have $k \geq 5$ and also using the fact that $j\colon A \to \hat{A}$ is a quasi-isomorphism by Lemma 4.2, and that $\hat{\epsilon}$ is an orientation by Lemma 4.5, it is immediate to check that $(\hat{A}, \hat{d}, \hat{\epsilon})$ satisfies the assumptions (4.1).

If $\mathcal{O}$ is $(k-1)$-half-acyclic for some $k \geq n/2+1$ then Lemmas 5.8, 5.10, and 5.15 imply that $\hat{\mathcal{O}}$ is k-half-acyclic. □

6. Proof of Theorem 1.1

We conclude the proof of our main theorem.

Proof of Theorem 1.1. If $n \leq 6$ then by [12] the CDGA (A, d) is formal and we can take its cohomology algebra as the Poincaré duality model.

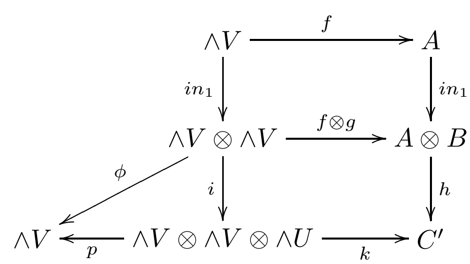
Suppose that $n \geq 7$. Since $H(A, d)$ is simply-connected, by taking a minimal Sullivan model we can suppose that A is of finite type, $A^0 = \mathbf{k}$, $A^1 = 0$, and $A^2 \subset \ker d$. Also there exists a chain map $\epsilon\colon A \to s^{-n}\mathbf{k}$ inducing a surjection in homology. So all the assumptions (4.1) are satisfied. Taking $k = n/2+1$ if n is even or $k = (n+1)/2+1$ if n is odd, the set of orphans $\mathcal{O}$ is $(k-1)$-half-acyclic because this condition is empty. An obvious induction using Proposition 5.1 yields a quasi-isomorphic oriented model $\hat{A}$ for which the set of orphans $\hat{\mathcal{O}}$ is $(n+1)$-half-acyclic. Propositions 3.3 and 3.6 imply that the quotient $A' = \hat{A}/\hat{\mathcal{O}}$ is a Poincaré CDGA quasi-isomorphic to A. □

7. Equivalence of differential Poincaré duality algebras

The next theorem shows that if we have two 3-connected quasi-isomorphic differential Poincaré duality algebras then they can be connected by quasi-isomorphisms involving only differential Poincaré duality algebras.

Theorem 7.1. *Suppose A and B are quasi-isomorphic finite type differential Poincaré duality algebras of dimension at least 7 such that $\mathrm{H}^{\leq 3}(A) = \mathrm{H}^{\leq 3}(B) = \mathbf{k}$ and $A^{\leq 2} = B^{\leq 2} = \mathbf{k}$. There exists a differential Poincaré duality algebra C and quasi-isomorphisms $A \to C$ and $B \to C$.*

Proof. Let $\wedge V$ be a minimal Sullivan model of A. Then there exist quasi-isomorphisms $f\colon \wedge V \to A$ and $g\colon \wedge V \to B$. Consider the factorization of the multiplication map $\phi\colon \wedge V \otimes \wedge V \to \wedge V$ into a cofibration $i\colon \wedge V \otimes \wedge V \to \wedge V \otimes \wedge V \otimes \wedge U$ followed by a quasi-isomorphism $p\colon \wedge V \otimes \wedge V \otimes \wedge U \to \wedge V$. Since $\mathrm{H}^{\leq 3}(\wedge V) = \mathrm{H}^{\leq 3}(A) = \mathbf{k}$, we can assume that $U^{\leq 2} = 0$ and that U is of finite type. Next consider the following diagram in which C' is defined to make the bottom square a pushout.



The maps in_1 denote inclusion into the first factor. Since p is a quasi-isomorphism and $\phi \circ in_1 = id$, $i \circ in_1$ is a quasi-isomorphism. Also since $f \otimes g$ is a quasi-isomorphism, i is a cofibration, the bottom square is a pushout, and the properness of CDGA [2, Lemma 8.13], k is a quasi-isomorphism. Finally since f, k and $i \circ in_1$ are quasi-isomorphisms, so too must $h \circ in_1$ be a quasi-isomorphism.

Since $A^{\leq 2} = B^{\leq 2} = \mathbf{k}$, the algebra $A \otimes B$ satisfies (i) and (ii) of (4.1). Also U is of finite type with $U^{\leq 2} = 0$, so $C' = A \otimes B \otimes \wedge U$ satisfies (i) and (ii) of (4.1). Since C' is quasi-isomorphic to A, it also satisfies (iii), and we can let $\epsilon\colon C' \to s^n\mathbf{k}$ be any orientation. Next by using induction and Propositions 5.1, 3.3 and 3.6 we get a quasi-isomorphism $l\colon C' \to C$ such that C is a differential Poincaré duality algebra. Clearly the map $l \circ h \circ in_1\colon A \to C$ is a quasi-isomorphism, and similarly $l \circ h \circ in_2\colon B \to C$ is a quasi-isomorphism, thus completing the proof of the theorem. □

Conjecture: The hypotheses $\mathrm{H}^{\leq 3}(A) = \mathbf{k}$, $A^{\leq 2} = B^{\leq 2} = \mathbf{k}$ and dimension of A at least 7 in Theorem 7.1 can be removed.

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