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OPEN BOOK DECOMPOSITIONS FOR ODD DIMENSIONAL MANIFOLDS

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AN OPEN book decomposition of a *PL* manifold M is a decomposition of M as $V_h \cup (\partial V \times D^2)$, where $V_h = V \times I/(x, 1) \sim (h(x), 0)$, h a *PL* homeomorphism of V which restricts to the identity on ∂V, and M is formed by joining V_h and $\partial V \times D^2$ by a *PL* homeomorphism of their boundaries. The terminology was introduced by H. Winkelnkemper, who proved that simply connected closed *PL* manifolds of dimension ≥ 7 possess open book decompositions if their index is zero (cf. [16, 17]). In particular, odd dimensional simply connected closed *PL* manifolds of dimension ≥ 7 always have open book decompositions. I. Tamura gave an independent proof[14] of the existence of open book decompositions (which he calls spinnable structures) in the odd dimensional simply connected case. Both Tamura[14] and Winkelnkemper[16] conjectured that the hypothesis of simple connectivity could be removed, but no proof of this conjecture has appeared. We wish to furnish a proof here. The reader can consult[5, 6, 10, 12–16, 18] for various applications of open book decompositions.

Our approach will be to start with a decomposition of M^{2k+1} as $W_1 \cup_E W_2$, where W_1 denotes the handles of index $\leq k$ and W_2 denotes the handles of index $\geq k+1$ in a handle decomposition, and show that after stabilization $M = W_1 \Pi(\Pi_1^n(S^k \times D^{k+1})_j) \cup_{E\#(\#_1^n(S^k+S^k)_j)} W_2 \Pi(\Pi_1^n(D^{k+1} \times S^k)_j) = W_1' \cup_{E'} W_2'$, we can imbed V in E' so that $W_i' = V \times I$. It is easy to get from this condition to the open book decomposition of M (cf. [16, 17]). Note that we get as a corollary that M is a double. The representation of a manifold as a double under various hypotheses was first given by Smale[11] (M simply connected, tors $H_k(M) = 0$, dim $M = 2k+1 \geq 7$), and has since been proved by Barden[3] (M orientable, dim $M = 2k+1 \geq 7$), Levitt[8] (dim $M = 4m+2 \geq 6$, M simply connected, tors $H_{2m+1}(M) = 0$), Winkelnkemper[15] (M simply connected, dim $M \geq 7$), and Alexander[1, 2] (dim $M \geq 7$, $\pi_1 M$ finite if dim M even). Unfortunately, none of the proofs besides Smale's is readily available in the literature. We are indebted to John Alexander for providing us with a copy of [2], which has influenced greatly our presentation here.

OPEN BOOK THEOREM. *Let M be an odd dimensional closed connected PL manifold of dimension ≥ 7. Then M has an open book decomposition.*

Proof. Our proof will consist of a number of steps, where we start with an initial decomposition and improve it in each step until it is in the required form.

Step 1. Our initial decomposition of M^{2k+1} will stem from a choice of handlebody decomposition, $M = W_{11} \cup_{E_1} W_{21}$, where W_{11} denotes the handles of index $\leq k$ and W_{21} denotes the handles of index $\geq k+1$. Note that W_{21} processes a dual handle decomposition with handles of index $\leq k$. There are k-dimensional *CW* complexes (essentially formed from displaced cores of handles) K_{i1} and simple homotopy equivalences $\bar{f}_{i1}$: $K_{i1} \to W_{i1}$, $i = 1, 2$. Note that $E_1 \subset W_{i1}$ is k-connected. In particular, this implies there is a map f_{11}: $K_{11} \to E_1$ with $K_{11} \xrightarrow{f_{11}} E_1 \subset W_{11}$ homotopic to $\bar{f}_{11}$. Let L be the $(k-1)$—skeleton of K_{11} and consider the compositions ϕ_1: $L \subset K_{11} \xrightarrow{f_{11}} E_1$ and ψ_1: $L \xrightarrow{\phi_1} E_1 \subset W_{21}$. Both are $(k-1)$—connected since $0 \approx \pi_i(W_{21}, E_1) \approx \pi_{i-1}(f_{11}) \approx$

$\pi_{i-1}(\psi_1)$, $i \le k$. Let us denote the common fundamental groups, which we will identify, by π_1.

Let $\mathbf{Z}\pi_1$ be coefficients for all homology groups unless otherwise indicated. Since W_{21} is homotopy equivalent to K_{21}, which is k-dimensional, $H_i(W_{21}) = 0$ for $i > k$ and $H^{k+1}(W_{21}; P) = 0$ for any $\mathbf{Z}\pi_1$-module P. Since L is $(k-1)$-dimensional, this implies $H_i(\psi_1) = 0$ for $i > k$ and $H^{k+1}(\psi_1; P) = 0$ for any $\mathbf{Z}\pi_1$-module P. Also, $\psi_1 (k-1)$-connected implies that $H_i(\psi_1) = 0$ for $i < k$. Now Theorem 4 of [7] implies that $H_k(\psi_1)$ is a finitely generated stably free $\mathbf{Z}\pi_1$-module.

In succeeding steps we will "stabilize" and change $W_{i1}, E_1, K_{i1}, \bar{f}_{i1}, f_{11}, \phi_1, \psi_1$ to $W_{in}, E_n, K_{in}, \bar{f}_{in}, f_{in}, \phi_n, \psi_n$, where $M = W_{1n} \cup_{E_n} W_{2n}$, $\bar{f}_{in}$: $K_{in} \to W_{in}$ is a simple homotopy equivalence (except $\bar{f}_{22}$), $L \subset K_{11} \subset K_{in} = K_{11} \vee (\vee_1^{Pn} S_j^k)$, $W_{in} = W_{11} \amalg (\amalg_1^{Pn} (S^k \times D^{k+1})_j)$, $W_{2n} = W_{21} \amalg (\amalg_1^{Pn} (D^{k+1} \times S^k)_j)$, $E_n = E_1 \# (\#_1^{Pn} (S^k \times S^k)_j)$, f_{in}: $K_{in} \to E_n$ with $\bar{f}_{in} = i_{E_n \subset W_{in}} f_{in}$ and ϕ_n: $L \to E_n$, $\psi_n = i_{E_n \subset W_{2n}} \phi_n$ are $(k-1)$-connected. The stabilization is achieved by using pairs of cancelling k and $(k+1)$-handles in a collar of the equator E_n, or equivalently, regarding M as $M \# S^{2k+1}$ and using the decomposition of S^{2k+1} as $S^k \times D^{k+1} \cup_{S^k \times S^k} D^{k+1} \times S^k$ and taking our connected sum carefully near equators. The change from $\bar{f}_{i1}$ to $\bar{f}_{in}$, etc., will be the obvious ones such as mapping additional factors of S^k to corresponding factors $S^k \times x$ or $x \times S^k$ unless otherwise indicated.

Step 2. We stabilize as indicated above to make $H_k(\psi_2)$ a free $\mathbf{Z}\pi_1$-module. Now choose a $\mathbf{Z}\pi_1$-module basis $e_1, \ldots, e_m$ for $H_k(\psi_2) \approx \pi_k(\psi_2)$. Attach m k-cells to L via $\partial e_i \in \pi_{k-1}(L)$ and use representatives for e_i to obtain an extension f_{22}: $K_{22} = L \cup (\cup_1^m e_j^k) \to W_{22}$. The exact sequence

$$0 \longrightarrow H_{k+1}(\bar{f}_{22}) \longrightarrow H_k(K, L) \xrightarrow{\approx} H_k(\psi_2) \longrightarrow H_{k+1}(\bar{f}_{22}) \longrightarrow 0$$

shows that $H_{k+1}(\bar{f}_{22}) = H_k(\bar{f}_{22}) = 0$. One easily sees that $H_i(\bar{f}_{22}) = 0$, $i \ne k, k+1$, so $\bar{f}_{22}$ is a homotopy equivalence. It may not be simple, however; we make it simple in Step 3. All other changes in Step 2 are the standard ones.

Step 3. Suppose the torsion τ of $\bar{f}_{22}$ is represented by an $n \times n$ matrix A. Let $B = A$ if k is odd, $B = A^{-1}$ if k is even. Note that $\bar{f}_{22} \vee i$: $K_{23} = K_{22} \vee (\vee_1^n S_j^k) \to W_{23} = W_{22} \amalg (\amalg_1^n (D^{k+1} \times S^k)_j)$ will again be a homotopy equivalence with torsion τ. Use the matrix B to define a map g: $K_{22} \vee (\vee_1^n S_j^k) \to K_{22} \vee (\vee_1^n S_j^k)$ with $g|K_{22} = 1|K_{22}$ and $S_j^k \to K_{22} \vee (\vee_1^n S_j^k)$ chosen to represent $\Sigma b_{ij} e_i$ in $\pi_k(K_{22} \vee (\vee_1^n S_j^k), K_{22})$, where e_i is a generator corresponding to a lift of S_i^k in the universal cover. One may check that g is a homotopy equivalence with torsion $-\tau$. Now replace $\bar{f}_{22}$ by $\bar{f}_{23} = (\bar{f}_{22} \vee i)g$. This will be a simple homotopy equivalence. Note that $L \subset K_{23}$ and $\bar{f}_{23}$ is unchanged on L, apart from stabilization. Use connectivity again to get f_{23}: $K_{23} \to E_3 = E_2 \# (\#_1^n (S^k \times S^k)_j)$, with $K_{23} \xrightarrow{f_{23}} E_3 \subset W_{23}$ a simple homotopy equivalence, and $f_{23}|L = f_{13}|L$. Complete the stabilization in the standard fashion, giving $M = W_{13} \cup_{E_3} W_{23}$, where there exist f_{i3}: $K_{i3} \to E_3$ with $K_{i3} \xrightarrow{f_{i3}} E_3 \subset W_{i3}$ a simple homotopy equivalence. Moreover $K_{i3} = L \cup (\cup_1^{r_i} e_j^k)$ and $f_{13}|L = f_{23}|L$. We claim $r_1 = r_2$. For using the handlebody decomposition of M to compute its Euler characteristic gives $0 = \chi(M) = \chi(W_{13}) - \chi(W_{23}) = \chi(K_{13}) - \chi(K_{23}) = \chi(L) + r_1 - \chi(L) - r_2 = r_1 - r_2$.

Step 4. Form $K = L \cup_{\alpha_1} (\cup_1^r e_j^k) \cup_{\alpha_2} (\cup_{r+1}^{2r} e_j^k) = K_{13} \cup_L K_{23}$, where $K_{13} = L \cup_{\alpha_1} (\cup_1^r e_j^k)$ and $K_{23} = L \cup_{\alpha_2} (\cup_{r+1}^{2r} e_j^k)$. First define f': $K \to E_3$ by $f'|K_{i3} = f_{i3}|K_{i3}$. We may assume f' sends a small disk in e_j^k to a base point. Now define ϕ: $K \to K \vee (\vee_1^{2r} S_j^k) = \bar{K}$ by pinching each cell e_j^k on its small disk and mapping e_j^k to $e_j^k \vee S_j^k$. Define g': $\bar{K} \to E_3 \#(\overset{r}{\underset{1}{\#}}(S^k \times S^k)_j) = E_4$ by sending K via f' and $S_j^k \to (x \times S^k)_j$, $j = 1, \ldots, r$, and $S_j^k \to (S^k \times x)_{j-r}$, $j = r+1, \ldots, 2r$. Let f: $K \to E_4$ be $g'p$. Stabilize again by $W_{14} = W_{13} \amalg (\overset{r}{\underset{1}{\amalg}}(S^k \times D^{k+1})_j)$, $W_{24} = W_{23} \amalg (\overset{r}{\underset{1}{\amalg}}(D^{k+1} \times S^k)_j)$, $M = W_{14} \cup_{E_4} W_{24}$. We claim that each composition f_i: $K \xrightarrow{f} E_4 \subset W_{i4}$ is a simple homotopy equivalence. By symmetry, we need only show it for $i = 1$. Consider the diagram

$$\begin{array}{ccc} K_{13} & \xrightarrow{\bar{f}_{13}} & W_{13} \\ \downarrow & & \downarrow \\ K & \xrightarrow{f_1} & W_{14} \end{array}$$

Note first that the diagram commutes up to homotopy since the only essential difference between the two compositions lies on the small disks that are mapped to $x \times S^k$ via f_1; but $x \times S^k$ bounds $x \times D^{k+1}$ in W_{14} and this disk may be used to construct a homotopy. Note also that the induced map $H_k(K, K_{13}) \to H_k(W_{14}, W_{13})$ is a based isomorphism of free $\mathbf{Z}\pi_1$-modules, where the bases come from the additional cells and handles, respectively. A chase in exact sequences, using the above fact and the fact that $\bar{f}_{13}$ is a simple homotopy equivalence together with Theorem 3.1 of [9], shows that f_1 is a simple homotopy equivalence.

Now use Stallings Embedding Theorem (cf. [4, Theorem 12.1]) to find an embedded subcomplex $K' \subset E_4$ with $K' \subset E_4 \subset W_{i4}$ a simple homotopy equivalence. Let V be a regular neighborhood of K' in E_4. Then the s-cobordism theorem implies $W_{i4} \approx V \times I$, giving the open book decomposition.

Let us now state as a standard corollary of the existence of open book decompositions (cf. [16]) the Double Theorem. A variant of this theorem (with the same hypotheses) is the main result in [2].

Double Theorem. *Let M be a closed connected PL manifold of dimension $2k + 1 \geq 7$. Then $M = W_1 \cup_E W_2$, where $W_1 \approx W_2$. Moreover, W_1 can be chosen to be of the simple homotopy type of a k-dimensional complex and there is a PL homeomorphism H: $M \to M$ isotopic to the identity fixing a codimension two submanifold of M contained in E with $H(W_1) = W_2$.*

We close with two immediate corollaries of our proof of the Open Book theorem.

Corollary 1. *Let M be a closed connected PL manifold of dimension $2k + 1 \geq 7$ decomposed as $M = W_1 \cup W_2$, where W_1 denotes the handles of index $\leq k$ and W_2 denotes the handles of index $\geq k + 1$ in a handle decomposition of M. Then there exists n with $W_1 \amalg (\overset{n}{\underset{1}{\amalg}}(S^k \times D^{k+1})_j) \approx W_2 \amalg (\overset{n}{\underset{1}{\amalg}}(S^k \times D^{k+1})_j)$.*

Corollary 2. *Suppose W_1, W_2 are connected handlebodies of dimension $2k + 1 \geq 7$ with handles of index $\leq k$ and PL homeomorphic connected boundaries. Then there exists n with $W_1 \amalg (\overset{n}{\underset{1}{\amalg}}(S^k \times D^{k+1})_j \approx W_2 \amalg (\overset{n}{\underset{1}{\amalg}}(S^k \times D^{k+1})_j)$.*

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