A Classification of Differentiable Knots*

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The classification, up to isotopy, of *knotted* n-spheres in m-space is an old problem on which there has been much recent progress. The classical situation of simple closed curves in 3-space is usually studied by means of the complement of the knot and, in general, when $m = n + 2$, this seems the correct approach (see [16] and [23]). Beyond this range, however, when $m \geqq n + 3$, the complements all look alike and one has to search for other invariants. In the topological and piecewise-linear case there are none, i.e., all knots are isotopic (see [23] and [31]); but in the differential case, with which we are here concerned, this is not usually true (see [3]).

It has been shown by Haefliger [3], using a result of Smale [22], that, if $n \geqq 5$, $m \geqq n + 3$, the isotopy classes of knotted n-spheres in S^m form a group $\Sigma^{m,n}$. Moreover, Haefliger has succeeded in showing that $\Sigma^{m,n}$ is trivial when $2m > 3(n + 1)$ (see [2]), but non-trivial when $2m = 3(n + 1)$ (see [3]).[1]

Instead of considering only knotted copies of S^n in S^m, we may more generally consider knotted homotopy n-spheres in S^m. This forms a larger group $\Theta^{m,n}$, which contains $\Sigma^{m,n}$ as a subgroup, and gives the additional information of which homotopy n-spheres imbed in S^m. It will be the aim of this work to show that the calculation of these groups reduces to standard homotopy group problems, about which a great deal is known. With only slightly extra effort, this paper will be independent of the work of Haefliger [2] and [3], except for a discussion of spherical modifications in [3, § 3], and will, in fact, recapture almost all of his results on $\Theta^{m,n}$.

The idea is to *unknot* gradually our knot in S^m by three stages. We look first at the normal bundle; if this is trivial we may put on a normal frame. Then we define an invariant of such framed knots which, as it turns out, is just the obstruction to bounding a framed submanifold in S^m. If the obstruction is zero we can try to simplify the resulting submanifold in the hope of making it a disk. This leads to a final obstruction in terms of certain well-known cobordism invariants of framed manifolds. To determine which of the various possible obstruction elements can actually be realized as obstructions is an exercise in the use of spherical modifications.

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[1] For a more detailed discussion, the reader is referred to the paper *Sphères Nouées* by A. Haefliger, Atti del Riunions del Groupe. de Math. d'expr. latine, Firenze-Bologna. 1961, p. 139-144.

The final results appear in what seems to be their most useful and elegant form as an inter-related family of exact sequences which can be considered non-stable versions of the (unpublished) exact cobordism sequences of Kervaire-Milnor.

The organization of the paper is as follows. In § 1 we fix certain preliminary definitions and notation. Then, in § 2, we write down those exact sequences whose derivation will be our main chore. The above referred to invariant of a framed knot is defined, discussed, and interpreted in § 3. The techniques of spherical modification which are needed are presented in § 4; and, in § 5, the sequences of § 2 are actually defined and proved exact. In § 6, the groups $\Theta^{m,n}$ and $\Sigma^{m,n}$, as well as several other geometrically-defined groups of interest, are studied with the aid of our results. Finally, in § 7, we present some tabulations of the orders of these groups for $n \leqq 11$.

1. Preliminary material

1.1. All manifolds will be (C^∞) differentiable and *oriented*; imbeddings will be differentiable. A diffeomorphism or imbedding of codimension zero will preserve orientation. A pair (V^m, M^n) will denote a manifold V and a submanifold M of dimensions m and n, respectively; in all cases the intersection of M and ∂V is normal (with respect to some riemannian metric) and $M \cap \partial V$ is a submanifold of ∂M. If $m = n$, the orientations of M and V agree. The normal bundle ν to M in V will be oriented by the equation

$$\tau(M) + \nu = \tau(V) \mid M ,$$

where $\tau(M)$, $\tau(V)$ are the tangent bundles of M and V. If $M = \partial V$, the boundary of V, then the orientation of M is prescribed by the equation

$$e + \tau(M) = \tau(V) \mid M ,$$

where e is a line bundle oriented positively by the vector pointing out from V. Let int $V = V - \partial V$.

1.2. A *normal frame* on a submanifold M of V is an ordered collection of orthonormal vector fields $\mathcal{F} = (f_1, \cdots, f_k)$ defined on M, spanning the normal space to M in V at each point, and giving the positive orientation of the normal bundle. We will say $(M, \mathcal{F})$ is a *framed submanifold* of V. We will say two framed submanifolds $(M, \mathcal{F})$ and $(M_0, \mathcal{F}_0)$ of V are *isotopic* if there is a continuous family of diffeomorphisms Λ_t, $0 \leqq t \leqq 1$, of V on itself so that Λ_0 is the identity, $\Lambda_1 \mid M$ is a diffeomorphism onto M_0, and $d\Lambda_1(\mathcal{F}) = \mathcal{F}_0$. We denote the isotopy class of $(M, \mathcal{F})$ by $[M, \mathcal{F}]$. Similarly we define the notion of isotopy of pairs (V, M) and (V, M_0) and denote the isotopy class of (V, M) by $[V, M]$.

We say that $(M_0, \mathcal{F}_0)$ and $(M_1, \mathcal{F}_1)$ are *cobordant* if there is a framed submanifold $(N, \mathcal{G})$ of $I \times V$ satisfying:

(i) $N \cap t \times V = t \times M_t$, $t = 0, 1$,

(ii) $\mathcal{G} \mid t \times M_t = t \times \mathcal{F}_t$, $t = 0, 1$,

(iii) $\overline{\partial N \cap ((\text{int } I) \times V)}$ is an h-cobordism (see [22] for a definition) between ∂M_0 and ∂M_1, and

(iv) $N \cap (I \times \partial V)$ is an h-cobordism between $M_0 \cap \partial V$ and $M_1 \cap \partial V$. Similarly we define cobordism between pairs (V, M_0) and (V, M_1).

Let $(M, \mathcal{F})$ be a framed submanifold of V and $(M_0, \mathcal{F}_0)$ a framed submanifold of V_0. The *connected sum* $(M, \mathcal{F}) \# (M_0, \mathcal{F}_0)$ is a framed submanifold of $V \# V_0$. If V, V_0, M and M_0 are closed, this is defined in [3, 1.2]. If M and M_0 are bounded (V and V_0 still closed) we modify the definition in [3] by imbedding a disk D^m in V such that $D^m \cap M = D^n_+$ (in the notation of [3, 1.2])—similarly for V_0. The isotopy class, respectively, cobordism class of $(M, \mathcal{F}) \# (M_0, \mathcal{F}_0)$ depends only on the isotopy classes, respectively, cobordism classes of $(M, \mathcal{F})$ and $(M_0, \mathcal{F}_0)$.

Let M^n be a submanifold of V^m, $m = n + k$, and $\mathcal{F} = (f_1, \cdots, f_{k-1})$ a field of orthonormal vectors on M, normal to M in V. Then there exists a unique normal frame $\mathcal{F}' = (f'_1, \cdots, f'_k)$ to M in V satisfying either $f'_i = f_i$, $1 \leqq i \leqq k - 1$, or $f'_i = f_{i-1}$, $2 \leqq i \leqq k$. We will say $\mathcal{F}'$ is, respectively, the *rear* or *front extension* of $\mathcal{F}$.

1.3. Let D^k be the closed unit disk in euclidean k-space E^k, with the natural orientation; let $S^{k-1} = \partial D^k$. We denote by G_k the space of maps $S^{k-1} \to S^{k-1}$ of degree $+1$; this has an H-space structure by composition of maps. Let SO_k be the subspace of G_k consisting of orthogonal maps. We may describe $\pi_n(G_k)$ as the group of homotopy classes of maps $S^n \times S^{k-1} \to S^{k-1}$ of degree $+1$ on the second factor. Addition in $\pi_n(G_k)$ is induced by the following addition. Let $\lambda_1, \lambda_2 \colon S^n \times S^{k-1} \to S^{k-1}$. Then

$$(\lambda_1 + \lambda_2)(x, y) = \lambda_1(x, \lambda_2(x, y)) \qquad x \in S^n,\ y \in S^{k-1}\,.$$

We may describe $\pi_n(G_k, SO_k)$ in an analogous manner. Consider pairs of maps (λ, μ), where $\lambda \colon D^n \times S^{k-1} \to S^{k-1}$ and $\mu \colon S^{n-1} \to SO_k$ satisfy the equation $\lambda(x, y) = \mu(x) \cdot y$ for $x \in S^{n-1}$, $y \in S^{k-1}$. With the obvious definition of a homotopy of such pairs, we may describe $\pi_n(G_k, SO_k)$ as the group of homotopy classes. Note that the correspondence $(\lambda, \mu) \to \mu$ induces the boundary homomorphism $\pi_n(G_k, SO_k) \to \pi_{n-1}(SO_k)$.

It is a consequence of the homotopy extension theorem that any map $S^n \times S^{k-1} \to S^{k-1}$ of degree $+1$ on the second factor is homotopic to a map $\lambda \colon S^n \times S^{k-1} \to S^{k-1}$ satisfying $\lambda(x, y) = y$ for all $x \in D^n_-$, $y \in S^{k-1}$. Let $h \colon D^n \to D^n_+$ be a diffeomorphism; consider the pair (λ', μ), where $\lambda' \colon D^n \times S^{k-1} \to S^{k-1}$ is defined by $\lambda'(x, y) = \lambda(h(x), y)$ and $\mu \colon S^{n-1} \to SO_k$ maps S^{n-1} onto the identity.

Then the correspondence $\lambda \to (\lambda', \mu)$ induces the inclusion homomorphism $\pi_n(G_k) \to \pi_n(G_k, SO_k)$.

There is a natural inclusion $S^{k-1} \subset S^k$; let D^k_+ and D^k_- denote the closures of the complementary domains such that $S^{k-1} = \partial D^k_+ = -\partial D^k_-$. Then $S^{k-1} \subset S^k$ induces an inclusion $G_k \subset G_{k+1}$, defined by suspension on the last coordinate (see [27, p. 206]).

Let F_k be the subspace of G_{k+1} of maps which preserve a base-point; then $G_k \subset F_k \subset G_{k+1}$. Recall [7, § 2] there is a fibre map $G_k \to S^{k-1}$ with F_{k-1} as fibre.

1.4. Let $(M^n, \mathcal{F})$ be a framed submanifold of V such that $M \cap \partial V = \partial M$. The *Thom construction* associates to $(M, \mathcal{F})$ an element $t(M, \mathcal{F}) \in \pi^k(V)$ ($=$ homotopy classes of maps $V \to S^k$); see [11, p. 346] for a definition. Conversely, let $\lambda\colon V \to S^k$ be a differentiable map, $y_0 \in S^k$ a regular value of λ, and ε_0 a positive tangent frame at y_0. Then $\lambda^{-1}(y_0) = M$ is a submanifold of V with a normal frame $\mathcal{F}$ defined by $d\lambda(\mathcal{F}(x)) = \varepsilon_0$ for each $x \in M$ which determines the orientation of M. We will say $\mathcal{F}$ is the *pull-back* by λ of ε_0. Then λ represents $t(M, \mathcal{F})$. If $(M_0, \mathcal{F}_0)$ is another framed submanifold of V, $M_0 \cap \partial V = \partial M_0$, then $t(M_0, \mathcal{F}_0) = t(M, \mathcal{F})$ if $(M, \mathcal{F})$ and $(M_0, \mathcal{F}_0)$ are cobordant, and conversely, if M is closed. Furthermore, if $V = \partial W$, then $t(M, \mathcal{F})$ extends to a map $W \to S^k$ if and only if $(M, \mathcal{F})$ *extends to a framed submanifold* of W, i.e., if there exists a framed submanifold $(N, \mathcal{G})$ of W such that $N \cap V = \partial N = M$ and $\mathcal{G} \mid M = \mathcal{F}$. These are all standard facts (see [11, 1.3]), except for the orientation properties, which are trivial.

1.5. In (1.3) we identified $\pi_n(G_k)$ with a subset of $\pi^{k-1}(S^n \times S^{k-1})$. Thus, for certain framed submanifolds $(M^n, \mathcal{F})$ of $S^n \times S^{k-1}$, we may regard $t(M, \mathcal{F}) \in \pi_n(G_k)$; we will see in (3.6) that the condition on $(M, \mathcal{F})$ is that M project onto the first factor of $S^n \times S^{k-1}$ with degree $+1$. Then $t(M, \mathcal{F}) = 0$ in $\pi_n(G_k)$ if and only if $(M, \mathcal{F})$ extends to a framed submanifold of $D^{n+1} \times S^{k-1}$.

Suppose $\alpha \in \pi_n(G_k)$. As remarked in (1.3), we may choose a representative $\lambda\colon S^n \times S^{k-1} \to S^{k-1}$ such that $\lambda(x, y) = y$ for $x \in D^n_0$, $y \in S^{k-1}$, where D^n_0 is any disk in S^n. As a consequence, for any given $x_0 \in S^n$, there is a framed submanifold $(M^n, \mathcal{F})$ of $S^n \times S^{k-1}$ such that M meets $x_0 \times S^{k-1}$ normally in a single point and $t(M, \mathcal{F}) = \alpha$.

The Thom construction may also be applied to the elements of $\pi_n(G_k, SO_k)$. If (λ, μ) is a pair of maps representing an element $\alpha \in \pi_n(G_k, SO_k)$, λ is differentiable and $y_0 \in S^{k-1}$ is a regular value of λ, then $M = \lambda^{-1}(y_0)$ has the property that $\partial M = M \cap (S^{n-1} \times S^{k-1})$ is the image of a differentiable cross-section of the projection $S^{n-1} \times S^{k-1} \to S^{n-1}$. Note that the map $S^{n-1} \to S^{k-1}$ defined by projecting this cross-section onto S^{k-1} is described by $x \to \mu^{-1}(x) \cdot y_0$. Now a differentiable homotopy of the pair (λ, μ) induces a cobordism of the associated framed submanifold $(M, \mathcal{F})$, since, on $S^{n-1} \times S^{k-1}$, we will have a family of cross-sections. Thus the cobordism class of $(M, \mathcal{F})$ depends only on α.

Suppose $\alpha, \alpha' \in \pi_n(G_k, SO_k)$ are represented by pairs (λ, μ), (λ', μ'), respectively. Let $D^n = D_1^n \cup D_2^n$, where the D_i^n are the intersections of D^n with the upper and lower half-spaces of n-space; then $D_1^n \cap D_2^n = D^{n-1}$. We may assume $\lambda \mid D_2^n \times S^{k-1}$ and $\lambda' \mid D_1^n \times S^{k-1}$ are projections on the second factor. If (λ'', μ'') is defined by

$$\lambda'' \mid D_1^n \times S^{k-1} = \lambda \mid D_1^n \times S^{k-1}$$

and

$$\lambda'' \mid D_2^n \times S^{k-1} = \lambda' \mid D_2^n \times S^{k-1} ,$$

then (λ'', μ'') represents $\alpha + \alpha'$. Suppose λ, λ' are differentiable with a regular value $y_0 \in S^{k-1}$; then so is λ''. If $(M, \mathcal{F})$, $(M_0, \mathcal{F}_0)$ and $(M_1, \mathcal{F}_1)$ are defined, by the Thom construction, from λ, λ' and λ'', respectively, then $M \cap (D_2^n \times S^{k-1})$ and $M_0 \cap (D_1^n \times S^{k-1})$ are disks while

$$(M_1, \mathcal{F}_1) \cap (D_1^n \times S^{k-1}) = (M, \mathcal{F}) \cap (D_1^n \times S^{k-1})$$

and

$$(M_1, \mathcal{F}_1) \cap (D_2^n \times S^{k-1}) = (M_0, \mathcal{F}_0) \cap (D_2^n \times S^{k-1}) .$$

1.6. Let M^n be a submanifold of V^m and $\mathcal{F}$ a normal frame defined on the complement of a point of M, called the *singularity* of $\mathcal{F}$. We define $O(M, \mathcal{F}) \in \pi_{n-1}(SO_k)$, where $k = m - n$. Let Δ be an n-simplex imbedded in M (orientation preserving), containing the singularity of $\mathcal{F}$ in its interior. Let $\mathcal{F}'$ be a normal frame to M in V defined on Δ. Then we define $\mu: \partial\Delta \to SO_k$ by the formula

$$\mathcal{F}(x) = \mu(x)\mathcal{F}'(x) ,$$

where SO_k is identified with the $(k \times k)$-orthogonal matrices with positive determinant (using the standard unit basis of k-space), and the action of such matrices on orthonormal k-frames is well-defined (see [11, 1.8]). Then $O(M, \mathcal{F})$ is the homotopy class of μ.

1.7. Let M^n be a submanifold of $S^n \times S^{k-1}$; we say M is *nuclear* in $S^n \times S^{k-1}$ if the inclusion $M \to S^n \times S^{k-1}$ followed by the projection $S^n \times S^{k-1} \to S^n$ defines a map $M \to S^n$ of degree $+1$. Note that this depends only on the cobordism class of $(S^n \times S^{k-1}, M)$.

A manifold K^n is an *n-sphere* if K is homotopy equivalent to S^n. If K is a submanifold of V, we say K is a *knotted n-sphere in V*; if K is diffeomorphic to S^n, we say K is a *knotted* S^n in V.

2. Some exact sequences

2.1. Throughout this paper, let m, n and k be integers satisfying $n \geqq 5$, $k \geqq 3$ and $m = n + k$. The set of isotopy classes of knotted n-spheres in S^m will be denoted by $\Theta^{m,n}$; the set of isotopy classes of framed knotted n-spheres in S^m will be denoted by $\Theta_f^{m,n}$.

It follows from [22, Th. 1.4] that $\Theta^{m,n}$ and $\Theta_f^{m,n}$ can be, equivalently, described as the h-cobordism classes of knotted spheres, respectively, framed knotted spheres, in S^m. As a consequence, the operation of connected sum imposes an abelian group structure on $\Theta^{m,n}$ and $\Theta_f^{m,n}$ (see [3, 1.3]).

We point out the following criteria for a knotted sphere K_0^n or a framed knotted sphere $(K_1^n, \mathcal{F})$ in S^m to represent the zero element in $\Theta^{m,n}$ or $\Theta_f^{m,n}$, respectively. $[S^m, K_0] = 0$ if and only if K_0 is the boundary of an $(n+1)$-disk imbedded in S^m or, equivalently, K_0 bounds an $(n+1)$-disk in D^{m+1}. $[K_1, \mathcal{F}] = 0$ if and only if there is a framed disk $(D_1, \mathcal{F}_1)$ in S^m such that $\partial D_1 = K_1$ and $\mathcal{F}$ is the front (or rear) extension of $\mathcal{F}_1 \mid K_1$. Also $[K_1, \mathcal{F}] = 0$ if and only if $(K_1, \mathcal{F})$ extends to a framed $(n+1)$-disk in D^{m+1}.

Let Θ^n be the group of diffeomorphism (or h-cobordism) classes of n-spheres (see [14, § 2]) and let

$$\theta(n, k)\colon \Theta^{m,n} \longrightarrow \Theta^n$$

be the natural homomorphism. We define $\Sigma^{m,n} = \operatorname{Ker} \theta(n, k)$ and $\Theta_k^n = \operatorname{Cok} \theta(n, k)$; $\Sigma^{m,n}$ is the group of knotted S^n in S^m and Θ_k^n is a measure of the n-spheres which cannot be imbedded in S^m.

2.2. Let P_n be defined for $n \geqq 5$ by

$$P_n = \begin{cases} Z & n \equiv 0 \bmod 4 \\ Z_2 & n \equiv 2 \bmod 4 \\ 0 & n \text{ odd} . \end{cases}$$

For each $k \geqq 3$, we will establish three exact sequences, defined for $n \geqq 5$:

$$(1)_k \qquad \cdots \longrightarrow \pi_n(SO_k) \xrightarrow{\omega_1} \Theta_f^{m,n} \xrightarrow{\varphi_1} \Theta^{m,n} \xrightarrow{\partial_1} \pi_{n-1}(SO_k) \xrightarrow{\omega_1} \Theta_f^{m-1,n-1} \longrightarrow \cdots$$

$$(2)_k \qquad \cdots \longrightarrow \Theta_f^{m,n} \xrightarrow{\omega_2} \pi_n(G_k) \xrightarrow{\varphi_2} P_n \xrightarrow{\partial_2} \Theta_f^{m-1,n-1} \xrightarrow{\omega_2} \pi_{n-1}(G_k) \longrightarrow \cdots$$

$$(3)_k \qquad \cdots \longrightarrow \Theta^{m,n} \xrightarrow{\omega_3} \pi_n(G_k, SO_k) \xrightarrow{\varphi_3} P_n \xrightarrow{\partial_3} \Theta^{m-1,n-1}$$
$$\xrightarrow{\omega_3} \pi_{n-1}(G_k, SO_k) \longrightarrow \cdots .$$

These sequences, together with the exact homotopy sequence of the pair (G_k, SO_k):

$$(4)_k \qquad \cdots \longrightarrow \pi_n(SO_k) \xrightarrow{\omega_4} \pi_n(G_k) \xrightarrow{\varphi_4} \pi_n(G_k, SO_k) \xrightarrow{\partial_4} \pi_{n-1}(SO_k) \longrightarrow \cdots$$

will be related by the following commutative diagram:

$$
(5)_k \qquad
\begin{array}{ccccccccc}
 & & \pi_n(SO_k) & \longrightarrow & \pi_n(G_k) & \longrightarrow & & & P_n \\
 & \nearrow & & \searrow & & \nearrow & \searrow & \nearrow & \\
\pi_{n+1}(G_k, SO_k) & & & & \Theta_f^{m,n} & & \pi_n(G_k, SO_k) & & \\
 & \searrow & & \nearrow & & \searrow & \nearrow & \searrow & \\
 & & P_{n+1} & \longrightarrow & \Theta^{m,n} & \longrightarrow & & & \pi_{n-1}(SO_k) .
\end{array}
$$

The definitions of the homomorphisms of (1)–(3) will be carried out in (5.1), and the commutativity (up to sign) of (5) will be verified in (5.2). Exactness of (1)–(3) is then proved in (5.4) and (5.5).

3. An invariant of framed knots

3.1. Let $(K^n, \mathcal{F})$ be a framed knotted sphere in S^m. By the tubular neighborhood theorem, there is an imbedding, unique up to isotopy, $\varphi\colon K \times D^k \to S^m$ such that $\varphi(x, 0) = x$ for all $x \in K$ and $d\varphi(\varepsilon) = \mathcal{F}$, where ε is the normal frame to $K \times 0$ in $K \times D^k$ pulled back by the projection $K \times D^k \to D^k$ from a positive frame at $0 \in D^k$. Recall that the map $S^{k-1} \to S^m - K$ defined by $y \to \varphi(x_0, y)$, for a fixed $x_0 \in K$, is a homotopy equivalence (see e.g. [17, p. 962]). Let $h\colon S^m - K \to S^{k-1}$ be a homotopy inverse; then consider the map $g\colon K \times S^{k-1} \to S^{k-1}$ defined by $g = h \cdot \varphi \mid K \times S^{k-1}$. It is clear that g has degree $+1$ on the second factor and its homotopy class depends only on $[K, \mathcal{F}]$. We denote by $v(K, \mathcal{F})$ the element of $\pi_n(G_k)$ represented by g.

3.2. Let $\mathcal{F}'$ be another normal frame on K in S^m, and let $\mu\colon K^n \to SO_k$ be defined by $\mathcal{F}'(x) = \mu(x)\mathcal{F}(x)$. Let

$$\omega_4(n, k)\colon \pi_n(SO_k) \longrightarrow \pi_n(G_k)$$

be the homomorphism induced by inclusion and $\alpha \in \pi_n(SO_k)$ the homotopy class of μ.

LEMMA. $v(K, \mathcal{F}') - v(K, \mathcal{F}) = \omega_4(n, k) \cdot \alpha$.

PROOF. Let $\varphi'\colon K \times D^k \to S^m$ be the imbedding defined by $\varphi'(x, y) = \varphi(x, \mu(x) \cdot y)$. Then $\varphi'(x, 0) = x$ and $d\varphi'(\varepsilon) = \mathcal{F}'$; therefore $v(K, \mathcal{F}')$ is represented by $g' = h \cdot \varphi' \mid K \times S^{k-1}$. Since $g'(x, y) = g(x, \mu(x) \cdot y)$, the lemma is immediate.

3.3. If K^n is a knotted sphere in S^m, we may define an invariant $v(S^m, K^n) \in \pi_n(G_k, SO_k)$ analogous to (3.1). Let D_1, D_2 be n-disks in M such that $D_1 \cap D_2$ is an $(n-1)$-sphere (see [22, Th. 5.1]). Let $\mathcal{F}_t$ be a normal frame on D_t in S^m, $t = 1, 2$, and let $\varphi^t\colon D_t \times D^k \to S^m$ be imbeddings satisfying $\varphi^t(x, 0) = x$ for $x \in D_t$ and $d\varphi^t(\varepsilon_t) = \mathcal{F}_t$, where ε_t is the pull-back by the projection $D_t \times D^k \to D^k$ of a positive frame at $0 \in D^k$. Furthermore, by the tubular neighborhood theorem, we may choose the φ^t so that $\varphi^2(x, y) = \varphi^1(x, \mu(x) \cdot y)$ for

$x \in D_1 \cap D_2$, $y \in D^k$ and some $\mu: D_1 \cap D_2 \longrightarrow SO_k$. It is clear that, if we orient $D_1 \cap D_2$ by ∂D_1, μ represents $O(K, \mathcal{F}_2) \in \pi_{n-1}(SO_k)$. Let $h: S^m - K^n \longrightarrow S^{k-1}$ be a homotopy inverse of $y \longrightarrow \varphi^1(x_0, y)$, $x_0 \in D_1$. By the homotopy extension theorem, we may assume $h \cdot \varphi^1(x, y) = y$ for all $x \in D_1$, $y \in S^{k-1}$. Now define $\lambda: D_2 \times S^{k-1} \longrightarrow S^{k-1}$ by $\lambda = h \cdot \varphi^2 \mid D_2 \times S^{k-1}$; then (λ, μ) defines an element of $\pi_n(G_k, SO_k)$ which we denote by $v(S^m, K^n)$. That $v(S^m, K^n)$ depends only on $[S^m, K^n]$ is a straightforward exercise. If

$$\partial_4(n, k): \pi_n(G_k, SO_k) \longrightarrow \pi_{n-1}(SO_k)$$

is the boundary homomorphism, we have proved the following:

LEMMA 1. *If $\mathcal{F}$ is any normal frame on the complement of a point of K^n in S^m, then:*

$$O(K^n, \mathcal{F}) = \partial_4(n, k) \cdot v(S^m, K^n) .$$

Moreover, if $(K^n, \mathcal{F})$ is a framed knotted sphere in S^m and

$$\varphi_4(n, k): \pi_n(G_k) \longrightarrow \pi_n(G_k, SO_k)$$

is the usual homomorphism, then by taking $\mathcal{F}_t = \mathcal{F} \mid D_t$ in the above definition, it is easy to see that the following is true.

LEMMA 2. $v(S^m, K^n) = \varphi_4(n, k) \cdot v(K^n, \mathcal{F})$.

3.4. LEMMA. (a) *If $(K_1^n, \mathcal{F}_1)$, $(K_2^n, \mathcal{F}_2)$ are framed knotted spheres in S^m, then:*

$$v\big((K_1, \mathcal{F}_1) \# (K_2, \mathcal{F}_2)\big) = v(K_1, \mathcal{F}_1) + v(K_2, \mathcal{F}_2) .$$

(b) *If K_1^n, K_2^n are knotted spheres in S^m, then*

$$v\big((S^m, K_1) \# (S^m, K_2)\big) = v(S^m, K_1) + v(S^m, K_2) .$$

PROOF. We prove (a) only. The proof of (b) is similar, and we leave it to the reader as an exercise.

Let φ^t be an imbedding of $K_t \times D^k$ in S^m and h_t a homotopy equivalence of $S^m - K_t$ on S^{k-1}, $t = 1, 2$, used in the definitions, as in (3.1). Let D_t be an m-disk in S^m, intersecting K_t in a subdisk and $\Lambda: D_1 \longrightarrow D_2$ an orientation reversing diffeomorphism such that $\Lambda(K_1 \cap D_1) = -K_2 \cap D_2$ and $d\Lambda(\mathcal{F}_1 \mid K_1 \cap D_1) = d\Lambda(\mathcal{F}_2 \mid K_2 \cap D_2)$. Now we may assume that

$$h_1 \mid (S^m - K_1) \cap D_1 = h_2 \cdot \Lambda \mid (S^m - K_1) \cap D_1 ,$$

by the homotopy extension theorem. If $(K, \mathcal{F}) = (K_1, \mathcal{F}_1) \# (K_2, \mathcal{F}_2)$, using Λ to form the connected sum, then there is a natural decomposition:

$$S^m - K = (S^m - K_1) \cup (S^m - K_2) ,$$

where $S^m - K_1$ and $S^m - K_2$ are identified on $(S^m - K_1) \cap D_1$ and $(S^m - K_2) \cap D_2$

by Λ. Also $K = K_1 \cup K_2$, identified on $K_1 \cap D_1$ and $K_2 \cap D_2$. With respect to these decompositions we may define φ: $K \times D^k \to S^m$ so that $\varphi \mid K_t \times S^{k-1} = \varphi^t \mid K_t \times S^{k-1}$, and h: $S^m - K \to S^{k-1}$ so that $h \mid S^m - K_t = h_t$. If $g = h \cdot \varphi \mid K \times S^{k-1}$, $g_t = h_t \cdot \varphi^t \mid K_t \times S^{k-1}$, then

$$g \mid (K_t - D_t) \times S^{k-1} = g_t \mid (K_t - D_t) \times S^{k-1} .$$

By one of the definitions of addition in $\pi_n(G_k)$, we have the desired result.

3.5. It will be useful to have another interpretation of $v(K^n, \mathcal{F})$ in terms of the Thom construction. A similar interpretation of $v(S^m, K^n)$ also exists, but this will not be needed.

Recall the well-known decomposition:

$$S^m = (S^n \times D^k) \cup (-1)^{n+1}(D^{n+1} \times S^{k-1}) .$$

This decomposition induces an imbedding of:

$$S^n \times S^{k-1} = \partial(D^{n+1} \times S^{k-1}) = (-1)^n \partial(S^n \times D^k)$$

into S^m; we will, in this way, consider $S^n \times S^{k-1}$ a submanifold of S^m. Let f denote the normal field to $S^n \times S^{k-1}$ in S^m, pointing out from $S^n \times D^k$. Note that, if $(M^n, \mathcal{F})$ is a framed submanifold of $S^n \times S^{k-1}$, and $\mathcal{F}'$ is the front extension of $\mathcal{F}$ to a normal frame to M in S^m, then the first vector in $\mathcal{F}'$ is $f \mid M$.

LEMMA. *Every framed knotted n-sphere in S^m is isotopic to one $(K^n, \mathcal{F})$, where K^n is nuclear in $S^n \times S^{k-1}$, and $\mathcal{F}$ is the front extension of a normal frame to K^n in $S^n \times S^{k-1}$.*

PROOF. Let $\mathcal{F} = (f_1, \cdots, f_k)$, and suppose L^{k-1} is a fibre of the normal sphere bundle to K, oriented so that the linking number of K and L in S^m is $(-1)^{n+1}$. Since L is unknotted, there is an isotopy of S^m which carries L onto $0 \times S^{k-1}$; therefore we may assume $L = 0 \times S^{k-1}$. Let T be a tubular neighborhood of K^n, disjoint from L; then L is a deformation retract of $\overline{S^m - T}$. Therefore, by [15, Th. 4.1], $\overline{S^m - T}$ is a tubular neighborhood of L. Now $(-1)^{n+1}D^{n+1} \times S^{k-1}$ is also a tubular neighborhood of L; therefore, by the tubular neighborhood theorem, another isotopy of S^m will insure that $\overline{S^m - T} = (-1)^{n+1}D^{n+1} \times S^{k-1}$. Thus $T = S^n \times D^k$ and, since the linking number of K and L is $(-1)^{n+1}$, K is homotopic to $S^n \times 0$ in T.

Now the normal field f_1 determines a trajectory along which we may isotopically deform K, in $S^n \times D^k$, to $S^n \times S^{k-1}$. Clearly K is then homotopic, in $S^n \times S^{k-1}$, to $S^n \times y_0$, $y_0 \in S^{k-1}$, and $f_1 = f \mid K$.

3.6. Let $(M^n, \mathcal{F})$ be a framed submanifold of $S^n \times S^{k-1}$; then $t(M, \mathcal{F}) \in \pi^{k-1}(S^n \times S^{k-1})$.

LEMMA. *M is a nuclear submanifold of $S^n \times S^{k-1}$ if and only if*

$t(M, \mathcal{F}) \in \pi_n(G_k)$.

PROOF. Suppose the intersection number of M and $x_0 \times S^{k-1}$ is a, $x_0 \in S^n$. We may then assume M intersects $x_0 \times S^{k-1}$ normally in $a + 2b$ isolated points, $a + b$ of them positively, and b of them negatively. Now look at the restriction of $t(M, \mathcal{F})$ to $x_0 \times S^{k-1}$. We have a map which is constant outside $a + 2b$ isolated disks and wraps these disks around S^{k-1}, $a + b$ of them positively, and b of them negatively. This demonstrates that $t(M, \mathcal{F})$ has degree a on the second factor. The lemma follows immediately.

3.7. We now relate the invariant v defined in (3.1) with the Thom construction. Let $(K^n, \mathcal{F})$ be a framed knotted nuclear sphere in $S^n \times S^{k-1}$; then, by (3.6), $t(K, \mathcal{F}) \in \pi_n(G_k)$. Since $S^n \times S^{k-1} \subset S^m$, let $\mathcal{F}'$ be the front extension of $\mathcal{F}$ to a normal frame in S^m.

LEMMA. $v(K^n, \mathcal{F}') + t(K^n, \mathcal{F}) = 0$.

PROOF. Let $(K_0, \mathcal{F}_0')$ be the framed knotted sphere obtained by isotopically deforming $(K, \mathcal{F}')$ radially into $S^n \times D^k$; then $(K, \mathcal{F}')$ is isotopic to $(K_0, \mathcal{F}_0')$. Now $S^n \times D^k$ is a tubular neighborhood of K_0, by [22, Th. 4.1], since K_0 is homotopic in $S^n \times D^k$ to $S^n \times y_0$. Thus there exists a diffeomorphism $\Lambda: K_0 \times D^k \to S^n \times D^k$, unique up to isotopy, such that $\Lambda(x, 0) = x$, for $x \in K_0$, and $d\Lambda(\varepsilon_0) = \mathcal{F}_0'$, where ε_0 is the pullback by the projection $K_0 \times D^k \to D^k$ of a positive tangent frame at 0. Then $K = \Lambda(K_0 \times y_0)$, for some $y_0 \in S^{k-1}$ and $d\Lambda(\varepsilon) = \mathcal{F}$, where ε is the pull-back by the projection $p: K_0 \times S^{k-1} \to S^{k-1}$, of a positive tangent frame at y_0. Thus it follows immediately that the map $g: S^n \times S^{k-1} \to S^{k-1}$, defined by $g = p \cdot \Lambda^{-1} | S^n \times S^{k-1}$; represents $t(K, \mathcal{F})$. In fact, y_0 is a regular value of g, $g^{-1}(y_0) = K$, and $\mathcal{F}$ is the pull-back by g of a positive tangent frame at y_0.

Let $p': S^n \times S^{k-1} \to S^{k-1}$ be the projection and define $g': K_0 \times S^{k-1} \to S^{k-1}$ by $g' = p' \cdot \Lambda | K_0 \times S^{k-1}$. Since Λ preserves orientation and has degree $+1$ on the first factor (this follows from K being nuclear) $\Lambda | K_0 \times S^{k-1}$ has degree $+1$ on the second factor. Therefore g' has degree $+1$ on the second factor. Now it follows immediately that g' represents $v(K_0, \mathcal{F}_0') = v(K, \mathcal{F}')$.

We now use these representatives to compute $t(K, \mathcal{F}) + v(K, \mathcal{F}')$. Let $h: S^n \to K_0$ be a map of degree $+1$; it follows from the definition of addition in $\pi_n(G_k)$, described in (1.3), that $t(K, \mathcal{F}) + v(K, \mathcal{F}')$ is represented by the composite map:

$$(x, y) \longrightarrow p'\Lambda\big(h(x), p \cdot \Lambda^{-1}(x, y)\big) .$$

By the covering homotopy theorem, $\Lambda^{-1} | S^n \times S^{k-1}$ is homotopic to a map $\lambda: S^n \times S^{k-1} \to K_0 \times S^{k-1}$ which carries $x \times S^{k-1}$ into $h(x) \times S^{k-1}$, for every $x \in S^n$. Replacing Λ^{-1} by λ in the above formula, we see that $t(K, \mathcal{F}) + v(K, \mathcal{F}')$ is represented by the map:

$$(x, y) \longrightarrow p'\Lambda\lambda(x, y) .$$

Since $\Lambda \cdot \lambda$ is homotopic to the identity, the proof of the lemma is completed.

3.8. We may now use the preceding results to show that $v(K, \mathcal{F})$ represents an obstruction to *extending* $(K, \mathcal{F})$ to a framed $(n+1)$-dimensional submanifold of S^m.

LEMMA. *Let $(K^n, \mathcal{F})$ be a framed knotted sphere in S^m. Then $v(K, \mathcal{F}) = 0$ if and only if there is a framed submanifold $(M^{n+1}, \mathcal{G})$ of S^m such that $\partial M = K$ and $\mathcal{F}$ is a front extension of $\mathcal{G} \mid K$.*

PROOF. Suppose $v(K, \mathcal{F}) = 0$. Let $\mathcal{F}'$ be the field of $(k-1)$-normal vectors on K of which $\mathcal{F}$ is a front extension. Then we define a new framed knotted sphere $(K_0, \mathcal{F}_0)$ in S^m by $K_0 = (-1)^{n+1}K$ and $\mathcal{F}_0 =$ front extension of $\mathcal{F}'$ on K_0. Then $v(K_0, \mathcal{F}_0) = 0$, since if $g\colon K \times S^{k-1} \rightarrow S^{k-1}$ is a representative map of $v(K, \mathcal{F})$, a representative map for $v(K_0, \mathcal{F}_0)$ is given by the composition $r \cdot g \cdot r'$, where r is a reflection of S^{k-1} and $r'\colon K_0 \times S^{k-1} \rightarrow K \times S^{k-1}$ is given by the identity on the first factor and reflection about the first coordinate on the second factor. By(3.5) we may assume that K_0 is nuclear in $S^n \times S^{k-1}$, and $\mathcal{F}'$ is a normal frame to K_0 in $S^n \times S^{k-1}$. By (3.7), $t(K_0, \mathcal{F}') = -v(K_0, \mathcal{F}_0) = 0$. Therefore $t(K_0, \mathcal{F}')$ extends to $D^{n+1} \times S^{k-1}$ and, by 1.4, this means there is a framed submanifold $(M_0, \mathcal{G})$ on $D^{n+1} \times S^{k-1}$ such that $\partial M_0 = K_0$ and $\mathcal{G} \mid K_0 = \mathcal{F}$. If we defined $M = (-1)^{n+1}M_0$, then $(M, \mathcal{G})$ is a framed submanifold of S^m and satisfies the desired conclusion.

Conversely, suppose $(M, \mathcal{G})$ exists as in the lemma. Let T be a tubular neighborhood of K so that $T \cap M$ is a collar of K in M. Define $M_0 = \overline{M - T \cap M}$, $\mathcal{G}_0 = \mathcal{G} \mid M_0$ and $K_0 = \partial M_0$, $\mathcal{F}_0 =$ front extension of $\mathcal{G}_0 \mid K_0$. As in (3.5) we may assume $T = S^n \times D^k$ and K is homotopic to $S^n \times 0$. Therefore K_0 is nuclear in $S^n \times S^{k-1}$. Now $((-1)^{n+1}M_0, \mathcal{G}_0)$ is a framed submanifold of $D^{n+1} \times S^{k-1}$ extending $((-1)^{n+1}K_0, \mathcal{G}_0 \mid K_0)$. Thus $t((-1)^{n+1}K_0, \mathcal{G}_0 \mid K_0) = 0$ and, by (3.7), if $\mathcal{F}_1$ is the front extension of $\mathcal{G}_0 \mid K_0$ on $(-1)^{n+1}K_0$, then $v((-1)^{n+1}K_0, \mathcal{F}_1) = 0$. As in the preceding paragraph, this implies $v(K_0, \mathcal{F}_0) = 0$. But $(K, \mathcal{F})$ is isotopic to $(K_0, \mathcal{F}_0)$, proving the lemma.

3.9. We conclude this section by determining the relation between the Thom construction in S^m and the invariant v on a framed knotted sphere in S^m. Let

$$\nu(n, k)\colon \pi_n(G_k) \longrightarrow \pi_m(S^k)$$

be the homomorphism defined by applying the Hopf construction (see [27, p. 208]) to a map $S^n \times S^{k-1} \rightarrow S^{k-1}$.

Let $(K^n, \mathcal{F})$ be a framed knotted sphere in S^m. Denote by ι_k the standard generator of $\pi_k(S^k)$ and let $\circ$ denote the composition operation in homotopy groups.

LEMMA. $(-1)^{n+1}t(K^n, \mathcal{F}) = ((-1)^{k+1}\iota_k)\circ(\nu(n, k)\cdot v(K^n, \mathcal{F}))$.

PROOF. By (3.5) we assume K^n is a nuclear submanifold of $S^n \times S^{k-1}$ and $\mathcal{F}$ is the front extension of a normal frame $\mathcal{F}'$ to K in $S^n \times S^{k-1}$. Then $v(K, \mathcal{F}) = -t(K, \mathcal{F}')$. Let $K_0 = (-1)^{k+1}K$ and $\mathcal{F}_0 =$ *rear* extension of $\mathcal{F}'$ on K_0; this insures that the last vector in $\mathcal{F}_0$ is actually $f \mid K_0$ (see (3.5) for definition of f). Now it is proved in [11, p. 348] that $t(K_0, \mathcal{F}_0) = (-1)^n\nu(n, k)\cdot t(K, \mathcal{F}')$. Since the first vector of $\mathcal{F}$ is $f \mid K$, we have the relation $\mathcal{F}(x) = \mu_0\cdot\mathcal{F}_0(x)$ for all $x \in K$, where μ_0 is the orthogonal transformation defined by:

$$(f_1, \cdots, f_{k-1}, f_k) \longrightarrow (f_k, f_1, f_2, \cdots, f_{k-1}) .$$

Since the degree of μ_0 is $(-1)^{k+1}$, it follows that $t(K^n, \mathcal{F}) = ((-1)^{k+1}\iota_k)\circ t(K_0^n, \mathcal{F}_0)$. This proves the lemma.

4. Simplifying a framed submanifold

4.1. The technique of spherical modifications will be essential in this work (see [20], [14] and [3]). We first discuss the notion of Arf invariant (see [12, § 1] and [14, § 8]) which will be needed.

Let $(M^n, \mathcal{F})$ be a framed submanifold of a π-manifold V^m, where ∂M is empty or a sphere. We assume $n = 4r + 2$, $r \geqq 1$, and M is $2r$-connected. Let $s \subset M$ be an imbedded S^{2r+1}, null-homotopic in V. We will describe a slight generalization of a construction in [3, 3.2]. Let $\mathcal{G}$ be a positive framing of $\tau(V) \mid V - y_0$, $y_0 \in V - M$. If $\mathcal{F} = (f_1, \cdots, f_k)$, consider the subbundle of $\tau(V) \mid s$ spanned by $\tau(s)$ and $f_1 \mid s$; this has a canonical framing $\varepsilon = (e_1, \cdots, e_{2r+2})$, as described in [3, 3.2(3)]. Now consider the vector framing

$$\mathcal{G}_0 = (e_1, \cdots, e_{2r+2}, f_2 \mid s, \cdots, f_k \mid s)$$

of a subbundle of $\tau(V) \mid s$. For each $x \in s$, there is an $((m - 2r - 1) \times m)$-matrix, with orthonormal row vectors, $\mu(x)$, such that (considering $\mathcal{G}$ and $\mathcal{G}_0$ as column vectors),

$$\mathcal{G}_0(x) = \mu(x)\mathcal{G}(x) .$$

We can identify the space of such matrices with the Stiefel manifold $V_{m,m-2r-1}$ of orthonormal $(m - 2r - 1)$-frames in E^m. Then μ defines an element $\varphi(M, \mathcal{F})\cdot s \in \pi_{2r+1}(V_{m,m-2r-1}) \approx Z_2$.

Suppose $V = E^m$; let $(\psi, \psi')\colon S^{2r+1} \longrightarrow E^m$ be the E^{m-2r-1}-immersion associated with $(s, \mathcal{F} \mid s)$ according to [4, Th. 3.3]. It is a direct consequence of the definitions that $\varphi(M, \mathcal{F})\cdot s$ is the same as $\tau(\psi')$, defined in [4, p. 258].

Since s is null-homotopic in $V - y_0$, $\varphi(M, \mathcal{F})\cdot s$ is independent of $\mathcal{G}$, for any two choices of $\mathcal{G}$ will be homotopic on s. Furthermore, $\varphi(M, \mathcal{F})\cdot s$ depends only on the homotopy class of s. In fact, if $s_0 \subset M$ is homotopic to s, it follows from the arguments of [4, § 8] that s and s_0 are regularly homotopic in M, for their self-intersection numbers are equal. But it is clear that $\varphi(M, \mathcal{F}) \cdot s$ is invariant under a regular homotopy of s.

Thus we have defined a function:

$$\varphi(M, \mathcal{F}): H_{2r+1}(M; Z) \longrightarrow Z_2$$

which depends only on the isotopy class of $(M, \mathcal{F})$.

4.2. We shall verify that $\varphi(M, \mathcal{F}) = \varphi$ satisfies the formula

$$\varphi(\alpha + \beta) = \varphi(\alpha) + \varphi(\beta) + (\alpha \cdot \beta)_2$$

for $\alpha, \beta \in H_{2r+1}(M)$, where $(\alpha \cdot \beta)_2$ is the mod 2 residue of the intersection number of α and β. The argument is similar to one in [25, p. 167–8]. First note that the construction in (4.1) may be carried out on an *immersed* sphere s. Let $B^m \subset V$ be an imbedded m-disk such that $B^m \cap M = B^n \subset B^{n+1}$, standard sub-disks of B^m, and $f_1 \mid s$ is tangent to B^{n+1}. Let $\psi: S^{2r+1} \to B^n$ be an immersion with self-intersection number one. Let $s = \psi(S^{2r+1})$ and (ψ, ψ') be the E^{2r+2}-immersion into B^{n+1} associated with $(s, f_1 \mid s)$. Then $\tau(\psi, \psi')$ coincides with the obstruction to extending ψ to an immersion of D^{2r+2} into B^n, which is non-zero, under the isomorphism $\pi_{2r+1}(V_{n,2r+1}) \to \pi_{2r+1}(V_{n+1,2r+2})$. Furthermore, it is clear that $\tau(\psi, \psi') \to \varphi(M, \mathcal{F}) \cdot s$, under the isomorphism $\pi_{2r+1}(V_{n+1,2r+2}) \to \pi_{2r+1}(V_{m,m-2r-1})$. Thus, $\varphi(M, \mathcal{F}) \cdot s \neq 0$.

Now let s_0, s_1 be imbedded $(2r + 1)$-spheres in M representing α, β, respectively. Then we can form the connected sum $s_0 \# s_1$, in M, to obtain an immersed sphere s, representing $\alpha + \beta$, with self-intersection number $(\alpha \cdot \beta)_2$. Now it is clear that

$$\varphi(M, \mathcal{F}) \cdot s = \varphi(M, \mathcal{F}) \cdot s_0 + \varphi(M, \mathcal{F}) \cdot s_1 .$$

By the above discussion it follows that, if s' is obtained from s by locally introducing $(\alpha \cdot \beta)_2$ self-intersections and then removing them by a regular homotopy, then

$$\varphi(M, \mathcal{F}) \cdot s' = \varphi(M, \mathcal{F}) \cdot s + (\alpha \cdot \beta)_2 .$$

Now the desired formula follows.

4.3. We define, in the usual way, (see [14, p. 535])

$$c(M, \mathcal{F}) = \sum_i \varphi(\alpha_i)\varphi(\beta_i)$$

where $\{\alpha_i; \beta_i\}$ is any symplectic basis of $H_{2r+1}(M)$.

If $r \neq 1$ or 3, it follows from [3, 3.4] that $\varphi(M, \mathcal{F})$ coincides with ψ_0 defined in [14, p. 534], since the homomorphism $\pi_{2r+1}(V_{m,m-2r-1}) \to \pi_{2r}(SO_{2r+1})$ is a monomorphism. Then, if $r \neq 1$ or 3, $c(M, \mathcal{F}) = c(M)$ [14, p. 535] and, therefore, is independent of $\mathcal{F}$ and the imbedding.

4.4. Let $(M^n, \mathcal{F})$ be a framed submanifold of V^m, as above, where $(V^m, \mathcal{G})$ is a framed submanifold of W^p. If $\mathcal{F} = (f_1, \cdots, f_k)$ and $\mathcal{G} = (g_1, \cdots, g_{p-m})$, we may define a normal frame $\mathcal{F}'$ to M in W by $\mathcal{F}' = (f_1, \cdots, f_k, g_1, \cdots, g_{p-m})$. Since the homomorphism $\pi_{2r+1}(V_{m,m-2r-1}) \to \pi_{2r+1}(V_{p,p-2r-1})$ is an isomorphism it follows from the definitions that

$$c(M, \mathcal{F}') = c(M, \mathcal{F}) .$$

As a consequence of this remark we may prove:

LEMMA. *$c(M, \mathcal{F})$ depends only on the cobordism class of $(M, \mathcal{F})$.*

PROOF. Since V is a π-manifold, we may imbed it in a high-dimensional sphere with a normal frame. Therefore, by the preceding remark, we may assume $V = S^m$ and m is large.

Suppose M is closed and $(M, \mathcal{F})$ extends to a framed submanifold $(N^{n+1}, \mathcal{G})$ of D^{m+1}. We shall show $c(M, \mathcal{F}) = 0$. By the arguments in [12, p. 260–1], we may assume N is $2r$-connected and $\{\alpha_i; \beta_i\}$ is a symplectic basis of $H_{2r+1}(M)$ such that α_i is null-homologous, and, therefore, null-homotopic, in N, for every i. If $\mathcal{F} = (f_1, \cdots, f_k)$, let $\mathcal{F}' = (f_1, \cdots, f_k, (-1)^{m+1}f)$ be the rear extension of $\mathcal{F}$ to a normal frame to M in D^{m+1}; f is the inward pointing radial vector of D^{m+1} on M. By the remark above, $c(M, \mathcal{F}') = c(M, \mathcal{F})$. Now define

$$\mathcal{F}'' = \big(f, f_2, \cdots, f_k, (-1)^m f_1\big) ,$$

another normal frame to M in D^{m+1}; then $\mathcal{F}''$ is homotopic to $\mathcal{F}'$ and so $c(M, \mathcal{F}') = c(M, \mathcal{F}'')$.

Let s be an imbedded sphere in M representing α_i. Then s extends to a mapping $\psi: D^{2r+2} \to N$; we may assume ψ is differentiable, $\psi(D^{2r+2})$ meets M only at s, and the inward pointing radial vector of D^{2r+2} on ∂D^{2r+2} is mapped onto $f \mid s$ by $d\psi$. It follows from [30, Th. 7] that we may assume ψ is an immersion ($\dim N = 2(2r + 2) - 1$); let $d = \psi(D^{2r+2})$. Consider the framing

$$\mathcal{G}_0 = \big(e_1, \cdots, e_{2r+2}, f_2 \mid s, \cdots, f_k \mid s, (-1)^m f_1 \mid s\big)$$

induced, as in (4.1), from $\mathcal{F}''$ on s. Now ε is just the restriction to s of a tangent framing of d. Furthermore $\big(f_2 \mid s, \cdots, f_k \mid s, (-1)^m f_1 \mid s\big)$ extends to a normal framing over d because $\mathcal{F}$ is defined on N. Thus $\mathcal{G}_0$ extends over d, giving an explicit null-homotopy of $\varphi(M, \mathcal{F}'') \cdot s$. This shows $\varphi(M, \mathcal{F}'') \cdot \alpha_i = 0$, for every i, which implies $c(M, \mathcal{F}'') = 0$.

Now suppose $(M_0, \mathcal{F}_0)$ is cobordant to $(M_1, \mathcal{F}_1)$. We may assume $V = D^m_+$ and $M_0 \cap S^{m-1} = \partial M_0$, for m large. By an isotopy we may assume, furthermore, that $\partial M_0 = \partial M_1$ and $\mathcal{F}_0 \mid \partial M_0 = \mathcal{F}_1 \mid \partial M_1$, since $(\partial M_0, \mathcal{F}_0 \mid \partial M_0)$ is h-cobordant to

$(\partial M_1, \mathcal{F}_1 \mid \partial M_1)$ (in S^{m-1}). Let $(M_2, \mathcal{F}_2)$ be the framed submanifold of D^m_- obtained by reflecting $(-M_1, \mathcal{F}_1)$ across S^{m-1}, and let $(M, \mathcal{F}) = (M_0, \mathcal{F}_0) \cup (M_2, \mathcal{F}_2)$ be the framed submanifold of S^m obtained by identifying boundaries. Then

$$c(M, \mathcal{F}) = c(M_0, \mathcal{F}_0) + c(M_2, \mathcal{F}_2) = c(M_0, \mathcal{F}_0) - c(M_1, \mathcal{F}_1) .$$

But it is easily seen that $(M, \mathcal{F})$ extends to a framed submanifold of D^{m+1}, using a cobordism between $(M_0, \mathcal{F}_0)$ and $(M_1, \mathcal{F}_1)$ which is *constant* on the boundary. Thus, by the preceding argument, $c(M, \mathcal{F}) = 0$.

This completes the proof of the lemma.

4.5. Let $(M^n, \mathcal{F})$ be a framed submanifold of a π-manifold V^m, where ∂M is empty or a sphere. Let us imbed V as a framed submanifold $(V, \mathcal{G})$ of S^p, for large p. Then $(M, \mathcal{F}')$ is defined, as in (4.4), as a framed submanifold of S^p. By spherical modifications [20], $(M, \mathcal{F}')$ is cobordant to $(M_0, \mathcal{F}_0)$, where M_0 is $2r$-connected ($n = 4r + 2$). We define

$$c(M, \mathcal{F}) = c(M_0, \mathcal{F}_0) .$$

According to (4.4) this is a well-defined extension of our first definition. It also follows that Lemma (4.4) is still true.

Let $(M^n, \mathcal{F})$ be a framed submanifold of a π-manifold V^m, where ∂M is empty or a sphere. We define an invariant $\gamma(M, \mathcal{F}) \in P_n$ as follows:

(i) $\gamma(M, \mathcal{F}) = 0$ if n is odd.

(ii) $\gamma(M, \mathcal{F}) = 1/8$ index M if $n \equiv 0 \bmod 4$.

(iii) $\gamma(M, \mathcal{F}) = c(M, \mathcal{F})$ if $n \equiv 2 \bmod 4$.

That index M is always a multiple of 8 follows from [18, Th. 1 and Lem. 3].

By Lemma (4.4) and a result of Thom, $\gamma(M, \mathcal{F})$ is an invariant of the cobordism class of $(M, \mathcal{F})$. We will also need the following additive property of $\gamma(M, \mathcal{F})$.

Let $(M_t^n, \mathcal{F}_t)$ be a framed submanifold of a π-manifold V_t^m, for $t = 0, 1$.

LEMMA. $\gamma((M_0, \mathcal{F}_0) \# (M_1, \mathcal{F}_1)) = \gamma(M_0, \mathcal{F}_0) + \gamma(M_1, \mathcal{F}_1)$.

PROOF. Suppose $n = 2r$. If r is even, this is a well-known property of the index.

Suppose r is odd; we may assume $V_t = S^m$, for large m, and M_t is $(r - 1)$-connected, since the desired formula is invariant under cobordism. But now this formula is immediate if we notice that a symplectic basis of $H_r(M_0 \# M_1)$ can be obtained by the union of symplectic bases of each $H_r(M_t)$ and that

$$\varphi\big((M_0, \mathcal{F}_0) \# (M_1, \mathcal{F}_1)\big) \mid H_r(M_t) = \varphi(M_t, \mathcal{F}_t) .$$

4.6. LEMMA. *Suppose $k \geq 2$ and $\gamma \in P_n$. Then there is a framed submanifold $(M^n, \mathcal{F})$ of S^m such that $\gamma(M, \mathcal{F}) = \gamma$ and ∂M is a sphere.*

PROOF. Suppose $n = 4r$. It follows from the results of [25] that there exists a $(2r-1)$-connected π-manifold M, with ∂M a sphere, such that index $M = 8\gamma$. According to [5, 4.3], M imbeds in S^{n+1}; consequently M imbeds in S^m with a normal frame, and the lemma follows for $n \equiv 0 \bmod 4$.

Suppose $n = 4r + 2$, $r \neq 1, 3$. It again follows from [25] that there exists a $2r$-connected π-manifold M, ∂M a sphere, such that $c(M) = \gamma$. By [5, 4.3], M imbeds in S^m with a normal frame $\mathcal{F}$. Then $c(M, \mathcal{F}) = c(M)$, as remarked in (4.3), and the lemma follows.

Suppose $n = 2r$, $r = 3$ or 7. Consider the homomorphism

$$p\colon \pi_r(SO_{m-r-1}) \longrightarrow \pi_r(V_{m-r-1,k-1}) \approx Z_2$$

induced by natural projection; since $\pi_r(SO_{r+1}) = 0$, p is an epimorphism. Choose $\xi_1, \xi_2 \in \pi_r(SO_{m-r-1})$ so that $p(\xi_1) = \gamma$, $p(\xi_2) = 1$.

For $t = 1, 2$, let $(D_t^{r+1}, \varepsilon_t)$ be disjoint framed disks in S^m. Let $\mu_t\colon \partial D_t \rightarrow SO_{m-r-1}$ represent ξ_t and define a normal frame $\mathcal{F}'_t = (f_1^t, \cdots, f_{m-r-1}^t)$ on ∂D_t by

$$\mathcal{F}'_t(x) = \mu_t(x)\varepsilon_t(x) .$$

By the tubular neighborhood theorem, there exists a submanifold M_t^n of S^m satisfying:

(i) M_t meets D_t normally along ∂D_t,

(ii) M_t is a tubular neighborhood of ∂D_t (in M_t),

(iii) $(f_1^t, \cdots, f_{k-1}^t)$ are normal to M_t.

Let $\mathcal{F}_t$ be a normal frame to M_t obtained by extending $(f_1^t, \cdots, f_{k-1}^t)$ over M_t and then taking the front extension. Clearly we may orient M_t so that the initial vector of $\mathcal{F}_t$ points radially into D_t.

Since $m > \dim \partial D_1 + \dim \partial D_2$, we may perform an isotopy on ∂D_1, say, so that the ∂D_t intersect at a single point at which the $\mathcal{F}_t$ coincide. Then we may assume $M_1 \cap M_2$ is an n-disk with corners, and $M_1 \cup M_2$ is a submanifold (with corners) of S^m. Let M be the result of straightening the angles of $M_1 \cup M_2$ and let $\mathcal{F}$ be the normal frame on M induced by $\mathcal{F}_1$ and $\mathcal{F}_2$.

Let $\alpha, \beta \in H_r(M)$ be the generators represented by $\partial D_1, \partial D_2$, respectively; clearly $\{\alpha; \beta\}$ is a symplectic basis of $H_r(M)$. It is straightforward, using the definition in (4.1) and the details of the construction of M, to see that

$$\varphi(M, \mathcal{F}) \cdot \alpha = p(\xi_1) = \gamma ; \qquad \varphi(M, \mathcal{F}) \cdot \beta = p(\xi_2) = 1 .$$

Therefore $c(M, \mathcal{F}) = \varphi(\alpha)\varphi(\beta) = \gamma$.

This completes the proof of the lemma.

4.7. The main result of this section is:

THEOREM. *Let $(M^n, \mathcal{F})$ be a framed submanifold of V where:*

(i) *$V = S^{m-1}$, and ∂M is a sphere, or*

(ii) $V = S^n \times S^{k-1}$, *and M is nuclear.*
Then $(M, \mathcal{F})$ is cobordant to a framed:
(i) *disk imbedded in $V = S^{m-1}$, or*
(ii) *knotted sphere in $V = S^n \times S^{k-1}$,*
if and only if $\gamma(M, \mathcal{F}) = 0$.

PROOF. Since $\gamma(M, \mathcal{F})$ is an invariant of cobordism, $\gamma(M, \mathcal{F}) = 0$ is certainly necessary. It remains to prove the sufficiency.

4.8. The technique for simplifying $(M, \mathcal{F})$ will be, as to be expected, that of framed spherical modifications, as presented in [3, § 3]. Let $\alpha \in \pi_r(M)$; we slightly re-formulate the hypotheses of [3, 3.2].

(1) α is represented by an imbedded sphere s in int M,

(2) s bounds an imbedded disk d in V such that $d \cap M = s$, and $f_1 | s$ points radially into d,

(3) the frame $(f_2, \cdots, f_{k-1}) | s$ extends to a (partial) normal frame on d, where $\mathcal{F} = (f_1, \cdots, f_{k-1})$.

The proof of [3, 3.3] shows that, if these hypotheses are satisfied, $(M, \mathcal{F})$ is cobordant to $(M_0, \mathcal{F}_0)$, where, in the notation of [14], M_0 is diffeomorphic to $\chi(M, i)$ for some imbedding $i: S^r \times D^{n-r} \to M$ representing α. We will say $(M_0, \mathcal{F}_0)$ is obtained from $(M, \mathcal{F})$ by a *spherical modification associated with α.*

Let us assume that M is $(r-1)$-connected, where $0 \leqq 2r \leqq n$. In [14], Kervaire and Milnor show how to kill $\pi_r(M)$ by a sequence of spherical modifications on M. Thus, our task is to show that these spherical modifications can be carried out on $(M, \mathcal{F})$.

4.9. We first show that hypotheses (1) and (2) of (4.8) can be satisfied for all $\alpha \in \pi_r(M)$. In fact, hypothesis (1) is a direct consequence of classical results of Whitney (also see [20, Lem. 6]). By the same arguments (see [29]) hypothesis (2) will be satisfied whenever s extends to a *singular* disk d' in V, meeting M only at $\partial d'$, and $V - M$ is 1-connected. But this will follow if $V - M$ is r-connected, which we shall now verify.

If $k \geqq 4$, then codim $M \geqq 3$ and the 1-connectedness of $V - M$ follows from the 1-connectedness of V by a general position argument. If $k = 3$ and M is bounded, then we can isotopically deform M into a small neighborhood of its $(n-1)$-skeleton (see e.g. [5, § 3]) and a general position argument will work again.

Suppose $k = 3$ and M is closed. $V - M$ is 0-connected, by general position, but it is not *a priori* true that $V - M$ is 1-connected. We must arrange that it be so by the following inductive procedure.

It is remarked in (1.5) that we may, initially, assume that, for some $x_0 \in S^n$,

M meets $x_0 \times S^2$ normally in a single point. We would like to preserve this property throughout our modifications. But, from the description of a spherical modification given in [3, § 3], we need only take care that s and d avoid $x_0 \times S^2$, and this follows from general position $((r+1)+2 < n+2)$.

We now check that $\pi_1(V - M) = 0$ if M satisfies the condition in the above paragraph. Let $\xi \in \pi_1(V - M)$ be the element represented by a fiber of the normal circle bundle to M. It follows from the van Kampen theorem, applied to the triad $(V; U, V - M)$, where U is a tubular neighborhood of M, that the extra relation $\xi = 0$ will kill $\pi_1(V - M)$. But ξ is represented by a circle in the contractible subspace $(V - M) \cap (x_0 \times S^2)$; thus ξ already is zero.

It now follows that we may assume $V - M$ is 1-connected.

To complete the verification of hypothesis (2), we will demonstrate that $H_i(V - M) = 0$ for $2 \leqq i \leqq r$; an application of the Hurewicz theorem then shows that $V - M$ is r-connected. By Lefschetz duality, $H_i(V - M) \approx H^{m-1-i}(V, M)$. Since M is $(r-1)$-connected and ∂M is empty or a sphere, $H^j(M) = 0$ for $j > n - r$, $j \neq n$; also $H^j(V) = 0$ for $m - 1 > j > k - 1$, $j \neq n$. When $j = n > k - 1$, the restriction homomorphism $H^n(V) \to H^n(M)$ is an isomorphism—if $V = S^m$ and M is bounded both groups are zero, while if $V = S^n \times S^{k-1}$, then M is nuclear. Therefore, $H^j(V) \to H^j(M)$ is an isomorphism when $m - 1 > j > \max\{k - 1, n - r\}$. Now considering the following exact sequence

$$\begin{aligned} H^{m-i-2}(V) \longrightarrow H^{m-i-2}(M) \longrightarrow H^{m-i-1}(V, M) \\ \longrightarrow H^{m-i-1}(V) \longrightarrow H^{m-i-1}(M)\ , \end{aligned}$$

we conclude that $H_i(V - M) \approx H^{m-i-1}(V, M) = 0$ for $0 < i < \min\{n-1, k+r-2\}$. Thus, since $k \geqq 3$ and $n \geqq 2r$, M is r-connected.

4.10. Now it remains to show that, by a sequence of spherical modifications on $(M, \mathcal{F})$ associated to elements $\alpha \in \pi_r(M)$ satisfying hypothesis (3) of (4.8), we may kill $\pi_r(M)$. Clearly the obstruction to extending $(f_2, \cdots, f_{k-1}) \mid s$ over d is represented by an element $\xi \in \pi_r(V_{m-r-2,k-2})$.

If $n > 2r$, this homotopy group is zero, so (3) is trivially satisfied for all $\alpha \in \pi_r(M)$. Thus we may perform a spherical modification on $(M, \mathcal{F})$ associated with any $\alpha \in \pi_r(M)$, and the procedures of [14] can be carried out. In case $n = 4t + 3$, $r = 2t + 1$, the arguments of [14, p. 522–6] require that we be more specific in choosing the imbedding $i: S^r \times D^{m-r} \to M$ representing α in performing the spherical modification. Thus a further argument is needed to be sure that we may correspondingly modify $(M, \mathcal{F})$. This point is dealt with in the proof of Lemma 3.1 of [6] to which we refer the reader. Thus we may assume M is r-connected for $n > 2r$. If n is odd, Theorem (4.7) follows by duality.

Suppose $n = 2r$, r is even and index $M = 0$. It follows from [20, Th. 4] or

[14] that $\pi_r(M)$ may be killed by a sequence of spherical modifications on M associated with elements of $\pi_r(M)$ whose self-intersection number is zero. That hypothesis (3) is satisfied for all such elements α follows from an argument in [3, 3.5]. In fact, if

$$\partial\colon \pi_r(V_{m-r-2,k-2}) \longrightarrow \pi_{r-1}(SO_{n-r})$$

is the boundary homomorphism of the fibration $SO_{m-r-2} \to V_{m-r-2,k-2}$, then $\partial(\xi)$ corresponds to the normal bundle of s, according to [3, 3.4]. Then, by [20, Lem. 7], $\partial(\xi) = 0$. Since ∂ is a monomorphism for $k \geqq 3$, this implies $\xi = 0$. This completes the proof of (4.7) in this case.

Suppose $n = 2r$, r is odd, $k \geqq 4$ and $c(M, \mathcal{F}) = 0$. It follows from the arguments of [14] that, if a sequence of spherical modifications can be performed on $(M, \mathcal{F})$ associated to all the α_i of a symplectic basis $\{\alpha_i; \beta_i\}$ of $H_r(M)$, we can kill $\pi_r(M)$. But, since $c(M, \mathcal{F}) = 0$, it follows from the proof of [14, Lem. 8.4] that there is a symplectic basis $\{\alpha_i; \beta_i\}$ of $H_r(M)$ such that $\varphi(M, \mathcal{F}) \cdot \alpha_i = 0$ for all i. But if $\xi_i \in \pi_r(V_{m-r-2,k-2})$ is the obstruction to satisfying (3) for α_i, it is easy to see [3, 3.3] that, under the natural homomorphism $\pi_r(V_{m-r-2,k-2}) \to \pi_r(V_{m-1,m-r-1})$, $\xi_i \to \varphi(M, \mathcal{F}) \cdot \alpha_i = 0$. Since this homomorphism is a monomorphism when $k \geqq 4$, then $\xi_i = 0$, and the necessary modifications can be carried out. This completes the proof of (4.7), except in the case $n = 2r$, r odd and $k = 3$.

4.11. When $k = 3$, the above argument shows that ξ_i is an even element of the infinite cyclic group $\pi_r(S^r)$. We will show that d may be replaced by a new disk d_2 for which (3) is satisfied. Denote α_i by α and ξ_i by ξ.

The orientations of d and V induce an identification of $\pi_r(S^r)$ with Z which allows us to identify ξ with an even integer $2a$. Let s_0 be an $(r + 1)$-sphere immersed in an m-ball in $V - (M \cup d)$ with self-intersection number $-a$. According to [15, Cor. 3.2] the obstruction to a normal field on s_0 is $-2a$ (again identifying $\pi_r(S^r)$ with Z). Since $V - (M \cup d \cup s_0)$ is 1-connected ($V - M$ is 1-connected by (4.9)) we may extend d and s_0 to an immersion of the connected sum $d_0 = d \# s_0$ such that the self-intersection number of d_0 is $-a$. The obstruction to satisfying (3) for d_0 is $2a + (-2a) = 0$, but, unfortunately, d_0 is not an imbedded disk.

Since α is a primitive element of $H_r(M)$, there exists $\beta \in H_{r+1}(V - M)$ such that the linking number of α with β is a. By [20, Lem. 6], let β be represented by an imbedded sphere s_1 in $V - M$, since $V - M$ is r-connected. We may assume s_1 meets d_0 transversely; thus the intersection number of d_0 with s_1 is a. Now we may extend d_0 and s_1 to an immersion of $d_1 = d_0 \# s_1$ whose only self-intersections are those of d_0 and s_1, and the intersections of d_0 and s_1. Thus the self-intersection number of d_1 is $-a + 0 + a = 0$. Since all self-intersections in $H_{r+1}(V)$

are zero, it follows from [20, Lem. 7] that the normal bundle of s_1 is trivial. Therefore the obstruction to satisfying (3) for d_1 is $0 + 0 = 0$.

We now construct d_2 from d_1 by performing a regular homotopy on d_1 in $V - M$ which removes all the self-intersections; this uses the techniques of [29] and the 1-connectedness of $V - M$. Then d_2 is an imbedded disk satisfying (3).

The proof of Theorem (4.7) is now complete.

5. Verifications of exactness

5.1. Now we are ready to define the homomorphisms in the sequences (1)–(3) of (2.2).

Let (K^n, ε) be a framed knotted sphere in S^m such that $[K, \varepsilon] = 0$ in $\Theta_f^{m,n}$. Let $\mu: K \to SO_k$ represent $\alpha \in \pi_n(SO_k)$ and define $\mathcal{F}$, a new normal frame on K, by

$$\mathcal{F}(x) = \mu(x)\varepsilon(x) .$$

We then define:

$$\omega_1 = \omega_1(n, k): \pi_n(SO_k) \longrightarrow \Theta_f^{m,n}$$

by $\omega_1(\alpha) = [K^n, \mathcal{F}]$. Clearly ω_1 is well-defined. Suppose $\alpha_t \in \pi_n(SO_k)$, $t = 1, 2$, and $\mu_t: K \to SO_k$ represents α_t with the property that $\mu_t(D_t^n) =$ identity element, where D_t is a disk imbedded in K such that $D_1 \cap D_2 = \partial D_1 = \partial D_2$ (orientation not being considered) and $K = D_1 \cup D_2$. Then $\alpha_1 + \alpha_2$ is represented by $\mu: K \to SO_k$ defined by $\mu \mid K - D_t = \mu_s$, $s \neq t$. Then $\mathcal{F}_t \mid D_t = \varepsilon \mid D_t$, where $(K, \mathcal{F}_t)$ represents $\omega_1(\alpha_t)$. It is easy to see that $(K, \mathcal{F})$ is isotopic to $(K, \mathcal{F}_1) \# (K, \mathcal{F}_2)$, where $\mathcal{F}$ is defined from μ. Thus ω_1 is a homomorphism.

Let $(K^n, \mathcal{F})$ be any framed knotted sphere in S^m. We define

$$\varphi_1 = \varphi_1(n, k): \Theta_f^{m,n} \longrightarrow \Theta^{m,n}$$

and

$$\omega_2 = \omega_2(n, k): \Theta_f^{m,n} \longrightarrow \pi_n(G_k) .$$

by $\varphi_1[K, \mathcal{F}] = [S^m, K^n]$ and $\omega_2[K, \mathcal{F}] = v(K, \mathcal{F})$. It is clear that φ_1 and ω_2 are well-defined, and that φ_1 is a homomorphism. That ω_2 is a homomorphism follows from Lemma 3.4 (a).

Let K^n be a knotted sphere in S^m. We define

$$\omega_3 = \omega_3(n, k): \Theta^{m,n} \longrightarrow \pi_n(G_k, SO_k)$$

and

$$\partial_1 = \partial_1(n, k): \Theta^{m,n} \longrightarrow \pi_{n-1}(SO_k) ,$$

by $\omega_3[S^m, K^n] = v(S^m, K^n)$ and $\partial_1[S^m, K^n] = O(K^n, \mathcal{F})$, where $\mathcal{F}$ is any normal frame on the complement of a point. Then ω_3 and ∂_1 are well-defined, since all such $\mathcal{F}$ are homotopic. By Lemma 3.4 (b), ω_3 is a homomorphism. But this implies ∂_1 is a homomorphism, by (3.3, Lemma 1).

We now define:

$$\varphi_2 = \varphi_2(n, k): \pi_n(G_k) \longrightarrow P_n$$

and

$$\varphi_3 = \varphi_3(n, k): \pi_n(G_k, SO_k) \longrightarrow P_n .$$

Let $\alpha \in \pi_n(G_k)$ and $(M^n, \mathcal{F})$ a framed nuclear submanifold of $S^n \times S^{k-1}$ such that $t(M, \mathcal{F}) = \alpha$. If $\alpha_0 \in \pi_n(G_k, SO_k)$, we let $(M_0^n, \mathcal{F}_0)$ be a framed submanifold of $D^n \times S^{k-1}$ derived by the Thom construction, as in (1.5), from α_0. Since the cobordism class of $(M_0, \mathcal{F}_0)$ depends only on α_0, according to (1.5), so does $\gamma(M_0, \mathcal{F}_0)$. Thus we may define $\varphi_2(\alpha) = \gamma(M, \mathcal{F})$ and $\varphi_3(\alpha_0) = \gamma(M_0, \mathcal{F}_0)$; it is clear that these are well-defined functions. That φ_3 is a homomorphism follows from (1.5) and Lemma (4.5). It is a consequence of (1.3) that $\varphi_2(n, k) = \varphi_3(n, k) \cdot \varphi_4(n, k)$; thus φ_2 is also a homomorphism.

Let $\gamma \in P_{n+1}$; by Lemma (4.6), $\gamma = \gamma(M^{n+1}, \mathcal{F})$ for some framed submanifold $(M, \mathcal{F})$ of S^m, where ∂M is a sphere. Let $\mathcal{F}'$ be the front extension of $\mathcal{F} \mid \partial M$; then we define

$$\partial_2 = \partial_2(n, k): P_{n+1} \longrightarrow \Theta_f^{m,n}$$
$$\partial_3 = \partial_3(n, k): P_{n+1} \longrightarrow \Theta^{m,n}$$

by $\partial_2(\gamma) = [\partial M, \mathcal{F}']$ and $\partial_3(\gamma) = [S^m, \partial M]$.

We must check that ∂_2 and ∂_3 are well-defined. It follows from Theorem (4.7 (i)) that $\partial_2(0)$ and $\partial_3(0)$ are well-defined (and equal to 0), since $(\partial M, \mathcal{F}')$ will be h-cobordant to the boundary of a framed $(n + 1)$-disk. But now it follows from the additivity of ∂_2 and ∂_3 with respect to connected sum (Lemma (4.5)) that ∂_2 and ∂_3 are well-defined on all of P_{n+1}, and homomorphisms.

5.2. PROPOSITION. *Diagram* $(5)_k$ *of* (2.2) *is commutative up to sign.*

PROOF. That $\omega_2(n, k) \cdot \omega_1(n, k) = \omega_4(n, k)$ is a direct consequence of Lemma (3.2). It has already been remarked that $\varphi_2(n, k) = \varphi_3(n, k) \cdot \varphi_4(n, k)$, and it is obvious from the definitions that $\partial_3(n, k) = \varphi_1(n, k) \cdot \partial_2(n, k)$. Furthermore, it follows from Lemma 1 of (3.3) that $\partial_1(n, k) = \partial_4(n, k) \cdot \omega_3(n, k)$, and from Lemma 2 of (3.3) that $\varphi_4(n, k) \cdot \omega_2(n, k) = \omega_3(n, k) \cdot \varphi_1(n, k)$.

To complete the proof, we will show that

$$\omega_1(n, k) \cdot \partial_4(n + 1, k) = (-1)^n \partial_2(n, k) \cdot \varphi_3(n + 1, k) .$$

Let (λ, μ) be a pair of maps representing $\alpha \in \pi_{n+1}(G_k, SO_k)$, as in (1.3), such that λ is differentiable and has $y_0 \in S^{k-1}$ as a regular value. Let $(M^{n+1}, \mathcal{F})$ be the framed submanifold of $D^{n+1} \times S^{k-1}$ derived, by the Thom construction, from λ; see (1.5). Then $((-1)^{n+1}M, \mathcal{F})$ becomes a framed submanifold of S^m; note that

$\gamma((-1)^{n+1}M, \mathcal{F}) = (-1)^{n+1}\varphi_3(\alpha)$. Therefore $((-1)^{n+1}\partial M, \mathcal{F}')$ is a framed knotted sphere in S^m, where $\mathcal{F}'$ is the front extension of $\mathcal{F} \mid (-1)^{n+1}\partial M$, and represents $(-1)^{n+1}\partial_2 \cdot \varphi_3(\alpha)$.

Let $K^n = (-1)^{n+1}\partial M$; then K^n is the image of the cross-section $S^n \to S^n \times S^{k-1}$ defined by $x \to \mu^{-1}(x) \cdot y_0$. If ε is a positive tangent frame to S^{k-1} at y_0, then $\mathcal{F}(x, y)$ is the frame which projects to zero by $S^n \times S^{k-1} \to S^n$ and projects to $d\mu^{-1}(x) \cdot \varepsilon$ by $S^n \times S^{k-1} \to S^{k-1}$. We will write simply:

$$\mathcal{F}(x) = d\mu^{-1}(x) \cdot \varepsilon \; .$$

Since $\mathcal{F}'$ is the front extension of $\mathcal{F}$ to a normal frame to K in $S^n \times D^k$ (and, therefore, in S^m), then $\mathcal{F}'(x) = d\mu^{-1}(x) \cdot \varepsilon'$, where ε' is the front extension of ε to a positive frame to D^k. Now by pushing K radially toward the 0-section $S^n \to S^n \times D^k$, we see that $(K, \mathcal{F}')$ is isotopic to the framed knotted sphere $(K_0^n, \mathcal{F}_0)$, where K_0 is the image of the 0-section, and $\mathcal{F}_0$ is defined by $\mathcal{F}_0(x) = d\mu^{-1}(x) \cdot \varepsilon_0$, for some positive tangent frame ε_0 at $0 \in D^k$. But since $d\mu^{-1}(x) = \mu^{-1}(x)$ at the origin, we may write

$$\mathcal{F}_0(x) = \mu_1^{-1}(x) \cdot \varepsilon_1 \qquad \text{for } x \in K_0^n \; ,$$

where ε_1 is the pull-back of ε_0 by the projection $S^n \times D^k \to D^k$, and $\mu_1 \colon K_0 \to SO_k$ is defined by $\mu_1(x, 0) = \mu(x)$ for $x \in S^n$. Now (K_0, ε_1) is a framed submanifold of $S^n \times D^k$, and ε_1 pulls back from a positive frame on D^k; it follows that $K_0 = +S^n \times 0$. Therefore μ_1^{-1} represents $-\partial_4(\alpha)$.

Since it is obvious that $[K_0, \varepsilon_1] = 0$, it follows from the definition that $\omega_1(-\partial_4(\alpha)) = [K_0, \mathcal{F}_0] = [K, \mathcal{F}'] = (-1)^{n+1}\partial_2 \cdot \varphi_3(\alpha)$. This completes the proof of the proposition.

5.3. It remains to prove the exactness of (1)–(3) of (2.2).

LEMMA. *The exactness of the sequences* $(1)_k$–$(4)_k$ *of* (2.2) *follows formally from the exactness of* $(1)_k$, $(2)_k$, *and* $(4)_k$, *the relation* $\varphi_3(n, k) \cdot \omega_3(n, k) = 0$, *for all* $n \geqq 5$, *and the commutativity* (*up to sign*) *of* $(5)_k$.

PROOF. This is just an exercise in *diagram chasing*. We must verify the following additional facts:

(i) $\operatorname{Ker} \varphi_3(n, k) \subset \operatorname{Im} \omega_3(n,k)$,

(ii) $\operatorname{Ker} \omega_3(n, k) = \operatorname{Im} \partial_3(n, k)$,

(iii) $\operatorname{Ker} \partial_3(n, k) = \operatorname{Im} \varphi_3(n + 1, k)$.

That $\omega_3 \cdot \partial_3$ and $\partial_3 \cdot \varphi_3$ are zero follows from $(5)_k$ and the assumptions that $\omega_2 \cdot \partial_2$ and $\varphi_1 \cdot \omega_1$ are zero.

Suppose $\varphi_3(\alpha) = 0$. Then $\omega_1 \cdot \partial_4(\alpha) = 0$ and, therefore, $\partial_4(\alpha) = \partial_1(\beta)$, for some β. Now $\partial_4(\alpha - \omega_3(\beta)) = 0$ and, thus, $\alpha = \omega_3(\beta) + \varphi_4(\gamma)$, for some γ. But $\varphi_2(\gamma) = \varphi_3(\alpha - \omega_3(\beta)) = 0$ and, consequently, $\gamma = \omega_2(\eta)$, for some η. It now follows that

$\omega_3(\beta + \varphi_1(\eta)) = \omega_3(\beta) + \varphi_4 \cdot \omega_2(\eta) = \alpha$. This proves (i).

Suppose $\omega_3(\alpha)=0$. Then $\partial_1(\alpha)=0$ and, therefore, $\alpha=\varphi_1(\beta)$. Since $\varphi_4 \cdot \omega_1(\beta)= \omega_3(\alpha) = 0$, $\omega_2(\beta) = \omega_4(\gamma)$. Now $\omega_2(\beta - \omega_1(\gamma)) = 0$ and so $\beta - \omega_1(\gamma) = \partial_2(\eta)$. It now follows that $\partial_3(\eta) = \varphi_1(\beta - \omega_1(\gamma)) = \varphi_1(\beta) = \alpha$. This completes the proof of (ii).

The completion of the proof of (iii) follows similar lines, and we omit the details.

5.4. THEOREM. *Sequences* $(1)_k$–$(3)_k$ *of* (2.2) *are exact,* $k \geqq 3$.

PROOF. We will prove below in (6.1) that $\varphi_3(n, k) \cdot \omega_3(n, k)=0$, for all $n \geqq 5$. By Lemma (5.3), the theorem will then follow if we prove the exactness of $(1)_k$ and $(2)_k$, since $(4)_k$ is already known to be exact.

Exactness of $(1)_k$. The exactness at $\Theta^{m,n}$ follows directly from the definitions, since it is clear that Ker $\partial_1(n, k)$ and Image $\varphi_1(n, k)$ both consist precisely of those isotopy classes $[S^m, K^n]$ where K^n has a trivial normal bundle in S^m. Exactness at $\Theta^{m,n}_f$ is also immediate; Image $\omega_1(n, k)$ and Ker $\varphi_1(n, k)$ consist precisely of those isotopy classes $[K^n, \mathcal{F}]$ where $[S^m, K^n] = 0$.

We now verify exactness at $\pi_{n-1}(SO_k)$. Suppose $\alpha \in$ Image $\partial_1(n, k)$. Let K^n be a knotted sphere in S^m such that $\partial_1[S^m, K^n] = \alpha$. We may choose K^n so that $K^n \cap D^m_+$ is an n-disk D^n_1; let $D^n_2 = K^n \cap D^m_-$. Let $\mathcal{F}_t$ be a normal frame on D_t in S^m. Then, according to [22, 1.4],

$$[\partial D_1, \mathcal{F}_1 \,|\, \partial D_1] = [\partial D_1, \mathcal{F}_2 \,|\, \partial D_1] = 0 \qquad \text{in } \Theta^{m-1,n-1}_f .$$

By the definition of ∂_1, we may write $\mathcal{F}_2(x) = \mu(x)\mathcal{F}_1(x)$ for all $x \in \partial D_1$, where $\mu\colon \partial D_1 \to SO_k$ represents α. But, since $[\partial D_1, \mathcal{F}_1 \,|\, \partial D_1]=0$, this means $[\partial D_1, \mathcal{F}_2 \,|\, \partial D_1]= \omega_1(\alpha)$. Thus $\omega_1(\alpha) = 0$.

Suppose $\alpha \in$ Ker $\omega_1(n-1, k)$. Let (K^{n-1}, ε) be a framed knotted sphere in S^{m-1} such that $[K, \varepsilon] = 0$, and $\mathcal{F}$ the normal frame on K defined by $\mathcal{F}(x) = \mu(x)\varepsilon(x)$, $x \in K$, where $\mu\colon K \to S0_k$ represents α. Then $[K, \mathcal{F}] = \omega_1(\alpha) = 0$. According to [22, 1.4], there exist framed disks (D^n_1, ε_1) in D^m_+ and $(D^n_2, \mathcal{F}_1)$ in D^m_- such that:

$$\partial D_1 = -\partial D_2 = K ,$$

$$\varepsilon_1 \,|\, \partial D_1 = \varepsilon ,$$

and

$$\mathcal{F}_1 \,|\, \partial D_1 = \mathcal{F} .$$

If we now define $K^n_0 = D_1 \cup D_2$, then K^n_0 is a knotted sphere in S^m and, by definition, $\partial_1[S^m, K^n_0] = \alpha$. This proves exactness at $\pi_{n-1}(SO_k)$.

5.5. *Exactness of* $(2)_k$. Exactness at $\Theta^{m,n}_f$ is precisely the content of Lemma (3.8), according to the definitions of ∂_3 and ω_3. Furthermore, it follows from Lemma (3.5) and (3.7) that exactness at $\pi_n(G_k)$ is just the statement that a framed nuclear submanifold $(M^n, \mathcal{F})$ of $S^n \times S^{k-1}$ is cobordant to a framed knotted sphere in $S^n \times S^{k-1}$ if and only if $\gamma(M, \mathcal{F}) = 0$. But this follows from Theorem (4.7).

It remains to prove exactness at P_n. Let $\alpha \in \pi_n(G_k)$ and $(M^n, \mathcal{F})$ be a framed nuclear submanifold of $S^n \times S^{k-1}$ such that $\alpha = t(M, \mathcal{F})$. We may assume, as in (1.5), that $M \cap (D^n_- \times S^{k-1}) = D^n_- \times y_0$, for some $y_0 \in S^{k-1}$. Let $M_0 = M \cap (D^n_+ \times S^{k-1})$, $\mathcal{F}_0 = \mathcal{F} \mid M_0$; note that $(\partial M_0, \mathcal{F}_0 \mid \partial M_0)$ may be extended to a framed disk in $D^n_+ \times S^{k-1}$, since it is so in $D^n_- \times S^{k-1}$. By identifying $D^n_+ \times S^{k-1}$ with $D^n \times S^{k-1}$, $(M_0, \mathcal{F}_0)$ determines a framed submanifold $(M_1, \mathcal{F}_1)$ of $D^n \times S^{k-1}$. As usual, we may regard $((-1)^n M_1, \mathcal{F}_1)$ as a framed submanifold of S^{m-1}. We now define a framed knotted sphere $(K^{n-1}, \mathcal{F}')$ in S^{m-1} by $K = (-1)^n \partial M_1$ and $\mathcal{F}' =$ front extension of $\mathcal{F}_1 \mid K$. Then

$$[K, \mathcal{F}'] = \partial_2 \cdot \gamma\big((-1)^n M_1, \mathcal{F}_1\big) = (-1)^n \partial_2 \cdot \gamma(M, \mathcal{F}) = (-1)^n \partial_2 \cdot \varphi_2(\alpha) ;$$

since $[K, \mathcal{F}'] = 0$, it follows that $\partial_2 \cdot \varphi_2 = 0$.

Suppose $\gamma \in P_n$ and $\partial_2(\gamma) = 0$. Then there exists a framed submanifold $(M^n, \mathcal{F})$ of S^{m-1} such that $\gamma(M, \mathcal{F}) = \gamma$ and $\partial M = K^{n-1}$ is a sphere where, if $\mathcal{F}'$ is the front extension of $\mathcal{F} \mid K$ to a normal frame to K in S^{m-1}, $[K, \mathcal{F}'] = 0$. Thus there is a framed disk $(D^n_0, \mathcal{F}_0)$ in S^{m-1} with $\partial D_0 = K$ and $\mathcal{F}_0 \mid K = \mathcal{F} \mid K$; in particular, D_0 and M are tangent along K. As in (3.8), there is a tubular neighborhood T of K such that $T \cap M$ and $T \cap D_0$ are collars of K in M and D_0; moreover, it follows from the tubular neighborhood theorem that we may choose D_0 so that $T \cap M = T \cap D_0$ and $\mathcal{F} \mid T \cap M = \mathcal{F}_0 \mid T \cap D_0$. As in (3.5), we can now assume $T = S^{n-1} \times D^k$ and K is homotopic to $S^{n-1} \times 0$. Let $(K_1, \mathcal{F}_1)$ be the framed knotted sphere in $S^{n-1} \times S^{k-1}$ defined by $K_1 = \partial(\overline{M - T \cap M})$, and $\mathcal{F}_1 = \mathcal{F} \mid K_1$; then $K_1 = \partial(\overline{D_0 - T \cap D_0})$ and $\mathcal{F}_1 = \mathcal{F}_0 \mid K_1$. Identifying $D^n \times S^{k-1}$, the complement of int T, with $D^n_+ \times S^{k-1}$ determines a framed submanifold $(M_2, \mathcal{F}_2)$ of $D^n_+ \times S^{k-1}$ from $(-1)^n \overline{M - (T \cap M)}$ and $\mathcal{F}$. Identifying $D^n \times S^{k-1}$ with $D^n_- \times S^{k-1}$ determines a framed disk $(D_3, \mathcal{F}_3)$ of $D^n_- \times S^{k-1}$ from $(-1)^n \overline{D_0 - (T \cap D_0)}$ and $\mathcal{F}_0$. Then $\partial M_2 = -\partial D_3 = K_1$ and $\mathcal{F}_2 \mid K_1 = \mathcal{F}_3 \mid K_1$. Now $(M_4, \mathcal{F}_4)$ is a framed submanifold of $S^n \times S^{k-1}$ defined by $M_4 = M_2 \cup D_3$ and $\mathcal{F}_4 = \mathcal{F}_2 \cup \mathcal{F}_3$. Notice that

$$\gamma(M_4, \mathcal{F}_4) = (-1)^n \gamma(M, \mathcal{F}) = (-1)^n \gamma .$$

Therefore $(-1)^n \gamma$ (and consequently γ) is in Image $\varphi_2(n, k)$. This completes the proof of exactness at P_n, and, thereby, of Theorem (5.4).

6. The structure of $\Theta^{m,n}$

6.1. Recall the natural inclusions $G_k \subset G_{k+1}$, $SO_k \subset SO_{k+1}$ and the induced homomorphisms

$$\pi_n(G_k) \longrightarrow \pi_n(G_{k+1}), \qquad \pi_n(SO_k) \longrightarrow \pi_n(SO_{k+1})$$

and

$$\pi_n(G_k, SO_k) \longrightarrow \pi_n(G_{k+1}, SO_{k+1}) ;$$

we refer to these as *suspensions*. There are also suspension homomorphisms $\Theta_f^{m,n} \to \Theta_f^{m+1,n}$ and $\Theta^{m,n} \to \Theta^{m+1,n}$ induced by the natural imbedding $S^m \subset S^{m+1}$ and the *rear* extension of a normal frame.

These suspensions, together with the identity map $P_n \to P_n$, define homomorphisms of sequences $(1)_k \to (1)_{k+1}$, $(2)_k \to (2)_{k+1}$, $(3)_k \to (3)_{k+1}$, and $(4)_k \to (4)_{k+1}$. The necessary commutativity relations follow readily; we leave them to the reader.

For any fixed value of n, the suspension homomorphisms are isomorphisms when k is large enough; this follows from classical results. If we define

$$G = \lim_{k\to\infty} G_k = \lim_{k\to\infty} F_k , \qquad SO = \lim_{k\to\infty} SO_k , \qquad \Theta_f^n = \lim_{k\to\infty} \Theta_f^{m,n}$$

and note that Θ^n may be identified with $\lim_{k\to\infty} \Theta^{m,n}$, then the sequences $(1)_k$–$(4)_k$, in passing to the limit $k \to \infty$, define four new exact sequences:

$$(1)_\infty \quad \cdots \longrightarrow \pi_n(SO) \xrightarrow{\omega_1} \Theta_f^n \xrightarrow{\varphi_1} \Theta^n \xrightarrow{\partial_1} \pi_{n-1}(SO) \longrightarrow \cdots$$

$$(2)_\infty \quad \cdots \longrightarrow \Theta_f^n \xrightarrow{\omega_2} \pi_n(G) \xrightarrow{\varphi_2} P_n \xrightarrow{\partial_2} \Theta_f^{n-1} \longrightarrow \cdots$$

$$(3)_\infty \quad \cdots \longrightarrow \Theta^n \xrightarrow{\omega_3} \pi_n(G, SO) \xrightarrow{\varphi_3} P_n \xrightarrow{\partial_3} \Theta^{n-1} \longrightarrow \cdots$$

$$(4)_\infty \quad \cdots \longrightarrow \pi_n(SO) \xrightarrow{\omega_4} \pi_n(G) \xrightarrow{\varphi_4} \pi_n(G, SO) \xrightarrow{\partial_4} \pi_{n-1}(SO) \longrightarrow \cdots$$

and a commutative diagram $(5)_\infty$. These are originally due to Kervaire and Milnor (unpublished).

We denote $\omega_1(n) = \lim_{k\to\infty} \omega_1(n, k)$, and similarly for the other eleven homomorphisms in $(1)_\infty$–$(4)_\infty$. We also define

$$\sigma_1(n, k)\colon \pi_n(SO_k) \longrightarrow \pi_n(SO) ,$$
$$\sigma_2(n, k)\colon \pi_n(G_k) \longrightarrow \pi_n(G) ,$$
$$\sigma_3(n, k)\colon \pi_n(G_k, SO_k) \longrightarrow \pi_n(G, SO) ,$$

as the suspension homomorphisms.

At this point we prove the equation $\varphi_3(n, k) \cdot \omega_3(n, k) = 0$. Since $\varphi_3(n, k) \cdot \omega_3(n, k) = \varphi_3(n) \cdot \omega_3(n) \cdot \theta(n, k)$, it suffices to prove $\varphi_3(n) \cdot \omega_3(n) = 0$. Now it follows from $(5)_\infty$ that $\varphi_3(n) \cdot \omega_3(n) \cdot \varphi_1(n) = \varphi_2(n) \cdot \omega_2(n) = 0$. But according to [14, Th. 3.1], $\partial_1(n) = 0$ and, therefore, $\varphi_1(n)$ is onto; this then implies $\varphi_3(n) \cdot \omega_3(n) = 0$.

A direct proof that $\varphi_3 \cdot \omega_3 = 0$ is also not difficult.

6.2. A useful tool in our considerations will be the following diagram, for any $k \geqq 2$:

$$
(6)_k \qquad
\begin{array}{ccccccccc}
 & & \pi_n(SO_{k-1}) & \xrightarrow{j} & \pi_n(SO_k) & & & & \pi_{n-1}(SO_{k-1}) \\
 & \nearrow^{\Delta'} & \downarrow{\scriptstyle \omega'} & & \downarrow{\scriptstyle \omega_4} & \searrow^{p} & & \nearrow^{\Delta'} & \downarrow{\scriptstyle \omega'} \\
\pi_{n+1}(S^{k-1}) & \xrightarrow{P'} & \pi_n(F_{k-1}) & \xrightarrow{j'} & \pi_n(G_k) & \xrightarrow{p'} & \pi_n(S^{k-1}) & \xrightarrow{P'} & \pi_{n-1}(F_{k-1}) \\
\downarrow{\scriptstyle P} & & \downarrow{\scriptstyle \nu'} & & \downarrow{\scriptstyle \nu} & & \downarrow{\scriptstyle P} & & \downarrow{\scriptstyle \nu'} \\
\pi_{m+1}(S^k; E_+, E_-) & \xrightarrow{\Delta} & \pi_{m-1}(S^{k-1}) & \xrightarrow{E} & \pi_m(S^k) & \xrightarrow{i} & \pi_m(S^k; E_+, E_-) & \xrightarrow{\Delta} & \pi_{m-2}(S^{k-1}) \\
\downarrow{\scriptstyle h} & & & & & & \downarrow{\scriptstyle h} & & \\
\pi_{m+1}(S^{2k-1}) & & & & & & \pi_m(S^{2k-1}) & &
\end{array}
$$

The top border is the exact homotopy sequence of the fibration $SO_{k-1} \to SO_k \to S^{k-1}$, and the first horizontal row is the exact homotopy sequence of the fibration $F_{k-1} \to G_k \to S^{k-1}$. The bottom horizontal row is the exact suspension sequence (see [9, § 1]) and h is defined as in [8, § 15]. The homomorphism ω' is induced by inclusion, and ν' is a Hurewicz isomorphism as in [26, 2.10]; ω_4 and ν have been defined elsewhere in this paper. We define:

$$P = P(n, k)\colon \pi_n(S^{k-1}) \longrightarrow \pi_m(S^k;\, E_+,\, E_-)$$

by $P(\alpha) = \{\alpha, \iota_{k-1}\}$, the triad Whitehead product [8, § 4], where ι_{k-1} is the canonical generator of $\pi_{k-1}(S^{k-1})$.

PROPOSITION. *Diagram* $(6)_k$ *is commutative, up to sign, and* $h \cdot P(n, k)$ *is the k-fold iterated suspension, as in* [27, p. 206].

PROOF. Since the homomorphisms ω_4 and ω' are induced by a bundle map $SO_k \to G_k$, covering the identity map of S^{k-1}, the commutativity of the squares and triangles involving these homomorphisms are immediate.

It follows from [8, 2.17 and 8.5] that $P \cdot p' = (-1)^k i \cdot \nu$. According to [26, 3.2] and [28, § 3], $\nu' \cdot P'(\alpha) = -[\alpha, \iota_{k-1}]$, for all α; thus, by [8, 2.3], we have $\Delta \cdot P = -\nu' \cdot P'$.

Consider the following diagram:

$$
\begin{array}{ccccc}
\pi_n(F_{k-1}) & \xrightarrow{j'} & \pi_n(G_k) & & \\
 & & & \searrow^{\nu''} & \\
\downarrow{\scriptstyle \nu'} & & \downarrow{\scriptstyle \nu} & & \pi_n(F_k) \\
 & & & \swarrow_{\nu_0'} & \\
\pi_{m-1}(S^{k-1}) & \xrightarrow{E} & \pi_m(S^k) & &
\end{array}
$$

where ν'' is induced by the inclusion and ν_0' is a Hurewicz isomorphism. According to [26, 3.10] and [28], $E \cdot \nu' = -\nu_0' \cdot \nu'' \cdot j'$; and, by [26], $\nu_0' \cdot \nu'' = \nu$. Therefore $E \cdot \nu' = -\nu \cdot j'$.

Finally, the fact that $h \cdot P(n, k)$ is the k-fold iterated suspension is proved in [10, 4.11].

This completes the proof of the proposition.

We have, incidentally, also proved that the composition $\nu' \cdot \omega' : \pi_n(SO_{k-1}) \to \pi_{m-1}(S^{k-1})$ coincides with $\nu(n, k-1) \cdot \omega_4(n, k-1)$, a fact which relates diagrams $(6)_{k-1}$ and $(6)_k$.

6.3. Let us denote by:

$$\xi(n, k) : \pi_n(G_k, SO_k) \longrightarrow \pi_n(G_{k+1}, SO_{k+1}) ,$$

the suspension homomorphism.

LEMMA. *The following sequence is exact:*

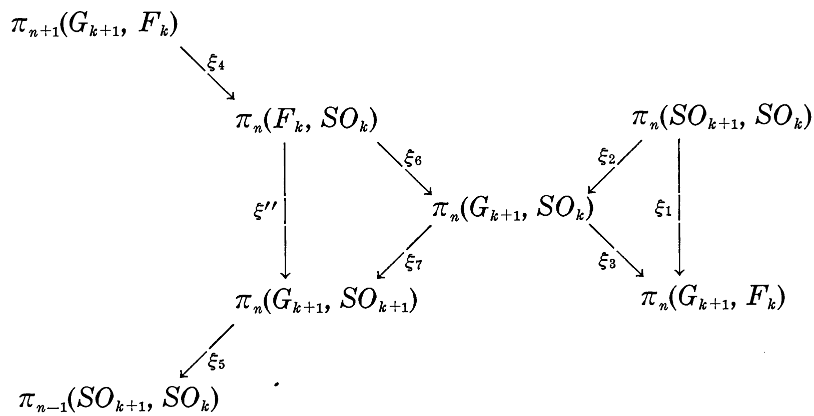
$$0 \longrightarrow \operatorname{Cok} \nu(n+1, k) \xrightarrow{\varphi'} \operatorname{Cok} \xi(n+1, k) \longrightarrow \operatorname{Ker} \nu(n, k)$$
$$\xrightarrow{\varphi''} \operatorname{Ker} \xi(n, k) \longrightarrow 0$$

where φ' is induced by $\varphi_4 \cdot j' \cdot (\nu')^{-1}$, and φ'' is induced by φ_4.

PROOF. Consider the factorization $\xi(n, k) = \xi'' \cdot \xi'(n, k)$:

$$\pi_n(G_k, SO_k) \xrightarrow{\xi'} \pi_n(F_k, SO_k) \xrightarrow{\xi''} \pi_n(G_{k+1}, SO_{k+1}) .$$

We will show that ξ'' is an isomorphism. Consider the following commutative diagram:



consisting of segments of the homotopy exact sequences of (G_{k+1}, F_k, SO_k) and $(G_{k+1}, SO_{k+1}, SO_k)$. Note that ξ_1 is an isomorphism since it is induced by a bundle map $SO_{k+1} \to G_{k+1}$, where SO_k and F_k are fibres, covering the identity map of the base S^k. As a consequence, ξ_2 is a monomorphism and ξ_3 an epimorphism. Since this holds for all n, exactness implies that ξ_6 is a monomorphism and ξ_7 is onto. Now we do a little *diagram chasing*.

Suppose $\xi''(\alpha) = 0$; then $\xi_6(\alpha) = \xi_2(\beta)$. But $\xi_1(\beta) = \xi_3\xi_2(\beta) = \xi_3\xi_6(\alpha) = 0$, and, therefore, $\beta = 0$. Since ξ_6 is a monomorphism, this implies $\alpha = 0$. Let $\alpha \in \pi_n(G_{k+1}, SO_{k+1})$; then, since ξ_7 is onto, $\alpha = \xi_7(\beta)$. Let $\xi_1(\gamma) = \xi_3(\beta)$; then

$\xi_3(\beta - \xi_2(\gamma)) = 0$. Therefore $\beta - \xi_2(\gamma) = \xi_6(\eta)$ and $\xi''(\eta) = \xi_7(\beta - \xi_2(\gamma)) = \xi_7(\beta) = \alpha$. This proves ξ'' is an isomorphism.

Therefore Ker $\xi(n, k) =$ Ker $\xi'(n, k)$, and ξ'' induces an isomorphism between Cok $\xi'(n, k)$ and Cok $\xi(n, k)$.

Now consider the following commutative diagram:

$$
\begin{array}{ccccccccc}
 & & \pi_n(G_k) & \xrightarrow{\varphi_4} & \pi_n(G_k, SO_k) & & & & \pi_{n-1}(G_k) \\
 & \overset{\omega_4}{\nearrow} & \downarrow \nu & & \downarrow \xi' & \overset{\partial_4}{\searrow} & & \overset{\omega_4}{\nearrow} & \downarrow \\
\pi_n(SO_k) & & & & & & \pi_{n-1}(SO_k) & & \\
 & \searrow & \downarrow & & \downarrow & \nearrow & & \searrow & \downarrow \\
 & & \pi_n(F_k) & \longrightarrow & \pi_n(F_k, SO_k) & & & & \pi_{n-1}(F_k)
\end{array}
$$

consisting of segments of the homotopy exact sequences of (G_k, SO_k) and (F_k, SO_k). It is a straight forward exercise to demonstrate that this diagram and its exactness properties formally implies the existence and exactness of the sequence:

$$
\begin{aligned}
0 \longrightarrow \operatorname{Cok} \nu(n+1, k) \longrightarrow \operatorname{Cok} \xi'(n+1, k) \longrightarrow \operatorname{Ker} \nu(n, k) \\
\longrightarrow \operatorname{Ker} \xi'(n, k) \longrightarrow 0 \, .
\end{aligned}
$$

The lemma now follows.

6.4. We are now ready to study $\Theta^{m,n}$.

THEOREM (Haefliger). *$\theta(n, k)$: $\Theta^{m,n} \to \Theta^n$ is an epimorphism for $2m \geqq 3(n+1)$ and a monomorphism for $2m > 3(n+1)$.*

PROOF. Recall that $\theta(n, k)$ and $\sigma_3(n, k)$, $n \geqq 5$, induce a homomorphism $(3)_k \to (3)_\infty$. By the five-lemma it suffices to prove that $\sigma_3(n, k)$:

$(*)$ is surjective for $2k \geqq n + 3$ and injective for $2k > n + 3$.

Clearly it is sufficient to prove that $(*)$ is satisfied by $\xi(n, k)$. By Lemma (6.3), it suffices to prove $(*)$ for $\nu(n, k)$. Finally, by diagram (6) and the five-lemma, it suffices to prove $(*)$ for $P(n, k)$.

The suspension theorem proves $(*)$ for $h \cdot P(n, k)$. Now, according to [9, 1.7], h is an isomorphism for $2k \geqq n + 2$, $n \neq 2k$; if $n = 2k$, h is onto and Ker $h \subset$ Im $P(n, k)$, by [9, 1.6 and 8.2]. It follows, then, that $(*)$ is also satisfied for $P(n, k)$.

6.5. We now show that, in general, the computation of $\Theta^{m,n}$, up to group extension, has been reduced to the computation of $\pi_n(G_k, SO_k)$, Θ^n and the homomorphism $\varphi_3(n)$, when $n \equiv 2 \bmod 4$, which we re-name:

$$c(n): \pi_n(G, SO) \longrightarrow Z_2 \qquad (n \equiv 2 \bmod 4)\, .$$

The computation of $c(n)$ is a well-known problem, and the only information known at present is that $c(n) = 0$ if $n = 10$ or 18; and $c(n) \neq 0$ if $n = 6$ or 14.[1]

[1] It has recently been proved, cf. Brown, Peterson, *The Kervaire iuvariant of* $(8k+2)$-*manifolds*, Bull. A.M.S., 71 (1965), 190–193, that $c(n) = 0$ when $n = 8k + 2$.

It is conjectured that $c(n) = 0$ if $n \neq 6, 14$ (see [14, p. 536]).

LEMMA.

$$\operatorname{Ker} \varphi_3(n, k) = \begin{cases} \operatorname{Ker} \partial_4(n) \cdot \sigma_3(n, k) & \textit{if } n \not\equiv 2 \bmod 4 \\ \operatorname{Ker} c(n) \cdot \sigma_3(n, k) & \textit{if } n \equiv 2 \bmod 4 . \end{cases}$$

PROOF. Since $\varphi_3(n, k) = \varphi_3(n) \cdot \sigma_3(n, k)$, the lemma is obvious if $n \equiv 2 \bmod 4$; clearly to prove the lemma for $n \not\equiv 2 \bmod 4$, it suffices to prove $\operatorname{Ker} \varphi_3(n) = \operatorname{Ker} \partial_4(n)$. If n is odd, this fact is a consequence of a result of Adams (see [14, p. 509]). If $n \equiv 0 \bmod 4$, it follows from the Hirzebruch index formula (see [19, p. 964–5]) that the index of a closed π-manifold is zero and, therefore, $\varphi_2(n) = 0$. Since $\partial_1(n) = 0$, as has been pointed out, it now is a consequence of a simple argument, using diagram $(5)_\infty$, that $\operatorname{Ker} \varphi_3(n) = \operatorname{Ker} \partial_4(n)$.

6.6. LEMMA. *If* $n \not\equiv 2 \bmod 4$ *and* $2m \leqq 3(n + 1)$, *then* $\partial_4(n) \cdot \sigma_3(n, k) = 0$.

PROOF. Suppose $n = 4r$; then $\sigma_1(n - 1, k) = 0$ if $k \leqq 2r$ (see e.g. [32, Lemma 1.1] and [19, Lemma 5]). Now $\operatorname{Image} \partial_4(n, k) \subset \operatorname{Image} \{j: \pi_{n-1}(SO_{k-1}) \to \pi_{n-1}(SO_k)\}$, since $p \cdot \partial_4(n, k) = p' \cdot \omega_4 \cdot \partial_4 = 0$. Thus

$$\operatorname{Image} \sigma_1(n - 1, k) \cdot \partial_4(n, k) \subset \operatorname{Image} \sigma_1(n - 1, k - 1) = 0$$

for $k \leqq 2r + 1$.

Since $\sigma_1(n - 1, k) \cdot \partial_4(n, k) = \partial_4(n) \cdot \sigma_3(n, k)$, this proves the lemma, when $n = 4r$. For n odd, $\partial_4(n) = 0$, as mentioned above.

COROLLARY. *If* $n \not\equiv 2 \bmod 4$ *and* $2m \leqq 3(n + 1)$, *then* $\varphi_3(n, k) = 0$.

This follows from the two preceding lemmas.

Now the desired reduction of the computation of $\Theta^{m,n}$ follows from theorem (6.4), lemma (6.5) and Lemma (6.6).

6.7. It follows from theorem (6.4) that $\Sigma^{m,n} = 0$ for $2m > 3(n + 1)$. We would like to compute $\Sigma^{m,n}$ for $2m \leqq 3(n + 1)$.

Let us define:

$$\Sigma_0^{m,n} = \Sigma^{m,n} \cap \operatorname{Image} \partial_3(n, k)$$
$$\tilde{\Sigma}^{m,n} = \Sigma^{m,n} / \Sigma_0^{m,n} .$$

$\Sigma_0^{m,n}$ is the subgroup of knotted spheres which bound framed submanifolds of S^m.

THEOREM. *If* $2m \leqq 3(n + 1)$, *then*:

$$\Sigma_0^{m,n} \approx \begin{cases} P_{n+1} & \textit{if } n \not\equiv 1 \bmod 4 \\ \operatorname{Image} c(n + 1)/\operatorname{Image} c(n + 1) \cdot \sigma_3(n + 1, k) & \textit{if } n \equiv 1 \bmod 4 \end{cases}$$

$$\tilde{\Sigma}^{m,n} \approx \operatorname{Ker} \sigma_3(n, k) .$$

PROOF. Consider the following commutative diagram:

$$\begin{array}{ccccccccc}
\pi_{n+1}(G_k, SO_k) & & & & \Theta^{m,n} & \xrightarrow{\omega_3} & \pi_n(G_k, SO_k) & & \\
 & \searrow^{\varphi_3} & & \nearrow^{\partial_3} & & & & \searrow^{\varphi_3} & \\
\Big\downarrow \sigma_3 & & P_{n+1} & & \Big\downarrow \theta & & \Big\downarrow \sigma_3 & & P_n \\
 & \nearrow_{\varphi_3} & & \searrow_{\partial_3} & & & & \nearrow_{\varphi_3} & \\
\pi_{n+1}(G, SO) & & & & \Theta^n & \xrightarrow[\omega_3]{} & \pi_n(G, SO) & &
\end{array}$$

where the upper row is a segment of $(3)_k$ and the lower a segment of $(3)_\infty$. As in (6.3), we derive from this an exact sequence.

$$(7)_{n,k} \qquad \begin{array}{l} 0 \longrightarrow \text{Cok}\,\theta(n+1, k) \longrightarrow \text{Cok}\,\sigma_3(n+1, k) \\ \xrightarrow{\theta'} \text{Ker}\,\theta(n, k) \longrightarrow \text{Ker}\,\sigma_3(n, k) \longrightarrow 0\ . \end{array}$$

Since θ' is induced by $\partial_3(n, k) \cdot \varphi_3(n)$ and $\Sigma^{m,n} = \text{Ker}\,\theta(n, k)$, it follows immediately that $\Sigma_0^{m,n} = \text{Image}\,\theta'$, $\widetilde{\Sigma}^{m,n} \approx \text{Ker}\,\sigma_3(n, k)$. Thus,

$$\Sigma_0^{m,n} \approx \text{Image}\,\varphi_3(n+1)/\text{Image}\,\varphi_3(n+1, k)\ .$$

The theorem now follows by Corollary (6.6) and [14, Th. 4.1].

COROLLARY. $\Sigma^{m,n}$ *(and, therefore,* $\Theta^{m,n}$*) is a finite group (recall* $n \geqq 5$, $k \geqq 3$*) unless* $n = 4r - 1$, $m \leqq 6r$, *in which case* $\Sigma^{m,n}$ *is finitely generated of rank one.*

PROOF. If $2m > 3(n+1)$, this follows from Theorem (6.4). If $2m \leqq 3(n+1)$, it follows from Theorem (6.7) if we prove $\text{Ker}\,\sigma_3(n, k)$ is finite.

We will prove $\pi_n(G_k, SO_k)$ is finite for $2m \leqq 3(n+1)$, by induction on k. If $k \leqq 2$, this is obvious. To show that $\pi_n(G_{k-1}, SO_{k-1})$ is finite implies $\pi_n(G_k, SO_k)$ is finite, we need to show that $\text{Cok}\,\xi(n, k-1)$ is finite. By Lemma (6.3), it suffices to prove that $\text{Cok}\,\nu(n, k-1)$ and $\text{Ker}\,\nu(n-1, k-1)$ are finite.

But $\pi_{m-1}(S^{k-1})$ is finite for $m \leqq 2n+1$, by [21, Prop. 5], and, therefore, so is $\text{Cok}\,\nu(n, k-1)$. Furthermore $\pi_{n-1}(S^{k-2})$ is finite for $2m \leqq 3(n+1)$, and $\pi_{n-1}(F_{k-2})$ is finite for $m \leqq 2n+1$ (also by [21]). Thus $\pi_{n-1}(G_{k-1})$ is finite (see $(6)_{k-1}$), and consequently so is $\text{Ker}\,\nu(n-1, k-1)$.

This completes the induction step.

6.8. It follows from Theorem (6.4) that $\Theta_k^n = 0$ for $2m \geqq 3(n+1)$. We would like to compute Θ_k^n when $2m < 3(n+1)$.

Let us define the following homomorphisms:

$$\begin{array}{lll} c(n, k)\colon \text{Cok}\,\sigma_3(n, k) \longrightarrow \text{Cok}\,c(n) \cdot \sigma_3(n, k) & & \text{for } n \equiv 2 \bmod 4 \\ \partial_4'(n, k)\colon \text{Cok}\,\sigma_3(n, k) \longrightarrow \pi_{n-1}(SO) & & \text{for } n = 4r,\ k \leqq 2r+1\ . \end{array}$$

Let $c(n, k)$ be induced by $c(n)$ and $\partial_4'(n, k)$ by $\partial_4(n)$, taking account of Lemma (6.6).

THEOREM. *If* $2m < 3(n+1)$, *then:*

$$\Theta_k^n \approx \begin{cases} \text{Cok}\ \sigma_3(n, k) & n\ odd \\ \text{Ker}\ \partial_4'(n, k) & n \equiv 0 \bmod 4 \\ \text{Ker}\ c(n, k) & n \equiv 2 \bmod 4\ . \end{cases}$$

PROOF. It follows from $(7)_{n-1,k}$ that Θ_k^n is isomorphic to the kernel of the map θ': Cok $\sigma_3(n, k) \longrightarrow \Theta^{m-1,n-1}$ induced by $\partial_3(n-1, k) \cdot \varphi_3(n)$. By Corollary (6.6), $\partial_3(n-1, k)$ is a monomorphism if $n \not\equiv 2 \bmod 4$. Using Lemma (6.5), it is now easy to see that Ker θ' is precisely as asserted in the theorem.

6.9. Let $N(n, k)$ and $N_0(n, k)$ be the subgroups of $\pi_{n-1}(SO_k)$ consisting of normal bundles to knotted spheres, respectively, knotted S^n, in S^m. Otherwise stated:

$$N(n, k) = \text{Image}\ \partial_1(n, k)\ ,$$
$$N_0(n, k) = \partial_1(n, k) \cdot \Sigma^{m,n}\ .$$

The following generalizes [6, Th. 1.2].

THEOREM.

$$N(n, k) = \begin{cases} \text{Ker}\ \omega_4(n-1, k) \cap \text{Ker}\ \sigma_1(n-1, k) & if\ n \not\equiv 2 \bmod 4 \\ \partial_4(n, k) \cdot \text{Ker}\ (c(n) \cdot \sigma_3(n, k)) & if\ n \equiv 2 \bmod 4 \end{cases}$$

$$N_0(n, k) = \partial_4(n, k) \cdot \text{Ker}\ \sigma_3(n, k)\ .$$

PROOF. It follows from $(5)_k$ that:

$$N(n, k) = \text{Image}\ \partial_4(n, k) \cdot \omega_3(n, k) = \partial_4(\text{Ker}\ \varphi_3(n, k))\ .$$

If $n \equiv 2 \bmod 4$, the assertion about $N(n, k)$ is immediate. If $n \not\equiv 2 \bmod 4$, it follows from Lemma (6.5) that:

$$N(n, k) = \partial_4(\text{Ker}\ \sigma_1(n-1, k) \cdot \partial_4(n, k)) = \text{Ker}\ \omega_4 \cap \text{Ker}\ \sigma_1\ .$$

To compute $N_0(n, k)$, we deduce from $(7)_{n,k}$ that:

$$N_0(n, k) = \partial_1(\Sigma^{m,n}) = \partial_4 \cdot \omega_3(\text{Ker}\ \theta(n, k)) = \partial_4(n, k) \cdot \text{Ker}\ \sigma_3(n, k)\ .$$

6.10. Let us define:

$$\theta_0(n, k)\colon \Theta^{m,n} \longrightarrow \Theta^{m+1,n}$$

to be the suspension homomorphism induced by the natural imbedding $S^m \subset S^{m+1}$.

PROPOSITION. Ker $\theta_0(n, k) \subset$ Ker $\partial_1(n, k)$.

PROOF. Suppose $\alpha \in$ Ker $\theta_0(n, k)$. It follows from the commutativity relation:

$$\xi(n, k) \cdot \omega_3(n, k) = \omega_3(n, k+1) \cdot \theta_0(n, k)\ ,$$

that $\omega_3(\alpha) \in$ Ker $\xi(n, k)$. But, by Lemma (6.3), Ker $\xi(n, k) \subset$ Image $\varphi_4(n, k)$. Therefore:

$$\partial_1(n, k) \cdot \alpha = \partial_4(n, k) \cdot \omega_3(n, k) \cdot \alpha \in \text{Image}\, \partial_4(n, k) \cdot \varphi(n, k) = 0 .$$

The geometric meaning of this proposition is that, if a knotted sphere in S^m has a non-trivial normal bundle, then its suspension into S^{m+1} is still non-trivially knotted.

COROLLARY (Kervaire). *If* $2m > 3n + 1$, $N_0(n, k) = 0$.

PROOF. This follows immediately from Theorem (6.4) and the proposition.

6.11. Let $T(n, k)$ and $T_0(n, k)$ be the subgroups of $\pi_m(S^k)$ of elements of the form $t(K^n, \mathcal{F})$, where $(K^n, \mathcal{F})$ is a framed knotted sphere, respectively, framed knotted S^n, in S^m. Let $H(n, k)$ be the Hopf construction subgroup of $\pi_m(S^k)$ (see [10, p. 76]). Let

$$\sigma(n, k)\colon \pi_m(S^k) \longrightarrow \pi_n(G)$$

be the iterated suspension.

THEOREM.

$$T(n, k) = \begin{cases} H(n, k) & \textit{if } n \not\equiv 2 \bmod 4 \\ H(n, k) \cap \text{Ker}\, c(n) \cdot \sigma(n, k) & \textit{if } n \equiv 2 \bmod 4 \end{cases}$$

$$T_0(n, k) = H(n, k) \cap \text{Ker}\, \varphi_4(n) \cdot \sigma(n, k) .$$

PROOF. It follows from Lemma (3.9) that

$$T(n, k) = \big((-1)^{k+1}\iota_k\big) \circ \big(\text{Image}\, \nu(n, k) \cdot \omega_2(n, k)\big) ,$$

and

$$T_0(n, k) = \big((-1)^{k+1}\iota_k\big) \circ \big(\nu \cdot \omega_2(\text{Ker}\, \theta(n, k) \cdot \varphi_1(n, k))\big) .$$

Since $\varphi_2(n, k) = \varphi_2(n) \cdot \sigma_2(n, k)$ and $\varphi_2(n) = 0$ for $n \not\equiv 2 \bmod 4$, as remarked in (6.5), $\omega_2(n, k)$ is onto for $n \not\equiv 2 \bmod 4$. If $n \equiv 2 \bmod 4$,

$$\begin{aligned} \text{Image}\, \nu \cdot \omega_2 &= \nu\big(\text{Ker}\, \varphi_2(n, k)\big) = \nu\big(\text{Ker}\, c(n) \cdot \sigma_2(n, k)\big) \\ &= \text{Image}\, \nu \cap \text{Ker}\, c(n) \cdot \sigma(n, k) . \end{aligned}$$

It follows from $(5)_k$ and $(5)_\infty$ that

$$\omega_2\big(\text{Ker}\, \theta(n, k) \cdot \varphi_1(n, k)\big) = \text{Ker}\, \varphi_4(n) \cdot \sigma_2(n, k) .$$

Furthermore, it is easy to see that

$$\nu\big(\text{Ker}\, \varphi_4(n) \cdot \sigma_2(n, k)\big) = \text{Image}\, \nu(n, k) \cap \text{Ker}\, \varphi_4(n) \cdot \sigma(n, k) .$$

Since Image $\nu(n, k) = H(n, k)$, by [10, Th. 1.7] (the conclusion of this theorem can be strengthened to read *a map* $S^{r-n} \times S^n \to S^n$ *of degree* $+1$ *on the second factor*, as is clear from the argument in the preceding paragraph), it only remains to show

$$\begin{aligned} T(n, k) &= \big((-1)^{k+1}\iota_k\big) \circ T(n, k) \\ T_0(n, k) &= \big((-1)^{k+1}\iota_k\big) \circ T_0(n, k) . \end{aligned}$$

In fact, if $(K^n, \mathcal{F})$ is a framed knotted sphere in S^m, where $\mathcal{F} = (f_1, \cdots, f_k)$ and $(K_0^n, \mathcal{F}_0)$ is defined by $K_0 = (-1)^{k+1}K$, $\mathcal{F}_0 = ((-1)^{k+1}f_1, f_2, \cdots, f_k)$, it is clear that

$$t(K_0, \mathcal{F}_0) = \big((-1)^{k+1}\iota_k\big) \cdot t(K, \mathcal{F}) \,.$$

From this the desired equalities are immediate.

It follows now from the theorem (or see [11, 1.8]) that, if we define

$$J(n, k) \colon \pi_n(SO_k) \longrightarrow \pi_m(S^k) \,,$$

by $J(n, k) = \nu(n, k) \cdot \omega_4(n, k)$, then Image $J(n, k) \subset T_0(n, k)$. We define

$$\tilde{T}_0(n, k) = T_0(n, k)/\text{Image } J(n, k)$$
$$T'(n, k) = \pi_m(S^k)/T(n, k) \,.$$

COROLLARY. (a) *If* $2m \geqq 3(n + 1)$, *then*

$$T'(n, k) \approx \begin{cases} 0 & \text{if } n \not\equiv 2 \bmod 4 \\ \text{Image } c(n) \cdot \sigma(n, k) & \text{if } n \equiv 2 \bmod 4 \,. \end{cases}$$

(b) *If* $2m > 3n + 1$, *then* $\tilde{T}_0(n, k) = 0$ (*see* [11, Lem. 8.1]).

PROOF. It is proved in (6.4) that $\nu(n, k)$ is an epimorphism for $2m \geqq 3(n + 1)$ and $\xi_3(n, k + 1)$ is a monomorphism for $2m > 3n + 1$. Since Image $\nu(n, k) = H(n, k)$, (a) follows from Theorem (6.11).

To prove (b) we proceed by downward induction on k. In the stable range this is clear. Suppose it true for $k + 1$. Consider the following commutative diagram

$$\begin{array}{ccccc} \pi_m(S^k) & \xleftarrow{\ \nu\ } & \pi_n(G_k) & \xrightarrow{\ \varphi_4\ } & \pi_n(G_k, SO_k) \\ \downarrow{\scriptstyle \xi''} & & \downarrow{\scriptstyle \xi'} & & \downarrow{\scriptstyle \xi} \\ \pi_{m+1}(S^{k+1}) & \xleftarrow{\ \nu\ } & \pi_n(G_{k+1}) & \xrightarrow{\ \varphi_4\ } & \pi_n(G_{k+1}, SO_{k+1}) \end{array}$$

where ξ' and ξ'' are suspensions. Given $\alpha \in T_0(n, k)$, we wish to find $\beta \in \text{Ker } \varphi_4(n, k)$ such that $\nu(\beta) = \alpha$. Suppose $\alpha = \nu(\beta')$, $\beta' \in \pi_n(G_k)$, and, by induction, $\xi''(\alpha) = \nu(\beta'')$ for some $\beta'' \in \text{Ker } \varphi_4(n, k + 1)$. Since

$$\xi'(\beta') - \beta'' \in \text{Ker } \nu(n, k + 1) \,,$$

it follows from Lemma (6.3) that $\varphi_4(\xi'(\beta') - \beta'') = \xi\varphi_4(\beta') \in \text{Ker } \xi(n, k + 1) = 0$. Therefore $\varphi_4(\beta') \in \text{Ker } \xi(n, k)$; by Lemma (6.3) again, $\varphi_4(\beta') = \varphi_4(\gamma)$ for some $\gamma \in \text{Ker } \nu(n, k)$. Now we may set $\beta = \beta' - \gamma$ and (b) is proved.

7. Computations in low dimensions

7.1. By strenuous use of Proposition (6.2), together with results of [1], [9], [10], [13] and [24], computations of many of the geometrically defined groups we have discussed can be carried out for low values of n. We present some tabulations of orders of such groups for $n \leqq 11$.

7.2.

order of $N_0(n, k)$—(see (6.9))

$k \backslash n$	5	6	7	8	9	10	11
3	1	1	1	1	1	1	1
4	1	1	4	2	2	b	5
5	1	1	1	1	1	1	5
6	1	1	1	1	1	1	5
7	1	1	1	1	1	1	1

where b is either 1 or 3.

7.3.

order of $\tilde{\Sigma}^{m,n}$—(see (6.7))

$k \backslash n$	5	6	7	8	9	10	11
3	2	2	3	15	1	2	24
4	1	1	60	2	2	$4b$	840
5	1	1	4	4	8	$96c$	3360
6	1	1	1	1	2	1	5040
7	1	1	1	1	1	1	5040
8	1	1	1	1	1	1	1

where b is as in (7.2), and c is 1 or 3 and satisfies $c \leqq b$.

7.4.

order of Θ^n_k—(see (2.1))

$k \backslash n$	8	9	10
3	2	2	3
4	1	1	b
5	1	1	c
6	1	1	1
7	1	1	1

where b and c are as in (7.2) and (7.3).

7.5.

order of $\widetilde{T}_0(n, k)$—(see (6.11))

$k \backslash n$	5	6	7	8	9	10	11
3	1	1	1	1	1	2	2
4	1	1	15	1	1	4	168
5	1	1	1	1	2	1	168
6	1	1	1	1	1	1	504
7	1	1	1	1	1	1	1

7.6.

order of $T'(n, k)$—(see (6.11))

$k \backslash n$	5	6	7	8	9	10	11
3	1	1	15	2	2	2	84
4	1	1	1	1	2	8	4
5	1	1	1	1	1	c	6
6	1	1	1	1	1	1	1
7	1	1	1	1	1	1	1

where c is as in (7.3)

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