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ON THE HOMOLOGY OF DOUBLE BRANCHED COVERS

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Dedicated to Frank Raymond on the occasion of his sixtieth birthday

ABSTRACT. If $\pi : \tilde{X} \to X$ is a double branched cover, with branching set F, we relate $H_*(\tilde{X} : \mathbb{Z}_2)$, $H_*(X : \mathbb{Z}_2)$, $H_*(X, F : \mathbb{Z}_2)$, and $H_*(F : \mathbb{Z}_2)$.

In this note we present a pair of observations on the homology of double branched covers. These observations arose in our investigations [LW1, LW2], but because of the specialized nature of those investigations and the potential general utility of these observations, we have chosen to present them separately here.

All (co)homology in this paper is to be taken with $\mathbb{Z}_2$ coefficients.

Our first result is a simple generalization of the Gysin sequence, which, however, we have not been able to find in the literature.

Theorem 1. *Let $\pi : \widetilde{X} \to X$ be a twofold cover of the simplicial complex X, branched over a subcomplex F of X. Let A be an arbitrary subcomplex of X, and set $\widetilde{A} = \pi^{-1}(A)$. If T_* denotes the transfer map on homology, then there is a long exact sequence*

$$\cdots \to H_i(X, F \cup A) \xrightarrow{T_*} H_i(\widetilde{X}, \widetilde{A}) \xrightarrow{\pi_*} H_i(X, A) \xrightarrow{d_*} H_{i-1}(X, F \cup A) \to \cdots .$$

Proof. If $T : C_*(X, A) \to C_*(\widetilde{X}, \widetilde{A})$ is the transfer map on chains, then there is a short exact sequence of chain complexes

$$0 \to C_*(X, F \cup A) \xrightarrow{T} C_*(\widetilde{X}, \widetilde{A}) \xrightarrow{\pi} C_*(X, A) \to 0. \quad \square$$

We suppose throughout the remainder of this paper that we are in the following situation (where we use the term "manifold" to mean connected, compact manifold):

Situation $(*)$. X is a smooth, and hence piecewise linear, n-manifold (possibly with boundary), and $\pi : \tilde{X} \to X$ is a double cover branched over a subcomplex F, where F is the union (not necessarily disjoint) of codimension 2 smooth

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submanifolds $F_1, \ldots, F_k$ which are pairwise mutually transverse. Also, F is transverse to ∂X, and hence $F \cap \partial X = \partial F$, which may be empty.

In Situation $(*)$, Theorem 1 yields the long exact sequence

$$\cdots \to H_i(X, F \cup \partial X) \xrightarrow{T_*} H_i(\tilde{X}, \partial \tilde{X}) \xrightarrow{\pi_*} H_i(X, \partial X) \xrightarrow{d_*} H_{i-1}(X, F \cup \partial X) \to \cdots.$$

Theorem 2. *In Situation $(*)$, let d_* be as in Theorem 1 and let ∂_* be the composition of the boundary map of the triple $(X, F \cup \partial X, \partial F)$ with the projection of*

$$H_*(F \cup \partial X, \partial F) \cong H_*(F, \partial F) \oplus H_*(\partial X, \partial F)$$

onto its first summand. Then the composite

$$H_n(X, \partial X) \xrightarrow{d_*} H_{n-1}(X, F \cup \partial X) \xrightarrow{\partial_*} H_{n-2}(F, \partial F)$$

satisfies

$$\partial_* d_*([X, \partial X]) = [F_1, \partial F_1] + \cdots + [F_k, \partial F_k].$$

Here $[X, \partial X]$ and $[F_i, \partial F_i]$ denote the fundamental homology classes of $(X, \partial X)$ and $(F_i, \partial F_i)$, and the sum lies in $H_{n-2}(F_1, \partial F_1) \oplus \cdots \oplus H_{n-2}(F_k, \partial F_k) \cong H_{n-2}(F, \partial F)$.

Proof. First we consider the case $(X, \partial X) = (D, S)$, where D is a 2-disk, S is its boundary circle, and $F = C$, the center of D. (Then we may identify $\pi : (\widetilde{X}, \partial \widetilde{X}) \to (X, \partial X)$ with the map $z \mapsto z^2$, $z \in \mathbb{C}$, $|z| \leq 1$.) Let R be a radius of D. Then it is easy to check directly that $d_*([D, S]) = [R] \in H_1(X, F \cup \partial X)$ and $\partial_*([R]) = [C] \in H_0(F)$ as claimed.

The exact sequence of Theorem 1 may be dualized to cohomology, and so we have a map $d^* : H^{i-1}(X, F \cup \partial X) \to H^i(X, \partial X)$. Then, if $\{\ \}$ denotes the fundamental cohomology class, we have in the above case

$$(**) \qquad \delta^*(\{C\}) = \{R\}, \qquad d^*(\{R\}) = \{D, S\}.$$

Now for the general case. Consider a submanifold F_i in F, and let E_i be the complement of an open regular neighborhood of $F_i \cap (F_1 \cup \cdots \cup F_{i-1} \cup F_{i+1} \cup \cdots \cup F_k)$ in F_i. Let P_i be a closed regular neighborhood of F_i in X. Then we may identify P_i with the total space of a D^2-bundle over F_i, and we assume that P_i is chosen small enough so that if N_i is the restriction of P_i to E_i, then $N_i \cap F = E_i$. Let $T_i = \partial P_i \cap N_i$, an S^1-bundle over E_i. Consider the diagram

$$\begin{array}{ccccc}
H^{n-2}(E_i, \partial E_i) & = & H^{n-2}(E_i, \partial E_i) & = & H^{n-2}(E_i, \partial E_i) \\
\downarrow \cup\{D,S\} & & \downarrow \cup\{R\} & & \downarrow \cup\{C\} \\
H^n(N_i, \partial N_i) & \xleftarrow{d^*} & H^{n-1}(N_i, E_i \cup \partial N_i) & \xleftarrow{\delta^*} & H^{n-2}(E_i, \partial E_i) \\
\downarrow & & \downarrow & & \downarrow \\
H^n(X, \partial X) & \xleftarrow{d^*} & H^{n-1}(X, F_i \cup \partial X) & \xleftarrow{\delta^*} & H^{n-2}(F_i, \partial F_i)
\end{array}$$

where we identify D with a fiber of N_i.

In this diagram we use $\{C\}$ to denote the unique class of $H^0(E_i)$ restricting to the class $\{C\} \in H^0(D_i)$, where D_i is the fiber of N_i over any point of $\text{Int}(E_i)$. This class is obviously just the nonzero element of $H^0(E_i)$. Similarly, we use $\{D, S\}$ to denote the unique class in $H^2(N_i, T_i)$ restricting to the class $\{D, S\} \in H^2(D_i, S_i)$, with D_i as above and $S_i = \partial D_i$. This class is simply the Thom class of the bundle N_i [B, II.2.3]. The vertical map in the upper left-hand corner is actually the composition of the isomorphism $H^{n-2}(E_i, \partial E_i) \cong H^{n-2}(N_i, N_i|\partial E_i)$ with the cup product with the class $\{D, S\}$ under the cup product map $H^{n-2}(N_i, N_i|\partial E_i) \times H^2(N_i, T_i) \to H^n(N_i, \partial N_i)$. Also, we use $\{R\}$ to denote the class in $H^1(N_i, E_i \cup T_i)$ which restricts to the class $\{R\}$ in $H^1(D_i, C_i \cup S_i)$, where C_i is the center of D_i, in each fiber, and the middle vertical map is a similar composition.

First, we observe that this diagram commutes. For the top half, this follows directly from $(**)$, and for the bottom half, this is clear.

Second, we observe that all four vertical maps on the outside of the diagram are isomorphisms. For the map in the upper left-hand corner, this is the Thom isomorphism [B, II.2.3], and for the other three maps, this is clear.

Thus we see that the composition $d^*\delta^*$ on the bottom line is an isomorphism, and since $\{F_i, \partial F_i\}$ (resp. $\{X, \partial X\}$) is the only nontrivial element of $H^{n-2}(F_i, \partial F_i)$ (resp. $H^n(X, \partial X)$), we have $d^*\delta^*(\{F_i, \partial F_i\}) = \{X, \partial X\}$, $i = 1, \dots, k$. Then, if $\langle\, ,\rangle$ denotes the Kronecker product of cohomology and homology,

$$\begin{aligned} 1 &= \langle \{X, \partial X\}, [X, \partial X] \rangle = \langle d^*\delta^*(\{F_i, \partial F_i\}), [X, \partial X] \rangle \\ &= \langle \{F_i, \partial F_i\}, \partial_* d_*([X, \partial X]) \rangle \end{aligned}$$

for $i = 1, \dots, k$, and hence $\partial_* d_*([X, \partial X]) = [F_1, \partial F_1] + \cdots + [F_k, \partial F_k]$ as claimed. □

Remark. Note that in the proof of Theorem 2 we removed a neighborhood of $F_i \cap (F_1 \cup \cdots \cup F_{i-1} \cup F_{i+1} \cup \cdots \cup F_k)$. Thus, while we assume F_i and F_j transverse for $i \neq j$, we need no assumption on triple (or higher) intersections. Similarly, while we assume F_i transverse to ∂X for each i, we need no assumption on double (or higher) intersections with ∂X. (Of course, $\tilde{X}$ will be a manifold if and only if $F_i \cap F_j = \varnothing$ for $i \neq j$.)

Let us indicate the use of Theorem 2 in conjunction with Theorem 1. For simplicity let us suppose that $\partial X = \varnothing$. Of course, $\pi_*([\tilde{X}]) = 0$, so $[X] \notin \mathrm{Im}(\pi_*)$. However, consider a class $y \in H_q(X)$ and the problem of deciding whether $y \in \mathrm{Im}(\pi_*)$ or, equivalently whether $d_*(y) = 0$. Suppose that y is represented by an embedded submanifold Y, i.e., $y = i_*([Y])$, where $i: Y \to X$ is the inclusion. Of course, if $Y \subset F$, then clearly $y \in \mathrm{Im}(\pi_*)$. Otherwise, we may suppose that for some $j \leq k$, Y intersects $F_1, \dots, F_j$ transversely and $F_{j+1}, \dots, F_k$ not at all and that $G_i = F_i \cap Y$, $i = 1, \dots, j$, are pairwise mutually transverse in Y. Let $G = F \cap Y = G_1 \cup \cdots \cup G_j$. Then we have a commutative diagram

$$\begin{array}{ccccc} H_q(X) & \xrightarrow{d_*} & H_{q-1}(X, F) & \xrightarrow{\partial_*} & H_{q-2}(F) \\ \uparrow{\scriptstyle i_*} & & \uparrow{\scriptstyle i_*} & & \uparrow{\scriptstyle i_i} \\ H_q(Y) & \xrightarrow{d_*} & H_{q-1}(Y, G) & \xrightarrow{\partial_*} & H_{q-2}(G) \end{array}$$

and

$$\begin{aligned} \partial_* d_*(y) &= \partial_* d_*(i_*([Y])) = i_*(\partial_* d_*([Y])) \\ &= i_*([G_1] + \cdots + [G_j]) = g \in H_{q-2}(F). \end{aligned}$$

Thus, if $g \neq 0$ in $H_{q-2}(F)$, then $d_*(y) \neq 0$ and $y \notin \operatorname{Im}(\pi_*)$. On the other hand, if $g = 0$ and $H_{q-1}(X) = 0$, then $d_*(y) = 0$ and $y \in \operatorname{Im}(\pi_*)$.

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