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An obstruction to slicing knots using the eta invariant

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0. *Introduction*

We establish a connection between the η invariant of Atiyah, Patodi and Singer ([**1**, **2**]) and the condition that a knot $K \subset S^3$ be slice. We produce a new family of metabelian obstructions to slicing K such as those first developed by Casson and Gordon in [**4**] in the mid 1970s. Surgery is used to turn the knot complement $S^3 - K$ into a closed manifold M and, for given unitary representations of $\pi_1(M)$, η can be defined. Levine has recently shown in [**11**] that η acts as an homology cobordism invariant for a certain subvariety of the representation space of $\pi_1(N)$, where N is zero-framed surgery on a knot concordance. We demonstrate a large family of such representations, show they are extensions of similar representations on the boundary of N and prove that for slice knots, the value of η defined by these representations must vanish.

The paper is organized as follows; Section 1 consists of background material on η and Levine's work on how it is used as a concordance invariant [**11**]. Section 2 deals with unitary representations of $\pi_1(M)$ and is broken into two parts. In 2·1, homomorphisms from $\pi_1(M)$ to a metabelian group Γ are developed using the Blanchfield pairing. Unitary representations of Γ are then considered in 2·2. Conditions ensuring that such two stage representations of $\pi_1(M)$ allow η to be used as an invariant are developed in Section 3 and $\mathscr{P}_k$, the family of such representations, is defined. Section 4 contains the main result of the paper, Theorem 4·3. Lastly, in Section 5, we demonstrate the construction of representations in $\mathscr{P}_k$.

Notation 0·1. Throughout this paper, let Λ be the ring of Laurent polynomials $\mathbb{Z}[\mathbb{Z}]$. We will frequently write $\mathbb{Z}$ multiplicatively as $\{t^i\}_{i\in\mathbb{Z}}$ so that $\Lambda = \mathbb{Z}[t, t^{-1}]$. Let S be the multiplicative set $\{p(t) \in \Lambda \mid p(1) = 1\}$.

In addition, let $R_k(G) = \{\theta: G \longrightarrow U(k)\}$ be the space of k-dimensional unitary representations of the group G.

1. *Background*

In [**1**] and [**2**], Atiyah, Patodi and Singer developed an invariant $\eta_\alpha(M)$ defined for any compact, oriented, odd dimensional Riemannian manifold M, with a unitary representation $\alpha: \pi_1(M) \longrightarrow U(k)$. This invariant appears in their *Index Theorem*,

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which relates $\eta_\alpha(M)$ to $\text{sign}_\beta(N)$, where $\partial N = M$ and $\beta: \pi_1(N) \longrightarrow U(k)$ restricts to α on $\pi_1(M)$.

A reduced η invariant is also defined in [**2**]; $\tilde{\eta}_\alpha(M) = \eta_\alpha(M) - k\eta_o(M)$, where o denotes the trivial representation. The main result of [**2**] is the following:

THEOREM 1·1 ([**2**], theorem 2·4). *$\tilde{\eta}_\alpha(M)$ is independent of the Riemannian metric and so is a differential invariant of M and α. If $M = \partial N$ and $\alpha: \pi_1(M) \longrightarrow U(k)$ extends to a unitary representation on $\pi_1(N)$, then $\tilde{\eta}_\alpha(M) = k \cdot \text{sign}\,(N) - \text{sign}_\alpha(N)$.*

In [**11**], Levine investigates the behaviour of $\tilde{\eta}_\alpha(M)$ under homology cobordism and applies it to study link concordance. We first review some of his results.

Definition 1·2. If G is a group, then a *G-manifold* is a pair (M, α) where M is a compact oriented manifold with components $\{M_i\}$ and α is a collection of homomorphisms $\alpha_i: \pi_1(M_i) \longrightarrow G$, where each α_i is defined up to inner automorphisms of G.

Thus for any such G-manifold (M, α) and any representation $\theta \in R_k(G)$, the composition $\theta\alpha: \pi_1(M) \longrightarrow U(k)$ determines $\tilde{\eta}_{\theta\alpha}(M) \in \mathbb{R}$. For $M = \coprod_{i=1}^n M_i$, $\tilde{\eta}_{\theta\alpha}(M) = \sum_{i=1}^n \tilde{\eta}_{\theta\alpha_i}(M_i)$.

Definition 1·3. Two odd-dimensional G-manifolds (M_i, α_i), $i = 0, 1$, are *homology G-bordant* if there exists a G-manifold (N, β) such that $\partial N = M_0 \amalg -M_1$, $\mathrm{H}_*(N, M_i) \cong 0$ and up to inner automorphisms of G, $\beta|\pi_1(M_i) = \alpha_i$.

Definition 1·4. Define $\rho(M, \alpha): R_k(G) \longrightarrow \mathbb{R}$ by $\rho(M, \alpha) \cdot \theta = \tilde{\eta}_{\theta\alpha}(M)$.

Levine shows [**11**], that if (M_0, α_0) and (M_1, α_1) are homology G-bordant, then $\rho(M_0, \alpha_0) \cdot \theta = \rho(M_1, \alpha_1) \cdot \theta$ for $\theta \in R_k(G)$ lying outside some *special subvariety*, defined in terms of *perfect modules*.

Definition 1·5. A module B over the group-ring $\mathbb{Z}G$ is a *perfect $\mathbb{Z}G$ module* if $\mathbb{Z} \otimes_{\mathbb{Z}G} B \cong 0$.

In addition, we will assume such modules are finitely presented. Note that this implies that for λ a presentation matrix for B, $\varepsilon(\lambda)$ is unimodular over $\mathbb{Z}$, where $\varepsilon: \mathbb{Z}G \longrightarrow \mathbb{Z}$ is augmentation. In particular, if λ is square, then $\det(\varepsilon(\lambda)) = \pm 1$.

Definition 1·6. A *special subvariety* is a subvariety of the form

$$\Sigma_A = \{\theta \in R_k(G) \mid \mathbb{C}^k \otimes_\theta A \neq 0\},$$

where $A = \mathbb{C} \otimes_{\mathbb{Z}} B$ for some perfect $\mathbb{Z}G$ module B.

Note that A is a finitely presented $\mathbb{C}G$ module with presentation matrix $(\lambda_{i,j})$ and Σ_A is the set of representations of G such that the image of $(\lambda_{i,j})$ is singular. Thus, Σ_A is a subvariety of $R_k(G)$.

The following proposition is due to Levine. The proof is included for completeness.

PROPOSITION 1·7 ([**11**], corollary 3·3). *For any homology G-bordant manifolds (M_i, α_i), $i = 0, 1$, there exists a special subvariety Σ such that $\rho(M_0, \alpha_0) = \rho(M_1, \alpha_1)$ on $R_k(G) - \Sigma$.*

Proof. Let (N,β) be a $2n$-dimensional homology G-bordism between (M_0,α_0) and (M_1,α_1). Then $\mathrm{H}_*(N,M_i)\cong 0$ for $i=0,1$, and so the intersection pairing

$$\mathrm{H}_n(N)\otimes\mathrm{H}_n(N)\longrightarrow\mathbb{Z}$$

is identically zero. Thus, $\mathrm{sign}\,(N)=0$.

We want to find a subvariety Σ such that for any $\theta\in R_k(G)-\Sigma$ we get $\mathrm{H}_*(N,M_i;\theta\beta)=0$ for some $i\in\{0,1\}$. Then $\mathrm{sign}_{\theta\beta}(N)=0$ and so, by Theorem 1·1 and the previous paragraph, $\rho(M_0,\alpha_0)=\rho(M_1,\alpha_1)$ on $R_k(G)-\Sigma$.

We proceed to show the above subvariety Σ exists. Let $C_*=C_*(N,M_i)$ for some $i\in\{0,1\}$. Since $\mathrm{H}_*(N,M_i)\cong 0$ and C_* is a chain complex of free $\mathbb{Z}$ modules, C_* is contractible. Let s_* be the contraction. Then $\partial s_{n-1}+s_n\partial=I_n$, where I_n is the identity on C_n.

Let $\widetilde{(N,M_i)}$ be the universal cover of (N,M_i) and consider $\widetilde{C}_*=C_*\widetilde{(N,M_i)}$. Since $\widetilde{C}_*$ is a chain complex of free $\mathbb{Z}\pi_1(N)$ modules, the contraction s_* lifts to a chain homotopy $\tilde{s}_*$ with $\tilde{\partial}\tilde{s}_{n-1}+\tilde{s}_n\tilde{\partial}=\psi_n$. Then $\psi_*:\widetilde{C}_*\longrightarrow\widetilde{C}_*$ is a chain homomorphism lifting I_*. By abuse of notation, we will denote the chain homomorphism on the complex of free $\mathbb{Z}G$ modules induced by the map $\beta:\pi_1(N)\longrightarrow G$ by ψ_* as well.

Let $B_n=\mathrm{cok}\,(\psi_n)$ for such a complex of $\mathbb{Z}G$ modules. Then B_n is a perfect $\mathbb{Z}G$ module, i.e. $\mathbb{Z}\otimes_{\mathbb{Z}G}B_n\cong 0$. Letting $A_n=\mathbb{C}\otimes_{\mathbb{Z}}B_n$, we get the exact sequence

$$C_n(N,M_i;\mathbb{C}G)\xrightarrow{\psi_n}C_n(N,M_i;\mathbb{C}G)\twoheadrightarrow A_n.$$

If we let

$$\Sigma_n=\{\theta\in R_k(G)\mid\mathbb{C}^k\otimes_\theta A_n\neq 0\},$$

then for $\theta\in R_k(G)-\Sigma_n$ we obtain the exact sequence

$$\mathbb{C}^k\otimes_\theta C_n(N,M_i;\mathbb{C}G)\xrightarrow{I\otimes\psi_n}\mathbb{C}^k\otimes_\theta C_n(N,M_i;\mathbb{C}G)\longrightarrow\mathbb{C}^k\otimes_\theta A_n\cong 0$$

where I is the identity matrix for $\mathbb{C}^k$. Thus $I\otimes\psi_n$ is an isomorphism of complex vector spaces. Letting $\Sigma=\bigcup\Sigma_n$, Σ is a special subvariety ([**11**], p. 94), and for $\theta\in R_k(G)-\Sigma$, $I\otimes\psi_*$ will be a chain isomorphism. Then $I\otimes\tilde{s}_*$ is a chain contraction on $\mathbb{C}^k\otimes_\theta C_*(N,M_i;\mathbb{C}G)$, and so $\mathrm{H}_*(N,M_i;\theta\beta)\cong 0$.

Terminology 1·8. We call the special subvariety Σ defined in the proof above the *special subvariety associated with the chain complex* $C_*(N,M_i;\mathbb{Z}G)$.

2. *Unitary representations of* $\pi_1(M)$

In the previous section, we saw how the η invariant can be used as a cobordism invariant over a group G. In order to make use of this for knot concordance, we need to make the knot complement into a closed manifold. By doing surgery on the knot concordance, i.e. removing a tubular neighbourhood of $S^1\times I$ and replacing it with $(D^2\times I)\times S^1$, the boundary becomes the disjoint union of zero framed surgeries on the knot complements.

Notation 2·1. We denote by N and M_i the zero framed surgery on the knot concordance and zero framed surgery on the knot K_i, respectively, with $\partial N=M_0\amalg -M_1$ for concordant knots K_0, K_1.

We also need a unitary representation of $\pi_1(\partial N)$ that extends to $\pi_1(N)$ *and* avoids the special subvarieties of the previous section. The problem is divided into two parts. First, a homomorphism from $\pi_1(\partial N)$ to an intermediate group Γ is constructed with the required extension criteria. Then representations of Γ are analysed, and are composed with the extendible homomorphism. The problem of avoiding the special subvarieties is tackled in Section 3.

2·1. *Homomorphisms from* $\pi_1(M)$ *to* Γ

Definition 2·2. $\Gamma = S^{-1}\Lambda/\Lambda \rtimes \mathbb{Z}$, where the action of an element $n \in \mathbb{Z}$ is multiplication in Λ by t^n.

We construct a family of non-abelian homomorphisms

$$\{\alpha_x : \pi_1(M) \longrightarrow \Gamma \mid x \in \mathrm{H}_1(M;\Lambda)\}$$

using the Blanchfield pairing. They will be dependent on $x \in \mathrm{H}_1(M;\Lambda)$, and well defined up to inner automorphisms of $\pi_1(M)$.

We begin with some well known results of Blanchfield [**3**] and provide sketches of proofs following [**10**]. We have not seen Corollary 2·9 in the literature, but believe it to be folklore.

PROPOSITION 2·3. *For M zero-surgery on a knot $K \subset S^3$,*

$$\mathrm{H}_1(M;\Lambda) \cong \mathrm{Hom}_\Lambda(\mathrm{H}_1(M;\Lambda), S^{-1}\Lambda/\Lambda).$$

Sketch of Proof. $\mathrm{H}_1(M;\Lambda) \cong \mathrm{H}^2(M;\Lambda)$ by Poincaré Duality. Using the Universal Coefficient Spectral Sequence, one shows $\mathrm{H}^2(M;\Lambda) \cong \mathrm{Ext}_\Lambda(H_1(M;\Lambda),\Lambda)$.

To see that $\mathrm{Ext}_\Lambda(\mathrm{H}_1(M;\Lambda),\Lambda) \cong \mathrm{Hom}_\Lambda(\mathrm{H}_1(M;\Lambda), S^{-1}\Lambda/\Lambda)$, consider the long exact sequence for the Λ module $\mathrm{H}_1(M;\Lambda)$

$$\cdots \longrightarrow \mathrm{Hom}_\Lambda(\mathrm{H}_1(M;\Lambda), S^{-1}\Lambda) \longrightarrow \mathrm{Hom}_\Lambda(\mathrm{H}_1(M;\Lambda), S^{-1}\Lambda/\Lambda)$$

$$\longrightarrow \mathrm{Ext}_\Lambda(\mathrm{H}_1(M;\Lambda),\Lambda) \longrightarrow \mathrm{Ext}_\Lambda(\mathrm{H}_1(M;\Lambda), S^{-1}\Lambda) \longrightarrow \cdots$$

obtained from the coefficient sequence

$$\Lambda \rightarrowtail S^{-1}\Lambda \twoheadrightarrow S^{-1}\Lambda/\Lambda.$$

Since $\mathrm{H}_1(M;\Lambda)$ is isomorphic to a subgroup of $\mathrm{H}_1(S^3 - K;\Lambda)$, which is S torsion (see [**12**]), $\mathrm{Hom}_\Lambda(\mathrm{H}_1(M;\Lambda), S^{-1}\Lambda)$ and $\mathrm{Ext}_\Lambda(\mathrm{H}_1(M;\Lambda), S^{-1}\Lambda)$ are both trivial. The result follows.

Definition 2·4. The isomorphism in Proposition 2·3 defines the nonsingular bilinear *Blanchfield pairing*

$$\langle\, , \rangle_M : \mathrm{H}_1(M;\Lambda) \otimes \mathrm{H}_1(M;\Lambda) \longrightarrow S^{-1}\Lambda/\Lambda.$$

One computes that for $a, b \in \mathrm{H}_1(M;\Lambda)$,

$$\langle a, b\rangle_M = \frac{1}{p(t)} \sum_{i\in\mathbb{Z}} (t^i a, c)_M \; t^i \quad (\mathrm{mod}\ \Lambda)$$

where $(\, ,)_M$ is the usual intersection pairing, and $c \in C_2(M;\Lambda)$ such that $\partial c = p(t)b$. Since $\mathrm{H}_1(M;\Lambda)$ is S-torsion, such a $p(t)$ exists for all $b \in \mathrm{H}_1(M;\Lambda)$.

The following is a relative version of Proposition 2·3. The proof follows that of Proposition 2·3 exactly.

PROPOSITION 2·5. *For N an homology cobordism between 0-surgeries on knots M_0 and M_1,*

$$\mathrm{H}_2(N, \partial N; \Lambda) \cong \mathrm{Hom}_\Lambda(\mathrm{H}_1(N; \Lambda), S^{-1}\Lambda/\Lambda).$$

This defines a bilinear Blanchfield style pairing on N,

$$\langle\, , \rangle_N : \mathrm{H}_2(N, \partial N; \Lambda) \otimes \mathrm{H}_1(N; \Lambda) \longrightarrow S^{-1}\Lambda/\Lambda,$$

however the pairing may be singular, as $\mathrm{H}_1(N; \Lambda)$ is not necessarily isomorphic to $\mathrm{Hom}_\Lambda(\mathrm{H}_2(N, \partial N; \Lambda), S^{-1}\Lambda/\Lambda)$.

Again, one computes that for $a \in \mathrm{H}_2(N, \partial N; \Lambda), b \in \mathrm{H}_1(N; \Lambda)$,

$$\langle a, b\rangle_N = \frac{1}{p(t)} \sum_{i \in \mathbb{Z}} (t^i a, c)_N \; t^i \quad (\mathrm{mod}\ \Lambda)$$

where $c \in C_2(N; \Lambda)$ such that $\partial c = p(t)b$, $p(t) \in S$.

Note that such pairings are additive over disjoint unions of manifolds by defining the pairings on the direct sum of the homology modules in the obvious way.

Definition 2·6. Let $D = \mathrm{im}\{\partial : \mathrm{H}_2(N, \partial N; \Lambda) \longrightarrow \mathrm{H}_1(\partial \mathrm{N}; \Lambda)\}$.

Notation 2·7. Let $D^\perp = \{a \in \mathrm{H}_1(\partial N; \Lambda) \mid \langle a, d\rangle = 0 \text{ for all } d \in D\}$ be the annihilator of D in $\mathrm{H}_1(\partial N; \Lambda)$.

PROPOSITION 2·8. *Let N be as above, $\partial N = M_0 \amalg -M_1$. Then:*

(i) $D \subseteq D^\perp$;

(ii) *$D^\perp$ is self annihilating under the Blanchfield pairing on ∂N, i.e. $D^\perp = D^{\perp\perp}$.*

Proof (*i*). Consider the exact sequence of pairs

$$\mathrm{H}_2(N, \partial N; \Lambda) \xrightarrow{\partial} \mathrm{H}_1(\partial \mathrm{N}; \Lambda) \xrightarrow{i_*} \mathrm{H}_1(\mathrm{N}; \Lambda),$$

and let $a, b \in D$. Then for any $a' \in \mathrm{H}_2(N, \partial N; \Lambda)$ such that $\partial a' = a$,

$$\langle a', i_*(b)\rangle_N = \langle a, b\rangle_{\partial N}.$$

Since $\ker\ i_* = \mathrm{im}\ \partial_*$,

$$\langle a, b\rangle_{\partial N} = \langle a', i_*(b)\rangle_N = \langle a', 0\rangle_N = 0,$$

and so $D \subseteq D^\perp$.

Proof (*ii*). It is clear from *Proof* (i) that $D^{\perp\perp} \subseteq D^\perp$. To see the converse, let $b \in D^\perp$. Then for any $a' \in \mathrm{H}_2(N, \partial N; \Lambda)$,

$$\langle a', i_*(b)\rangle_N = \langle \partial_*(a'), b\rangle_{\partial N} = 0.$$

But this Blanchfield style pairing on N may be singular, and so $i_*(b)$ may *not* be trivial in $\mathrm{H}_1(N; \Lambda)$. Following the proof of Proposition 2·3, Poincaré duality yields $\mathrm{H}_1(N; \Lambda) \cong \mathrm{H}^3(N, \partial N; \Lambda)$, and the Universal Coefficient Spectral Sequence gives us the short exact sequence

$$\mathrm{Ext}^2_\Lambda(\mathrm{H}_1(N, \partial N; \Lambda), \Lambda) \rightarrowtail \mathrm{H}^3(\mathrm{N}, \partial\mathrm{N}; \Lambda) \twoheadrightarrow \mathrm{Ext}_\Lambda(\mathrm{H}_2(\mathrm{N}, \partial\mathrm{N}; \Lambda), \Lambda).$$

Thus, $\mathrm{H}_1(N; \Lambda)/\mathrm{Ext}^2_\Lambda(\mathrm{H}_1(N, \partial N; \Lambda), \Lambda) \cong \mathrm{Ext}_\Lambda(\mathrm{H}_2(N, \partial N; \Lambda), \Lambda)$. Levine shows in [**10**] that the functors Ext_Λ and Ext^2_Λ pick out the $\mathbb{Z}$ torsion free and

$\mathbb{Z}$ torsion parts (respectively) of Λ modules such as $\mathrm{H}_1(N, \partial N; \Lambda), \Lambda)$ and $\mathrm{H}_2(N, \partial N; \Lambda), \Lambda)$. Since $\mathrm{Ext}_\Lambda(\mathrm{H}_2(N, \partial N; \Lambda), \Lambda)$ is $\mathbb{Z}$ torsion-free, the $\mathbb{Z}$ torsion part of $\mathrm{H}_1(N; \Lambda)$ is isomorphic to $\mathrm{Ext}^2_\Lambda(\mathrm{H}_1(N, \partial N; \Lambda), \Lambda)$.

If we combine all this with another result from the proof of Proposition 2·3, that $\mathrm{Hom}_\Lambda(\mathrm{H}_2(N, \partial N; \Lambda), S^{-1}\Lambda/\Lambda) \cong \mathrm{Ext}_\Lambda(\mathrm{H}_2(N, \partial N; \Lambda), \Lambda)$, we see that

$$\mathrm{Hom}_\Lambda(\mathrm{H}_2(N, \partial N; \Lambda), S^{-1}\Lambda/\Lambda) \cong \mathrm{H}_1(N; \Lambda)/\mathrm{Ext}^2_\Lambda(\mathrm{H}_1(N, \partial N; \Lambda), \Lambda).$$

Thus, the elements that give us a zero homomorphism under this Blanchfield style pairing are exactly the $\mathbb{Z}$ torsion elements of $\mathrm{H}_1(N, \partial N; \Lambda)$.

Let $a, b \in D^\perp$, and let $i_*(b) \in \mathrm{H}_1(N; \Lambda)$, be $\mathbb{Z}$-torsion of order n. Then $nb \in D$ and so

$$0 = \langle a, nb\rangle_{\partial N} = n\langle a, b\rangle_{\partial N}.$$

Thus $\langle a, b\rangle_{\partial N} = 0$ for all $a, b \in D^\perp$ and the result follows.

COROLLARY 2·9. *The homomorphism* $\phi \in \mathrm{Hom}_\Lambda(\mathrm{H}_1(\partial N; \Lambda), S^{-1}\Lambda/\Lambda)$ *extends to* $\phi' \in \mathrm{Hom}_\Lambda(\mathrm{H}_1(N; \Lambda), S^{-1}\Lambda/\Lambda)$ *if and only if there exists an* $x \in D$ *such that* $\phi(y) = \langle x, y\rangle_{\partial N}$ *for all* $y \in \mathrm{H}_1(\partial N; \Lambda)$.

Proof. Consider the diagram

$$\begin{array}{ccccc}
\mathrm{H}_2(N, \partial N; \Lambda) & \xrightarrow{\partial} & \mathrm{H}_1(\partial N; \Lambda) & \xrightarrow{i_*} & \mathrm{H}_1(N; \Lambda) \\
\downarrow \cong \text{(Prop. 2·5)} & & \downarrow \cong \text{(Prop. 2·3)} & & \\
\mathrm{Hom}_\Lambda(\mathrm{H}_1(N; \Lambda), S^{-1}\Lambda/\Lambda) & \xrightarrow{\hat{\partial}} & \mathrm{Hom}_\Lambda(\mathrm{H}_1(\partial N; \Lambda), S^{-1}\Lambda/\Lambda) & &
\end{array}$$

where $\hat{\partial}$ is defined by restricting $\phi' \in \mathrm{Hom}_\Lambda(\mathrm{H}_1(N; \Lambda), S^{-1}\Lambda/\Lambda)$ to $i_*(\mathrm{H}_1(\partial N; \Lambda))$. A diagram chase confirms that it commutes. Let $\phi \in \mathrm{Hom}_\Lambda(\mathrm{H}_1(\partial N; \Lambda), S^{-1}\Lambda/\Lambda)$. By the definition of the Blanchfield pairing, there exists an $x \in \mathrm{H}_1(\partial N; \Lambda)$ such that $\phi(y) = \langle x, y\rangle_{\partial N}$ for all $y \in \mathrm{H}_1(\partial N; \Lambda)$.

Let ϕ' be an extension of ϕ. Then there exists an $x' \in \mathrm{H}_2(N, \partial N; \Lambda)$ such that $\phi'(z) = \langle x', z\rangle_N$ for all $z \in \mathrm{H}_1(N; \Lambda)$. Then

$$\phi(y) = \hat{\partial}\phi'(y) = \langle x', i_*(y)\rangle_N = \langle \partial(x'), y\rangle_{\partial N},$$

as in the proof of Proposition 2·8. Since the Blanchfield pairing is nonsingular, $\partial x' = x$ and so $x \in D$.

Conversely, let $x \in D$. Then there exists an $x' \in \mathrm{H}_2(N, \partial N; \Lambda)$ such that $\partial x' = x$. Define $\phi' \in \mathrm{Hom}_\Lambda(\mathrm{H}_1(N; \Lambda), S^{-1}\Lambda/\Lambda)$ as $\phi'(z) = \langle x', z\rangle_N$ for all $z \in \mathrm{H}_1(N; \Lambda)$. Then for any $y \in \mathrm{H}_1(\partial N; \Lambda)$,

$$\hat{\partial}\phi'(y) = \phi'(i_*(y)) = \langle x', i_*(y)\rangle_N = \langle \partial(x'), y\rangle_{\partial N} = \langle x, y\rangle_{\partial N} = \phi(y),$$

and so ϕ' extends ϕ.

We use this extension property of the Blanchfield pairing to construct homomorphisms from $\pi_1(\partial N)$ to the metabelian group Γ that extend over $\pi_1(N)$. Our first step is getting from $\pi_1(\partial N)$ to $\mathrm{H}_1(\partial N; \Lambda)$, in order to apply the Blanchfield pairing.

Notation 2·10. Let $\pi = \pi_1(M)$, π' be the commutator subgroup $[\pi, \pi]$, and $\pi'' = [\pi', \pi']$.

Note that $\mathrm{H}_1(\pi) \cong \mathrm{H}_1(M) \cong \mathbb{Z}$, generated by the image of any meridian of K. Since π'' is normal in both π and π', and $\mathbb{Z}$ is free, we get the following split exact sequence.

$$\pi'/\pi'' \rightarrowtail \pi/\pi'' \underset{a}{\overset{s}{\rightleftarrows}} \mathbb{Z}.$$

A choice of splitting $s: \mathbb{Z} \longrightarrow \pi/\pi''$ determines a unique isomorphism

$$\psi: \pi/\pi'' \longrightarrow \pi'/\pi'' \rtimes \mathbb{Z}$$

given by $\psi(g) = \big(g \cdot sa(g^{-1}), a(g)\big)$ with inverse $\psi^{-1}(h, k) = h \cdot s(k)$.

Recall that given a splitting $s: \mathbb{Z} \longrightarrow \pi/\pi''$, the action of $\mathbb{Z}$ on π'/π'' is given by conjugation in π/π'' by $s(1)$. Thus π'/π'' is a Λ module. Since $\pi' \cong \pi_1(M_\infty, *)$, the $\mathbb{Z}$ cover of M, $\pi'/\pi'' \cong \mathrm{H}_1(M_\infty)$, and a choice of base point in the $\mathbb{Z}$ cover determines the Λ module structure for the group, giving an isomorphism between $\mathrm{H}_1(M_\infty)$ and $\mathrm{H}_1(M; \Lambda)$ (see [**12**], section 7·*D*). This induces an isomorphism

$$\phi: \pi'/\pi'' \rtimes \mathbb{Z} \longrightarrow \mathrm{H}_1(\mathrm{M}; \Lambda) \rtimes \mathbb{Z}.$$

Definition 2·11. For $x \in \mathrm{H}_1(M; \Lambda)$, define $B_x: \mathrm{H}_1(M; \Lambda) \rtimes \mathbb{Z} \longrightarrow \Gamma = S^{-1}\Lambda/\Lambda \rtimes \mathbb{Z}$ by $B_x(y, k) = (\langle y, x\rangle_M, k)$ where $\langle y, x\rangle_M$ is the Blanchfield pairing on $\mathrm{H}_1(M; \Lambda)$.

Definition 2·12. Given a splitting $s: \mathbb{Z} \longrightarrow \pi/\pi''$ and an isomorphism ϕ as above, we define $\alpha_x: \pi \longrightarrow \Gamma$ by the composition

$$\pi \xrightarrow{q} \pi'/\pi'' \xrightarrow{\psi} \pi'/\pi'' \rtimes \mathbb{Z} \xrightarrow{\phi} \mathrm{H}_1(\mathrm{M}; \Lambda) \rtimes \mathbb{Z} \xrightarrow{B_x} \Gamma,$$

where q is the quotient map and ψ is the isomorphism defined by our choice of splitting s.

PROPOSITION 2·13. *α_x is well defined up to inner automorphisms of $\pi = \pi_1(M)$.*

Proof. The Blanchfield pairing is well defined and so B_x is well defined, q is a quotient homomorphism and therefore well defined. This leaves ψ and ϕ. Since ψ is uniquely determined by the choice of a meridian of K for the image of $s(1)$ and ϕ by the choice of base point in M_∞, ψ and ϕ are well defined up to inner automorphisms of their domains. But these inner automorphisms lift to inner automorphisms of π, giving the result.

2·2. *Unitary Representations of* Γ

In order to have unitary representations of $\pi_1(M)$ for the η invariant, we will compose the homomorphisms from Section 2·1 with unitary representations of the group Γ. The results of this section are used in the proof of Theorem 4·3 and in the construction of examples in Section 5.

LEMMA 2·14 ([**6**], lemma 3·4). *Let $\{p_1, \ldots, p_r\}$ be the set of distinct primes dividing k, and d_i be the largest power of p_i dividing k for each $1 \leqslant i \leqslant r$. Then there is a map $\Psi: S^{-1}\mathbb{Z}[\mathbb{Z}_k] \longrightarrow \bigoplus_{i=1}^{r} \mathbb{Z}_{(p_i)}[\mathbb{Z}_{d_i}]$ such that the following diagram is Cartesian, where*

Δ *is the diagonal map.*

$$\begin{array}{ccc} S^{-1}\mathbb{Z}[\mathbb{Z}_k] & \xrightarrow{\ \varepsilon\ } & \mathbb{Z} \\ \Big\downarrow{\scriptstyle\Psi} & & \Big\downarrow{\scriptstyle\Delta} \\ \bigoplus_{i=1}^{r} \mathbb{Z}_{(p_i)}[\mathbb{Z}_{d_i}] & \xrightarrow{\ \oplus\varepsilon\ } & \bigoplus_{i=1}^{r} \mathbb{Z}_{(p_i)} \end{array}$$

Notation 2·15. We will henceforth denote $\mathbb{Z}[\mathbb{Z}_k]$ by Λ_k.

Corollary 2·16. *$S^{-1}\Lambda_k/\Lambda_k$ is a torsion $\mathbb{Z}$ module.*

Proof. Since the above diagram is Cartesian, the sequence

$$S^{-1}\Lambda_k \xrightarrow{(\Psi,\varepsilon)} \left(\bigoplus_{i=1}^{r} \mathbb{Z}_{(p_i)}[\mathbb{Z}_{d_i}]\right) \oplus \mathbb{Z} \xrightarrow{\oplus\varepsilon-\Delta} \bigoplus_{i=1}^{r} \mathbb{Z}_{(p_i)}$$

of additive groups is exact. Modding the first term by Λ_k and the subsequent terms by the appropriate images of Λ_k gives the result.

Definition 2·17. For $\theta \in R_k(\Gamma)$, let θ' be θ restricted to $S^{-1}\Lambda/\Lambda < \Gamma$. Then define $H_\theta = \mathrm{im}\,(\theta') < U(k)$.

Definition 2·18. For $g \in \Gamma$ an element that generates $\mathbb{Z}$ under abelianization, let $u = \theta(g)$.

Using these two definitions, we see that $\mathrm{im}\,(\theta)$ is generated by H_θ and u.

Note that the choice of $g \in \Gamma$ corresponds to a choice of splitting $s\colon \mathbb{Z} \longrightarrow \Gamma$ of the abelianization of Γ and so is well defined up to inner automorphisms of Γ. In practice, we let g be the image under α_x of a meridian of the knot K and so these inner automorphisms of Γ will lift back to inner automorphisms of $\pi_1(M)$ (see Proposition 2·13).

Note also that since $S^{-1}\Lambda/\Lambda$ is abelian, H_θ is abelian. Thus, it is contained in some maximal torus of $U(k)$. Since all maximal tori of $U(k)$ are conjugate, there exists an element $a \in U(k)$ such that $aH_\theta a^{-1} \subseteq T^k$, the maximal torus of diagonal elements in $U(k)$. We can thus assume, up to conjugacy in $U(k)$, that $H_\theta \subseteq T^k$.

Lemma 2·19. *Any matrix $h \in H_\theta$ has entries which are roots of unity. In particular, H_θ is a torsion abelian group.*

Proof. Let $W = N(H_\theta)/Z(H_\theta)$ be the Weyl group of $H_\theta \subseteq T^k$. Then W is isomorphic to a subgroup of the symmetric group on k letters (see e.g. [**7**]). Since $u \in N(H_\theta)$, there exists an $n \in \mathbb{Z}$ such that $u^n \in Z(H_\theta)$. Thus conjugating by u^n is the identity in $\mathrm{Aut}\,(H_\theta)$. Since the $\mathbb{Z}$ action in Γ is multiplication by t, $\theta((t^n - 1)S^{-1}\Lambda/\Lambda)$ is the identity in $U(k)$. This, and the exactness of localization, tells us that θ factors through $S^{-1}\Lambda_n/\Lambda_n \rtimes \mathbb{Z}$. By Corollary 2·16, $S^{-1}\Lambda_n/\Lambda_n$ is a torsion $\mathbb{Z}$ module, i.e. all elements are of finite order. Since every element of H_θ is the image of an element in $S^{-1}\Lambda_n/\Lambda_n$, H_θ is a torsion $\mathbb{Z}$ module and the result follows.

Remark 2·20. At this point it is important to note that, of the unitary representations of $\pi_1(\partial N)$ constructed in this section, those which extend to $\pi_1(N)$ are a larger set than they first appear. For $x \in D$ (Definition 2·6), we have shown that the Blanchfield pairing, and thus the representation $\theta\alpha_x$, extends to a unitary representation of $\pi_1(N)$ (Corollary 2·9). In fact, for any x in the self-annihilating submodule $D^\perp$ that contains D, we also get this extension property.

LEMMA 2·21. *Let $x \in D^\perp$. Then for all $\theta \in R_k(\Gamma)$, $\theta\alpha_x$ extends to a $U(k)$ representation of $\pi_1(N)$.*

Proof. We have seen in the proof of Proposition 2·8 that $i_*(x)$ is $\mathbb{Z}$-torsion for $x \in D^\perp$. Then there exists $n \in \mathbb{Z}$ such that $0 = ni_*(x) = i_*(nx)$ in $\mathrm{H}_1(N;\Lambda)$. Thus, $nx \in D$, and so α_{nx} extends to a homomorphism $\alpha'\colon \pi_1(N) \longrightarrow \Gamma$ based on the Blanchfield style pairing on N, defined as in Definition 2·12. Thus, the composite representation $\theta\alpha_{nx}$ extends to the representation $\theta\alpha'$ of $\pi_1(N)$.

Recall that $\pi_1(N)$ is finitely generated and let $\{g_1, \ldots, g_n\}$ be a generating set. Then $\alpha'(g_i) = (p_i, m_i) \in \Gamma = S^{-1}\Lambda/\Lambda \rtimes \mathbb{Z}$. Under θ, these are elements $h_i \cdot u^{m_i} \in U(k)$, a generating set for the image of $\pi_1(N)$ in $U(k)$, with each $h_i \in H_\theta$.

As shown in Lemma 2·19, H_θ is a torsion abelian group whose elements are diagonal matrices in $U(k)$ with entries that are roots of unity. Thus, we can write

$$h_i = \begin{pmatrix} \exp\{2\pi i q_1\} & & \\ & \ddots & \\ & & \exp\{2\pi i q_k\} \end{pmatrix}$$

for $0 < q_1 \ldots q_k \leqslant 1$ in $\mathbb{Q}$.

In general, for $p \in S^{-1}\Lambda/\Lambda$, $\theta(np) = (\theta(p))^n$ in $U(k)$. Thus, we define the representation $\chi\colon \pi_1(N) \longrightarrow U(k)$ by $\chi(g_i) = h_i^{1/n} \cdot u^{m_i}$ where

$$h_i^{1/n} = \begin{pmatrix} \exp\{2\pi i q_1/n\} & & \\ & \ddots & \\ & & \exp\{2\pi i q_k/n\} \end{pmatrix}.$$

Note that by the linearity of the Blanchfield pairing, and the use of the generating set $\{g_i\}$, χ extends $\theta\alpha_x$. However, it does not factor through Γ, since $S^{-1}\Lambda/\Lambda$ does not admit division over $\mathbb{Z}$.

3. *Representations and special subvarieties*

To use the η invariant as a concordance invariant, we compose the representations of the previous two sections in order to miss the special subvariety associated with the chain complex $C_*(N, M_i; \mathbb{Z}G)$ (see Section 1). This section develops the required machinery.

Definition 3·1. A group Π is called $\mathbb{Z}$-*primary* if $\Pi \cong P \rtimes \mathbb{Z}$ for some finite p group P and the projection $\Pi \longrightarrow \mathbb{Z}$ is an isomorphism on first homology.

Notation 3·2. Throughout this section, we assume Π is a $\mathbb{Z}$-primary group. Then the commutator subgroup of Π is the p-group P. Thus, a $\mathbb{Z}\Pi$ module A is also a $\mathbb{Z}P$ module. Similarly, a choice of splitting $\mathbb{Z} \longrightarrow \Pi$ of the abelianization of Π makes

A a $\mathbb{Z}[\mathbb{Z}]$ module. For a ring R, $R \otimes_{\mathbb{Z}} A$ is an $R\Pi$ module which we will denote RA. Note that if R is a field, RA is a module over the p.i.d. $R[\mathbb{Z}]$.

We begin with a finitely presented $\mathbb{Z}\Pi$ module A.

PROPOSITION 3·3. *If A is a perfect $\mathbb{Z}\Pi$ module (see Definition 1·5), then for any splitting $\mathbb{Z} \longrightarrow \Pi$, $\mathbb{Q}A$ is a torsion $\mathbb{Q}[\mathbb{Z}]$ module.*

The proof of Proposition 3·3 is delayed; the idea is to show that if $\mathbb{Q}A$ is not a torsion $\mathbb{Q}[\mathbb{Z}]$ module, then A has an infinitely generated free summand as a $\mathbb{Z}$ module. Lemma 3·5 shows $\mathbb{Z}_pA$ is a finite module and thus can have no infinitely generated free summand over $\mathbb{Z}_p$. Thus $\mathbb{Q}A$ is a finite dimensional vector space and is therefore $\mathbb{Q}[\mathbb{Z}]$ torsion. Lemma 3·4 is used to prove Lemma 3·5.

LEMMA 3·4 (cf. lemma I·4·3 of [**11**]). *If P is a finite p-group and B a $\mathbb{Z}_p[P]$ module with $\mathbb{Z}_p \otimes_{\mathbb{Z}_p[P]} B$ finite, then B is finite.*

Proof (*Case* 1). Let $P \cong \mathbb{Z}_p$, generated by x. Then

$$B/(x-1)B \cong \mathbb{Z}_p \otimes_{\mathbb{Z}_p[P]} B$$

and so $B/(x-1)B$ is finite. Now consider the map

$$(x-1)^{k-1}B \xrightarrow{x-1} (x-1)^kB/(x-1)^{k+1}B.$$

This map is surjective and factors through $(x-1)^{k-1}B/(x-1)^kB$. Induction then gives us $(x-1)^kB/(x-1)^{k+1}B$ finite for all k.

The exact sequence

$$(x-1)^kB/(x-1)^{k+1}B \rightarrowtail B/(x-1)^{k+1}B \twoheadrightarrow B/(x-1)^kB$$

gives us another inductive step; for if the first and last terms are finite, the centre term shall be and so $B/(x-1)^kB$ is finite for all k. Now, in $\mathbb{Z}_p[P] \cong \mathbb{Z}_p[\mathbb{Z}_p]$, $(x-1)^p = x^p - 1 = 0$ and so $B/(x-1)^pB \cong B$ is finite.

(*Case* 2). Let $|P| = p^i$, $i > 1$. Then we can find P' normal in P with $|P'| = p^{i-1}$. Let $B' = \mathbb{Z}_p[P/P'] \otimes_{\mathbb{Z}_p[P]} B$. Since $P/P' \cong \mathbb{Z}_p$, we get

$$\mathbb{Z}_p \otimes_{\mathbb{Z}_p[P/P']} B' \cong \mathbb{Z}_p \otimes_{\mathbb{Z}_p[P]} B,$$

which is finite. Case 1 implies B' is finite.

Since $P' < P$, the $\mathbb{Z}_p[P]$ module B is also a $\mathbb{Z}_p[P']$ module. As $\mathbb{Z}_p[P']$ modules, $\mathbb{Z}_p \otimes_{\mathbb{Z}_p[P']} B \cong \mathbb{Z}_p[P/P'] \otimes_{\mathbb{Z}_p[P]} B = B'$ and this is finite when $\mathbb{Z}_p \otimes_{\mathbb{Z}_p[P]} B$ is. Continuing this inductive process along the composition series for P shows B is finite.

LEMMA 3·5. *If A is a perfect $\mathbb{Z}\Pi$ module, then $\mathbb{Z}_pA$ is finite.*

Proof. First recall that for a perfect $\mathbb{Z}\Pi$ module A (Definition 1·5), $\varepsilon(\lambda)$ is unimodular over $\mathbb{Z}$, where λ is the $\mathbb{Z}\Pi$ presentation matrix for A. Since $\mathbb{Z}_pA$ has presentation matrix $\lambda' = \mathrm{id}_{\mathbb{Z}_p} \otimes \lambda$ over $\mathbb{Z}_p\Pi$, $\varepsilon(\lambda') = \varepsilon(\lambda)(\mathrm{mod}\ p)$ is also unimodular and so $\mathbb{Z}_p \otimes_{\mathbb{Z}_p\Pi} \mathbb{Z}_pA \cong 0$.

Since $\Pi \cong P \rtimes \mathbb{Z}$, A is a $\mathbb{Z}[P]$ module and a $\mathbb{Z}[\mathbb{Z}]$ module. Let $A' = \mathbb{Z}_p \otimes_{\mathbb{Z}_p[P]} \mathbb{Z}_pA$. Then A' is a $\mathbb{Z}_p[\mathbb{Z}]$ module where $t^i(x \otimes y) = x \otimes t^iy$. We get

$$\mathbb{Z}_p \otimes_{\mathbb{Z}_p[\mathbb{Z}]} A' = \mathbb{Z}_p \otimes_{\mathbb{Z}_p[\mathbb{Z}]} (\mathbb{Z}_p \otimes_{\mathbb{Z}_p[P]} \mathbb{Z}_pA) \cong \mathbb{Z}_p \otimes_{\mathbb{Z}_p\Pi} \mathbb{Z}_pA \cong 0.$$

This implies that A' is a torsion module over the p.i.d. $\mathbb{Z}_p[\mathbb{Z}]$, since a free $\mathbb{Z}_p[\mathbb{Z}]$ summand of A' would not vanish in $\mathbb{Z}_p \otimes_{\mathbb{Z}_p[\mathbb{Z}]} A'$.

Since A is a finitely generated $\mathbb{Z}\Pi$ module and $[\mathbb{Z}: \Pi] = |P|$ is finite, A' is a finitely generated module over $\mathbb{Z}_p[\mathbb{Z}]$. Since A' is $\mathbb{Z}_p[\mathbb{Z}]$ torsion, $A' \cong \oplus_{i=1}^n \mathbb{Z}_p[\mathbb{Z}]/(p_i(t))$ for some n and some set of monic polynomials $\{p_i(t)\} \subset \mathbb{Z}_p[\mathbb{Z}]$, where t is a generator of $\mathbb{Z}$. Thus, A' is a $\mathbb{Z}_p$ vector space with dimension $\sum_1^n \deg(p_i)$ and so is finite.

Since $A' = \mathbb{Z}_p \otimes_{\mathbb{Z}_p[P]} \mathbb{Z}_p A$ is finite, Lemma 3·4 completes the proof.

Proof of Proposition 3·3. Since A is finitely presented, A is finitely generated over $\mathbb{Z}\Pi$. Since $[\mathbb{Z}: \Pi] = |P|$ is finite, A is a finitely generated $\mathbb{Z}[\mathbb{Z}]$ module and so $\mathbb{Q}A$ is a finitely generated $\mathbb{Q}[\mathbb{Z}]$ module.

If $\mathbb{Q}A$ is not a torsion $\mathbb{Q}[\mathbb{Z}]$ module, then $\mathbb{Q}A$ has a free $\mathbb{Q}[\mathbb{Z}]$ summand, since $\mathbb{Q}[\mathbb{Z}]$ is a p.i.d. Then there exists a nonzero $\mathbb{Q}[\mathbb{Z}]$ homomorphism $f: \mathbb{Q}A \longrightarrow \mathbb{Q}[\mathbb{Z}]$. Let f' be the restriction of f to A. Then f' is a nonzero $\mathbb{Z}[\mathbb{Z}]$ homomorphism into $\mathbb{Q}[\mathbb{Z}]$. Since A is finitely generated as a $\mathbb{Z}[\mathbb{Z}]$ module, we can find $n \in \mathbb{Z}$ such that nf' is a nonzero $\mathbb{Z}[\mathbb{Z}]$ homomorphism into $\mathbb{Z}[\mathbb{Z}]$. Then the ideal $nf'(A) \subset \mathbb{Z}[\mathbb{Z}]$ is infinitely generated over $\mathbb{Z}$. Thus, A has a split free summand as a $\mathbb{Z}[\mathbb{Z}]$ module, and so $\mathbb{Z}_p A = \mathbb{Z}_p \otimes_{\mathbb{Z}} A$ has a split free summand as a $\mathbb{Z}_p[\mathbb{Z}]$ module, contradicting Lemma 3·5. Thus, $\mathbb{Q}A$ is a torsion $\mathbb{Q}[\mathbb{Z}]$ module.

COROLLARY 3·6. *Given A as in Proposition* 3·3, *there is a polynomial $p \in \mathbb{Q}[\mathbb{Z}]$ such that p annihilates $\mathbb{Q}A$.*

The corollary follows immediately, upon observing that such modules are finitely generated.

We now show how $\mathbb{Z}$-primary groups allow us to avoid the special subvarieties associated with particular chain complexes.

Definition 3·7. Let G be a group such that $\mathrm{H}_1(G) \cong \mathbb{Z}$. A *transcendental $\mathbb{Z}$-primary representation* $\theta: G \longrightarrow U(k)$ is a non-abelian representation satisfying:

(i) *$\mathbb{Z}$-primary* – There exists an epimorphism $G \twoheadrightarrow \Pi$, where Π is a non-abelian $\mathbb{Z}$-primary group, such that θ factors through Π;

(ii) *transcendental* – For some $g \in G$ such that g generates $H_1(G)$, $\theta(g)$ has eigenvalues which are transcendental over $\mathbb{Q}$.

We denote these representations by $\mathscr{P}_k(G) \subset R_k(G)$.

Remark 3·8. Note that an epimorphism $G \twoheadrightarrow \Pi$ will induce an isomorphism on H_1, since the induced map will be an epimorphism of $\mathbb{Z}$ onto itself.

We will see in Section 5 that $\mathbb{Z}$-primary transcendental representations of knot groups are quite easy to construct.

PROPOSITION 3·9. *Let $\gamma: \Pi \longrightarrow U(k)$ be a transcendental representation of a $\mathbb{Z}$-primary group Π. Then for any perfect $\mathbb{Z}\Pi$ module A, $\mathbb{C}^k \otimes_\gamma A \cong 0$.*

Proof. Note that $\mathbb{C}^k \otimes_\gamma A \cong \mathbb{C}^k \otimes_\gamma \mathbb{Q}A$, so it will suffice to show that $x \otimes_\gamma y = 0$ for $x \in \mathbb{C}^k$ and $y \in \mathbb{Q}A$. Since A is a perfect $\mathbb{Z}\Pi$ module, $\mathbb{Q}A$ is a torsion $\mathbb{Q}[\mathbb{Z}]$ module, by Proposition 3·3. By Corollary 3·6, we are able to find a nonzero polynomial $p(t) \in \mathbb{Q}[\mathbb{Z}]$ that annihilates $\mathbb{Q}A$, i.e. $p(t) \cdot y = 0$ for all $y \in \mathbb{Q}A$.

Let $g \in \Pi$ be such that g generates $\mathrm{H}_1(\Pi)$ and $\gamma(g)$ has transcendental eigenvalues $\{g_i\}$. Let $h \in U(k)$ be such that $h\gamma(g)h^{-1}$ is diagonal. Then $hp(\gamma(g))h^{-1} = p(h\gamma(g)h^{-1})$, which is a diagonal matrix with entries $p(g_i)$. Since each g_i is transcendental, this diagonal matrix is nonsingular, implying that $p(\gamma(g))$ is nonsingular.

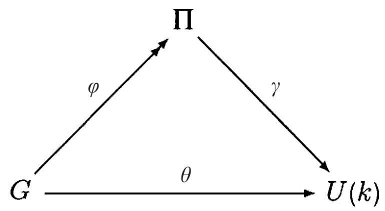
Let $\delta = p(\gamma(g))$. Since the action of g on the $\mathbb{Q}[\mathbb{Z}] \cong \mathbb{Q}[t, t^{-1}]$ modules is multiplication by t, $p(g)$ annihilates $\mathbb{Q}A$. Thus, for any $x \otimes y \in \mathbb{C}^k \otimes_\gamma \mathbb{Q}A$,

$$0 = \delta^{-1}x \otimes p(g)y = (\delta\delta^{-1}x) \otimes y = x \otimes y.$$

The result follows.

COROLLARY 3·10. *Let $\theta \in \mathscr{P}_k(G)$, i.e. θ is a $\mathbb{Z}$-primary transcendental representation of a group G. Then for A a perfect $\mathbb{Z}G$ module, $\mathbb{C}^k \otimes_\theta A \cong 0$.*

Proof. Since θ is $\mathbb{Z}$-primary and transcendental, it factors as $\gamma\varphi$, where γ is transcendental, and the following diagram commutes. Thus, $\mathbb{C}^k \otimes_\theta A \cong \mathbb{C}^k \otimes_\gamma (\mathbb{Z}\Pi \otimes_{\mathbb{Z}G} A)$, and so $\mathbb{Z}\Pi \otimes_{\mathbb{Z}G} A$ is a perfect module over $\mathbb{Z}\Pi$. Proposition 3·9 finishes the proof.



The main result of this section demonstrates that such representations lie outside of the special subvariety Σ associated with a connected homology cobordism N with $\partial N = M_0 \amalg M_1$.

THEOREM 3·11. *Let N be a connected homology cobordism with M one of the boundary components. Let $G = \pi_1(N)$ and let Σ be the special subvariety associated with $C_*(N, M; \mathbb{Z}G)$ (see Section 1). If $\theta \in \mathscr{P}_k(G)$, then $\theta \notin \Sigma$.*

Proof. Since $\mathrm{H}_*(N, M) \cong 0$, there exists a chain contraction on $C_*(N, M)$. Since $\mathbb{Z} \otimes_{\mathbb{Z}G} C_*(N, M; \mathbb{Z}G) \cong C_*(N, M)$, this chain contraction lifts to a chain homotopy on $C_*(N, M; \mathbb{Z}G)$ between a chain homomorphism ψ_* and the zero homomorphism, as in the proof of Proposition 1·7. Note that $\varepsilon(\psi_n) = I_n$, the identity on $C_n(N, M)$, for all n.

Let $A_n = \operatorname{cok}(\psi_n)$. Then A_n is a perfect $\mathbb{Z}G$ module. Since $\theta \in \mathscr{P}_k(G)$, $\mathbb{C}^k \otimes_\theta A \cong 0$ by Corollary 3·10. Thus, θ lies outside the special subvariety Σ_n for all n and so θ lies outside the special subvariety $\Sigma = \bigcup_n \Sigma_n$ of $R_k(G)$ associated with $C_*(N, M; \mathbb{Z}G)$.

4. *Main Theorem*

Definition 4·1. Define $\eta_K \colon \mathrm{H}_1(M; \Lambda) \times R_*(\Gamma) \longrightarrow \mathbb{R}$ by

$$\eta_K(x, \theta) = \rho(M, \alpha_x) \cdot \theta = \tilde{\eta}_{\theta\alpha_x}(M),$$

for any $\theta \in R_k(\Gamma)$ and for any k.

LEMMA 4·2. *Let M_O be zero-framed surgery on the trivial knot O (the unknot), i.e. $M_O = S^1 \times S^2$. Then for any group G and any $\alpha \colon \pi_1(M_O) = \mathbb{Z} \longrightarrow G$, $\rho(M_O, \alpha) = 0$. In particular, $\eta_O(x, \theta) = 0$ for all $x \in \mathrm{H}_1(M_O; \Lambda)$ and all $\theta \in R_k(\Gamma)$.*

Proof. Let $N = S^1 \times D^3$. Then $\partial N = M_O$. Since N is homotopy equivalent to S^1, $\mathrm{H}_2(N) \cong 0 \cong \mathrm{H}_2(N;\theta\alpha)$ for any $\theta \in R_k(G)$. Thus, sign$\,(N) = 0 = \mathrm{sign}_{\theta\alpha}(N)$, and so $\rho(M_O,\alpha) = 0$ on all of $R_k(G)$ by Theorem 1·1.

THEOREM 4·3 (Main Theorem). *Let M be zero-framed surgery on a* slice *knot K. Then there exists $P \subseteq \mathrm{H}_1(M;\Lambda)$ such that $P = P^\perp$, and for all $x \in P$ and $\theta \in R_k(\Gamma)$ such that $\theta\alpha_x \in \mathscr{P}_k(\pi_1(M))$, $\eta_K(x,\theta) = 0$.*

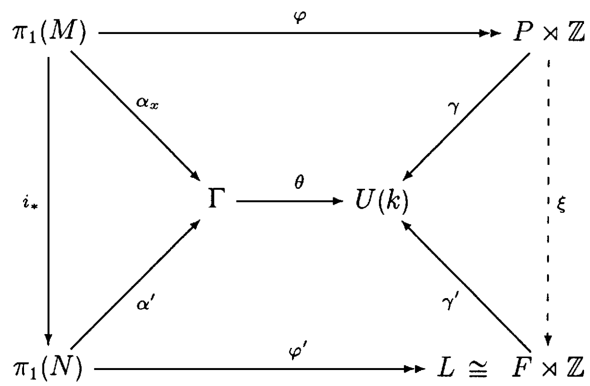
Note that the self annihilating submodule P is the module $D^\perp$ of Section 2·1.

Proof. We have seen in Lemma 2·21 that for an homology cobordism N between M and $M_O = S^1 \times S^2$ (zero-framed surgery on the unknot O), there exists a unitary representation β extending $\theta\alpha_x$. We need to show that β is in $\mathscr{P}_k(\pi_1(N))$ and so lies outside the special subvariety Σ of $R_k(\pi_1(N))$ associated with $C_*(N,M;\mathbb{Z}\pi_1(N))$. Since K is concordant to the unknot O, Proposition 1·7 and Lemma 4·2 imply $\eta_K(x,\theta) = \eta_O(x,\theta) = 0$.

Let N be as above, obtained by zero-framed surgery on a concordance, and suppose $x \in D = \mathrm{im}\{\partial\colon \mathrm{H}_2(N,\partial N;\Lambda) \longrightarrow \mathrm{H}_1(\partial \mathrm{N};\Lambda)\} \subseteq \mathrm{P}$ (see Section 2·1). Since $\partial N = M \amalg (S^1 \times S^2)$, we have $\mathrm{H}_1(\partial N;\Lambda) \cong \mathrm{H}_1(M;\Lambda) \oplus \mathrm{H}_1(S^1 \times S^3;\Lambda)$. Since the $\mathbb{Z}$ cover of $S^1 \times S^2$ is $\mathbb{R} \times S^2$, $\mathrm{H}_1(\partial N;\Lambda)$ is just $\mathrm{H}_1(M;\Lambda)$. By Corollary 2·9, $\alpha_x\colon \pi_1(M) \longrightarrow \Gamma$ extends to a homomorphism $\alpha'\colon \pi_1(N) \longrightarrow \Gamma$.

Since N is compact, $\pi_1(N)$ is finitely generated, and so $L = \mathrm{im}\,(\theta\alpha')$ is a finitely generated subgroup of $H_\theta \rtimes \mathbb{Z} = \mathrm{im}\,(\theta)$. The inclusion of L into $H_\theta \rtimes \mathbb{Z}$ is a first homology isomorphism, and so the commutator subgroup $[L,L] < H_\theta$. H_θ is a torsion abelian group by Lemma 2·19, so $[L,L] = F$ is a finite abelian group. Thus, $L \cong F \rtimes \mathbb{Z}$.

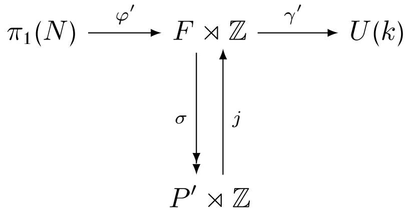
By our choice of θ, $\theta\alpha_x \in \mathscr{P}_k(\pi_1(M))$ and so factors onto a group of the form $P \rtimes \mathbb{Z}$ for some nontrivial p-group P. Then in the following diagram, all solid arrows commute.



Since $\mathrm{im}\,(\theta\alpha_x) \subseteq \mathrm{im}\,(\theta\alpha')$, there exists a homomorphism $\xi\colon P \rtimes \mathbb{Z} \longrightarrow F \rtimes \mathbb{Z}$ (the dashed arrow above), making the whole diagram commute. Since i_* and both surjections φ and φ' are isomorphisms on first homology, ξ is as well. Since P and F are the kernels of the respective abelianizations, $\xi(P) \subseteq F$.

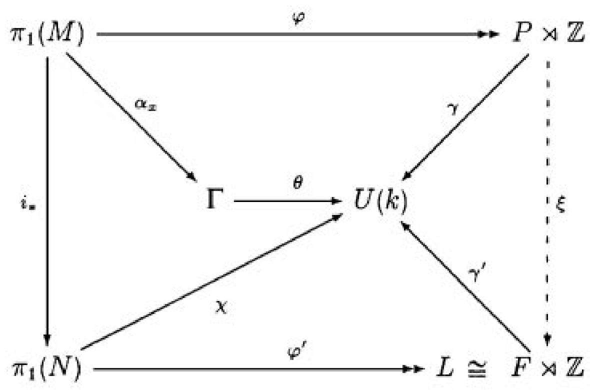
Let P' be the Sylow p-subgroup of F containing the image of P. Since $\theta\alpha_x \in \mathscr{P}_k\big(\pi_1(M)\big)$, it is a non-abelian representation. Thus the image of P under ξ is nontrivial and so P' is nontrivial. Since F is finite abelian, the inclusion of P' into F

splits. Since the action of $\mathbb{Z}$ preserves the order of the elements of F, it restricts to an action on P' and so $P' \rtimes \mathbb{Z}$ is a subgroup of $F \rtimes \mathbb{Z}$. Let $j\colon P' \rtimes \mathbb{Z} \longrightarrow F \rtimes \mathbb{Z}$ be this inclusion. The splitting $F \longrightarrow P'$ of $P' \subseteq F$ determines a splitting $\sigma\colon F \rtimes \mathbb{Z} \longrightarrow P' \rtimes \mathbb{Z}$. This gives the following non-commutative diagram, which can be added to the diagram above.



Let $\beta = \gamma' j\sigma\varphi'$. Then $\beta i_* = \gamma' j\sigma\varphi' i_* = \gamma' j\sigma\xi\varphi$. Since $\mathrm{im}\,(\xi) \subseteq \mathrm{im}\,(j)$, $\beta i_* = \gamma'\xi\varphi$, and by the commutativity of the diagram, $\beta i_* = \theta\alpha_x$. Note that as a representation, β is $\mathbb{Z}$-primary since it factors through $P' \rtimes \mathbb{Z}$ for the p group P'. To see that β is transcendental, let μ be a meridian of the knot K. Then μ generates $\mathrm{H}_1(M)$ under abelianization and so $i_*(\mu)$ generates $\mathrm{H}_1(N)$. Thus, $\beta i_*(\mu) = \theta\alpha_x(\mu)$ has transcendental eigenvalues and so $\beta \in \mathscr{P}_k(\pi_1(N))$. This implies $\beta \notin \Sigma$, the special subvariety of $R_k(\pi_1(N))$ associated with $C_*(N, M; \mathbb{Z}\pi_1(N))$. Thus, $\eta_{_K}(x, \theta) = 0$.

To deal with the case where $x \in P$ but is *not* in D, we consider $nx \in D$, as in Lemma 2·21. So the representation $\theta\alpha_x$ can still be extended to χ, by extending α_{nx} to α' as above and taking the n^{th} roots of the generators of the image of $\theta\alpha'$ in $U(k)$, as we did in the proof of Lemma 2·21. Although this extension does not factor through Γ anymore, the image of χ is still of the form $F \rtimes \mathbb{Z}$, with F a finite subgroup containing P. The large diagram above becomes the diagram below, and β can be constructed as before, extending $\theta\alpha_x$ and contained in $\mathscr{P}_k(\pi_1(N))$.



5. *Examples*

Notation 5·1. We shall use the ordered pair notation for elements of a semi-direct product (see e.g. [**8**]). We also let H_θ and u be defined as they are in Section 2·2 and assume $H_\theta \subseteq T^k$ in $U(k)$.

Consider a genus 1 knot K with Seifert pairing $A = \begin{pmatrix} 0 & -7 \\ -8 & 5 \end{pmatrix}$ over some basis $\{e_1, e_2\}$ for $\mathrm{H}_1(F)$, where F is the Seifert surface of K. Such a knot is algebraically slice. We compute representations such that the associated η invariants vanish if such a knot is slice.

By [**9**], the Blanchfield pairing is given by

$$(1-t)(tA-A^T)^{-1} = \frac{t-1}{(8t-7)(7t-8)}\begin{pmatrix} 5(t-1) & 8t-7 \\ 7t-8 & 0 \end{pmatrix}$$

over the Λ module basis $\{f_1, f_2\}$ of $\mathrm{H}_1(X_\infty) = \mathrm{H}_1(S^3 - K; \Lambda)$, where X_∞ is the infinite cyclic cover of the knot complement $S^3 - K$. The generators f_i are the Alexander duals to the basis elements e_i in $\mathrm{H}_1(S^3 - F)$, lifted to a fixed fundamental domain in the $\mathbb{Z}$ cover of $S^3 - K$. Note that $\mathrm{H}_1(S^3 - K; \Lambda) = \mathrm{H}_1(M; \Lambda)$, since the addition of the two-handle in our zero-framed surgery bounds a parallel to the knot, which is already null-homologous. Thus, the above pairing is also the Blanchfield pairing on $\mathrm{H}_1(M; \Lambda)$.

A calculation shows that the self annihilating submodules for this pairing are P_1, generated by $x = f_2$, and P_2, generated by $y = -3f_1 + f_2$. Then the homomorphisms $\alpha_x, \alpha_y \colon \mathrm{H}_1(M; \Lambda) \rtimes \mathbb{Z} \longrightarrow \Gamma$ are determined by the images of the generators $(f_1, 0)$ and $(f_2, 0)$. We get

$$\alpha_x(f_1, 0) = \left(\frac{t-1}{8t-7}, 0\right) \qquad \alpha_x(f_2, 0) = (0, 0)$$

and

$$\alpha_y(f_1, 0) = \left(\frac{1-t}{7t-8}, 0\right) \qquad \alpha_y(f_2, 0) = \left(\frac{3(1-t)}{7t-8}, 0\right).$$

We now begin the construction of a set of representations $\mathcal{Q} \subseteq \coprod_k R_k(\Gamma)$ such that if K is slice then either $\eta_K(x, \theta) = 0$ for all $\theta \in \mathcal{Q}$ or $\eta_K(y, \theta) = 0$ for all $\theta \in \mathcal{Q}$. We will concern ourselves only with irreducible representations, since the value of an η invariant defined by a reducible representation is determined by its irreducible constituents. Though we only compute representations for $k = 2$ and 3, higher dimensional representations are computed analogously.

We will show by example that such representations are not hard to construct. The general technique is given in the following paragraph, with the details for $k = 2$ and 3 following.

One first observes, using the arguments of Lemma 2·19, that if $\theta \in R_k(\Gamma)$ is irreducible, then u_*, the action generated by u, is the action of a k-cycle. Then $\theta \colon S^{-1}\Lambda/\Lambda \rtimes \mathbb{Z} \longrightarrow U(k)$ factors through a representation $\theta' \colon S^{-1}\Lambda_k/\Lambda_k \rtimes \mathbb{Z} \longrightarrow U(k)$. By Lemma 2·14, $S^{-1}\Lambda_k/\Lambda_k$ is a subring of a sum of group rings. Fixing u, the possible representations θ' are then computed.

For irreducible $\theta \in R_2(\Gamma)$, u_* is a 2-cycle, and so $u = \begin{pmatrix} 0 & \zeta_1 \\ \zeta_2 & 0 \end{pmatrix}$ where $|\zeta_i| = 1$. The representation is transcendental if and only if the eigenvalues of u, $\pm\sqrt{\zeta_1\zeta_2}$, are transcendental over $\mathbb{Q}$. This is equivalent to $\det(u) = -\zeta_1\zeta_2$ being transcendental over $\mathbb{Q}$.

θ factors through $S^{-1}\Lambda_2/\Lambda_2 \rtimes \mathbb{Z}$. By Lemma 2·14, this is isomorphic to a subgroup of $\mathfrak{R}_2[\mathbb{Z}_2] \rtimes \mathbb{Z}$, where $\mathfrak{R}_2 = \mathbb{Z}_{(2)}/\mathbb{Z}$. A representation $\theta' \colon S^{-1}\Lambda_2/\Lambda_2 \rtimes \mathbb{Z} \longrightarrow U(2)$ restricts to a $\mathbb{Z}_2$ equivariant homomorphism $S^{-1}\Lambda_2/\Lambda_2 \longrightarrow T^2 \subseteq U(2)$ where $\mathbb{Z}_2$ acts on T^2 via conjugation by u, which permutes the diagonal entries. Clearly such a homomorphism extends to a $\mathbb{Z}_2$ invariant homomorphism $\mathfrak{R}_2[\mathbb{Z}_2] \longrightarrow T^2$ and any

$\mathbb{Z}_2$ invariant homomorphism $\rho\colon \mathfrak{R}_2[\mathbb{Z}_2] \longrightarrow T^2$ together with a choice of u as above, determines an irreducible representation $\theta'\colon S^{-1}\Lambda_2/\Lambda_2 \rtimes \mathbb{Z} \longrightarrow U(2)$, and thus an irreducible representation $\theta \in R_2(\Gamma)$.

Note that $\mathbb{Z}_2$ invariant homomorphisms $\rho\colon \mathfrak{R}_2[\mathbb{Z}_2] \longrightarrow T^2 \cong (\mathbb{R}/\mathbb{Z})^2$ lie in one-to-one correspondence to commuting diagrams of $\mathbb{Z}[\mathbb{Z}_2]$ homomorphisms

$$\begin{array}{ccc} \mathbb{Z}[\mathbb{Z}_2] & \longrightarrow & \mathbb{Z}[\mathbb{Z}_2] \\ \downarrow & & \downarrow \\ \mathbb{Z}_{(2)}[\mathbb{Z}_2] & \longrightarrow & \mathbb{R}[\mathbb{Z}_2] \end{array}$$

and these lie in one-to-one correspondence to $\mathrm{Hom}_{\mathbb{Z}[\mathbb{Z}_2]}(\mathbb{Z}[\mathbb{Z}_2], \mathbb{Z}[\mathbb{Z}_2])$ and are given by multiplication by a polynomial $q(t) = q_0 + q_1 t \in \mathbb{Z}[\mathbb{Z}_2]$. The representation determined by $q_0 + q_1 t$ is given by

$$\rho\left(\frac{a}{m}t + \frac{b}{n}\right) = \begin{pmatrix} \exp\{2\pi i(q_0 a/m + q_1 b/n)\} & 0 \\ 0 & \exp\{2\pi i(q_1 a/m + q_0 b/n)\} \end{pmatrix}.$$

In the group $\mathfrak{R}_2[\mathbb{Z}_2]$, $(7t-8)^{-1} = -(7t+8)/15$, and $(8t-7)^{-1} = (8t+7)/15$. Thus, the image of $\alpha_x(f_1, 0) = ((1-t)/15, 0)$ and $\alpha_x(f_2, 0) = (0,0)$ and the image of $\alpha_y(f_1, 0) = ((t-1)/15, 0)$ and $\alpha_y(f_2, 0) = ((t-1)/5, 0)$ in $\mathfrak{R}_2[\mathbb{Z}_2] \rtimes \mathbb{Z}$.

Then

$$\theta\alpha_x(f_1, 0) = \theta'\left(\frac{1-t}{15}, 0\right) = \begin{pmatrix} \exp\{-2\pi i(q_0 - q_1)/15\} & 0 \\ 0 & \exp\{2\pi i(q_0 - q_1)/15\} \end{pmatrix}$$

$$\theta\alpha_x(f_2, 0) = \theta'(0,0) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

and

$$\begin{aligned} \theta\alpha_y(f_1, 0) &= \theta'\left(\tfrac{t-1}{15}, 0\right) = \begin{pmatrix} \exp\{2\pi i(q_0 - q_1)/15\} & 0 \\ 0 & \exp\{-2\pi i(q_0 - q_1)/15\} \end{pmatrix} \\ \theta\alpha_y(f_2, 0) &= \theta'\left(\tfrac{t-1}{5}, 0\right) = \begin{pmatrix} \exp\{2\pi i(q_0 - q_1)/5\} & 0 \\ 0 & \exp\{-2\pi i(q_0 - q_1)/5\} \end{pmatrix}. \end{aligned}$$

Clearly for $F \rtimes \mathbb{Z}$, the image of θ composed with one of either α_x or α_y, $F \cong \mathbb{Z}_{15}$, $\mathbb{Z}_5$ or $\mathbb{Z}_3$, depending on our choice of $q(t)$. Thus, the representations θ that have either $\theta\alpha_x$ or $\theta\alpha_y$ in $\mathscr{P}_2(\pi_1(M))$ are the ones such that $q_0 - q_1$ is a multiple of either 5 or 3.

The computations are similar for $U(3)$ representations. Without loss of generality, let

$$u = \begin{pmatrix} 0 & 0 & \zeta_1 \\ \zeta_2 & 0 & 0 \\ 0 & \zeta_3 & 0 \end{pmatrix},$$

so that u_* is the 3-cycle $(1\,2\,3)$. Again, if $\det(u) = \zeta_1\zeta_2\zeta_3$ is transcendental over $\mathbb{Q}$, θ will be a transcendental representation.

The group $S^{-1}\Lambda_3/\Lambda_3$ is isomorphic to a subgroup of $\mathfrak{R}_3[\mathbb{Z}_3]$, where $\mathfrak{R}_3 = \mathbb{Z}_{(3)}/\mathbb{Z}$, and in this group $(7t-8)^{-1} = (-49t^2 - 56t - 64)/169$. Then the image of α_y becomes $((15 - 8t - 7t^2)/169, 0)$ and $(3(15 - 8t - 7t^2)/169, 0)$ for $(f_1, 0)$ and $(f_2, 0)$, respectively.

Since the automorphism θ' is multiplication by $q = q_0 + q_1 t + q_2 t^2$ in $\mathfrak{R}_3[\mathbb{Z}_3]$, we get

$$\frac{(15q_0 - 8q_2 - 7q_1) + (15q_1 - 8q_0 - 7q_2)t + (15q_2 - 8q_1 - 7q_0)t^2}{169}$$

and

$$\frac{3(15q_0 - 8q_2 - 7q_1) + 3(15q_1 - 8q_0 - 7q_2)t + 3(15q_2 - 8q_1 - 7q_0)t^2}{169}$$

respectively. The simplest example, where $q(t) = 1$, gives us the result

$$\theta\alpha_y(f_1, 0) = \begin{pmatrix} \exp\{2\pi i(-7/169)\} & 0 & 0 \\ 0 & \exp\{2\pi i(-8/169)\} & 0 \\ 0 & 0 & \exp\{2\pi i(15/169)\} \end{pmatrix}$$

and

$$\theta\alpha_y(f_2, 0) = \begin{pmatrix} \exp\{2\pi i(-21/169)\} & 0 & 0 \\ 0 & \exp\{2\pi i(-24/169)\} & 0 \\ 0 & 0 & \exp\{2\pi i(45/169)\} \end{pmatrix}.$$

Clearly, the image is isomorphic to $\mathbb{Z}_{13^2} \rtimes \mathbb{Z}$ and so $\theta\alpha_y \in \mathscr{P}_3(\pi_1(M))$. Similar computations give, for $q(t) = 1$,

$$\theta\alpha_x(f_1, 0) = \begin{pmatrix} \exp\{2\pi i(-8/169)\} & 0 & 0 \\ 0 & \exp\{2\pi i(-7/169)\} & 0 \\ 0 & 0 & \exp\{2\pi i(15/169)\} \end{pmatrix}$$

and

$$\theta\alpha_x(f_2, 0) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The computations are left to the reader.

As a final example, we give an interesting $U(6)$ representation of the knot with Seifert pairing

$$\begin{pmatrix} 0 & -6 & 0 & 0 \\ -7 & 1 & 0 & 0 \\ 0 & 0 & 0 & -5 \\ 0 & 0 & -6 & 1 \end{pmatrix}.$$

Note that this can be thought of as the connected sum of two genus one knots, each having Seifert pairing given by the obvious 2×2 diagonal blocks. Our interest in such a knot stems from the fact that we have 13-torsion in both the $U(2)$ and $U(3)$ irreducible representations. The knot with Seifert pairing $\begin{pmatrix} 0 & -6 \\ -7 & 1 \end{pmatrix}$ has 13-torsion appearing in its $U(2)$ representations and the knot with Seifert pairing $\begin{pmatrix} 0 & -5 \\ -6 & 1 \end{pmatrix}$ has both 7-torsion and 13-torsion in its $U(3)$ representations.

The Blanchfield pairing for this knot is given by

$$\begin{pmatrix} \frac{(1-t)^2}{(6-7t)(7-6t)} & \frac{1-t}{6-7t} & 0 & 0 \\ \frac{1-t}{7-6t} & 0 & 0 & 0 \\ 0 & 0 & \frac{(1-t)^2}{(5-6t)(6-5t)} & \frac{1-t}{5-6t} \\ 0 & 0 & \frac{1-t}{6-5t} & 0 \end{pmatrix}$$

with respect to the Λ module basis $\{f_1, f_2, f_3, f_4\}$ of $\mathrm{H}_1(M;\Lambda)$. One obvious self annihilating submodule is $P = \mathrm{span}\{f_2, f_4\}$ (there are several others). We shall look at the representations $\theta\alpha_x$ where $x = f_2 + f_4 \in P$. Since both f_2 and f_4 vanish when paired with x, We need only concern ourselves with the images of the generators $(f_1, 0)$ and $(f_3, 0)$ of $\mathrm{H}_1(M\Lambda) \rtimes \mathbb{Z}$. We get

$$\alpha_x(f_1,0) = \left(\frac{1-t}{7-6t}, 0\right) \quad \alpha_x(f_3,0) = \left(\frac{1-t}{6-5t}, 0\right).$$

Let $\theta \in U(6)$ be irreducible, with u_* a 6-cycle. Without loss of generality, let

$$u = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \zeta_1 \\ \zeta_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & \zeta_3 & 0 & 0 & 0 & 0 \\ 0 & 0 & \zeta_4 & 0 & 0 & 0 \\ 0 & 0 & 0 & \zeta_5 & 0 & 0 \\ 0 & 0 & 0 & 0 & \zeta_6 & 0 \end{pmatrix},$$

where $|\zeta_i| = 1$. Again, if $\det(u) = -\prod \zeta_i$ is transcendental, then the representation will be transcendental. Now $\theta\colon S^{-1}\Lambda/\Lambda \rtimes \mathbb{Z} \longrightarrow U(6)$ factors through $S^{-1}\Lambda_6/\Lambda_6 \rtimes \mathbb{Z}$ and $S^{-1}\Lambda_6/\Lambda_6$ is isomorphic to a subgroup of $\mathfrak{R}_2[\mathbb{Z}_2] \oplus \mathfrak{R}_3[\mathbb{Z}_3]$, by [**6**]; so the image of α_x in $(\mathfrak{R}_2[\mathbb{Z}_2] \oplus \mathfrak{R}_3[\mathbb{Z}_3]) \rtimes \mathbb{Z}$ is

$$\left(\left(\frac{1-t}{13}, \frac{13-7t-6t^2}{127}\right), 0\right) \quad \text{and} \quad \left(\left(\frac{1-t}{11}, \frac{11-6t-5t^2}{91}\right), 0\right)$$

for $(f_1, 0)$ and $(f_3, 0)$, respectively.

Note that an automorphism θ' on $\mathfrak{R}_2[\mathbb{Z}_2] \oplus \mathfrak{R}_3[\mathbb{Z}_3]$ can be represented by a pair $(q(t), r(t)) \in \mathbb{Z}[\mathbb{Z}_2] \oplus \mathbb{Z}[\mathbb{Z}_3]$. For simplicity in our example, we will assume $q(t) = r(t) = 1$.

The Chinese Remainder Theorem allows us to find an element in $\mathbb{Z}[\mathbb{Z}_6]$ that projects to both $127(1-t) \in \mathbb{Z}[\mathbb{Z}_2]$ and $13(13-7t-6t^2) \in \mathbb{Z}[\mathbb{Z}_3]$, and another that projects to both $91(1-t) \in \mathbb{Z}[\mathbb{Z}_2]$ and $11(11-6t-5t^2) \in \mathbb{Z}[\mathbb{Z}_3]$. Letting $\mathfrak{R}_6 = \mathbb{Z}_{(6)}/\mathbb{Z}$, where $\mathbb{Z}_{(6)} = \mathbb{Z}_{(2)} \bigcap \mathbb{Z}_{(3)}$, we get the image of α_x in $\mathfrak{R}_6[\mathbb{Z}_6] \rtimes \mathbb{Z}$ is

$$\left(\frac{205-91t-78t^2-36t^3}{13 \times 127}, 0\right) \quad \text{and} \quad \left(\frac{146-66t-55t^2-25t^3}{7 \times 11 \times 13}, 0\right)$$

for $(f_1, 0)$ and $(f_2, 0)$, respectively. Thus, since θ' is the identity, we get

$$\theta\alpha_x(f_1,0) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \exp\left\{\dfrac{2\pi i(-36)}{1651}\right\} & 0 & 0 & 0 \\ 0 & 0 & 0 & \exp\left\{\dfrac{2\pi i(-78)}{1651}\right\} & 0 & 0 \\ 0 & 0 & 0 & 0 & \exp\left\{\dfrac{2\pi i(-91)}{1651}\right\} & 0 \\ 0 & 0 & 0 & 0 & 0 & \exp\left\{\dfrac{2\pi i(205)}{1651}\right\} \end{pmatrix}$$

and

$$\theta\alpha_x(f_2,0) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \exp\left\{\dfrac{2\pi i(-25)}{1001}\right\} & 0 & 0 & 0 \\ 0 & 0 & 0 & \exp\left\{\dfrac{2\pi i(-55)}{1001}\right\} & 0 & 0 \\ 0 & 0 & 0 & 0 & \exp\left\{\dfrac{2\pi i(-66)}{1001}\right\} & 0 \\ 0 & 0 & 0 & 0 & 0 & \exp\left\{\dfrac{2\pi i(146)}{1001}\right\} \end{pmatrix}.$$

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