Proceedings of a Symposium in Recognition of the Retirement of Terry Lawson from Tulane University

April 20-21, 2007

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Introduction

On the weekend of April 20-21, 2007, a group of mathematicians gathered in New Orleans to honor Terry Lawson on the occasion of his impending retirement from Tulane University.

The event was organized by Terry's Tulane colleague Sławomir Kwasik.

The symposium began on Friday afternoon with an appreciation of Terry's work in topology presented by Reinhard Schultz (University of California, Riverside) in the form of a general colloquium talk for the whole mathematics department at Tulane.

That evening a group went out for dinner with Terry at a local restaurant.

On Saturday four friends and colleagues of Terry's gave lectures on topological topics of current interest. These speakers at the symposium included

- Nikolai Saveliev (University of Miami)
- Allan Edmonds (Indiana University)
- Daniel Ruberman (Brandeis University)
- Ron Fintushel (Michigan State University)

On Saturday evening the visiting topologists joined members of the Tulane Math Department at a party in honor of Terry and his wife Barb.

Terry and Barb will both be sorely missed by colleagues and friends in New Orleans. We wish them all the best in their new adventures upon their move to Portland, Oregon.

Between lower and higher dimensions

(in the work of Terry Lawson)

Reinhard Schultz

Introduction

There are several approaches to summarizing a mathematician's research accomplishments, and each has its advantages and disadvantages. This article is based upon a talk given at Tulane that was aimed at a fairly general audience, including faculty members in other areas and graduate students who had taken the usual entry level courses. As such, it is meant to be relatively nontechnical and to emphasize qualitative rather than quantitative issues; in keeping with this aim, references will be given for some standard topological notions that are not normally treated in entry level graduate courses.

Since this was an hour talk, it was also not feasible to describe every single piece of published mathematical work that Terry Lawson has ever written; in particular, some papers like [**42**] and [**50**] would require lengthy digressions that are not easily related to the central themes in his main lines of research. Instead, we shall focus on some ways in which Terry's work relates to an important thread in geometric topology; namely, the passage from studying problems in a given dimension to studying problems in the next dimensions. Qualitatively speaking, there are fairly well-developed theories for very low dimensions and for all sufficiently large dimensions, but between these ranges there are some dimensions in which the answers to many fundamental questions are extremely unclear. Much of Terry's work, and most of his best known results and papers, are directly related to such questions.

Acknowledgments. I am grateful to Sławomir Kwasik for inviting me to speak on this topic at Tulane and for assistance with some questions which arose in preparing this writeup. Also, I would like to thank Elmar Winkelnkemper for some helpful comments regarding the theory and applications of Open Book Decompositions.

Lower versus higher dimensions

Of course, the concept of *dimension* is central to many geometrical questions, and in the physical world one can have objects of dimension n for $n = 0, 1, 2, 3$. During the nineteenth century, several mathematicians recognized that the methods of coordinate geometry lead to a theory of n-dimensional geometrical objects, where n is an arbitrary nonnegative integer. In particular, the vector space structure on $\mathbb{R}^n$, including the standard inner product, provide a setting in which one can describe an n-dimensional analog of classical Euclidean plane or solid geometry. Higher dimensional objects are more than just intellectual curiosities, for they have multiple uses in many contexts, including a many areas in the mathematical sciences, several branches of physics, and even in other subjects like mathematical economics.

Many important n-dimensional geometrical objects are examples of *topological n-manifolds*; formally, these are Hausdorff topological spaces in which every point has an open neighborhood which is homeomorphic to $\mathbb{R}^n$. Objects of this sort were introduced in the middle of the nineteenth century, and as noted above they arise naturally in a wide range of topics, both within the mathematical sciences and in their applications to other fields. We shall deal mainly with topological manifolds in this article, but in some cases we must restrict attention to *differential* or *smooth n*-manifolds (see [**46**] or [**33**]), which have the additional structure needed to discuss differentiation and integration on the space.

In classical Euclidean geometry, clearly some things become more complicated when one passes from line geometry to plane geometry or from plane geometry to solid geometry, and it is normal to expect a similar pattern when one goes from n-dimensional objects to $(n+1)$-dimensional objects. This is true in many cases, but one also has the following somewhat unanticipated fact:

> *Sometimes the answers to basic geometrical questions become simpler if the dimension n is sufficiently large. In other words, there are instances where general patterns of results exist if one excludes finitely many exceptional dimensions.*

An example from euclidean geometry. The classification of solid regular polyhedra in Euclidean n-space up to similarity illustrates this phenomenon fairly well. If $n = 2$ then the possibilities are given by the usual regular k-gons, where k is an arbitrary integer ≥ 3. On the other hand, if $n = 3$ then the theory is simpler in some ways but more complicated in others. There are only finitely many possibilities, and they are given by the classical *Platonic solids*; namely, the regular triangular pyramid (or tetrahedron), the cube, the regular octahedron (which can be constructed by taking the centers of the six faces of a cube), the regular dodecahedron and the regular icosahedron (compare [**15**] and [**32**]).

If we pass to higher dimensions, then purely algebraic considerations show that for every $n \geq 4$ one can construct a *hypercube* given by all $\mathbf{x} \in \mathbb{R}^n$ whose coordinates lie between 0 and 1, an *n-simplex* which is analogous to an equilateral triangle or regular tetrahedron, and a third object which is dual to the hypercube, with vertices given by the centers of the faces of the hypercube; such objects are analogous to the regular octahedron in 3 dimensions. Further information on these figures can be found in either **[32]** or **[15]**.

One immediate question is whether there are any other examples, and this was answered by results of Ludwig Schläfli **[94]** which date back to the mid-nineteenth century. In particular, he showed that there are *three additional examples* if $n = 4$, but *no additional examples* if $n \geq 5$. All but one of the examples for $n = 4$ are analogs of Platonic solids (again see **[32]** or **[15]**).

The illustrates the earlier comment about simplifications for sufficiently large dimensions; if we agree that the 2- and 3-dimensional cases are understood, then we see that the 4-dimensional case is more complicated than the 3-dimensional situation and in all dimensions $n \geq 5$ there is a uniform pattern of behavior which is simpler to describe than in either dimension 3 or 4.

Similar patterns in algebra. Such patterns also arise very often in group theory. For example, for each integer n consider the alternating group A_n of all even permutations on n letters. A basic result of group theory states that A_n has no nontrivial normal subgroups for all n except $n = 4$. For lower values of n there is no room to squeeze in any nonzero proper subgroups at all, while if $n \geq 5$ there is enough room to perform certain algebraic constructions which force a nontrivial normal subgroup to be the whole group.

Still further examples arise at deeper levels of group theory. In each case there is a very systematic conclusion provided one avoids a finite list of exceptional values; however, in general the latter are not contained in $\{1, 2, 3, 4\}$. For example, one can consider the automorphism group of the symmetric group Σ_n on n letters; one natural question is whether this group has automorphisms besides the standard inner automorphisms; in this case there are no other automorphisms unless $n = 6$, in which case there is an additional "outer" automorphism (for example, see **[88]**). Another illustration of systematic behavior with finitely many exceptions is the classification of compact simply connected Lie groups, which can be written down very directly provided a numerical invariant called the **rank** is greater than 8 **[4]** (a quick but accurate summary is available online at http://en.wikipedia.org/wiki/Compact_Lie_Group), and yet another such pattern is the classification of finite simple groups (see **[109]** for a summary and **[102]** for a more detailed discussion; this result involves 26 exceptional or *sporadic* examples — the orders of the latter are often astronomical, so the notion of "sufficiently large" is not in the very small ranges we have seen thus far).

Counterparts in geometric topology. Here is a basic question that is simple to formulate:

> *For a fixed value of n, which finite abelian groups can arise as the fundamental groups of compact (unbounded) n-manifolds?*

If $n \leq 2$ one can answer this using the well-known classification theory for manifolds in these dimensions (*e.g.*, see [**70**] for the 2-dimensional case); no finite groups can be realized if $n = 1$, and only finite groups of orders 1 and 2 can be realized if $n = 2$. Fundamental results of C. D. Papakyriakopoulos in 3-dimensional topology [**85**] imply that a finite abelian group G can be realized if $n = 3$ if and only if G is cyclic (see Chapter 9 of [**31**] for further information). On the other hand, if $n \geq 4$ then by results of A. A. Markov (see [**74**] or [**75**]) one has enough geometric "room" to show that *every* finite abelian group can be realized.

Similar patterns appear elsewhere in geometric topology. Often one sees that everything can be described fairly systematically if $n \geq M$ for some small value of M (which is generally equal to 4, 5 or 6), and for all sufficiently small values of n (usually $n \leq 2$) everything is fairly well understood but usually for entirely different reasons. In particular, if $n = 1$ everything is usually extremely straightforward (for example, see the relevant sections of [**35**]), and our understanding geometric topology in dimension 2 is fairly complete based upon advances from the first part of the twentieth century (compare [**70**], [**95**], or [**112**]). If $n = 3$, there are many new phenomena to consider (including some highly pathological ones as in [**7**] or [**93**] in addition to new regular patterns discussed in [**31**] and [**79**]), but it appears that 3-dimensional topology will be in a fairly definitive (but still incomplete) form within the next ten years.

As in the case of regular polyhedra (but for entirely different reasons), many basic phenomena in geometric topology become much easier to analyze if $n \geq 5$, As noted in a survey article by L. Siebenmann [**99**] several breakthroughs involving work from the nineteen forties to seventies have laid a very solid foundation for studying n-manifolds with a few loose ends remaining if $n = 5$ (see [**40**] for additional information; some later developments are covered in [**87**]). The results in [**99**] and [**40**] also imply that some basic results in higher dimensions cannot be extended to dimensions 3 and 4 (see [**98**]) Our present understanding of the case $n = 4$ is still only partial despite some revolutionary advances during the past three decades, particularly in the work of M. H. Freedman (see [**24**] and [**25**]) and S. K. Donaldson (see [**16**], [**17**], [**18**]); when R. Kirby compiled a list of open questions in 4-dimensional topology during the past decade [**41**], the result was a massive work of more than 350 pages. A good qualitiative description of the situation is given at the beginning of of A. Scorpan's long and very readable survey of 4-dimensional topology [**97**]: Dimension 4 has enough room for wild things to happen, but not enough room to tame and undo them.

REMARK. Since $n = 4$ is exceptional in both geometric topology and the structure of alternating groups, it seems worthwhile to stress that the similarities are qualitative and (presumably) the appearance of the same number 4 in both contexts is basically coincidental.

Much of Terry's mathematical work has been devoted to issues involving the relation of 4-manifold theory to the theory of manifolds in higher dimensions. I shall concentrate on two themes runing through many of his papers; the first mainly involves work up to the early nineteen eighties, and the second mainly involves work after that point.

Higher dimensional shadows: Stabilization and bisection

We have already noted one basic fact from higher dimensional topology which in fact holds for all $n \geq 4$ (all finite abelian groups arise as fundamental groups of compact n-manifolds). During the nineteen sixties it was known that reasonably simple modifications of certain other basic results for $n \geq 5$ were also true if $n = 4$, and one recurrent (but often unstated) motivation for much of the research during the sixties and seventies was to see how much insight into 4-dimensional topology could be obtained using the methods and results from higher dimensions (*cf.* **[72]**).

We shall be particularly interested in the following problem, which is important for its own sake and has many far-reaching implications throughout the topology of manifolds:

Cylinder recognition question. *Suppose that we are given a compact connected unbounded n-manifold M^n. Can one describe elementary criteria under which a topological space X is equivalent to the cylinder $M^n \times [0,1]$?*

If $n = 1$ this question has a very elegant answer given by the classical theory of surfaces. The first step is to generalize the concept of n-manifold to include *manifolds with boundaries*. For example, the unit disk in $\mathbb{R}^n$ should be an n-manifold whose boundary is the $(n-1)$-dimensional unit sphere, and a **standard cylinder** $M^n \times [0,1]$ should be an $(n+1)$-manifold whose boundary is two disjoint copies of M; more generally, an $(n+1)$-manifold with boundary W will then have a closed subset ∂W (called the *boundary* of W) such that ∂W is an n-manifold without boundary and the *interior* $W - \partial W$ is an $(n+1)$-manifold without boundary. More information on manifolds with boundary can be found in the standard textbooks by S. Lang **[46]** and M. Hirsch **[33]**.

Standard results in classical surface theory (see **[70]**) imply that *a compact connected 2-manifold with boundary W is topologically equivalent to the standard circular cylinder $S^1 \times [0,1]$ if and only if*

(i)**:** *the boundary of W has two components, say V_0 and V_1,*

(ii)**:** *the inclusion of either boundary component is a homotopy equivalence.*

More generally, manifolds with boundary that satisfy these properties are called *h-cobordisms*, and the following *h-cobordism Theorem*, which was shown by S. Smale [**101**] around 1960, is one of the cornerstones of high-dimensional geometric topology. The standard source for the proof in the category of smooth manifolds is Milnor's book [**77**]; the first proof in the topological case was given a few years later by E. H. Connell [**14**] and predates the results presented in [**40**].

THEOREM. Let $n \geq 5$, and let W be a simply connected compact $(n+1)$-manifold with boundary $V_0 \amalg V_1$ such that conditions (i) and (ii) above are satisfied. Then W is topologically equivalent to the cylinders $V_0 \times [0,1]$ and $V_1 \times [0,1]$.

This result extends to manifolds with free abelian fundamental groups, but it does not extend to the general case. Instead, one has the following result, known as the *s-cobordism Theorem* [**43**] (original sources include [**89**] and [**40**]):

THEOREM. Let $n \geq 5$, and let W be a connected compact $(n+1)$-manifold with boundary $V_0 \amalg V_1$ such that conditions (i) and (ii) above are satisfied. Then W is topologically equivalent to the cylinders $V_0 \times [0,1]$ and $V_1 \times [0,1]$ if and only if a Whitehead torsion invariant $\tau(W, V_0)$ in the algebraically defined Whitehead group $\mathrm{Wh}(\pi_1(V_0))$ is equal to zero.

Elements of the Whitehead group are represented by invertible matrices over a certain ring associated to $\pi_1(M)$, and the Whitehead torsion can be defined entirely in terms of algebraic topology (see [**13**]); the Whitehead group is trivial if $\pi_1(V_0)$ is a free abelian group, and it follows from our previous remarks that the *s*-cobordism theorem is also true if $n = 1$; thus the result is true provided $n \neq 2, 3, 4$. It is not known whether the result remains true for arbitrary topological manifolds if $n = 4$, but the analogous result for smooth 5-dimensional *h*-cobordisms was shown to be false in the nineteen eighties by S. Donaldson [**18**]. If a basic statement about 3-manifolds known as the *Thurston Geometrization Conjecture* [**80**] is true (as most workers in the area expect), then the *s*-cobordism Theorem will also hold if $n = 2$, but if $n = 3$ then there are *s*-cobordisms that are not cylinders (the first examples are described in [**11**]). Finally, we should note that

> *the topological h-cobordism Theorem for* simply connected *manifolds is true in* EVERY *positive dimension*.

If $n = 4$ this follows from the work of Freedman [**25**] in the nineteen eighties, if $n = 2$ this follows from the recent solution of the 3-dimensional Poincaré Conjecture by G. Perelman [**80**], and if $n = 3$ this follows by combining Perelman's result with certain parts of Freedman's work.

The techniques which prove the *s*-cobordism theorem yield weak analogs of the latter if $n = 4$ by results of D. Barden [**5**] and C. T. C. Wall [**107**]. In particular, Wall's results are part of a general pattern.

Many basic results concerning manifolds of dimension ≥ 5 *have "stabilized" analogs in dimension* 4*.*

Roughly speaking, the advantage of stabilization is that it provides some extra room in which to make key constructions. The alternating groups A_n provide a simple but fundamentally important example of **algebraic stabilization**. One crucial step in proving the simplicity of A_n for $n \geq 5$ is showing that it is generated by cyclic permutations of three letters. If $n = 4$, then there is not enough room in A_4 to express some even permutations in this manner, but if one stabilizes by passing to A_5 then there is enough working room to write an even permutation of four letters as a product of such cyclic permutations.

There are several ways of viewing the **geometric stablization** process. Given a manifold M^n, one can adopt the viewpoint of algebraic geometry and "blow up" a finite number of points topologically in a suitable manner (the mental picture is the nonexplosive inflation of a balloon). More precisely, one finds a manifold N^n and a map $f : N^n \to M^n$ such that f is a homeomorphism (or diffeomorphism of smooth manifolds) on some set $f^{-1}[A]$, where A is a finite subset of M, and the inverse images of points in A all have some prescribed topological type (the classical process of blowing up points is described in detail, with extensive illustrations, on pages 286–290 of [**97**]). For one of Wall's result when $n = 4$, these exceptional sets are all homeomorphic to unions of two 2-dimensional spheres with exactly one point in common; alternatively, one can view these stabilizations as connected sums [**92**] with finitely many copies of $S^2 \times S^2$, and if there are k exceptional points we shall say that N^4 is a k-fold stabilization of M^4 by $S^2 \times S^2$.

One then has the following analog of the s-cobordism theorem when $n = 4$ for finite stabilizations by $S^2 \times S^2$ (see [**107**]).

THEOREM. Let $n = 4$, and let W be a connected compact smooth $(n+1)$-manifold with boundary $V_0 \amalg V_1$ such that conditions (i) and (ii) above are satisfied (hence W is an h-cobordism). Then there is some $k \geq 0$ such that the k-fold stabilizations of V_0 and V_1 by $S^2 \times S^2$ are diffeomorphic.

There are also several interesting and important results involving stabilizations by other 4-manifolds (*e.g.*, see [**73**] or page 151 of [**97**]), but for our purposes it will suffice to consider only stabilizations by $S^2 \times S^2$.

Numerous other results involving stabilizations by $S^2 \times S^2$ were obtained by many topologists during the nineteen sixties and seventies (for example, [**9**], [**10**], [**23**], and [**100**]), and Terry was also one of the contributors ([**51**], [**52**], [**53**], [**54**], [**55**], [**58**]). In some instances his work also shed light on related questions about higher dimensional manifolds; for example, his paper with A. Hatcher [**30**] proves a strong analog of Wall's result in higher dimensions and also gives a very nice 1-parameter analog. The latter can also be viewed as one aspect of Terry's work on fiber bundles (see **[48]**, **[49]**, **[52]**), which contains several interesting results but was not covered in my talk at the miniconference due to time constraints.

One of the more important and easily stated contributions in Terry's work is his extension of Wall's result to a *stabilized h-cobordism theorem* **[55]** which gives deeper insight into the structure of a 5-dimensional h-cobordism and shows that such an object becomes a product if one performs a 1-parameter version of the stabilization construction described above.

Twisted doubles and open books. Certain other results from around this time concern special structures on manifolds that are highly significant, both for the insights they yield into the structure theory of manifolds and for their usefulness in studying various sorts of flexible geometrical structures on manifolds. The underlying concept is given as follows:

DEFINITION. Let W be a manifold with boundary V, and let $h : V \to V$ be a homeomorphism. The **twisted double** $W \cup_h W$ is the space formed by taking two disjoint copies W_1 and W_2 of W and gluing them together such that each point $x \in \partial W_1 \cong V$ is identified to the corresponding point $h(x) \in W_2 \cong V$.

A result of M. Brown (the Collar Neighborhood Theorem **[8]**) implies that $W \cup_h W$ is a topological manifold without boundary. Furthermore, if W has a smooth structure and h is a diffeomorphism, then the twisted double has a smooth structure, and frequently other special properties of h translate into corresponding special properties of $W \cup_h W$.

For each positive integer n, the n-dimensional sphere $S^n \subset \mathbb{R}^{n+1}$ has a standard description as a twisted double, where W is the unit disk and the images of W_1 and W_2 correspond to northern and southern hemispheres, given by points for which the last coordinate x_n is either nonnegative or nonpositive. Of course, the common boundary corresponds to the equator, which is merely S^{n-1}, and in this case one can take h to be the identity map (*i.e.*, the sphere is an *untwisted* double). More generally, if W is any manifold with boundary we can form the untwisted double

$$\mathcal{D}(W) \ = \ W \cup_{\text{identity}} W \, .$$

The only compact 1-manifold (without boundary) is the circle S^1, and we have seen that the latter is an untwisted double. In the case of 2-manifolds, the theory of surfaces yields three important facts about twisted doubles.

Dependence on h**:** Different choices of h generally yield manifolds that are not homeomorphic (or even homotopy equivalent). For example, the 2-dimensional torus is homeomorphic to the untwisted double of $S^1 \times [0,1]$, but if one forms the twisted double using the homeomorphism of

$$\partial S^1 \times [0,1] \ = \ S^1 \times \{0\} \ \cup \ S^1 \times \{1\}$$

which sends (x, y, ε) to $(x, (-1)^{\varepsilon} y, \varepsilon)$, then one obtaines the Klein bottle [**110**].

Most surfaces are doubles: A compact unbounded 2-manifold is (homeomorphic to) a twisted double if and only if it is NOT homeomorphic to the real projective plane $\mathbb{RP}^2$ (see [**81**], p. 372), and every **oriented** surface is in fact (homeomorphic to) an untwisted double. (See pp. 234–236 of [**29**] for a discussion of orientations.)

Converse statement: The manifold $\mathbb{RP}^2$ is not (homeomorphic to) a twisted double.

In 3-dimensional topology, twisted double structures always exist (*cf.* Chapter 2 of [**31**]); the standard examples are called *Heegaard splittings* because the existence of such structures on arbitrary compact unbounded 3-manifolds was discovered (in the smooth case, at least) by P. Heegaard just before the end of the nineteenth century.

What happens in higher dimensions? There are systematic infinite families of manifolds in all even dimensions ≥ 2 which cannot be realized as twisted doubles (for example, the even- dimensional complex projective spaces $\mathbb{CP}^{2n}$, where $n \geq 1$; these are defined on pp. 90–93 of [**6**]). On the other hand, in **odd** dimensions such structures always exist, and for sufficiently large odd dimensions this was shown in the unpublished doctoral dissertations of D. Barden [**5**] and J. P. Alexander [**2**]. In the early nineteen seventies, H. E. Winkelnkemper [**113**] and (independently) I. Tamura ([**104**] and [**105**]) described a very special type of twisted double structure called an *open book decomposition* [**27**], which has proven to be extremely useful in the theory of foliations on manifolds (see [**47**]) and also in recent work on contact geometry. A detailed discussion of these matters would require substantial digressions (see the survey by Winkelnkemper [**114**] for more information, and see [**115**] for a purely algebraic approach to the 3-dimensional case). For our purposes it will suffice to state the Open Book Theorem for simply connected manifolds as follows:

THEOREM. Let $n \geq 6$, and let M be a simply connected compact smooth n-manifold (without boundary). Then M has an open book decomposition if and only if either n is NOT divisible by 4 or if n is divisible by 4 and an integer valued invariant called the **signature** of M (see [**78**]) is equal to zero.

Terry's results establish a nontrivial extension of the Open Book Theorem to arbitrary odd-dimensional manifolds in dimensions ≥ 7 [**56**], and in another paper the existence of twisted double structures for 5-manifolds is shown [**57**]. If one combines these results with the previous remarks on low-dimensional cases, then one has the following unified conclusion.

THEOREM. If n is an odd positive integer, then every compact n-manifold can be realized as a twisted double.

In addition to its intrinsic interest and applications, this result reflects a relationship between $2k$-manifolds and $(2k+1)$-manifolds that plays a central role in the classification theory of manifolds; for example, in Wall's theory of nonsimply connected surgery **[108]** one has a parallel relationship between the surgery obstruction groups in dimensions $2k$ and $2k+1$ (in more technical terms, the common thread is that $(2k+1)$-dimensional objects correspond to automorphisms of $2k$-dimensional objects that are represent zero in some appropriate group of equivalence classes).

Incidentally, if one has a nonsimply connected $2k$-manifold, the existence of an open book structure implies additional numerical conditions beyond the vanishing of the signature, and further work is needed. Subsequent work of F. Quinn **[86]** gives a definitive formulation of the necessary conditions and shows that they are also sufficient for the existence of open book decompositions on arbitrary $2k$-dimensional compact manifolds if $k > 2$.

From the preceding discussion it is clear that Terry's work on some of these problems during the nineteen seventies is closely related to the research of several other topologists, and in fact there are cases of overlapping, independently obtained results; we shall not try to tabulate such instances for the sake of relative brevity (in particular, there is no conscious effort to ignore or denigrate the contributions of others). In cases where there is overlap with the contributions of others, usually Terry's work is particularly noteworthy because (i) he always added some fresh insights of his own, (ii) he was very effective at writing up his results in a clear and thorough form. At the time, geometric topology was an extremely active field with an enormous amount of competition, and in the rush for recognition many pieces of work were written up too hastily (or never even published!) and did not always meet the high standards for mathematical writing that are implicit in Terry's papers (related concerns are stated emphatically and but perhaps excessively in **[83]**).

Stabilization revisited. The work of Terry described above was done during the nineteen seventies. However, during the nineteen eighties he wrote one more paper on the subject, and it reflected some important breakthroughs that had taken place in 4-dimensional topology during the intervening years and yielded the following results on 5-dimensional h-cobordisms.

THEOREM. Let W be a simply connected compact 5-manifold with boundary $V_0 \cup V_1$ that is an h-cobordism. Then W is topologically equivalent to the cylinders

$V_0 \times [0,1]$ and $V_1 \times [0,1]$. However, there are examples of smooth simply connected compact 5-dimensional h-cobordisms that are NOT smoothly equivalent to cylinders because V_0 and V_1 are not diffeomorphic.

The first part of this follows from the work of M. Freedman [**25**], while the second follows from the work of S. Donaldson [**18**]. Further work of many topologists and geometers yielded large families of examples similar to Donaldson's (see [**26**] for a survey of the earliest examples, and [**97**] for an extensive survey of work through the middle of 2004), and one particularly noteworthy family involves a class of objects related to algebraic geometry which are called *Dolgachev surfaces* (see pp. 310-316 of [**97**]). By Wall's earlier work, if such 4-manifolds are h-cobordant then certain stabilizations of them are diffeomorphic, and the central question in [**65**] concerns the number of stabilizations that are needed. We know that this number must be positive, and [**65**] gives simple conditions on Dolgachev surfaces for which one or two stabilizations will suffice. In some cases this yielded new classification theorems for smooth h-cobordisms between nondiffeomorphic Dolgachev surfaces.

The preceding results reflect the emergence of gauge theory as an important tool for studying questions about smooth 4-manifolds, and as such they provide a natural transition to the second theme in Terry's work to be discussed here.

Gauge theory and surfaces in 4-manifolds

Gauge theory was first studied by physicists, and in the late nineteen seventies mathematicians began to discover some striking results on the relationship of gauge theories to geometry [**3**]. In the early nineteen eighties the potential of gauge theory to be a powerful tool in topology became undeniably obvious in monumental work of Donaldson (see [**16**] and [**17**]), including his totally unanticipated discovery of smooth manifolds that are homeomorphic to ordinary Euclidean 4-space but not smoothly equivalent to it. We shall not attempt to discuss the details of gauge theory here, for our emphasis will be on its applications to topological questions in Terry's work during the nineteen eighties and nineties. Much of the work involves questions regarding smooth nonsingular surfaces embedded in a smooth 4-manifold.

Questions about embedded surfaces play important roles in the structure theory of n-manifolds if $n \neq 1$ (in which case everything can be worked our directly). The reasons for this may be summarized as follows.

$n = 2$**:** The quickest justification is that "a surface IS a surface."

$n \geq 5$**:** Fundamental methods due to H. Whitney [**111**] show it is possible to construct embedded surfaces which can be used to replace certain geometric configurations with much simpler ones (in fact, this property essentially characterizes topological manifolds in sufficiently large dimensions [**87**]).

$n = 3$**:** The work of Papakyriakopoulos [**85**] (see also Chapter 4 of [**31**] and later results of other topologists (*e.g.*, W. Haken [**28**], F. Waldhausen [**106**], K. Johannson [**38**], W. Jaco and P. Shalen [**37**]) show that one can often detect embedded surfaces from relatively weak algebraic data, and these surfaces can often be used to cut a 3-manifold into relatively manageable pieces.

$n = 4$**:** Under suitable restrictions, the work of Freedman yields *locally flat* topological surfaces (see [**93**], p. 33) which behave like Whitney's surfaces when $n \geq 5$.

In several respects, our understanding of 4-manifolds is limited by our lack of understanding embedded surfaces. The first example of a breakdown was discovered by M. Kervaire and J. Milnor around 1960 [**39**], and it concerns smoothly embedded copies of S^2 in $S^2 \times S^2$. Up to homotopy, continuous mappings from S^2 to $S^2 \times S^2$ are classified by an ordered pair of integers known as the *degrees of the projections onto the factors* (see Hatcher's book [**29**] for the concept of degree). It is not difficult to show that a degree pair (a, b) can be realized if either a or b is equal to 0 or ± 1 (the other can be arbitrary). In contrast, the result of Kervaire and Milnor showed that the pair $(2, 2)$ cannot be realized by a smoothly embedded sphere (however, one can realize every pair by a **piecewise smooth** embedded sphere). Several further results on nonembeddings of surfaces in 4-manifolds were obtained by others before the emergence of gauge theory in the early nineteen eighties Their methods and results were extended by others (*e.g.*, see W.-C. Hsiang and R. Szczarba [**36**]; in a somewhat different direction see [**12**]). One early application of gauge theory was a complete determination of the pairs (a, b) that could be realized by a theorem first published by K. Kuga [**45**] (see also [**103**]):

THEOREM. A pair of integers (a, b) is realized by a smooth embedding of S^2 into $S^2 \times S^2$ if and only if one of a or b is equal to 0 or ± 1.

This result also illustrates one of the many ways in which the structure theories of topological and smooth 4-manifolds differ, for it is known that many ordered pairs of integers (a, b) can be realized by locally flat topologically embedded spheres; if a and b are nonzero and relatively prime, this is true by Corollary 1 of [**24**], and the results of [**71**] provide considerably more detailed information for other ordered pairs. Incidentally, there is a much closer relationship between smooth and locally flat embeddings in higher dimensions (*cf.* Theorem 2 in [**96**]).

More generally, for every compact, unbounded, smooth, simply connected 4-manifold M and every continuous mapping from S^2 to M, one can assign a *multidegree* — *i.e.*, a sequence of k integers $(d_1, \cdots, d_k)$, where k depends upon the underlying topological space of M — which generalizes the notion of degree pair when $M = S^2 \times S^2$, and one can then ask which multidegrees are realized by smooth embeddings of S^2.

For the most basic choices of M, there are relatively short lists of multidegrees which can be realized by well-known constructions. The preceding theorem implies that no others can be realized if $M = S^2 \times S^2$, and a similar conclusion holds for the complex projective plane $\mathbb{CP}^2$. In **[64]** Terry considered some of the next few cases from a somewhat different viewpoint involving results of R. Fintushel and R. Stern **[21]**, and he obtained new results for the manifolds $M(1,1)$ and $M(1,2)$ given by taking connected sums of $\mathbb{CP}^2$ with 1 or 2 copies of the oppositely oriented manifold $\overline{\mathbb{CP}^2}$ (in the previously used language of algebraic geometry **[26]**, this corresponds to blowing up one or two points). The results for $M(1,1)$ are complete, while the results for $M(1,2)$ apply to exactly half of the possible multidegrees.

Several other papers by Terry address further questions involving the methods of Fintushel and Stern as well as the applications of their techniques. To describe this work, we first recall that gauge theory analyzes topological questions by first constructing certain associated "moduli spaces of instantons" whose elements are equivalence classes of appropriate types of geometric structures, and then studying the properties of such spaces. Especially in the early work, compactness questions involving such spaces played a fundamental role, and a pair of Terry's papers (**[6]** and **[66]**) — one of which was joint with Fintushel — show that earlier compactness results of Fintushel and Stern **[22]** could be generalized extensively.

In some related papers such as **[63]** and **[68]**, Terry considered another question arising from work of Fintushel and Stern **[21]**. It is known that every compact 3-manifold M^3 bounds a smooth compact manifold W^4, and a central problem in low-dimensional topology is to make W^4 as simple as possible. The results of **[63]** yield lower limits on the amount of simplification that can be done for certain fundamental 3-manifolds called *Seifert homology* 3-*spheres* (see **[84]**), and the precise conclusions are stated in terms of certain trigonometric expressions. Terry extended the the earlier results of **[21]** on such questions in two ways, using his compactness results and analyzing the trigonometric expressions by number-theoretic methods from work of W. Neumann and D. Zagier **[82]**.

An entirely different class of contributions appear in **[62]**, which consider smooth embeddings of the real projective plane $\mathbb{RP}^2$ into simply connected 4-manifolds. Terry's interest in such issues was already evident in earlier papers about embeddings of $\mathbb{RP}^2$ in S^4 (**[59]**, **[60]**, **[61]**). In general, if we are given a smooth embedding of $\mathbb{RP}^2$ into a simply connected 4-manifold, then there is an integer called the *twisted Euler number* which describes small neighborhoods of the embedded submanifold, and the goal of **[62]** is to describe the possible Euler numbers for certain choices of M. When $M = S^4$, the answer to this question was found in the late nineteen sixties **[76]**. Using the methods described above for the given 4-manifolds, Terry proves a numerical congruence mod 4 and determines a lower bound for the twisted Euler number in a substantially more general situation; there is also a natural conjecture for the upper bound, but this remains an open question.

In all these cases, Terry's results yielded strong new results on questions that had seemed totally beyond reach in 1980 (the beginning of the decade when the papers were written). Equally important, his work was also significant because it provided models for applying the recently developed machinery of gauge theory in a systematic manner that did not require extensive work with the deep and complicated details of gauge theory itself. Terry's work marked a major step in reducing many topologists' apprehensiveness about the powerful and effective new methods that had already made such an enormous impact on the subject.

The results of **[63]** on Seifert homology 3-spheres led to some highly original joint work with S. Kwasik **[44]** on symmetries of certain compact 4-manifolds with boundary. References are given for several specialized terms which appear in the statement of the main result.

THEOREM. There are infinitely many finite group actions **[91]** on compact, smooth, contractible 4-manifolds with boundary W^4 (see **[81]**, p. 330) such that

(i) each action is free **[90]** on the complement of a single fixed point in the interior of W^4,

(ii) the restrictions of each action to the interior and boundary are smoothable,

(iii) none of these actions are globally smoothable.

The results of **[44]** also yielded some new implications about the differences between the structure theories for smooth and topological 4-manifolds which are unique to dimension 4.

During the nineteen nineties, gauge theory underwent some major changes that were motivated by work in theoretical physics due to N. Seiberg and E. Witten (*e.g.*, see **[19]**). This new and improved version of gauge theory depends strongly on geometric properties called **Spin** and **Spin**^{c} structures, which are essentially higher order analogs of orientations on a manifold. In **[1]**, written jointly with D. Acosta, the role of these conditions in the case of 4-manifolds is analyzed carefully, and the result is a clear description of issues which, as noted in the summary of **[1]** in *Mathematical Reviews*, "can be confusing even to the initiated."

Finally, no discussion of Terry's papers on gauge theory would be complete without mentioning two excellent and very highly regarded survey articles of results on smoothly embedded surfaces in compact simply connected 4-manifolds. The first of these [**68**] deals with embedded spheres, while the second [**69**] concerns more general oriented surfaces and lower bounds for a basic numerical invariant (the **genus**) of such a smoothly embedded surface.

Closing remarks

Terry Lawson has worked productively on a variety of problems that really matter in geometric topology, he has been willing and able to move with the subject, and he has done an excellent job of presenting both his results and related material. Each of these qualities is indispensable for the successful development of a mathematical subject, and I have very much appreciated Terry's contributions in all these directions.

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On the homology cobordism group

Nikolai Saveliev

Introduction

A *homology sphere* Σ is an oriented 3-manifold such that $H_*(\Sigma) = H_*(S^3)$. According to the recently solved Poincaré conjecture, the only simply connected homology sphere is S^3. Other examples of homology spheres include Brieskorn homology spheres $\Sigma(p,q,r) = \{ (x,y,z) \in \mathbb{C}^3 \mid x^p + y^q + z^r = 0 \} \cap S^5$ with $\gcd(q,r) = \gcd(p,r) = \gcd(p,q) = 1$, and more generally, Seifert fibered homology spheres $\Sigma(a_1, \dots, a_n)$ with $\gcd(a_i, a_j) = 1$ for all $i \neq j$.

A *homology cobordism* between Σ_0 and Σ_1 is an oriented smooth cobordism W with $\partial W = -\Sigma_0 \cup \Sigma_1$ such that the induced maps $H_*(\Sigma_i) \to H_*(W)$ are isomorphisms for $i = 0, 1$.

The *homology cobordism group* Θ^3 is the abelian group of the equivalence classes of homology cobordant homology spheres with connected sum as operation. The *Rohlin invariant* is the homomorphism $\rho : \Theta^3 \to \mathbf{Z}/2$ defined by $\rho(\Sigma) = \operatorname{sign} W/8 \pmod 2$, for any choice of smooth compact spin manifold W with $\partial W = \Sigma$.

Here is a list of what is known about the homology cobordism group and the Rohlin invariant:

- ρ is surjective, in fact, $\rho(\Sigma(2,3,5)) = 1$ because the singularity at zero of $x^2 + y^3 + z^5 = 0$ has a resolution with the intersection form E_8.

- $\Sigma(2,3,5)$ has infinite order in Θ^3. This follows from Donaldson's diagonalization theorem **[5]**, whose proof relies on the study of ASD moduli spaces on closed 4-manifolds (recall that a connection A is called ASD if $F_A^+ = 0$, where $F_A^+ = F_A + *F_A$ stands for the self-dual part of its curvature).

- Fintushel and Stern **[7]** studied ASD moduli spaces on closed orbifolds and came up with an invariant $R(a_1, \dots, a_n) \in \mathbf{Z}$ having the property that $R(a_1, \dots, a_n) > 0$ implies that $\Sigma(a_1, \dots, a_n)$ has infinite order in Θ^3.

- Fintushel and Lawson **[6]** included $R(a_1, \dots, a_n)$ into a family of invariants $R(a_1, \dots, a_n; \ell)$ with the same significance.

- Furuta **[9]** showed that Θ^3 is infinitely generated; in fact, $\Sigma(2,3,6m-1)$, $m \geq 1$, all have infinite order and are linearly independent over $\mathbf{Z}$.

• Seiberg–Witten techniques on orbifolds and the $\bar{\mu}$–invariant of Neumann and Siebenmann were used by Fukumoto–Furuta **[8]** and Saveliev **[17]** to prove the following result. Suppose that Σ is homology cobordant to a $\Sigma(a_1, \ldots, a_n)$. If $\rho(\Sigma) = 1$ then Σ has infinite order in Θ^3 (similar techniques were used earlier by Danny Acosta **[1]** to study embedded surfaces in 4–manifolds).

• Frøyshov (unpublished) showed that there are homology spheres which are not homology cobordant to any $\Sigma(a_1, \ldots, a_n)$. The homology sphere $\Sigma(2,9,17) \# (-3)\,\Sigma(2,3,5)$ is one such example.

It is still not known if Θ^3 has any torsion, and the last two results on the above list demonstrate that Seifert fibered homology spheres are of limited use in answering this question. This torsion problem is of great importance because of its link to the triangulation conjecture.

THEOREM. *(Matumoto* **[13]** *and Galewski–Stern* **[11]***) Every closed topological n–manifold, $n \geq 5$, is homeomorphic to a simplicial complex iff there is a homology sphere Σ of order two in Θ^3 such that $\rho(\Sigma) = 1$.*

Below, I will outline our joint work with Daniel Ruberman on two possible approaches to showing that all order two homology spheres in Θ^3 must have trivial Rohlin invariant.

An approach via Donaldson theory

Let W be a homology cobordism from a homology sphere Σ to itself, and $X = W/\sim$ the furled up manifold obtained by identifying the two copies of Σ in the boundary of W by the identity map.

Let $D(X)$ be a (properly defined) signed count of the conjugacy classes of irreducible representations $\pi_1(X) \to SU(2)$. If the number of such conjugacy classes is infinite, choose a metric on X and note that all ASD connections in a trivial $SU(2)$ bundle over X are flat, so that the holonomy provides an identification between the ASD moduli space and the $SU(2)$ representation variety. We define $D(X)$ as a signed count of irreducible solutions of the perturbed ASD equation $F_A^+ = \varepsilon$. Of course, this is just a version of the degree zero Donaldson polynomial.

THEOREM. *(Furuta–Ohta* **[10]***, Ruberman–Saveliev* **[16]***) $D(X)$ is well defined and $D(X) = 0 \pmod 4$.*

CONJECTURE. *(Furuta–Ohta* **[10]***) Let $\lambda_{\mathrm{FO}}(X) = D(X)/4$ then $\lambda_{\mathrm{FO}}(X) = \rho(\Sigma) \pmod 2$.*

THEOREM. *(Ruberman–Saveliev* **[16]***) The above conjecture is true for the mapping tori X of all orientation preserving diffeomorphisms $\tau : \Sigma \to \Sigma$ of finite order (in fact, λ_{FO} in this case equals the equivariant Casson invariant, compare with* **[4]***, which can be given by an explicit formula in terms of classical invariants).*

Our general approach to the conjecture is via torus surgery on X. Despite some serious progress we have made to date, see survey [**14**], a lot remains to be done.

The relevance to the 2–torsion problem in Θ^3 is as follows. Let us suppose that the Furuta–Ohta conjecture is true. Let V be a homology cobordism with $\partial V = \Sigma \cup \Sigma$ (compare with $\partial([0,1] \times \Sigma) = -\Sigma \cup \Sigma$) and glue V to $-V$ in such a fashion that $X = -V \cup V$ is closed and has a free orientation reversing involution. Then $\lambda_{\mathrm{FO}}(X) = \lambda_{\mathrm{FO}}(-X) = -\lambda_{\mathrm{FO}}(X)$ (this last equality is another conjecture) hence $\rho(\Sigma) = \lambda_{\mathrm{FO}}(X) = 0 \pmod 2$.

An approach via Seiberg–Witten theory

Let W be a spin cobordism (not necessarily a homology cobordism) from Σ to itself, and $X = W/\sim$ the furled up manifold. Choose a smooth compact spin manifold Z with $\partial Z = \Sigma$ and consider the end–periodic manifold $Z \cup W \cup W \cup \ldots$ as in Taubes [**18**].

THEOREM. *(Ruberman–Saveliev* [**15**]*) Vanishing of* $\operatorname{sign} X$ *is a necessary and sufficient condition for the Dirac operator* D *on* $Z \cup W \cup W \cup \ldots$ *to be Fredholm for a generic metric on* X.

The intended application of this result is as follows. Let V be a spin cobordism with $\partial V = \Sigma \cup \Sigma$ and consider the (non-orientable) furled up manifold $X' = V/\sim$ and its orientation double cover $X = -V \cup V$. Choose a metric g on X as provided by the above theorem, and define the (metric dependent) invariant

$$\tilde{\beta} = \operatorname{ind}_{\mathbb{C}} D^+(Z \cup (-V \cup V) \cup \ldots) + \frac{1}{8} \operatorname{sign} Z - \frac{1}{16} \operatorname{sign} V,$$

where D^+ is the chiral Dirac operator. Observe that $\tilde{\beta}$ is independent of the choice of Z (by excision property of index) and reduces modulo 2 to the *Cappell–Shaneson invariant* [**3**] (a diffeomorphism invariant)

$$\beta(X') = \rho(\Sigma) - \frac{1}{16} \operatorname{sign} V \pmod 2.$$

If the metric g on X is invariant with respect to the natural orientation reversing involution on X, we can write

$$\begin{aligned}\tilde{\beta} &= \operatorname{ind} D^+((Z \cup -V) \cup (V \cup -V) \cup \ldots) + \frac{1}{8} \operatorname{sign}(Z \cup -V) + \frac{1}{16} \operatorname{sign} V \\ &= \operatorname{ind} D^+(-Z \cup (V \cup -V) \cup \ldots) - \frac{1}{8} \operatorname{sign} Z + \frac{1}{16} \operatorname{sign} V = -\tilde{\beta}\end{aligned}$$

(we replaced $-Z \cup V$ with $-Z$ in the second line). Therefore, $\tilde{\beta} = 0$ and $\beta(X') = 0 \pmod 2$.

In particular, if V is a homology cobordism with $\partial V = \Sigma \cup \Sigma$ and there exists an invariant metric g as above then $\beta(X') = \rho(\Sigma)$ must vanish. Unfortunately, there is an obstruction to making g invariant, and this obstruction is precisely $\rho(\Sigma)$!

One way to fix the problem we are currently pursuing is to add a certain count of solutions of the Seiberg–Witten equations on X to the above $\tilde{\beta}$ to make it metric independent and see if the resulting invariant is still good enough to prove that $\rho(\Sigma) = 0$.

Metrics of positive scalar curvature

There is at least one instance when an invariant metric g as in the previous section can be found : this is when X' possesses a metric of positive scalar curvature. The above argument then exhibits $\beta(X')$ as an obstruction to the existence of a metric of positive scalar curvature on X'.

For example, let $S^1 \tilde{\times} S^3$ be the non-orientable S^3 bundle over S^1 and consider the manifold $X' = (S^1 \tilde{\times} S^3) \# K3$. By cutting X' along $\Sigma = S^3$, we calculate $\beta(X') = 1 \pmod 2$, hence X' cannot have metric of positive scalar curvature. Note that the orientation double cover of X' is $(S^1 \tilde{\times} S^3) \#_{22} (S^2 \times S^2)$ and hence admits a metric of positive scalar curvature. The first examples of this nature in dimension four were found by Claude LeBrun **[12]** using Seiberg–Witten theory.

Observe that X' is homeomorphic but not diffeomorphic to $(S^1 \tilde{\times} S^3) \#_{11} (S^1 \times S^2)$ because the latter has $\beta = 0$. Thus X' can be viewed as an exotic $(S^1 \tilde{\times} S^3) \#_{11} (S^1 \times S^2)$. In fact, Akbulut **[2]** and Fintushel and Stern constructed examples of exotic $(S^1 \tilde{\times} S^3) \#_k (S^1 \times S^2)$ for all $k \geq 1$. They are distinguished from the standard $(S^1 \tilde{\times} S^3) \#_k (S^1 \times S^2)$ by the non-vanishing β hence none of them has a metric of positive scalar curvature.

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Finite group actions on tori

Allan Edmonds

For Terry

Introduction

Tori of the form $T^n = \mathbb{R}^n/\mathbb{Z}^n$ constitute the most accessible family of aspherical manifolds, whose topology is largely determined by fundamental group, yet have many natural symmetries. We consider questions of the finite group actions on tori, addressable in quite fine detail, motivated by a low-dimensional perspective.

Simple examples of group actions arise from subgroups $G \subset \mathrm{GL}(n, \mathbb{Z})$ acting on $\mathbb{R}^n$ while preserving the integer lattice $\mathbb{Z}^n$. For example it follows that any finite group acts on some torus. The motivating question throughout is how nearly does an arbitrary action resemble such a linear model?

Actions on tori in dimensions less than four

In low dimensions geometry dominates topology. One naturally expects that every action is equivalent to a standard geometric model. One can work out all possible finite group actions on the 2-torus T^2. All are equivalent to standard actions that one naturally writes down in coordinates. Similarly, it has been shown directly that every free action of a finite group on the 3-torus is equivalent to a standard geometric action. This involves basic 3-manifold theory, in particular, the theory of Haken manifolds. There are exactly ten possible orbit spaces. This is due to Hempel **[5]** in the case of cyclic groups, to K.B. Lee et al. **[7]** for abelian groups, and to K.Y. Ha et al. **[4]** in the case of general finite groups.

Indeed, it would be a consequence of Thurston's Geometrization Theorem that any smooth, PL, or topological locally linear action of a finite group on T^3, free or not, is equivalent to a standard action. Of course there are topological actions with wildly embedded fixed point sets.

Linear Models

A *crystallographic group of rank n* is a uniform discrete subgroup π of the group $E(n)$ of rigid motions of $\mathbb{R}^n$. According to the classical Bieberbach theorems,

(1) A group is isomorphic to a crystallographic group of rank n if and only if it contains a normal free abelian group of rank n of finite index that is maximal abelian.
(2) In each dimension there are only finitely many isomorphism classes of crystallographic groups.
(3) Two crystallographic groups are isomorphic if and only if they are conjugate in the affine group $A(n)$.

Thus a crystallographic group Γ fits into an extension

$$0 \to \pi \to \Gamma \to G \to 1$$

where $\pi \approx \mathbb{Z}^n$ is maximal abelian and normal, the quotient G is finite, and acts faithfully on $\pi \approx \mathbb{Z}^n$. Note that there is an induced action of G on $T^n = \mathbb{R}^n/\mathbb{Z}^n$.

J. Wolf gives a full discussion in his book *Spaces of Constant Curvature* [**10**]. See also Thurston's book [**8**] on three-dimensional geometry for a topological perspective in low dimensions.

There are 17 crystallographic groups in dimension 2. This classification was certainly known, if not proved rigorously, in the 19th century. There are exactly 230 isomorphism classes of crystallographic groups in dimension 3, up to orientation preserving affine equivalence. There are exactly 32 finite subgroups of $O(3)$ that occur as images of crystallographic groups of rank 3, the so-called point groups. With improved algorithms and computer assistance, there are now known to be exactly 4783 crystallographic groups in dimension 4. In higher dimensions no one has succeeded in enumerating all crystallographic groups, although partial results do exist.

THEOREM (K.B. Lee-F. Raymond (1981) and probably others). *If G is a finite group acting on T^n, then there is a geometric realization of G acting effectively and isometrically on T^n, inducing the same action on homology.*

A given action of G on T^n gives rise to an extension Γ consisting of all lifts of all elements of G to the universal covering, which can be shown to satisfy the requirements for being isomorphic to a crystallographic group. When one has realized Γ as an actual crystallographic group, then there is an induced action on the torus obtained by quotienting out the subgroup of translations. It follows from Bieberbach's theorems that the induced action on the torus is unquely determined up to affine equivalence.

General Conjectures and Questions

Could it be that a finite group action on a torus T^n is equivalent to its linear model action? We have seen that this is true in dimensions ≤ 3, as long as the actions are locally linear.

This cannot be true in dimensions ≥ 5, since the fixed point set of G acting on T^n could then contain a non-torus, e.g. the connected sum of a torus and a suitable lens space (Y. W. Lee, unpublished, mentioned by Lee-Raymond [**6**]). It is not true in dimension 4 either, since one could connect sum the given action with an action on S^4 fixing a *knotted* 2-sphere.

The study of free actions on tori is essentially a question of the homeomorphism classification of the orbit spaces, which are themselves aspherical manifolds. The so-called Borel Conjecture states that if M^n and N^n are closed, aspherical, topological n-manifolds, then any homotopy equivalence $M^n \to N^n$ is homotopic to a homeomorphism.

This conjecture is true for tori, and more generally, flat manifolds, of dimensions at least five, by work of Farrell and Hsiang. The surgery-theoretic Farrell-Hsiang argument can be carried over to dimension four, since the fundmental groups in question are "good" in the sense of Freedman and Quinn. The result also holds in dimension three "'modulo the Poincaré Conjecture" by Waldhausen's theory of sufficiently large manifolds. Of course the Poincaré Conjecture is now generally accepted by the work of Perelman on Ricci flow.

It follows that a free action of a finite group on a torus is determined up to equivalence (and automorphisms of the group) by the the induced extension by $\mathbb{Z}^n$.

It remains to consider more closely how nearly an action with fixed points resembles its associated linear model. In this direction we also have an equivariant analogue of the Borel Conjecture: if M^n and N^n are closed, aspherical, topological n-manifolds, with actions of a finite group G, then any G-homotopy equivalence $M^n \to N^n$ is G-homotopic to a G-homeomorphism.

Connelly and Kosniewski [**1**] proved that it is true for an action of a finite group of odd order acting affinely on a torus with "small gaps" of at least 3 dimensions between fixed point sets included in one another (i.e., when one of the manifolds is as described). Connolly and Kosniewski ConnollyKosniewski1991 also gave example of non-rigidity for crystallographic groups.

Weinberger [**9**] has observed that although the (naive) equivariant Borel conjecture is most definitely false, it still serves as a good guiding principle. One needs to replace G-homotopy equivalence by a suitably stratified version, say requiring "isovariance" and something about the neighborhoods of fixed point sets.

Basic results about actions on tori

Homologically trivial actions. The following result is the core observation needed in the construction of linear model actions alluded to earlier.

THEOREM (Lee-Raymond [**6**]). *Let G be a finite group and let $0 \to C \to E \to G \to 1$ be a central extension of G such that E is torsion free. Then E and G are abelian.*

COROLLARY. *Let G be a finite group acting effectively on the n-torus T^n. If the action is trivial on homology, then the action is free, G is abelian of rank $\leq n$, and the orbit space T^n/G is a (homotopy) torus.*

In this case, one identifies the orbit space T^n/G with T^n by rigidity for tori, and the orbit map then is identified with a standard covering map.

Actions of p-groups.

THEOREM. *If a finite p-group G acts on the n-torus, then each component of the fixed point set has the mod p homology of a k-torus for some k.*

This may be proved by lifting the action to the universal covering, applying Smith Theory there, and then interpreting the result in the base space.

THEOREM. *If a p-group G acts on the n-torus, then all components of the fixed point set have the same dimension.*

This is really an addendum to the previous theorem. One can identify the group of deck transformations preserving the upstairs fixed point set with the invariant elements $\pi_1(T^n, x)^G$. The dimension of F_x then is the mod p cohomological dimension of this group, which is independent of the choice of fixed base point.

COROLLARY. *Suppose a finite p-group G acts in an orientation-preserving fashion on the 4-torus T^4, with fixed point set F. Then F consists of isolated points or 2-tori, not both.*

Is it necessary to assume the group is a p-group?

There is also an interesting analysis possible of orientation reversing actions of C_2 on T^n with codimension one fixed point set, but we do not discuss it further here.

Normal representations. The next step is to understand the possible actions in a neighborhood of the fixed point set.

CONJECTURE. *If a finite group G acts on a torus T^n, then the normal representations to all components of the fixed point set of G are equivalent.*

We will, however, propose a possible example where this might not be true.

Related to this we also have the question of non-equivariant type of the neighborhood of the fixed point set.

CONJECTURE. *If a finite group G acts on a torus T^n, then the normal bundle to the fixed point set is trivial.*

To what extent do the results of this section have analogues for actions on other aspherical manifolds and for groups of order that is not a prime power?

Homologically faithful actions

Since homologically trivial actions are understood, we now turn to the opposite extreme of homologically faithful actions, ignoring the issues that come with group actions that involve elements of both. We concentrate on such actions by cyclic groups C_p of prime order, and further assume actions are orientation preserving.

We further specialize to the case $n = 4$, where we only have $p = 2, 3, 5$.

One may enumerate all the possible representations on homology and use the Lefschetz Fixed Point Theorem together with the spectral sequence of the Borel fibering, to find that in all these cases the fixed point set must agree with that of a corresponding linear model.

One may go further and consider the G-Signature Theorem in a case by case manner. The conclusion is that in all but possibly a single case the fixed point set and its normal representation must agree with those of the linear models.

If C_5 acts linearly on T^4 in such a way that the action on $H_1(T^n)$ has no fixed vectors, then the action has exactly five fixed points. This standard linear action has equivalent local tangential representations at all five fixed points, namely the one with rotation numbers $(1, 3)$. That is, up to equivalence and choice of generator of C_5 the action in a neighborhood of each fixed point is equivalent to the action given in complex coordinates by $T(z, w) = (e^{2\pi i/5}z, e^{6\pi i/5}w)$.

THEOREM. *If C_5 acts locally linearly on T^4 in such a way that the action on $H_1(T^n)$ has no fixed vectors, then the action has exactly five fixed points. In addition to the linear fixed point data, there is essentially one other family of fixed point data that satisfies the g-signature formula for such an action, having the triple of rotation numbers $(1, 4), (1, 3), (2, 2)$, together with an arbitrary "cancelling pair" (that contributes nothing to the g-signature formula, namely, $(1, 1), (1, 4)$ or $(1, 2), (1, 3)$ or $(2, 2), (2, 3)$.*

Can this non-linear data be realized by a locally linear or smooth action of C_5?

Addendum. It turns out that this data cannot be realized. One can argue that the normal representations are G-homotopy equivalent to those of the linear model, at least when the fixed set is of sufficiently large codimension. And free G-homotopy equivalent representations of C_5 are (weakly) equivalent. It follows that, in contrast to the simply connected case, we have data satisfying the G-signature theorem (and a certain Reidemeister torsion condition) that is not realizable by an action on the torus.

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Casson and Gordon meet Heegaard and Floer

Daniel Ruberman*

To Terry, with admiration and thanks

The computation of the classical knot concordance groups $\mathcal{C}_{\text{top}}$ (topological) and $\mathcal{C}_{\text{smooth}}$ (smooth) remains, despite some 50 years of activity [**5**], one of the central problems in low-dimensional topology. The work of Levine [**14**] gave a complete calculation of the knot concordance group in high dimensions, via an isomorphism $\Phi : \mathcal{C} \to \mathbf{Z}^\infty \oplus \mathbf{Z}_2^\infty \oplus \mathbf{Z}_4^\infty$. (Except for a minor issue in ambient dimension 5, there isn't any difference between $\mathcal{C}_{\text{top}}$ and $\mathcal{C}_{\text{smooth}}$ in high dimensions.) Subsequently, Casson and Gordon [**1, 2**] showed that while Φ is onto in the classical dimension, it is not injective.

In addition to their intrinsic interest, questions about concordance are intimately related to surgery theory and other questions about 4-manifolds. Indeed, the original paper of Fox and Milnor [**6**] explains that the problem of representing a 2-dimensional homology class in a 4-manifold by an embedded sphere often reduces to asking if some knot is slice. Conversely, the ability to represent certain homology classes by topologically embedded spheres means that methods of surgery theory will show certain knots to be topologically slice. The most famous instance of this is Freedman's proof that Alexander-polynomial 1 knots are slice; there are more recent results along these lines by Friedl-Teichner [**7**].

More recently, gauge theory has provided very strong obstructions to a homology class in a 4–manifold being represented by an embedded sphere, or more generally by a surface of low genus. Terry was an early contributor to this study–see [**11**] and the wonderful survey articles [**12, 13**]. The most recent 'gauge-theoretic' tool, the Heegaard-Floer homology of Ozsváth and Szabó is no exception: many of the genus bounds proved via Donaldson and Seiberg-Witten theory have new (and in some sense easier) proofs via the new theory. These results can be

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*My talk at Tulane, and this note on which it was based, represent joint work with Eli Grigsby and Sašo Strle.

translated into new obstructions to knots being slice, but the Ozsváth-Szabó theory provides a more direct route to such obstructions. They [**21, 20**], simultaneously with Rasmussen [**22**], introduced 'knot Floer homology' groups $HFK(K)$ for a knot $K \subset S^3$, from which they derived a numerical invariant $\tau(K) \in \mathbf{Z}$.

The invariant $\tau(K)$ in fact vanishes on slice knots, and provides a homomorphism $\mathcal{C}_{\text{smooth}} \to \mathbf{Z}$ that is distinct from all classical invariants, including the signature. It has a nice property that is, on the other hand, reminiscent of signature invariants of knots: the τ invariant can in fact be defined for a null-homologous knot in a rational homology sphere Y; in the case at hand Y will be the 2-fold branched cover of a knot K in S^3, and we will consider $\tau(Y, \tilde{K})$ of the branch set $\tilde{K} \subset Y$. As we will explain below, $\tau(Y, \tilde{K})$ is a function from $H^2(Y) \to \mathbb{Z}$, where $H^2(Y)$ parameterizes the spin^c structures on Y. Our idea was that τ could be considered as analogous to the Casson-Gordon invariant $\tau(K, \chi)$ which is a function of $\chi \in \text{Hom}(H_1(Y), \mathbb{Q}/\mathbb{Z})$. The main theorem says that if K is slice, then $\tau(Y, \tilde{K}, s)$ vanishes for appropriately chosen $s \in \text{Spin}^c(Y)$, much as $\tau(K, \chi)$ must vanish for appropriate characters χ.

To give a precise statement, we need to quickly review some notions from the realm of Heegaard-Floer theory. Let Y be a rational homology sphere, and let $s \in \text{Spin}^c(Y)$ be a spin^c structure. A Heegaard decomposition of Y (or equivalently, a Morse function) defines a chain complex $\widehat{\text{CF}}(Y, s)$ whose homology is the Heegaard-Floer group $\widehat{\text{HF}}(Y, s)$. Because we are working on a rational homology sphere, there is a rational-valued 'Maslov grading', ie a function $\tilde{\text{gr}} : \widehat{\text{HF}}(Y, s) \to \mathbb{Q}$. There is a canonical summand $\widehat{\text{HF}}_U(Y, s)$ of $\widehat{\text{HF}}(Y, s)$, and the *correction term* for a spin^c structure s, denoted $d(Y, s)$, is the absolute $\mathbb{Q}$ homological grading, $\tilde{\text{gr}}$, of $\widehat{\text{HF}}_U(Y, s)$.

The d-invariant has the important property that $d(Y, s) = 0$ whenever $Y = \partial W$ where W is a rational homology ball and the spin^c structure s extends over W. Because the 2-fold cover of S^3 branched along a slice knot bounds a rational homology ball (the branched cover of the 4-ball over the slicing disk) the d-invariant gives a new obstruction to a knot being slice. This has been investigated by Manolescu-Owens [**17**] and Jabuka-Naik [**10**]). One point about the use of the d-invariant that is common with the original Casson-Gordon invariants is that one does not know *a priori* which spin^c structures on Y extend over W, so that in applying this obstruction one may have to do a great deal of computation.

Our idea was to strengthen the application of Heegaard-Floer homology by using observation that (with notation as in the last paragraph) not only does $Y = \partial W$, but the preimage of K in Y is slice in W. So we can use another concordance invariant: the τ invariant, which arises from the knot homology theory $HFK(K)$. Very briefly, a null-homologous knot K gives rise to a $\mathbb{Z}$-grading on the Heegaard-Floer chain complex $\widehat{\text{CF}}(Y, s)$. The minimal grading of an element that projects non-trivially to the aforementioned group $\widehat{\mathrm{HF}}_U(Y, s)$ is by definition $\tau(Y, K, s)$. When Y is the 3-sphere, $\tau(K)$ is a concordance invariant, and in fact gives a lower bound for the genus of an oriented surface bounded by K in the 4-ball. Our main technical result states something similar for all of the spin^c structures on Y that extend over W.

THEOREM. *Let K be a knot in S^3, and Y the 2-fold cover of S^3 branched along K. Denote by $\widetilde{K}$ the preimage of K in Y. If K is slice, then there exists a subgroup $G < H^2(Y;\mathbb{Z})$ with $|G|^2 = |H^2(Y;\mathbb{Z})|$ such that $d(Y, s) = 0$ and $\tau(Y, \widetilde{K}, s) = 0$ for all $s \in s_0 + G$, where s_0 is the unique spin structure on Y.*

An only slightly more elaborate statement holds with 2 replaced by p^r for any prime p.

Having been raised to believe that a theorem is not worth much unless it can be applied to some interesting examples, we went looking for knots for which we could compute the invariants $\tau(Y, \widetilde{K}, s)$. There have been important recent advances **[18]** in the computation of Heegaard-Floer groups, and these have brought the computation of $\tau(Y, \widetilde{K}, s)$ within reach for at least one class of knots, the 2-bridge knots. Recall that these are knots $K_{p,q}$ whose double branched cover is the lens space $L(p, q)$. Eli Grigsby **[9]** showed how to compute the groups

$$\widehat{\mathrm{HFK}}(L(p, q), \widetilde{K}_{p,q})$$

purely combinatorially, and with some extra work one can extract combinatorial calculations of the corresponding τ invariants. The question of which 2-bridge knots are smoothly slice has been definitively answered by Paolo Lisca **[16]**. However, there remain further questions about this category of knots, in particular the question of showing that a particular 2-bridge knot has *infinite order* in the concordance group.

One issue that arises in the Casson-Gordon invariants, and indeed in all concordance obstructions based on branched covers, is that one does not know *a priori* the restriction map $H^2(W;\mathbb{Z}) \to H^2(Y;\mathbb{Z})$ where W and Y are as above the branched covers of the 4-ball and 3-sphere respectively. In the original setting and also in the initial gauge-theoretic extensions **[4, 19, 23]**, this comes out as a lack of information about which cyclic covers of Y extend over W, whereas in the Seiberg-Witten and Heegaard-Floer setting, it is a question about extension of spin^c structures. In principle, to use an obstruction such as that in Theorem to show that some knot isn't slice, one might have to test whether the conclusions fail for *every* $G < H^2(Y;\mathbb{Z})$ with $|G|^2 = |H^2(Y;\mathbb{Z})|$! This clearly gets out of hand when the order of $H^2(Y;\mathbb{Z})$ is large. For example, to use the theorem as stated to determine if the sum of 4 copies of the 2-bridge knot $45/17$ is slice would require examination of $9,745,346$ such subgroups. As it turns out, there are subgroups on which the

d-invariant vanishes, so that invariant alone would not suffice to determine the concordance order of 45/17.

To get around this problem, we developed a way to package the functions $d(\cdot) : \mathrm{Spin}^c(Y) \to \mathbb{Q}$ and $\tau(Y, \widetilde{K}, \cdot) : \mathrm{Spin}^c(Y) \to \mathbb{Z}$ that does not require examination of subgroups. It is easiest to explain in the case that $H^2(Y;\mathbb{Z})$ is cyclic, which we now assume. We define two invariants $\mathcal{T}_p$ (resp. $\mathcal{D}_p$) to be the absolute value of

$$\sum_{\{\mathfrak{s} \in \mathrm{Spin}^c(Y) | \mathfrak{s} \text{ has order p}\}} \tau_{\mathfrak{s}}(\widetilde{K}) \qquad (\text{resp. } d_{\mathfrak{s}}(Y)).$$

We then showed

THEOREM. *Let $K \subset S^3$ be a knot and $p \in \mathbb{Z}_+$ prime or* 1*. If there exists a positive $n \in \mathbb{Z}$ such that $\#_n K$ is smoothly slice, then $\mathcal{T}_p(K) = \mathcal{D}_p(K) = 0$.*

To finish the discussion of the example for K the 2-bridge knot 45/17, the invariants $\mathcal{D}_3(K)$ and $\mathcal{D}_5(K)$ both vanish, but $\mathcal{T}_3(K) = \mathcal{T}_5(K) = -1$. Hence we conclude that this knot has infinite order in the smooth concordance group. It would of course be of great interest to know if it has finite order in the topological concordance group.

Postscript: There have been some interesting recent developments in applying gauge theory to knot concordance since we posted the preprint on which this talk was based. Lisca [**15**] showed how to apply Donaldson's diagonalization theorem to deduce that every 2-bridge knot, other than those already known to be ribbon [**1, 2, 3**] has infinite concordance order. Combining this essentially algebraic technique with the d-invariant, Greene and Jabuka [**8**] extended this result to pretzel knots all of whose twisting numbers are odd. It seems likely that much more can be said with these techniques.

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Reverse engineering families of 4-manifolds

Ronald Fintushel

To Terry, a true friend

Introduction

We describe joint work with Ron Stern and Doug Park which introduces a general procedure called 'reverse engineering' that can be used to construct infinite families of smooth 4-manifolds in a given homeomorphism type. As one of the applications of this technique, we produce an infinite family of pairwise non-diffeomorphic 4-manifolds homeomorphic to $\mathbf{CP}^2 \# 3\overline{\mathbf{CP}}^2$.

Reverse engineering is a process for constructing infinite families of distinct smooth structures on a given simply connected 4-manifold. Starting with a model manifold M which has a nontrivial Seiberg-Witten invariant, one tries to find b_1 essential tori that carry generators of H_1 and to surger each of these tori in order to kill H_1 (and, in favorable circumstances, π_1). The final step is to compute Seiberg-Witten invariants. In each of the first $b_1 - 1$ surgeries, one needs to preserve the fact that the Seiberg-Witten invariant is nonzero. The Morgan, Mrowka, Szabó formula roughly states that if $1/n$-surgery is performed on a torus $S^1 \times \alpha$ in a manifold X to obtain X_n then $\mathcal{SW}_{X_n} = \mathcal{SW}_X + n\,\mathcal{SW}_{X_0}$ where X_0 is the result of 0-surgery. To get an infinite family, we only need to assure that $\mathcal{SW}_{X_0} \neq 0$. The terminology 'reverse engineering' arises from the fact that the above process is contrived so that if X is the final manifold in the above process, with $b_1 = 0$, then $\mathcal{SW}_{X_0}$ is forced to be nonzero.

Infinite families

One of the key questions in smooth 4-manifold topology is whether a fixed homeomorphism type containing a smooth 4-manifold must actually contain infinitely many diffeomorphism types. We wish to state and prove a general theorem pointing in this direction.

Let T be a torus of self-intersection 0 in a 4-manifold X with tubular neighborhood N_T. Let α and β generate $\pi_1(T^2)$ and let S^1_α and S^1_β be loops in $T^3 = \partial N_T$ homologous in N_T to α and β respectively. Let μ_T denote a meridional circle to T

in X. By *p/q-surgery* on T with respect to β we mean

$$X_{T,\beta}(p/q) = (XN_T) \cup_\varphi (S^1 \times S^1 \times D^2),$$
$$\varphi : S^1 \times S^1 \times \partial D^2 \to \partial(XN_T)$$

where the gluing map satisfies $\varphi_*([\partial D^2]) = q[S^1_\beta] + p[\mu_T]$ in $H_1(\partial(XN_T); \mathbf{Z})$. The new manifold $X_{T,\beta}(p/q)$ has a 'core torus' $T' = S^1 \times S^1 \times \{0\}$.

We have the following situations for surgery on T:
1. Suppose that T_0 is primitive in $H_2(X)$ and that $S^1_\beta \neq 0$ in $H_1(XN_{T_0}; \mathbf{R})$, then T'_0 is nullhomologous in $X_{T_0,\beta}(p)$.
2. Suppose that T_1 is nullhomologous and that and that S^1_β bounds in XN_{T_1}, then T'_1 is primitive in $X_{T_1,\beta}(0)$. Furthermore, in this case, $H_1(X_{T_1,\beta}(1/n); \mathbf{Z}) = H_1(X; \mathbf{Z})$ and the core torus in $X_{T_1,\beta}(1/n)$ is nullhomologous.

In situation (1) there is a loop λ on T'_0 with pushoff S^1_λ which is the image of μ_{T_0} under the gluing map which defines the surgery manifold $X_{T_0,\beta}(p/q)$. Then S^1_λ is plainly nullhomologous in $X_{T_0,\beta}(p/q)N_{T'_0} = XN_{T_0}$ and 0-surgery on λ gives back X. Similarly in situation (2), the loop on $\partial N_{T'_1}$ which is the image of μ_{T_1} is essential, and 0-surgery on it gives back X. Thus these two situations are dual.

If X is a symplectic manifold and T is any Lagrangian torus, then there is a canonical framing, called the *Lagrangian framing*, of N_T. This framing is uniquely determined by the property that pushoffs of T in this framing remain Lagrangian. If one performs $1/n$ surgeries with respect to the pushoff in this framing of any nontrivial curve on T, then the result is also a symplectic manifold.

An easy-to-state simplified version of our theorem (joint with Ron Stern and B. Doug Park) is:

THEOREM (**[FPS]**). *Let X be a smooth 4-manifold with an embedded nullhomologous torus Λ containing a nontrivial loop λ with pushoff S^1_λ which is nullhomologous in XN_Λ. Suppose that $X_0 = X_{\Lambda,\lambda}(0)$ has, up to sign, just one Seiberg-Witten basic class. Then the manifolds $X_n = X_{\Lambda,\lambda}(1/n)$, $n = 1, 2, 3, \ldots$ are pairwise nondiffeomorphic.*

This is particularly interesting when X is simply connected and if it can be shown that the $\{X_{\Lambda,\lambda}(1/n)\}$ are also simply connected; for then the $\{X_{\Lambda,\lambda}(1/n)\}$ are all homeomorphic.

Here is outline of a proof of the theorem: The hypothesis puts us in situation (2) above; so, letting $X_0 = X_{\Lambda,\lambda}(0)$ and $\Lambda_0 = \Lambda'$, after surgery, we have $(X_0, \Lambda_0, S^1_\beta)$ as in situation (1). Similarly, set $X_n = X_{\Lambda,\lambda}(1/n)$ and let Λ_n be the core torus of the surgery in X_n. As above, Λ_n is nullhomologous in X_n.

Let $k_0 \in H_2(X_0; \mathbf{Z})$ be the (up to sign) unique basic class of X_0. The adjunction inequality implies that k_0 is orthogonal to Λ_0. Thus, there are (unique, because Λ (resp. Λ_n) are nullhomologous) corresponding homology classes k_n and k in $H_2(X_n; \mathbf{Z})$ and $H_2(X; \mathbf{Z})$, respectively, where k agrees with the restriction of k_0 in

$H_2(XN_\Lambda, \partial; \mathbf{Z})$ in the diagram:

$$\begin{array}{ccc} H_2(X;\mathbf{Z}) & \longrightarrow & H_2(X, N_\Lambda; \mathbf{Z}) \\ & & \downarrow \cong \\ & & H_2(XN_\Lambda, \partial; \mathbf{Z}) \\ & & \uparrow \cong \\ H_2(X_0;\mathbf{Z}) & \longrightarrow & H_2(X_0, N_T; \mathbf{Z}) \end{array}$$

and similarly for k_n.

It follows from **[MMS]** that

$$\mathcal{SW}_{X_n}(k_n) = \mathcal{SW}_X(k) + n\,\mathcal{SW}_{X_0}(k_0)$$

and that $\pm k_n$ are the only basic classes of X_n. (When $b_1(Y) \neq 0$, the expression $\mathcal{SW}_Y(\kappa)$ denotes the sum of the Seiberg-Witten invariants of all $spin^c$ structures on Y whose c_1 is Poincaré dual to κ.) Thus, if $b^+ > 1$, the manifolds X_n, $n \geq 2$, are pairwise nondiffeomorphic. This is still true when $b^+ = 1$, but a few details need to be checked. For this see **[FPS]**.

Fake $\mathbf{CP}^2 \# 3\overline{\mathbf{CP}}^2$'s

In order to illustrate reverse engineering, we show how to construct an infinite family of 4-manifolds homeomorphic but not diffeomorphic to $\mathbf{CP}^2 \# 3\overline{\mathbf{CP}}^2$. The first examples of such manifolds were obtained via a different process by Akhmedov and Park, and Baldridge and Kirk **[AP, BK]**. For the model manifold take $M = Sym^2(\Sigma_3)$, the 2-fold symmetric power of a genus 3 surface. Since $\pi_1(Sym^2(\Sigma_3)) = H_1(\Sigma_3; \mathbf{Z})$, $b_1(M) = 6$ and also $\mathrm{e}(M) = 6$, $\mathrm{sign}(M) = -2$, in agreement with the characteristic numbers for $\mathbf{CP}^2 \# 3\overline{\mathbf{CP}}^2$.

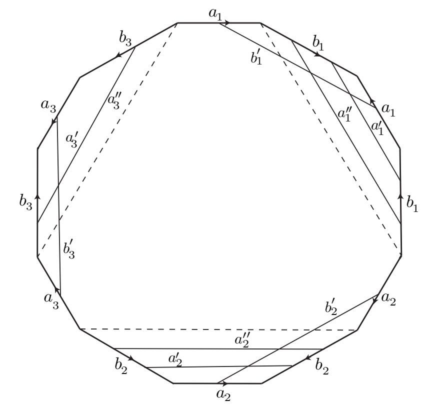
Let $\{a_i, b_i\}$, $i = 1, 2, 3$ denote standard generators for $\pi_1(\Sigma_3)$ as in the figure below. Then the tori $a_i \times a_j$, $b_i \times b_j$, $a_i \times b_j$, and $b_i \times a_j$, $i < j$, in $\Sigma_3 \times \Sigma_3$ descend to twelve Lagrangian tori in M. In the figure we see loops a_i, a_i', and a_i''. We also have based loops (with basepoint the vertex x) which we denote by the same symbols.

The abelian group $\pi_1(M) = \mathbf{Z}^6$ is generated by the $a_i = a_i \times \{x\}$ and $b_j = b_j \times \{x\}$. With respect to the Lagrangian framings these surgeries are:

$$(a_1' \times a_2', a_2', -1),\ \ (a_1'' \times b_2', b_2', -1),\ \ (a_1' \times a_3', a_1', -1),$$
$$(b_1' \times a_3'', b_1', -1),\ \ (a_2' \times a_3', a_3', -1),\ \ (a_2'' \times b_3', b_3', -1).$$

The (Lagrangian) framing circles are $S^1_{a_i'} = a_i$ and $S^1_{b_i'} = b_i$.

Performing these surgeries iteratively, we obtain manifolds M_i, $i = 1, \dots, 6$, and our comment above about surgeries on Lagrangian tori implies that each M_i is a symplectic manifold. Let $X = M_6$. A simple computation shows that X is simply connected (see **[FPS]**).



In M_5, there is a primitive torus T corresponding to $a_2'' \times b_3'$ and a loop β corresponding to b_3' whose Lagrangian pushoff $S^1_\beta = b_3$ is essential in $M_5 N_T$. We are in situation (1) of the previous section, and the surgery giving $M_6 = X$ puts us in situation (2). There is a nullhomologous torus Λ in X and a loop λ on Λ so that 0-surgery on (Λ, λ) with respect to the appropriate framing (not Lagrangian!) gives back $M_5 = X_{\Lambda,\lambda}(0)$.

In order to apply our theorem, we wish to see that M_5 has just one basic class, up to sign. The model manifold, $M = Sym^2(\Sigma_3)$ is a surface of general type, and so its only basic class (up to sign) is its canonical class, K_M, which we now describe. The three tori $a_i \times b_i$ in $\Sigma_3 \times \Sigma_3$ descend to tori T_i of square -1 in M, and $\{\mathrm{pt}\} \times \Sigma_3 \cup \Sigma_3 \times \{\mathrm{pt}\}$ descends to a genus 3 surface which represents a homology class b with self-intersection $+1$. We have $K_M = 3b + T_1 + T_2 + T_3$. Consider $M_1 = M_{a_1' \times a_2', a_2'}(-1)$, the result of the first surgery on M. In $Z = M_{a_1' \times a_2', a_2'}(0)$ the surface $\Sigma_3 \times \{\mathrm{pt}\}$, which represents b, has its genus reduced by one because of the surgery. Applying the adjunction inequality to this situation, we see that any basic class of Z has the form $\pm b \pm T_1 \pm T_2 \pm T_3$. Since the square of a basic class must be $3\,\mathrm{sign}(Z) + 2\,\mathrm{e}(Z) = 6$, in fact none of these classes can be basic; so the Seiberg-Witten invariant of Z vanishes. The result of this argument is that the Morgan, Mrowka, Szabó formula tells that the only basic classes of the manifold M_1 are $\pm$ its canonical class. The very same argument works for each surgery and finally shows that M_5 and $X = M_6$ have just one basic class up to sign.

Thus the theorem shows that the manifolds X and $X_{\Lambda,\lambda}(1/n)$, $n \geq 2$ are pairwise nondiffeomorphic. Since one can also check that the manifolds $X_{\Lambda,\lambda}(1/n)$, $n \geq 2$ are simply connected, they are all homeomorphic to $\mathbf{CP}^2 \# 3\overline{\mathbf{CP}}^2$, and we get our desired examples.

We conclude with a question:

- Is there a nullhomologous torus T in $\mathbf{CP}^2 \# 3\overline{\mathbf{CP}}^2$ so that surgeries on T give these manifolds?

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