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Bordism groups of Poincaré E_∞-coalgebras and symmetric L-groups

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Abstract. A Poincaré E_∞-coalgebra construction over involutive algebras is introduced in this paper. Various types of bordism between Poincaré E_∞-coalgebras are defined and the relations between the corresponding bordism groups are studied. It is shown in particular that the Thom bordism groups of closed non-oriented smooth manifolds and the rational Wall groups of a unitary group have a common algebraic origin, that is, they are obtained by the same construction considered over the fields $\mathbb{Z}/2$ and $\mathbb{Q}$, respectively.

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In [1] Wall defined obstruction groups to surgery on non-simply-connected smooth oriented manifolds up to homotopic equivalence. On the other hand Mishchenko [2] showed that rational Wall groups can be regarded as rational bordism groups of algebraic Poincaré complexes. Ranicki [3] completely overcame the rationality condition. By using this technique Solov'ev [4] brought into consideration the so-called signature realizable subgroups of Wall groups. Investigating the structure of these subgroups turned out to be very complicated. The fact is that from an algebraic point of view the chain complex of an oriented smooth non-simply-connected manifold is a very mysterious object to study. Thus there arose the problem of describing the algebraic nature of the chain complexes for smooth manifolds and the various types of bordism between such complexes. A first step towards solving this problem is a description of the algebraic structure of chain complexes of non-simply-connected Poincaré spaces and the various bordism relations between them.

In [5], [6] Smirnov showed that one can introduce the structure of an E_∞-coalgebra on the chain complex of an arbitrary topological space and, in the case of rational coefficients, the structure of a commutative A_∞-coalgebra. Thus the chain complex of a Poincaré space has both the indicated structure and, in addition, a fundamental cycle where the operation of intersection with this cycle induces the duality isomorphism. The formalization of these conditions leads to the concept of a Poincaré E_∞-coalgebra (respectively commutative A_∞-coalgebra).

The concept of a Poincaré E_∞-coalgebra (respectively commutative A_∞-coalgebra) over an involutive algebra is introduced in the present paper.

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Various types of bordism are defined between Poincaré coalgebras and the relations between the corresponding bordism groups are studied. In particular, it is shown that the Thom bordism groups of closed non-oriented smooth manifolds and the rational Wall groups of a unitary group have a common algebraic origin, that is, they are obtained by the same construction considered over the fields $\mathbb{Z}/2$ and $\mathbb{Q}$, respectively.

§ 1. Necessary information about algebraic Poincaré complexes and symmetric L-groups

In this section we consider the basic concepts and constructions of the theory of algebraic Poincaré complexes developed by Mishchenko in the cycle of papers [2], [7], [8]. The homotopy version of this theory was constructed by Ranicki [9]. In this account we shall adhere to the terminology and techniques of [9].

Let K be a commutative ring with an identity and let A be an associative unital K-algebra with an involution $*$. We recall that an *involution* of the K-algebra A is an anticommutative automorphism $* : A \to A$, $a \mapsto a^*$ which has order two and is trivial on the scalar ring K. Let X be a chain complex of right A-modules. Then, using the involution on the algebra A, we introduce the structure of a left A-module on X in the standard way: $ax = xa^*$. We let X^* denote the dual cochain complex of right A-modules where the right A-modular structure is given by the rule

$$(fa)(x) = a^* f(x), \qquad f \in \hom_A(X;K), \quad a \in A, \quad x \in X.$$

If X is a chain complex of finitely generated projective right A-modules, then there is a natural isomorphism of chain complexes of right A-modules

$$X \to X^{**}, \qquad x \to (f \to f(x))^*,$$

which is taken as an identification. Let X be a chain complex of right A-modules; then there is a chain map of K-modules

$$X \otimes_A X \to \hom_A(X^*;X), \qquad x_1 \otimes x_2 \to x_1 f(x_2)^*,$$

which is an isomorphism if X is a chain complex of finitely generated projective A-modules.

A chain complex of right A-modules is said to be *n-dimensional* if it is chain equivalent over A to a chain complex

$$0 \longleftarrow X_0 \longleftarrow X_1 \longleftarrow \cdots \longleftarrow X_n \longleftarrow 0 \longleftarrow \cdots$$

of finitely generated projective A-modules. Let X be a given A-module chain complex and let W be the canonical resolution of the trivial $K[\mathbb{Z}_2]$-module K:

$$0 \longleftarrow K[\mathbb{Z}_2] \xleftarrow{1-T} K[\mathbb{Z}_2] \xleftarrow{1+T} K[\mathbb{Z}_2] \longleftarrow \cdots,$$

where T is the generator of $\mathbb{Z}_2$. For the complex X we consider the so-called *Q-groups*

$$Q^n(X) = H_n\big(\hom_{K[\mathbb{Z}_2]}(W; X \otimes_A X)\big),$$

where the right $K[\mathbb{Z}_2]$-module structure on $X \otimes_A X$ is given by

$$(x_1 \otimes x_2)T = (-1)^\epsilon x_2 \otimes x_1, \qquad \epsilon = \dim x_1 \dim x_2, \quad T \in \mathbb{Z}_2.$$

An arbitrary A-module chain map $f : X \to Y$ induces a homomorphism of Q-groups, which is denoted by $f^n : Q^n(X) \to Q^n(Y)$, $n \geqslant 0$. If f is a chain equivalence over A, then the homomorphisms f^n are isomorphisms. The homology class $\varphi^X \in Q^n(X)$ is the equivalence class of the chains

$$\{\varphi_s^X \in (X \otimes_A X)_{n+s},\ s \geqslant 0\},$$

for which

$$d(\varphi_s^X) = (-1)^n(\varphi_{s-1}^X + (-1)^s\varphi_{s-1}^X T), \qquad s \geqslant 0.$$

Next, let X be a finite-dimensional chain complex of right A-modules. Then the $\mathbb{Z}_2$-equivariant chain map

$$X \otimes_A X \to \hom_A(X^*; X), \qquad x_1 \otimes x_2 \to x_1 f(x_2)^*,$$

is a chain equivalence over $K[\mathbb{Z}_2]$, where the right $K[\mathbb{Z}_2]$-module structure on $\hom_A(X^*; X)$ is defined by the formula

$$(f)T = (-1)^{pq} f^*, \qquad f\colon X^p \to X_q, \quad T \in \mathbb{Z}_2.$$

This implies that the induced chain map of K-modules

$$\hom_{K[\mathbb{Z}_2]}(W; X \otimes_A X) \to \hom_{K[\mathbb{Z}_2]}(W; \hom_A(X^*; X)),$$

is also a chain equivalence and consequently for any $n \geqslant 0$ we have an isomorphism

$$Q^n(X) = H^n(\hom_{K[\mathbb{Z}_2]}(W; X \otimes_A X)) \to H_n(\hom_{K[\mathbb{Z}_2]}(W; \hom_A(X^*; X))),$$

taken as an identification. Therefore it may be assumed that the homology class $\varphi^X \in Q^n(X)$ is represented by the collection of A-module homomorphisms

$$\{\varphi_s^X : X^{n-i+s} \to X_i,\ i \geqslant 0,\ s \geqslant 0\},$$

for which

$$d(\varphi_s^X) = (-1)^n(\varphi_{s-1}^X + (-1)^s\varphi_{s-1}^X T), \qquad s \geqslant 0,$$

where

$$d(\varphi_s^X) = d^X \circ \varphi_s^X + (-1)^{n+s-1}\varphi_s^X \circ d^{X^*},$$
$$d^{X^*}(g) = (-1)^{p+1} g \circ d^X, \qquad g \in X^p.$$

An n-dimensional chain complex X of A-modules, when considered together with an element $[\varphi_s^X] \in Q^n(X)$ is called an *n-dimensional algebraic Poincaré complex* over A if the A-module chain map

$$\varphi_0^X : X^\bullet \to X_{n-\bullet}$$

induces an isomorphism of A-modules

$$H(\varphi_0^X)\colon H^i(X) \to H_{n-i}(X)$$

for any $0 \leqslant i \leqslant n$. A morphism (respectively *homotopy equivalence*)

$$f\colon (X, [\varphi_s^X]) \to (Y, [\varphi_s^Y])$$

of n-dimensional algebraic Poincaré complexes over A is an A-module chain map (respectively chain equivalence) $f : X \to Y$ such that $f^n([\varphi_s^X]) = [\varphi_s^Y]$. Given n-dimensional algebraic Poincaré complexes $(X, [\varphi_s^X])$ and $(Y, [\varphi_s^Y])$ over A we define the direct sum

$$\bigl(X, [\varphi_s^X]\bigr) \oplus \bigl(Y, [\varphi_s^Y]\bigr) = \bigl(X \oplus Y, [\varphi_s^X \oplus \varphi_s^Y]\bigr),$$

which is an n-dimensional algebraic Poincaré complex over A. By a change of orientation of the algebraic Poincaré complex $(X, [\varphi_s^X])$ we mean the following operation:

$$\bigl(-(X, [\varphi_s^X])\bigr) = (X, [-\varphi_s^X]).$$

Suppose that we are given an arbitrary chain map $f : X \to Y$ of right A-modules. We consider the relative Q-groups

$$Q^{n+1}(f) = H_{n+1}\bigl(\hom_{K[\mathbb{Z}_2]}(W; C(f \otimes_A f))\bigr),$$

where $C(f \otimes_A f)$ is the mapping cone of the chain map $f \otimes_A f$ and the right $K[\mathbb{Z}_2]$-module structure on $C(f \otimes_A f)$ is given by the formula

$$(y_1 \otimes y_2, x_1 \otimes x_2)T = \bigl((y_1 \otimes y_2)T, (x_1 \otimes x_2)T\bigr), \qquad T \in \mathbb{Z}_2.$$

The homology class $\varphi^f \in Q^{n+1}(f)$ is the equivalence class of chains

$$\bigl\{(\delta\varphi_s^Y, \varphi_s^X) \in C(f \otimes_A f)_{n+1+s} = (Y \otimes_A Y)_{n+1+s} \oplus (X \otimes_A X)_{n+s},\ s \geqslant 0\bigr\},$$

for which we have:

$$\begin{aligned} d(\delta\varphi_s^Y) &= (-1)^{n+1}\bigl(\delta\varphi_{s-1}^Y + (-1)^s \delta\varphi_{s-1}T\bigr) + (-1)^{n+s}(f \otimes_A f)(\varphi_s^X), \\ d(\varphi_s^X) &= (-1)^{n+1}\bigl(\varphi_{s-1}^X + (-1)^s \varphi_{s-1}T\bigr). \end{aligned}$$

Now let $f : X \to Y$ be an arbitrary A-module chain map of finite-dimensional complexes. We consider the $\mathbb{Z}_2$-equivariant chain map

$$\begin{gathered} \hom_A(f^*; f)\colon \hom_A(X^*; X) \to \hom_A(Y^*; Y), \\ (\hom_A(f^*; f))(g) = f \circ g \circ f^*. \end{gathered}$$

Since the complexes X and Y are finite-dimensional, there is a $K[\mathbb{Z}_2]$-module chain equivalence

$$C(f \otimes_A f) \to C(\hom_A(f^*; f))$$

between the mapping cones of the maps $f \otimes_A f$ and $\hom_A(f^*; f)$. This implies that the induced chain map of K-modules

$$\hom_{K[\mathbb{Z}_2]}\big(W; C(f \otimes_A f)\big) \to \hom_{K[\mathbb{Z}_2]}\big(W; C(\hom_A(f^*; f))\big)$$

is also a chain equivalence and consequently for any $n \geqslant 0$ we have an isomorphism

$$Q^{n+1}(f) = H_{n+1}\big(\hom_{K[\mathbb{Z}_2]}(W; C(f \otimes A f))\big) \to H_{n+1}\big(\hom_{K[\mathbb{Z}_2]}(W; C(\hom_A(f^*; f)))\big),$$

taken as an identification. Therefore we can assume that the homology class φ^f in $Q^{n+1}(f)$ is represented by a pair $(\delta\varphi_s^Y, \varphi_s^X)$ of families of A-module homomorphisms

$$\begin{aligned} &\{\delta\varphi_s^Y : Y^{n+1-i+s} \to Y_i,\ i \geqslant 0,\ s \geqslant 0\}, \\ &\{\delta\varphi_s^X : X^{n-i+s} \to X_i,\ i \geqslant 0,\ s \geqslant 0\} \end{aligned}$$

for which

$$\begin{aligned} d(\delta\varphi_s^Y) &= (-1)^{n+1}\big(\delta\varphi_{s-1}^Y + (-1)^s \delta\varphi_{s-1}^Y T\big) + (-1)^{n+s} f \circ \varphi_s^X \circ f^*, \\ d(\varphi_s^X) &= (-1)^{n+1}\big(\varphi_{s-1}^X + (-1)^s \varphi_{s-1}^X T\big), \end{aligned}$$

where

$$\begin{aligned} d(\delta\varphi_s^Y) &= d^Y \circ \delta\varphi_s^Y + (-1)^{n+s} \delta\varphi_s^Y \circ d^{Y^*}, \\ d(\varphi_s^X) &= d^X \circ \varphi_s^X + (-1)^{n+s-1} \varphi_s^X \circ d^{X^*}. \end{aligned}$$

An A-module chain map $f : X \to Y$ from an n-dimensional complex into an $(n+1)$-dimensional complex considered together with an element $[(\delta\varphi_s^Y, \varphi_s^X)] \in Q^{n+1}(f)$ is called an *$(n+1)$-dimensional algebraic Poincaré pair* over A if the chain A-module map

$$\begin{aligned} (\delta\varphi_0^Y, \varphi_0^X)&: C^\bullet(f) \to Y_{n+1-\bullet}, \\ (\delta\varphi_0^Y, \varphi_0^X)(g,h) &= \delta\varphi_0^Y(g) + f(\varphi_0^X(h)), \\ (g,h) \in Y^i \oplus X^{i-1} &= (Y_i)^* \oplus (X_{i-1})^*, \end{aligned}$$

induces an isomorphism of A-modules

$$H(\delta\varphi_0^Y, \varphi_0^X) : H^i(f) \to H_{n+1-i}(Y)$$

for any $0 \leqslant i \leqslant n+1$.

Suppose that we are given an $(n+1)$-dimensional algebraic Poincaré pair

$$\big(f : X \to Y, [(\delta\varphi_s^Y, \varphi_s^X)]\big)$$

over A. Then $(X, [(-1)^s \varphi_s^X])$ is an n-dimensional algebraic Poincaré complex over A, called the *boundary* of the pair

$$\big(f : X \to Y, [(\delta\varphi_s^Y, \varphi_s^X)]\big).$$

The n-dimensional algebraic Poincaré complexes $(X, [\varphi_s^X])$ and $(Y, []\varphi_s^Y])$ over A are said to be *bordant* if there is an $(n+1)$-dimensional algebraic Poincaré pair over A such that its boundary is

$$(X, [\varphi_s^X]) \oplus \big(-(Y, [\varphi_s^Y])\big) = \big(X \oplus Y, [\varphi_s^X \oplus -\varphi_s^Y]\big).$$

Bordism is an equivalence relation between n-dimensional algebraic Poincaré complexes over A. We denote by $L^n(A)$ the set of bordism classes of n-dimensional algebraic Poincaré complexes over A. The operations of forming the direct sum and changing the orientation of the algebraic Poincaré complexes give $L^n(A)$ the structure of an Abelian group. The groups $L^n(A)$, $n \geqslant 0$, are called the *symmetric L-groups* of the K-algebra A. An important property of symmetric L-groups is that homotopically equivalent n-dimensional algebraic Poincaré complexes over A determine the same element in the group $L^n(A)$.

We now consider separately the situation when $K = \mathbb{Q}$. In this case the chain complex

$$\hom_{\mathbb{Q}[\mathbb{Z}_2]}(W; X \otimes_A X)$$

is homotopically equivalent to the chain complex $(X \otimes_A X)^{\mathbb{Z}_2}$ of invariants of the action of the group $\mathbb{Z}_2$ on $X \otimes_A X$. Therefore for any $n \geqslant 0$ the groups $Q^n(X)$ and $H_n((X \otimes_A X)^{\mathbb{Z}_2})$ are isomorphic. This implies that the homology class

$$\varphi^X \in Q^n(X) = H_n\big((X \otimes_A X)^{\mathbb{Z}_2}\big)$$

is an n-dimensional cycle $\xi^X \in (X \otimes_A X)_n$ such that $\xi^X T = \xi^X$, $T \in \mathbb{Z}_2$. If X is an n-dimensional chain complex over A, then, taking into account the homotopy equivalence

$$X \otimes_A X \to \hom_A(X^*; X),$$

we can suppose that the homology class $\varphi^X \in Q^n(X)$ is the family of A-module homomorphisms

$$\{\xi_i^X : X^{n-i} \to X_i,\ i \geqslant 0\},$$

for which

$$d_i^X \circ \xi_i^X + (-1)^{i-1} \xi_{i-1}^X \circ (d_{n-i+1}^X)^* = 0,$$
$$(\xi_i^X)^* = (-1)^{i(n-i)} \xi_{n-i}^X.$$

Thus, if $K = \mathbb{Q}$, then an n-dimensional algebraic Poincaré complex (X, φ^X) over A is an n-dimensional A-module chain complex X considered together with a homology class

$$\varphi^X = [\xi_i^X] \in Q^n(X),$$

for which the homomorphisms

$$H(\xi_{n-i}^X) \colon H^i(X) \to H_{n-i}(X)$$

are A-module isomorphisms for any i, $0 \leqslant i \leqslant n$.

Similarly, if $K = \mathbb{Q}$, then given an A-module chain map $f : X \to Y$ the complex

$$\hom_{\mathbb{Q}[\mathbb{Z}_2]}(W; C(f \otimes_A f))$$

is homotopically equivalent to the chain complex $C(f \otimes_A f)^{\mathbb{Z}_2}$ of invariants of the action of the group $\mathbb{Z}_2$ on the cone $C(f \otimes_A f)$. Therefore for any $n \geqslant 0$ the groups $Q^{n+1}(f)$ and $H_{n+1}(C(f \otimes_A f)^{\mathbb{Z}_2})$ are isomorphic. This implies that the homology class

$$\varphi^f \in Q^{n+1}(f) = H_{n+1}\big(C(f \otimes_A f)^{\mathbb{Z}_2}\big)$$

is represented by an $(n+1)$-dimensional cycle

$$(\xi^Y, \xi^X) \in C(f \otimes_A f)_{n+1} = (Y \otimes_A Y)_{n+1} \oplus (X \otimes_A X)_n$$

such that $(\xi^Y, \xi^X)T = (\xi^Y, \xi^X)$, where $T \in \mathbb{Z}_2$. If $f : X \to Y$ is a chain A-module map from an n-dimensional complex into an $(n+1)$-dimensional complex then, taking into account the homotopy equivalences

$$X \otimes_A X \to \hom_A(X^*; X), \qquad Y \otimes_A Y \to \hom_A(Y^*; Y),$$

the homology class $\varphi^f \in Q^{n+1}(f)$ is represented by a pair (ξ_i^Y, ξ_i^X) of families of A-module homomorphisms

$$\begin{gathered}\{\xi_i^Y : Y^{n+1-i} \to Y_i,\ i \geqslant 0\},\\ \{\xi_i^X : X^{n-i} \to X_i,\ i \geqslant 0\},\end{gathered}$$

for which

$$\begin{gathered}d_i^Y \circ \xi_i^Y + (-1)^{i-1}\xi_{i-1}^Y \circ (d_{n-i+2}^Y)^* = (-1)^n f \circ \xi_{i-1}^X \circ f^*,\\ d_i^X \circ \xi_i^X + (-1)^{i-1}\xi_{i-1}^X \circ (d_{n-i+1}^X)^* = 0,\\ (\xi_i^Y)^* = (-1)^{i(n+1-i)}\xi_{n+1-i}^Y, \qquad (\xi_i^X)^* = (-1)^{i(n-i)}\xi_{n-i}^X.\end{gathered}$$

Thus, if $K = \mathbb{Q}$, then an $(n+1)$-dimensional algebraic Poincaré pair $(f : X \to Y, \varphi^f)$ over A is an A-module chain map $f : X \to Y$ from an n-dimensional complex into an $(n+1)$-dimensional complex considered together with a homology class

$$\varphi^f = \big[(\xi_i^Y, \xi_i^X)\big] \in Q^{n+1}(f),$$

for which the A-module homomorphisms

$$\begin{gathered}(\xi_{n+1-i}^Y, \xi_{n+1-i}^X) \colon C^i(f) = Y^i \oplus X^{i-1} \to Y_{n+1-i},\\ (\xi_{n+1-i}^Y, \xi_{n+1-i}^X)(g, h) = \xi_{n+1-i}^Y(g) + f(\xi_{n+1-i}^X(h)),\end{gathered}$$

induce an isomorphism of A-modules

$$H(\xi_{n+1-i}^Y, \xi_{n+1-i}^X) \colon H^i(f) \to H_{n+1-i}(Y)$$

for any $0 \leqslant i \leqslant n+1$. Interest was aroused in symmetric L-groups of involutive Q-algebras mainly by the following circumstance. Let $L_n(\pi)$, $n \geqslant 0$, be the Wall group (see [1]) of a finitely determined group π, and let $\mathbb{Q}[\pi]$ be the rational group algebra for π endowed with the involution $g^* = g^{-1}$, $g \in \pi$. Then the groups $L^n(\mathbb{Q}[\pi]) \otimes \mathbb{Q}$ and $L_n(\pi) \otimes \mathbb{Q}$ are isomorphic for any $n \geqslant 0$ (see [8]).

We now describe the relation between the bordism groups of smooth manifolds and symmetric L-groups. Let M be a closed oriented smooth manifold of dimension n with fundamental group π, let $\widetilde{M}$ be its universal cover and let $[M] \in H_n(M;\mathbb{Z})$ be the fundamental class. By applying the functor $H_n(-\otimes_{\mathbb{Z}[\pi]}\mathbb{Z})$ to the diagonal chain approximation

$$\Delta\colon C_\bullet(\widetilde{M};\mathbb{Z}) \to \hom_{\mathbb{Z}[\mathbb{Z}_2]}(W; C_\bullet(\widetilde{M};\mathbb{Z})^{\otimes 2}),$$

where $C_\bullet(\widetilde{M};\mathbb{Z})$ is the singular chain complex for the space $\widetilde{M}$ with coefficients in $\mathbb{Z}$, we get the n-dimensional algebraic Poincaré complex

$$\bigl(C_\bullet(\widetilde{M};\mathbb{Z}); (\Delta)_*([M])\bigr)$$

over $\mathbb{Z}[\pi]$, which we denote by $\sigma(M)$. Similarly, an $(n+1)$-dimensional smooth oriented manifold M with boundary ∂M determines an $(n+1)$-dimensional algebraic Poincaré pair over $\mathbb{Z}[\pi]$, which we denote by $\sigma(M;\partial M)$. The boundary of $\sigma(M;\partial M)$ is $\sigma(\partial M)$. This implies that we have a well-defined homomorphism

$$\sigma\colon \Omega_n^{\mathrm{SO}}(B\pi) \to L^n(\mathbb{Z}[\pi]), \qquad n \geqslant 0,$$

where $\Omega_n^{\mathrm{SO}}(B\pi)$ is the singular bordism group of closed oriented n-dimensional smooth manifolds for the classifying space $B\pi$ of the group π. It is clear that the ring embedding $\mathbb{Z} \to \mathbb{Q}$ induces the homomorphism

$$\sigma\colon \Omega_n^{\mathrm{SO}}(B\pi) \to L^n(\mathbb{Q}[\pi]), \qquad n \geqslant 0.$$

In the case of non-oriented smooth manifolds we obtain, analogously to the above, a well-defined homomorphism

$$\sigma\colon \Omega_n^{\mathrm{O}}(B\pi) \to L^n\bigl((\mathbb{Z}/2)[\pi]\bigr), \qquad n \geqslant 0,$$

where $\Omega_n^{\mathrm{O}}(B\pi)$ is the singular bordism group of the closed non-oriented n-dimensional smooth manifolds of the space $B\pi$.

§ 2. Necessary information about operads and coalgebras over operads

In this section we consider the basic concepts and constructions of the operad theory of chain complexes developed by Smirnov in [5], [6], [10], [11].

Let K be a commutative ring with an identity and let A be an associative unital K-algebra. All the chain complexes that are considered in this section and the next are assumed to be non-negatively-graded.

A *symmetric family* $\mathcal{E} = \{\mathcal{E}(j)\}_{j\geqslant 1}$ is a family of chain complexes $\mathcal{E}(j)$ over K on which the corresponding symmetric groups Σ_j act on the right. A *morphism* of symmetric families $f : \mathcal{E}' \to \mathcal{E}''$ is a family of chain maps

$$f(j)\colon \mathcal{E}'(j) \to \mathcal{E}''(j), \qquad j \geqslant 1,$$

that commute with the action of the symmetric groups Σ_j.

Two morphisms of symmetric families $f, g : \mathcal{E}' \to \mathcal{E}''$ are said to be *chain homotopic*, and we denote this by $f \simeq g$, if there is a family $h = \{h(j)\}$, $j \geqslant 1$, of Σ_j-equivariant homotopies $h(j)$ between the maps $f(j)$ and $g(j)$. The symmetric families $\mathcal{E}'$ and $\mathcal{E}''$ are said to be *chain equivalent*, and we denote this by $\mathcal{E}' \simeq \mathcal{E}''$, if there are morphisms of the symmetric families $f : \mathcal{E}' \to \mathcal{E}''$, $g : \mathcal{E}'' \to \mathcal{E}'$ such that $g \circ f \simeq 1$ and $f \circ f \simeq 1$. The symmetric family $\mathcal{E} = \{\mathcal{E}(j)\}_{j\geqslant 1}$ is said to be *Σ-free* (respectively *acyclic*) if every chain complex $\mathcal{E}(j)$ is Σ_j-free (respectively acyclic).

Given two symmetric families $\mathcal{E}'$ and $\mathcal{E}''$, we consider the symmetric family

$$\mathcal{E}' \times \mathcal{E}'' = \{(\mathcal{E}' \times \mathcal{E}'')(j)\}, \qquad j \geqslant 1,$$

where $(\mathcal{E}' \times \mathcal{E}'')(j)$ is the quotient complex of the free Σ_j-complex generated by the chain complex

$$\prod_{k\geqslant 1} \sum_{j_1+\cdots+j_k=j} \mathcal{E}'(k) \otimes \mathcal{E}''(j_1) \otimes \cdots \otimes \mathcal{E}''(j_k),$$

with respect to the equivalence $\sim$ defined by the relations

$$\begin{gathered} x'\delta \otimes x_1'' \otimes \cdots \otimes x_k'' \sim x' \otimes x''_{\delta^{-1}(1)} \otimes \cdots \otimes x''_{\delta^{-1}(k)}\delta(j_1, \dots, j_k), \\ x' \otimes x_1''\delta_1 \otimes \cdots \otimes x_k''\delta_k \sim x' \otimes x_1'' \otimes \cdots \otimes x_k''(\delta_1 \times \cdots \times \delta_k), \end{gathered}$$

where $\delta(j_1, \dots, j_k)$ is the permutation of j elements obtained by partitioning this set of elements into k blocks of $j_1, \dots, j_k$ elements and by letting δ act on these blocks; and where $\delta_1 \times \cdots \times \delta_k$ is the image of the element $(\delta_1, \dots, \delta_k)$ under the embedding $\Sigma_{j_1} \times \cdots \times \Sigma_{j_k} \to \Sigma_j$. It is clear that the $\times$-product of symmetric families under consideration is associative, that is, for symmetric families $\mathcal{E}, \mathcal{E}', \mathcal{E}''$ we have an isomorphism

$$\mathcal{E} \times (\mathcal{E}' \times \mathcal{E}'') \approx (\mathcal{E} \times \mathcal{E}') \times \mathcal{E}''.$$

A symmetric family $\mathcal{E}$ is an *operad* if there is given a morphism of symmetric families $\gamma : \mathcal{E} \times \mathcal{E} \to \mathcal{E}$, called the operad multiplication, such that $\gamma(\gamma \times 1) = \gamma(1 \times \gamma)$. In addition there is an element $1 \in \mathcal{E}(1)_0$ such that

$$\begin{aligned} \gamma(1 \otimes e_j) &= e_j, \qquad e_j \in \mathcal{E}(j), \quad j \geqslant 1, \\ \gamma(e_k \otimes 1 \otimes \cdots \otimes 1) &= e_k, \qquad e_k \in \mathcal{E}(k), \quad k \geqslant 1. \end{aligned}$$

An *operad morphism* $f : \mathcal{E}' \to \mathcal{E}''$ is a morphism of symmetric families that commutes with the operad multiplications γ' and γ'' on $\mathcal{E}'$ and $\mathcal{E}''$, respectively.

Example 1. Let X be a chain complex of right A-modules and let $X^{\otimes j}$ be the tensor power over K with the diagonal structure of a right A-module. We define the operad $\mathcal{E}^X = \{\mathcal{E}^X(j)\}_{j\geqslant 1}$ by putting

$$\mathcal{E}^X(j) = \hom_A(X; X^{\otimes j}),$$

where $\hom_A(X; X^{\otimes j})$ is the K-module chain complex of A-homorphisms and where

$$\mathcal{E}^X(j) = \hom_A(X; X^{\otimes j})_0$$

is the K-module of chain maps over A. The action of the symmetric group Σ_j on $\mathcal{E}^X(j)$ is defined by the action of Σ_j on $X^{\otimes j}$, where Σ_j acts by permuting the factors with the usual agreement about signs. The structure of an operad on $\mathcal{E}^X$, that is, the operad multiplication $\gamma^X : \mathcal{E}^X \times \mathcal{E}^X \to \mathcal{E}^X$ is given by the formula

$$\gamma^X(g \otimes g_1 \otimes \cdots \otimes g_k) = (g_1 \otimes \cdots \otimes g_k) \circ g,$$

where $g_i \in \mathcal{E}^X(j_i)$, $1 \leqslant i \leqslant k$, $g \in \mathcal{E}^X(k)$.

Example 2. We consider the symmetric family $C = \{C(j)\}_{j\geqslant 1}$, where $C(j)$ is the free K-module with one zero-dimensional generator $c(j)$ and trivial action of the group Σ_j. The structure of the operad $\gamma : C \times X \to C$ is defined by

$$\gamma\big(c(k) \otimes c(j_1) \otimes \cdots \otimes c(j_k)\big) = c(j), \qquad j = \sum_{i=1}^{k} j_i.$$

Example 3. For an arbitrary symmetric family $\mathcal{E} = \{\mathcal{E}(j)\}_{j\geqslant 1}$ we let $\Sigma\mathcal{E}$ denote the symmetric family for which $(\Sigma\mathcal{E})(j)$ is the free Σ_j-module generated by $\mathcal{E}(j)$. If $\mathcal{E}$ is an operad, then the operad multiplication in $\mathcal{E}$ defines the structure of the operad in $\Sigma\mathcal{E}$. By applying the functor Σ to the operad C we obtain from Example 2 the Σ-free operad ΣC.

Example 4. Let $\mathcal{E} = \{\mathcal{E}(j)\}_{j\geqslant 1}$ be a symmetric family such that $\mathcal{E}(1) = K$. We consider the operad $T\mathcal{E}$ defined as the quotient family of the symmetric family

$$\sum_{n\geqslant 1} \mathcal{E}^n, \qquad \mathcal{E}^n = \underbrace{\mathcal{E} \times \cdots \times \mathcal{E}}_{n\,\text{times}},$$

with respect to the equivalence generated by the relations

$$1 \otimes e_k \sim e_k, \quad e_k \otimes 1 \otimes \cdots \otimes 1 \sim e_k, \qquad e_k \in \mathcal{E}(k), \quad k \geqslant 1, \quad 1 \in \mathcal{E}(1).$$

The operad multiplication in $T\mathcal{E}$ is defined by the natural maps $\mathcal{E}^n \times \mathcal{E}^m \to \mathcal{E}^{n+m}$. The operad $T\mathcal{E}$ is the free operad generated by the family $\mathcal{E}$. If $\mathcal{E}$ is an operad, then the operad multiplication $\gamma : \mathcal{E} \times \mathcal{E} \to \mathcal{E}$ induces an operad morphism $T\gamma : T\mathcal{E} \to \mathcal{E}$.

Example 5. In this example we describe an inductive procedure for constructing an operad CA_∞ which is a free acyclic operad augmented over the operad C in Example 2. For each $n \geqslant 0$ we shall find a symmetric family $M^{(n)}$ and a free operad $CA^{(n)}$ generated by $M^{(n)}$ that is a free resolution of the operad C up to dimension $(n-1)$. We put $M^{(0)} = C$ and $CA^{(0)} = TA$. We assume that the symmetric family $M^{(n)}$ is defined such that for the operad $CA^{(n)} = TM^{(n)}$ there is an exact sequence

$$(CA^{(n)})_n \xrightarrow{d_n} (CA^{(n)})_{n-1} \longrightarrow \cdots \longrightarrow (CA^{(n)})_1 \xrightarrow{d_1} (CA^{(n)})_0 \xrightarrow{d_0=\epsilon} C \longrightarrow 0.$$

We define $M^{(n+1)}$ and $CA^{(n+1)}$ by putting

$$M_i^{(n+1)} = M_i^{(n)}, \quad i \neq n+1,$$
$$M_{n+1}^{(n+1)} = \ker\bigl(d_n\colon (CA^{(n)})_n \to (CA^{(n)})_{n-1}\bigr), \qquad CA^{(n+1)} = TM^{(n+1)}.$$

The differential in $CA^{(n+1)}$ is determined by the differential in $CA^{(n)}$ and is given on elements $x_{n+1} \in M_{n+1}^{(n+1)}$ by the formula $d_{n+1}(x_{n+1}) = i(x_{n+1})$, where $i : M_{n+1}^{(n+1)} \to (CA^{(n)})_n$ is the embedding. In this way $CA^{(n+1)}$, as a graded operad, is a free augmented operad over the operad, C and it is acyclic up to dimension n. If we iterate this process and put

$$M = \varinjlim M^{(n)}, \qquad CA_\infty = \varinjlim CA^{(n)},$$

then we obtain a symmetric family M and an operad CA_∞ which, as a graded operad, coincides with the graded operad TM and in which the differential determines the long exact sequence

$$\cdots \longrightarrow (CA_\infty)_n \xrightarrow{d_n} (CA_\infty)_{n-1} \longrightarrow \cdots \longrightarrow (CA_\infty)_1 \xrightarrow{d_1} (CA_\infty)_0 \xrightarrow{\epsilon} C \longrightarrow 0.$$

Thus we see that CA_∞ is a free acyclic operad and there is an operad morphism $\epsilon : CA_\infty \to C$ which is a chain equivalence.

If we take ΣC instead of C in the process of constructing the operad CA_∞ considered above, then as a result we get an operad which is denoted by A_∞. By construction, the operad A_∞ is a free Σ-free operad augmented over the operad ΣC, that is, there is a chain equivalence of operads $\epsilon : A_\infty \to \Sigma C$.

Example 6. If in the procedure of constructing the operad CA_∞ we replace everywhere the functor T by the functor $T \circ \Sigma$, then we get a sequence of symmetric families $N^{(n)}$ and a sequence of Σ-free operads $E^{(n)}$ generated by $N^{(n)}$ which are free Σ-free resolutions of the Σ-trivial operad C up to dimension $(n-1)$. If we put

$$N = \varinjlim N^{(n)}, \qquad E_\infty = \varinjlim E^{(n)},$$

then we obtain a symmetric family N and an operad E_∞ which, as a graded operad, coincides with the graded operad $T\Sigma N$ and in which the differential determines the long exact sequence

$$\cdots \longrightarrow (E_\infty)_n \xrightarrow{d_n} (E_\infty)_{n-1} \longrightarrow \cdots \longrightarrow (E_\infty)_1 \xrightarrow{d_1} (E_\infty)_0 \xrightarrow{\epsilon} C \longrightarrow 0.$$

Thus, E_∞ is a free Σ-free operad and there is an operad morphism $\epsilon : E_\infty \to C$ which is a chain equivalence. In particular, we note that the chain complex $E_\infty(j)$, $j \geqslant 1$, is a free resolution of the trivial $K[\Sigma_j]$-module K, where $K[\Sigma_j]$ is the group K-algebra of the symmetric group Σ_j. For example, $E_\infty(2)$ is a free acyclic chain complex with generators U_i of dimension i and differential

$$dU_i = U_{i-1} + (-1)^i U_{i-1} T, \qquad T \in \Sigma_2 = \mathbb{Z}_2.$$

Example 7. We describe the Smirnov operad $E = \{E(j)\}_{j\geqslant 1}$. Let $\overline{\Delta}[n]$ be the chain complex over K of the standard n-dimensional simplex, that is, the free simplicial set generated by a single n-dimensional generator (see [12]). The graded family $\overline{\Delta}[*] = \{\overline{\Delta}[n]\}_{n\geqslant 0}$ is a cosimplicial object in the category of chain complexes over K with respect to the coboundary and codegeneration operators. We let $\overline{\Delta}[*]^{\otimes j}$ denote the cosimplicial chain complex that is the jth tensor power over K of the cosimplicial object $\overline{\Delta}[*]$. We consider the operad $E = \{E(j)\}_{j\geqslant 1}$, where $E(j)$ is the chain realization of the cosimplicial object $\overline{\Delta}[*]^{\otimes j}$, that is, $E(j) = \hom(\overline{\Delta}[*]; \overline{\Delta}[*]^{\otimes j})$ is the chain complex of cosimplicial K-homomorphisms $\overline{\Delta}[*] \to \overline{\Delta}[*]^{\otimes j}$ and where $E(j)_0 = \hom(\overline{\Delta}[*]; \overline{\Delta}[*]^{\otimes j})_0$ is the K-module of cosimplicial chain maps. The structure of an operad on E is defined exactly as in Example 1.

Example 8. In this example we give a description of the inductive construction of an operad morphism

$$\varphi_\infty : E_\infty \to E,$$

where E_∞ and E are the operads in Examples 6 and 7, respectively. Let

$$\nabla : \overline{\Delta}[m] \to \overline{\Delta}[m] \otimes_K \overline{\Delta}[m]$$

be the chain map given on the K-module generators $x \in \overline{\Delta}[m]_n$ by

$$\nabla(x) = \sum_{i=0}^{n} \partial_{i+1} \dots \partial_n(x) \otimes \partial_0 \dots \partial_0(x).$$

Then this map induces the chain map of cosimplicial objects

$$\nabla : \overline{\Delta}[*] \to \overline{\Delta}[*] \otimes_K \overline{\Delta}[*].$$

We consider the chain maps of the cosimplicial objects

$$\nabla(j) : \overline{\Delta}[*] \to \overline{\Delta}[*]^{\otimes j}, \qquad j \geqslant 1,$$

obtained by iterating the maps ∇, that is,

$$\nabla(1) = 1, \qquad \nabla(2) = \nabla, \qquad \nabla(j) = \big(\nabla(j-1) \otimes 1\big)\nabla, \quad j \geqslant 2.$$

We define the operad morphism

$$\varphi^{(0)}_{\bullet} : E^0 = T\Sigma C \to E,$$

by putting $\varphi^{(0)}(j)(c(j)) = \nabla(j)$ on the generators. We assume that the operad morphism $\varphi^{(n)} : E^{(n)} \to E$ has been constructed for the operad $E^{(n)}$. We now define the operad morphism $\varphi^{(n+1)} : E^{(n+1)} \to E$. Let $x \in N^{(n+1)}_{n+1}$; then $d(x) \in E^{(n)}$ and $d\varphi^{(n)}(dx) = 0$. Since the operad E is acyclic, there is a map $f : N^{(n+1)}_{n+1} \to (E)_{n+1}$ such that $d \circ f = \varphi^{(n)} \circ d$. The map $f : N^{(n+1)} \to E$ induces the operad map

$$\varphi^{(n+1)} \colon E^{(n+1)} = T\Sigma N^{(n+1)} \to E.$$

In this way the operad morphism $\varphi^{(n)} : E^{(n)} \to E$ can be lifted to an operad morphism $\varphi^{(n+1)} : E^{(n+1)} \to E$. By setting $\lim \varphi^{(n)} = \varphi_\infty$ we obtain the operad morphism $\varphi_\infty : E_\infty \to E$. The acyclicity of the operad E implies that any two morphisms from E_∞ into E that are constructed by the method indicated are chain homotopic.

Example 9. Here we consider the situation when K is a field of characteristic zero and we describe in this case the inductive construction of the operad morphism

$$\psi_\infty \colon CA_\infty \to E,$$

where CA_∞ and E are the operads in Examples 5 and 7, respectively. Let $\nabla(j)$ in $E(j)_0$ be as in Example 8. We define the operad morphism

$$\psi^0 \colon CA^0 = TC \to E,$$

by putting

$$\psi^0(j)(c(j)) = \frac{1}{j!} \sum_\sigma \sigma \circ \nabla(j)$$

on the generators, where the summation is carried out over all $\sigma \in \Sigma_j$. The rest of the construction of $\psi_\infty : CA_\infty \to E$ is completely analogous to the construction of the morphism $\varphi_\infty : E_\infty \to E$ in Example 8. In this way we obtain the operad morphism $\psi_\infty : CA_\infty \to E$ and any two morphisms from CA_∞ into E that are constructed by the method indicated are chain homotopic.

We now consider the definition of a coalgebra over the operad $\mathcal{E}$ or, more briefly, an $\mathcal{E}$-coalgebra. Let $(\mathcal{E}, \gamma)$ be an operad and let X be a chain complex of right A-modules. Then X is called an *$\mathcal{E}$-coalgebra* over A if we are given an operad morphism $\alpha : \mathcal{E} \to \mathcal{E}^X$ where $\mathcal{E}^X$ is the operad in Example 1. It is clear that if X is an $\mathcal{E}$-coalgebra over A, then the morphism α determines A-module Σ_j-equivariant chain maps

$$\alpha^X_j : X \otimes_K \mathcal{E}(j) \to X^{\otimes j}, \qquad j \geqslant 1,$$

such that $\alpha^X_1(x \otimes 1) = x$, $1 \in \mathcal{E}(1)$, and the diagrams

$$\begin{array}{ccc}
X \otimes \mathcal{E}(k) \otimes \mathcal{E}(j_1) \otimes \cdots \otimes \mathcal{E}(j_k) & \xrightarrow{1\otimes\gamma} & X \otimes \mathcal{E}(j) \\
{\scriptstyle \alpha^X_k \otimes 1}\big\downarrow & & \big\downarrow{\scriptstyle \alpha^X_j} \\
X^{\otimes k} \otimes \mathcal{E}(j_1) \otimes \cdots \otimes (X \otimes \mathcal{E}(j_k)) & \xrightarrow{g} & X^{\otimes j}
\end{array}$$

is commutative,

$$g = \left(\bigotimes_{s=1}^{k} \alpha_{j_s}^X\right) \circ U,$$

$$U\colon (X^{\otimes k} \otimes \mathcal{E}(j_1)) \otimes \cdots \otimes \mathcal{E}(j_k) \to (X \otimes \mathcal{E}(j_1)) \otimes \cdots \otimes (X \otimes \mathcal{E}(j_k))$$

being the obvious rearrangement map. By a *morphism* of $\mathcal{E}$-coalgebras $f : X \to Y$ over A we mean a chain map of A-modules $f : X \to Y$ such that the diagrams

$$\begin{array}{ccc} X \otimes \mathcal{E}(j) & \xrightarrow{\alpha_j^X} & X^{\otimes j} \\ {\scriptstyle f\otimes 1}\downarrow & & \downarrow{\scriptstyle f^{\otimes j}} \\ Y \otimes \mathcal{E}(j) & \xrightarrow{\alpha_j^Y} & Y^{\otimes j} \end{array}$$

are commutative for any $j \geqslant 1$.

A morphism of $\mathcal{E}$-coalgebras $f : X \to Y$ over A is called a *quasi-isomorphism* if $f_* : H_*(X) \to H_*(Y)$ is an isomorphism of A-modules. A *homotopy category* of $\mathcal{E}$-coalgebras over A is a localization of categories (see [13]) of $\mathcal{E}$-coalgebras over A with respect to the family of quasi-isomorphisms. A morphism of $\mathcal{E}$-coalgebras over A is called a *homotopy equivalence* if it is invertible in the homotopy category of $\mathcal{E}$-coalgebras over A.

Example 10. Giving the structure of a C-coalgebra over A on a chain A-module complex X, where C is the operad in Example 2, is equivalent to giving a chain A-module map $\nabla : X \to X \otimes_K X$ such that

$$(\nabla \otimes 1)\nabla = (1 \otimes \nabla)\nabla, \qquad T \circ \nabla = \nabla,$$

where T is the map of interchanging the factors. Thus, the category of C-coalgebras over A is isomorphic to the category of commutative K-coalgebras over A.

Similarly, giving the structure of a ΣC-coalgebra over A on a chain A-module complex X, where ΣC is the operad from Example 3, is equivalent to giving a chain A-module map $\nabla : X \to X \otimes_K X$ such that

$$(\nabla \otimes 1)\nabla = (1 \otimes \nabla)\nabla.$$

Thus, the category of ΣC-coalgebras over A is isomorphic to the category of K-coalgebras over A.

Example 11. Let the chain complex X be an A_∞-coalgebra over A, where A_∞ is the operad indicated at the end of Example 5. Since the operad A_∞ is Σ-free and also augmented over ΣC, giving the structure of an A_∞-coalgebra on X is equivalent to giving a family of A-module maps

$$\nabla_X^n \colon X \to X^{\otimes(n+1)}, \qquad n \geqslant 1,$$

that raise the dimension by $(n-1)$ and are connected by the relations

$$d\nabla_X^n = \sum_{m=1}^{n-1} \sum_{i=1}^{n-m+1} (-1)^{m(n+i+1)+1}(1 \otimes \cdots \otimes \nabla_X^m \otimes \cdots \otimes 1)\nabla_X^{n-m},$$

where d is the boundary operator of the chain complex

$$\hom_A(X; X^{\otimes(n+1)})_\bullet, \qquad n \geqslant 1.$$

Example 12. The CA_∞-coalgebras over A, where CA_∞ is the operad in Example 5, are called *commutative A_∞-coalgebras.* Giving the structure of a commutative A_∞-coalgebra over A on the chain complex X is equivalent to giving a family of A-module maps

$$\nabla_X^n \colon X \to X^{\otimes(n+1)}, \qquad n \geqslant 1,$$

that raise the dimension by $(n-1)$ and are connected by the relation

$$d\nabla_X^n = \sum_{m=1}^{n-1} \sum_{i=1}^{n-m+1} (-1)^{m(n+i+1)+1}(1 \otimes \cdots \otimes \nabla_X^m \otimes \cdots \otimes 1)\nabla_X^{n-m}, \qquad n \geqslant 1,$$

and, in addition, the conditions

$$\sum_{p+q=n+1} (-1)^\mu \sigma(p,q) \circ \nabla_X^n = 0, \qquad n \geqslant 1$$

are satisfied, where the sum is taken over all (p,q)-rearrangements (see [14]) and μ is the sign of the rearrangement.

Example 13. Let X be an E_∞-coalgebra over the K-algebra A, where E_∞ is the operad in Example 6, and let $K = \mathbb{Z}/p$, where p is an arbitrary prime number. Then the graded A-module $H^*(X;\mathbb{Z}/p)$ is (see [11], [15]) an algebra over the algebra A_p of Steenrod operations, and moreover the A-module structure commutes with the action of the operations in A_p.

Example 14. Let X be a topological space having a universal cover $\widetilde{X}$ and let $\pi = \pi_1(X)$ be the fundamental group of X. We consider the singular chain complex $C_\bullet(\widetilde{X};K)$ which is the chain complex of right $K[\pi]$-modules where $K[\pi]$ is the group K-algebra of the group π. The structure of an E-coalgebra over $K[\pi]$, where E is the operad in Example 7, is defined in a natural way on the chain complex $C_\bullet(\widetilde{X};K)$. We describe the structure. The singular chain complex $C_\bullet(\widetilde{X};K)$ is the chain complex of the free simplicial $K[\pi]$-module generated by the simplicial set of singular simplexes of the space $\widetilde{X}$ on which the group π acts freely and simplicially. Let $x \in C_n(\widetilde{X};K)$ be a $K[\pi]$-generator, let $\tau \in E(j) = \hom(\overline{\Delta}[*];\overline{\Delta}[*]^{\otimes j})$ and let

$$\overline{x} \colon \overline{\Delta}[n] \to C_n(\widetilde{X};K)$$

be a chain K-module map taking the n-dimensional generator $u_n \in \overline{\Delta}[n]$ into x. We define the structure of an E-coalgebra on $C_\bullet(\widetilde{X};K)$ over $K[\pi]$

$$\alpha_j \colon C_\bullet(\widetilde{X};K) \otimes_K E(j) \to C_\bullet(\widetilde{X};K)^{\otimes j}, \qquad j \geqslant 1,$$

by setting on the $K[\pi]$-generators $x \in C_n(\widetilde{X};K)$

$$\alpha_j(x \otimes \tau) = (\overline{x} \otimes \cdots \otimes \overline{x})\tau(u_n).$$

The $K[\pi]$-module Σ_j-equivariant chain maps α_j, $j \geqslant 1$, determine an operad morphism

$$\alpha\colon E \to \mathcal{E}^{C_\bullet(\widetilde{X};K)},$$

that is, $C_\bullet(\widetilde{X};K)$ is an E-coalgebra over $K[\pi]$. Let $f : X \to Y$ be a continuous map and let $\widetilde{f} : \widetilde{X} \to \widetilde{Y}$ be the corresponding map of covering spaces. Then $C_\bullet(\widetilde{X};K)$ and $C_\bullet(\widetilde{Y};K)$ are chain complexes of right $K[\pi]$-modules, where the $K[\pi]$-module structure on $C_\bullet(\widetilde{Y};K)$ is determined by the K-algebra homomorphism $K[\pi_1(f)]$. The map f induces a morphism of E-coalgebras

$$C_\bullet(\widetilde{f};K)\colon C_\bullet(\widetilde{X};K) \to C_\bullet(\widetilde{Y};K)$$

over $K[\pi]$. If f is a homotopy equivalence of topological spaces, then $C_\bullet(\widetilde{f};K)$ is a homotopy equivalence of E-coalgebras over $K[\pi]$.

The chain complex $C_\bullet(\widetilde{X};K)$ has the natural structure of an E_∞-coalgebra over $K[\pi]$ which is given as a composition of operad morphisms

$$E_\infty \xrightarrow{\varphi_\infty} E \xrightarrow{\alpha} \mathcal{E}^{C_\bullet(\widetilde{X};K)},$$

where φ_∞ is the operad morphism considered in Example 8.

When K is a field of characteristic zero, the chain complex $C_\bullet(\widetilde{X};K)$ has the natural structure of a commutative A_∞-coalgebra over $K[\pi]$ which is determined by the composition of operad morphisms

$$CA_\infty \xrightarrow{\psi_\infty} E \xrightarrow{\alpha} \mathcal{E}^{C_\bullet(\widetilde{X};K)},$$

where ψ_∞ is the operad morphism in Example 9.

We now consider the definitions of a comonad and coalgebra over a comonad that are dual to the corresponding concepts of a monad and an algebra over a monad. A *comonad* in the category $\mathcal{X}$ is a covariant functor $G : \mathcal{X} \to \mathcal{X}$ considered together with natural transformations of functors

$$\nabla\colon G \to G^2 = G \circ G, \qquad p\colon G \to 1,$$

for which we have the commutative diagrams

$$\begin{tikzcd} G & G^2 \arrow[l, "Gp"'] \arrow[r, "pG"] & G \\ & G \arrow[ul, equal] \arrow[u, "\nabla"] \arrow[ur, equal] & \end{tikzcd} \qquad \begin{tikzcd} G^3 & G^2 \arrow[l, "\nabla G"'] \\ G^2 \arrow[u, "G\nabla"] & G. \arrow[l, "\nabla"] \arrow[u, "\nabla"'] \end{tikzcd}$$

A coalgebra over the comonad G in the category $\mathcal{X}$ is an object X of the category $\mathcal{X}$ considered together with a morphism $\eta : X \to GX$ of the category $\mathcal{X}$ for which we have the commutative diagrams

$$\begin{tikzcd} X & GX \arrow[l, "p(X)"'] \\ & X \arrow[ul, equal] \arrow[u, "\eta"'] \end{tikzcd} \qquad \begin{tikzcd} G^2X & GX \arrow[l, "\nabla(X)"'] \\ GX \arrow[u, "G\eta"] & X. \arrow[l, "\eta"] \arrow[u, "\eta"'] \end{tikzcd}$$

Example 15. Let $(\mathcal{E}, \gamma)$ be an operad and let X be a chain complex of right A-modules. We consider the A-module chain complex

$$\overline{\mathcal{E}}X = \prod_{j\geqslant 1} \hom_{K[\Sigma_j]}\big(\mathcal{E}(j); X^{\otimes j}\big),$$

where

$$\hom_{K[\Sigma_j]}\big(\mathcal{E}(j); X^{\otimes j}\big)_0$$

is an A-module of chain maps over $K[\Sigma_j]$ for each $j \geqslant 1$. $\overline{\mathcal{E}}$ is a covariant functor on the category of A-module chain complexes. The operad multiplication $\gamma : \mathcal{E} \times \mathcal{E} \to \mathcal{E}$ determines the A-module chain map

$$\nabla\colon \overline{\mathcal{E}}X \to \overline{\mathcal{E}}^2 X.$$

The element $1 \in \mathcal{E}(1)$ gives the evaluation map: $p(f) = f(1) \in X$

$$p\colon \overline{\mathcal{E}}X \to X,$$

which is a chain map of A-modules. The triple $(\overline{\mathcal{E}}, \nabla, p)$ is a comonad in the category of chain complexes of right A-modules. In addition, $\overline{\mathcal{E}}X$ is a free $\mathcal{E}$-coalgebra over A, that is, for any chain map $f : Z \to X$ of right A-modules, where Z is an arbitrary $\mathcal{E}$-coalgebra over A, there is a natural morphism of $\mathcal{E}$-coalgebras $g : Z \to \overline{\mathcal{E}}X$ over A such that $p \circ g = f$.

Example 16. If the chain complex X is an $\mathcal{E}$-coalgebra over A, then the structure morphism

$$\alpha\colon \mathcal{E} \to \mathcal{E}^X, \qquad \alpha = \{\alpha(j)\}, \quad j \geqslant 1,$$
$$\alpha(j)\colon \mathcal{E}(j) \to \hom_A(X; X^{\otimes j})$$

determines a chain map

$$\eta\colon X \to \prod_{j\geqslant 1} \hom_{K[\Sigma_j]}\big(\mathcal{E}(j); X^{\otimes j}\big) = \overline{\mathcal{E}}X$$

of right A-modules. The pair $(X; \eta)$ is a coalgebra over the comonad $\overline{\mathcal{E}}$.

§ 3. Poincaré E_∞-coalgebras

Let A be an associative unital K-algebra with an involution, where K is an arbitrary commutative ring with an identity. In addition, we shall assume that A is an augmented K-algebra, that is, there is a fixed K-algebra homomorphism $\epsilon : A \to K$. It is clear that if a K-algebra A is augmented, then the ring K with the homomorphism ϵ can be regarded as a trivial A-module.

Let X be an arbitrary E_∞-coalgebra over A. Then we define the intersection operation for X

$$\cap\colon H^i(X) \otimes_K H_n(X \otimes_A K) \to H_{n-i}(X), \qquad 0 \leqslant i \leqslant n,$$

induced by the chain map

$$\alpha^X(U_0) \otimes_A K\colon X \otimes_A K \to (X \otimes_K X) \otimes_A K = X \otimes_A X,$$

where $\alpha^X : E_\infty \to E^X$ is the structure morphism of the E_∞-coalgebra X over A and $U_0 \in E_\infty(2)_0$. The E_∞-coalgebra X over A is said to be *n-dimensional* if X is an n-dimensional chain complex of right A-modules.

Definition 1. An n-dimensional E_∞-coalgebra X over A is called an *n-dimensional Poincaré E_∞-coalgebra* over A if there exists $[X] \in H_n(X \otimes_A K)$ such that

$$\cap [X] \colon H^i(X) \to H_{n-i}(X)$$

is an isomorphism of A-modules for all $0 \leqslant i \leqslant n$. The homology class $[X]$ is called the *fundamental class of the n-dimensional Poincaré E_∞-coalgebra* X over A.

By a *morphism of n-dimensional Poincaré E_∞-coalgebras*

$$f \colon (X, [X]) \to (Y, [Y])$$

over A we mean a morphism $f : X \to Y$ of E_∞-coalgebras over A such that

$$(f \otimes_A K)_*[X] = [Y].$$

A morphism f of n-dimensional Poincaré E_∞-coalgebras is called a *homotopy equivalence* if f is a homotopy equivalence of E_∞-coalgebras over A.

Given n-dimensional Poincaré E_∞-coalgebras $(X, [X])$ and $(Y, [Y])$ over A we define their *direct sum* by putting

$$(X, [X]) \oplus (Y, [Y]) = (X \oplus Y, [X] \oplus [Y]),$$

where the structure of the E_∞-coalgebra on $X \oplus Y$ over A is given by

$$\alpha_j^{X\oplus Y}\big((x, y) \otimes e\big) = \big(\alpha_j^X(x \otimes e), \alpha_j^Y(y \otimes e)\big) \in X^{\otimes j} \oplus Y^{\otimes j} \subset (X \oplus Y)^{\otimes j}, \qquad j \geqslant 1.$$

By the *change of orientation* of a Poincaré E_∞-coalgebra $(X; [X])$ over A we mean the operation

$$(-(X, [X])) = (X, -[X])$$

which leads to a Poincaré E_∞-coalgebra over A.

The n-dimensional Poincaré E_∞-coalgebra $(X, [X])$ over A uniquely determines an n-dimensional algebraic Poincaré complex $(X, [\varphi_s^{[X]}])$ over A. In fact we put

$$\varphi_s^{[X]} = (\alpha_2^X \otimes_A K)(\zeta_n \otimes U_s), \qquad s \geqslant 0,$$

where ζ_n is an arbitrary representative of the fundamental class $[X]$, and $(\alpha_2^X \otimes_A K)$ is the chain map

$$(X \otimes_A K) \otimes_K E_\infty(2) = (X \otimes_K E_\infty(2)) \otimes_A K \to X^{\otimes 2} \otimes_A K = X \otimes_A X,$$

determined by the chain A-module map

$$\alpha_2^X : X \otimes_K E_\infty(2) \to X \otimes_K X,$$

that enters into the structure of the E_∞-coalgebra on X. Since

$$\begin{aligned} d(\varphi_s^{[X]}) &= d(\alpha_2^X \otimes_A K)(\zeta_n \otimes U_s) \\ &= (\alpha_2^X \otimes_A K)((-1)^n \zeta_n \otimes dU_s) \\ &= (-1)^n (\alpha_2^X \otimes_A K)\big(\zeta_n \otimes (U_{s-1} + (-1)^s U_{s-1} T)\big) \\ &= (-1)^n \big(\varphi_{s-1}^{[X]} + (-1)^s \varphi_{s-1}^{[X]} T\big), \qquad s \geqslant 0, \end{aligned}$$

it follows that the collection of chains

$$\{\varphi_s^{[X]} \in (X \otimes X)_{n+s},\ s \geqslant 0\}$$

defines an element $[\varphi_s^{[X]}] \in Q^n(X)$. The homomorphism of A-modules

$$H(\varphi_0^X)\colon H^i(X) \to H_{n-i}(X), \qquad 0 \leqslant i \leqslant n,$$

coincides, by definition, with the homomorphism

$$\cap\,[X]\colon H^i(X) \to H_{n-i}(X), \qquad 0 \leqslant i \leqslant n,$$

and is therefore an A-module isomorphism. Suppose now that ζ_n and ζ_n' are homological representatives of the fundamental class $[X]$, that is, $dx = \zeta_n - \zeta_n'$ for some $x \in (X \otimes_A K)_{n+1}$. We consider the element

$$\rho^{[X]} = \{\rho_s^{[X]} = (\alpha_2^X \otimes_A K)(x \otimes U_s)\} \in \hom_{K[\mathbb{Z}_2]}(W; X \otimes_A X)_{n+1},$$

where $W = E_\infty(2)$ is the canonical resolution of the trivial $K[\mathbb{Z}_2]$-module K. If we compute the left and right-hand sides of the equality

$$d\rho^{[X]} = d \circ \rho^{[X]} + (-1)^n \rho^{[X]} \circ d$$

on the element $U_s \in E_\infty(2)_s$, then we obtain

$$(d\rho^{[X]})(U_s) = (\alpha_2^X \otimes_A K)(dx \otimes U_s) = \varphi_s^{[X]} - \varphi_s'^{[X]}, \qquad s \geqslant 0.$$

Thus, $[\varphi_s^{[X]}] = [\varphi'^{[X]}] \in Q^n(X)$.

Definition 2. The n-dimensional Poincaré E_∞-coalgebras $(X, [X])$ and $(Y, [Y])$ over A will be called *algebraically bordant* if the n-dimensional algebraic Poincaré complexes $(X, [\varphi_s^{[X]}])$ and $(Y, [\varphi_s^{[Y]}])$ that they determine are bordant.

In the same way as in [9] we obtain the following assertion.

Lemma 1. *Algebraic bordism between n-dimensional Poincaré E_∞-coalgebras over A is an equivalence relation. Homotopy equivalent Poincaré E_∞-coalgebras over A are algebraically bordant.*

We let $(LE_\infty)^n(A)$ denote the set of classes of algebraically bordant n-dimensional Poincaré E_∞-coalgebras over A. The direct sum operation and the change of orientation on Poincaré E_∞-coalgebras over A give the structure of an Abelian group on $(LE_\infty)^n(A)$. It is clear that $(LE_\infty)^n(A)$ is a subgroup of the symmetric L-group $L^n(A)$, $n \geqslant 0$.

Definition 3. Two n-dimensional Poincaré E_∞-coalgebras $(X, [X])$ and $(Y, [Y])$ over A are said to be *chainwise bordant* if there is an $(n+1)$-dimensional algebraic Poincaré pair

$$(f\colon X \oplus Y \to C, \varphi^f \in Q^{n+1}(f))$$

such that

$$(f \otimes_A K)_*([X] \oplus -[Y]) = 0 \in H_n(C \otimes_A K),$$

and the boundary of this pair is

$$(X, [\varphi_s^{[X]}]) \oplus (-(Y, [\varphi_s^{[Y]}])).$$

By analogy with Lemma 1 we have the following assertion.

Lemma 2. *Chainwise bordism of n-dimensional Poincaré E_∞-coalgebras over A is an equivalence relation. Homotopy equivalent Poincaré E_∞-coalgebras over A are chainwise bordant.*

We let $(L^c E_\infty)^n(A)$ denote the set of classes of chainwise bordant n-dimensional Poincaré E_∞-coalgebras over A. The direct sum operation and the change of orientation on Poincaré E_∞-coalgebras define the structure of an Abelian group on $(L^c E_\infty)^n(A)$. In addition we have the obvious epimorphism

$$S\colon (L^c E_\infty)^n(A) \to (LE_\infty)^n(A)$$

which is induced by forgetting the 'zeroing' of the image of the fundamental class in the definition of the chain bordism relation.

Let $f : X \to Y$ be a morphism of E_∞-coalgebras over A. We consider the cone $C(f)$ of the chain map f. We give the chain A-module complex $C(f)$ the structure

$$\alpha_j^{C(f)}\colon C(f) \otimes_K E_\infty(j) \to C(f)^{\otimes j}, \qquad j \geqslant 1,$$

of an E_∞-coalgebra over A by putting

$$\begin{aligned}\alpha_j^{C(f)}((y,x)\otimes e) &= \big(\alpha_j^Y(y\otimes e), (-1)^\nu \alpha_j^X(x\otimes e)\big)\\ &\in (Y^{\otimes j})_{n+\nu} \oplus (X^{\otimes j})_{n-1+\nu} \xrightarrow{\text{in}} C(f)^{\otimes j}_{n+\nu},\end{aligned}$$

where $e \in E_\infty(j)$, $\nu = \dim e$, $(y,x) \in C(f)_n$, and $\xrightarrow{\text{in}}$ is the chain A-module Σ_j-equivariant map of the embedding. Given a morphism $f : X \to Y$ of E_∞-coalgebras over A we consider the intersection operation

$$\cap\colon H^i(f) \otimes_K H_{n+1}\big(C(f)\otimes_A K\big) \to H_{n+1-i}(Y), \qquad 0 \leqslant i \leqslant n+1,$$

that is determined by the chain map

$$\begin{aligned}\alpha^{C(f)}(U_0)\otimes_A K = \big(\alpha^Y(U_0)\otimes_A K, \alpha^X(U_0)\otimes_A K\big)&\colon\\ C(f)_\bullet \otimes_A K = (Y\otimes_A K)_\bullet \oplus (X\otimes_A K)_{\bullet-1}&\\ \to (Y_\bullet^{\otimes 2} \oplus X_{\bullet-1}^{\otimes 2})\otimes_A K = (Y\otimes_A Y)_\bullet \oplus (X\otimes_A X)_{\bullet-1}&.\end{aligned}$$

A morphism of E_∞-coalgebras $f : X \to Y$ is called an $(n+1)$-*dimensional E_∞-pair* over A if f is an $(n+1)$-dimensional pair of chain complexes of right A-modules.

Definition 4. An $(n+1)$-dimensional E_∞-pair $f : X \to Y$ over A is called an $(n+1)$-*dimensional Poincaré E_∞-pair* over A if there is an element $[f] \in H_{n=1}(C(f)\otimes_A K)$ such that

$$\cap[f]\colon H^i(f) \to H_{n+1-i}(Y)$$

is an isomorphism of A-modules for all i, $0 \leqslant i \leqslant n+1$. The homology class $[f]$ is called the *fundamental class of the $(n+1)$-dimensional Poincaré E_∞-pair* $f : X \to T$ over A.

The following assertion is obtained in the same way as in [2].

Lemma 3. *Suppose that we are given an $(n+1)$-dimensional Poincaré E_∞-pair $(f : X \to Y, [f])$ over A. Then the n-dimensional E_∞-coalgebra X over A considered together with the element $[X] = \delta[f]$, where δ is the connecting homomorphism in the long exact homology sequence of the map $f \otimes_A K$, is an n-dimensional Poincaré E_∞-coalgebra over A.*

Definition 5. The n-dimensional Poincaré E_∞-coalgebra $(X, [X] = \delta[f])$ over A is called the *boundary* of the $(n+1)$-dimensional Poincaré E_∞-pair $(f : X \to Y, [f])$ over A.

Definition 6. Two n-dimensional Poincaré E_∞-coalgebras $(X, [X])$ and $(Y, [Y])$ over A are said to be *E_∞-bordant* if there is an $(n+1)$-dimensional Poincaré E_∞-pair over A such that its boundary is $(X, [X]) \oplus (-(Y, [Y]))$.

Theorem 1. *The relations of chain bordism and E_∞-bordism for n-dimensional Poincaré E_∞-coalgebras over A are equivalent.*

Proof. It is clear that if the Poincaré E_∞-coalgebras over A are E_∞-bordant, then they are chainwise bordant. We shall now show that if Poincaré E_∞-coalgebras over A are chainwise bordant, then they are E_∞-bordant. Let $(f : X \to Y, [(\delta\varphi_s^Y, \varphi_s^{[X]})])$ be an $(n+1)$-dimensional algebraic Poincaré pair over A, where $(X, [X])$ is an n-dimensional Poincaré E_∞-coalgebra over A,

$$\varphi_s^{[X]} = (\alpha_2^X \otimes_A K)(\zeta_n \otimes U_s) \in (X^{\otimes 2} \otimes_A K)_{n+s} = (X \otimes_A X)_{n+s},$$

$U_s \in E_\infty(2)_s$, $s \geqslant 0$, let ζ_n be an arbitrary representative of the class $[X]$ and, in addition, let $(f \otimes_A K)_*[X] = 0$. We first construct an $(n+1)$-dimensional algebraic Poincaré pair over A

$$(\widetilde{f} : X \to \widetilde{Y}, [(\delta\varphi_s^{\widetilde{Y}}, \varphi_s^{[X]})]),$$

where $\widetilde{Y}$ is an $(n+1)$-dimensional E_∞-coalgebra over A, and f is a morphism of E_∞-coalgebras over A. We consider the short exact sequence of chain complexes

$$0 \longrightarrow \overline{E}^1 Y \xrightarrow{\ i\ } \overline{E}_\infty Y \xrightarrow{\ p\ } Y \longrightarrow 0,$$

where $(\overline{E}_\infty, \nabla, p)$ is the comonad in the category of A-module chain complexes determined by the operad E_∞, $\overline{E}_\infty^1 Y$ is the quotient complex of the complex $\overline{E}_\infty Y$ with respect to the direct summand that is isomorphic to Y and i is the chain embedding. Since i is a morphism of E_∞-coalgebras over A, it follows that the cone $C(i)$ of the chain map i is also an E_∞-coalgebra over A, which we denote by $\widetilde{Y}$. In addition, the obvious embedding

$$\mathrm{in}\colon (\overline{E}_\infty Y)_\bullet \to \widetilde{Y}_\bullet = C(i)_\bullet = (\overline{E}_\infty Y)_\bullet \oplus (\overline{E}_\infty^1 Y)_{\bullet-1}$$

is a morphism of E_∞-coalgebras over A. We now define the chain map $\widetilde{f} : X \to \widetilde{Y}$ as the composition

$$\widetilde{f}\colon X \xrightarrow{\ \eta^X\ } \overline{E}_\infty X \xrightarrow{\overline{E}_\infty(f)} \overline{E}_\infty Y \xrightarrow{\ \mathrm{in}\ } \widetilde{Y},$$

where (X,η^X) is the coalgebra over the comonad $(\overline{E}_\infty,\nabla,p)$ determined by the E_∞-coalgebra over A. Since η^X, $\overline{E}_\infty(f)$ and $\xrightarrow{\text{in}}$ are morphisms of E_∞-coalgebras over A, it follows that $\widetilde{f}$ is also a morphism of E_∞-coalgebras over A. The chain maps f and $\widetilde{f}$ are related by $f=\widetilde{p}\circ\widetilde{f}$, where $\widetilde{p}:\widetilde{Y}\to Y$ is the chain A-module map, $\widetilde{p}(g,h)=p(g)$ and $(g,h)\in\widetilde{Y}$. If we consider the cone $C(i)=\widetilde{Y}$ as a bicomplex and if we apply the spectral sequence of a bicomplex to $C(i)$, then we find that $\widetilde{p}$ is a chain equivalence of right A-modules. Therefore the E_∞-coalgebra over A is $(n+1)$-dimensional. The chain equivalence $\widetilde{p}$ determines a chain equivalence $q:C(\widetilde{f})\to C(f)$ of cones. Consequently the induced map of relative Q-groups

$$q^{n+1}\colon Q^{n+1}(\widetilde{f})\to Q^{n+1}(f)$$

is an isomorphism. The chain equivalence inverse to $\widetilde{p}$ is $\lambda:Y\to\widetilde{Y}$ which is a chain A-module embedding as a direct summand. We now put

$$\delta\varphi_s^{\widetilde{Y}}=(\lambda^{\otimes 2}\otimes_A K)(\delta\varphi_s^Y).$$

Since $\widetilde{p}\circ\lambda=\mathrm{id}$, it follows that

$$q^{n+1}\big[(\delta\varphi_s^{\widetilde{Y}},\varphi_s^{[X]})\big]=\big[(\delta\varphi_s^{Y},\varphi_s^{[X]})\big].$$

This implies that the diagram

$$\begin{array}{ccc}
C^i(\widetilde{f}) & \xleftarrow{(q)^*} & C^i(f)\\
{\scriptstyle\varphi_0^{\widetilde{f}}}\Big\downarrow & & \Big\downarrow{\scriptstyle\varphi_0^{f}}\\
\widetilde{Y}_{n+1-i} & \xrightarrow{\widetilde{p}} & Y_{n+1-i},
\end{array}$$

where $\varphi_0^{\widetilde{f}}=(\delta\varphi_0^{\widetilde{Y}},\varphi_0^{[X]})$, $\varphi_0^{f}=(\delta\varphi_s^{Y},\varphi_s^{[X]})$, is homotopy commutative. Since φ_0^f, q and $\widetilde{p}$ are chain equivalences over A, it follows that

$$H(\varphi_0^{\widetilde{f}})\colon H^i(\widetilde{f})\to H_{n+1-i}(\widetilde{Y})$$

is an isomorphism of A-modules for any $0\leqslant i\leqslant n+1$. Thus we have constructed an $(n+1)$-dimensional algebraic Poincaré pair

$$(\widetilde{f}\colon X\to\widetilde{Y},[(\delta\varphi_s^{\widetilde{Y}},\varphi_s^{[X]})])$$

over A, where $\widetilde{Y}$ is an $(n+1)$-dimensional E_∞-coalgebra over A and $\widetilde{f}$ is a morphism of E_∞-coalgebras over A.

We now turn to the construction, for an $(n+1)$-dimensional E_∞-pair $\widetilde{f}:X\to\widetilde{Y}$ over A, of the fundamental class

$$[\widetilde{f}]\in H_{n+1}(C(\widetilde{f})\otimes_A K)$$

such that $\delta[\widetilde{f}] = X$. We first consider the elements

$$\varphi_e^{[X]} = (\alpha_j^X \otimes_A K)(\zeta_n \otimes e) \in (X^{\otimes j} \otimes_A K)_{n+\nu},$$

where

$$\alpha_j^X : X \otimes_K E_\infty(j) \to X^{\otimes j}, \qquad j \geqslant 1,$$

are the structure maps of the E_∞-coalgebra X over A, ζ_n is a representative of the fundamental class of X, and $e \in E_\infty(j)$, $\dim(e) = \nu$. It is clear that $\varphi_{U_s}^{[X]} = \varphi_s^{[X]}$. The operad E_∞ is a free Σ-free acyclic operad, consequently for every $j \geqslant 0$ there is an Σ_j-equivariant contracting chain homotopy

$$s(j)\colon E_\infty(j)_\bullet \to E_\infty(j)_{\bullet+1}.$$

We consider the set of chains

$$\{\delta\varphi_e^Y \in (Y^{\otimes j} \otimes_A K)_{n+\nu+1},\ e \in E_\infty(j)_\nu,\ j \geqslant 1\},$$

where $\delta\varphi_{U_s}^Y = \delta\varphi_s^Y$, $s \geqslant 0$, and $\delta\varphi_e^Y = (f^{\otimes j} \otimes_A K)(\varphi_{s(j)e}^{[X]})$, $j \neq 2$. Since, by the hypothesis, $(f \otimes_A K)_*[X] = 0$, there is an $(n+1)$-dimensional chain $y \in (Y \otimes_A K)_{n+1}$ such that $dy = (f \otimes_A K)(\zeta_n)$. We define an element

$$\begin{aligned}\beta^f \in (\overline{E}_\infty Y)_{n+1} \otimes_A K &= \Big(\prod_{j\geqslant 1} \hom_{\Sigma_j}\big(E_\infty(j); Y^{\otimes j}\big)\Big)_{n+1} \otimes_A K \\ &= \Big(\prod_{j\geqslant 1} \hom_{\Sigma_j}\big(E_\infty(j); Y^{\otimes j} \otimes_A K\big)\Big)_{n+1},\end{aligned}$$

by putting

$$\begin{aligned}\beta^f(1) &= (-1)^n y \in (Y \otimes_A K)_{n+1}, &\qquad 1 &\in E_\infty(1)_0,\\ \beta^f(e) &= \delta\varphi_e^Y \in (Y^{\otimes j} \otimes_A K)_{n+1+\nu}, & e &\in E_\infty(j)_\nu.\end{aligned}$$

We now consider the element

$$\begin{aligned}y^{\widetilde{f}} &= \big((\beta^f, 0), \zeta_n\big) \in (C(\widetilde{f}) \otimes_A K)_{n+1} = (\widetilde{Y} \otimes_A K)_{n+1} \oplus (X \otimes_A K)_n \\ &= \big((\overline{E}_\infty Y \otimes_A K)_{n+1} \oplus (\overline{E}_\infty^1 Y \otimes_A K)_n\big) \oplus (X \otimes_A K)_n.\end{aligned}$$

We check that $y^{\widetilde{f}}$ is an $(n+1)$-dimensional cycle of the complex $C(\widetilde{f}) \otimes_A K$. We recall that

$$(d(\beta^f))(e) = d(\beta^f(e)) + (-1)^n \beta^f(de).$$

We now have

$$\begin{aligned}d(y^{\widetilde{f}}) &= \big(d(\beta^f, 0) + (-1)^{n+1}\widetilde{f}(\zeta_n), d\zeta_n\big) \\ &= \big(d\beta^f + (-1)^{n+1}(\overline{E}_\infty(f)\eta^X \otimes_A K)(\zeta_n), 0, 0\big),\end{aligned}$$

where $\eta^X : X \to \overline{E}_\infty X$ is the structure map of the coalgebra X over the comonad $(\overline{E}_\infty, \nabla, p)$. For arbitrary $e \in E_\infty(j)$, $j > 1$, we have

$$
\begin{aligned}
&\big(d\beta^f + (-1)^{n+1}(\overline{E}_\infty(f)\eta^X \otimes_A K)(\zeta_n)\big)(e) \\
&\quad = d(\beta^f(e)) + (-1)^n\beta^f(de) + (-1)^{n+1}(f^{\otimes j} \otimes_A K)(\alpha_j^X \otimes_A K)(\zeta_n \otimes e) \\
&\quad = d\delta\varphi_e^Y + (-1)^n\delta\varphi_{de}^Y + (-1)^{n+1}(f^{\otimes j} \otimes_A K)(\alpha_j^X \otimes_A K)(\zeta_n \otimes e) \\
&\quad = (-1)^n(f^{\otimes j} \otimes_A K)(\varphi_{ds(j)(e)}^{[X]}) + (-1)^n(f^{\otimes j} \otimes_A K)(\varphi_{s(j)de}^{[X]}) \\
&\qquad\quad + (-1)^{n+1}(f^{\otimes j} \otimes_A K)(\alpha_j^X \otimes_A K)(\zeta_n \otimes e) \\
&\quad = (-1)^n(f^{\otimes j} \otimes_A K)(\varphi_e^{[X]}) + (-1)^{n+1}(f^{\otimes j} \otimes_A K)(\alpha_j^X \otimes_A K)(\zeta_n \otimes e) \\
&\quad = (-1)^n(f^{\otimes j} \otimes_A K)(\varphi_e^{[X]}) + (-1)^{n+1}(f^{\otimes j} \otimes_A K)(\varphi_e^{[X]}) = 0.
\end{aligned}
$$

For $1 \in E_\infty(1)_0$ we have

$$
\begin{aligned}
&\big(d\beta^f + (-1)^{n+1}(\overline{E}_\infty(f)\eta^X \otimes_A K)(\zeta_n)\big)(1) \\
&\quad = d(\beta^f(1)) + (-1)^n\beta^f(d1) + (-1)^{n+1}(f \otimes_A K)(\alpha_1^X \otimes_A K)(\zeta_n \otimes 1) \\
&\quad = (-1)^n dy + (-1)^{n+1}(f \otimes_A K)(\zeta_n) \\
&\quad = (-1)^n dy + (-1)^{n+1} dy = 0.
\end{aligned}
$$

Hence $d(y^{\widetilde{f}}) = 0$. We let $[\widetilde{f}] \in H_{n+1}(C(\widetilde{f}) \otimes_A K)$ denote the homology class of the cycle $y^{\widetilde{f}}$. For the free E_∞-coalgebra $\overline{E}_\infty Y$ over A we consider the chain A-module map

$$\alpha_2^{\overline{E}_\infty Y}(U_0)\colon \overline{E}_\infty Y \to (\overline{E}_\infty Y) \otimes_K (\overline{E}_\infty Y)$$

given by the formula

$$\big(\alpha_2^{\overline{E}_\infty Y}(U_0)(g)\big)(e_{j_1} \otimes e_{j_2}) = g\big(\gamma(U_0 \otimes e_{j_1} \otimes e_{j_2})\big) \in Y^{\otimes j_1} \otimes_K Y^{\otimes j_2},$$

where $g \in \overline{E}_\infty Y$, $e_{j_1} \in \overline{E}_\infty(j_1)$, $e_{j_2} \in \overline{E}_\infty(j_2)$, $j_1, j_2 \geqslant 1$. In exactly the same way we are given the A-module chain map

$$\alpha_2^{\overline{E}^1_\infty Y}(U_0)\colon \overline{E}^1_\infty Y \to (\overline{E}^1_\infty Y) \otimes_K (\overline{E}^1_\infty Y).$$

We note that

$$p^{\otimes 2}\big(\alpha_2^{\overline{E}_\infty Y}(U_0)(g)\big) = \big(\alpha_2^{\overline{E}^1_\infty}(U_0)(g)\big)(1 \otimes 1) = g\big(\gamma(U_0 \otimes 1 \otimes 1)\big) = g(U_0) \in Y^{\otimes 2},$$

where $g \in \overline{E}_\infty Y$, $p\colon \overline{E}_\infty Y \to Y$, $p(w) = w(1)$. Similarly, if

$$g \in (\overline{E}_\infty(Y \otimes_A K)) = (\overline{E}_\infty Y) \otimes_A K,$$

then

$$(p \otimes_A K)^{\otimes 2}\big(\alpha_2^{\overline{E}_\infty Y \otimes_A K}(U_0)(g)\big) = g(U_0) \in Y^{\otimes 2} \otimes_A K.$$

We now consider the chain map

$$\alpha_2^{C(\widetilde{f})}(U_0)\otimes_A K=\big(\alpha_2^{\overline{E}_\infty Y}(U_0)\otimes_A K,\alpha_2^{\overline{E}^1_\infty Y}(U_0)\otimes_A K,\alpha_2^{X}(U_0)\otimes_A K\big)\colon$$
$$C(\widetilde{f})\to\big((\overline{E}_\infty Y)^{\otimes 2}\otimes_A K\big)\oplus\big((\overline{E}_\infty Y)^{\otimes 2}\otimes_A K\big)\oplus(X^{\otimes 2}\otimes_A K)\subset C(\widetilde{f})^{\otimes 2}\otimes_A K,$$

which induces the chain intersection operation

$$\cap\colon C^i(\widetilde{f})\otimes_K\big(C(\widetilde{f})\otimes_A K\big)_{n+1}\to\widetilde{Y}_{n+1-i},\qquad 0\leqslant i\leqslant n+1.$$

If we apply to the element

$$\begin{aligned}&\big(\alpha_2^{C(\widetilde{f})}(U_0)\otimes_A K\big)(y^{\widetilde{f}})\\&\quad=\big(\alpha_2^{C(\widetilde{f})}(U_0)\otimes_A K\big)(\beta^f,0,\zeta_n)\\&\quad=\big((\alpha_2^{\overline{E}_\infty Y}(U_0)\otimes_A K)(\beta^f),0,(\alpha_2^{X}(U_0)\otimes_A K)(\zeta_n)\big)\in(C(\widetilde{f})^{\otimes 2}\otimes_A K)_{n+1}\end{aligned}$$

the map

$$(q^{\otimes 2}\otimes_A K)\colon C(\widetilde{f})^{\otimes 2}\otimes_A K\to C(f)^{\otimes 2}\otimes_A K,$$

where $q(w_1,w_2,x)=\big(\widetilde{p}(w_1,w_2),x\big)=\big(p(w_1),x\big)=\big(w_1(1),x\big)$, then we obtain

$$\begin{aligned}&(q^{\otimes 2}\otimes_A K)\big(\alpha_2^{C(\widetilde{f})}(U_0)\otimes_A K\big)(y^{\widetilde{f}})\\&\quad=(q^{\otimes 2}\otimes_A K)\big((\alpha_2^{\overline{E}_\infty Y}(U_0)\otimes_A K)(\beta^f),0,(\alpha_2^{X}(U_0)\otimes_A K)(\zeta_n)\big)\\&\quad=\big((p^{\otimes 2}\otimes_A K)(\alpha_2^{\overline{E}_\infty Y}(U_0)\otimes_A K)(\beta^f),(\alpha_2^{X}(U_0)\otimes_A K)(\zeta_n)\big)\\&\quad=\big(\beta^f(U_0),(\alpha_2^{X}(U_0)\otimes_A K)(\zeta_n)\big)=(\delta\varphi^Y_{U_0},\varphi^{[X]}_{U_0})=(\delta\varphi^Y_0,\varphi^{[X]}_0).\end{aligned}$$

This implies that the diagram

$$\begin{tikzcd} C^i(\widetilde{f}) \arrow[d, "\cap y^{\widetilde{f}}"'] & C^i(f) \arrow[l, "(q)^*"'] \arrow[d, "(\delta\varphi_0^Y,\varphi_0^{[X]})"] \\ \widetilde{Y}_{n+1-i} \arrow[r, "\widetilde{p}"] & \widetilde{Y}_{n+1-i} \end{tikzcd}$$

is commutative for any $0\leqslant i\leqslant n+1$. Since the A-module maps $\widetilde{p},q^*,(\delta\varphi_0^Y,\varphi_0^{[X]})$ induce A-module isomorphisms in homology, it follows that the map

$$\cap[\widetilde{f}]\colon H^i(\widetilde{f})\to H_{n+1-i}(\widetilde{Y})$$

is an A-module isomorphism for any $0\leqslant i\leqslant n+1$, where $[\widetilde{f}]=[y^{\widetilde{f}}]\in H_{n+1}(\widetilde{f})$. In addition,

$$\delta[\widetilde{f}]=\delta\big[(\beta^f,0),\zeta_n\big]=[\zeta_n]=[X].$$

Thus, the n-dimensional Poincaré E_∞-coalgebra $(X,[X])$ over A is the boundary of the $(n+1)$-dimensional Poincaré E_∞-pair $(\widetilde{f}:X\to\widetilde{Y},[\widetilde{f}])$ over A.

Corollary 1. *The relation of E_∞-bordism for n-dimensional Poincaré E_∞-coalgebras over A is an equivalence relation. Homotopy equivalent n-dimensional Poincaré E_∞-coalgebras over A are E_∞-bordant.*

We let $(\Omega E_\infty)^n(A)$ denote the set of classes of E_∞-bordant n-dimensional Poincaré E_∞-coalgebras over A. The direct sum and the change of orientation for the Poincaré E_∞-coalgebras determine the structure of an Abelian group on $(\Omega E_\infty)^n(A)$.

Corollary 2. *The groups $(\Omega E_\infty)^n(A)$ and $(L^c E_\infty)^n(A)$ are isomorphic for any $n \geqslant 0$.*

Let M be a closed oriented smooth manifold of dimension n with fundamental group π, let $\widetilde{M}$ be its universal cover and let $[M] \in H_n(M;\mathbb{Z})$ be the fundamental class. Then (see Example 14) the singular chain complex $C_*(\widetilde{M};\mathbb{Z})$ when considered together with the fundamental class

$$[M] \in H_n(M;\mathbb{Z}) = H_n\big(C_\bullet(\widetilde{M};\mathbb{Z}) \otimes_{\mathbb{Z}[\pi]} \mathbb{Z}\big)$$

is an n-dimensional Poincaré E_∞-coalgebra over $\mathbb{Z}[\pi]$, which we denote by $\theta(M)$. Similarly, an $(n+1)$-dimensional smooth oriented manifold M with boundary ∂M determines, via the singular-complex functor, an $(n+1)$-dimensional Poincaré E_∞-pair over $\mathbb{Z}[\pi]$, which we denote by $\theta(M, \partial M)$. The boundary of $\theta(M,\partial M)$ is $\theta(\partial M)$. This implies that we have a well-defined homomorphism

$$\theta\colon \Omega_n^{\mathrm{SO}}(B\pi) \to (\Omega E_\infty)^n(\mathbb{Z}[\pi]), \qquad n \geqslant 0.$$

Moreover, for any $n \geqslant 0$ the diagram

$$\begin{array}{ccc} \Omega_n^{\mathrm{SO}}(B\pi) & \xrightarrow{\ \sigma\ } & (LE_\infty)^n(\mathbb{Z}[\pi]) \\ {\scriptstyle\theta}\big\downarrow & & \big\uparrow{\scriptstyle s} \\ (\Omega E_\infty)^n(\mathbb{Z}[\pi]) & = & (L^c E_\infty)^n(\mathbb{Z}[\pi]) \end{array}$$

is commutative, where σ is the homomorphism considered in §1. In the case of non-oriented manifolds we obtain in exactly the same way as above a well-defined homomorphism

$$\theta\colon \Omega_n^{\mathrm{O}}(B\pi) \to (\Omega E_\infty)^n\big((\mathbb{Z}/2)[\pi]\big), \qquad n \geqslant 0,$$

and for any $n \geqslant 0$ we obtain a commutative diagram

$$\begin{array}{ccc} \Omega_n^{\mathrm{O}}(B\pi) & \xrightarrow{\ \sigma\ } & (LE_\infty)^n\big((\mathbb{Z}/2)[\pi]\big) \\ {\scriptstyle\theta}\big\downarrow & & \big\uparrow{\scriptstyle s} \\ (\Omega E_\infty)^n\big((\mathbb{Z}/2)[\pi]\big) & = & (L^c E_\infty)^n\big((\mathbb{Z}/2)[\pi]\big). \end{array}$$

If we take into account Example 13 and also use the concept of an n-dimensional Poincaré algebra in the sense of Adams (see [13], [14]) and the bordism groups of such algebras (see [14]), then we obtain the following lemma.

Lemma 4. *Let π be the trivial group. Then the homomorphism*

$$\theta\colon \Omega_n^{\mathrm{O}}(pt) \to (\Omega E_\infty)^n(\mathbb{Z}/2)$$

is an embedding for any $n \geqslant 0$, that is, $\operatorname{Im}\theta = \Omega_n^{\mathrm{O}}(pt)$.

§ 4. Rational commutative Poincaré A_∞-coalgebras

Let A be an associative unital augmented K-algebra with an involution, where $K = \mathbb{Q}$ is the field of rational numbers. Given an arbitrary commutative A_∞-coalgebra X over A, that is, given a CA_∞-coalgebra over A (see Examples 5 and 12), there is an intersection operation

$$\cap\colon H^i(X) \otimes_K H_n(X \otimes_A K) \to H_{n-i}(X), \qquad 0 \leqslant i \leqslant n,$$

that is induced by the chain map

$$\nabla^1_X \otimes_A K\colon X \otimes_A K \to (X \otimes_K X) \otimes_A K = X \otimes_A X,$$

where ∇^1_n belongs to the CA_∞-structure $\{\nabla^m_X\}_{m\geqslant 1}$ of the CA_∞-coalgebra X over A. The CA_∞-coalgebra X over A is said to be *n-dimensional* if X is an n-dimensional chain complex of right A-modules.

Definition 7. An n-dimensional CA_∞-coalgebra X over A will be called an *n-dimensional rational commutative Poincaré A_∞-coalgebra* or, more briefly, a *Poincaré CA_∞-coalgebra*, over A if there is an element $[X] \in H_n(X \otimes_A K)$ such that

$$\cap[X]\colon H^i(X) \to H_{n-i}(X)$$

is an isomorphism of A-modules for all $0 \leqslant i \leqslant n$. The homology class $[X]$ is called the *fundamental class of the n-dimensional Poincaré CA_∞-coalgebra* X over A.

The concepts of morphism, homotopy equivalence, direct sum and reversal of orientation are defined for n-dimensional Poincaré CA_∞-coalgebras over A analogously to the corresponding concepts which were introduced in §3 for Poincaré E_∞-coalgebras over A.

The n-dimensional Poincaré CA_∞-coalgebra $(X;[X])$ over A uniquely determines an n-dimensional algebraic Poincaré complex $(X,[\xi^{[X]}_i])$ over A. In fact, we consider the n-dimensional cycle

$$(\nabla^1_X \otimes_A K)(\zeta_n) = \sum_{i=0}^{n} x_i \otimes x_{n-i} \in \sum_{i=0}^{n} X_i \otimes_A X_{n-i} = (X \otimes_A X)_n,$$

where ζ_n is an arbitrary representative of the fundamental class $[X]$. We define A-module homomorphisms $\xi^n_i : X^{n-i} \to X_i$, $i \geqslant 0$, by putting

$$\xi^X_i(f) = x_i f(x_{n-i})^*, \qquad f \in X^{n-i}.$$

Since $T \circ \nabla^1_X = \nabla^1_X$, where $T(x_p \otimes x_q) = (-1)^{pq} x_q \otimes x_p$, it follows that

$$(\xi^X_i)^* = (-1)^{i(n-i)} \xi^X_{n-i}, \qquad i \geqslant 0.$$

The condition $\delta\big((\nabla \otimes_A K)(\zeta_n)\big) = 0$ implies that

$$d^X_i \circ \xi^X_i + (-1)^{i-1} \xi^X_{i-1} \circ (d^X_{n-i+1})^* = 0.$$

If ζ_n and ζ'_n are homologous representatives of the fundamental class $[X]$, then the corresponding families of A-module homomorphisms $\{\xi_i^{[X]}\}$ and $\{\xi'^{[X]}_i\}$ determine the same element in the group $Q^n(X)$. The A-module homomorphism

$$H(\xi_{n-i}^X)\colon H^i(X) \to H_{n-i}, \qquad 0 \leqslant i \leqslant n,$$

coincides, by definition, with the homomorphism

$$\cap[X]\colon H^i(X) \to H_{n-i}(X), \qquad 0 \leqslant i \leqslant n,$$

and therefore it is an A-mdoule isomorphism.

Definition 8. Two n-dimensional Poincaré CA_∞-coalgebras $(X,[X])$ and $(Y,[Y])$ over A are said to be *algebraically bordant* if the n-dimensional algebraic Poincaré complexes $(X,[\xi_i^{[X]}])$ and $(Y,\xi_i^{[Y]}])$ that they determine are bordant.

We obtain the following assertion in the same way as in [9].

Lemma 5. *Algebraic bordism for n-dimensional Poincaré CA_∞-coalgebras over A is an equivalence relation. Homotopy equivalent Poincaré CA_∞-coalgebras over A are algebraically bordant.*

We let $(LCA_\infty)^n(A)$ denote the set of classes of algebraically bordant n-dimensional Poincaré CA_∞-coalgebras over A. The operations of direct sum and reversing the orientation for Poincaré CA_∞-coalgebras over A define the structure of a commutative group on $(LCA_\infty)^n(A)$. It is clear that $(LCA_\infty)^n(A)$ is a subgroup of the symmetric L-group $L^n(A), n \geqslant 0$.

Definition 9. Two n-dimensional Poincaré CA_∞-coalgebras $(X;[X])$ and $(Y;[Y])$ over A are said to be *chainwise bordant* if there is an $(n+1)$-dimensional algebraic Poincaré pair over A

$$(f\colon X \oplus Y \to C, \xi^f \in Q^{n+1}(f))$$

such that

$$(f \otimes_A K)_*([X] \oplus -[Y]) = 0 \in H_n(C \otimes_A K),$$

and the boundary of this pair is

$$(X,[\xi_i^{[X]}]) \oplus (-(Y,[\xi_i^{[Y]}])).$$

We obtain the next assertion in the same way as Lemma 5.

Lemma 6. *Chainwise bordism between n-dimensional Poincaré CA_∞-coalgebras over A is an equivalence relation. Homotopy equivalent Poincaré CA_∞-coalgebras over A are chainwise bordant.*

We let $(L^cCA_\infty)^n(A)$ denote the set of classes of chainwise bordant n-dimensional Poincaré CA_∞-coalgebras over A. The operations of direct sum and change of orientation for Poincaré CA_∞-coalgebras induce the structure of an Abelian group on $(L^cCA_\infty)^n(A)$. In addition we can define the obvious epimorphism

$$S\colon (L^cCA_\infty)^n(A) \to (LCA_\infty)^n(A), \qquad n \geqslant 0.$$

Lemma 7. *Let $A = K = \mathbb{Q}$. Then the group homomorphism*

$$S\colon (L^cCA_\infty)^n(\mathbb{Q}) \to (LCA_\infty)^n(\mathbb{Q})$$

is an isomorphism for any $n \geqslant 0$.

Proof. It is obvious that the homomorphism S is epimorphic. We now show that it is monomorphic. Let the n-dimensional Poincaré CA_∞-coalgebra $(X,[X])$ over $\mathbb{Q}$ be algebraically bordant to zero, that is, the n-dimensional algebraic Poincaré complex $(X,[\xi_i^{[X]}])$ over $\mathbb{Q}$ is the boundary of some $(n+1)$-dimensional algebraic Poincaré pair

$$(f\colon X \to Y, [(\xi_i^Y;\xi_i^{[X]})])$$

over $\mathbb{Q}$. We claim that the n-dimensional rational CA_∞-coalgebra $(X,[X])$ is chain-wise bordant to zero, that is, there is an $(n+1)$-dimensional algebraic Poincaré pair

$$(g\colon X \to C, [(\xi_i^C;\xi_i^{[X]})])$$

over $\mathbb{Q}$ such that $(g)_*[X] = 0 \in H_n(C)$. Let ζ_n be a representative of the fundamental class $[X]$. If $f(\zeta_n)$ is a boundary element from Y_n, then in place of

$$(g\colon X \to C, [(\xi_i^C;\xi_i^{[X]})])$$

one can take

$$(f\colon X \to Y, [(\xi_i^Y;\xi_i^{[X]})]).$$

Therefore we shall assume that the cycle $f(\zeta_n)$ is not a boundary. Let $\alpha : \mathbb{Q} \to Y_n$ be the homomorphism over $\mathbb{Q}$ given by $\alpha(1) = f(\zeta_n)$. We consider the rational $(n+1)$-dimensional chain complex (C, d^c), where

$$C_0 = Y_0 \oplus \mathbb{Q}, \qquad C_{n+1} = Y_{n+1} \oplus \mathbb{Q}, \qquad C_i = Y_i, \quad i \neq 0, n+1,$$
$$d_1^C = \begin{pmatrix} d_1^Y \\ 0 \end{pmatrix}, \qquad d_{n+1}^C = \left(d_{n+1}^Y, (-1)^n\alpha\right), \qquad d_i^C = d_i^Y, \quad i \neq 1, n+1.$$

Let $[\varepsilon] \in H^0(X)$ be the element that is taken under the isomorphism

$$H(\xi_n^{[X]})\colon H^0(X) \to H_n(X)$$

to $[X]$. We take a representative cocycle

$$\varepsilon \in X^0 = \hom_{\mathbb{Q}}(X_0;\mathbb{Q})$$

of the class $[\varepsilon]$ such that $(\xi_n^{[X]}(\varepsilon) = \zeta_n$. We define the chain map

$$g = \{g_i\colon X_i \to C_i,\ i \geqslant 0\}$$

by putting

$$g_0 = \begin{pmatrix} f_0 \\ \varepsilon \end{pmatrix}, \qquad g_{n+1} = \begin{pmatrix} f_{n+1} \\ 0 \end{pmatrix}, \qquad g_i = f_i, \quad i \neq 0, n+1.$$

Let $1 : \mathbb{Q}^* \to \mathbb{Q}$ be the standard identification. We consider the family of rational homomorphisms

$$\{\xi_i^C : C^{n+1-i} \to C_i,\ i \geqslant 0\},$$

where

$$\xi_0^C = \begin{pmatrix} \xi_0^Y & 0 \\ 0 & 1 \end{pmatrix}, \qquad \xi_{n+1}^C = \begin{pmatrix} \xi_{n+1}^Y & 0 \\ 0 & 1 \end{pmatrix}, \qquad \xi_i^C = \xi_i^Y, \quad i \neq 0, n+1.$$

It is clear that $(\xi_i^C)^* = (-1)^{i(n+1-i)}\xi_{n+1-i}^C$ for any $i \geqslant 0$. The relations

$$d_i^C \circ \xi_i^C + (-1)^{i-1}\xi_{i-1}^C \circ (d_{n-i+2}^C)^* = (-1)^n g \circ \xi_{i-1}^{[X]} \circ g^*$$

are obviously true for $i \neq 1$ or $n+1$, while for $i = 1$ or $n+1$ they are implied by the commutativity of the diagram

$$\begin{tikzcd} \mathbb{Q} \arrow[r, "\varepsilon^*"] \arrow[d, "\alpha"'] & X^0 \arrow[d, "\xi_n^{[X]}"] \\ Y_n & X_n. \arrow[l, "f"'] \end{tikzcd}$$

Since

$$g(\zeta_n) = f(\zeta_n) = \alpha(1) = d_{n+1}^C \begin{pmatrix} 0 \\ (-1)^n \end{pmatrix},$$

it follows that $(g)_*[X] = 0$. Thus, we have constructed an $(n+1)$-dimensional algebraic Poincaré pair

$$(g : X \to C, [(\xi_i^C; \xi_i^{[X]})])$$

over $\mathbb{Q}$, where $(g)_*[X] = 0$, which has boundary $(X; [X])$. Consequently the n-dimensional Poincaré CA_∞-coalgebra $(X; [X])$ over $\mathbb{Q}$ is chainwise bordant to zero.

Let $f : X \to Y$ be an arbitrary morphism of CA_∞-coalgebras over A. We give the cone $C(f)$ of the chain A-module map f the structure

$$\nabla_{C(f)}^m : C(f) \to C(f)^{\otimes(m+1)}, \qquad m \geqslant 1,$$

of a CA_∞-coalgebra over A by putting

$$\nabla_{C(f)}^m(y, x) = \big(\nabla_Y^m(y), (-1)^{m-1}\nabla_X^m(x)\big) \in \big(Y_{n+m-1}^{\otimes(m+1)}\big) \oplus \big(X_{n+m-2}^{\otimes(m+1)}\big) \subset \big(C(f)_{n+m-1}^{\otimes(m+1)}\big),$$

where $(y, x) \in C(f)_n$. Given a morphism of CA_∞-coalgebras $f : X \to Y$ we consider the intersection operation

$$\cap : H^i(f) \otimes_K H_{n+1}(C(f) \otimes_A K) \to H_{n+1-i}(Y), \qquad 0 \leqslant i \leqslant n,$$

that is determined by the chain map

$$\nabla_{C(f)}^1 \otimes_A K : C(f)_\bullet \otimes_A K \to C(f^{\otimes 2}) \otimes_A K = (Y \otimes_A Y)_\bullet \oplus (X \otimes_A X)_{\bullet-1}.$$

A morphism of CA_∞-coalgebras $f : X \to Y$ is called an $(n+1)$-*dimensional* CA_∞-*pair* over A if X is an n-dimensional CA_∞-coalgebra over A and Y is an $(n+1)$-dimensional CA_∞-coalgebra.

Definition 10. An $(n+1)$-dimensional CA_∞-pair $f : X \to Y$ over A is called an $(n+1)$-*dimensional Poincaré CA_∞-pair* over A if we are given

$$[f] \in H_{n+1}(C(f) \otimes_A K)$$

such that

$$\cap[f] \colon H^i(f) \to H_{n+1-i}(Y)$$

is an isomorphism of A-modules for all $0 \leqslant i \leqslant n+1$. The homology class $[f]$ is called the *fundamental class of the $(n+1)$-dimensional Poincaré CA_∞-pair $f : X \to Y$* over A.

We obtain the following assertion in the same way as in [2].

Lemma 8. *Suppose that we are given an $(n+1)$-dimensional Poincaré CA_∞-pair $(f : X \to Y, [f])$ over A. Then the CA_∞-coalgebra X over A when considered together with the element $[X] = \delta[f]$, where δ is the connecting homomorphism in the long exact homology sequence of the map $(f \otimes_A K)$, is an $(n+1)$-dimensional Poincaré CA_∞-coalgebra over A.*

By the *boundary* of the $(n+1)$-dimensional Poincaré CA_0-pair $(f : X \to Y, [f])$ we mean the n-dimensional CA_∞-coalgebra $(X, [X] = \delta[f])$ over A.

Definition 11. Two n-dimensional Poincaré CA_∞-coalgebras $(X; [X])$ and $(Y; [Y])$ over A are said to be *CA_∞-bordant* if there is an $(n+1)$-dimensional Poincaré CA_∞-pair over A such that its boundary is

$$(X; [X]) \oplus -(Y; [Y]).$$

The next theorem is obtained in the same way as in §3.

Theorem 2. *The relations of chainwise bordism and CA_∞-bordism between n-dimensional Poincaré CA_∞-coalgebras over A are equivalent.*

Corollary 3. *The relation of CA_∞-bordism between n-dimensional Poincaré CA_∞-coalgebras over A is an equivalence relation. Homotopy equivalent n-dimensional Poincaré CA_∞-coalgebras over A are CA_∞-bordant.*

Corollary 4. *The groups $(\Omega CA_\infty)^n(A)$ and $(L^c CA_\infty)^n(A)$ are isomorphic for any $n \geqslant 0$.*

Let M be a closed oriented smooth manifold of dimension n with fundamental group π, let $\widetilde{M}$ be its universal cover and let $[M] \in H_n(M; \mathbb{Q})$ be its rational fundamental class. Then (see Example 14) the singular chain complex $C_*(\widetilde{M}; \mathbb{Q})$ when considered together with the fundamental class

$$[M] \in H_n(M; \mathbb{Q}) = H_n(C_\bullet(\widetilde{M}; \mathbb{Q}) \otimes_{\mathbb{Q}[\pi]} \mathbb{Q}),$$

is an n-dimensional Poincaré CA_∞-coalgebra over $\mathbb{Q}[\pi]$, which we denote by $\theta(M)$. Similarly, the $(n+1)$-dimensional smooth oriented manifold M with boundary ∂M determines via the rational singular complex functor an $(n+1)$-dimensional

Poincaré CA_∞-pair over $\mathbb{Q}[\pi]$, which we denote by $\theta(M, \partial M)$. The boundary of $\theta(M, \partial M)$ is $\theta(\partial M)$. This implies that we have a well-defined homomorphism

$$\theta\colon \Omega_n^{\mathrm{SO}}(B\pi) \to (\Omega CA_\infty)^n(\mathbb{Q}[\pi]), \qquad n \geqslant 0,$$

and, in addition, the diagram

$$\begin{array}{ccc} \Omega_n^{\mathrm{SO}}(B\pi) & \xrightarrow{\sigma} & (LCA_\infty)^n(\mathbb{Q}[\pi]) \\ {\scriptstyle\theta}\big\downarrow & & \big\uparrow{\scriptstyle S} \\ (\Omega CA_\infty)^n(\mathbb{Q}[\pi]) & = & (L^cCA_\infty)^n(\mathbb{Q}[\pi]) \end{array}$$

is commutative, where σ is the homomorphism considered in §1. If the group π is trivial, then $\sigma \otimes \mathbb{Q}$ is the epimorphic signature homomorphism. Therefore the groups

$$(LCA_\infty)^n(\mathbb{Q}) \otimes \mathbb{Q} \quad \text{and} \quad L^n(\mathbb{Q}) \otimes \mathbb{Q}$$

are isomorphic, $n \geqslant 0$.

Corollary 5.

$$\operatorname{Im}\theta \otimes \mathbb{Q} = (\Omega CA_\infty)^n(\mathbb{Q}) \otimes \mathbb{Q} = (L^cCA_\infty)^n(\mathbb{Q}) \otimes \mathbb{Q} = \begin{cases} \mathbb{Q}, & n = 4k, \\ 0, & n \neq 4k. \end{cases}$$

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