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Extending Groups by Monoids

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A study is made of the problem of extending groups by monoids. Given a group K and a monoid Q, a category $\mathrm{Ext}(Q, K)$ is defined and studied by means of functors, natural transformations, factor systems, and various homological concepts which generalize from group extension theory. Comparisons are made between group-by-monoid extensions and both group extensions and $\mathscr{H}$-coextensions of monoids.

In this paper we study normal extensions of a group K by a monoid Q. By a normal extension of K by Q is meant a monoid S containing K in its group of units $G(S)$ so that K is a normal subgroup of S and $S/K \cong Q$. We study such extensions by reducing various questions we may ask about them to questions involving functors, natural transformations, and the computations of certain cocycle and cohomology groups. In this aspect our study will be quite similar to classical group cohomology theory as found in Chap. 4 of [8].

As was shown in [5], the part of monoid theory corresponding to group extension theory is $\mathscr{H}$-coextension theory. For $\mathscr{H}$-coextensions, most of the concepts of group extension theory and group cohomology generalize. However, there is only modest success in generalizing many of the cohomological results to $\mathscr{H}$-coextensions. This is not the case for the subject of this paper. Almost all of the results of classical group cohomology can be successfully generalized. Hence group-by-monoid extension theory is in some sense a halfway point between group extension theory and $\mathscr{H}$-coextension theory. Thus this extension theory should prove to be useful in establishing the structural similarities and differences between the Schreier–Eilenberg–MacLane theory of group extensions and $\mathscr{H}$-coextension theory for monoids.

The first section of this paper is comprised mainly of definitions and some basic lemmas. In particular, the definition is given of the extension category, $\mathrm{Ext}(Q, K)$. In the second section, which consists of the main part of this paper, we study this category by means of a generalized Schreier–Eilenberg–

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MacLane theory. In the third section we look at the case where the monoid Q is idempotent generated and make some remarks about central extensions of groups by monoids. The results obtained there are useful in finding some differences between the extension theory of this paper and group extension theory. In the last section we study the ways in which group-by-monoid extension theory fits into $\mathscr{H}$-coextension theory.

In order to save space and avoid a lot of repetition we assume some familiarity with the basic results of [5]. Nonetheless, we have repeated a minimal amount of material from [5] in order to hopefully ensure some smoothness in the exposition. In our notation we follow both [1] and [5].

Finally, we make a few historical remarks about some of the previous work on the subject. Normal extensions of groups by monoids were first studied by the author in his unpublished dissertation (UCLA, 1969). Modified versions of Lemma 1.3, Lemma 1.6, Theorem 2.2, Lemma 2.5, and Theorem 2.13 first appeared there. In 1971, Fulp and Stepp in a pair of papers [2, 3] studied (using our terminology) central extensions of abelian groups by monoids under the restriction that all groups and monoids under consideration were compact and all homomorphisms were continuous. Some of our results in the third section are mild generalizations of the algebraic (i.e., discrete) analogues of their results. In 1971–1972 the author studied the $\mathscr{H}$-coextension problem and his results (which appear in [5]) have strongly influenced the approach taken in this paper. Quite recently, Grillet has shown in [4] the relevance of obstructions in the third cohomology group to our etension theory. His result, together with those of this paper, completes, to a large extent, the task of generalizing classical group extension theory to both group-by-monoid extension theory and $\mathscr{H}$-coextension theory.

1. Normal Subgroups and Normal Extensions

Definition 1.1. If K is a group and S is a monoid, then K is called a *normal subgroup* of S if K is a subgroup of $G(S)$, the group of units of S, and if for all $x \in S$, $xK = Kx$. Under these conditions we also say that K is normal in S.

1.2. If K is a subgroup of the group of units of S, then as for groups we may talk about left cosets of K in S, i.e., subsets of the form xK. As in the case for groups, left cosets form a partition of the monoid S. Likewise, we may talk about right cosets of K in S and the right coset partition of S. When K is normal in S, left cosets and right cosets coincide, i.e., $xK = Kx$, and the usual subset multiplication on 2^K yields

$$xKyK = xyKK = xyK.$$

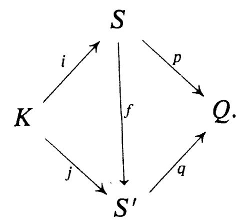
Hence the coset decomposition of S by K is a congruence decomposition. The corresponding quotient monoid, i.e., the monoid of cosets, is denoted S/K.

Lemma 1.3. *Let K be a normal subgroup of S and let $\pi: S \to S/K$ be the canonical morphism: $x \to xK$. Then the coset congruence on S is contained in the Green's relation $\mathscr{H}$ and $\pi: S \to S/K$ is an $\mathscr{H}$-coextension.*

Proof. $xK \subseteq xG \subseteq \mathscr{R}(x)$ and $Kx \subseteq Gx \subseteq \mathscr{L}(x)$. Hence $xK = Kx \subseteq \mathscr{R}(x) \cap \mathscr{L}(x) = \mathscr{H}(x)$.

1.4. Hence we can factor out a normal subgroup K to obtain S/K. If $S/K \cong Q$, then this isomorphism induces an epimorphism $p: S \to Q$ whose corresponding congruence on S is precisely the coset congruence induced by K on S. Thus we call S a *normal extension of K by Q*. In a more formal fashion we define a normal extension of K by Q to be a short exact sequence $K \to^i S \to^p Q$, where i is a monomorphism from K into $G(S)$ such that iK is normal in S, p is an epimorphism of S onto Q whose induced congruence on S is precisely the coset congruence on S induced by iK. Extensions will be denoted by triples of the form $\langle i, S, p\rangle$.

1.5. It is clear that we actually have a category of normal extensions of K by Q, $\mathrm{Ext}(Q, K)$, if we define a morphism from $\langle i, S, p\rangle$ to $\langle j, S', q\rangle$ to be a monoid morphism, $f: S \to S'$ such that the following diagram commutes:



In particular, if f is an isomorphism, then we say that the two extensions are isomorphic since it is clear that f^{-1} must also be a morphism of extensions. The first observation to be made about the morphisms of $\mathrm{Ext}(Q, K)$ is the following analogue of the short five lemma.

Lemma 1.6. *Let $f: \langle i, S, p\rangle \to \langle j, S', q\rangle$ be a morphism of extensions in $\mathrm{Ext}(Q, K)$. Then f is an epimorphism. Moreover, if $g: \langle j, S', q\rangle \to \langle i, S, p\rangle$ is also a morphism, then f and g are both isomorphisms. In particular, all endomorphisms of extensions are automorphisms.*

Proof. For each $x \in Q$ let $\bar{x} \in S$ be chosen so that $p(\bar{x}) = x$. Then $q \circ f(\bar{x}) = p(\bar{x}) = x$ and setting $\bar{\bar{x}} = f(\bar{x})$ we get $q(\bar{\bar{x}}) = x$. Now $S = \bigcup \bar{x}(iK)$ and $S' = \bigcup \bar{\bar{x}}(jK)$. Hence $fS = f[\bigcup \bar{x}(iK)] = \bigcup f(\bar{x}) f(iK) = \bigcup \bar{\bar{x}}(jK) = S'$, and f is surjective. To prove the rest of the lemma, we need only prove that endomorphisms of extensions are automorphisms. Let f be an endomorphism of $\langle i, S, p\rangle$. Then f must be an epimorphism. Suppose $f(\bar{x}k_1^i) = f(\bar{y}k_2^i)$. Then $x = p \circ f(\bar{x}k_1^i) = p \circ f(\bar{y}k_2^i) = y$. Since $f \circ i = i$, the above equation now becomes $f(\bar{x})\, k_1^i = f(\bar{x})\, k_2^i$. Since $p \circ f = p$, we have $f(\bar{x}) = k^i\bar{x}$ and hence our equation becomes $k^i\bar{x}k_1^i = k^i\bar{x}k_2^i$. Cancelling k^i yields $\bar{x}k_1^i = \bar{x}k_2^i$. Hence f is bijective.

The second observation to be made about $\operatorname{Ext}(Q, G)$ is:

LEMMA 1.7. *The map* $\operatorname{Ext}(Q, G) \to \mathscr{H}(\mathbf{Mon}_Q)$ *defined by* $\langle i, S, p\rangle \to \langle S, p\rangle$ *and* $f \to f$ *is a well defined forgetful functor.*

As a consequence of Lemma 1.7 we can use the results of [5] to give a description of the objects and morphisms of $\operatorname{Ext}(Q, K)$. This is done in the next section, and as a consequence of results presented there we may deduce the main theorem on the morphism structure of $\operatorname{Ext}(Q, K)$ which we state here:

THEOREM 1.8. *All morphisms of* $\operatorname{Ext}(Q, K)$ *are epimorphisms. All automorphism groups are abelian. If* $\operatorname{Hom}(\langle i, S, p\rangle, \langle j, S' q\rangle)$ *is nonempty, then the action of* $\operatorname{Aut}(j, S', q)$ *on this hom-set given by morphism composition is simply transitive. In this situation, for each automorphism* θ *of* $\langle i, S, p\rangle$ *there exists a unique automorphism* θ' *of* $\langle j, S', q\rangle$ *such that for all extension morphisms* $f: S \to S'$, $\theta' \circ f = f \circ \theta$. *Finally, all endomorphisms of* $\operatorname{Ext}(Q, K)$ *are automorphisms.*

1.9. We have already seen the first and last statements of the above theorem. The last statement also must follow from the simple transitivity assertion, since then every endomorphism must differ from the identity morphism by a unique automorphism. From the simple transitivity, epimorphism, and abelian assertions everything else must also follow. For by simple transitivity, for given f and θ there exists a unique θ' such that $\theta' \circ f = f \circ \theta$. If another extension morphism g is given, then $g = u \circ f$ for some automorphism u and hence $\theta' \circ g = \theta' \circ u \circ f = u \circ \theta' \circ f = u \circ f \circ \theta = g \circ \theta$ and the map $\theta \to \theta'$ is invariant of f. Thus we need only show abelianess and simple transitivity.

1.10. When Q is a group, then all normal extensions of K by Q are groups. In this case Theorem 1.8 reduces to the well known fact that $\operatorname{Ext}(Q, K)$ is a groupoid (i.e., a category of all whose morphisms are isomorphisms), all of whose automorphism groups are abelian since they are all isomorphic with various groups of one dimensional cocycles of Q with coefficients in $\mathbf{Z}K$, the center of K. Here the map $\theta \to \theta'$ is just the map $\theta \to f\theta f^{-1}$, where f is any isomorphism in the hom-set.

2. THE STRUCTURE OF $\operatorname{Ext}(Q, K)$

In this section we use the results of [5] to study $\operatorname{Ext}(Q, K)$. This is done by using certain group valued functors to construct a full subcategory of $\operatorname{Ext}(Q, K)$ such that every object of $\operatorname{Ext}(Q, K)$ is isomorphic with an object of this subcategory. We begin with a construction.

2.1. Let $\mathscr{L}(Q)$ and $\mathscr{R}(Q)$ be the $\mathscr{L}$- and $\mathscr{R}$-quasiorders on Q. Let **Gr** denote the category of groups. We call a pair of functors $F: \mathscr{L}(Q) \to$ **Gr** and $G: \mathscr{R}(Q) \to$ **Gr** *compatible* if for all $x \in Q$ we have $F(x) = G(x)$, and denote compatible pairs of functors by $(F, G): (\mathscr{L}(S), \mathscr{R}(S)) \to$ **Gr**. We call the pair (F, G) *surjective* if for all morphisms $x \geqslant_{\mathscr{L}} y$ $[x \geqslant_{\mathscr{R}} y]$ we have $F(x \geqslant_{\mathscr{L}} y)$ $[G(x \geqslant_{\mathscr{R}} y)]$ being an epimorphism from $F(x)$ onto $F(y)$. In what follows, $F(x \geqslant_{\mathscr{L}} y)$ will be denoted by F_x^y and $G(x \geqslant_{\mathscr{R}} y)$ will be denoted by G_x^y. If (F, G) is a compatible pair of functors, then a *factor system* for (F, G) is a function $\alpha: Q \times Q \to \bigcup_{x \in Q} F(x)$ such that for all $x, y \in Q$, $\alpha(x, y) \in F(xy)$. Throughout the paper, $\langle F, G, \alpha\rangle$ denotes a compatible pair of functors together with a factor system. Given a triple $\langle F, G, \alpha\rangle$ one can construct a multiplicative system $Q \times_\alpha (F, G)$ as follows: the underlying set is $\{\langle x, a\rangle \mid x \in Q, a \in F(x)\}$ and the multiplication is defined by $\langle x, a\rangle\langle y, b\rangle = \langle xy, G_x^{xy}(a)\, \alpha(x, y)\, F_y^{xy}(b)\rangle$. This multiplication need not be associative, but when it is, $Q \times_\alpha (F, G)$ is a monoid whose identity element is $\langle 1, \alpha(1, 1)^{-1}\rangle$. When associativity holds, then α is called an *associative factor system* and $\langle F, G, \alpha\rangle$ is called an *associative triple.* It is shown in [5] that there is no loss in generality in assuming that $\alpha(1, 1) = 1_{F(1)}$, and if α is associative, this forces $\alpha(x, 1) = \alpha(1, x) = 1_{F(x)}$ for all $x \in S$. Such an α is called *unitary* and $\langle F, G, \alpha\rangle$ is called a *unitary triple.*

We are now ready to construct objects in $\operatorname{Ext}(Q, K)$.

THEOREM 2.2. *Let* $\langle F, G, \alpha\rangle$ *be a unitary associative triple such that* (F, G) *is surjective and* $F(1) = K$. *Define* $i: K \to Q \times_\alpha (F, G)$ *by* $i(k) = \langle 1, k\rangle$ *and define* $\pi: Q \times_\alpha (F, G) \to Q$ *by* $\pi(x, a) = x$. *Then*

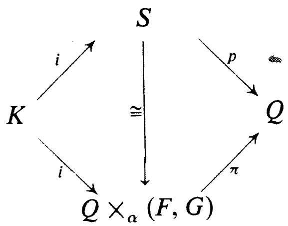
$$K \xrightarrow{\ i\ } Q \times_\alpha (F, G) \xrightarrow{\ \pi\ } Q$$

is a normal extension of K *by* Q. *Moreover, every normal extension of* K *by* Q *is isomorphic to an extension constructed in this fashion.*

Proof. By Lemma 1.7 we know that if (i, S, p) is a normal extension of K by Q, then in particular (S, p) is an $\mathscr{H}$-coextension of Q. Thus by Theorem 3.9 of [5] there is an isomorphism over Q of (S, p) with a pair $(Q \times_\alpha (F, G), \pi)$ for some associative unitary triple. Since

$$K \cong p^{-1}(1) \cong \pi^{-1}(1) \cong F(1)$$

we may assume that $F(1)=G(1)=K$. If $i: K \to Q \times_\alpha (F, G)$ is defined as in the theorem, then it is clear that we have



commuting, so that $\langle i, Q \times_\alpha (F, G), \pi \rangle$ is in $\operatorname{Ext}(Q, K)$ and is isomorphic with $\langle i, S, p \rangle$ in this category. Thus all parts of the theorem follow when we prove that an associative unitary triple $\langle F, G, \alpha \rangle$ with $F(1)=K$ yields an object $\langle i, Q \times_\alpha (F, G), \pi \rangle$ in $\operatorname{Ext}(Q, K)$ if and only if (F, G) is a surjective pair.

Asserting that $\langle i, Q \times_\alpha (F, G), \pi \rangle \in \operatorname{Ext}(Q, K)$ is clearly equivalent to asserting that for all $x \in Q$, $a \in F(x)$ we have

$$K\langle x, a \rangle = \{x\} \times F(x) = \pi^{-1}(x) = \langle x, a \rangle K.$$

But $K\langle x, a \rangle = \{\langle x, G_1^x(k)a \rangle : k \in K\}$. Clearly this equals $\pi^{-1}(x)$ if and only if G_1^x is surjective. Dually $\pi^{-1}(x) = \langle x, a \rangle K$ if and only if F_1^x is surjective. Thus $\langle i, Q \times_\alpha (F, G), \pi \rangle$ is a normal extension if and only if for all $x \in Q$, both F_1^x and G_1^x are surjective. But the latter imply that both F and G are surjective. For all $x, y \in Q$ we have $F_x^{yx} \circ F_1^x = F_1^{yx}$ and $G_x^{xy} \circ G_1^x = G_1^{xy}$. But if F_1^{yx} and G_1^{xy} are surjective, then so are F_x^{yx} and G_x^{xy}. Thus both F and G are surjective. This proves the theorem.

2.3. Let $F: \mathscr{L}(Q) \to \mathbf{Gr}$. Then $C^1(f)$ denotes the set $\{\phi \in \prod_{x \in Q} F(x): \phi(1) = 1\}$. Two unitary triples $\langle F, G, \alpha \rangle$ and $\langle F', G', \beta \rangle$ are said to be *equivalent* if $F = F'$, and there exists $\phi \in C^1(F)$ such that

(i) $G'^{y}_{x} = (\ )^{\phi(y)} \circ G_x^y \circ (\ )^{\phi(x)^{-1}}$ for $x \geqslant_{\mathscr{L}} y$,

(ii) $\beta(x, y) = \phi(xy)^{-1}\, G_x^{xy}(\phi(x))\, \alpha(x, y)\, F_y^{xy}(\phi(y))$,

where $(\ )^b$ denotes the conjugation $b^{-1}(\ )b$. We denote this relationship by $\langle F', G', \beta \rangle = \langle F, G, \alpha \rangle^\phi = \langle F, G_\phi, \alpha_\phi \rangle$. When (i) and (ii) hold, then there is induced an isomorphism of multiplicative systems $\phi^*: Q \times_\alpha (F, G) \cong Q \times_\beta (F', G')$ defined by $\phi^*(x, a) = \langle x, \phi(x)^{-1}a \rangle$. Since (F', G') is surjective if and only if (F, G) is, and β is unitary if and only if α is, we see that $\langle F, G, \alpha \rangle$ satisfies the conditions of Theorem 2.2 if and only if $\langle F', G', \beta \rangle$ does, in which case it is an easy check to see that ϕ^* is an isomorphism of normal extensions. We call these isomorphisms, *equivalences*. Note that if we define $-\phi \in C^1(F)$ by $-\phi(x) = \phi(x)^{-1}$, then $\langle F, G, \alpha \rangle = \langle F', G', \beta \rangle^{-\phi}$ and $(-\phi)^* = \phi^{*-1}$.

2.4. As in [5] we define a natural transformation, $\sigma: \langle F, G, \alpha \rangle \to \langle F', G', \beta \rangle$ to be a collection of morphisms $\{\sigma(x): F(x) \to F'(x)\}$ such that for all $u, x, v \in Q$ the following diagram commutes:

$$\begin{array}{ccccc} F(xv) & \xleftarrow{G_x^{xv}} & F(x) & \xrightarrow{F_x^{ux}} & F(ux) \\ \downarrow{\scriptstyle \sigma(xv)} & & \downarrow{\scriptstyle \sigma(x)} & & \downarrow{\scriptstyle \sigma(ux)} \\ F'(xv) & \xleftarrow{G'^{xv}_x} & F'(x) & \xrightarrow{F'^{ux}_x} & F'(ux) \end{array}$$

and for all $x, y \in Q$, $\beta(x, y) = \sigma(xy)[\alpha(x, y)]$. Moreover, in this paper we make the added restriction that $\sigma(1)$ is the identity map on $F(1) = F'(1) = K$. σ induces a map on algebraic structures, $\sigma^*: Q \times_\alpha (F, G) \to Q \times_\beta (F', G')$ defined by $\sigma^*(x, g) = \langle x, \sigma(x)[g] \rangle$. σ^* is a homomorphism and the following lemma yields the important facts about σ^*.

Lemma 2.5. *Let $\langle i, Q \times_\alpha (F, G), \pi \rangle$ and $\langle i, Q \times_\beta (F', G'), \pi \rangle$ be objects of $\operatorname{Ext}(Q, K)$. If $\sigma: \langle F, G, \alpha \rangle \to \langle F', G', \beta \rangle$ is a natural transformation, then it is unique and σ^* is a morphism in $\operatorname{Ext}(Q, K)$ such that $\sigma^*(x, 1) = \langle x, 1 \rangle$. Moreover, if $f: Q \times_\alpha (F, G) \to Q \times_\beta (F', G')$ is an extension morphism such that for all x, $f(x, 1) = \langle x, 1 \rangle$ then f is the only morphism with this property and $f = \sigma^*$ for some unique natural transformation, σ. Finally, both f and σ are surjective ($\sigma(x): F(x) \to F'(x)$ is surjective for all x).*

Proof. If σ is a natural transformation then for all $x \in S$ the following diagram commutes:

$$\begin{array}{ccccc} F(1) & = & K & = & F'(1) \\ \downarrow{\scriptstyle F_1^x} & & & & \downarrow{\scriptstyle F'^x_1} \\ F(x) & & \xrightarrow{\sigma(x)} & & F'(x). \end{array}$$

Since F_1^x is surjective, $\sigma(x)$ is uniquely determined. Since $F'^x_1 = \sigma(x) \circ F_1^x$ is surjective, so is $\sigma(x)$. The rest of the lemma is an immediate consequence of these two facts and Lemma 3.31 of [5] which is the $\mathscr{H}(\mathbf{Mon}_Q)$ version of this lemma.

From our proof we also get:

Proposition 2.6. *If $F, F': \mathscr{L}(Q) \to \mathbf{Gr}$ are surjective functors such that $F(1) = K = F'(1)$, then if there exists a natural transformation $\sigma: F \to F'$ such that $\sigma(1) = \mathrm{id}_K$, then σ is uniquely determined.*

2.7. If such a natural transformation exists, then we denote this by $F \twoheadrightarrow_K F'$. If $G: \mathscr{R}(Q) \to \mathbf{Gr}$ is a surjective functor such that (F, G) is a compatible pair, then the question arises as to whether or not there exists a surjective functor $G'\colon \mathscr{R}(Q) \to \mathbf{Gr}$ such that (F', G') is a compatible pair and the unique natural transformation $\sigma\colon F \twoheadrightarrow_K F'$ becomes a natural transformation of compatible pairs $\sigma\colon (F, G) \to (F', G')$. Clearly a necessary and sufficient condition for this is that for all $x, y \in Q$, $G_x^{xy}\,\mathrm{Ker}(\sigma(x)) \subseteq \mathrm{Ker}(\sigma(xy))$. Then when this occurs, $G_x'^{xy}$ is defined for all $x, y \in Q$ to be the unique morphism which makes the following diagram commute:

$$\begin{array}{ccccc} \mathrm{Ker}(\sigma(x)) & \longrightarrow & F(x) & \xrightarrow{\sigma(x)} & F'(x) \\ \big\downarrow{\scriptstyle G_x^{xy}} & & \big\downarrow{\scriptstyle G_x^{xy}} & & \big\downarrow{\scriptstyle G_x'^{xy}} \\ \mathrm{Ker}(\sigma(xy)) & \longrightarrow & F(xy) & \xrightarrow{\sigma(xy)} & F'(xy). \end{array}$$

If σ can be extended to (F, G) and α is an associative factor system for (F, G), then if we define α_σ by $\alpha_\sigma(x, y) = \sigma(xy)[\alpha(x, y)]$ then α_σ is an associative factor system for (F', G') and $\sigma^*\colon Q \times_\alpha (F, G) \to Q \times_{\alpha_\sigma} (F', G')$ thus is a morphism of extensions. We denote the triple $\langle F', G', \alpha_\sigma\rangle$ thus constructed by $\langle F, G, \alpha\rangle^\sigma$. Clearly all morphisms of Lemma 2.5 are morphisms of this type.

2.8. We are now in a position to describe all morphisms from $\langle i, Q \times_\sigma (F, G), \pi\rangle$ to $\langle i, Q \times_\beta (F', G'), \pi\rangle$. If f is a morphism, then f determines an element $\phi \in C^1(F')$ defined by the equation $f(x, 1) = \langle x, \phi(x)\rangle$. From 2.3 we see that $\phi^* \circ f$ is a morphism from $\langle i, Q \times_\alpha (F, G), \pi\rangle$ to $\langle i, Q \times_{\beta_\phi} (F', G'_\phi), \pi\rangle$ such that $\phi^* \circ f(x, 1) = \langle x, 1\rangle$. Hence by Lemma 2.5 we have $\phi^* \circ f = \sigma^*$, where σ is the unique natural transformation from (F, G, α) to $(F', G', \beta)^\phi$. From 2.7 we must have $(F', G', \beta)^\phi = (F, G, \alpha)^\sigma$. We obtain the following theorem:

THEOREM 2.9. *There exists an extension morphism from $Q \times_\alpha (F, G)$ to $Q \times_\beta (F', G')$ if and only if*

(i) $F \twoheadrightarrow_K F'$.

(ii) *If $\sigma\colon F \to F'$ is the unique morphism such that $\sigma(1) = \mathrm{id}_K$, then the triple $\langle F, G, \alpha\rangle^\sigma$ can be formed.*

(iii) *There exists $\phi \in C^1(F')$ such that $\langle F, G, \alpha\rangle^\sigma = \langle F', G', \beta\rangle^\phi$.*

Under these conditions, the homomorphism set is

$$\{\phi^{*-1} \circ \sigma^* \mid \phi \in C^1(F') \text{ and } \langle F, G, \alpha\rangle^\sigma = \langle F', G', \beta\rangle^\phi\}.$$

Finally, the decomposition of each morphism into $\phi^{*-1} \circ \sigma^*$ is unique.

We are now ready to compute the automorphism group of an extension.

THEOREM 2.10. *Let $\langle i, Q \times_\alpha (F, G), \pi\rangle$ be a normal extension of K by Q. Let $Z^1(F, G)$ denote the subset of $C^1(F)$ consisting of all ϕ such that*

(i) *for all $x \in Q$, $\phi(x) \in \mathcal{Z}(F(x))$.*

(ii) *for all $x, y \in Q$, $\phi(xy) = G_x^{xy}(\phi(x))\, F_y^{xy}(\phi(y))$.*

Then $Z^1(F, G)$ is an abelian group under pointwise addition, i.e., $(\phi + \psi)(x) = \phi(x)\,\psi(x)$. Moreover, $Z^1(F, G) \cong \mathrm{Aut}\langle i, Q \times_\alpha (F, G), \pi\rangle$ under the map $\phi \to \phi^*$.

Proof. We first note that since $\phi(x)$ and $\phi(y)$ lie in the centers of their respective groups, $(\ )^{\phi(y)}$ and $(\ )^{\phi(x)}$ are the identity automorphisms, and if $x \geqslant_{\mathscr{R}} y$, then in 2.3(i) $G^\phi = G$. Since epimorphisms of groups take centers into centers, $G_x^{xy}(\phi(x))$ and $F_y^{xy}(\phi(y))$ lie in the center of $F(xy)$. Hence by (ii) above, 2.3(ii) becomes $\alpha_\phi = \alpha$. Thus $\langle F, G, \alpha\rangle^\phi = \langle F, G, \alpha\rangle$ and $\phi^* \in \mathrm{Aut}\langle i, Q \times_\alpha (F, G), \pi\rangle$. If $\phi, \psi \in Z^1(F, G)$, then $\psi^* \circ \psi^*(x, a) = \langle x, \psi(x)^{-1}\,\phi(x)^{-1}a\rangle = \langle x, (\phi + \psi)(x)^{-1}a\rangle = (\phi + \psi)^*(x, a)$ so that $\phi \to \phi^*$ is a monomorphism of groups. By Theorem 2.9 above, we must have $\mathrm{Aut}\langle i, Q \times_\alpha (F, G), \pi\rangle = \{\phi^* \in C^1(F) \mid \langle F, G, \alpha\rangle^\phi = \langle F, G, \alpha\rangle\}$. For such ϕ, 2.3(i) tells us that $G_1^x = (\ )^{\phi(x)} \circ G_1^x$. Since G_1^x is surjective, $(\ )^{\phi(x)} = \mathrm{id}_{F(x)}$. Thus for all $x \in Q$ we have $\phi(x) \in \mathcal{Z}(F(x))$. Again, surjectivity of F and G implies $G_x^{xy}(\phi(x))$ and $F_y^{xy}(\phi(y))$ lie in the center of $F(xy)$. Hence 2.3(ii),

$$\alpha(x, y) = \phi(xy)^{-1}\, G_x^{xy}(\phi(x))\, \alpha(x, y)\, F_y^{xy}(\phi(y)),$$

becomes $\phi(xy)^{-1}\, G_x^{xy}(\phi(x))\, F_y^{xy}(\phi(y)) = 1$. Hence $\phi \in Z^1(F, G)$.

COROLLARY 2.11. *If $\sigma\colon \langle F, G, \alpha\rangle \to \langle F', G', \beta\rangle$, then*

(i) $\mathrm{Hom}(\langle i, Q \times_\alpha (F, G), \pi\rangle, \langle i, Q \times_\beta (F', G'), \pi\rangle) = Z^1(F, G)^* \circ \sigma^*$.

Moreover, σ induces a homomorphism $\sigma^1\colon Z^1(F, G) \to Z^1(F', G')$ defined by $\sigma^1(\phi)(x) = \sigma(x)[\phi(x)]$ such that for all $\phi \in Z^1(F, G)$, $\sigma^* \circ \phi^* = \sigma^1(\phi)^* \circ \sigma^*$. Finally, if $\tau\colon \langle F', G', \beta\rangle \to \langle F'', G'', \gamma\rangle$, $\phi \in Z^1(F', G')$ and $\psi \in Z^1(F'', G'')$, then

(ii) $(\psi^* \circ \tau^*) \circ (\phi^* \circ \sigma^*) = (\psi + \tau^1\phi)^* \circ (\tau\sigma)^*$.

Proof. The hom-set of (i) above must consist of those $\phi^* \circ \sigma^*$ for which $\langle F', G', \beta\rangle^\phi = \langle F', G', \beta\rangle$, i.e., those $\phi^* \circ \sigma^*$ with $\phi \in Z^1(F', G')$. The rest of the corollary follows from straightforward computations. (Or else, see 3.35 of [5].)

2.12. If $f\colon \langle i, S, p\rangle \to \langle j, S', q\rangle$ is a morphism of normal extensions, then it is possible to find triples $\langle F, G, \alpha\rangle$ and $\langle F', G', \beta\rangle$, a natural transformation $\sigma\colon \langle F, G, \alpha\rangle \to \langle F', G', \beta\rangle$, and a pair of isomorphisms such that the diagram

$$\begin{array}{ccc} \langle i, S, p\rangle & \cong & \langle i, Q\times_\alpha (F, G), \pi\rangle \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \sigma^*} \\ \langle j, S', q\rangle & \cong & \langle i, Q\times_\beta (F', G'), \pi\rangle \end{array}$$

commutes. Hence many of the results of this section may be stated in full generality. From this diagram and Theorem 2.10 and Corollary 2.11(i) we obtain Theorem 1.8.

We now turn our attention to the statements of other analogues of results from group extension theory. We shall not furnish some proofs, since they are furnished elsewhere.

THEOREM 2.13. *Let* $\langle i, Q\times_\alpha (F, G), \pi\rangle \in \mathrm{Ext}(Q, K)$. *Then* (F, G) *are restrictions of a group-valued functor defined on* $\mathscr{D}(Q)$ *if and only if for all* $x, y\in Q$, $\alpha(x, y)\in ZF(xy)$. *Suppose* (F, G) *are restrictions of such a functor, denoted by F also. Then the set of all associative factor systems for F forms an abelian group, denoted* Z^2F, *where addition is defined pointwise, i.e.,* $(\alpha+\beta)(x, y) = \alpha(x, y)\,\beta(x, y)$. *Let* B^2F *denote the set of all factor systems* $\delta\phi$ *defined by*

$$\delta\phi = G_x^{xy}(\phi(x))\,\phi(xy)^{-1}\,F_y^{xy}(\phi(y)),$$

where $\phi\in C^1(F)$ *is such that* $\phi(x)\in ZF(x)$ *for all* $x\in Q$. *Then* B^2F *is a subgroup of* Z^2F, *and two extensions* $Q\times_\alpha F$ *and* $Q\times_\beta F$ *are isomorphic if and only if* $\alpha-\beta\in B^2F$. *Finally, if F is abelian group-valued, an extension* $Q\times_\alpha F$ *splits if and only if* $\alpha\in B^2F$. (See 3.47, 3.53, 5.18 of [5].)

Naturally one calls the second cohomology group $H^2F = Z^2F/B^2F$ the group of F-extensions of K by Q.

2.14. Suppose we are given a compatible pair of surjective functors (F, G) for which $F(1) = K$. The question arises as to whether or not there exists an associative factor system α belonging to (F, G) so that one can form an extension $Q\times_\alpha (F, G)$. From 3.9 I of [5] it is clear that a necessary condition for an α to exist is that for all u, x, $v\in Q$, $F_{xv}^{uxv}\circ G_x^{xv}$ and $G_{ux}^{uxv}\circ F_x^{ux}$ differ only by an inner automorphism of $F(uxv)$. Grillet has called such pairs *weakly coherent* (as compared with coherent pairs (F, G) where $F_{xv}^{uxv}\circ G_x^{xv} = G_{ux}^{uxv}\circ F_x^{ux}$ for all $u, x, v\in Q$, i.e., F, G can be extended to a functor $F\colon \mathscr{D}(Q)\to \mathbf{Gr}$). If (F, G) is a weakly coherent pair, then the pair (ZF, ZG), where $ZF(x) = ZG(x) =$ the center of $F(x)$, is a coherent pair of functors with a unique extension to all of $\mathscr{D}(Q)$ denoted ZF and called the *center* of (F, G). Note that the group $Z^1(F, G)$ of 2.10 is really the group of 1-cocycles of the center of the pair (F, G) in that theorem. Likewise the groups B^2F, Z^2F, and H^2F are really the groups of 2-coboundaries, 2-cocycles, and the second cohomology group of ZF. Grillet has shown that if (F, G) is a weakly coherent pair of functors, then one can form an *obstruction*, $\mathrm{Obs}(F, G)$, lying in the third cohomology group of ZF. He obtains the following generalization of a classical result of group extension theory.

THEOREM 2.15 (Grillet [4]). *Let* (F, G) *be a weakly coherent pair of functors and let* $ZF\colon \mathscr{D}(Q)\to \mathbf{Ab}$ *be the center of* (F, G). *Let* $\mathrm{Obs}(F, G)\in H^3ZF$ *denote the obstruction formed from* (F, G). *Then* (F, G) *possesses an associative factor system if and only if* $\mathrm{Obs}(F, G) = 0$.

2.16. Suppose that (F, G) has an associative unitary factor system. We can talk about the set of extensions $\mathrm{Ext}(F, G) = \{\langle i, Q\times_\alpha (F, G), \pi\rangle\}$. If ZF is the center of (F, G), then there exists a well defined group action

$$\mathrm{Ext}(F, G)\times Z^2(ZF)\to \mathrm{Ext}(F, G),$$

given by $(\langle i, Q\times_\alpha (F, G), \pi\rangle, \beta)\to \langle i, Q\times_{\alpha+\beta} (F, G), \pi\rangle$. Indeed, if $\beta\in Z^2(ZF)$, then it is an easy check using 3.9 I of [5] to see that $\alpha+\beta$ is also an associative unitary factor system for (F, G). Notice that this action is semiregular, i.e., $\alpha+\beta = \alpha$ if and only if $\beta = 0$. It turns out that this action is also transitive. For suppose that α, α' are associative unitary factor systems for (F, G). Then the epimorphisms $F_x^{xy}\circ G_1^y$ and $G_x^{xy}\circ F_1^y$ are related by

$$\begin{aligned} F_y^{xy}\circ G_1^y &= (\ )^{\alpha(x,y)}\circ G_x^{xy}\circ F_1^x \\ &= (\ )^{\alpha'(x,y)}\circ G_x^{xy}\circ F_1^x. \end{aligned}$$

Hence there exists a 2-chain $\beta(x, y)$ of ZF defined by $\alpha'(x, y) = \alpha(x, y) + \beta(x, y)$. Plugging $\alpha(x, y) + \beta(x, y)$ into 3.9 I of [5] it is easy to see that $\beta\in Z^2(ZF)$. Hence our action is regular, or simply transitive. Using Theorem 2.9, it is easy to see that two extensions $\langle i, Q\times_\alpha (F, G), \pi\rangle$ and $\langle i, Q\times_{\alpha'} (F, G), \pi\rangle$ are isomorphic if and only if α and α' differ by a coboundary in $B^2(ZF)$. To sum up:

THEOREM 2.17. *Let* (F, G) *be a weakly coherent pair of functors for which* $\mathrm{Ext}(F, G)$ *is not empty. Then the action of* $Z^2(ZF)$ *on* $\mathrm{Ext}(F, G)$ *is simply transitive. Two extensions of* $\mathrm{Ext}(F, G)$ *are isomorphic if and only if they lie in the same orbit of* $B^2(ZF)$. *Hence there is a bijection between the isomorphism classes of* $\mathrm{Ext}(F, G)$ *and* $H^2(ZF)$.

2.18. From Theorems 2.15 and 2.17 it follows that if (F, G) is a weakly coherent pair for which $F(x)$ is centerless for $x\in Q$, $\mathrm{Ext}(F, G)$ consists of precisely one extension. Note that this does not follow if we just assume that $F(1) = K$ is centerless.

We conclude this section with a description of the functorial aspects of the above theorem.

2.19. Let $\sigma: (F, G) \to (F', G')$ be a natural transformation of compatible pairs of surjective functors such that $\sigma(1) = \mathrm{id}_K$. As in Lemma 2.5, $\sigma(x): F(x) \to F'(x)$ must be surjective. It is easy to see that if (F, G) is a weakly coherent pair, then so is (F', G') weakly coherent. For suppose $A(u, x, v) \in F(uxv)$ is defined for all $u, x, v \in Q$ such that

(i) $(\ )^{A(u,x,v)} \circ F^{uxv}_{xv} \circ G^{xv}_{x} = G^{uxv}_{ux} \circ F^{ux}_{x}$.

Then setting $B(u, x, v) = \sigma(uxv)[A(u, x, v)] \in F'(uxv)$ it is easy to show that

(ii) $(\ )^{B(u,x,v)} \circ F'^{uxv}_{xv} \circ G'^{xv}_{x} = G'^{uxv}_{ux} \circ F'^{ux}_{x}$.

Indeed composing (i) with $\sigma(uxv)$ on the left and pulling σ to the right yields (ii) composed with $\sigma(x)$ on the right. Since $\sigma(x)$ is surjective, cancellation yields (ii). Likewise, if α is an associative unitary factor system for (F, G) and one sets $\alpha_\sigma(x, y) = \sigma(xy)[\alpha(x, y)]$, then it is easy to show that α_σ is an associative unitary factor system for (F', G') and that $\sigma^*: \langle i, Q \times_\alpha (F, G), \pi\rangle \to \langle i, Q \times_{\alpha_\sigma} (F', G'), \pi\rangle$ is a morphism in $\mathrm{Ext}(Q, K)$. Finally, if (F, G) and hence (F', G') are weakly coherent, then under restriction σ becomes a natural transformation $\sigma: \mathbf{Z}F \to \mathbf{Z}F'$. This is a consequence of the fact that $\sigma(x)$ is surjective and thus $\sigma(x)[\mathbf{Z}F(x)] \subseteq \mathbf{Z}F'(x)$. Thus we obtain induced maps $\sigma^2: Z^2(\mathbf{Z}F) \to Z^2(\mathbf{Z}F')$ and $H^2(\sigma): H^2(\mathbf{Z}F) \to H^2(\mathbf{Z}F')$.

THEOREM 2.20. *Let $\sigma: (F, G) \to (F', G')$ be a natural transformation of surjective pairs such that $\sigma(1) = \mathrm{id}_K$. If (F, G) is weakly coherent then so is (F', G'). If* $\mathrm{Ext}(F, G)$ *is nonempty then so is* $\mathrm{Ext}(F', G')$ *and σ induces a map* $\mathrm{Ext}(\sigma)$ *between these sets defined by $Q \times_\alpha (F, G) \to Q \times_{\alpha_\sigma} (F', G')$. Moreover, $\sigma^*: Q \times_\alpha (F, G) \to Q \times_{\alpha_\sigma} (F', G')$ is a morphism of extensions. Finally, the following diagram commutes; i.e.,* $(\mathrm{Ext}(\sigma), \sigma^2)$ *is a morphism of actions,*

$$\begin{array}{ccccc} \mathrm{Ext}(F, G) & \times & Z^2(\mathbf{Z}F) & \longrightarrow & \mathrm{Ext}(F, G) \\ {\scriptstyle \mathrm{Ext}(\sigma)}\downarrow & & \downarrow{\scriptstyle \sigma^2} & & \downarrow{\scriptstyle \mathrm{Ext}(\sigma)} \\ \mathrm{Ext}(F', G') & \times & Z^2(\mathbf{Z}F') & \longrightarrow & \mathrm{Ext}(F', G'), \end{array}$$

and $\mathrm{Ext}(\sigma)$ *is isomorphism class preserving.*

3. Central Extensions and the Case where Q is Idempotent Generated

LEMMA 3.1. *Let K be a normal subgroup of S. Then K centralizes the submonoid of S generated by its idempotents, $\langle E(S)\rangle$.*

Proof. Let $\alpha \in E(S)$. Then $K\alpha = \alpha K \subseteq G(\alpha)$, the maximal subgroup of α. Hence $K\alpha = \alpha K \alpha = \alpha K$, and for all $k \in K$ we have $k\alpha = \alpha k \alpha = \alpha k$. Hence K centralizes $E(S)$, and thus $\langle E(S)\rangle$.

THEOREM 3.2. *Let $\mathscr{T}(Q)$ be the $\mathscr{T}$-quasiorder on Q and let $F: \mathscr{T}(Q) \to$ **Gr** be a surjective functor such that $F(1) = K$. Let α be an associative unitary factor system for $(F \mid \mathscr{L}(Q), F \mid \mathscr{R}(Q))$. Set $Q \times_\alpha F = Q \times_\alpha (F \mid \mathscr{L}, F \mid \mathscr{R})$. Then $\alpha \in Z^2(\mathbf{Z}F)$. If Q is idempotent generated, then each normal extension of K by Q is isomorphic to one constructed in this fashion, and moreover, for all such F, $Z^1(F)$ is trivial. Thus if Q is idempotent generated, all hom-sets of* $\mathrm{Ext}(Q, K)$ *are either singletons or empty. In particular, all automorphism groups are trivial.*

Proof. The first part is a consequence of Theorem 2.13. For the second part, let Q be idempotent generated and let K be a normal subgroup of S with $S/K \cong Q$ under p. We construct a lifting $x \to \bar{x}$ Q to S as follows. For each $e \in E(Q)$, $p^{-1}(e)$ is a subgroup of S with idempotent $\bar{e}$. Set $\bar{e} = \bar{e}$. For each $x \in Q$, choose one expression of x as a product of idempotents, $x = e_1 \cdots e_n$. Set $\bar{x} = \overline{e_1} \cdots \overline{e_n}$. Clearly $\{\bar{x} \mid x \in Q\} \subseteq \langle E(S)\rangle$ and hence by Lemma 3.1, $\{\bar{x} \mid x \in Q\}$ and K commute elementwise. Let Σ be the right structure functor on $\mathscr{D}(S)$ corresponding to the coset congruence of K. Consider $\Sigma \mid \mathscr{D}(\langle E(S)\rangle)$. Since $\langle E(S)\rangle$ is idempotent generated 1 is an initial object in $\mathscr{D}(S)$, and hence for all $u, v \in \langle E(S)\rangle$ we have $\Sigma(u, 1, v) = \Sigma(uv, 1, 1) = \Sigma(1, 1, uv)$. (See [7].) Let $u, u', x, v, v' \in \langle E(S)\rangle$ be such that $uxv = u'xv'$. Then

$$\Sigma(u, x, v) \circ \Sigma(x, 1, 1) = \Sigma(ux, 1, v) = \Sigma(uxv, 1, 1) = \Sigma(u'xv', 1, 1)$$
$$= \Sigma(u'x, 1, v') = \Sigma(u', x, v') \circ \Sigma(x, 1, 1).$$

Since $\Sigma(x, 1, 1)$ is an epimorphism we can cancel to obtain $\Sigma(u, x, v) = \Sigma(u', x, v')$. Hence $\Sigma \mid \mathscr{D}(\langle E(S)\rangle)$ must factor through $\mathscr{T}(\langle E(S)\rangle)$. Let $\alpha(x, y) \in \Sigma(\overline{xy})$ be such that $\bar{x}\bar{y} = \overline{xy} \cdot \alpha(x, y)$, where $x, y \in Q$. Then for all $k \in K$ we have (using the exponential notation of 3.5 of [5]),

$$\overline{xy} \cdot \alpha(x, y)\, k^{(\overline{xy},1,1)} = (\overline{xy} \cdot \alpha(x, y))\, k = \bar{x}\bar{y}k = k\bar{x}\bar{y}$$
$$= k\overline{xy} \cdot \alpha(x, y) = \overline{xy} \cdot k^{(1,1,\overline{xy})}\alpha(x, y)$$
$$= \overline{xy} \cdot k^{(\overline{xy},1,1)}\alpha(x, y).$$

Since $\Sigma(\overline{xy}, 1, 1)$ is surjective, $\alpha(x, y)$ must lie in the center of $\Sigma(\overline{xy})$. Thus by 3.9 of [5], the (F, G, α) used to parametrize (i, S, p) must be such that (F, G) are restrictions of a functor defined on $\mathscr{T}(Q)$ and $\alpha \in Z^2(F)$. Denoting this extension by F also, we are done with the second part. For the final part, let $\phi \in Z^1(F)$. Then for all $e \in E(Q)$ we have,

$$\phi(e) = \phi(e^2) = F_e^{ee}(\phi(e))\, F_e^{ee}(\phi(e)) = \phi(e)^2.$$

Hence $\phi(e) = 1_{F(e)}$. Finally let $x = e_1 e_2 \cdots e_n$. Then

$$\phi(x) = \prod_{i=1}^{n} F_{e_i}^{x} \phi(e_i) = 1_{F(x)},$$

so that $Z^1(F)$ is the trivial group.

3.3. Just because the first cohomology group of a $\mathscr{D}(S)$-domained, abelian group-valued functor of an idempotent generated monoid always vanishes, one should not assume that the second cohomology group must vanish. The homological reason for this lies in the fact that our cohomology comes from $\{\mathrm{Ext}^n(\mathbb{Z},\ )\}$ in the abelian category $\mathbf{Ab}^{D(S)}$ and not from $\{\mathrm{Ext}^n(\mathbb{Z},\ )\}$ in the category $\mathbf{Ab}^{\mathscr{D}(S)}$. (See the first chapter of [5].) Hence although $\mathrm{Ext}^1(\mathbb{Z},\ )$ of $\mathscr{D}(S)$-domained functors may vanish, $\mathrm{Ext}^1(\mathbb{Z},\ )$ need not vanish on all of $\mathbf{Ab}^{D(S)}$. Likewise, using the appropriate examples of idempotent generated monoids, it is easy to see that it is not always possible to embed a normal extension of an abelian group into a split normal extension of some perhaps bigger abelian group. (See [6].) The homological reason for this is similar to that above. Although it is possible to embed a functor of $\mathbf{Ab}^{\mathscr{D}(S)}$ into a functor of $\mathbf{Ab}^{D(S)}$ for which the second cohomology group vanishes, e.g., an injective functor, such a functor need not be in $\mathbf{Ab}^{\mathscr{D}(S)}$.

3.4. The above theorem leads us to the following considerations. We call an extension $\langle i, S, p\rangle$ a *central extension* if $S = K^i C_S(K^i)$, where $C_S(K^i)$ denotes the centralizer of $K^i = iK$ in S. Equivalently, $\langle i, S, p\rangle$ is a central extension if there exists a unitary lifting $x \to \bar{x}$ from Q to S such that for all $x \in Q$, $\bar{x} \in C_S(K^i)$. From such a cross section one can construct a surjective functor $F: \mathscr{T}(Q) \to \mathbf{Gr}$ and an associative unitary factor system α such that $\langle i, S, p\rangle \cong \langle i, Q \times_\alpha F, \pi\rangle$. Conversely, normal extensions of K by Q constructed from $\mathscr{T}(Q)$-domained surjective and their factor systems must be central extensions. The proofs of these facts are quite similar to much of what is found in the proof of Theorem 3.2 and so we shall not give them.

3.5. Surjective functors $F: \mathscr{T}(Q) \to \mathbf{Gr}$ for which $F(1) = K$ have an important subclass which we shall describe. First we let $\Omega(K)$ denote the lattice of normal subgroups of K. We let $(\mathscr{T}(Q), \Omega(K))$ denote the set of all maps, μ, from $\mathscr{T}(Q)$ to $\Omega(K)$ such that $\mu(1) = \{1_K\}$ and for all $x \geqslant_{\mathscr{T}} y$ in Q, $\mu(x) \subseteq \mu(y)$. This collection of maps becomes a lattice if we define $\mu < \nu$ to mean $\mu(x) \subseteq \nu(x)$ for all $x \in Q$. We can use these order morphisms to construct functors from $\mathscr{T}(Q)$ to $\mathbf{Gr}$, as well as natural transformations between them. Indeed, if $u: \mathscr{T}(Q) \to \Omega(K)$ is an order morphism, then define $F_\mu: \mathscr{T}(Q) \to \mathbf{Gr}$ by:

(i) $F_\mu(x) = K/\mu(x)$.

(ii) If $x \geqslant_{\mathscr{T}} y$ then $F_{\mu x}^{y}: F_\mu(x) \to F_\mu(y)$ is the canonical epimorphism induced by the inclusion of pairs, $(K, \mu(x)) \subseteq (K, \mu(y))$.

If $\mu, \nu: \mathscr{T}(Q) \to \Omega(K)$, then it is easy to see that there exists a (necessarily unique by Proposition 2.6) natural transformation $\sigma: F_\mu \to F_\nu$ such that $\sigma(1) = \mathrm{id}_K$ if and only if $\mu \leqslant \nu$, in which case for all x, $\sigma(x)$ is the canonical epimorphism from $K/\mu(x) \to K/\nu(x)$. We denote the subcategory of $\mathbf{Gr}^{\mathscr{T}(Q)}$ constructed above by $(\mathscr{T}(Q), \Omega^*(K))$, where $\Omega^*(K)$ denotes the category of all canonical images of K and canonical epimorphisms between them.

3.6. If $F: \mathscr{T}(Q) \to \mathbf{Gr}$ is a surjective functor for which $F(1) = K$, then we define $\mu_F: \mathscr{T}(Q) \to \Omega(K)$ by $\mu_F(x) = \mathrm{Ker}(F_1^x)$. It is easy to see that there exists a unique natural isomorphism $\sigma_F: F \to F_{\mu_F}$ such that $\sigma_F(1) = \mathrm{id}_K$. It follows that $(\mathscr{T}(Q), \Omega^*(K))$ is a skeletal subcategory of the category of all surjective $\mathscr{T}(Q)$-domained functors which are K at 1, and that $F \to F_{\mu_F}$ is a retraction onto this subcategory. Moreover, the full subcategory of the category of central extensions of K by Q whose objects are those $\langle i, Q \times_\alpha F, \pi\rangle$ where $F \in (\mathscr{T}(Q), \Omega^*(K))$ is almost a skeletal subcategory of the category of central extensions. It is skeletal modulo equivalence isomorphisms. This leads to the following construction.

3.7. To each $F_\mu \in (\mathscr{T}(Q), \Omega^*(K))$ associate the cohomology group $H^2(ZF_\mu)$. To each canonical natural transformation $\sigma: F_\mu \to F_\nu$ associate the map $H^2(\sigma): H^2(ZF_\mu) \to H^2(ZF_\nu)$. Since $(\mathscr{T}(Q), \Omega^*(K))$ naturally forms a lattice, we obtain a functor $H^2: (\mathscr{T}(Q), \Omega^*(K)) \to \mathbf{Ab}$. We use this functor to form a semilattice of groups (see 4.11 of [1]). We denote this semilattice of groups by $H^2(\mathscr{T}(Q), \Omega^*(K))$ and call it *the semigroup of central extensions of K by Q*. Clearly its elements are bijective correspondence with the isomorphism classes of central extensions of K by Q. Finally, if the $\mathscr{T}$-class of 1 in Q is $\{1\}$, this semigroup must have a zero. For setting $\mu_0(1) = \{1\}$ and $\mu_0(x) = K$ for $x \neq 1$, clearly $\mu \leqslant \mu_0$ for all μ, and $H^2(ZF_{\mu_0}) = 0$.

THEOREM 3.8. *Let Q be an idempotent generated monoid and let K be a group. Then all normal extensions of K by Q are central. Moreover, the isomorphism classes of extensions naturally form an abelian inverse monoid with zero.*

Proof. Immediate from Theorem 3.2 and 3.4–3.7 and the fact that the $\mathscr{T}$-class of 1 in an idempotent generated monoid is trivial (see [7]).

4. Relationships between Normal Extensions and $\mathscr{H}$-Coextensions

If Q is a monoid, then we define $\operatorname{Ext}(Q, \mathbf{Gr})$ to be the full subcategory of $\mathscr{H}(\mathbf{Mon}_S)$ whose objects consist of these $\mathscr{H}$-coextensions which are obtained as normal extensions of arbitrary groups by Q. Notice that $\operatorname{Ext}(Q, \mathbf{Gr})$ is more than just the big union of $\operatorname{Ext}(Q, K)$ as K ranges over $\mathbf{Gr}$. $\operatorname{Ext}(Q, \mathbf{Gr})$ is usually a proper subcategory of $\mathscr{H}(\mathbf{Mon}_S)$. In this section we examine various relationships between $\mathscr{H}(\mathbf{Mon}_S)$ and its subcategory $\operatorname{Ext}(Q, \mathbf{Gr})$. We begin with a lemma.

Lemma 4.1. *Let Q be such that for all $u, x, v \in Q$, $ux = x = xv$ if and only if $u = v = 1$. Then $\mathscr{H}(\mathbf{Mon}_Q) = \operatorname{Ext}(Q, \mathbf{Gr})$. In particular all $\mathscr{H}$-coextensions of Q are in* $\operatorname{Ext}(Q, \mathbf{Gr})$ *whenever Q is a cancellative monoid.*

Proof. Let $Q \times_\alpha (F, G)$ be an $\mathscr{H}$-coextension of Q for which α is a unitary factor system. Then condition 3.9 II of [5] reduces to

$$F_1^x(F(1)) = F(x) = G_1^x(F(1)).$$

Hence F and G are surjective so that $Q \times_\alpha (F, G)$ is a normal extension of $F(1)$ by Q.

4.2. We let $(\mathbf{Mon}, Q)$ denote the comma category of monoids over Q. The objects are pairs $\langle S, p\rangle$ where $p: S \to Q$ is a monoid morphism. A morphism $f: \langle S, p\rangle \to \langle S', q\rangle$ is a monoid morphism $f: S \to S'$ such that $q \circ f = p$. $\mathbf{Mon}_Q$ denotes the full subcategory determined by those pairs for which the p is an epimorphism onto. Clearly we have the following containments of full subcategories:

$$\operatorname{Ext}(Q, \mathbf{Gr}) \subseteq \mathscr{H}(\mathbf{Mon}_Q) \subseteq \mathbf{Mon}_Q \subseteq (\mathbf{Mon}, Q).$$

4.3. Clearly pullbacks exist in $\mathbf{Mon}$, with the usual construction being that of the fibred product. So let $\tau: T \to Q$ be an object of $(\mathbf{Mon}, Q)$. Then τ induces a functor $\tau^*: (\mathbf{Mon}, Q) \to (\mathbf{Mon}, T)$. Here $\tau^*\langle S, p\rangle = \langle T \times_Q S, \pi_T\rangle$ and $\tau^*(f) = \mathrm{id} \times f \mid T \times_Q S$. It is easy to see that $\tau^*: \mathbf{Mon}_Q \to \mathbf{Mon}_T$. Suppose that $\langle S, p\rangle = \langle Q \times_\alpha (F, G), \pi\rangle$. Then $\langle T \times_Q S, \pi_T\rangle \cong \langle T \times_{\alpha(T \times T)} (F \circ \mathscr{L}(\tau), G \circ \mathscr{R}(\tau)), \pi\rangle$ under the map $(t, \langle x, g\rangle) \to \langle t, g\rangle$. If (F, G) is a surjective pair of functors, then so is $(F \circ \mathscr{L}(\tau), G \circ \mathscr{R}(\tau))$. Hence $\tau^*: \operatorname{Ext}(Q, \mathbf{Gr}) \to \operatorname{Ext}(T, \mathbf{Gr})$. In general $\tau^*: \mathscr{H}(\mathbf{Mon}_Q) \nrightarrow \mathscr{H}(\mathbf{Mon}_T)$. To see this, let T be any cancellative monoid for which there exists an epimorphism τ from T onto Q (e.g., T is the free monoid on the *set* $Q - \{1\}$ and $\tau: T \to Q$ is the obvious epimorphism). Then $(F \circ \mathscr{L}(\tau), G \circ \mathscr{R}(\tau))$ is a surjective pair if and only if (F, G) is a surjective pair. Hence for this particular $\langle T, \tau\rangle$, if $\langle S, p\rangle \in \mathscr{H}(\mathbf{Mon}_Q)$, then $\tau^*\langle S, p\rangle \in \mathscr{H}(\mathbf{Mon}_T)$ if and only if $\langle S, p\rangle \in \operatorname{Ext}(Q, \mathbf{Gr})$. Hence we have the following theorem:

Theorem 4.4. *Let $\langle S, p\rangle \in \mathbf{Mon}_Q$. Then the following are equivalent:*

(i) $\langle S, p\rangle \in \operatorname{Ext}(Q, \mathbf{Gr})$.

(ii) *For all* $\langle T, \tau\rangle \in (\mathbf{Mon}, Q)$, $\tau^*\langle S, p\rangle \in \operatorname{Ext}(T, \mathbf{Gr})$.

(iii) *For all* $\langle T, \tau\rangle \in (\mathbf{Mon}, Q)$, $\tau^*\langle S, p\rangle \in \mathscr{H}(\mathbf{Mon}_T)$.

(iv) *For all* $\langle T, \tau\rangle \in \mathbf{Mon}_Q$, $\tau^*\langle S, p\rangle \in \operatorname{Ext}(T, \mathbf{Gr})$.

(v) *For all* $\langle T, \tau\rangle \in \mathbf{Mon}_Q$, $\tau^*\langle S, p\rangle \in \mathscr{H}(\mathbf{Mon}_T)$.

Proof. From the above we have (i) implying (ii), (iii), (iv), and (v). Using $\tau = \mathrm{id}_Q$ we have (ii) and (iv) implying (i). In both (iii) and (v) we have $\tau = \mathrm{id}_Q$ implying $\langle S, p\rangle \in \mathscr{H}(\mathbf{Mon}_Q)$. Now letting $\langle T, \tau\rangle$ be an epimorphism from a cancellative monoid onto Q, we have $\langle S, p\rangle \in \operatorname{Ext}(Q, \mathbf{Gr})$ by the above.

4.5. Since $\mathbf{Mon}$ has pullbacks, $(\mathbf{Mon}, Q)$ has products. Here $\langle S, p\rangle \times \langle S', q\rangle = \langle S \times_Q S', p \circ \pi_S\rangle$. These products become products in $\mathbf{Mon}_Q$. Suppose $Q \times_\alpha (F, G)$ and $Q \times_\beta (F', G')$ are given. Define $F \times F'$ and $G \times G'$ to be the pointwise products of the functors involved, e.g., $F \times F'(x) = F(x) \times F'(x)$ and $(F \times F')_x^y = F_x^y \times F_x'^y$. Define $\alpha \times \beta$ by $\alpha \times \beta(x, y) = (\alpha(x, y), \beta(x, y))$. Clearly we have

$$\langle Q \times_\alpha (F, G), \pi\rangle \times \langle Q \times_\beta (F', G'), \pi\rangle \cong \langle Q \times_{\alpha \times \beta} (F \times F', G \times G'), \pi\rangle.$$

under the map $(\langle x, a\rangle, \langle x, b\rangle) \to \langle x, (a, b)\rangle$. In general $\mathscr{H}(\mathbf{Mon}_Q)$ is not closed under this product. However,

Theorem 4.6. *Let $\langle S, p\rangle \in \mathscr{H}(\mathbf{Mon}_Q)$ and $\langle S', q\rangle \in \operatorname{Ext}(Q, \mathbf{Gr})$. Then $\langle S, p\rangle \times \langle S', q\rangle \in \mathscr{H}(\mathbf{Mon}_Q)$.*

Proof. Let $\langle T, q'\rangle = p^*\langle S', q\rangle \in \operatorname{Ext}(S, \mathbf{Gr})$. Then $\langle T, p \circ q'\rangle \in \mathscr{H}(\mathbf{Mon}_Q)$ since $\mathscr{H}$-coextensions of $\mathscr{H}$-coextensions are $\mathscr{H}$-coextensions. But $\langle T, p \circ q'\rangle = \langle S, p\rangle \times \langle S', q\rangle$, and we are done.

Corollary 4.7. $\operatorname{Ext}(Q, \mathbf{Gr})$ *is closed under products.*

4.8. We can use a construction like that of 5.16 of [5] to give an example of an $\mathscr{H}$-coextension $\langle S, p\rangle$ such that $\langle S, p\rangle \times \langle S, p\rangle$ is not an $\mathscr{H}$-coextension.

4.9. A question which one might ask is whether or not $\mathscr{H}$-coextensions of Q can be embedded in objects of $\operatorname{Ext}(Q, \mathbf{Gr})$, i.e., given $\langle S, p\rangle \in \mathscr{H}(\mathbf{Mon}_S)$ does there exist $\langle S', q\rangle \in \operatorname{Ext}(Q, \mathbf{Gr})$ together with a monomorphism

$f: S \to S'$ such that $q \circ f = p$? In general, the answer is no. To see this, let Q be an idempotent generated monoid. Let $f: \langle S, p \rangle \to \langle S', q \rangle$ be a monomorphism from an $\mathscr{H}$-coextension to an object of Ext(Q, **Gr**). As in the proof of Theorem 3.2 we choose a lifting $x \to \bar{x}$ from Q to S such that $\bar{e} \in E(S)$ for all $e \in E(Q)$, and $\bar{x} = \bar{e}_1 \cdots \bar{e}_n$ if $e_1 \cdots e_n$ is a fixed representation of x as a product of idempotents. Setting $\bar{\bar{x}} = f(\bar{x})$, then $x \to \bar{\bar{x}}$ is a lifting from Q to S' with the same properties. Let $\langle F, G, \alpha \rangle$ and $\langle F', G', \beta \rangle$ be the triples which parametrize S and S' and are obtained by these liftings. Then there exists a natural transformation $\sigma: \langle F, G, \alpha \rangle \to \langle F', G', \beta \rangle$ such that σ^* is the parametrization of f. Clearly each $\sigma(x): F(x) \to F'(x)$ is a monomorphism. But by Theorem 3.2 we have $\beta(x, y) \in \mathbf{Z}(F'(xy))$ for all $x, y \in Q$. Hence $\alpha(x, y) \in \mathbf{Z}(F(xy))$ for all $x, y \in Q$. Since it is easy to construct $\mathscr{H}$-coextensions of idempotent semigroups which do not possess this property, such $\mathscr{H}$-coextensions cannot be embedded in objects of Ext(Q, **Gr**). For example, let S be a completely simple semigroup for which $E(S)$ is not a subsemigroup. Suppose also that the structure group of S is nontrivial and centerless. Then S^1 is an $\mathscr{H}$-coextension of $Q = S^1/\mathscr{H}$ which cannot be embedded in a normal extension of some group by Q.

4.10. Let $S = \mathscr{M}(G; I, \Lambda, P)$ be a completely simple Rees matrix semigroup with structure group G and sandwich matrix P. We attach G as a group of units to S by extending the multiplication of G and S to $G \cup S$ by

$$g * (h)_{i\lambda} = (gh)_{i\lambda} \qquad \text{and} \qquad (h)_{i\lambda} * g = (hg)_{i\lambda}.$$

The only place where associativity may break down is at $(h)_{i\lambda} * g * (h')_{ju}$. Multiplying both ways yields $(hgP_{\lambda j}h')_{iu}$ and $(hP_{\lambda j}gh')_{iu}$. Clearly $*$ is associative (and thus $G \cup S$ is a monoid) if and only if $P_{\lambda j} \in \mathbf{Z}G$ for all λ, j. In this case $G \cup S$ is a normal extension of G by the band $(I \times \Lambda)^1$. It is easy to show that if S is a completely simple semigroup, then the following are equivalent:

(i) S^1 can be embedded in an object of Ext(Q, **Gr**), where $Q = S^1/\mathscr{H}$.

(ii) If $e, f \in E(S)$, then $ef \in \mathbf{Z}\mathscr{H}(ef)$.

(iii) There exists a Rees matrix parametrization of S whose sandwich matrix P takes its values in the center of the structure group of this parametrization.

(iv) A normal faithful group of units can be attached to S, i.e., there exist a group G and an associative multiplication on $G \cup S$ extending the multiplication on S and on G such that G is a normal group of units and for all maximal subgroups $\mathscr{H}(e)$, the actions $G \times \mathscr{H}(e) \to \mathscr{H}(e)$ and $\mathscr{H}(e) \times G \to \mathscr{H}(e)$ given by multiplication are simply transitive. Under these conditions we obtain $G \cong \mathscr{H}(e)$ under $g \to ge$.

References

1. A. H. Clifford and G. B. Preston, "The Algebraic Theory of Semigroups," Vol. 1, Mathematical Surveys No. 7, Amer. Math. Soc., Providence, R.I., 1961.
2. R. O. Fulp and J. W. Stepp, Semigroup extensions, *J. Reine Angew. Math.* **284** (1971), 28–41.
3. R. O. Fulp and J. W. Stepp, Structure of the semigroup of semigroup extensions, *Trans. Amer. Math. Soc.* **158** (1971), 63–73.
4. P. A. Grillet, Obstruction to coset extensions of semigroups, preprint.
5. J. E. Leech, $\mathscr{H}$-coextensions of monoids, *Mem. Amer. Math. Soc.* **157** (1975), 1–66.
6. J. E. Leech, The structure of a band of groups, *Mem. Amer. Math. Soc.* **157** (1975), 67–95.
7. J. E. Leech, Initial objects and idempotent generated semigroups, in preparation.
8. S. MacLane, "Homology Theory," Springer-Verlag, Berlin, 1963.