Differentiable structures on manifolds

Timothy Lance

Prologue.

Suppose M is a closed smooth manifold. Is the "smoothness" of the underlying topological manifold unique up to diffeomorphism?

The answer is no, and the first, stunningly simple examples of distinct smooth structures were constructed for the 7-sphere by John Milnor as 3-sphere bundles over S^4.

THEOREM 1.1. (Milnor [52]) *For any odd integer $k = 2j + 1$ let M_k^7 be the smooth 7-manifold obtained by gluing two copies of $D^4 \times S^3$ together via a map of the boundaries $S^3 \times S^3$ given by $f_j : (u, v) \to (u, u^{1+j} v u^{-j})$ where the multiplication is quaternionic. Then M_k^7 is homeomorphic to S^7 but, if $k^2 \not\equiv 1$ mod 7, is not diffeomorphic to S^7.*

This paper studies smooth structures on compact manifolds and the role surgery plays in their calculation. Indeed, one could reasonably claim that surgery was created in the effort to understand these structures. Smooth manifolds homeomorphic to spheres, or homotopy spheres, are the building blocks for understanding smoothings of arbitrary manifolds. Milnor's example already hints at surgery's role. M_k^7 is the boundary of the 4-disk bundle over S^4 constructed by gluing two copies of $D^4 \times D^4$ along $S^3 \times D^4$ using the same map f_j. Computable invariants for the latter manifold identify its boundary as distinct from S^7.

Many homotopy spheres bound manifolds with trivial tangent bundles. Surgery is used to simplify the bounding manifold so that invariants such as Milnor's identify the homotopy sphere which is its boundary. We will encounter obstructions lying in one of the groups $0, \mathbf{Z}/2$, or $\mathbf{Z}$ (depending on dimension), to simplifying the bounding manifold completely to a contractible space, so that its boundary will be the usual sphere. We call these groups the Wall groups for surgery on simply connected manifolds.

Except in the concluding §7, no advanced knowledge of topology is required. Some basic definitions are given below, and concepts will be introduced, intuitively or with precision, as needed, with many references to the literature. Expanded presentations of some of this material are also available, e.g. [40] or Levine's classic paper [45].

§1 Topological and smooth manifolds.

A topological n-manifold (perhaps with boundary) is a compact Hausdorff space M which can be covered by open sets V_α, called coordinate neighborhoods, each of which is homeomorphic to $\mathbf{R}^n$ (or $\mathbf{R}^{n-1} \times [0, \infty)$) via some "coordinate map" $\varphi_\alpha : V_\alpha \to \mathbf{R}^n$, with any points of the boundary ∂M carried to $\mathbf{R}^{n-1} \times 0$ via the maps φ_α (M is closed if no such points exist). M is a smooth manifold if it has an "atlas" of coordinate neighborhoods and maps $\{(V_\alpha, \varphi_\alpha)\}$ such that the composites $\varphi_\alpha \circ \varphi_{\alpha'}^{-1}$ are smooth bijections between subsets of Euclidean space where defined (i.e., on the sets $\varphi_{\alpha'}(V_\alpha \bigcap V_{\alpha'})$.)

Similarly, M is piecewise linear, or PL, if an atlas exists such that the composites $\varphi_\alpha \circ \varphi_{\alpha'}^{-1}$, when defined, are piecewise linear. For any PL manifold there is a polyhedron $P \subset \mathbf{R}^q$ for some large q and a homeomorphism $T : P \to M$, called a triangulation, such that each composite $\varphi_\alpha \circ T$ is piecewise linear. Any smooth manifold M may be triangulated and given the structure of a PL manifold, and the underlying PL-manifold is unique up to a PL-isomorphism.

The triangulation T may be chosen so that the restriction to each simplex is a smooth map. Any PL manifold clearly has an underlying topological structure. A deep result of Kirby and Siebenmann [39] (see also §7) shows that most topological manifolds may be triangulated.

We assume that all manifolds are also orientable. If M is smooth this means that coordinate maps φ_α can be chosen so that the derivatives of the composites $\varphi_\alpha \circ \varphi_{\alpha'}^{-1}$ have positive determinants at every point. The determinant condition ensures the existence, for each coordinate neighborhood V_α, of a coherent choice of orientation for $\varphi_\alpha(V_\alpha) = \mathbf{R}^n$. Such a choice is called an orientation for M, and the same manifold with the opposite orientation we denote $-M$. Orient the boundary (if non-empty) by choosing the orientation for each coordinate neighborhood of ∂M which, followed by the inward normal vector, yields the orientation of M.

The sphere S^n, consisting of all vectors in $\mathbf{R}^{n+1}$ of length 1, is an example of an orientable smooth n-manifold. S^n has a smooth structure with two coordinate neighborhoods V_n and V_s consisting of all but the south (north) pole, with φ_n carrying a point $x \in V_n$ to the intersection with $\mathbf{R}^n \times 0$ of the line from the south pole to x, and similarly for φ_s. S^n is the boundary of the smooth $(n+1)$-manifold D^{n+1}.

If M is a closed, smooth, oriented manifold, then the question regarding the uniqueness of "smoothness" means the following: given another set of coordinate neighborhoods U_β and maps ψ_β, does there exist a homeomorphism Φ of M such that the composites $\varphi_\alpha \circ \Phi \circ \psi_\beta^{-1}$ and their inverses are smooth bijections of open subsets of $\mathbf{R}^n$ which preserve the chosen orientations?

One might also ask whether a topological or PL manifold has at least one smooth structure. The answer is again no, with the first examples due to Kervaire [37] and Milnor [52]. In this paper we assume that all manifolds have a smooth structure. But we shall see in 4.5 and again in 4.8 examples (including Kervaire's and Milnor's) of topological manifolds which have smooth structures everywhere except a single point. If a neighborhood of that point is removed, the smooth boundary is a homotopy sphere.

§2 The groups of homotopy spheres.

Milnor's example inspired intensive study of the set Θ_n of h-cobordism classes of manifolds homotopy equivalent to the n-sphere, culminating in Kervaire and Milnor's beautiful *Groups of homotopy spheres: I* [38]. Two manifolds M and N are homotopy equivalent if there exist maps $f : M \to N$ and $g : N \to M$ such that the composites $g \circ f$ and $f \circ g$ are homotopic to the identity maps on M and N, respectively. They are h-cobordant if each is a deformation retraction of an oriented $(n+1)$-manifold W whose boundary is the disjoint union $M \sqcup (-N)$.

For small values of $n \neq 3$ the set Θ_n consists of the h-cobordism class of S^n alone. This is clear for $n = 1$ and 2 where each topological manifold has a unique smooth structure, uniquely determined by its homology. The triviality of Θ_4, due to Cerf [24], is much harder, requiring a meticulous study of singularities. The structure of Θ_3 is unknown, depending as it does on the Poincaré conjecture. But each topological 3-manifold has a unique differentiable structure ([65], [98]), so if a homotopy 3-sphere is homeomorphic to S^3 it is diffeomorphic to it. The vanishing of Θ_5 and Θ_6 will use surgery theory, but depends as well on the h-cobordism theorem of Smale.

<u>Theorem 2.1</u>. (Smale [79]) *Any n-dimensional simply connected h-cobordism W, $n > 5$, with $\partial W = M \sqcup (-N)$, is diffeomorphic to $M \times [0,1]$.*

Smale's proof is a striking demonstration of reflecting geometrically the algebraic simplicity of the triple (W, M, N), that is, $H_*(W, M) \cong H_*(W, N) \cong 0$. One can find a smooth real valued function $f : (W, M, N) \to ([a, b], \{a\}, \{b\})$ such that, around each point $x \in W$ where the derivative of f vanishes, there is a coordinate neighborhood $(V_\alpha, \varphi_\alpha)$ such that the composite $f \circ \varphi_\alpha^{-1} : \mathbf{R}^n \to \mathbf{R}$ equals $(x_1, \ldots x_n) \to -x_1^2 - x_2^2 - \ldots - x_\lambda^2 + x_{\lambda+1}^2 + \ldots + x_n^2$. We call x a non-degenerate singularity of index λ, and f a Morse function for W. The singularities are necessarily isolated, and f can be adjusted so that $[a, b] = [-1/2, n + 1/2]$ and $f(x) = \lambda$ for any singularity of index λ. Morse functions for W not only exist, but are plentiful ([56], [57]). If f could be found with no singularities, then the integral curves of this function (roughly, orthogonal trajectories to the level sets of f, whose existence and uniqueness follow by standard differential equations arguments) yield a diffeomorphism $W \cong M \times [0,1]$. This is always possible given the above assumptions about trivial homology of (W, M) and (W, N).

To check this, let $W_\lambda = f^{-1}((-\infty, \lambda + 1/2])$, and let $M_{\lambda-1}$ be the level set $f^{-1}(\lambda - 1/2)$ (the level set of any value between $\lambda - 1$ and λ would be equivalent). Let x_α be an index λ critical point. Then x_α together with the union of all integral curves beginning in $M_{\lambda-1}$ and approaching x_α form a disk $D^\lambda_{\alpha,L}$, called the left-hand disk of x_α, with bounding left-hand sphere $S^{\lambda-1}_{\alpha,L} \subset M_{\lambda-1}$. W_λ is homotopy equivalent to the union of $W_{\lambda-1}$ and all left hand disks associated to critical points of index λ, so that $C_\lambda = H_\lambda(W_\lambda, W_{\lambda-1})$ is a free abelian group with a generator for each such singularity. We can similarly define, for any index $(\lambda-1)$ critical point y_β, the right-hand disk $D^{n-\lambda+1}_{\beta,R}$ and right-hand sphere $S^{n-\lambda}_{\beta,R} \subset M_{\lambda-1}$.

If the intersection number $S^{\lambda-1}_{\alpha,L} \cdot S^{n-\lambda}_{\beta,R} = \pm 1$, we can move $S^{\lambda-1}_{\alpha,L}$ by a homotopy so that it intersects $S^{n-\lambda}_{\beta,R}$ transversely in a single point, and change f to a new Morse function g with the same critical points and the newly positioned left hand sphere for x_λ. (The dimension restriction $n > 5$ is critical here, providing enough room to slide $S^{\lambda-1}_{\alpha,L}$ around to remove extraneous intersection points.) With this new Morse function there is a single integral curve from y_β to x_α. By a result of Morse, g can be further altered in a neighborhood of this trajectory to eliminate both critical points x_α and y_β.

This cancellation theorem is the key tool in proving the h-cobordism theorem. The groups C_λ form a chain complex with $\partial_\lambda : C_\lambda \to C_{\lambda-1}$, the boundary map of the triple $(W_\lambda, W_{\lambda-1}, W_{\lambda-2})$, given explicitly by intersection numbers: the y_β coefficient of $\partial_\lambda(x_\alpha)$ equals $S^{\lambda-1}_{\alpha,L} \cdot S^{n-\lambda}_{\beta,R}$. But $H_*(C) \cong H_*(W, M) \cong 0$. Thus for each λ, $kernel(\partial_\lambda)$ is the isomorphic image under $\partial_{\lambda+1}$ of some subgroup of $C_{\lambda+1}$. Thus the matrices for the boundary maps ∂_λ corresponding to bases given by critical points can, by elementary operations, be changed to block matrices consisting of identity and trivial matrices. These operations can be reflected by correspondingly elementary changes in the Morse function. By the above cancellation theorem all critical points can thus be removed, and $W \cong M \times [0,1]$.

As an immediate consequence of 2.1, two homotopy spheres are h-cobordant if and only if they are orientation preserving diffeomorphic. The h-cobordism theorem also fixes the topological type of a homotopy sphere in dimensions ≥ 6. If Σ is a homotopy n-sphere, $n \geq 6$, and W equals Σ

with the interiors of two disks removed, what remains is an h-cobordism which, by 2.1, is diffeomorphic to $S^{n-1} \times [0,1]$. Since this product may be regarded as a boundary collar of one of the two disks, it follows that Σ may be obtained by gluing two disks D^n via some diffeomorphism f of the boundaries S^{n-1} of the two disks. If Σ' is constructed by gluing n-disks via a diffeomorphism f' of S^{n-1}, we may try to construct a diffeomorphism $\Sigma \to \Sigma'$ by beginning with the identity map of the "first" disk in each sphere. This map induces a diffeomorphism $f' \circ f^{-1}$ of the boundaries of the second disks, which extends radially across those disks. Such an extension is clearly a homeomorphism, and smooth except perhaps at the origin. If $n = 5$ or 6, then Σ bounds a contractible 6- or 7-manifold [38], and by the above argument is diffeomorphic to S^5 or S^6.

Corollary 2.2. ([79], [81], [100]) *If $n \geq 5$, any two homotopy n-spheres are homeomorphic by a map which is a diffeomorphism except perhaps at a single point.*

Θ_n has a natural group operation $\#$, called connected sum, defined as follows. If Σ_1 and Σ_2 are homotopy n-spheres, choose points $x_i \in \Sigma_i, i = 1, 2$, and let D_i be a neighborhood of x_i which maps to the disk D^n under some coordinate map φ_i which we may assume carries x_i to 0. Define $\Sigma_1 \# \Sigma_2$ as the identification space of the disjoint union $(\Sigma_1 - x_1) \sqcup (\Sigma_2 - x_2)$ in which we identify $\varphi_1^{-1}(tu)$ with $\varphi_2^{-1}((1-t)u)$ for every $u \in S^{n-1}$ and $0 < t < 1$.

Give $\Sigma_1 \# \Sigma_2$ an orientation agreeing with those given on $\Sigma_1 - x_1$ and $\Sigma_2 - x_2$ (which is possible since the map of punctured disks $tu \to (1-t)u$ induced by the gluing is orientation preserving). Intuitively, we are cutting out the interiors of small disks in Σ_1 and Σ_2 and gluing along the boundaries, appropriately oriented.

Connected sum is well defined. By results of Cerf [23] and Palais [69], given orientation preserving embeddings $g_1, g_2 : D^n \to M$ into an oriented n-manifold, then $g_2 = f \circ g_1$ for some diffeomorphism f of M. (One may readily visualize independence of the choice of points x_i. Given x_1 and x_1' in Σ_1, there is an n-disk $D \subset \Sigma_1$ containing these points in the interior and a diffeomorphism carrying x_1 to x_1' which is the identity on ∂D.) Connected sum is clearly commutative and associative, and S^n itself is the identity.

The inverse of any homotopy sphere Σ is the oppositely oriented $-\Sigma$. If we think of $\Sigma \# (-\Sigma)$ as two disks D^n glued along their common boundary S^{n-1}, then we may intuitively visualize a contractible $(n+1)$-manifold W bounding $\Sigma \# (-\Sigma)$ by rotating one of the disks 180° around the boundary S^{n-1} till it meets the other — rather like opening an n-dimensional awning with S^{n-1} as the hinge. Removing the interior of a disk from the interior of W yields an h-cobordism from $\Sigma\#(-\Sigma)$ to S^n.

Theorem 2.3. (Kervaire, Milnor [38]) *For $n \neq 3$ the group Θ_n is finite.*

We shall see below that in almost all dimensions, Θ_n is a direct sum of two groups: one is a cyclic group detected, much as in Milnor's example, from invariants of manifolds which the spheres bound; the second is a quotient group of the stable n^{th} homotopy of the sphere.

The above definition of # applies to arbitrary closed n-manifolds M_1 and M_2. Though not a group operation in this case, it does define a group action of Θ_n on h-cobordism classes of n-manifolds. For bounded manifolds the analogous operation, connected sum along the boundary, is defined as follows.

Suppose $M_i = \partial W_i$, $i = 1, 2$. Choose a disk D^{n+1} in W_i such that the southern hemisphere of the bounding sphere lies in M_i. Remove the interior of D^{n+1} from W_i, and the interior of the southern hemisphere from $\partial W_i = M_i, i = 1, 2$. What remains of these $(n+1)$-disks are the northern hemispheres of their bounding spheres. Glue the two resulting manifolds together along these hemispheres D^n to form $W_1\#W_2$. Restricted to the boundaries this operation agrees with # defined above, and again respects h-cobordism classes.

§3 An exact sequence for smoothings.

To compute the group Θ_n, we consider the tangent bundles of homotopy spheres and the manifolds they bound. Let M be any compact smooth m-manifold. We may suppose M is a differentiable submanifold of $\mathbf{R}^k$ via a differentiable inclusion $\Phi : M \to \mathbf{R}^k$ for some k sufficiently large. (In fact, this is a fairly direct consequence of the definition of smooth manifold). For any $x \in M$, coordinate neighborhood V_α containing x, and coordinate map $\varphi : V_\alpha \to \mathbf{R}^m$, define the tangent space to M at x, $\tau(M)_x$, to be the image of the derivative of $\Phi \circ \varphi_\alpha^{-1}$ at $\varphi_\alpha(x)$. Change of variables in calculus shows that the m-dimensional subspace $\tau(M)_x$ of $\mathbf{R}^k$ is independent of the choice of V_α and φ_α. Define the tangent bundle $\tau(M)$ to be the set $\{(x, v) \in \mathbf{R}^k \times \mathbf{R}^k | v \in \tau(M)_x\}$ together with the map $p : \tau(M) \to M$ induced by projection to the first coordinate. The fiber $p^{-1}(x)$ is the m-dimensional vector space $\{x\} \times \tau(M)_x$.

The tangent bundle is a special case of an n-dimensional vector bundle ξ consisting of a total space E, base space B, and map $p : E \to B$ which locally is a projection map of a product. Thus we assume there are open sets U_β covering B (or M in the case of $\tau(M)$) and homeomorphisms : $\psi_\beta : U_\beta \times \mathbf{R}^n \to p^{-1}(U_\beta)$ such that $\psi_\beta^{-1} \circ \psi_{\beta'}$ is a linear isomorphisms on $x \times \mathbf{R}^n$ for every $x \in U_\beta \cap U_{\beta'}$, and $p \circ \psi_\beta$ is the projection onto U_β. When the base space is a smooth manifold, we will assume the maps ψ_β are diffeomorphisms. Any operation on vector spaces defines a corresponding operation on bundles. For example, using direct sum of spaces we define the Whitney sum $\oplus$ as follows.

If ξ_1 and ξ_2 are m- and n-plane bundles, with total spaces E_1 and E_2 and common base B, then $\xi_1 \oplus \xi_2$ is the $(m+n)$-plane bundle with base B and total space the fiber product $\{(x_1, x_2) | p_1(x_1) = p_2(x_2)\}$. Bundles over a manifold "stabilize" once the fiber dimension exceeds that of the manifold. That is, if ξ_1 and ξ_2 are bundles over an m-manifold M of fiber dimension $k > m$, and if $\xi_1 \oplus \epsilon_M^j \cong \xi_2 \oplus \epsilon_M^j$, where ϵ_M^j is the product bundle $M \times \mathbf{R}^j \to M$, then $\xi_1 \cong \xi_2$.

We will need other vector bundles associated with M. If M is embedded as a submanifold of an n-dimensional smooth manifold N, where for simplicity we assume both M and N are closed, contained in $\mathbf{R}^k$ for some k, and $m < n$, the $(n-m)$-dimensional normal bundle of M in N, $\nu(M, N)$, has as fiber at $x \in M$ the elements of $\tau(N)_x$ which are orthogonal to $\tau(M)_x$. Here orthogonality can be defined using dot product in $\mathbf{R}^k$. We denote the normal bundle of M in $\mathbf{R}^k$ by $\nu(M)$.

We call a manifold M parallelizable if $\tau(M)$ is trivial, that is, isomorphic to $M \times \mathbf{R}^m \to M$. The sphere S^n is parallelizable precisely when n equals 1, 3, or 7, a magical fact proved by Bott and Milnor [10] (and independently by Kervaire) who also show that these are the only spheres which support multiplications (complex, quaternionic, and Cayley). Recall that Milnor used the quaternionic multiplication on S^3 in his first construction of homotopy spheres.

A somewhat weaker condition on $\tau(M)$ is stable parallelizability, that is, the bundle $\tau(M) \oplus \epsilon_M^1 \cong \epsilon_M^{n+1}$. More generally, two vector bundles ξ_1 and ξ_2 over a base B are stably isomorphic if $\xi_1 \oplus \epsilon_B^j \cong \xi_2 \oplus \epsilon_B^k$ where, if B is a complex of dimension r, the total fiber dimension of these Whitney sums exceeds r. Such bundles are said to be in the "stable range". A connected, compact m-manifold M with non-trivial boundary is parallelizable iff it is stably parallelizable, since it has the homotopy type of an $(m-1)$-complex and thus $\tau(M)$ is already in the stable range.

Though few spheres are parallelizable, all are stably parallelizable. In fact, if we envision the fiber of $\nu(S^n)$ at x, in the usual embedding of $S^n \subset \mathbf{R}^{n+1}$, as generated by x, then $\nu(S^n) \cong S^n \times \mathbf{R}$. Thus $\tau(S^n) \oplus \epsilon_{S^n}^1$ is isomorphic to the restriction of the trivial tangent bundle of $\mathbf{R}^{n+1}$ to S^n. Far less obvious is the following result of Kervaire and Milnor ([38], 3.1), which follows from obstruction theory and deep computations of Adams about the J-homomorphism:

<u>THEOREM 3.1</u>. *Every homotopy sphere Σ^n is stably parallelizable.*

As an immediate corollary, if Σ^n is embedded in $\mathbf{R}^k$ where $k > 2n+1$, then $\nu(\Sigma^n) \cong \epsilon^{k-n}_{\Sigma^n}$. For $\tau(\Sigma^n) \oplus \epsilon^1_{\Sigma^n} \oplus \nu(\Sigma^n)$ equals the restriction of $\mathbf{R}^k \oplus \epsilon^1_{\mathbf{R}^k}$ restricted to Σ^n. But $\tau(\Sigma^n) \oplus \epsilon^1 \cong \epsilon^{n+1}_{\Sigma^n}$ since the tangent bundle is stably parallelizable, so $\nu(\Sigma^n)$ is trivial by stability.

Given an isomorphism $\varphi : \nu(\Sigma^n) \cong \Sigma \times \mathbf{R}^{k-n}$ we define a continuous map $S^k \to S^{k-n}$ as follows. Regard $\Sigma \times \mathbf{R}^{k-n}$ as a subset of S^k, and S^{k-n} as the disk D^{k-n} with its boundary identified to a point $*$. Then send the pair $\varphi^{-1}(x, y)$, where $(x, y) \in \Sigma \times D^{k-n}$ to the point in S^{k-n} corresponding to y, and send all other points of S^k to $*$. Following [38], let $p(\Sigma^n, \varphi)$ denote the homotopy class of this map in the stable homotopy group of the sphere $\Pi_n(S) = \pi_k(S^{k-n})$.

Generally, if (M, φ) is any n-manifold with framing $\varphi : \nu(M) \xrightarrow{\cong} (M \times \mathbf{R}^{k-n})$ of the normal bundle in $\mathbf{R}^k$, the same definition yields a map $p(M, \varphi) \in \Pi_n(S)$. This is the Pontrjagin-Thom construction. If $(M_1, \varphi_1) \sqcup (M_2, \varphi_2) \subset \mathbf{R}^k$ form the framed boundary of an $(n+1)$-manifold $(W, \partial W, \Phi) \subset (\mathbf{R}^k \times [0, \infty), \mathbf{R}^k \times 0)$, we say that they are framed cobordant.

<u>THEOREM 3.2</u>. (Pontrjagin [72], Thom [89]) *For any manifold M with stably trivial normal bundle with framing φ, there is a homotopy class $p(M, \varphi)$ dependent on the framed cobordism class of (M, φ). If $p(M) \subset \Pi_n(S)$ is the set of all $p(M, \varphi)$ where φ ranges over framings of the normal bundle, it follows that $0 \in p(M)$ iff M bounds a parallelizable manifold.*

The set $p(S^n)$ has an explicit description. Any map $\alpha : S^n \to SO(r)$ induces a map $J(\alpha) : S^{n+r} \to S^r$ by writing $S^{n+r} = (S^n \times D^r) \cup (D^{n+1} \times S^{r-1})$, sending $(x, y) \in S^n \times D^r$ to the equivalence class of $\alpha(x)y$ in $D^r/\partial D^r = S^r$, and sending $D^{n+1} \times S^{r-1}$ to the (collapsed) ∂D^r. Let $J : \pi_n(SO) \to \Pi_n(S)$ be the stable limit of these maps as $r \to \infty$. Then $p(S^n) = \text{image}(J(\pi_n(SO)) \subseteq \Pi_n(S)$.

Let bP_{n+1} denote the set of those h-cobordism classes of homotopy spheres which bound parallelizable manifolds. In fact, bP_{n+1} is a subgroup of Θ_n. If $\Sigma_1, \Sigma_2 \in bP_{n+1}$, with bounding parallelizable manifolds W_1, W_2, then $\Sigma_1 \# \Sigma_2$ bounds the parallelizable manifold $W_1 \# W_2$ where the latter operation is connected sum along the boundary.

<u>THEOREM 3.3</u>. *For $n \neq 3$, there is a split short exact sequence*

$$0 \to bP_{n+1} \to \Theta_n \to \Theta_n / bP_{n+1} \to 0$$

where the left hand group is finite cyclic and Θ_n/bP_{n+1} injects into

$$\Pi_n(S)/J(\pi_n(SO))$$

via the Pontrjagin-Thom construction. The right hand group is isomorphic to $\Pi_n(S)/J(\pi_n(SO))$ when $n \neq 2^j - 2$.

Injectivity of $\Theta_n/bP_{n+1} \to \Pi_n(S)/J(\pi_n(SO))$ follows from 3.2; see [38] for details. Since the stable homotopy groups are finite, so is Θ_n/bP_{n+1}. In the next two sections we examine how surgery is used to calculate bP_{n+1} and show that the sequence splits. In particular, we will get an exact order for the group bP_{n+1} for most n, and verify the finiteness asserted in 2.1.

§4 Computing bP_{n+1} using surgery.

Suppose $\Sigma^n \in bP_{n+1}$ bounds a parallelizable manifold W whose homotopy groups $\pi_i(W)$ vanish below dimension j for some $j < n/2$. With this latter restriction, any element of $\pi_j(W)$ may be represented by an embedding $f : S^j \to \text{interior}(W)$. Since W is parallelizable, the restriction of $\tau(W)$ to $f(S^j)$ is trivial and hence, by stability, so is the normal bundle $\nu(f(S^j), W)$. Let $F : S^j \times D^{n+1-j} \to \text{interior}(W)$ be an embedding which extends f and frames the normal bundle. Let $W(F)$ denote the quotient space of the disjoint union $(W \times [0,1]) \sqcup (D^{j+1} \times D^{n+1-j})$ in which $(x, y) \in S^j \times D^{n+1-j}$ is identified with $(F(x, y), 1) \in W \times 1$. Think of the $(n+2)$-manifold $W(F)$ as obtained from $W \times [0,1]$ by attaching a $(j+1)$-handle $D^{j+1} \times D^{n+1-j}$ via F. This manifold seems to have non-smooth corners near the gluing points $S^j \times S^{n-j}$, but a straightforward argument shows how to smoothly straighten the angle on this set. The resulting manifold has boundary $(W \times \{0\}) \cup (\Sigma^n \times [0,1]) \cup W'$ where W', the "upper boundary" of $W(F)$, is obtained by cutting out the interior of $F(S^j \times D^{n+1-j})$, leaving a boundary equal to $S^j \times S^{n-j}$, and gluing $D^{j+1} \times S^{n-j}$ to it along its boundary.

We say that W' is obtained from W by doing surgery via the framed embedding F. Since this process attaches a $(j+1)$-disk via f and $j < n/2$, it follows that $\pi_i(W') \cong \pi_i(W)$ for $i < j$, and $\pi_j(W') \cong \pi_j(W)/\Lambda$ for some group Λ containing the homotopy class of f. The surgery can be done in such a way that the tangent bundle $\tau(W')$ is again trivial. The restriction of $\tau(W(F))$ to the image of f has two trivializations, one coming from the parallelizability of $W \times [0,1]$, the other from the triviality of any bundle over $D^{j+1} \times D^{n+1-j}$, a contractible space. Comparing them gives a map $\alpha : S^j \to SO(n+2)$. Since $j < n - j$, this factors as a composite $S^j \xrightarrow{\beta} SO(j+1) \xrightarrow{\subseteq} SO(n+2)$, where the second map is the natural inclusion. (This is an elementary argument using exact homotopy sequences of fibrations $SO(r) \to SO(r+1) \to S^r$ for $r \geq j$.) It follows that the $(n+2)$-manifold $W(F_{\beta^{-1}})$ is parallelizable, where $\beta^{-1} : S^j \to SO(n-j)$ carries x to $(\beta(x))^{-1}$, and $F_{\beta^{-1}}(x,y) = F(x, \beta^{-1}(x)y)$. The restriction of the tangent bundle of the "upper boundary" $W'_{\beta^{-1}}$ of $W(F_{\beta^{-1}})$ is isomorphic to $\tau(W'_{\beta^{-1}}) \oplus \epsilon^1_{W'_{\beta^{-1}}}$, with the trivial subbundle $\epsilon^1_{W'_{\beta^{-1}}}$ generated by the inward normal vectors along the boundary. Thus $W'_{\beta^{-1}}$ is stably parallelizable and, since $\partial W'_{\beta^{-1}} = \Sigma \neq \emptyset$, parallelizable.

Though surgery kills the homotopy class represented by f, it opens up an $(n-j)$-dimensional "hole" represented by the homotopy class of $F|_{x \times S^{n-j}}$ for any $x \in D^{j+1}$. But no matter. Our strategy is to start with a generator g of the lowest non-zero homotopy group $\pi_j(W)$. As long as $j < n/2$ we can do surgery to kill g, adding no new homotopy in dimension j or lower, and leaving $\partial M = \Sigma$ fixed. Thus working inductively on the finite number of generators in a given dimension j, and on the dimension, we obtain:

<u>Proposition 4.1</u>. *If Σ^n bounds a parallelizable manifold, it bounds a parallelizable manifold W such that $\pi_j(W) = 0$ for $j < n/2$.*

Suppose that $n = 2k$. The first possible non-zero homotopy (and hence homology) group of the manifold W of 4.1 occurs in dimension k. By Poincaré duality, all homology and cohomology of W is concentrated in dimensions k and $k+1$. If by surgery we can kill $\pi_k(W)$, the resulting manifold W' will have trivial homology. Removing a disk from the interior of W' thus yields an h-cobordism between Σ and S^n.

But if we do surgery on W using a framed embedding $F : S^k \times D^{k+1} \to$ interior(W) to kill the homotopy class of $f = F|_{S^k \times 0}$, it is possible that the homotopy class of $f' = F|_{0 \times S^k}$ might be a "new" non-zero element of $\pi_k(W')$. If there were an embedding $g : S^{k+1} \to W$ whose image intersected that of f transversely in a single point, then f' would be null-homotopic. For we may suppose that image$(g) \cap$ image$(F) = F(x \times D^{k+1})$ for some $x \in S^k$. Then f' is homotopic to $\tilde{f} = F|_{x \times S^k}$, and $\tilde{f}$ deforms to a constant in the disk formed by the image of g lying outside $F(S^n \times int(D^{n+1}))$.

By moving to homology, we get criteria which are easier to fulfill and insure the triviality of f'. Let $\lambda \in H_k(W)$ and $\lambda' \in H_k(W')$ be the homology classes corresponding to f and f' under the Hurewicz isomorphism, and suppose also that λ generates a free summand in $H_k(W)$. By Poincaré duality there is $\mu \in H_{k+1}(M)$ such that $\lambda \cdot \mu = 1$ where $\cdot$ denotes intersection number. The element μ plays the role of the transverse sphere. A straightforward argument involving homology exact sequences of the pairs (W, W_0) and (W', W_0), where $W_0 = W \backslash int(F(S^k \times D^{k+1}))$ shows that

$\lambda' = 0$, even when the framing F is replaced by $F_{\beta^{-1}}$ to ensure parallelizability. Thus, by a sequence of surgeries, we can reduce $H_k(W)$ to a torsion group T.

Here the argument becomes more technical and delicate. Kervaire and Milnor show that if k is even, surgery always changes the free rank of $H_k(W)$, so if λ is a generator of T, surgery on λ reduces $|T|$ at the cost of introducing non-zero $\mathbf{Z}$ summands, which are then killed by subsequent surgeries. If k is odd, special care must be taken to choose a framing which both reduces the size of T and preserves parallelizability. In both cases, $H_k(W)$ can be eliminated by surgery and we obtain:

Theorem 4.2. (Kervaire-Milnor [38]) *For any* $k \geq 1, bP_{2k+1} = 0$.

§5 The groups bP_{2k}.

Suppose W is a parallelizable $2k$-manifold with boundary the homotopy sphere Σ^{2k-1}, $k > 2$. As in §4, we may assume, after performing surgery on W leaving ∂W fixed, that W is $(k-1)$-connected. By Poincaré duality the homology of W is free and concentrated in dimension k. Once again, the homotopy class of an embedding $f : S^k \to W$ can be killed by surgery without adding new non-trivial homotopy classes if the geometry near it is nice — i.e., if $f(S^k)$ has a trivial normal bundle and there is an embedding $g : S^k \to W$ whose image intersects $f(S^k)$ transversely in a single point. Of course, we have no assurance that such transverse spheres exists and, since $\nu(f(S^k), W)$ is just below the stable range, parallelizability of W does not guarantee triviality of this normal bundle. But there is a simple criterion for ensuring homological intersection conditions which enable elimination of $H_k(W)$ by surgery.

Let $k = 2m$. The intersection number defines a symmetric bilinear map $H_{2m}(W) \times H_{2m}(W) \to \mathbf{Z}$. Since ∂W is homeomorphic to a sphere, we can view W as a topologically closed manifold and hence, by Poincaré duality, the intersection pairing is non-singular. If this pairing is diagonalized over $\mathbf{R}$, define the signature $\sigma(W)$ to be the number of positive diagonal entries minus the number of negative ones.

Theorem 5.1. ([38], [54]) *The homotopy (and hence homology) groups of* W *can be killed by surgery if and only if* $\sigma(W) = 0$.

The intersection form is also even (for any homology class $\lambda \in H_{2m}(W)$, the self-intersection number $\lambda \cdot \lambda$ is an even integer), and hence the signature must be divisible by 8 ([15], [76]). A $(2m-1)$-connected parallelizable $4m$-manifold W_0^{4m} with boundary a homotopy sphere and signature $\sigma(W_0^{4m})$ precisely equal to 8 can be constructed as follows. Let $E_1, E_2, \ldots, E_8$ be disjoint copies of the subset of $\tau(S^{2m})$ of vectors of length ≤ 1. We glue E_1 to E_2 as follows. The restriction of E_i to a $2m$-disk in the base is diffeomorphic to $D^{2m}_{b,i} \times D^{2m}_{f,i}$ where the subscripts b and f denote base and fiber (i.e., for $x \in D^{2m}_{b,i}$, an element of the base of E_i, $x \times D^{2m}_{f,i}$ is the fiber over x). Then we identify $(x, y) \in D^{2m}_{b,1} \times D^{2m}_{f,1}$ with $(y, x) \in D^{2m}_{b,2} \times D^{2m}_{f,2}$. Thus the disk in the base of E_1 maps onto a fiber in E_2, transversely crossing the base in a single point. We say that we have attached these disks by "plumbing". As before, there are corners, but these can be easily smoothed by straightening the angle. We similarly attach E_2 to E_3, E_3 to $E_4, \ldots$, E_6 to E_7, and E_8 to E_5. The resulting manifold W_0^{4m} has boundary a homotopy sphere Σ_0^{4m-1} and homology intersection form given by the following matrix with determinant 1 and signature 8 (see [15] or [60] for nice expositions):

$$A = \begin{pmatrix} 2 & 1 & & & & & & \\ 1 & 2 & 1 & & & & & \\ & 1 & 2 & 1 & & & & \\ & & 1 & 2 & 1 & & & \\ & & & 1 & 2 & 1 & 0 & 1 \\ & & & & 1 & 2 & 1 & 0 \\ & & & & 0 & 1 & 2 & 0 \\ & & & & 1 & 0 & 0 & 2 \end{pmatrix}$$

where all omitted entries are 0. The 2's on the main diagonal are the self-intersections of the 0-section of $\tau(S^{2m})$ in $E_1, \ldots E_8$. Note that even though the 0-section of each E_i has a sphere intersecting transversely in a single point, it cannot be killed by surgery since its normal bundle in W_0^{4m} is non-trivial.

By taking connected sums along the boundary (as described in §2) we obtain, for any j, a parallelizable manifold with signature $8j$ and boundary a homotopy sphere. If $(W_0^{4m})^{\#j}$ equals the j-fold sum $W_0 \# \ldots \# W_0$ if $j > 0$, and the $(-j)$-fold sum $(-W_0)\# \ldots \#(-W_0)$ if $j < 0$, then $\sigma((W_0^{4m})^{\#j}) = 8j$. By 2.2, the boundary $(\Sigma_0^{4m-1})^{\#j}$ is a homotopy sphere, homeomorphic to S^{4m-1}. We use this construction to compute the cyclic group bP_{4m}.

<u>Theorem 5.2</u>. *For $m > 1$ the homomorphism $\sigma : \mathbf{Z} \to bP_{4m}$ given by*

$$\sigma(j) = \partial(W_0^{4m})^{\#j} = (\Sigma_0^{4m-1})^{\#j}$$

is a surjection with kernel all multiples of

$$\sigma_m = a_m 2^{2m-2}(2^{2m-1} - 1)\text{numerator}(B_m/4m)$$

Here $a_m = 2$ or 1 depending on whether m is odd or even, and the rational Bernoulli numbers B_m are defined by the power series

$$\frac{z}{e^z - 1} = 1 - \frac{z}{2} + \frac{B_1}{2!}z^2 - \frac{B_2}{4!}z^4 + \frac{B_3}{6!}z^6 - + \dots$$

This lovely result, announced in *Groups of Homotopy Spheres I* , is a confluence of earlier work of Kervaire and Milnor, the signature theorem of Hirzebruch ([31], [62]) and J-homomorphism computations of Adams ([1] – [4]). We sketch a proof.

Suppose W^{4m} is an oriented, closed, smooth manifold with a framing φ of the stable tangent bundle in the complement of a disc. By the signature theorem,

$$\sigma(W) = \Big\langle \frac{2^{2m}(2^{2m-1} - 1)B_m}{(2m)!} p_m(W), [W] \Big\rangle$$

where $p_m(W)$ is the m^{th} Pontrjagin class and $[W] \in H_{4m}(W)$ is the orientation class. There is an obstruction $O(W, \varphi) \in \pi_{4m-1}(SO) \cong \mathbf{Z}$ to extending to all W the given framing on W less a disk. Milnor and Kervaire [61] showed that the Pontrjagin number $\langle p_m(W), [W]\rangle \in \mathbf{Z}$ corresponds to $\pm a_m(2m-1)!O(W, \varphi)$ under this identification of groups. This shows that $O(W, \varphi)$ is independent of the choice of φ, and that an almost parallelizable W is stably parallelizable iff $\sigma(W) = 0$. A straightforward argument using the Pontrjagin-Thom construction shows that an element $\gamma \in \pi_{j-1}(SO)$ occurs as an obstruction $O(W)$ to framing an almost parallelizable W^j iff $J(\gamma) = 0$. A hard computation of Adams [4] showed that the order$(J(\pi_{4m-1}(SO)))$ = denominator$(B_m/4m)$, up to (perhaps) multiplication by 2 in half the dimensions. In their solutions to the Adams conjecture, Quillen [73] and Sullivan [87] showed that this multiplication by 2 is unnecessary, completing the proof.

<u>Corollary 5.3</u>. *Let $\widehat{W}_j^{4m}$ denote the space obtained from $(W_0^{4m})^{\#j}$ by attaching a cone on the boundary. If $j \not\equiv 0$ mod σ_m, then $\widehat{W}_j^{4m}$ is a closed topological $4m$-manifold with a smooth structure in the complement of a point, but no smooth structure overall.*

A second application of Adams' J-homomorphism computation yields the exact order of Θ_{4n-1}. By the Pontrjagin-Thom construction, any element of the stable homotopy group $\Pi_{4n-1}(S)$ corresponds uniquely to a framed cobordism class of $(4n-1)$-manifolds. From the same argument that showed that $bP_{4m-1} = 0$, any such class is represented by a homotopy sphere. Thus the injection of $\Theta_{4m-1}/p(S^{4m-1}) \to \Pi_{4n-1}(S)/J(\pi_{4m-1}(SO))$ of 3.3 is a bijection and we have:

THEOREM 5.4. [38] *For $m > 1$, Θ_{4m-1} has order*

$$a_m 2^{2m-4}(2^{2m-1}-1)B_m(\text{order}(\Pi_{4m-1}(S)))/m.$$

Brieskorn ([11], [12]), Hirzebruch [33], and others ([34], [71]) have studied these homotopy spheres and their bounding manifolds in a very different context. Let $a = (a_0, a_1, \ldots, a_n)$ be an $(n+1)$-tuple of integers $a_j \geq 2$, $n \geq 3$. Define a complex polynomial $f_a(z_0, \ldots, z_n) = z_0^{a_0} + \ldots + z_n^{a_n}$. The intersection $f_a^{-1}(0) \cap S^{2n+1}$ of the affine variety $f_a^{-1}(0) \subset \mathbf{C}^{n+1}$ with the sphere is a smooth $(2n-1)$-manifold M_a. For small $\epsilon > 0$, M_a is diffeomorphic to $f_a^{-1}(\epsilon) \cap S^{2n+1}$, and this in turn bounds the parallelizable $2n$-manifold $f_a^{-1}(\epsilon) \cap D^{2n+2}$. Brieskorn [12] shows that if $a = (3, 6j-1, 2, \ldots, 2)$, with 2 repeated $2m-1$ times (so that $n = 2m$), then $\sigma(f_a^{-1}(\epsilon) \cap D^{2n+2}) = (-1)^m 8j$. In particular, M_a is diffeomorphic to $(\Sigma_0^{4m-1})^{\#(-1)^m j}$. It follows from the work above, and is shown directly in [12] or [34], that $\sigma(f_a^{-1}(\epsilon) \cap D^{2n+2})$ is diffeomorphic to $(W_0^{4m})^{\#(-1)^m j}$.

Finally, we consider bP_{n+1} when $n = 4m+1$, even more delicate and still not computed for all m. Suppose $\Sigma = \partial W^{4m+2}$ where W is parallelizable with framing φ. By surgery, we may assume W is $2m$-connected. The obstruction to continuing this framed surgery to obtain a contractible space, the Kervaire invariant $c(W, \varphi) \in \mathbf{Z}/2$, derives from the Arf-invariant for non-singular $\mathbf{Z}/2$ quadratic forms. If V is a $\mathbf{Z}/2$ vector space, we say that $\xi : V \to \mathbf{Z}/2$ is a quadratic form if $\xi(x+y) - \xi(x) - \xi(y) = (x, y)$ is bilinear, and ξ is non-singular if the associated bilinear form is. Suppose V is finite dimensional, and choose a symplectic basis $\{\alpha_i, \beta_i | i = 1 \ldots r\}$ where $(\alpha_i, \alpha_i) = (\beta_i, \beta_i) = 0$ and $(\alpha_i, \beta_j) = \delta_{i,j}$. Define the Arf invariant of ξ by $A(\xi) = \sum_{i=1}^r \xi(\alpha_i)\xi(\beta_i) \in \mathbf{Z}/2$. A theorem of Arf [6] (see also [15], pp 54-55) states that two non-singular quadratic forms on a finite dimensional $\mathbf{Z}/2$ vector space are equivalent iff their Arf invariants agree. The Arf invariant has been used by Kervaire [37], Kervaire and Milnor [38], Browder [16], Brown [17], Brown and Peterson [18], and others to study surgery of spheres and other simply-connected $(4m+2)$-manifolds, and by Wall ([95], [96]) to extend this work to the non-simply connected case.

THEOREM 5.5. *There is a non-singular quadratic form*

$$\psi : H^{2k+1}(W, \partial W; \mathbf{Z}/2) \to \mathbf{Z}/2$$

with associated quadratic form $(x, y) \to \langle x \cup y, [W] \rangle$. Let $c([W, \partial W], \varphi)$, the Kervaire invariant, be the Arf invariant of ψ, which depends on the *framed cobordism class of* $([W, \partial W], \varphi)$*. Then* $([W, \partial W], \varphi)$ *is framed cobordant to a contractible manifold iff* $c([W, \partial W], \varphi) = 0$*. In particular,* bP_{4k+2} *is isomorphic to* $\mathbf{Z}/2$ *or 0.*

In [38], ψ is defined as a cohomology operation which detects $[\ \iota,\ \iota](x \cup y)$, where $[\ ,\ ]$ is the Whitehead product; Browder [16] defines ψ using functional cohomology operations. Using the Poincaré duality isomorphism $H^{2k+1}(W, \partial W; \mathbf{Z}/2) \cong H_{2k+1}(W; \mathbf{Z}/2)$, an alternative description of ψ may be given using homology. From the Hurewicz theorem, any integral homology class reducing to $w \in H_{2k+1}(W, \mathbf{Z}/2)$ can be represented by a map $\omega : S^{2k+1} \to W$. By [28] or [77], this map is homotopic to a framed immersion. Define $\psi_0(\omega)$ to be the self-intersection number of this immersion mod 2. Then $A(\psi) = A(\psi_0) = c(W, \varphi)$.

If $\dim(W) = 6$ or 14, we can find a symplectic basis represented by framed embeddings, and $bP_6 = bP_{14} = 0$. Thus we suppose that $m \neq 1, 3$, and that W^{4m+2} is a framed manifold with boundary a homotopy sphere. As in the case of the signature, $c(W, \varphi)$ is independent of the choice of framing. We write $c(W)$ for the Kervaire invariant, which vanishes iff the quadratic form vanishes on more than half the elements of $H^{2m+1}(W, \partial W; \mathbf{Z}/2)$ (see, e.g. [15]). But computing this invariant has proved extraordinarily hard.

By plumbing together two copies of $\tau(S^{2m+1})$, we obtain a $(4m+2)$-manifold W_0 with $c(W_0) = 1$ and ∂W_0 a homotopy sphere. If $\partial W_0 = S^{4m+1}$, then by attaching a disk we obtain a closed, almost framed $(4m+2)$-manifold of Kervaire invariant 1. Adams [2] showed that if $m \not\equiv 3 \bmod 4$, the J-homomorphism $\pi_m(SO) \to \Pi_m(S)$ is injective, so the almost framed manifold can be framed. Thus $bP_{4m+2} = \mathbf{Z}/2$ precisely when the Kervaire invariant vanishes for framed, closed $(4m+2)$-manifolds — that is, when ∂W_0^{4m+2} is non-trivial. Kervaire [37] showed this for dimensions 10 and 18, Brown and Peterson [18] in dimensions $8k+2$. Browder extended this to show that the Kervaire invariant vanishes in all dimensions $\neq 2^i - 2$, and is non-zero in one of those dimensions precisely when a certain element in the Adams spectral sequence survives to E^∞. Combining this with calculations of Mahowald and Tangora [48], and of Barratt, Jones, and Mahowald [8], we have:

<u>THEOREM 5.6</u>. $bP_{4m+2} = \mathbf{Z}/2$ *if* $4m+2 \neq 2^i - 2$*, and vanishes if* $4m+2 = 6, 14, 30,$ *and* 62.

It is interesting to compare the results of this section with the h-cobordism theorem. Suppose, for example, that $j > 0$ is chosen so that $\partial((W_0^{4m})^{\# j})$ is the usual sphere S^{4m-1}. Removing a disk from the interior, we obtain a cobordism W of S^{4m-1} to itself. Giving W a Morse function f and following the proof of 2.1, it is possible to replace f with a new Morse function f' on W with critical points of index $2m$ only, with each left hand disk corresponding to one of the $2m$-spheres used to construct $(W_0^{4m})^{\#j}$. These disks are embedded in W, but their bounding spheres S^{2m-1} in the "lower" boundary component S^{4m-1} link according to the same rules for intersections of the spheres plumbed together in constructing $(W_0^{4m})^{\#j}$.

§6 Computation of Θ_n and number theory.

Let Ω_k^{framed} denote the family of framed cobordism classes of k-manifolds with a framing φ of the stable trivial normal bundle in Euclidean space. The Pontrjagin-Thom construction gives an equivalence $\Omega_k^{\text{framed}} \cong \Pi_k(S)$ which generates the injection $\Theta_n/bP_{n+1} \to \Pi_n(S)/J(\pi_n(SO))$ of §3. In particular, Ω_k^{framed} is finite group, with disjoint union as the group operation.

By placing different restrictions on the normal bundle, we obtain other cobordism groups. For example, Ω_k^U denotes the class of manifold where the stable normal bundle has the structure of a complex vector bundle. Milnor [53] showed that the groups Ω_k^U are torsion free, so that the canonical map $\Omega_k^{\text{framed}} \to \Omega_k^U$ must be trivial. Thus for any $\Sigma \in \Theta_k$ there is a U-manifold W^k with $\partial W = \Sigma$, even though Σ may not bound a parallelizable manifold. When $k = 4m-1$, Brumfiel [19] shows that W^{4m} may be chosen with all decomposable Chern numbers vanishing. In this case, $\sigma(W)$ is again divisible by 8, and independent mod $8\sigma_m$ of the choice of such W. Define a homomorphism $\alpha_m : \Theta_{4m-1} \to \mathbf{Z}/\sigma_m$ by sending the h-cobordism class of Σ^{4m-1} to $\sigma(W)/8$ mod σ_m. Then α_m is a splitting map for the exact sequence 3.3 in dimension $n = 4m-1$. By similar arguments, Brumfiel ([20], [21]) defines splittings in all dimensions $n = 4m+1$ not equal to $2^j - 3$. Combining this, the bijection $\Theta_n/bP_{n+1} \to \Pi_n(S)/J(\pi_n(SO))$, and the calculations of bP_n for n even we obtain:

<u>Theorem 6.1</u>. *If $n = 4m+1 \neq 2^j - 3$, then*

$$\Theta_{4m+1} \cong \mathbf{Z}/2 \oplus \Pi_{4n+1}/J(\pi_{4m+1}(SO)).$$

If $n = 4m-1 \geq 7$, then $\Theta_{4m-1} \cong \mathbf{Z}/\sigma_m \oplus \Pi_{4n+1}/J(\pi_{4m+1}(SO))$, where $\sigma_m = a_m 2^{2m-2}(2^{2m-1}-1)$numerator$(B_m/4m)$.

The calculation of Θ_n is thus reduced to determination of

$$\Pi_n(S)/J(\pi_n(SO)),$$

the cokernel of the J-homomorphism, a hard open problem in stable homotopy theory, and calculating bP_{n+1}. Surgery techniques yield $bP_{n+1} = 0$

when n is even, $\mathbf{Z}/2$ most of the time when $n+1 \equiv 2 \bmod 4$ (and a hard open homotopy theory problem if $n+1 = 2^j - 2$), and the explicit formula $bP_{4m} = \mathbf{Z}/\sigma_m$ for $m > 1$.

Even for the latter, there are intricacies and surprises. For any given m, it is possible (with patience) to display bP_{4m} explicitly. When $m = 25$, for example, we get a cyclic group of order

62,514,094,149,084,022,945,360,663,993,469,995,647,144,254,828,014,731,264,

generated by the boundary of the parallelizable manifold W_0^{100} of signature 8.

The integer σ_m increases very rapidly with m, with the fastest growing contribution made by numerator$(B_m)/m$. For all $m > \pi e$,

$$\text{numerator}\left(\frac{B_m}{m}\right) > \frac{B_m}{m} > \frac{4}{\sqrt{e}}\left(\frac{m}{\pi e}\right)^{2m-\frac{1}{2}} > 1$$

where the first three terms are asymptotically equal as $m \to \infty$ (see [62], Appendix B, or [66]). As noted in §5, denominator$(B_m/4m)$ equals the image of the J-homomorphism ([4], [47]). Unlike the numerator, it is readily computable. In 1840 Clausen [26] and von Staudt [83] showed that denominator(B_m) is the product of all primes p with $(p-1)$ dividing $2m$, and the next year von Staudt showed that p divides the denominator of B_m/m iff it divides the denominator of B_m. Thus for any such prime p, if p^μ is the highest power dividing m, then $p^{\mu+1}$ is the highest power of p dividing the denominator of B_m/m.

Such results suggest that it might be better to compute one prime p at a time. Let $\mathbf{Z}_{(p)}$, the integers localized at p, denote the set of rational numbers with denominators prime to p. Then for any finite abelian group G, $G \otimes \mathbf{Z}_{(p)}$ is the p-torsion of G. We investigate the p-group $bP_{4m} \otimes \mathbf{Z}_{(p)}$.

Let p be a fixed odd prime (the only 2-contribution in σ_m comes from the factor $a_m 2^{2m-4}$), and suppose $k \in \mathbf{Z}$ generate the units in $\mathbf{Z}/p^2$. Define sequences $\{\eta_m\}$, $\{\zeta_m\}$, $\{\tilde{\sigma}_m\}$, and $\{\beta_m\}$ by $\eta_m = (-1)^{m+1}(k^{2m}-1)B_m/4m$, $\zeta_m = 2^{2m-1} - 1$, $\tilde{\sigma}_m = (-1)^m 2^{2m}(k^{2m} - 1)(2^{2m-1} - 1)B_m/2m$, and $\beta_m = (-1)^m B_m/m$ if $m \not\equiv 0 \bmod (p-1)/2$ and 0 otherwise. The first three are sequences in $\mathbf{Z}_{(p)}$ since, for any generator k of the units in $\mathbf{Z}/p^2$, $\nu_p(k^{2m} - 1) = \nu_p(\text{denominator}(B_m/4m))$, where $\nu_p(x)$ denotes the exponent p in a prime decomposition of the numerator of $x \in Q$ ([2], §2 or [62], Appendix B). The last sequence lies in $\mathbf{Z}_{(p)}$ from Clausen's and von Staudt's description of the denominator of B_m/m.

The sequence η isolates p-divisibility in the numerator of B_m/m: $\nu_p(\eta_m) = \nu_p(\text{numerator}(B_m/m))$, and one is a unit in $\mathbf{Z}_{(p)}$ times the other.

Similarly, $\nu_p(\sigma_m) = \nu_p(\tilde{\sigma}_m)$ where σ_m from 5.2 is the order of bP_{4m}. These sequences come from maps, described in §7, of the classifying space BO for stable bundles. The homology of these maps yields congruences between terms of the sequences, and descriptions of the growth of p-divisibility of those terms satisfied for many primes p:

<u>Theorem 6.2</u>. [42] *Let $\lambda = (\lambda_1, \lambda_2, \ldots)$, denote any of the above sequences.*

1) *$\lambda_m \equiv \lambda_n$ mod p^{k+1} whenever $m \equiv n$ mod $p^k(p-1)/2$.*
2) *Suppose $m \equiv n$ mod $(p-1)/2$ are prime to p, and j is minimal such that $\nu_p(\lambda_{mp^j}) \leq j$. Then $\nu_p(np^i) = j$ for all $i \geq j$.*

For the sequence β, the congruences are the familiar congruences of Kummer: $(-1)^m B_m/m \equiv (-1)^n B_n/n \bmod p^{k+1}$ if $m \equiv n \bmod p^k(p-1)$ and $m, n \not\equiv 0 \bmod (p-1)/2$.

For $\lambda = \eta, \zeta$, or $\tilde{\sigma}$, 6.2 gives tools for mapping out the p-torsion in the groups bP_m. If $\nu_p(\lambda_m) = 0$, that is, λ_m is a unit in $\mathbf{Z}_{(p)}$, the same is true for any λ_n where $n \equiv m \bmod (p-1)/2$. Applying this to $\tilde{\sigma}$, it follows that a given group bP_{4m} has p-torsion iff bP_{4n} does for every $n \equiv m$ mod $(p-1)/2$. Thus to map out where all p-torsion occurs, it suffices to check p-divisibility of the coefficients $\tilde{\sigma}_1, \tilde{\sigma}_2, \ldots, \tilde{\sigma}_{(p-1)/2}$ Furthermore, the growth of p-torsion is likely quite well behaved.

<u>Conjectures 6.3</u>. *Let $\lambda = \eta, \zeta$, or $\tilde{\sigma}$, and suppose $m_0 \in \{1, 2, \ldots (p-1)/2\}$ is such that p divides λ_{m_0} (there could be several such m_0).*

1. *For any $n \equiv m_0$ mod $(p-1)/2$ which is prime to p, the exponents of p in the subsequence $\lambda_n, \lambda_{pn}, \lambda_{p^2n}, \ldots$ are given by j, $\nu_p(\lambda_{m_0}), \nu_p(\lambda_{m_0}), \ldots$, where j may be any integer $\geq \nu_p(\lambda_{m_0})$.*
2. *$\nu_p(\lambda_{m_0} - \lambda_{m_0+(p-1)/2}) = \nu_p(\lambda_{m_0})$.*
3. *η_{m_0} and η_{pm_0} are non-zero mod p^2.*

These have been verified by computer for many primes. By the congruences in 6.2, conjectures 1 and 2 are actually equivalent. When $\lambda = \zeta$, 1 and 2 are not conjectures but true globally and easily proved. The statement for ζ analogous to 3 fails, however. There exist primes p, albeit not many, such that p^2 divides $2^{p-1} - 1$. For primes less than a million, $p = 1093$ and 3511 satisfy this. However, there is no 1093 torsion in the groups bP_{4m}, since $2^j \not\equiv 1$ mod 1093 for any odd exponent j. The prime 3511 is more interesting, with possible values of m_0 equal to 708 and 862 (where 3511 but not 3511^2 divides η_{m_0} and $\tilde{\sigma}_{m_0}$), and $m_0 = 878$ (where 3511^2 but not 3511^3 divides $2^{1755} - 1$ and $\tilde{\sigma}_{m_0}$).

The p-divisibility of these sequences have long been of interest because of their relationship to Fermat's Last Theorem, recently proved by Andrew Wiles ([99], [88]).

<u>Theorem 6.4</u>. *1.* (Kummer [41]) *If p does not divide the numerator of B_m/m for $m = 1, 2, \dots, (p-3)/2$, then there is no integral solution to $x^p + y^p = z^p$.*

2. (Wieferich [97]) *If $2^{p-1} \not\equiv 1$ mod p^2, then there is no integral solution to $x^p + y^p = z^p$ where xyz is prime to p.*

A prime p satisfying the condition in 1 is said to be regular. Thus p is regular iff it is prime to the sequence η. The smallest irregular prime is 37, which divides the numerator of B_{16}. There are infinitely many irregular primes, with the same statement unknown for regular primes. Extensive computations suggest rough parity in the number of each (about 40% are irregular).

The condition in 2 is almost equivalent to p^2 not dividing the sequence ζ — almost, but not quite. The prime $p = 1093$ is prime to ζ even though 1093^2 divides $(2^{1092} - 1)$ because $2^j \not\equiv 1$ mod 1093 for any odd factor of 1092. Vandiver [91], Miramanoff [63], and others (see [92] for an extensive summary) have shown that for primes $r \le 43$, $r^{p-1} \not\equiv 1$ mod p^2 implies that $x^p + y^p = z^p$ has no integral solutions with xyz prime to p.

All this work attempted to verify Fermat's last theorem. It would be wonderful to know whether Wiles's result could be used to establish any of the conjectures 6.3, potentially giving complete information about the p-torsion in the groups bP_{4m}. Other fairly recent algebraic results have yielded partial information. For example, by translating Ferrero and Washington's proof of the vanishing of the Iwasawa invariant [27] into the equivalent formulation using Bernoulli numbers [36], it follows that p^2 does not divide $(B_n/n) - B_{n+(p-1)/2}/(n + (p-1)/2)$.

<u>§7 Classifying spaces and smoothings of manifolds</u>.

By comparing the linear structures on a "piecewise linear" bundle (discussed below), we are able to define a space PL/O whose homotopy groups equal Θ_n. Specifically, PL/O is the fiber of a map $BO \to BPL$ where BO and BPL, the spaces which classify these bundle structures, are defined as follows.

For any positive integers n and k, let $G_n(\mathbf{R}^{n+k})$ be the compact Grassmann nk-manifold $O(n+k)/O(n) \times O(k)$ where $O(j)$ is the orthogonal group. We may think of this as the space of n-planes in $(n+k)$-space. There are natural maps $G_n(\mathbf{R}^{n+k}) \to G_n(\mathbf{R}^{n+k+1})$, and we write $BO(n) = \lim_{k\to\infty} G_n(\mathbf{R}^{n+k})$. The elements of the individual n-planes in

$G_n(\mathbf{R}^{n+k})$ form the fibers of a canonical $\mathbf{R}^n$-bundle γ^n, and given any n-bundle ξ over a compact base B, there is a map $g : B \to BO(n)$ unique up to homotopy such that ξ is isomorphic to the pullback bundle $f^*(\gamma^n)$. Set $BO = \lim_{n\to\infty} BO(n) \cong \lim_{n\to\infty} G_n(\mathbf{R}^{2n})$. The set of homotopy classes $[M, BO(n)]$ and $[M, BO]$ then correspond to n-dimensional and stable bundles over the compact manifold M.

For PL manifolds the object corresponding to the vector bundle is the block bundle [74]. (Alternatively, one may use Milnor's microbundles [55]). We omit the definition, but note that the vector bundle tools used for surgery on a smooth manifold are also available for block bundles. Given any embedding $M \to N$ of PL manifolds, for example, there is a normal block bundle of M in N. One may construct a classifying space $BPL(n)$ for n-dimensional block bundles, which is the base space for a universal block bundle γ^n_{PL}. (We abuse notation slightly; BPL is often denoted $\widetilde{BPL}$ in the literature, with BPL used for its equivalent in the semisimplicial category.)

Set $BPL = \lim_{n\to\infty} BPL(n)$. Piecewise differentiable triangulation of the canonical vector bundles γ^n yields γ^n_{PL}, classifying maps $BO(n) \to BPL(n)$, and the limit map $BO \to BPL$. Regarding this map as a fibration, we define PL/O to be its fiber.

Products and Whitney sums of block bundles are defined analogously to $\times$ and $\oplus$ for vector bundles. Using these constructions, we obtain commutative H-space structures $\mu^\oplus : BO \times BO \to BO$ and $\mu^\oplus_{PL} : BPL \times BPL \to BPL$ under which the map $BO \to BPL$ is an H-map, and defines an H-space structure on the fiber PL/O as well.

Let $\mathcal{S}(M)$ denote the set of concordance classes of smoothings of a PL manifold M, where two smoothings of M are concordant if there is a smoothing of $M \times [0,1]$ which restricts to the given smoothings on $M \times 0$ and $M \times 1$. If M is the smooth triangulation of a smooth manifold M_α, we think of $\mathcal{S}(M_\alpha)$ as the concordance classes of smoothings of M with a given preferred smoothing M_α. Note that $\mathcal{S}(S^n) = \Theta_n$ except for $n = 3$. The unique smooth and PL structure on a topological S^3 dictates that $\mathcal{S}(S^3)$ consists of a single element.

If a PL manifold M has a smooth structure, then the normal block bundle of the diagonal Δ in $M \times M$ is actually the normal vector bundle. In fact, such a linearization is sufficient for existence of a smoothing. For any PL manifold M and submanifold $K \subset M$, a linearization of (M, K) is a piecewise differentiable vector bundle $p : E \to K$ where E is a neighborhood of K in M, and M induces a compatible PL structure on E. Let $\mathcal{L}(M, K)$ denote the set of all equivalence classes of such linearizations, and $\mathcal{L}_s(M, K)$

the classes of stable linearizations (i.e., the direct limit of $\mathcal{L}(M,K) \to \mathcal{L}(M \times \mathbf{R}^1, K) \to \mathcal{L}(M \times \mathbf{R}^2, K) \to \ldots$, where the maps are defined by Whitney sum with a trivial bundle).

<u>Theorem 7.1</u>. ([55], [30], [44]) *A closed PL-manifold M has a smooth structure iff $\mathcal{L}(M \times M, \Delta) \neq \emptyset$ iff $\mathcal{L}_s(M \times M, \Delta) \neq \emptyset$, and there is a bijection $\mathcal{S}(M) \to \mathcal{L}_s(M \times M, \Delta)$.*

This description uses block bundles and follows the notation of [30], but essentially identical results using microbundles are true. The theorem suggests that "smoothability" is a stable phenomenon. This is true; the natural map $\mathcal{S}(M) \to \mathcal{S}(M \times \mathbf{R}^m)$ is a bijection (see [29], [50] for smoothing products with $\mathbf{R}$, or [90], [65] for the $M \times [0,1]$ analogue). By 7.1, a PL manifold M supports a smooth structure iff the classifying map $M \to BPL$ for the stable normal block bundle of the diagonal $\Delta \subset M \times M$ lifts to BO. But the homotopy classes of such lifts are in turn classified by maps into the fiber of $BO \to BPL$:

<u>Theorem 7.2</u>. ([30], [44]) *Let M be closed PL-manifold which can be smoothed, and let M_α be some fixed smooth structure on it. Then there is a bijection $\Psi_\alpha : \mathcal{S}(M) \to [M, PL/O]$ which carries the concordance class of M_α to the trivial homotopy class.*

Despite apparent dependence on a particular smooth structure and, given M_α, on a choice of smooth triangulation, there is a great deal of naturality in the bijection Ψ_α. If N is another smoothable PL manifold with chosen smooth structure N_β, and if $f : M_\alpha \to N_\beta$ is both a diffeomorphism and PL-homeomorphism, then $\Psi_\beta \circ f^* = f^\# \circ \Psi_\alpha$ where $f^* : \mathcal{S}(N) \to \mathcal{S}(M)$ and $f^\# : [N, PL/O] \to [M, PL/O]$ are the natural maps. The bijections Ψ_α can be used to reformulate 7.2 as a well defined homotopy functor defined on "resmoothings" of a smooth manifold [30].

The H-multiplication $PL/O \times PL/O \to PL/O$ gives $[M, PL/O]$ the structure of an abelian group. Given a smoothing M_α of M, the bijection Ψ_α gives $\mathcal{S}(M)$ the structure of a group, which we denote $\mathcal{S}(M_\alpha)$. For any smoothing M_β of M, let $[M_\beta]$ denote its concordance class, a group element of $\mathcal{S}(M_\alpha)$

<u>Theorem 7.3</u>. *The group operation $*$ in $\mathcal{S}(M_\alpha)$ is given by $[M_\beta] * [M_\gamma] = [M_\omega]$ where M_ω is the unique (up to concordance) smoothing such that the germ of the smooth manifold $M_\alpha \times M_\omega$ along the diagonal equals that of $M_\beta \times M_\gamma$. In particular, $[M_\alpha]$ is the identity element. If $M = S_0^n$ denotes the n-sphere regarded as a PL manifold given the usual smoothing, the resulting bijection Ψ_0 is a group isomorphism $\Theta_n \to \pi_n(PL/O)$ for $n \neq 3$.*

Using 7.2, the H-space structure on PL/O, induced by Whitney sum of vector- and block-bundles, allowed us to define (isomorphic) group structures on $\mathcal{S}(M)$ via the bijections Ψ_α.

It is interesting to note that the finite group structures on Θ_n (with 0 substituted for the unknown Θ_3) can in turn be used to describe the H-multiplication on PL/O. See [30] for a proof. Theorem 7.2 also provides a homotopy theoretic description of smoothings for an arbitrary smoothable PL manifold, one which recasts the obstruction theories for smoothings of Munkres [65] and Hirsch [29] in terms of classical obstruction theory.

We examine the homotopy theory of PL/O, studying it one prime at a time, just as we did for the coefficients in §6. For any prime p and well-behaved space X (for example, any CW complex or any H-space), there is a space $X_{(p)}$, the localization of X at p, and map $X \to X_{(p)}$ which on homotopy groups is the algebraic localization $\pi_n(X) \to \pi_n(X) \otimes \mathbf{Z}_{(p)}$. Similarly, $H_*(X_{(p)}) \cong H_*(X) \otimes \mathbf{Z}_{(p)}$. We will see below that the localization $PL/O_{(p)}$ is a product, reflecting homotopy theoretically the splitting of the exact sequence of 3.3.

Suppose first that p is an odd prime. The sequences $\eta, \zeta, \tilde{\sigma}$, and β of §6 all arise from self-maps of the p-localizations of BO and BU (the analogue of BO which classifies stable complex vector bundles) which are reflections of geometric operations on bundles. An important example is the Adams map $\psi^k : BU_{(p)} \to BU_{(p)}$, which arises from the K-theory operation

$$\psi^k(x) = \sum_{w(\alpha)=k} (-1)^{|\alpha|+k} (k/|\alpha|)\{\alpha\}(\wedge^1(x)^{\alpha_1} \ldots \wedge^j(x)^{\alpha_j})\,,$$

where $x \in K(X)$, $\wedge^i$ is the exterior power, the sum is taken over all j-tuples of non-negative integers $\alpha = (\alpha_1, \ldots, \alpha_j)$ of weight $w(\alpha) = \alpha_1 + 2\alpha_2 + \ldots + j\alpha_j = k$, and $\{\alpha\}$ is the multinomial coefficient $(\alpha_1 + \ldots + \alpha_j)!/\alpha_1! \ldots \alpha_j!$. The reader may recognize this as the Newton polynomial applied to exterior operators. The induced map on the homotopy group $\pi_{2m}(BU_{(p)}) = \mathbf{Z}_{(p)}$ is multiplication by k^m. The Adams map on $BU_{(p)}$ induces one on $BO_{(p)}$ by the following:

<u>Theorem 7.4</u>. (Adams [5], Peterson [70]). *There are H-space equivalences*

$$BU_{(p)} \to W \times \Omega^2 W \times \ldots \Omega^{2p-4} W \quad and \quad BO_{(p)} \to W \times \Omega^4 W \times \ \ldots \Omega^{2p-6} W$$

where $\pi_{2j(p-1)}(W) = \mathbf{Z}_{(p)}, j = 1, 2, \ldots,$ and $\pi_i(W) = 0$ otherwise. In particular, $BU_{(p)} \cong BO_{(p)} \times \Omega^2 BO_{(p)}$. Any H-map $f : BO_{(p)} \to BO_{(p)}$

induces H-maps f_{4j} of $\Omega^{4j}W$, and under this equivalence f becomes a product $f_0 \times f_4 \times \ldots \times f_{2p-6}$, with an analogous decomposition for a self H-map of $BU_{(p)}$.

In fact, these are infinite loop space equivalences. Peterson constructs W as the bottom space of a spectrum associated to a bordism theory with singularities. The maps f_{4j} allow us to write the fiber F of f as a product $F_0 \times F_4 \times \ldots \times F_{2p-6}$, where F_{4j} can be seen to be indecomposable by examining the action of the Steenrod algebra on it.

Returning to the Adams map, associated to the K-theory operation $x \to \psi^k(x) - x$ is an H-map $\psi^k - 1$ of $BU_{(p)}$ and, by 7.4, $\psi^k - 1 : BO_{(p)} \to BO_{(p)}$. The induced homomorphism on $\pi_{4m}(BO_{(p)}) \cong \mathbf{Z}_{(p)}$ is multiplication by $k^{2m} - 1$. If k generates the units in $\mathbf{Z}/p^2$, the fiber J of $\psi^k - 1$ (sometimes called "Image J") is independent, up to infinite loop space equivalence, of the choice of k, and has homotopy groups equal to the p component of the image of the J-homomorphism. Since $k^{2j} \not\equiv 1 \bmod p$ unless $j \equiv 0 \bmod (p-1)/2$, using 7.4 we may also describe J as the fiber of $(\psi^k - 1)_0 : W \to W$.

The numbers $k^{2j} - 1$ are an example of a "characteristic sequence". If $f : BO_{(p)} \to BO_{(p)}$ induces multiplication by $\lambda_j \in \mathbf{Z}_{(p)}$ on $\pi_{4m}(BO_{(p)})$, it is determined up to homotopy by the characteristic sequence $\lambda = (\lambda_1, \lambda_2, \ldots)$ (with a similar statement for self-maps of BU). The elements of λ satisfy the congruences and p-divisibility conditions of 6.2 for any self-map of $BO_{(p)}$, and analogous statements for almost any map (including all H-maps) of $BU_{(p)}$ ([42], [25]).

We can use the Adams map and other bundle operations to realize the sequences of §6 as characteristic sequences. To construct these maps we must depart from the more self-contained material of the first six sections, and in particular require p-local versions of oriented bundle theory and the Thom isomorphism Φ. We refer the reader to May [49] or the Adams $J(X)$ papers [1] - [4] for beautiful presentations of this material, and present the constructions without greater detail simply to show that these sequences, so rich with number theoretic information, all arise geometrically.

ζ. Since p is an odd prime, ψ^2 is a homotopy equivalence. Then

$$(1/2)(\psi^2)^{-1} \circ (\psi^4 - 2\psi^2) : BO_{(p)} \to BO_{(p)}$$

has characteristic sequence ζ, where $\zeta_m = 2^{2m-1} - 1$.

η. For an oriented bundle ξ define $\rho^k(\xi) = (\Phi)^{-1}\psi^k\Phi(1) \in KO(B)$ where B is the base of the bundle ξ and k generates the units in $\mathbf{Z}/p^2$ as above. The resulting H-map $\rho^k : BO^{\oplus}_{(p)} \to BO^{\otimes}_{(p)}$, the so called Adams-Bott cannibalistic class ([2], [9], [49]), has characteristic sequence η given by $\eta_m = (-1)^{m+1}(k^{2m}-1)B_m/4m$. The superscripts $\oplus$ and $\otimes$ indicate that $BO(p)$ carries the H-multiplication coming from Whitney sum and tensor product, respectively.

$\tilde{\sigma}$. In the $J(X)$ papers ([1] to [4]) Adams conjectured, and Sullivan [87] and Quillen [73] proved, that for any $x \in K(X)$, where X is a finite complex, the underlying spherical fiber space of $k^q(\psi^k(x) - x)$ is stably trivial for large enough q. Localized at p, this means that the map $BO_{(p)} \xrightarrow{\psi^k - 1} BO_{(p)} \to BG_{(p)}$ is null-homotopic, where BG is the classifying space for stable spherical fiber spaces. (Its loop space $G = \Omega BG$ has homotopy equal to the stable homotopy of spheres.) Thus $\psi^k - 1$ lifts to a map $\gamma^k : BO_{(p)} \to (G/O)_{(p)}$, where G/O is the fiber of $BO \to BG$. In his thesis [84] Sullivan showed that, when localized at an *odd* prime p, the fiber G/PL of $BPL \to BG$ is H-space equivalent to $BO^{\otimes}_{(p)}$. Let θ^k denote the composite $BO^{\oplus}_{(p)} \xrightarrow{\gamma^k} (G/O)_{(p)} \to (G/PL)_{(p)} \xrightarrow{\approx} BO^{\otimes}_{(p)}$. Then θ^k has characteristic sequence $\tilde{\sigma}$ where $\tilde{\sigma}_m = (-1)^m 2^{2m}(k^{2m}-1)(2^{2m-1}-1)B_m/2m$.

β. For $j = 1, 2, \ldots, (p-3)/2$, define $b_j : \Omega^{4j}W \to \Omega^{4j}W$ to be $\rho^k \circ (\psi^k - 1)^{-1}$, and let b_0 be the constant map on W. Taking the product of these maps and applying 7.4, the resulting map $b : BO_{(p)} \to BO_{(p)}$ has characteristic sequence β where $\beta_m = (-1)^m B_m/m$.

Let bP denote the fiber of θ^k. Clearly there a map $\iota : bP \to (PL/O)_{(p)}$. There is also a p-local space C, the so-called "cokernel of J", whose homotopy is the p-component of the cokernel of the J-homomorphism in $\Pi_*(S)$ (see e.g. [49]). These spaces yield a p-local splitting of PL/O (see [49] for a beautiful presentation).

<u>Theorem 7.5</u>. *There is a map $\kappa : C \to (PL/O)_{(p)}$ such that the composite*

$$bP \times C \xrightarrow{\iota \times \kappa} (PL/O)_{(p)} \times (PL/O)_{(p)} \xrightarrow{mult} (PL/O)_{(p)}$$

is an equivalence of H-spaces (indeed, of infinite loop spaces).

The homotopy equivalence $(PL/O)_{(p)} \approx bP \times C$ now yields a p-local isomorphism $\Theta_n \cong bP_{n+1} \oplus \Pi_n(S)/J(\pi_n(SO))$, not just when $n = 4m - 1$ (6.1), but in all dimensions $\neq 3$ (where $\pi_3(PL/O)$ vanishes). Hidden in this is the representability of framed cobordism classes by homotopy spheres. In dimension $4m$ this is a consequence of the signature theorem and finiteness of $\Pi_*(S)$. In dimension $4m+2$ there may be a Kervaire invariant obstructions to such a representative, but the obstruction lies in $\mathbf{Z}/2$ and hence vanishes when localized at odd p.

Let L and M denote the (p-local) fibers of the maps $2\psi^2 - \psi^4$ and ρ. Comparing characteristic sequences, it follows that θ^k is homotopic to $\rho^k \circ (2\psi^2 - \psi^4)$, and there is a fibration $L \to bP \to M$. If the factor spaces L_{4j} and M_{4j} are both non-trivial for some $j = 1, \ldots, (p-3)/2$, the induced fibration $L_{4j} \to bP_{4j} \to M_{4j}$ cannot be a product. Otherwise, we do have a homotopy equivalence $bP \approx L \times M$ by 7.4.

This is usually the case; bP can be written this way, for example, for all primes $p < 8000$ except $p = 631$. When $p = 631$, both ρ^k_{452} and $(2\psi^2 - \psi^4_{452})$ are non-trivial, and the indecomposable space bP_{452} cannot be written as a product.

In sections 4 and 5, we saw that it was possible to kill a framed homotopy class x by surgery with no new homotopy introduced if there was a framed sphere crossing a representing sphere for x transversely at a single point. The same kind of criterion provides a tool for computing the group $[M, bP]$ of smoothings classified by bP. Suppose M is a smoothable PL n-manifold, with smooth handle decomposition $\emptyset \subseteq M_0 \subseteq M_1 \subseteq \ldots \subseteq M_n = M$, where each M_j is obtained from M_{j-1} by attaching handles $D^j \times D^{n-j}$ via embeddings $\varphi_\alpha : S^{j-1} \times D^{n-j} \to \partial M_{j-1}$. Thus with each attachment of a j-handle h^j_α we are performing surgery on a homotopy class in ∂M_{j-1}. For any j-handle, we refer to the image of $D^j \times 0$ as the left hand disk of the handle.

Handles give us a way of trying to build new smoothings. Given a j-handle h^j_α, and a homotopy j-sphere Σ regarded as the union of two j-disks attached by a diffeomorphism $f_{j,\beta} : S^{j-1} \to S^{j-1}$, form a new smooth manifold $M_j \# \Sigma$ by attaching the handle using the map $\varphi_\alpha \circ (f_{j,\beta} \times 1)$. We describe circumstances under which such resmoothings extend to all of M and give a tool for explicitly calculating smoothings.

Suppose that the homology of M and of its suspension ΣM is p-locally Steenrod representable. Thus given any x in the p-local homology of M or of ΣM, there is an orientable smooth manifold X and map $X \to M$ which carries the orientation class of X to x. Suppose in addition that the odd prime p satisfies the conjecture 6.3. Then there is a set of manifolds $\{X_\alpha\}$, and maps $X_\alpha \to M$ with the top handle D^{j_α} of X_α mapped homeomorphically onto the left hand disk of some j_α-handle $h^{j_\alpha}_\alpha$ (the rest mapping to $M_{j_\alpha - 1}$) satisfying the following: any resmoothing of M corresponding to a homotopy class in $[M, bP]$ is formed by extensions to M of smoothings of M_{j_α} of the form $M_{j_\alpha} \# \Sigma^{j_\alpha}_\beta$. This is a sort of "characteristic variety theorem" for smoothings classified by bP. See [43] for details.

We conclude with a few remarks about the prime 2, where life is very different. At the outset we have the problem posed by Kervaire in 1960, and still not completely settled, on the existence of a framed, closed $(4m+2)$-manifolds W with Kervaire invariant 1. This prevents the algebraic splitting $\Theta_{4m+1} \cong \mathbf{Z}/2 \oplus \Theta_{4m+1}/bP_{4m+2}$ for some values of m. Furthermore, not all parallelizable manifolds are representable by homotopy spheres, so we may not in general identify Θ_{4m+1}/bP_{4m+2} with the cokernel of the J-homomorphism.

Many of the results above at odd primes depend on the solution of the Adams conjecture — a lift $\gamma^k : BSO_{(2)} \to (G/O)_{(2)}$ of $\psi^k - 1$. Such a solution exists at 2, but cannot be an H-map, a condition needed to define the H-space structure for bP.

Finally, at odd p Sullivan defined an equivalence of H-spaces

$$(G/PL)_{(p)} \to (BO^{\otimes})_{(p)}.$$

At 2, G/PL is equivalent to a product

$$S \times \prod_{j \geq 1} (K(\mathbf{Z}/2, 4j+2) \times K(\mathbf{Z}_{(2)}, 4j+4)),$$

where $K(G,n)$ denotes the Eilenberg-Maclane space with a single homotopy group G in dimension n, and where S is a space with two non-zero homotopy groups, $\mathbf{Z}/2$ in dimension 2, and $\mathbf{Z}_{(2)}$ in dimension 4. There is a non-trivial obstruction in $\mathbf{Z}/2$ to S being a product, the first k-invariant of S. For the 2-localization of the analogous space G/TOP (the fiber of $BTOP \to BG$) that obstruction vanishes. This is a consequence of the extraordinary work of Kirby and Siebenmann:

<u>Theorem 7.6</u>. [39] *The fiber TOP/PL of the fibration $G/PL \to G/TOP$ is an Eilenberg-MacLane space $K(\mathbf{Z}/2, 3)$, and the following homotopy exact sequence does not split:*

$$0 \to \pi_4(G/PL) \to \pi_4(G/TOP) \to \pi_3(TOP/PL) \to 0\,.$$

<u>Epilogue</u>.

Surgery techniques, first developed to study smooth structures on spheres, have proved fruitful in an extraordinary array of topological problems. The Browder-Novikov theory of surgery on normal maps of simply connected spaces, for example, attacked the problem of finding a smooth manifold within a homotopy type. This was extended by Wall to the non-simply connected space. Surgery theory has been used to study knots and links, to describe manifolds with special restrictions on their structure (e.g., almost complex manifolds, or highly connected manifolds), and to understand group actions on manifolds. The articles in this volume expand on some of these topics and more, and further attest to the rich legacy of surgery theory.

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Mathematics Department
University at Albany
Albany, NY 12222

Email: lance@math.albany.edu