Actes, Congrès intern. Math., 1970. Tome 2, p. 95 à 98.

THE ROLE OF THE SEIFERT MATRIX IN KNOT THEORY

by J. LEVINE

1. I will be interested in the following situation.

$K^n \subset \mathbf{R}^{n+2}$ is an imbedded oriented sphere (topological) – the imbedding may be smooth or *PL* locally flat. If n is odd, say $n = 2q - 1$, one can associate to K^n a matrix, called the *Seifert matrix*, in the following way. Let $M^{n+1} \subset \mathbf{R}^{n+2}$ be a submanifold bounded by K^n – one always exists. Then define a pairing Φ : $Hq(M) \otimes Hq(M) \to Z$ as follows. If $\alpha, \beta \in Hq(M)$, choose representative cycles α', β'. Translate α' off M in the positive normal direction (defined from the orientation of K) and define $\Phi(\alpha \otimes \beta)$ to be the linking number of the translated α' with β'. Φ is bilinear and satisfies : $\Phi + (-1)^q \Phi^T = - B$ where Φ^T is the transpose of Φ and B is the intersection pairing of M. Any representative matrix A of Φ is called a Seifert matrix of K. $A \pm A^T$ is unimodular, since B is. Conversely any A satisfying this property is the Seifert matrix of a knot – unless $n = 3$, in which case the additional condition – signature $(A + A^T) \equiv 0 \bmod 16$ – is needed and even then it is not quite true. See [1] for details.

Naturally K does not determine a unique Seifert matrix – any congruent matrix is clearly also a Seifert matrix. But even more so, the size of A can be changed by altering M. For example adding a handle to M of index q which links the other q-cycles in M will enlarge A to one of the forms :

$$\left(\begin{array}{c|cc} A & \xi & 0 \\ \hline 0 & 0 & 1 \\ & 0 & 0 \end{array}\right) \quad \text{or} \quad \left(\begin{array}{c|cc} A & 0 & \\ \hline \xi & 0 & 0 \\ & 1 & 0 \end{array}\right)$$

where ξ is, respectively, a column or row vector. We call these, respectively, right and left enlargements.

These types of enlargements, together with congruence, generate an equivalence relation we call *S-equivalence* (in [1], it is called *equivalence*). This was first considered by Trotter [3] and Murasugi [2].

THEOREM [1]. – *Any two Seifert matrices of a knot are S-equivalent.*

This was proved by Murasugi [2], when $n = 1$. The proof proceeds by considering a cobordism $V \subset I \times \mathbf{R}^{n+2}$ between two choices of M, stationary on K. By considering a handle-body decomposition of V, the transition between the Seifert matrices can be broken down into steps of the sort used to define S-equivalence.

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(1) This work was done while the author was partially supported by NSF GP 21510.

Thus S-equivalence is the correct relation on Seifert matrices. It cannot be expected that the S-equivalence class of its Seifert matrices is a complete invariant of knot type, since it contains no information pertaining to other than the middle dimension. On the other hand, it seems to contain all the information on the middle dimensional behavior.

THEOREM [1]. – *Two knots with S-equivalent Seifert matrices are of the same knot type if they satisfy the conditions :* (1) $\Pi_i(\mathbf{R}^{n+2} - K) \approx \Pi_i(S^1)$ *for* $i < q$, (i.e. *the complement looks like a circle up to dimension* q) *and* (2) $n > 1$.

2. We now look at the algebraic problem.

One problem is the variable size of the matrix. There are two ways of dealing with such a situation. The usual approach is to *stabilize* ie. consider infinite matrices. This does not seem to be of much use here. Another approach is to find minimal representatives and restrict attention to these. This seems to be more fruitful.

LEMMA [3]. – Any matrix, satisfying $A \pm A^T$ unimodular, is S-equivalent to a non-singular matrix. Moreover the rank and determinant are invariants of the S-equivalence class.

This is the algebraic analogue of a minimal spanning surface of a knot of dimension one, ie. a surface of minimal genus. Actually a minimal spanning surface need not give rise to a non-singular Seifert matrix.

We may now restrict our attention to non-singular matrices of a fixed rank. One question that arises is whether S-equivalence may coincide with congruence (among non-singular matrices). This corresponds algebraically to asking whether minimal spanning surfaces are unique. We shall see the answer is *NO*. One approach to this problem is to find ways of generating all the matrices S-equivalent to a given one (or rather congruence classes) in a finite number of steps, and to recognize when we are through. We show how this can be done. Proofs will appear in a future work.

The first step is :

THEOREM 1. – *Any two S-equivalent non-singular matrices can be joined by a sequence of the following two types of moves :*

(i) *right enlargement, then left reduction*

(ii) *left enlargement, then right reduction.*

Moreover, we can do all of type (i) *first and then all of type* (ii).

Thus we never have to deal with matrices much larger than the original one.

The next step would be to examine a single move of the type (i) or (ii) and be able to write down all the matrices obtained by such a move from a given one. A priori this may seem improbable since the vectors ξ used in the enlargment may vary over an infinite number of choices. But, in fact, only a finite number of distinct (up to congruence) enlargements occur and these can be constructed in a finite number of steps.

Suppose A has rank r. Consider the free abelian group of rank r, written as column vectors, and the subgroup generated by the columns of A. Let the quotient group be denoted $V(A)$. Then $V(A)$ is a finite group with det A elements. Let $O(A) = \{P \text{ unimodular} : PAP' = A\}$ be the orthogonal group of A. Then $O(A)$ acts on $V(A)$ by left multiplication. Given A clearly one can completely write down this situation.

THEOREM 2. – *If ξ, η are two column vectors, the right enlargements of A given by ξ and η are congruent if and only if the corresponding elements of $V(A)$ lie in the same orbit of $O(A)$.*

Thus we can write down all the enlargements of A. To handle the reductions :

LEMMA. – Two non-singular right (left) reductions of a matrix are congruent.

Thus we can effectively write down all the matrices obtained from A by one step of the form (i) or (ii) in Theorem 1.

One useful observation one can already make at this stage is :

COROLLARY [3]. – Two *unimodular* matrices are S-equivalent if and only if they are already congruent.

This follows immediately from the above results and $V(A) = 0$. For example fibered knots have unimodular Seifert matrices.

As illustration I would like to give some examples.

Example 1. – $A = \begin{pmatrix} 2 & 1 \\ 0 & 3 \end{pmatrix}$ – this seems to be the simplest non-trivial example.

$V(A) \approx Z_6$ generated by $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$. $O(A) = \pm I$. Thus there are four orbits and right enlargement gives four different matrices. Upon left reduction this reduces to three : A together with $\begin{pmatrix} 2 & 0 \\ 1 & 3 \end{pmatrix} = A'$ and $\begin{pmatrix} 1 & 1 \\ 0 & 6 \end{pmatrix}$, all of which are non-congruent. Repeating these steps always yields the same three matrices. Since A' appears here, we have also considered left enlargement and right reduction of A. It is easy to then conclude that these matrices comprise the S-equivalence class of A (up to congruence).

Example 2. – $\begin{pmatrix} a\lambda^2 & 1 \\ 0 & b \end{pmatrix}$ is S-equivalent to $\begin{pmatrix} a & 1 \\ 0 & b\lambda^2 \end{pmatrix}$ – by right enlargement by $\begin{pmatrix} 0 \\ \lambda \end{pmatrix}$ – but they are congruent only if $\lambda = \pm 1$ or $a = b = 1$. This gives examples of S-equivalence classes containing arbitrarily large numbers of congruence classes. For example $\begin{pmatrix} x^{2i} & 1 \\ 0 & x^{2j} \end{pmatrix}$ for all $i + j = k$ gives k non-congruent, but S-equivalent, matrices, for $x > 1$.

Finally we would like to know how many steps are needed to obtain *all* matrices S-equivalent to A. We must first ask whether it is finite i.e. are there only a finite number of matrices (up to congruence) S-equivalent to A ? If so, how many ?

Of course we can tell when we are done by noticing that no new matrices are produced at a given step, but it would be nicer to have a number given a priori from A which would serve as an upper bound for the number of steps required. This would also give an upper bound a priori for the number of matrices (up to congruence) S-equivalent to A.

THEOREM 3. – If B is obtained from A by a sequence of steps of type (i) in Theorem 1, then no more than (rank A) of such steps are needed. Similarly for steps of type (ii).

COROLLARY. – If $d = |\det A|$ and $r = \text{rank } A$, then there are at most d^{2r} congruence classes of non-singular matrices S-equivalent to A.

Question. – Is A always S-equivalent to A' ?

This is related to the existence of non-invertible knots of dimensions > 1.

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