Journal of Pure and Applied Algebra 58 (1989) 251-260
North-Holland

METABOLIC AND HYPERBOLIC FORMS FROM KNOT THEORY

J.P. LEVINE
Department of Mathematics, Brandeis University, Waltham, MA 02254, U.S.A.

Communicated by J.D. Stasheff
Received 12 June 1987
Revised 27 January 1988

In codimension two knot theory various *signature* invariants arise. We show that they all vanish on *double slice* knots but only certain linear combinations vanish on *slice* knots. This is done in the context of constructing complete sets of invariants of ± hermitian torsion forms over any Laurent polynomial ring $F[t,t^{-1}]$ (F a field) to detect metabolic or hyperbolic forms. Finally a precise formula relating two different versions of these signature invariants is given which shows that one is strictly weaker than the other.

Introduction

In the study of codimension two knots, various bilinear forms arise as algebraic invariants. These forms can be used to detect whether a knot is slice (null-cobordant) or double slice (double null-cobordant), according to whether they are, respectively, metabolic or hyperbolic (see [2,4,10]). In this note we study and compare these latter properties for the Blanchfield pairing and the Seifert form *over* a *field*.

The Blanchfield pairing β, with coefficients in a field F, is classified in [5,7] by certain associated Hermitian forms over finite extension fields of F. We will establish the precise conditions on these associated forms which corresponds to β being metabolic or hyperbolic, extending results of [1]. When $F=\mathbb{R}$, this condition is expressed in terms of signature invariants $\sigma_{\theta,r}$.

The Seifert form S gives rise to a signature invariant $\sigma: C\to\mathbb{Z}$, where C is the unit circle in the complex plane, which is an invariant of the congruence class of S over $\mathbb{R}$. In [4] it is shown that $\sigma=0$ away from the roots of the Alexander polynomial, when S is null-cobordant. We observe here that $\sigma=0$ *everywhere* when S is double null-cobordant. This gives a particularly easy way to detect slice knots which are not double slice.

Finally, we make a precise comparison between the signature invariants σ and $\sigma_{\theta,r}$. As a consequence, we see that σ gives a complete criterion for null-cobordism over $\mathbb{R}$, but not for double null-cobordism.

1. The Blanchfield pairing

We will consider the following general situation. Let F be a field and $\Lambda = F[t, t^{-1}]$ equipped with the involution induced by $t \mapsto t^{-1}$ – for any $\lambda \in \Lambda$, we denote by $\bar{\lambda}$ the image of λ under this involution. We consider forms $\beta: H \times H' \to S(\Lambda) = Q(\Lambda)/\Lambda$ where H, H' are finitely-generated Λ-torsion modules and $Q(\Lambda)$ is the quotient field of Λ; the involution of Λ extends over $Q(\Lambda)$ with the same notation. We say β is *bilinear* (often called sesqui-linear elsewhere) if $\beta(\lambda x, y) = \beta(x, \bar{\lambda} y) = \lambda\beta(x, y)$ for all $\lambda \in \Lambda$, $x \in H$, $y \in H'$ and β is additive in both variables. If $H = H'$ we say β is *ε-Hermitian* ($\varepsilon = \pm 1$) if $\beta(x, y) = \varepsilon\overline{\beta(y, x)}$. We say β is *non-singular* if the adjoints $H \to \bar{H}'^*$, $H' \to \bar{H}^*$ are both isomorphisms. $\bar{H}$ denotes the Λ-module obtained by changing the Λ-module structure on H via the involution; $H^* = \mathrm{Hom}_\Lambda(H, S(\Lambda))$ with the usual Λ-module structure

$$(\lambda\phi)(x) = \phi(\lambda x) = \lambda\phi(x).$$

A non-singular bilinear ε-Hermitian form will be referred to as an *ε-form*.

We will need some preliminary facts.

Lemma 1.1. *The natural map $H \to H^{**}$ is an isomorphism.*

This follows easily from the fact that Λ is a PID and so $S(\Lambda)$ is injective, and H is finitely-generated torsion. Note that this lemma implies that a bilinear form is non-singular if either adjoint is an isomorphism, or if both adjoints are injective (i.e. β is non-degenerate).

If $\beta: H_1 \times H_2 \to S(\Lambda)$ is a bilinear form and $K \subset H_1$, then $K^\perp \subset H_2$ consists of all $x \in H_2$ such that $\beta(y, x) = 0$ for all $y \in K$. Similarly, if $K \subseteq H_2$ we define $K^\perp \subseteq H_1$.

Lemma 1.2. *Suppose $\beta: H \times H' \to S(\Lambda)$ is a non-singular bilinear form and $K \subseteq H$, then*

(i) $K = (K^\perp)^\perp$

(ii) *The induced pairings*

$$K \times H'/K^\perp \to Q(\Lambda)/\Lambda \quad \text{and} \quad H/K \times K^\perp \to Q(\Lambda)/\Lambda$$

are non-singular.

Proof. (i) Clearly $K \subseteq (K^\perp)^\perp$. If $x \in (K^\perp)^\perp$, but $x \notin K$, choose $\phi: H \to S(\Lambda)$, $\phi(K) = 0$ and $\phi(x) \neq 0$. This uses the fact that Λ is a principal ideal domain and injectivity of $S(\Lambda)$. By non-singularity, $\exists y \in H'$ such that $\beta(a, y) = \phi(a)$ for all $a \in H$. Then $y \in K^\perp$ but $\beta(x, y) = \phi(x) \neq 0$, contradicting $x \in (K^\perp)^\perp$..

(ii) This follows easily from (i) and the non-singularity of β. □

Suppose β is an ε-form on H. We say β is *metabolic* if there exists a submodule $K \subseteq H$ (a *metabolizer*) such that $K = K^\perp$. We say β is *hyperbolic* if there exists a direct sum decomposition $H = K_1 \oplus K_2$ such that $K_i = K_i^\perp$ $(i = 1, 2)$.

If (H, β) is the Blanchfield pairing of a knotted $(2q-1)$-sphere K in S^{2q+1} (so $\varepsilon = (-1)^{q+1}$), then β is metabolic (resp. hyperbolic) if K is slice (resp. double slice) (see [2, 3, 10]).

Let (H, β) be an ε-form and $p \in \Lambda$ an irreducible polynomial satisfying $p(t) = up(t^{-1})$ for some unit $u \in \Lambda$. (We say p is *symmetric*.) It is easy to see that either $p(t) = t+1$, $t-1$ or $p(t) = p(t^{-1})$, after multiplication by a suitable unit.

If $K_{p,i} \subseteq H$ is the submodule of elements x satisfying $p^i x = 0$, we take

$$\Delta_{p,i} = \frac{K_{p,i}}{K_{p,i-1} + pK_{p,i+1}} \quad \text{for } i \geq 1.$$

Then $\Delta_{p,i}$ is a vector space over $\Lambda/(p) \approx F(\xi)$, where ξ is a root of $p(t)$, and β induces an ε-Hermitian pairing $\beta_{p,i}$ on $\Delta_{p,i}$ defined by the formula

$$\beta_{p,i}(x, y) = p^{i-1}\beta(\tilde{x}, \tilde{y}) \qquad x, y \in \Delta_{p,i}$$

where $\tilde{x}, \tilde{y}$ are any lifts of x, y to $K_{p,i}$ and $\Lambda/(p)$ is identified with a subset of $S(\Lambda)$ by the imbedding $\lambda \mapsto \lambda/p$ (see [4, 7]). Note that $\Lambda/(p)$ has an involution induced by that on Λ. It follows from Lemma 1.2 that $\beta_{p,i}$ is non-singular.

Theorem 1.3. *The isomorphism class of β is determined by the isomorphism classes of all $\{\beta_{p,i}$: p symmetric, $i \geq 1\}$.*

Proof. See [5, 7]. □

We now establish the criteria on $\{\beta_{p,i}\}$ which correspond to β being metabolic or hyperbolic.

Theorem 1.4. *β is metabolic ⇔ $\bigoplus_{i\, odd} \beta_{p,i}$ is metabolic for every symmetric p.*

Theorem 1.5. *If $\mathrm{ch}\, F \neq 2$ or $p \neq t-1$, then β is hyperbolic ⇔ $\beta_{p,i}$ is hyperbolic for every $i \geq 1$ and symmetric p.*

To precisely compare these situations we need to understand the relation between metabolic and hyperbolic for $\beta_{p,i}$. Let γ be any non-singular ε-Hermitian form over a field E with involution. The notions of metabolic and hyperbolic are defined exactly as for ε-forms.

Theorem 1.6. *Unless $\mathrm{ch}\, E = 2$ and the involution is trivial, the notions of metabolic and hyperbolic are coincident for ε-Hermitian forms over E.*

Thus in our situation, where $E = \Lambda/(p)$, this excludes only the case $\mathrm{ch}\, F = 2$, $p(t) = t+1$, and this case does not arise in the context of knot theory.

We take note of the particular case $F = \mathbb{R}$. If the involution is non-trivial, $\beta_{p,i}$ is a non-singular complex ε-Hermitian form and so has a signature $\sigma_{p,i}$ which is zero exactly when $\beta_{p,i}$ is metabolic. If the involution is trivial, then $p(t)=t-1$ or $t+1$ (which cannot happen in the context of knot theory) and $\beta_{p,i}$ is an ε-symmetric real quadratic form. If $\varepsilon=+1$, there is a signature $\sigma_{p,i}$, but if $\varepsilon=-1$, $\beta_{p,i}$ is always metabolic. Note that the sign of $\sigma_{p,i}$ depends upon the choice of p, not just the roots.

Corollary 1.7. *If $F=\mathbb{R}$, then β is metabolic (resp. hyperbolic) iff $\sum_{i\ odd}\sigma_{p,i}=0$ (resp. $\sigma_{p,i}=0$ for all $i\geq 0$) for every symmetric p, except when $p=t-1$ or $t+1$ and $\varepsilon=-1$.*

2. The Seifert form

A *Seifert* ε-form over a field F is a bilinear form S on a vector space V such that $S-\varepsilon S^{\mathrm{T}}$ is non-singular (S^{T} = transpose of S). We will say S is *metabolic* if there exists a subspace $W\subseteq V$ (a *metabolizer*) such that $W=W^{\perp}$, where $W^{\perp}=\{x\in V: \beta(x,y)=0=\beta(y,x)$ for all $y\in W\}$. We say S is *hyperbolic* if there exists a direct sum decomposition $V=W_1\oplus W_2$, where W_i are metabolizers of S.

One associates to any Seifert ε-form (S,V) an ε-form (H,β) as follows: First define a non-singular bilinear form B on the vector space $\tilde{V}=V\otimes_F Q(\Lambda)$ by $B(v,w)=tS(v,w)-\varepsilon S(w,v)$ for $v,w\in V$, and extending over $\tilde{V}$ to satisfy $B(\lambda v,w)=B(v,\bar{\lambda}w)=\lambda B(v,w)$, for any $\lambda\in Q(\Lambda)$. Let $V_0=V\otimes_F\Lambda\subseteq\tilde{V}$ and $V_1=\{v\in\tilde{V}: B(v,w)\in\Lambda$ for all $w\in V_0\}$, both Λ-submodules of $\tilde{V}$. Then $V_0\subseteq V_1$ and we define $H=V_1/V_0$; let β be the ε-Hermitian form defined on H be the formula

$$\beta(x,y)=(t^{-1}-1)B(\bar{x},\bar{y})\quad\text{for } x,y\in H \text{ with lifts } \bar{x},\bar{y}\in V_1.$$

We note that $H\otimes_\Lambda F=0$, since $S-\varepsilon S^{\mathrm{T}}$ is non-singular. Thus $t-1$ is an automorphism of H, and so β is non-singular.

Proposition 2.1. *S is metabolic (hyperbolic) if and only if β is metabolic (hyperbolic).*

Recall that a knotted $(2q-1)$-sphere K in S^{2q+1} gives rise to a class of Seifert $(-1)^{q+1}$-forms over $\mathbb{Q}$. In fact S is integral and $S-\varepsilon S^{\mathrm{T}}$ is non-singular over $\mathbb{Z}$. If K is slice, then any associated Seifert form is metabolic. If K is double slice, then there is an associated Seifert form which is hyperbolic – but not *every* associated form is (see [4, 10]).

Suppose $F=\mathbb{R}$ and A is a representative matrix for S. Then, for any unit complex number $z\neq 1$, $A(z)=(zA-\varepsilon A^{\mathrm{T}})/(z-1)$ is an ε-Hermitian form – let $A(1)=i(A-\varepsilon A^{\mathrm{T}})$. Then

$$\sigma_S(z)=\text{signature}\begin{cases} A(z), & \varepsilon=+1,\\ iA(z), & \varepsilon=-1\end{cases}$$

is an integer-valued function defined on the unit circle C in the complex plane which is locally constant on the complement C_S of those roots of the polynomial $\Delta(t)=\det(tA-\varepsilon A^{\mathrm{T}})$ which lie on C (see [4]). Note that $\sigma_S(1)=0$. If S is metabolic, then $\sigma_S(z)=0$ for all $z\in C_S$.

Proposition 2.2. *If S is hyperbolic, then $\sigma_S(z)=0$ for all $z\in C$.*

We relate σ_S to the signature $\sigma_{p,i}$ associated to the ε-form β associated to S.

Theorem 2.3. *Let ξ be a root of $\Delta(t)$ on C, p the minimal polynomial of ξ (over $\mathbb{R}$), ξ_+ and ξ_- points of C_S such that ξ is the only root of $\Delta(t)$ lying on an arc of C connecting them. Then, if we choose $p(t)$ so that $p(\xi_+)>0$, we have:*

$$\sigma_S(\xi_+)-\sigma_S(\xi_-)=2\sum_{r\ odd}\sigma_{r,p},$$

$$\sigma_S(\xi)=\tfrac{1}{2}(\sigma_S(\xi_+)+\sigma_S(\xi_-))-\sum_{r\ even}\sigma_{r,p}.$$

Compare this to the results of [1, 6, 9].

Corollary 2.4. (i) *$\sigma_S=0$ on C_S if and only if β is metabolic (over $\mathbb{R}$).*
(ii) *$\sigma_S\equiv 0$ on C if and only if*

$$\sum_{odd\ i}\sigma_{p,i}=0=\sum_{even\ i}\sigma_{p,i}\quad\textit{for all symmetric } p.$$

Finally we give some examples.

Theorem 2.5. *Let $\tau_i,\ldots,\tau_k$ be any collection of integers, and let $p=p(t)=t-1+t^{-1}$. Then there is a metabolic Seifert form S such that*

$$\sigma_{i,p}=\begin{cases}\tau_r & \text{if } i=2^r,\\ 0 & \text{otherwise,}\end{cases}$$

and $\sigma_{i,q}=0$ for any $q\neq p$.

In fact the Seifert forms produced correspond to knots i.e. S is integral satisfying $\det(S-\varepsilon S^{\mathrm{T}})=\pm 1$. Combining Theorem 2.5 with Corollary 2.4 we see that there exist metabolic Seifert forms S with $\sigma_S\not\equiv 0$ on C, or even with $\sigma_S\equiv 0$ on C, but whose associated ε-form is not hyperbolic over $\mathbb{R}$, and these forms can be realized by slice knots (which are, therefore, not double slice).

3. Proofs of results in Section 1.

We now begin the proofs of the theorems and propositions.

Proof of Theorem 1.4. This is indicated in [4] but we provide some more detail here. First of all notice that H decomposes as a direct sum $H=\bigoplus_p H_p$ where p ranges over irreducible elements of Λ, and H_p is the p-primary component of H. H_p is orthogonal to H_q, unless $p=u\bar{q}$, for some unit $u\in\Lambda$, and $H_p, H_{\bar{p}}$ are dually paired under β. Thus H decomposes as an orthogonal direct sum of terms of two types:

(i) $H_p \oplus H_{\bar{p}}$, if $p, \bar{p}$ are relatively prime,

(ii) H_p, if p is symmetric.

Summands of type (i) are hyperbolic and so it suffices to consider the case of H p-primary for some symmetric p. The next observation is that H_p can be further decomposed as an orthogonal direct sum:

$$H_p = H_{p,1} \oplus H_{p,2} \oplus \cdots \oplus H_{p,n}$$

where $H_{p,i}$ is a free module over $\Lambda/(p^i)$ – see [7]. It is easy to see that $\Delta_{p,i}= H_{p,i}/pH_{p,i}$.

We need the following familiar lemma:

Lemma 3.1. *Suppose β is a non-singular ε-form on H and $K\subseteq H$ satisfies $K\subseteq K^\perp$ (i.e. $\beta|_K=0$). Then β is metabolic if and only if the non-singular form β' induced by β on $K^\perp/K$ is metabolic.*

Proof. If $L'\subset K^\perp/K$ is a metabolizer for β', then the lift of L' to $K^\perp$ is a metabolizer for β. Conversely, suppose L is a metabolizer for β. Let L' be the projection of $L\cap K^\perp$ into $K^\perp/K$ – we show that L' is a metabolizer for β'. Clearly $\beta'|_{L'}=0$, so we need to show $(L')^\perp \subset L'$.

Now suppose $x\in K^\perp$ such that $x\in (L\cap K^\perp)^\perp$; we need to show that $x\in L+K$. Consider $\beta(x,\cdot)$ as a homomorphism $K^\perp/L\cap K^\perp \to S(\Lambda)$; since $S(\Lambda)$ is injective, this extends to a homomorphism $H/L\to S(\Lambda)$ which, by non-singularity of β is of the form $\beta(y,\cdot)$. Thus $y\in L^\perp = L$ and $x-y\in (K^\perp)^\perp = K$, as desired. This proves the lemma. □

Now consider the submodule of H_p given by

$$K=\bigoplus_i p^i H_{p,2i} \oplus \bigoplus_i p^{i+1} H_{p,2i+1}.$$

It is easy to see that

$$K^\perp = \bigoplus p^i H_{p,2i} \oplus \bigoplus p^i H_{p,2i+1}$$

and so $K^\perp/K \approx \bigoplus_{i\,\text{odd}} \Delta_{i,p}$. The induced form on $K^\perp/K$ is easily seen to be isometric to $\bigoplus_{i\,\text{odd}} \beta_{p,i}$. This proves Theorem 1.4. □

Proof of Theorem 1.5. If β is hyperbolic, and $H=K_1\oplus K_2$ is a hyperbolic splitting, then $H_p = K_{1p}\oplus K_{2p}$ and $\Delta_{p,i}=\Delta_{1,p,i}\oplus \Delta_{2,p,i}$. This is clearly a hyperbolic splitting for $\beta_{p,i}$.

Conversely, if $\beta_{p,i}$ is hyperbolic we show that $\beta|_{H_{p,i}}$ is hyperbolic. We can translate this into a matrix problem. Let

$$A=\begin{pmatrix} A & B \\ \varepsilon\bar{B}^{\mathrm{T}} & C \end{pmatrix}$$

be a square ε-Hermitian matrix over $\Lambda/(p^i)=R$ where A, C are divisible by p and B is non-singular over R – we must show that this matrix is congruent to one of this form with $A=C=0$. Suppose, inductively that A, C are divisible by p^k – we may also assume that $B=I$ after a congruence. Now consider a congruent matrix

$$A'=\begin{pmatrix} I & p^k X \\ 0 & I \end{pmatrix} A \begin{pmatrix} I & 0 \\ p^k \bar{X}^{\mathrm{T}} & I \end{pmatrix}.$$

The diagonal blocks in A' are

$$A' = A + p^k(\varepsilon X + \bar{X}^{\mathrm{T}}) + p^{2k}(XC\bar{X}^{\mathrm{T}})$$

and $C'=C$. We now need:

Lemma 3.2. *Let E be any field with involution. Unless* char $E=2$ *and the involution is trivial, there exist $x\in E$ such that $x+\bar{x}=1$.*

Proof. If $\xi+\bar{\xi}\neq 0$ for some $\xi\in E$, we may take $x=\xi/(\xi+\bar{\xi})$. If char $E\neq 2$, we may choose $\xi=1$; if char $E=2$ and the involution is non-trivial, then choose ξ so that $\xi\neq\bar{\xi}$. □

From this lemma it follows that any ε-Hermitian matrix over E can be written in the form $X+\varepsilon\bar{X}^{\mathrm{T}}$, for some matrix X over E. Now consider $E=\Lambda/(p)$; the conditions on F, p imposed by Theorem 1.5 correspond to the conditions of E in Lemma 3.2. Thus we may choose X over $\Lambda/(p^i)$ so that $A\equiv -p^k(X+\varepsilon\bar{X}^{\mathrm{T}}) \bmod p^{k+1}$ and, thus, A' will be divisible by p^{k+1}. We can then change A', by a congruence of a similar type, to make both diagonal blocks divisible by p^{k+1}, completing the inductive step. □

Proof of Theorem 1.6. The proof is a simpler form of the argument we just went through. A metabolic form over E has a matrix representation

$$A=\begin{pmatrix} 0 & B \\ \varepsilon\bar{B}^{\mathrm{T}} & C \end{pmatrix}.$$

We may assume $B=I$ after a congruence and use Lemma 3.2 to write $C=X+\varepsilon\bar{X}^{\mathrm{T}}$, for the same matrix X over E. Then, the following congruence:

$$\begin{pmatrix} I & X \\ 0 & I \end{pmatrix} A \begin{pmatrix} I & 0 \\ \varepsilon\bar{X}^{\mathrm{T}} & I \end{pmatrix} = \begin{pmatrix} 0 & I \\ \varepsilon I & 0 \end{pmatrix}$$

converts A to a hyperbolic form. □

4. Proofs of results in Section 2

Proof of Proposition 2.1. We prove Proposition 2.1, using methods of [11] as suggested by the referee.

Suppose, first of all that S is non-singular. Then we obtain an F-isomorphism $\tau: V \to H$ defined by $\tau(v) = (tS - \varepsilon S^{\mathrm{T}})^{-1} \cdot v$. The action of t on H corresponds, under τ, to the automorphism T of V defined by $T = \varepsilon S^{\mathrm{T}} S^{-1}$. To carry over the form β, we use the 'trace' function $\chi: Q(\Lambda)/\Lambda \to F$, defined in [11], and consider the non-singular ε-symmetric form $\beta': H \times H \to F$ defined by $\beta'(v, w) = \chi\beta(v, w)$. This corresponds, under τ, to the form $\beta'': V \times V \to F$ with matrix representative $D = (S - \varepsilon S^{\mathrm{T}})^{-1}$. Using the equation $S = (I - T)^{-1} D^{-1}$, we see that the metabolizers of S correspond, under τ, to the metabolizers of β.

Now suppose S is singular. By a sequence of elementary reductions (see [11]) S can be changed into a non-singular form without changing the associated (H, β). Thus it suffices to prove that a sequence of elementary enlargements of a non-singular metabolic (hyperbolic) Seifert form is again metabolic (hyperbolic). As an inductive hypothesis, we assume S has the form:

$$S = \begin{pmatrix} 0 & X \\ Y & N \end{pmatrix},$$

where X is non-singular and square (Y and N are square). Then an elementary enlargement of S has the form

$$\begin{pmatrix} & S & & 0 & 0 \\ & & & \vdots & \vdots \\ & & & 0 & 0 \\ 0 & \cdots & 0 & 0 & 1 \\ \xi & & \eta & 0 & 0 \end{pmatrix}$$

where ξ, η are row vectors of the same length. Using the non-singularity of X, this is congruent to

$$\begin{pmatrix} & S & & 0 & 0 \\ & & & \vdots & \vdots \\ & & & 0 & 0 \\ 0 & \cdots & 0 & 0 & 1 \\ \xi & & 0 & 0 & 0 \end{pmatrix}$$

which is then congruent to

$$\begin{pmatrix} 0 & \bar{X} \\ \bar{Y} & \bar{N} \end{pmatrix}: \quad \bar{X} = \begin{pmatrix} & X & & 0 \\ & & & \vdots \\ & & & 0 \\ 0 & \cdots & 0 & 1 \end{pmatrix}, \quad \bar{N} = \begin{pmatrix} & N & & 0 \\ & & & \vdots \\ & & & 0 \\ 0 & \cdots & & 0 \end{pmatrix}.$$

Note that $\bar{X}$ is non-singular and $\bar{N} = 0$ if $N = 0$. An induction completes the argument. The other type of elementary enlargement is handled by a similar argument. □

Proof of Proposition 2.2. S has a matrix representation

$$A = \left(\begin{array}{c|c} 0 & A_1 \\ \hline A_2 & 0 \end{array}\right)$$

where $A_1 - \varepsilon A_2^{\mathrm{T}}$ is non-singular. Then

$$A(z) = \left(\begin{array}{c|c} 0 & B(z) \\ \hline \overline{B(z)}^{\mathrm{T}} & 0 \end{array}\right) \quad \text{where } B(z) = \frac{zA_1 - \varepsilon A_2^{\mathrm{T}}}{z - 1}.$$

For any value of $z \neq 1$, $A(z)$ is a complex Hermitian matrix with zero signature; if $z = 1$, this is already observed above. □

Proof of Theorem 2.3. Given a Seifert ε-form (S, V), suppose we perform the construction of (H, β) as in Section 2, except that we replace Λ by the discrete valuation ring $\Lambda_p \subseteq Q(\Lambda)$ consisting of all fractions with denominator prime to a symmetric $p \in \Lambda$. If H_p is the resulting Λ_p-module and β_p the ε-form, which takes values in $Q(\Lambda)/\Lambda_p = S_p(\Lambda)$, then there is a natural map $H \to H_p$, which corresponds to β followed by the natural map $S(\Lambda) \to S_p(\Lambda)$.

We now need:

Lemma 3.3. *Let M be an ε-Hermitian matrix over Λ_p. Then M is congruent to a block sum of matrices $\{p^r M_r: r = 0, 1, 2, \ldots\}$ where M_r is non-singular over Λ_p.*

Proof. Choose the largest r such that every entry of M is divisible by p^r, and write $M = p^r N$. Let $\bar{N}$ be the (non-zero) ε-Hermitian matrix over $\Lambda/(p) = E$ defined by N. By a congruence over E we can convert $\bar{N}$ to a block sum of a non-singular matrix and a zero matrix. By lifting this to a congruence over Λ_p, we represent N as a block sum $M_r \oplus N'$, where M_r is non-singular over Λ_p and N' is divisible by p. An induction completes the proof. □

Note that $\gamma_S(Z) = \operatorname{sig} B(z)$, where $B(t) = (tA - \varepsilon A^{\mathrm{T}})/(t-1)$ is unchanged by any congruence of $B(t)$ over Λ_p, except at a finite number of values of z away from the roots of p. Also note that the isometry class of β_p is unaffected by this congruence.

Now $B(t)$, after a congruence over Λ_p, is a block sum of matrices $p^r M_r(t)$, where $M_r(t)$ is a non-singular matrix over Λ_p.

Suppose p is a divisor of $\Delta(t)$, with root ξ. Let ξ_+, ξ_- be values of $z \in C$ near ξ such that $p(\xi_+) > 0$, $p(\xi_-) < 0$. Then

$$\sigma_S(\xi_-) = \sum_r (-1)^r \tau_{r,p}, \qquad \sigma_S(\xi_+) = \sum_r \tau_{r,p}, \qquad \sigma_S(\xi) = \tau_{0,p},$$

where $\tau_{r,p}$ = signature $M_r(z)$, $z = \xi_+$, ξ_- or ξ (this is well-defined if ξ_+, ξ_- are near enough to ξ).

Putting $B(t)$ in the above form also makes the structure of H_p, β_p more transparent. H_p is the orthogonal direct sum of the free $\Lambda_p/p^r\Lambda_p$-module $H_{p,r}$ and

$\beta_p|_{H_{p,r}}$ has representative matrix $p^{-r}M_r$. Thus $\beta_{p,r}$ has representative matrix M_r and so $\sigma_{p,r}$ = signature $M_r(\xi)=\tau_{r,p}$ for $r\geq 1$.

The formulae of Theorem 3.2 now follow. □

Proof of Theorem 2.5. Let $A_0=\begin{pmatrix}1&1\\0&1\end{pmatrix}$ and recursively define

$$A_{n+1}=\begin{pmatrix}0 & A_n\\ A_n & B_n\end{pmatrix}$$

where B_n is to be specified. We also demand that $A_n-A_n^{\mathrm{T}}$ be non-singular over $\mathbb{Z}$, and $B_n=B_n^{\mathrm{T}}$. We consider the Seifert form S_n defined by A_n and the associated $+1$-form β_n which is defined on the Λ-module H_n. Then H_0 is the cyclic module of order $p=p(t)$ and we would like to choose B_n so that H_n is the cyclic module of order p^{2^n}. It will then follow that

$$\sigma_{q,r}(\beta_n)=\begin{cases}\pm 1, & \text{if } r=2^n,\ q=p,\\ 0, & \text{otherwise.}\end{cases}$$

By taking block sums of arbitrary numbers of $\{\pm A_n\}$ we obtain Theorem 2.5.

Suppose, inductively, that H_n is cyclic of order p^{2^n}. Since $P_n=tA_n-A_n^{\mathrm{T}}$ is a presentation-matrix of H_n we have $\det P_n=p^{2^n}$ and P_n has a minor P_n', obtained by eliminating say, the ith row and jth column, whose determinant Δ is prime to p. Now define B_n to have all entries 0 except the (i,j)-entry which is one. Consider the minor P_{n+1}' obtained by eliminating the ith row and jth column of P_{n+1}. One checks that the determinant of P_{n+1}' is $\pm(t-1)\Delta^2$, which is prime to p. This now implies that H_{n+1} is cyclic of order $p^{2^{n+1}}$ (which is $\det P_{n+1}$). □

References

[1] C. Kearton, Hermitian signatures and double null-cobordism of knots, J. London Math. Soc. 23 (1981) 563–576.

[2] C. Kearton, Simple knots which are double null-cobordant, Proc. Amer. Math. Soc. 52 (1975) 471–472.

[3] C. Kearton, Cobordism of knots and Blanchfield duality, J. London Math. Soc. 10 (1975) 406–408.

[4] J. Levine, Knot Cobordism group in codimension 2, Comment. Meth. Helv. 44 (1969) 229–244.

[5] J. Levine, Algebraic Structure of Knot Module, Lecture Notes in Mathematics 772 (Springer, Berlin, 1980).

[6] T. Matumoto, On the signature invariants of a non-singular complex sesqui-linear form, J. Math. Soc. Japan 29 (1977) 67–71.

[7] J. Milnor, On isometries of linear product spaces, Invent. Math. 8 (1969) 83–97.

[8] N. Stoltzfus, Algebraic computations of the integral concordance and double null concordance group of knots, knot theory, Proc. Knot Theory Conference: Les Plans-sur-Bex, Lecture Notes in Mathematics 685 (Springer, Berlin, 1978) 274–290.

[9] N. Stoltzfus, The Bass map in knot theory and relations among signature computations of $W(K(X),J)$, Preprint.

[10] D. Sumners, Invertible knot cobordism, Comment. Math. Helv. 46 (1971) 240–256.

[11] H. Trotter, On S-equivalence of Seifert matrices, Invent. Math. 20 (1973) 173–207.

Journal of Pure and Applied Algebra 58 (1989) 261–265
North-Holland

ON THE MORDELL–WEIL RANK OF AN ABELIAN VARIETY OVER A NUMBER FIELD

Takeshi OOE

Department of Mathematics, Faculty of Science, University of Tokyo, Hongo, Tokyo, 113 Japan

Jaap TOP*

Rijksuniversiteit Utrecht, Mathematisch Instituut, Postbus 80.010, 3508 TA Utrecht, The Netherlands

Communicated by F. Oort
Received 22 March 1988
Revised 20 May 1988

Let K be a number field and A an abelian variety over K. The K-rational points of A are known to constitute a finitely generated abelian group (Mordell–Weil theorem). The problem studied in this paper is to find an explicit upper bound for the rank r of its free part in terms of other invariants of A/K. This is achieved by a close inspection of the classical proof of the so-called 'weak Mordell–Weil theorem'.

1. Introduction

Let K be a number field and A an abelian variety over K. The K-rational points of A are known to constitute a finitely generated abelian group (Mordell-Weil theorem) and it is an interesting question to give an explicit upper bound for the rank r of its free part in terms of other invariants of A/K.

In case A is an elliptic curve and $K=\mathbb{Q}$ there are already some theorems in this direction. For example, Tate proved the following (cf. [2, Chapter 6]):

"Let E be an elliptic curve over $\mathbb{Q}$ given by an equation $y^2=x^3+ax^2+bx$ with $a,b\in\mathbb{Z}$. Then $r\leq s+t+1$ where s and t are the numbers of prime divisors of b and a^2-4b respectively. (Note that the discriminant of this model of E is $2^4b^2(a^2-4b)$.)"

A somewhat sharper bound for elliptic curves over $\mathbb{Q}$ having $\mathbb{Q}$-rational (not necessarily 2-) torsion points can be found in [5], and for elliptic curves over $\mathbb{Q}$ having no rational 2-torsion points a similar bound is obtained in [1].

Under the assumption of very powerful conjectures (Birch and Swinnerton–Dyer, Taniyama–Weil and the generalized Riemann hypothesis), Mestre proves in [6] and

* Supported by the Dutch Organization for the Advancement of Pure Research (Z.W.O.).