NOVIKOV HOMOLOGIES IN KNOT THEORY

A. Yu. Lazarev

1. Introduction. In the early 1980s Novikov constructed an analog of Morse theory for closed 1-forms. Let us briefly recall the basic ideas, limiting ourself, for the sake of simplicity, to the case of rational forms. Suppose a closed Morse 1-form is defined on a closed smooth manifold M. We will suppose that this form is rational, i.e., there exists a regular Z-covering on which the form becomes exact and on which a Morse function is defined. As is usual in Morse theory the incidence coefficient between the critical points of neighboring indices may be determined. The only difference is that an infinite number of points with index $p-1$ may occur at the algebraic boundary of the point with index p. Further, a Novikov complex

$$C_0 \leftarrow - C_1 \leftarrow - \ldots C_{n-1} \leftarrow - C_n$$

may be constructed by analogy with a Morse complex. Here C_i is a free module with number of generators equal to the number of critical points of index i over a ring of integral Laurent series infinite in only one direction. The homologies of this complex are called Novikov homologies. They are homotopically invariant, and possess Morse-type inequalities and an analog of Smale's theorem concerning the precision of these inequalities (cf. [2, 5]).

In the present article, we compute Novikov homologies for the complementary space to a knot. We will also provide an explanation of the relation between Alexander's polynomial and homologies with coefficients in a one-dimensional local system.

2. Basic Definitions. Suppose K is a knot in a three-dimensional sphere S^3. Since $\pi_1(S^3\backslash K)/[\pi_1(S^3\backslash K), \pi_1(S^3\backslash K)] = \mathbf{Z}$, there is a standard $\mathbf{Z}$-covering corresponding to the commutant $\pi_1(S^3\backslash K)$. We partition $S^3\backslash K$ into cells; this partition is understood as an etale space that transforms it into a CW-complex. We let $S^3\backslash K = M$ and let $\hat{M}$ be an etale space. $C_*(\hat{M})$ is, obviously, a free complex over $\mathbf{Z}[t, t^{-1}]$. The Alexander polynomial $\Delta(t)$ is, by definition, of order $H_1(\hat{M})$ as a module over $\mathbf{Z}[t, t^{-1}]$. This means that if this module is specified by n generators and k relations, $\Delta(t)$ is the greatest common divisor of the minors of a relational matrix of order n; and if there are fewer relations than generators, $\Delta(t) = 0$. Note, however, that the latter possibility is actually not realized, since it is well known that for any knot, $\Delta(1) = \pm 1$.

The equivalence of this definition to the generally accepted definition is proved by Milnor in [6].

Now suppose we are given the representation $\pi_1(M) \to C^*$. We denote this representation by ρ_t if a generator in $H_1(M)$ is carried into t. (It is clear that any such representation factors through $H_1(M)$.) We set $b_k(M, t) = \mathrm{rk}_c H_k(\hat{M}, \mathbf{C})$, i.e., $\dim_c[H_k(C_*(\hat{M})_{z[t,t^{-1}]} \otimes \mathbf{C}]$. We set $\Lambda = \mathbf{Z}[t, t^{-1}]$, $S = \{P \in \Lambda\colon\ P(0) = 1\}$. Here it is supposed that P does not have any negative exponents. The Novikov numbers $\beta_k(M)$ and $q_k(M)$ are, by definition, the torsion ranks and torsion numbers of the module $H_k(\hat{M})$ over $S^{-1}\Lambda$. Farber proved in [5] that the Novikov numbers thus introduced coincide with those which were defined in [1].

3. Relation Between Alexander Polynomials and Homologies with Local Coefficients. Novikov [3] proved that the Betti numbers $b_k{}^\rho(M)$ with local coefficients of an arbitrary smooth manifold M are almost always constant, and that discontinuities occur on certain algebraic manifolds (over $\mathbf{Z}$) in the space of representations $\pi_1(M)$ in $GL(N, \mathbf{C})$.

Pazhitnov investigated this problem for one-dimensional representations.

LEMMA 1. Let us consider $H_k(\hat{M}, C)$ as a module over $\mathbf{Q}[z, z^{-1}]$. This constitutes a ring of principal ideals and possesses the decomposition $H_k(\hat{M}, \mathbf{Q}) = \Sigma \mathbf{Q}[z^{\pm 1}] \oplus \mathbf{Q}[z^{\pm 1}]/(P_j)$, where (P_j) is the ideal generated by the polynomial P_j. Then if t is transcendental, then

Moscow University. Translated from Matematicheskie Zametki, Vol. 51, No. 3, pp. 53-57, March, 1992. Original article submitted July 18, 1989.

0001-4346/92/5134-0259$12.50 © 1992 Plenum Publishing Corporation

$\beta_k(M) = b_k(M, t)$ and is equal to the number of free generators in $H_k(\hat{M}, Q)$, and if t is algebraic, $b_k(M, t) = \beta_k(M) + I_k + I_{k-1}$, where I_k is the number of polynomials P_j that annihilate t.

A proof may be given following [4] with only slight changes.

It is easily seen that if t is transcendental, C is a $Q[z, z^{-1}]$-planar module. Further, $H_k^{\rho_t}(M, C)_{\mathbf{Z}}\otimes \mathbf{Q} = H_k(C_*(\hat{M})_{\mathbf{Q}[z^{\pm1}]}\otimes \mathbf{C}) = H_k(C_*(\hat{M}))_{\mathbf{Q}[z^{\pm1}]}\otimes \mathbf{C}$. On the other hand, if the module $H_k(\hat{M})_\Lambda \otimes S^{-1}\Lambda$ is additionally localized with respect to integral constants, it will obviously have rank equal to $\beta_k(M)$ and equal to the rank of $H_k(\hat{M}, \mathbf{Q})$ over $\mathbf{Q}[z, z^{-1}]$. The first part of the lemma is proved.

Now suppose t is algebraic. By the formula for universal coefficients, we have the following decomposing triple:

$$0 \dashrightarrow H_k(\hat{M}, \mathbf{Q})_{\mathbf{Q}[z^{\pm1}]} \otimes \mathbf{C} \dashrightarrow H_k(C_*(\hat{M})_{\mathbf{Q}[z^{\pm1}]} \otimes \mathbf{C} \dashrightarrow$$
$$\dashrightarrow \mathrm{Tor}_1^{\mathbf{Q}[z^{\pm1}]}(H_{k-1}(\hat{M}), \mathbf{C}) \dashrightarrow 0.$$

We have the resolvent

$$0 \dashrightarrow \sum_1^{q_{k-1}} \mathbf{Q}[z^{\pm1}] \overset{P_j}{\dashrightarrow} \sum_1^{p_{k-1}+q_{k-1}} \mathbf{Q}[z^{\pm1}] \dashrightarrow$$
$$\dashrightarrow \sum_1^{q_k} \mathbf{Q}[z^{\pm1}] \oplus \sum_1^{q_{k-1}} \mathbf{Q}[z^{\pm1}]/(P_j) = H_{k-1}(\hat{M}).$$

Multiplying it as a tensor by $\mathbf{C}$, we obtain

$$0 \dashrightarrow \sum_1^{q_{k-1}} \mathbf{Q}[z^{\pm1}]_{\mathbf{Q}[z^{\pm1}]} \otimes \mathbf{C} \dashrightarrow \sum_1^{p_{k-1}+q_{k-1}} \mathbf{Q}[z^{\pm1}]_{\mathbf{Q}[z^{\pm1}]} \otimes \mathbf{C} \dashrightarrow \ldots$$

It is clear that every factor in P_j which vanishes in $\mathbf{C}$, adds one to the dimension of Tor a. A corresponding contribution is also made to $H_k(\hat{M}, \mathbf{Q})_{\mathbf{Q}[z^{\pm1}]}\hat{\otimes}\mathbf{C}$. The lemma is proved.

Proposition 1. Suppose $A = \{\text{roots of } \Delta(t)\} \cup \{1\}$. Then for $t \notin A$, we have $b_k(M, t) = \beta_k(M) = 0$.

Since M is an open manifold, it is homotopically equivalent to a complex with dimension 2. Therefore, in $\hat{M}$ it is homotopically equivalent to a complex of dimension 2. That is, $H_3(\hat{M}) = 0$.

Since $C_3(\hat{M}) = 0$, there are no boundaries in the group $C_2(\hat{M}, \mathbf{Q})$, consequently $H_2(\hat{M}, \mathbf{Q}) = Z_2(\hat{M}, \mathbf{Q})$, where $Z_2(\hat{M}, \mathbf{Q})$ are two-dimensional cycles, and since $C_*(\hat{M}, \mathbf{Q})$ is a free complex over the ring of principal ideals $\mathbf{Q}[z^{\pm1}]$, we find that $H_2(\hat{M}, \mathbf{Q}) = \sum \mathbf{Q}[z]$ is a free module.

Now let $H_1(\hat{M}, \mathbf{Q}) = \sum \mathbf{Q}[z^{\pm1}] + \sum \mathbf{Q}[z^{\pm1}]/(P_j)$.

We consider $H_1(\hat{M}, \mathbf{Z})$ as a module over $\mathbf{Z}[t^{\pm1}]$. The relations in $H_1(\hat{M}, \mathbf{Q})$ over $\mathbf{Q}[z^{\pm1}]$ are obtained from the relations in $H_1(\hat{M}, \mathbf{Z})$ by discarding those of the relations that are linearly dependent (over $\mathbf{Q}$) on the others. Next the relations are reduced to standard diagonal form. It is clear that if $\Delta(t) \neq 0$ (and this is, in fact, the case), $H_1(\hat{M}, \mathbf{Q})$ will be a torsion module and $\Delta(t) = \Pi_j P_j$ (to within a rational factor).

Let us turn to the null group of homologies. It is clear that multiplication by z represents an identity automorphism of $H_0(\hat{M})$. Therefore, $H_0(\hat{M}, \mathbf{Q}) = \mathbf{Q}[z^{\pm1}]/(Z - 1)$, and, by the lemma, the unit is always a singularity (point of discontinuity) of the Betti numbers.

Thus, we have proved that the set of singularities of the Betti numbers of the homology group with coefficients in the representation ρ_t coincides with the set $\{\Delta(t) = 0\} \cup \{1\}$. Moreover, from the foregoing argument it follows that, for a general position relation,

$$0 = b_0(M, t) = \beta_0(M) = b_1(M, t) = \beta_1(M) = b_3(M, t) = \beta_3(M).$$

It remains for us to prove that $b_2(M, t) = 0$, where $t \notin \{\Delta(t) = 0\} \cup \{1\}$. By Alexander–Pontyagin duality $H_2(M) = \tilde{H}^0(S^1) = 0$, where $\tilde{H}^*(\cdot)$ are reduced cohomologies. Therefore, the Euler characteristic $\chi(M) = 0$. Note the equality $0 = \chi(M) = b_2(M, t) - b_1(M, t) + b_0(M, t)$.

Since $b_1(M, t) = b_0(M, t) = 0$, $b_2(M, t) = 0$ as well. The proposition is proved.

4. Calculation of One-Dimensional Novikov Homologies. By the proposition, there is only a single nonzero Novikov number, namely $q_1(M)$. This number, it turns out, may be computed in terms of the Alexander matrix.

THEOREM. Suppose $\Delta(t) = I_1 \cdot I_2$, where I_1 is the product of polynomials irreducible over $\mathbf{Z}$ that occur in the decomposition of $\Delta(t)$ and that have free term 1, and I_2 is the product of the remaining cofactors, while $\Delta(t)$ is a normalized Alexander polynomial with no negative powers.

Then the former group of Novikov homologies may be represented in the form $\Sigma_j S^{-1}\Lambda/(P_j)$, where $\Pi_j P_j = I_2$.

LEMMA 2. A system of generators may be selected in the $\mathbf{Z}[t, t^{-1}]$-module $H_1(\hat{M})$ such that the Alexander matrix, i.e., the relational matrix, becomes quadratic.

Proof of the Lemma. Let us suppose that $\mathbf{Z}[t^{\pm 1}]$, a module of $H_1(\hat{M})$, is specified by n generators and k relations. It is clear that $n \le k$ and that we have to prove that $k \le n$. Without loss of generality, it may be assumed that $k = n + 1$. Since in the passage to $\mathbf{Q}[t^{\pm 1}]$, the matrix may be diagonalized, one of its rows, say the last row, is a linear combination of the first n rows with rational coefficients. We write it thus:

$$m_{n+1}e_{n+1} = m_n e_n + m_{n-1}e_{n-1} + \ldots + m_1 e_1,$$

where m_i is an integer. If all the m_i are divisible by m_{n+1}, the entire lemma is proved and the last row is an integral linear combination of the first n rows. Suppose this is not the case; for example, suppose that m_1 is not divisible by m_{n+1}. Let us consider the minor obtained by discarding the first row. It is obviously equal to $m_1 \cdot \Delta(t)/m_{n+1}$. By the condition $\Delta(1) = \pm 1$, this cannot be. We have obtained a contradiction.

The lemma is proved.

LEMMA 3. There are no $\mathbf{Z}$-torsions in the $\mathbf{Z}[t, t^{-1}]$-module $H_1(\hat{M})$ (which is equivalent to asserting that $H_1(\hat{M})$ is a free abelian group).

Proof. Suppose the $\mathbf{Z}[t, t^{-1}]$-group module $H_1(\hat{M})$ is specified by n generators and k relations. The presence of $\mathbf{Z}$-torsion asserts that an element may be found in the submodule of relations divisible by p in $H_1(\hat{M})$, but not divisible by p in the submodule of relations. This element may be represented in the form $m_1 e_1 + \ldots + m_n e_n$, where e_i is a generator in the submodule of relations corresponding to the i-th row of the matrix. Let us show a contradiction by analogy to the proof of the preceding lemma; suppose m_1 is not divisible by p. We replace the first row of the relational matrix by a linear combination of rows with coefficients $m_1, \ldots, m_n$. The determinant is thus multiplied by m_1. But since the first row is divisible by p, this new determinant will be divisible by p. We have arrived at a contradiction. That is, our element is actually divisible by p in the relational submodule. That is, there is no p-torsion.

These lemmas are, apparently, not new, though I do not know where they have been published previously.

Let us now prove the theorem.

By Lemma 3 the Novikov numbers do not vary in the passage to a rational field, i.e., they are equal to the torsion ranks and torsion numbers over $S^{-1}\Lambda$ of the module $\Sigma\mathbf{Q}[z^{\pm 1}]/(P_j)_\Lambda \otimes S^{-1}\Lambda$, where $\Pi_j P_j = \Delta(t)$. If there are polynomials among the P_j that have absolute term 1 (as always, it is assumed that the absolute term is the trailing term), following localization these polynomials will vanish. The theorem is proved, that is, those irreducible cofactors in $\Delta(t)$, and only those cofactors, that do not have absolute term 1 contribute to torsion.

Remark. The Alexander polynomial uniquely determines a Novikov number only if the irreducible cofactors that contribute to the torsion occur linearily in the expansion of the Alexander polynomials. In the general case this is not so. It is, however, possible to refine the theorem by introducing polynomials $\Delta_k(t)$ equal to the greatest common divisor of all minors of order $n - k$ of the Alexander matrix, which may be supposed to be quadratic, of dimension $n \times n$. In this notation $\Delta(t) = \Delta_0(t)$.

Using the polynomials $\Delta_k(t)$, we may obtain any information concerning the Novikov homologies. Suppose, for example, that an irreducible cofactor P_j ($P_j(0) \neq 1$) occurs in the decomposition of $\Delta(t)$ raised to the power i; we wish to determine the contribution it makes to the torsion. The answer: If and only if k is defined in such a way that $\Delta_k(t)$ is divisible by P_j whereas $\Delta_{k+1}(t)$ is not divisible by P_j, the contribution of P_j is equal to $\Sigma_{s=1}^{k} S^{-1}\Lambda/(P_j^{l_s})$. There is also an obvious procedure for finding ℓ_s. These results may all be easily found by passing to a rational field and diagonalizing the Alexander matrix.

Thus, the final result is as follows. All the groups of Novikov homologies of the complementary space to a knot in S^3, other than the first, are null, while the first is a torsion module over $S^{-1}\Lambda$ without p-torsions, and the torsion number may be effectively computed in terms of the knot polynomials, i.e., essentially in terms of the knot diagram.

In conclusion, I wish to express my appreciation to S. P. Novikov for having formulated the problem, and to A. V. Pazhitnov, for a host of useful suggestions.

LITERATURE CITED

1. S. P. Novikov, "Many-valued functions and functionals. An analog of Morse's theorem," Dokl. Akad. Nauk SSSR, 260, No. 1, 31-34 (1981).
2. S. P. Novikov, "Hamiltonian formalism and many-valued analog of Morse's theorem," Usp. Mat. Nauk, 37, No. 5, 3-49 (1982).
3. S. P. Novikov, "Bloch homologies. Critical points of functions in closed 1-forms," Dokl. Akad. Nauk SSSR, 287, No. 6, 1321-1324 (1986).
4. A. V. Pazhitnov, "Analytic proof of the real part of Novikov's inequality," Dokl. Akad. Nauk SSSR, 283, No. 6. 1305-1307 (1987).
5. M. Sh. Farber, "Precision of Novikov's inequalities," Funkts. Anal. Prilozhen., 19, No. 1, 49-59 (1985).
6. J. Milnor, "Infinite cyclic coverings," in: Conference on the Topology of Manifolds; Hocking, Prindle, Weber, and Smidt (eds.), Boston (1968), pp. 115-133.
7. R. Crowell and R. Foks, Introduction to Knot Theory [in Russian], Mir, Moscow (1967).

GROUPS WITH THE MAXIMAL CONDITION

S. V. Larin

An example of a group with the maximal condition is an almost polycyclic group (the terminology is taken from [1]). It is natural to try to generalize this example by considering groups having a subnormal series such that each section is either finite or infinite cyclic. However, as our Lemma 1 shows, the expected extension of the class of almost polycyclic groups does not result. Lemma 2 establishes that for an almost polycyclic group to be nilpotent it is sufficient that each finite factor group be nilpotent. A theorem dividing the set of all groups with the maximal condition into three nonoverlapping classes is proved. Using this classification, we establish two criteria for a group with the maximal condition to be nilpotent.

In this paper we will employ the following notation, taken from [1]: $H^m = \langle h^m | h \in H\rangle$; $|G:H| < \infty$ means that the index in G of the subgroup H is finite; $|G:H| = \infty$ means that the index in G of the subgroup H is infinite; $\gamma_m(G)$ is the m-th central of the group G; H' is the commutant of the group H; $\Phi(H)$ is the Frattini subgroup of H; $N_F(M)$ is the normalizer of the set M in the group F; $b^a = a^{-1}ba$ and $[a, b] = a^{-1}b^{-1}ab$, where a, b are elements of a group; $|a|$ is the order of the element a; C(A) is the center of the group A.

LEMMA 1. If a group G has a subnormal series for which each section is either finite or infinite cyclic, then G is almost polycyclic and torsion-free.

Krasnoyarsk Pedagogic Institute. Translated from Matematicheskie Zametki, Vol. 51, No. 3, pp. 58-62, March, 1992. Original article submitted February 27, 1990.

 0001-4346/92/5134-0262$12.50